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Howe correspondence and Springer correspondence for dual pairs over a finite field

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Abstract. We study the Howe correspondence for the unipotent representations of the irreducible dual reductive pairs $(\mathrm{G}', \mathrm{G}) = (\mathrm{GL}_{n'}(\mathbb{F}_q), \mathrm{GL}_n(\mathbb{F}_q))$ with $n' \leq n$, and $(\mathrm{G}', \mathrm{G}) = (\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{O}^+_{2n})(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field with q elements (q odd), and O^+_{2n} is the $\mathbb{F}_q$-split orthogonal group. We show how to extract a "preferred" irreducible representation of G from the image by the (conjectural in the second case) correspondence of a given irreducible representation of G'.

Contents

1. Introduction

In case of a dual pair $(\mathrm{G}', \mathrm{G})$ defined over a finite field, the integral

$$\int_{\mathrm{G}'} \Theta(g'g)\, \Theta_{\Pi'^c}(g')\, dg' \qquad (g \in \mathrm{G}),$$

where Θ is the character of the Weil representation and Π'^c is the representation contragredient to Π', is a finite sum which obviously converges and defines a class function on G. This class function decomposes into a sum of several irreducible characters Θ_Π. In other words Howe correspondence often does not associate a single irreducible representation of G to a given irreducible representation Π' of G' and the situation is quite complex.

Then the following question arises naturally: is there a "preferred" representation among the irreducible representations of G which correspond to Π'? It is the aim of this article to propose a candidate for such a preferred irreducible representation, assuming that Π' is unipotent.

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Let $\mathbb{F}_q$ be a finite field of q elements of characteristic p. As a consequence of our main result (assuming here for the simplicity of the exposition that p of is large enough), we obtain that, in the following situations

(1) the dual pair is of type II, *i.e.*, $(\mathrm{G}', \mathrm{G}) = (\mathrm{GL}_{n'}(\mathbb{F}_q), \mathrm{GL}_n(\mathbb{F}_q))$, and Π' is unipotent;
(2) the dual pair is $(\mathrm{G}', \mathrm{G}) = (\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{O}_{2n}^+(\mathbb{F}_q))$, where $\mathrm{O}_{2n}^+(\mathbb{F}_q)$ denotes the split orthogonal group, and Π' is unipotent and belongs to the principal series of G',

the preferred representation is an irreducible representation Π_{pref} of G that corresponds to Π' by Howe correspondence and is the unique such representation the wave front set of which contains the wave front set of any irreducible representation of G which correspond to Π' (see Corollary 14).

More generally, we consider an irreducible dual pair $(\mathrm{G}', \mathrm{G})$ over $\mathbb{F}_q$, with p odd (without further assumption on it). As shown in [**AM93**], Howe correspondence for this pair induces a (non-bijective) correspondence between unipotent representations of G' and G. This correspondence between unipotent representations has been described in [**AMR96**, Théorème 5.5] in the case of $(\mathrm{G}', \mathrm{G}) = (\mathrm{GL}_{n'}(\mathbb{F}_q), \mathrm{GL}_n(\mathbb{F}_q))$. Recall that unipotent representations of G' are parametrized by partitions of n'. Assume that $n' \leq n$. We will prove that the unipotent representation of G, say Π_{pref}, that is parametrized by the *joint partition* $\mu' \cup (n - n')$ (see Definition 1), occurs in the image by the correspondence of the representation of G', say Π', that is parametrized by μ'. Moreover, every representation Π of G which occurs in the image of Π' is parametrized by a partition of n which is larger than $\mu' \cup (n - n')$ for the usual order on partitions. It follows that the closure of the unipotent support of each such representation Π contains the unipotent support of Π_{pref}.

In the case of ortho-symplectic dual pairs, the correspondence between unipotent representations has been described conjecturally in [**AMR96**, Conjecture 3.11] in terms of a (in general non-bijective) correspondence between irreducible representations of two Weyl groups.

In [**KS05**, Theorem 5.15], Kable and Sanat have proved the validity of the conjecture for the dual pair $(\mathrm{Sp}_4(\mathbb{F}_q), \mathrm{SO}_{2n}^+(\mathbb{F}_q))$ in the case of unipotent representations that belong to the principal series. Let $\mathrm{O}_{2n}^-(\mathbb{F}_q)$ denote the non-split orthogonal group, and let $\varepsilon = \pm$. The conjecture for the dual pair $(\mathrm{Sp}_{2n'}(\mathbb{F}_q), \mathrm{O}_{2n}^\varepsilon(\mathbb{F}_q))$ has been also confirmed computationally in [**AMR96**] for $n, n' \leq 11$ up-to a slight ambiguity in the principal series case $\mathrm{O}_{2n}^\varepsilon(\mathbb{F}_q)$ (in which case we have only the restriction of Weil Representation to $\mathrm{Sp}_{2n'}(\mathbb{F}_q) \cdot \mathrm{SO}_{2n}^\varepsilon(\mathbb{F}_q)$).

We will compute explicitly that correspondence between representations of Weyl groups in the case where one of them is $\mathsf{W}_2 = \mathsf{W}(\mathrm{B}_2)$ (see Proposition 8), give its translation into a correspondence between u-symbols and extract a bijective correspondence which behaves well with respect to unipotent classes (see Theorem 10).

For instance, for unipotent representations in the principal series of split groups, assuming the validity of the conjectural description of the correspondence in this case, and that we have $n \geq 3$ and $n' = 2$, we prove that the representation of G, say Π_{pref}, that is parametrized by the pair of partitions $(\xi', \eta' \cup (n - n'))$ of n, occurs in the image by the correspondence of the representation of G', say Π', that is parametrized by the pair of partitions (ξ', η') of n'. Moreover, every representation of G which occurs in the image of Π' is such that the closure of its unipotent support contains the unipotent support of Π_{pref} (see Corollary 12).

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2. Unipotent representations

Let G be the group of $\mathbb{F}_q$-rational points of a connected algebraic reductive group $\mathbf{G}$ over $\bar{\mathbb{F}}_q$, defined over $\mathbb{F}_q$, and let $F\colon \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius map so that $\mathrm{G} = \mathbf{G}^F$ (fixed points by F).

To each G-conjugacy class of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an F-stable maximal torus in $\mathbf{G}$ and θ is an irreducible character of $T = \mathbf{T}^F$, Deligne and Lusztig attached a virtual character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of G, [**DL76**].

Recall that the *uniform class functions* on G are, by definition, the complex linear combinations of Deligne-Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. Recall also that an irreducible representation of G is called *unipotent* if its character has non-zero scalar product with $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some $\mathbf{T}$. If $\mathrm{G} = \mathrm{GL}_n(\mathbb{F}_q)$, then the uniform class functions span the space of all class functions on G. For G arbitrary, it is not the case in general: for instance, the character of the cuspidal unipotent representation θ_{10} of the symplectic group $\mathrm{Sp}_4(\mathbb{F}_q)$ defined by Srinivasan is not uniform.

Because we will need to include the case of orthogonal groups, it is necessary to extend the definition of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ to the case when $\mathbf{G}$ is a disconnected reductive algebraic group. In this case, we put $R_{\mathbf{T}}^{\mathbf{G}}(\theta) := \mathrm{Ind}_{\mathrm{G}^\circ}^{\mathrm{G}}(R_{\mathbf{T}}^{\mathbf{G}^\circ}(\theta))$, where $\mathbf{G}^\circ$ denotes the identity connected component of $\mathbf{G}$ and $\mathrm{G}^\circ := (\mathbf{G}^\circ)^F$. We will call *uniform class functions* all the linear combinations of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. An irreducible representation of G is called *unipotent* if its character has non-zero scalar product with $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some $\mathbf{T}$.

Since the cyclic group ${\mathbb{F}_q}^*$ is of even order, $|{\mathbb{F}_q}^*/{\mathbb{F}_q}^{*2}| = 2$. Therefore there are exactly two non-equivalent non-degenerate symmetric bilinear forms on the vector space ${\mathbb{F}_q}^{2n}$, see [**Jac74**, Theorem 6.9], one is split and the other one is not split. Let $\mathrm{O}_{2n}^+(q) = \mathrm{O}_{2n}^+(\mathbb{F}_q)$ (resp. $\mathrm{O}_{2n}^-(q) = \mathrm{O}_{2n}^-(\mathbb{F}_q)$) denotes the corresponding split (resp. non-split) orthogonal group. See [**DM91**, sec. 15.3] for more details. Also, we shall write $\mathrm{Sp}_{2n}(q) := \mathrm{Sp}_{2n}(\mathbb{F}_q)$.

We recall some results from [**Lus80**]. The group $\mathrm{Sp}_{2n}(q)$ has a unipotent cuspidal irreducible representation if and only if n is a triangular number, that is, $n = k^2+k$ for some $k \in \mathbb{N}$. The group $\mathrm{Sp}_{2(k^2+k)}(q)$ has a unique unipotent cuspidal representation. Similarly, the group $\mathrm{SO}_{2n}^\varepsilon(q)$, with $\varepsilon \in \{-,+\}$, has a unipotent cuspidal irreducible representation if and only if n is a square, that is, $n = k^2$ for some $k \in \mathbb{N}$. The group $\mathrm{SO}_{2k^2}^\varepsilon(q)$ has a unique unipotent cuspidal representation, say Π_k. It follows that $\mathrm{O}_{2n}^\varepsilon(q)$ admits unipotent cuspidal representations if and only if $n = k^2$ for some $k \in \mathbb{N}$, and that $\mathrm{O}_{2k^2}^\varepsilon(q)$ has exactly two unipotent cuspidal representations, Π_k^{I} and Π_k^{II}. (Indeed, we have $\mathrm{Ind}_{\mathrm{SO}_{2k^2}^\varepsilon(q)}^{\mathrm{O}_{2k^2}^\varepsilon(q)} \Pi_k = \Pi_k^{\mathrm{I}} + \Pi_k^{\mathrm{II}}$. Both Π_k^{I} and Π_k^{II} have the same restriction to $\mathrm{SO}_{2k^2}^\varepsilon(q)$ and thus differ by tensoring with the determinant character of $\mathrm{O}_{2k^2}^\varepsilon(q)$.) See [**Lus77**, Theorem 8.2] or [**AM93**, Theorem 5.1] for the details.

Let $\mathbf{M} := \mathrm{Sp}_{2(k^2+k)} \times \mathbf{T}$ (resp. $\mathbf{M} := \mathrm{O}_{2k^2}^\varepsilon \times \mathbf{T}$), where $\mathbf{T}$ is a split torus of $\mathbf{G} = \mathrm{Sp}_{2n}$ (resp. $\mathbf{G} = \mathrm{O}_{2n}^\varepsilon$), and let $\Pi^{\mathbf{M}}$ be a unipotent cuspidal irreducible representation of $\mathbf{M}$. The representation $\Pi^{\mathbf{M}}$ is the tensor product of the unipotent cuspidal representation of $\mathrm{Sp}_{2(k^2+k)}$ (resp. Π_k^{I} or Π_k^{II}) with the trivial representation of $\mathbf{T}$. On the other hand, $\mathbf{M}$ is an $\mathbb{F}_q$-rational Levi subgroup of an $\mathbb{F}_q$-rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ and the commuting algebra $\mathrm{End}_{\mathrm{G}}(\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\Pi^{\mathbf{M}}))$ (where the cuspidal representation $\Pi^{\mathbf{M}}$ of $\mathbf{M}$ is trivially extended to the unipotent radical of $\mathbf{P}$, that is, $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\Pi^{\mathbf{M}})$, also denoted by $R_{\mathbf{M}}^{\mathbf{G}}(\Pi^{\mathbf{M}})$, is the usual Harish-Chandra induction) is an Iwahori-Hecke algebra of type $\mathrm{B}_{\tilde{n}}$, with $\tilde{n} := n - (k^2 + k)$ (resp. $\tilde{n} := n - k^2$), see for instance [**Lus80**] or [**AMR96**, § 3.A]. Hence the irreducible representations of G which occur in $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\Pi^{\mathbf{M}})$ are in bijection with $\mathrm{Irr}(\mathsf{W}_{\tilde{n}})$, where $\mathsf{W}_{\tilde{n}} = \mathsf{W}(\mathrm{B}_{\tilde{n}}) = (\mathbb{Z}/2\mathbb{Z})^{\tilde{n}} \rtimes \mathfrak{S}_{\tilde{n}}$ (*cf.* [**Car93**, Chapter 10] or [**Lus84**, Corollary 8.7]).

We will denote by $\Pi^{\mathrm{G}}_{\Pi^{\mathbf{M}},\rho}$ the irreducible representation of G which corresponds to $\rho \in \mathrm{Irr}(\mathsf{W}_{\tilde{n}})$ by this bijection.

We put

$$\mathrm{Sp} := \{\mathrm{Sp}_{2n}(q) \, : \, n \in \mathbb{N}\} \quad \text{and} \quad \mathrm{O}^{\varepsilon} := \{\mathrm{O}^{\varepsilon}_{2n}(q) \, : \, n \in \mathbb{N}\}.$$

We call Sp (resp. O^{ε}) a *Witt tower of symplectic* (resp. *orthogonal*) *type*. Let $\mathcal{T}$, $\mathcal{T}'$ be two Witt towers, one is of symplectic type and the other one is of orthogonal type.

For a finite group H let $\mathfrak{R}(H)$ denote the free abelian group generated by the irreducible characters of H. Thus the subset of the irreducible characters $\mathrm{Irr}(H) \subseteq \mathfrak{R}(H)$ is a base of $\mathfrak{R}(H)$ over $\mathbb{Z}$. Let $\mathrm{G}'_{m'}$ be an element of $\mathcal{T}'$ and let G_m be an element of $\mathcal{T}$. Denote by $\omega_{m',m}$ the projection onto the space of the uniform class functions on $\mathrm{G}'_{m'} \times \mathrm{G}_m$ of the pullback of the character of the oscillator representation (determined by one fixed character of the field $\mathbb{F}_q$) via the map $\mathrm{G}'_{m'} \times \mathrm{G}_m \ni (g', g) \to g'g \in \mathrm{Sp}_{4m'm}(q)$. By Howe correspondence for the dual pair $(\mathrm{G}'_{m'}, \mathrm{G}_m)$ we shall understand the map

$$\theta^{\mathrm{G}_m} \colon \mathfrak{R}(\mathrm{G}'_{m'}) \to \mathfrak{R}(\mathrm{G}_m) \tag{1}$$

defined by

$$\omega_{m',m} = \sum_{\Pi' \in \mathrm{Irr}(\mathrm{G}'_{m'})} \Pi' \otimes \theta^{\mathrm{G}_m}(\Pi'). \tag{2}$$

(See [**AMR96**, (1.4)], where $\theta^{\mathrm{G}_m}(\Pi')$ was denoted by $\Theta_{\mathrm{G}_m}(\Pi')$.)

Let Π' be a cuspidal irreducible representation of an element $\mathrm{G}'_{m'}$ of $\mathcal{T}'$. Then there exists $\mathrm{G}_m \in \mathcal{T}$ such that $\theta^{\mathrm{G}_m}(\Pi')$ is a cuspidal irreducible representation of G_m, see [**AMR96**, Theorem 3.7]. Moreover (see *loc. cit.*), the image by Howe correspondence for the dual pair $(\mathrm{G}'_{m'+l'}, \mathrm{G}_{m+l})$, with $l', l \in \mathbb{N}$, of each component of the Harish-Chandra parabolic induced representation $R^{\mathbf{G}'_{m'+l'}}_{\mathbf{G}'_{m'} \times \mathbf{T}'}(\Pi')$ (where $\mathbf{T}'$ is a split torus in $\mathbf{G}'_{m'+l'}$) belongs to the Harish-Chandra parabolic induced representation $R^{\mathbf{G}_{m+l}}_{\mathbf{G}_m \times \mathbf{T}}(\theta^{\mathrm{G}_m}(\Pi'))$ (where $\mathbf{T}$ is a split torus in $\mathbf{G}_{m+l}$).

Using the description of the uniform part of the restriction of $\mathrm{G}_m \times \mathrm{G}'_{m'}$ of the Weil representation obtained by Srinivasan in [**Sri79**], Adams and Moy proved that Howe correspondence sends unipotent representations to unipotent representations, [**AM93**, Theorem 3.5], and that the unique cuspidal unipotent representation of the group $\mathrm{Sp}_{2(k^2+k)}(q)$ corresponds to the representation Π^{II}_k of $\mathrm{O}^{\varepsilon}_{2k^2}(q)$ if ε is the sign of $(-1)^k$ and to the representation Π^{I}_{k+1} of $\mathrm{O}^{\varepsilon}_{2(k+1)^2}(q)$, where ε is the sign of $(-1)^{k+1}$, otherwise, [**AM93**, Theorem 5.2]. (In fact this defines the representations Π^{I}_k and Π^{II}_k.)

3. Dual pairs of type II

In this section we will consider the case of the dual pair $(\mathrm{G}', \mathrm{G}) = (\mathrm{GL}_{n'}(q), \mathrm{GL}_n(q))$. We assume that $n' \leq n$. The characters of the unipotent irreducible representations of G are in bijection with the irreducible characters of the symmetric group $\mathfrak{S}_n$, hence in bijection with the partitions of n. Let ρ_μ denote the irreducible representation of $\mathfrak{S}_n$ which corresponds to the partition μ of n. Define

$$R_{\rho_\mu} := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \rho_\mu(\sigma)\, R_{\mathbf{T}_\sigma}^{\mathrm{GL}_n}(1), \tag{3}$$

where $\mathbf{T}_\sigma$ is a maximal torus of type σ. Then R_{ρ_μ} is a unipotent irreducible character of G, and each such character is of this form for some partition μ of n (see for instance [**DM91**, sec. 15.4]).

Recall that a *partition* of a positive integer n is a finite sequence $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0]$ of integers λ_i such that $\sum_{i=1}^k \lambda_i = n$. Let $\mathrm{ht}(\lambda)$ denote the *hight* of the partition λ (that is, the largest i with $\lambda_i \neq 0$). Flipping a Young diagram of a partition λ of n over its main diagonal (from upper left to lower right), we obtain the Young diagram of another partition ${}^t\lambda$ of n, which is called the *conjugate partition* of λ. Thus, for $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k]$, we have ${}^t\lambda = [{}^t\lambda_1 \geq {}^t\lambda_2 \geq \cdots \geq {}^t\lambda_l]$, where $l = \lambda_1$ and ${}^t\lambda_j = |\{i : 1 \leq i \leq k, \lambda_i \geq j\}|$ for $1 \leq j \leq l$.

If $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k]$ and $\mu = [\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h]$ are any partitions, we write $\mu \subset \lambda$ if the followings holds: $\mathrm{ht}(\mu) \leq \mathrm{ht}(\lambda)$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \mathrm{ht}(\mu)$. If we identify λ and μ with their Young diagrams, this means that the diagram of μ is contained in those of λ. Removing the boxes of λ which belong to μ, we obtain a *skew diagram* which we denote by $\lambda - \mu$.

We will also need to consider the *intersection partition* of λ and μ:

$$\lambda \cap \mu := \left[\min(\lambda_1, \mu_1), \ldots, \min(\lambda_{\min(k,h)}, \mu_{\min(k,h)})\right].$$

We have $\mu \subset \lambda$ if and only if $\lambda \cap \mu = \mu$.

Let $\nu = [\nu_1 \geq \nu_2 \geq \cdots \geq \nu_m] \subset \lambda \cap \mu$. Then we denote by $p_{\lambda=\mu}(\nu)$ the partition $(\nu_i)_{\{i : \lambda_i = \mu_i\}}$ and we put

$$\lambda \cap^= \mu := p_{\lambda=\mu}(\lambda \cap \mu).$$

We will say that λ and μ are *close* if for each i we have $|\lambda_i - \mu_i| \leq 1$.

For later use, we will now introduce some more notation. If $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k]$ is a partition of n and $\mu = [\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h]$ is a partition of m, by adding zero parts if necessary we can assume that $h = k$, and we define the partition $\lambda \oplus \mu$ of $n + m$ as

$$(\lambda \oplus \mu)_i := \lambda_i + \mu_i, \quad \text{for } 1 \leq i \leq k.$$

For a partition λ and for any integer i, let $n_i(\lambda)$ be the numbers of $j \geq 1$ such that $\lambda_j = i$.

Definition 1. *Let $\lambda \cup \mu$ be the unique partition of $n + m$ such that*

$$n_i(\lambda \cup \mu) = n_i(\lambda) + n_i(\mu), \quad \textit{for each } i \geq 1.$$

We observe that ${}^t(\lambda \oplus \mu) = {}^t\lambda \cup {}^t\mu$ and ${}^t(\lambda \cup \mu) = {}^t\lambda \oplus {}^t\mu$. Also, for any $L \in \mathrm{N}$, we have $(L) \cup \mu = [\mu_1 \geq \cdots \geq L \geq \cdots \geq \mu_l]$ (or $[L \geq \mu_1 \geq \cdots \geq \mu_l]$ or $[\mu_1 \geq \cdots \geq \mu_l \geq L]$).

Consider $\mathfrak{R}(\mathfrak{S}) := \bigoplus_{n\geq 0} \mathfrak{R}(\mathfrak{S}_n)$ (it is a free $\mathbb{Z}$-module with basis $\bigcup_{n\geq 0} \mathrm{Irr}(\mathfrak{S}_n)$), and define a map $\theta^{\mathfrak{S}} : \mathfrak{R}(\mathfrak{S}) \to \mathfrak{R}(\mathfrak{S})$ by

$$\rho_{\mu'} \mapsto \sum_{{}^t\mu \text{ close to } {}^t\mu'} f({}^t\mu' \cap^{=} {}^t\mu)\, \rho_\mu,$$

where, if $\nu = [r^{a_1}, \ldots, 1^{a_r}]$, we have put $f(\nu) = \prod_i a_i$, and where the empty partition is sent by f to 1 (in accordance with [**AMR96**, proof of Lemma 5.4]).

Theorem 2. [**AMR96**, Théorème 5.5] *Howe correspondence between unipotent characters of* $\mathrm{GL}_{n'}(q)$ *and* $\mathrm{GL}_n(q)$ *is given by the map*

$$R^{\mathrm{GL}_{n'}}_{\rho_{\mu'}} \mapsto R^{\mathrm{GL}_n}_{\theta^{\mathfrak{S}}(\rho_{\mu'})}.$$

The following result is a direct consequence of Theorem 2.

Theorem 3. *Let* (n', n) *be a pair of positive integers with* $n' \leq n$. *Let* μ' *be a partition of* n'. *The unipotent representations of* $\mathrm{GL}_{n'}(q)$ *and* $\mathrm{GL}_n(q)$ *with characters* $R_{\rho_{\mu'}}$ *and* $R_{\rho_{\mu'\cup(n-n')}}$, *respectively, correspond by Howe correspondence.*

Moreover, any representation of $\mathrm{GL}_n(q)$ *which belongs to the image of* $R_{\rho_{\mu'}}$ *by Howe correspondence is of the form* R_{ρ_μ} *where* $\mu \geq (\mu' \cup (n-n'))$, *where* $\geq$ *denotes the usual order on partitions.*

Proof. We note that the unipotent representation $R_{\rho_{\mu'\cup(n-n')}}$ occurs in the image of $R_{\rho_{\mu'}}$ by Howe correspondence. Indeed we have

$${}^t(\mu' \cup (n-n')) = {}^t(\mu') \oplus 1^{n-n'},$$

the partitions ${}^t(\mu')$ and ${}^t(\mu' \cup (n-n'))$ are close, and, since

$${}^t(\mu') \cap^{=} {}^t(\mu' \cup (n-n')) = \emptyset,$$

we have $f({}^t(\mu') \cap^{=} {}^t(\mu' \cup (n-n'))) = 1$.

Now, ${}^t(\mu' \cup (n-n'))$ is the largest partition in the set of partitions of n which are close to ${}^t(\mu')$. Hence, if R_{ρ_μ} belongs to the image of $R_{\rho_{\mu'}}$ by Howe correspondence we have ${}^t\mu \leq {}^t(\mu' \cup (n-n'))$, *i.e.*, $\mu \geq \mu' \cup (n-n')$. $\square$

4. Ortho-symplectic dual pairs

4.1. A correspondence between Weyl groups. Let (n_1, n_2) be a pair of positive integers. Let k be an integer such that $0 \leq k^2 + k \leq n_1$ and $k^2 \leq n_2$, and let Π_k^{sp} denote the unipotent cuspidal representation of $\mathrm{Sp}_{2(k^2+k)}$. We denote by ε_k the sign of $(-1)^k$.

It follows that:

- Howe correspondence for the dual pair $(\mathrm{Sp}_{2n_1}(q), \mathrm{O}^{\varepsilon_k}_{2n_2}(q))$ induces a correspondence between irreducible components of $R^{\mathbf{Sp}_{2n_1}}_{\mathbf{Sp}_{2(k^2+k)}\times\mathbf{T}}(\Pi_k^{\mathrm{sp}} \otimes 1)$ and irreducible components of $R^{\mathbf{O}^{\varepsilon_k}_{2n_2}}_{\mathbf{O}^{\varepsilon_k}_{2k^2}\times\mathbf{T}}(\Pi_k^{\mathrm{II}} \otimes 1)$,
- Howe correspondence for the dual pair $(\mathrm{Sp}_{2n_1}(q), \mathrm{O}^{\varepsilon_{k+1}}_{2n_2}(q))$ induces a correspondence between irreducible components of $R^{\mathbf{Sp}_{2n_1}}_{\mathbf{Sp}_{2(k^2+k)}\times\mathbf{T}}(\Pi_k^{\mathrm{sp}} \otimes 1)$ and irreducible components of $R^{\mathbf{O}^{\varepsilon_{k+1}}_{2n_2}}_{\mathbf{O}^{\varepsilon_{k+1}}_{2(k+1)^2}\times\mathbf{T}}(\Pi_{k+1}^{\mathrm{I}} \otimes 1)$.

All these irreducible components are unipotent, see [**Lus84**, (8.5.1)].

Let $k_2 \in \{k, k+1\}$. We set

$$\Pi^{\mathrm{or}}_{k_2} := \begin{cases} \Pi^{\mathrm{II}}_{k} & \text{if } k_2 = k, \\ \Pi^{\mathrm{I}}_{k+1} & \text{if } k_2 = k+1, \end{cases} \tag{4}$$

$$\tilde{n}_1(k) := n_1 - (k^2 + k), \quad \tilde{n}_2(k_2) := n_2 - (k_2)^2. \tag{5}$$

Then Howe correspondence for the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_{2n_1}(q), \mathrm{O}^{\varepsilon_{k_2}}_{2n_2}(q))$ induces a correspondence, $\Theta^{\mathrm{G},\mathrm{G}'}_k$, between $\mathrm{Irr}(\mathsf{W}_{\tilde{n}_1(k)})$ and $\mathrm{Irr}(\mathsf{W}_{\tilde{n}_2(k_2)})$, defined as follows.

Definition 4. *We will say that the representations $\rho \in \mathrm{Irr}(\mathsf{W}_{\tilde{n}_1(k)})$ and $\rho' \in \mathrm{Irr}(\mathsf{W}_{\tilde{n}_2(k_2)})$ correspond by $\Theta^{\mathrm{G},\mathrm{G}'}_k$ if the character of $\Pi^{\mathrm{G}}_{\Pi^{\mathrm{sp}}_k \otimes 1, \rho} \otimes \Pi^{\mathrm{G}'}_{\Pi^{\mathrm{or}}_{k_2} \otimes 1, \rho'}$ has a non-zero scalar product with ω_{n_1,n_2}.*

In particular, taking $k = k_2 = 0$, we obtain a correspondence between $\mathrm{Irr}(\mathsf{W}_{n_1})$ and $\mathrm{Irr}(\mathsf{W}_{n_2})$.

Let $\mathrm{sgn}_{\mathrm{CD},\tilde{n}} \colon \mathsf{W}_{\tilde{n}} \to \{\pm 1\}$ denote the unique character whose restriction to the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^{\tilde{n}}$ of $\mathsf{W}_{\tilde{n}}$ is the product of the sign characters and that is trivial on the subgroup $\mathfrak{S}_{\tilde{n}}$. The kernel of $\mathrm{sgn}_{\mathrm{CD},\tilde{n}}$ is isomorphic to the Weyl group $\mathsf{W}(\mathrm{D}_n)$. The restriction of the character $\mathrm{sgn}_{\mathrm{CD},\tilde{n}}$ to the subgroup $\mathsf{W}_{\tilde{n}-1}$ of $\mathsf{W}_{\tilde{n}}$ equals the character $\mathrm{sgn}_{\mathrm{CD},\tilde{n}-1}$. Because of this, we will denote $\mathrm{sgn}_{\mathrm{CD},\tilde{n}}$ simply by $\mathrm{sgn}_{\mathrm{CD}}$.

A conjectural description of the correspondence $\Theta^{\mathrm{G},\mathrm{G}'}_k$ was stated in [**AMR96**]. It can be formulated as follows:

Conjecture 1. *The representations $\rho \in \mathrm{Irr}(\mathsf{W}_{\tilde{n}_1(k)})$ and $\rho' \in \mathrm{Irr}(\mathsf{W}_{\tilde{n}_2(k_2)})$ correspond by $\Theta^{\mathrm{G},\mathrm{G}'}_k$ if and only if $\rho \otimes \rho'$ has a non-zero scalar product with*

$$\sum_{0 \le r \le N'} \sum_{\rho \in \mathrm{Irr}(\mathsf{W}_r)} \mathrm{Ind}^{W_{\tilde{n}_1(k)}}_{W_r \times W_{\tilde{n}_1(k)-r}} (\sigma \otimes \mathrm{sgn}_{\mathrm{CD}}) \otimes \mathrm{Ind}^{W_{\tilde{n}_2(k_2)}}_{W_r \times W_{\tilde{n}_2(k_2)-r}} (\sigma \otimes \mathrm{sgn}_{\mathrm{CD}})$$

$$(\textit{resp.} \quad \sum_{0 \le r \le N'} \sum_{\sigma \in \mathrm{Irr}(\mathsf{W}_r)} \mathrm{Ind}^{W_{\tilde{n}_1(k)}}_{W_r \times W_{\tilde{n}_1(k)-r}} (\sigma \otimes 1) \otimes \mathrm{Ind}^{W_{\tilde{n}_2(k_2)}}_{W_r \times W_{\tilde{n}_2(k_2)-r}} (\sigma \otimes \mathrm{sgn}_{\mathrm{CD}})),$$

where $k_2 = k$ (resp. $k_2 = k+1$).

We put

$$N' := \min(\tilde{n}_1(k), \tilde{n}_2(k_2)) \quad N := \max(\tilde{n}_1(k), \tilde{n}_2(k_2)), \quad \text{and} \quad L := N - N'. \tag{6}$$

In Conjecture 1, G stands for a symplectic group and G′ for an orthogonal group. However we would like to consider Howe's correspondence (1) in any of the two directions. Therefore we will consider the following cases and keep in mind that Conjecture 1 applies to any of them:

Case 1:

(a) $\mathrm{G}' = \mathrm{O}^{\varepsilon_k}_{2(k^2+N')}(q)$ and $\mathrm{G} = \mathrm{Sp}_{2(k^2+k+N))}(q)$.
(Here we have $N' = \tilde{n}_2(k)$ and $N = \tilde{n}_1(k)$.)

(b) $\mathrm{G}' = \mathrm{Sp}_{2(k^2+k+N')}(q)$ and $\mathrm{G} = \mathrm{O}^{\varepsilon_k}_{2(k^2+N)}(q)$.
(Here we have $N' = \tilde{n}_1(k)$ and $N = \tilde{n}_2(k)$.)

Case 2: $\mathrm{G}' = \mathrm{Sp}_{2(k^2+k+N')}(q)$ and $\mathrm{G} = \mathrm{O}^{\varepsilon_{k+1}}_{2((k+1)^2+N)}(q)$.
(Here we have $N' = \tilde{n}_1(k)$ and $N = \tilde{n}_2(k+1)$.)
Case 3: $\mathrm{G}' = \mathrm{O}^{\varepsilon_{k+1}}_{2((k+1)^2+N')}(q)$ and $\mathrm{G} = \mathrm{Sp}_{2(k^2+k+N)}(q)$.
(Here we have $N' = \tilde{n}_2(k+1)$, $N = \tilde{n}_1(k)$.)

Let (ξ, η) be a pair of partitions of N, *i.e.*, ξ and η are two partitions with $|\xi| + |\eta| = N$. The irreducible representations of W_N are parameterized by the pairs of partitions of N (see [**Lus77**]). The trivial representation of W_N corresponds to $((N), \emptyset)$ while the sign representation corresponds to $(\emptyset, (1^N))$ and the representation afforded by the character $\mathrm{sgn}_{\mathrm{CD}} = \mathrm{sgn}_{\mathrm{CD},N}$ corresponds to $(\emptyset, (N))$.

Definition 5. *We define* $\underline{\theta}^{N',N} \colon \mathrm{Irr}(\mathsf{W}_{N'}) \to \mathrm{Irr}(\mathsf{W}_N)$ *by*

$$\underline{\theta}^{N',N}(\rho_{\xi',\eta'}) := \begin{cases} \rho_{\xi',(L)\cup\eta'} & \text{in Cases 1 and 2,} \\ \rho_{(L)\cup\xi',\eta'} & \text{in Case 3.} \end{cases}$$

Theorem 6. *If Conjecture* 1 *holds, then the representations* $\rho_{\xi',\eta'} \in \mathrm{Irr}(\mathsf{W}_{N'})$ *and* $\underline{\theta}^{N',N}(\rho_{\xi',\eta'})$ *correspond by* $\Theta_k^{\mathrm{G},\mathrm{G}'}$.

In order to prove Theorem 6, we will need to introduce some more combinatorics.

Removing the boxes of λ which belong to μ, we obtain a *skew diagram* which we denote by $\lambda - \mu$. Then a *generalized tableau* of shape $\lambda - \mu$ is a filling of the boxes of $\lambda - \mu$ with positive integers such that the entries are weakly increasing from the left to the right along each row and strictly increasing down each column. Tableaux of shape λ are examples of generalized tableaux. Let T be a generalized tableau. Let $n_i = n_i(T)$ denote the number of occurrences of the integer i in T. The *weight* of T is defined as the sequence $(n_1, n_2, \ldots)$. The *word* $w(T)$ of T is the sequence obtained by reading the entries of T from right to left in successive rows, starting with the top row. On the other hand, any sequence $\mathbf{a} = (a_1, a_2, \ldots, a_l)$ with $a_i \in \{1, 2, \ldots, N\}$ is called a *lattice permutation* if, for $1 \le j \le l$ and $1 \le i \le N-1$, the number of occurrences of i in $(a_1, a_2, \ldots, a_j)$ is not less than the number of occurrences of $i+1$.

Let λ, μ, ν be partitions such that $|\lambda| = |\mu| + |\nu|$. The *Littlewood-Richardson coefficient* $c^{\lambda}_{\mu,\nu}$ is defined as the number of generalized tableaux T of shape $\lambda - \mu$ and weight ν such that $w(T)$ is a lattice permutation.

The Littlewood-Richardson rule (*cf.* for instance [**GP00**, 6.1.1, 6.1.6]) says that

$$\mathrm{Ind}_{\mathfrak{S}_\ell \times \mathfrak{S}_{N-\ell}}^{\mathfrak{S}_N}(\rho_\mu \otimes \rho_\nu) = \sum_{\lambda} c^{\lambda}_{\mu,\nu}\, \rho_\lambda,$$

where the sum runs over all partitions λ of N.

A similar rule occurs in the group $\mathsf{W}_n = \mathsf{W}(\mathrm{B}_n)$ (*cf.* [**GP00**, 6.1.3]):

$$\mathrm{Ind}_{\mathsf{W}_\ell \times \mathsf{W}_{N-\ell}}^{\mathsf{W}_N}(\rho_{\xi_1,\eta_1} \otimes \rho_{\xi_2,\eta_2}) = \sum_{(\xi,\eta)} c^{\xi}_{\xi_1,\xi_2}\, c^{\eta}_{\eta_1,\eta_2}\, \rho_{\xi,\eta}, \tag{7}$$

where the sum runs over all pairs of partitions (ξ, η) with $|\xi| = |\xi_1| + |\xi_2|$ and $|\eta| = |\eta_1| + |\eta_2|$.

Proposition 7. *We have*

$$\mathrm{Ind}_{\mathsf{W}_\ell \times \mathsf{W}_{N-\ell}}^{\mathsf{W}_N}(\rho_{\xi_1,\eta_1} \otimes 1) = \sum_{\xi} \rho_{\xi,\eta_1}, \tag{8}$$

where the sum is over all partitions ξ of $N - |\eta_1| = N - \ell + |\xi_1|$ whose Young diagram is obtained from that of ξ_1 by adding $N - \ell$ boxes, with no two boxes in the same column. In particular $\rho_{(N-l)\cup\xi_1,\eta_1}$ occurs in (8).

In a similar way, we have:

$$\mathrm{Ind}_{\mathsf{W}_\ell \times \mathsf{W}_{N-\ell}}^{\mathsf{W}_N}(\rho_{\xi_1,\eta_1} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \sum_{\eta} \rho_{\xi_1,\eta}, \tag{9}$$

where the sum is over all partition η of $N-|\xi_1| = N-\ell+|\eta_1|$ whose Young diagram is obtained from that of η_1 by adding $N - \ell$ boxes, with no two boxes in the same column. . In particular $\rho_{\xi_1,(N-l)\cup\eta_1}$ occurs in (9).

Proof. As already mentioned, the trivial character of $\mathsf{W}_{N-\ell}$ corresponds to the pair of partitions $((N - \ell), \emptyset)$. Hence we have to consider certain generalized tableaux T of shape $\xi - \xi_1$ and weight $(N - \ell)$. The integer 1 occurs $n_1(T) = N - \ell$ times in T. It follows that all the entries of T are equal to 1 and so the condition of $w(T)$ is empty. On the other hand, the fact that the entries of T have to be strictly increasing down each column implies that there is at most one box in each column of T. The first equality follows. The second equality is proved in an analogous way, using the fact that $\mathrm{sgn}_{\mathrm{CD},n-\ell} = \rho_{\emptyset,(N-\ell)}$. □

The following special cases of Proposition 7 will be used in the proof of Proposition 8.

Example 1. *Assume $\ell = 1$ and $N \geq 2$. We obtain*

$$\mathrm{Ind}_{\mathsf{W}_1 \times \mathsf{W}_{N-1}}^{\mathsf{W}_N}(\rho_{(1),\emptyset} \otimes 1) = \rho_{(N),\emptyset} \oplus \rho_{(N-1,1),\emptyset},$$
$$\mathrm{Ind}_{\mathsf{W}_1 \times \mathsf{W}_{N-1}}^{\mathsf{W}_N}(\rho_{\emptyset,(1)} \otimes 1) = \rho_{(N-1),(1)},$$
$$\mathrm{Ind}_{\mathsf{W}_1 \times \mathsf{W}_{N-1}}^{\mathsf{W}_N}(\rho_{(1),\emptyset} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \rho_{(1),(N-1)},$$
$$\mathrm{Ind}_{\mathsf{W}_1 \times \mathsf{W}_{N-1}}^{\mathsf{W}_N}(\rho_{\emptyset,(1)} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \rho_{\emptyset,(N)} \oplus \rho_{\emptyset,(N-1,1)}.$$

Example 2. *Assume $\ell = 2$ and $N \geq 3$. We obtain*

$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{(2),\emptyset} \otimes 1) = \rho_{(N),\emptyset} \oplus \rho_{(N-1,1),\emptyset} \oplus \rho_{(N-2,2),\emptyset},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{(1^2),\emptyset} \otimes 1) = \rho_{(N-1,1),\emptyset} \oplus \rho_{(N-2,1^2),\emptyset},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{(1),(1)} \otimes 1) = \rho_{(N-1),(1)} \oplus \rho_{(N-2,1),(1)},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{\emptyset,(2)} \otimes 1) = \rho_{(N-2),(2)},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{\emptyset,(1^2)} \otimes 1) = \rho_{(N-2),(1^2)},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{(2),\emptyset} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \rho_{(2),(N-2)},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{(1^2),\emptyset} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \rho_{(1^2),(N-2)},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{(1),(1)} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \rho_{(1),(N-1)} \oplus \rho_{(1),(N-2,1)},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{\emptyset,(2)} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \rho_{\emptyset,(N)} \oplus \rho_{\emptyset,(N-1,1)} \oplus \rho_{\emptyset,(N-2,2)},$$
$$\mathrm{Ind}_{\mathsf{W}_2 \times \mathsf{W}_{N-2}}^{\mathsf{W}_N}(\rho_{\emptyset,(1^2)} \otimes \mathrm{sgn}_{\mathrm{CD}}) = \rho_{\emptyset,(N-1,1)} \oplus \rho_{\emptyset,(N-2,1^2)}.$$

Proof of Theorem 6. It follows easily from the description given in Conjecture 1, combined with Proposition 7. □

In the case when $N' = 2$ we will describe Conjecture 1 in a more explicit manner. For $j \in \{1,2,3\}$, and $N \geq 2$, let $\theta_j^{2,N}\colon \mathrm{Irr}(W_2) \to \mathbb{Z}\,\mathrm{Irr}(\mathsf{W}_N)$ be the maps defined by (where in each case, the underlined representation $\underline{\rho_{\xi,\eta}}$ is equal to $\underline{\theta}^{2,N}(\rho_{\xi',\eta'})$):

$$\theta_1^{2,2}\colon \begin{array}{rcc} \rho_{(2),\emptyset} & \mapsto & \underline{\rho_{(2),\emptyset}} \\ \rho_{(1),(1)} & \mapsto & \underline{\rho_{(1),(1)}} \oplus \rho_{(1),(1)} \\ \rho_{(1^2),\emptyset} & \mapsto & \underline{\rho_{(1^2),\emptyset}} \\ \rho_{\emptyset,(2)} & \mapsto & \underline{\rho_{\emptyset,(2)}} \oplus 2\rho_{\emptyset,(2)} \oplus \rho_{\emptyset,(1^2)} \\ \rho_{\emptyset,(1^2)} & \mapsto & \rho_{\emptyset,(2)} \oplus \underline{\rho_{\emptyset,(1^2)}} \oplus \rho_{\emptyset,(1^2)} \end{array},$$

$$\theta_2^{2,2}\colon \begin{array}{rcc} \rho_{(2),\emptyset} & \mapsto & \underline{\rho_{(2),\emptyset}} \oplus \rho_{(1),(1)} \oplus \rho_{\emptyset,(2)} \\ \rho_{(1),(1)} & \mapsto & \underline{\rho_{(1),(1)}} \oplus \rho_{\emptyset,(2)} \oplus \rho_{\emptyset,(1^2)} \\ \rho_{(1^2),\emptyset} & \mapsto & \underline{\rho_{(1^2),\emptyset}} \oplus \rho_{(1),(1)} \\ \rho_{\emptyset,(2)} & \mapsto & \underline{\rho_{\emptyset,(2)}} \\ \rho_{\emptyset,(1^2)} & \mapsto & \underline{\rho_{\emptyset,(1^2)}} \end{array},$$

$$\theta_3^{2,2}\colon \begin{array}{rcc} \rho_{(2),\emptyset} & \mapsto & \underline{\rho_{(2),\emptyset}} \\ \rho_{(1),(1)} & \mapsto & \rho_{(2),\emptyset} \oplus \underline{\rho_{(1),(1)}} \oplus \rho_{(1^2),\emptyset} \\ \rho_{(1^2),\emptyset} & \mapsto & \underline{\rho_{(1^2),\emptyset}} \\ \rho_{\emptyset,(2)} & \mapsto & \rho_{(2),\emptyset} \oplus \rho_{(1),(1)} \oplus \underline{\rho_{\emptyset,(2)}} \\ \rho_{\emptyset,(1^2)} & \mapsto & \rho_{(1),(1)} \oplus \underline{\rho_{\emptyset,(1^2)}} \end{array},$$

$$\theta_1^{2,N}\colon \begin{array}{rcc} \rho_{(2),\emptyset} & \mapsto & \underline{\rho_{(2),(N-2)}} \\ \rho_{(1),(1)} & \mapsto & 2\rho_{(1),(N-1)} \oplus \underline{\rho_{(1),(N-2,1)}} \\ \rho_{(1^2),\emptyset} & \mapsto & \underline{\rho_{(1^2),(N-2)}} \\ \rho_{\emptyset,(2)} & \mapsto & 3\rho_{\emptyset,(N)} \oplus 2\rho_{\emptyset,(N-1,1)} \oplus \underline{\rho_{\emptyset,(N-2,2)}} \\ \rho_{\emptyset,(1^2)} & \mapsto & \rho_{\emptyset,(N)} \oplus 2\rho_{\emptyset,(N-1,1)} \oplus \underline{\rho_{\emptyset,(N-2,1^2)}} \end{array}, \quad \text{if } N \geq 3,$$

$$\theta_2^{2,N}\colon \begin{array}{rcc} \rho_{(2),\emptyset} & \mapsto & \rho_{\emptyset,(N)} \oplus \underline{\rho_{(2),(N-2)}} \oplus \rho_{(1),(N-1)} \\ \rho_{(1),(1)} & \mapsto & \rho_{(1),(N-1)} \oplus \underline{\rho_{(1),(N-2,1)}} \oplus \rho_{\emptyset,(N)} \oplus \rho_{\emptyset,(N-1,1)} \\ \rho_{(1^2),\emptyset} & \mapsto & \underline{\rho_{(1^2),(N-2)}} \oplus \rho_{(1),(N-1)} \\ \rho_{\emptyset,(2)} & \mapsto & \rho_{\emptyset,(N)} \oplus \rho_{\emptyset,(N-1,1)} \oplus \underline{\rho_{\emptyset,(N-2,2)}} \\ \rho_{\emptyset,(1^2)} & \mapsto & \rho_{\emptyset,(N-1,1)} \oplus \underline{\rho_{\emptyset,(N-2,1^2)}} \end{array}, \quad \text{if } N \geq 3,$$

$$\theta_3^{2,N}\colon \begin{array}{rcc} \rho_{(2),\emptyset} & \mapsto & \rho_{(N),\emptyset} \oplus \rho_{(N-1,1),\emptyset} \oplus \underline{\rho_{(N-2,2),\emptyset}} \\ \rho_{(1),(1)} & \mapsto & \rho_{(N),\emptyset} \oplus \rho_{(N-1,1),\emptyset} \oplus \rho_{(N-1),(1)} \oplus \underline{\rho_{(N-2,1),(1)}} \\ \rho_{(1^2),\emptyset} & \mapsto & \rho_{(N-1,1),\emptyset} \oplus \underline{\rho_{(N-2,1^2),\emptyset}} \\ \rho_{\emptyset,(2)} & \mapsto & \rho_{(N),\emptyset} \oplus \rho_{(N-1),(1)} \oplus \underline{\rho_{(N-2),(2)}} \\ \rho_{\emptyset,(1^2)} & \mapsto & \rho_{(N-1),(1)} \oplus \underline{\rho_{(N-2),(1^2)}} \end{array}, \quad \text{if } N \geq 3.$$

PROPOSITION 8. *We assume that $N'=2$, $N\geq 2$ and that Conjecture 1 holds. Then Howe correspondence for the dual pair (G',G) is given by the map*

$$\begin{cases} \theta_1^{2,N} & \textit{if } \mathrm{G}'=\mathrm{O}^{\varepsilon_k}_{2(k^2+2)}(q) \textit{ and } \mathrm{G}=\mathrm{Sp}_{2(k^2+k+N)}(q),\\ \theta_1^{2,N} & \textit{if } \mathrm{G}'=\mathrm{Sp}_{2(k^2+k+2)}(q) \textit{ and } \mathrm{G}=\mathrm{O}^{\varepsilon_k}_{2(k^2+N)}(q),\\ \theta_2^{2,N} & \textit{if } \mathrm{G}'=\mathrm{Sp}_{2(k^2+k+2)}(q) \textit{ and } \mathrm{G}=\mathrm{O}^{\varepsilon_{k+1}}_{2((k+1)^2+N)}(q),\\ \theta_3^{2,N} & \textit{if } \mathrm{G}'=\mathrm{O}^{\varepsilon_{k+1}}_{2((k+1)^2+2)}(q) \textit{ and } \mathrm{G}=\mathrm{Sp}_{2(k^2+k+N)}(q).\end{cases}$$

PROOF. We will consider the three cases listed after Conjecture 1 separately.

Case 1:

The combination of Conjecture 1 and Example 1 gives

$$\begin{array}{lll}
\rho_{(2),\emptyset} & \mapsto & \mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(2),\emptyset}\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{(1),(1)} & \mapsto & \mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(1\otimes\mathrm{sgn}_{\mathrm{CD}})\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(1),(1)}\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{(1^2),\emptyset} & \mapsto & \mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(1^2),\emptyset}\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{\emptyset,(2)} & \mapsto & \rho_{\emptyset,(N)}\oplus\mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(\mathrm{sgn}_{\mathrm{CD}}\otimes\mathrm{sgn}_{\mathrm{CD}})\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{\emptyset,(2)}\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{\emptyset,(1^2)} & \mapsto & \mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(\mathrm{sgn}_{\mathrm{CD}}\otimes\mathrm{sgn}_{\mathrm{CD}})\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{\emptyset,(1^2)}\otimes\mathrm{sgn}_{\mathrm{CD}}).
\end{array}$$

Using the computations done in Examples 1, 2, we obtain the map $\theta_1^{2,N}$.

Case 2:

The combination of Conjecture 1 and Example 1 gives

$$\begin{array}{lll}
\rho_{(2),\emptyset} & \mapsto & \mathrm{sgn}_{\mathrm{CD}}\oplus\mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(1\otimes\mathrm{sgn}_{\mathrm{CD}})\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(1\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{(1),(1)} & \mapsto & \mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(\mathrm{sgn}_{\mathrm{CD}}\otimes\mathrm{sgn}_{\mathrm{CD}})\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(1),(1)}\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{(1^2),\emptyset} & \mapsto & \mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(1\otimes\mathrm{sgn}_{\mathrm{CD}})\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(1^2),\emptyset}\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{\emptyset,(2)} & \mapsto & \mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{\emptyset,(2)}\otimes\mathrm{sgn}_{\mathrm{CD}})\\
\rho_{\emptyset,(1^2)} & \mapsto & \mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{\emptyset,(1^2)}\otimes\mathrm{sgn}_{\mathrm{CD}}).
\end{array}$$

Using the computations done in Examples 1, 2, we obtain $\theta_2^{2,N}$.

Case 3:

The combination of Conjecture 1 and Example 1 gives

$$\begin{array}{lll}
\rho_{(2),\emptyset} & \mapsto & \mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(2),\emptyset}\otimes 1)\\
\rho_{(1),(1)} & \mapsto & \mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(1\otimes 1)\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(1),(1)}\otimes 1)\\
\rho_{(1^2),\emptyset} & \mapsto & \mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{(1^2),\emptyset}\otimes 1)\\
\rho_{\emptyset,(2)} & \mapsto & \rho_{(N),\emptyset}\oplus\mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(\mathrm{sgn}_{\mathrm{CD}}\otimes 1)\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{\emptyset,(2)}\otimes 1)\\
\rho_{\emptyset,(1^2)} & \mapsto & \mathrm{Ind}_{W_1\times W_{N-1}}^{W_N}(\mathrm{sgn}_{\mathrm{CD}}\otimes 1)\oplus\mathrm{Ind}_{W_2\times W_{N-2}}^{W_N}(\rho_{\emptyset,(1^2)}\otimes 1).
\end{array}$$

Using the computations done in Examples 1, 2, we get $\theta_3^{2,N}$. □

4.2. Symbols and u-symbols. We will recall part of the formalism of symbols due to Lusztig. (See [**Lus84**] and references there.)

A *symbol* is an ordered pair $\Lambda = \begin{pmatrix} A \\ B \end{pmatrix}$ of finite subsets (including the empty set $\emptyset$) of $\{0, 1, 2, \ldots\}$. The *rank* of Λ is defined to be

$$\operatorname{rank}(\Lambda) := \sum_{a \in A} a + \sum_{b \in B} b - \left\lfloor \left(\frac{|A| + |B| - 1}{2} \right)^2 \right\rfloor,$$

where for any real number r we denote by $\lfloor r \rfloor$ the largest integer not greater than r. The *defect* of Λ, to be denoted by $\operatorname{def}(\Lambda)$, is defined to be the absolute value of $|A| - |B|$. There is an equivalence relation on such pairs generated by the shift

$$\begin{pmatrix} A \\ B \end{pmatrix} \sim \begin{pmatrix} \{0\} \cup (A+1) \\ \{0\} \cup (B+1) \end{pmatrix}.$$

We shall identify a symbol with its equivalence class. The functions $\operatorname{rank}(\Lambda)$ and $\operatorname{def}(\Lambda)$ are invariant under the shift operation, hence are well-defined on the set of symbol classes. A symbol $\Lambda = \begin{pmatrix} A \\ B \end{pmatrix}$ is said to be *degenerate* if $A = B$, and *non-degenerate* otherwise. The entries appearing in exactly one row of Λ are called *singles*.

There is also a notion of u-symbols due to Lusztig related to unipotent classes.

Let (ξ, η) be a pair of partitions of N. We ensure that ξ has exactly one more part than η by adding zeros as parts where necessary. Let m denote the number of parts of η. We then attach to (ξ, η), where $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_m \geq \xi_{m+1})$ and $\eta = (\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m)$, a *symbol* $\Lambda = \Lambda_{\xi,\eta}$ of defect 1 and two *u-symbols* $\Lambda^{\mathrm{u,sp}}_{\xi,\eta}$ and $\Lambda^{\mathrm{u,or}}_{\xi,\eta}$ to be defined by

$$\Lambda_{\xi,\eta} := \left(\begin{array}{ccccccccccc} \xi_{m+1} & & \xi_m + 1 & & \xi_{m-1} + 2 & & \cdots & \cdots & \cdots & & \xi_1 + m \\ & \eta_m & & \eta_{m-1} + 1 & & \eta_{m-2} + 2 & & \cdots & & \eta_1 + m - 1 & \end{array} \right),$$

$$\Lambda^{\mathrm{u,sp}}_{\xi,\eta} := \left(\begin{array}{ccccccccc} \xi_{m+1} & & \xi_m + 2 & & \xi_{m-1} + 4 & \cdots & \cdots & & \xi_1 + 2m \\ & \eta_m + 1 & & \eta_{m-1} + 3 & & \cdots & & \eta_1 + 2(m-1) + 1 & \end{array} \right),$$

$$\Lambda^{\mathrm{u,or}}_{\xi,\eta} := \begin{pmatrix} \xi_{m+1} & \xi_m + 2 & \xi_{m-1} + 4 & \cdots & \cdots & \xi_1 + 2m \\ \eta_{m+1} & \eta_m + 2 & \eta_{m-1} + 4 & & \cdots & \eta_1 + 2m \end{pmatrix},$$

where in the orthogonal case we arranged for the two partitions to have the same length $m+1$. The symbol $\Lambda_{\xi,\eta}$ is called *special* if

$$\xi_{m+1} \leq \eta_m \leq \xi_m + 1 \leq \eta_{m-1} + 1 \leq \xi_{m-1} + 2 \leq \cdots \leq \eta_1 + m - 1 \leq \xi_1 + m.$$

Similarly, $\Lambda^{\mathrm{u,sp}}_{\xi,\eta}$ is called *distinguished* if

$$\xi_{m+1} \leq \eta_m + 1 \leq \xi_m + 2 \leq \eta_{m-1} + 3 \leq \cdots \leq \eta_1 + 2(m-1) + 1 \leq \xi_1 + 2m,$$

and $\Lambda^{\mathrm{u,or}}_{\xi,\eta}$ is called distinguished if

$$\xi_{m+1} \leq \eta_{m+1} \leq \xi_m + 2 \leq \eta_m + 2 \leq \cdots \leq \xi_1 + 2m \leq \eta_1 + 2m.$$

We observe that the fact that $\Lambda_{\xi,\eta}$ is special implies the distinguishness of $\Lambda^{\mathrm{u,sp}}_{\xi,\eta}$.

The set of all the symbols (resp. u-symbols) which contain the same entries with the same multiplicities as a given symbol (resp. u-symbol) is called the *similarity class* of the latter. If Λ, Λ' belong to the same similarity class, we will write $\Lambda \sim_{\mathrm{sim}} \Lambda'$. Each similarity class of symbols (resp. u-symbols) contains exactly one special (resp. distinguished) element.

We will now recall the algorithm described in [**Car93**, §13.3]. To each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ we attach the *sequence of β-numbers*

(10)
$$\lambda^* = (\lambda_1^* < \lambda_2^* < \cdots < \lambda_k^*), \text{ defined by } \lambda_j^* := \lambda_{k-j+1} + j - 1, \text{ for } 1 \leq j \leq k.$$

For instance, we have

$$(N)^* = (N), \quad (N-1,1)^* = (1,N), \quad (N-2,2)^* = (2,N-1), \quad (N-2,1,1)^* = (1,2,N).$$

Recall that a partition λ is called *symplectic* (resp. *orthogonal*) if each odd (resp. even) row occurs with even multiplicity. For N a given integer, let $\mathcal{P}^{\mathrm{sp}}(N)$ (resp. $\mathcal{P}^{\mathrm{or}}(N)$) denote the set of symplectic (resp. orthogonal) partitions of N.

Consider a symplectic or orthogonal partition λ of N and the corresponding group G_N. We ensure that the number of parts of λ has same parity as the defining module of G_N, by calling the last part 0 if necessary. Thus $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2k}$ (resp. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2k+1}$) if $\mathrm{G}_N = \mathrm{Sp}_{2N}(\bar{\mathbb{F}}_q)$ or $\mathrm{O}_{2N}(\bar{\mathbb{F}}_q)$ (resp. $\mathrm{G}_N = \mathrm{O}_{2N+1}(\bar{\mathbb{F}}_q)$). We then divide λ^* into its odd and even parts. Let the odd parts and the even parts of λ^* be

$$2\xi_1^* + 1 < 2\xi_2^* + 1 < \cdots < 2\xi_k^* + 1 \ \ (\text{resp. } 2\xi_{k+1}^* + 1) \quad \text{and} \quad 2\eta_1^* < 2\eta_2^* < \cdots < 2\eta_k^*,$$

respectively. Then we have

$$0 \leq \xi_1^* < \xi_2^* < \cdots < \xi_k^* \ \ (\text{resp. } \xi_{k+1}^*) \quad \text{and} \quad 0 \leq \eta_1^* < \eta_2^* < \cdots < \eta_k^*.$$

Next we define $\xi_i := \xi_{k-i+1}^* - (k-i)$ and $\eta_i := \eta_{k-i+1}^* - (k-i)$ for each i. We then have $\xi_i \geq \xi_{i+1} \geq 0$, $\eta_i \geq \eta_{i+1} \geq 0$, and $|\xi| + |\eta| = n$.

Thus we obtain a map

(11)
$$\varphi \colon \lambda \mapsto (\xi, \eta)$$

from $\mathcal{P}^{\mathrm{sp}}(2N)$ or $\mathcal{P}^{\mathrm{or}}(2N)$ (resp. $\mathcal{P}^{\mathrm{or}}(2N+1)$) to the set of pairs of partitions of N, which is injective.

A pair of partitions (ξ_0, η_0) of N is in the image of the map (11) of a symplectic partition, say $\lambda_{\xi_0,\eta_0}^{\mathrm{sp}}$ (resp. an orthogonal partition, say $\lambda_{\xi_0,\eta_0}^{\mathrm{or}}$) if and only if the u-symbol $\Lambda_{\xi_0,\eta_0}^{\mathrm{u,sp}}$ (resp. $\Lambda_{\xi_0,\eta_0}^{\mathrm{u,or}}$) is distinguished, see [**Car93**, page 420].

Definition 9. *If (ξ, η) is not in the image of the map φ defined by (11), we put $\lambda_{\xi,\eta}^{\mathrm{sp}} := \lambda_{\xi_0,\eta_0}^{\mathrm{sp}}$ (resp. $\lambda_{\xi,\eta}^{\mathrm{or}} := \lambda_{\xi_0,\eta_0}^{\mathrm{or}}$), where $\Lambda_{\xi_0,\eta_0}^{\mathrm{u,sp}}$ (resp. $\Lambda_{\xi_0,\eta_0}^{\mathrm{u,or}}$) is the distinguished u-symbol in the similarity class of $\Lambda_{\xi,\eta}^{\mathrm{u,sp}}$ (resp. $\Lambda_{\xi,\eta}^{\mathrm{u,or}}$).*

We will use the computations done in the following exeamples in the proof of Theorem 10.

Example 3. *Let $N \geq 2$ and $1 \leq h \leq 2$. We have*

$$\Lambda_{(N-h,h),\emptyset}^{\mathrm{u,sp}} = \begin{pmatrix} h & & N-h+2 \\ & 1 & \end{pmatrix} \quad \textit{and} \quad \Lambda_{(N-h,h),\emptyset}^{\mathrm{u,or}} = \begin{pmatrix} h & N-h+2 \\ 0 & 2 \end{pmatrix}.$$

- $\Lambda_{(N-1,1),\emptyset}^{\mathrm{u,sp}}$ *is distinguished. Because $(N-1,1)^* = (1,N)$ and $\emptyset^* = (0,1)$, we obtain $\lambda_{(N-1,1),\emptyset}^{\mathrm{sp},*} = (0,2,3,2N{+}1)$, that gives $\lambda_{(N-1,1),\emptyset}^{\mathrm{sp}} = (2N{-}2,1^2)$.*
- $\Lambda_{(N-2,2),\emptyset}^{\mathrm{u,sp}}$ *is not distinguished. The distinguished u-symbol in its similarity class is*

$$\begin{pmatrix} 1 & & N \\ & 2 & \end{pmatrix} = \Lambda_{(N-2,1),(1,0)}^{\mathrm{u,sp}}.$$

Because $(N-2,1)^* = (1,N-1)$ *and* $(1,0)^* = (0,2)$, *we obtain* $\lambda^{\mathrm{sp},*}_{(N-2,1),(1,0)} = (0,3,4,2N-1)$, *that gives* $\lambda^{\mathrm{sp}}_{(N-2,2),\emptyset} = \lambda^{\mathrm{sp}}_{(N-2,1),(1)} = (2N-4,2^2)$.

- *For* $h=1,2$, *the distinguished* u*-symbol in the similarity class of* $\Lambda^{\mathrm{u,or}}_{(N-h,h),\emptyset}$ *is*
$$\begin{pmatrix} 0 & 2 \\ h & N-h+2 \end{pmatrix} = \Lambda^{\mathrm{u,or}}_{\emptyset,(N-h,h)}.$$
Because $(0^2)^* = (0,1)$ *and* $(N-h,h)^* = (h,N-h+1)$, *we obtain*
$$\lambda^{*,\mathrm{or}}_{(0^2),(N-h,h)} = \begin{cases} (1,2,3,2N) & \text{if } h=1, \\ (1,3,4,2N-2) & \text{if } h=2, \end{cases}$$
that is,
$$\lambda^{\mathrm{or}}_{(N-h,h),\emptyset} = \lambda^{\mathrm{or}}_{(0^2),(N-h,h)} = \begin{cases} (2N-3,1^3) & \text{if } h=1, \\ (2N-5,2^2,1) & \text{if } h=2. \end{cases}$$

Example 4.
$$\Lambda^{\mathrm{u,sp}}_{(N-2,1^2),\emptyset} = \begin{pmatrix} 1 & & 3 & & N+2 \\ & 1 & & 3 & \end{pmatrix} \quad \textit{and} \quad \Lambda^{\mathrm{u,or}}_{(N-2,1^2),\emptyset} = \begin{pmatrix} 1 & 3 & N+2 \\ 0 & 2 & 4 \end{pmatrix}.$$

- $\Lambda^{\mathrm{u,sp}}_{(N-2,1^2),\emptyset}$ *is distinguished. Because* $(N-2,1^2)^* = (1,2,N)$ *and* $(0^3)^* = (0,1,2)$, *we obtain* $\lambda^{\mathrm{sp},*}_{(N-2,1^2),\emptyset} = (0,2,3,4,5,2N+1)$, *that gives* $\lambda^{\mathrm{sp}}_{(N-2,1^2),\emptyset} = (2N-4,1^4)$.
- $\Lambda^{\mathrm{u,or}}_{(N-2,1^2),\emptyset}$ *is not distinguished. If* $N \geq 3$, *the distinguished* u*-symbol in its similarity class is*
$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & N+2 \end{pmatrix} = \Lambda^{\mathrm{u,or}}_{\emptyset,(N-2,1^2)}.$$
We have $\lambda^{\mathrm{or},*}_{\emptyset,(N-2,1^2)} = (1,2,3,4,5,2N)$, *that gives* $\lambda^{\mathrm{or}}_{(N-2,1^2),\emptyset} = (2N-5,1^5)$.

Example 5.
$$\Lambda^{\mathrm{u,sp}}_{(N-2),(1^2)} = \begin{pmatrix} 0 & & 2 & & N \\ & 2 & & 4 & \end{pmatrix} \quad \textit{and} \quad \Lambda^{\mathrm{u,or}}_{(N-2),(1^2)} = \begin{pmatrix} 0 & N \\ 1 & 3 \end{pmatrix}.$$

- $\Lambda^{\mathrm{u,sp}}_{(N-2),(1^2)}$ *is distinguished if* $N \geq 4$, $(N-2,0)^* = (0,N-1)$, $(1^2)^* = (1,2)$. *Hence* $\lambda^{\mathrm{sp},*}_{(N-2),(1^2)} = (1,2,4,2N-1)$ *and* $\lambda^{\mathrm{sp}}_{(N-2),(1^2)} = (2N-4,2,1^2)$.
- $\Lambda^{\mathrm{u,or}}_{(N-2),(1^2)}$ *is not distinguished if* $N \geq 4$. *The distinguished* u*-symbol in its similarity class is*
$$\begin{pmatrix} 0 & 3 \\ 1 & N \end{pmatrix} = \Lambda^{\mathrm{u,or}}_{(1),(N-2,1)}.$$
$(1,0)^* = (0,2)$, $(N-2,1)^* = (1,N-1)$. *Hence* $\lambda^{\mathrm{or},*}_{(1),(N-2,1)} = (1,2,5,2N-2)$, *and* $\lambda^{\mathrm{or}}_{(N-2),(1^2)} = (2N-5,3,1^2)$.

Example 6.
$$\Lambda^{\mathrm{u,sp}}_{(N-2),(2)} = \begin{pmatrix} 0 & & N \\ & 3 & \end{pmatrix} \quad \textit{and} \quad \Lambda^{\mathrm{u,or}}_{(N-2),(2)} = \begin{pmatrix} N-2 \\ 2 \end{pmatrix}.$$

- $\lambda^{\mathrm{sp},*}_{(N-2),(2)} = (4,2N-3)$, *and* $\lambda^{\mathrm{sp}}_{(N-2),(2)} = (2N-4,4)$, *when* $N \geq 3$.

- $\lambda^{\mathrm{or},*}_{(N-2),(2)} = (5, 2N-4)$, *and* $\lambda^{\mathrm{or}}_{(N-2),(2)} = (2N-5, 5)$.

EXAMPLE 7.

$$\Lambda^{\mathrm{u,sp}}_{(N-1),(1)} = \begin{pmatrix} 0 & & N+1 \\ & 2 & \end{pmatrix} \quad \textit{and} \quad \Lambda^{\mathrm{u,or}}_{(N-1),(1)} = \begin{pmatrix} N-1 \\ 1 \end{pmatrix}.$$

- $\Lambda^{\mathrm{u,sp}}_{(N-1),(1)}$ *is distinguished,* $\lambda^{\mathrm{sp},*}_{(N-1),(1)} = (2, 2N-1)$, *and* $\lambda^{\mathrm{sp}}_{(N-1),(1)} = (2N-2, 2)$.
- $\Lambda^{\mathrm{u,or}}_{(N-1),(1)}$ *is not distinguished. The distinguished u-symbol in its similarity class is* $\Lambda^{\mathrm{u,or}}_{(1),(N-1)}$. *We have* $\lambda^{\mathrm{or},*}_{(N-1),(1)} = (3, 2N-2)$, *and* $\lambda^{\mathrm{or}}_{(N-1),(1)} = (2N-3, 3)$.

EXAMPLE 8.

$$\Lambda^{\mathrm{u,sp}}_{(2),(N-2)} = \begin{pmatrix} 0 & & 4 \\ & N-1 & \end{pmatrix} \quad \textit{and} \quad \Lambda^{\mathrm{u,or}}_{(2),(N-2)} = \begin{pmatrix} 2 \\ N-2 \end{pmatrix}.$$

- $\Lambda^{\mathrm{u,sp}}_{(2),(N-2)}$ *is not distinguished if* $N \geq 6$. *Then the distinguished u-symbol in its similarity class is*

$$\begin{pmatrix} 0 & & N-1 \\ & 4 & \end{pmatrix} = \Lambda^{\mathrm{u,sp}}_{(N-3),(3)}.$$

We have $(N-3, 0)^* = (0, N-2)$ *and* $(3, 0)^* = (0, 4)$. *Hence* $\lambda^{\mathrm{sp},*}_{(2),(N-2)} = (0, 1, 8, 2N-3)$, *and* $\lambda^{\mathrm{sp}}_{(2),(N-2)} = (2N-6, 6)$.
- $\lambda^{\mathrm{or},*}_{(2),(N-2)} = (5, 2N-4)$, *and* $\lambda^{\mathrm{or}}_{(2),(N-2)} = (2N-5, 5)$, $N \geq 5$.

EXAMPLE 9.

$$\Lambda^{\mathrm{u,sp}}_{(N-2,1),(1)} = \begin{pmatrix} 1 & & N \\ & 2 & \end{pmatrix} \quad \textit{and} \quad \Lambda^{\mathrm{u,or}}_{(N-2,1),(1)} = \begin{pmatrix} 1 & N \\ 0 & 3 \end{pmatrix}.$$

- $\Lambda^{\mathrm{u,sp}}_{(N-2,1),(1)}$ *is distinguished if* $N \geq 2$, $(N-2, 1)^* = (1, N-1)$ *and* $(1, 0)^* = (0, 2)$. *Hence* $\lambda^{\mathrm{sp},*}_{(N-2,1),(1)} = (0, 3, 4, 2N-1)$, *and* $\lambda^{\mathrm{sp}}_{(N-2,1),(1)} = (2N-4, 2^2)$.
- $\Lambda^{\mathrm{u,or}}_{(N-2,1),(1)}$ *is not distinguished if* $N \geq 4$. *The distinguished u-symbol in its similarity class is*

$$\begin{pmatrix} 0 & 3 \\ 1 & N \end{pmatrix} = \Lambda^{\mathrm{u,or}}_{(1),(N-2,1)}.$$

Hence $\lambda^{\mathrm{or},*}_{(1),(N-2,1)} = (1, 2, 5, 2N-2)$, *and* $\lambda^{\mathrm{or}}_{(1),(N-2,1)} = (2N-5, 3, 1^2)$.

EXAMPLE 10.

$$\Lambda^{\mathrm{u,sp}}_{(1),(N-2,1)} = \begin{pmatrix} 0 & & 2 & & 5 \\ & 2 & & N+1 & \end{pmatrix}$$

is not distinguished if $N \geq 5$. *The distinguished u-symbol in its similarity class is*

$$\Lambda^{\mathrm{u,sp}}_{(N-3),(2,1)} = \begin{pmatrix} 0 & & 2 & & N+1 \\ & 2 & & 5 & \end{pmatrix}.$$

We have $\lambda^{\mathrm{sp},*}_{(N-3),(2,1)} = (1, 2, 6, 2N-3)$, *and hence* $\lambda^{\mathrm{sp}}_{(1),(N-2,1)} = (2N-6, 4, 1^2)$.

Example 11.

$$\Lambda^{\mathrm{u,sp}}_{\emptyset,(N-2,2)} = \begin{pmatrix} 0 & & 2 & & 4 \\ & 3 & & N+1 & \end{pmatrix}$$

is not distinguished if $N \geq 4$. The distinguished u-symbol in its similarity class is

$$\Lambda^{\mathrm{u,sp}}_{(N-3,1),(1^2)} = \begin{pmatrix} 0 & & 3 & & N+1 \\ & 2 & & 4 & \end{pmatrix}.$$

We have $\lambda^{\mathrm{sp},*}_{(N-3,1),(1^2)} = (2,3,4,2N-3)$, and $\lambda^{\mathrm{sp}}_{\emptyset,(N-2,2)} = (2N-6,2^3)$.

Example 12.

$$\Lambda^{\mathrm{u,sp}}_{\emptyset,(N-2,1^2)} = \begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 2 & & 4 & & N+3 & \end{pmatrix}$$

is not distinguished if $N \geq 3$. The distinguished u-symbol in its similarity class is

$$\Lambda^{\mathrm{u,sp}}_{(N-3),(1^3)} = \begin{pmatrix} 0 & & 2 & & 4 & & N+3 \\ & 2 & & 4 & & 6 & \end{pmatrix}.$$

Hence $\lambda^{\mathrm{sp},*}_{\emptyset,(N-2,1^2)} = (1,2,3,4,6,2N-1)$, and $\lambda^{\mathrm{sp}}_{\emptyset,(N-2,1^2)} = (2N-6,2,1^4)$.

Example 13.

$$\Lambda^{\mathrm{u,sp}}_{(1^2),(N-2)} = \begin{pmatrix} 1 & & 3 \\ & N-1 & \end{pmatrix}$$

is not distinguished if $N \geq 5$. The distinguished u-symbol in its similarity class is $\Lambda^{\mathrm{u,sp}}_{(N-3,1),(2)}$. We have $\lambda^{\mathrm{sp},*}_{(N-3,1),(2)} = (0,3,6,2N-3)$. Hence $\lambda^{\mathrm{sp}}_{(1^2),(N-2)} = (2N-6,4,2)$.

4.3. Howe correspondence and unipotent orbits. Let $\bar{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$, and let $\mathcal{N}(\mathbf{G})$ denote the set of unipotent classes of $\mathbf{G}$. This set is partially ordered by the relation $\mathcal{O}_1 \leq \mathcal{O}_2$ meaning that $\mathcal{O}_1$ is contained in $\overline{\mathcal{O}}_2$, the closure of $\mathcal{O}_2$. The unipotent orbits in the corresponding algebraic groups over $\bar{\mathbb{F}}_q$ are parameterized by partitions λ of the dimension of the defining module. The partition λ is symplectic (resp. orthogonal) if $G = \mathrm{Sp}_{2N}(\bar{\mathbb{F}}_q)$ (resp. $\mathrm{O}_{2N}(\bar{\mathbb{F}}_q)$ or $\mathrm{O}_{2N+1}(\bar{\mathbb{F}}_q)$).

To the representation $\rho_{\xi,\eta}$ of W_N we shall associate the u-symbol $\Lambda^{\mathrm{u,sp}}_{\xi,\eta}$ (resp. $\Lambda^{\mathrm{u,or}}_{\xi,\eta}$) of the group $\mathbf{G} = \mathrm{Sp}_{2N}(\bar{\mathbb{F}}_q)$ (resp. $\mathbf{G} = \mathrm{O}^{\varepsilon_{k_2}}_{2N}(\bar{\mathbb{F}}_q)$).

For the groups $\mathrm{Sp}_{2N}(\bar{\mathbb{F}}_q)$ and $\mathrm{O}_{2N+1}(\bar{\mathbb{F}}_q)$, we associate the representation $\rho_{\xi,\eta}$ to the orbit $\mathcal{O}(\lambda)$, where $(\xi,\eta) := \varphi(\lambda)$, with φ defined by (11).

Consider the group $\mathrm{O}_{2N}(\bar{\mathbb{F}}_q)$. In this case the unipotent orbits are parameterized by partitions λ of $2N$ where the even rows occur with even multiplicities. We attach to such a partition λ the ordered pair of partitions (ξ,η) defined by $(\xi,\eta) := \varphi(\lambda)$. Then we associate to $\mathcal{O}(\lambda)$ the representation $\rho_{\eta,\xi}$.

Let $\mathbf{S}(\mathbf{G})$ denote the set of u-symbols attached to $\mathbf{G}$. Let

$$\vartheta^{2,2}_{1,a}\colon \mathbf{S}(\mathrm{O}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{Sp}_4(\bar{\mathbb{F}}_q)) \quad \text{and} \quad \vartheta^{2,2}_{1,b}\colon \mathbf{S}(\mathrm{Sp}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{O}_4(\bar{\mathbb{F}}_q))$$

TABLE 1. Unipotent classes and the corresponding similarity classes of the u-symbols for $\mathrm{Sp}_4(\bar{\mathbb{F}}_q)$

$$\begin{array}{ccc}
\mathcal{O}(4) & \leftrightarrow & \Lambda^{\mathrm{u,sp}}_{(2),\emptyset} \\
| & & \\
\mathcal{O}(2^2) & \leftrightarrow & \{\Lambda^{\mathrm{u,sp}}_{(1),(1)}, \Lambda^{\mathrm{u,sp}}_{\emptyset,(2)}\} \\
| & & \\
\mathcal{O}(2,1^2) & \leftrightarrow & \Lambda^{\mathrm{u,sp}}_{(1^2),\emptyset} \\
| & & \\
\mathcal{O}(1^4) & \leftrightarrow & \Lambda^{\mathrm{u,sp}}_{\emptyset,(1^2)}
\end{array}$$

TABLE 2. Unipotent classes and and the corresponding similarity classes of the u-symbols for $\mathrm{O}_4(\bar{\mathbb{F}}_q)$

$$\begin{array}{ccc}
\mathcal{O}(3,1) & \leftrightarrow & \{\Lambda^{\mathrm{u,or}}_{(2),\emptyset}, \Lambda^{\mathrm{u,or}}_{\emptyset,(2)}\} \\
| & & \\
\mathcal{O}(2^2) & \leftrightarrow & \Lambda^{\mathrm{u,or}}_{(1),(1)} \\
| & & \\
\mathcal{O}(1^4) & \leftrightarrow & \{\Lambda^{\mathrm{u,or}}_{(1^2),\emptyset}, \Lambda^{\mathrm{u,or}}_{\emptyset,(1^2)}\}
\end{array}$$

be defined by (where in each case, if the input symbol is indexed by ξ', η', then the underlined orbit in the output corresponds to the representation $\underline{\theta}^{2,N}(\rho_{\xi',\eta'})$)

$$\vartheta^{2,2}_{1,a}\colon \begin{array}{ccc}
\Lambda^{\mathrm{u,or}}_{(2),\emptyset} & \mapsto & \underline{\mathcal{O}(4)} \\
\Lambda^{\mathrm{u,or}}_{(1),(1)} & \mapsto & \underline{\mathcal{O}(2^2)} \\
\Lambda^{\mathrm{u,or}}_{(1^2),\emptyset} & \mapsto & \underline{\mathcal{O}(2,1^2)} \\
\Lambda^{\mathrm{u,or}}_{\emptyset,(2)} & \mapsto & \{\underline{\mathcal{O}(2^2)}, \mathcal{O}(1^4)\} \\
\Lambda^{\mathrm{u,or}}_{\emptyset,(1^2)} & \mapsto & \{\mathcal{O}(2^2), \underline{\mathcal{O}(1^4)}\}
\end{array}, \qquad
\vartheta^{2,2}_{1,b}\colon \begin{array}{ccc}
\Lambda^{\mathrm{u,sp}}_{(2),\emptyset} & \mapsto & \{\underline{\mathcal{O}(3,1)}, \mathcal{O}(2^2)\} \\
\Lambda^{\mathrm{u,sp}}_{(1),(1)} & \mapsto & \{\mathcal{O}(3,1), \underline{\mathcal{O}(2^2)}, \mathcal{O}(1^4)\} \\
\Lambda^{\mathrm{u,sp}}_{(1^2),\emptyset} & \mapsto & \{\mathcal{O}(2^2), \underline{\mathcal{O}(1^4)}\} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(2)} & \mapsto & \underline{\mathcal{O}(3,1)} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(1^2)} & \mapsto & \underline{\mathcal{O}(1^4)}
\end{array}.$$

Let

$$\vartheta^{2,2}_2\colon \mathbf{S}(\mathrm{Sp}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{O}_4(\bar{\mathbb{F}}_q)) \quad \text{and} \quad \vartheta^{2,2}_3\colon \mathbf{S}(\mathrm{O}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{Sp}_4(\bar{\mathbb{F}}_q))$$

be defined by

$$\vartheta^{2,2}_2\colon \begin{array}{ccc}
\Lambda^{\mathrm{u,sp}}_{(2),\emptyset} & \mapsto & \{\underline{\mathcal{O}(3,1)}, \mathcal{O}(2^2), \mathcal{O}(3,1)\} \\
\Lambda^{\mathrm{u,sp}}_{(1),(1)} & \mapsto & \{\underline{\mathcal{O}(2^2)}, \mathcal{O}(3,1), \mathcal{O}(1^4)\} \\
\Lambda^{\mathrm{u,sp}}_{(1^2),\emptyset} & \mapsto & \{\underline{\mathcal{O}(1^4)}, \mathcal{O}(2^2)\} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(2)} & \mapsto & \underline{\mathcal{O}(3,1)} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(1^2)} & \mapsto & \underline{\mathcal{O}(1^4)}
\end{array}, \qquad
\vartheta^{2,2}_3\colon \begin{array}{ccc}
\Lambda^{\mathrm{u,or}}_{(2),\emptyset} & \mapsto & \{\mathcal{O}(4), \mathcal{O}(2^2)\} \\
\Lambda^{\mathrm{u,or}}_{(1),(1)} & \mapsto & \{\mathcal{O}(4), \underline{\mathcal{O}(2^2)}, \mathcal{O}(2,1^2)\} \\
\Lambda^{\mathrm{u,or}}_{(1^2),\emptyset} & \mapsto & \{\mathcal{O}(2^2), \underline{\mathcal{O}(1^4)}\} \\
\Lambda^{\mathrm{u,or}}_{\emptyset,(2)} & \mapsto & \underline{\mathcal{O}(4)} \\
\Lambda^{\mathrm{u,or}}_{\emptyset,(1^2)} & \mapsto & \underline{\mathcal{O}(2,1^2)}
\end{array}.$$

If $N \geq 3$, let $\vartheta^{2,N}_{1,a} : \mathbf{S}(\mathrm{O}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{Sp}_{2N}(\bar{\mathbb{F}}_q))$ be defined by

$$\vartheta^{2,N}_{1,a} : \begin{array}{ccc} \Lambda^{\mathrm{u,or}}_{(2),\emptyset} & \mapsto & \underline{\mathcal{O}(2N-6,6)} \\ \Lambda^{\mathrm{u,or}}_{(1),(1)} & \mapsto & \{\mathcal{O}(2N-4,4), \underline{\mathcal{O}(2N-6,4,1^2)}\} \\ \Lambda^{\mathrm{u,or}}_{\emptyset,(2)} & \mapsto & \{\mathcal{O}(2N-2,2), \mathcal{O}(2N-4,2,1^2), \underline{\mathcal{O}(2N-6,2^3)}\} \\ \Lambda^{\mathrm{u,or}}_{(1^2),\emptyset} & \mapsto & \{\mathcal{O}(2N-2,2), \mathcal{O}(2N-4,2,1^2), \underline{\mathcal{O}(2N-6,2,1^4)}\} \\ \Lambda^{\mathrm{u,or}}_{\emptyset,(1^2)} & \mapsto & \underline{\mathcal{O}(2N-6,4,2)} \end{array}.$$

If $N \geq 3$, let $\vartheta^{2,N}_{1,b} : \mathbf{S}(\mathrm{Sp}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{O}_{2N}(\bar{\mathbb{F}}_q))$ be defined by

$$\vartheta^{2,N}_{1,b} : \begin{array}{ccc} \Lambda^{\mathrm{u,sp}}_{(2),\emptyset} & \mapsto & \underline{\mathcal{O}(2N-5,5)} \\ \Lambda^{\mathrm{u,sp}}_{(1),(1)} & \mapsto & \{\mathcal{O}(2N-3,3), \underline{\mathcal{O}(2N-5,3,1^2)}\} \\ \Lambda^{\mathrm{u,sp}}_{(1^2),\emptyset} & \mapsto & \underline{\mathcal{O}(2N-5,3,1^2)} \\ \Lambda^{\mathrm{u,sp}}_{\emptyset,(2)} & \mapsto & \{\mathcal{O}(2N-1,1), \mathcal{O}(2N-3,1^3), \underline{\mathcal{O}(2N-5,2^2,1)}\} \\ \Lambda^{\mathrm{u,sp}}_{\emptyset,(1^2)} & \mapsto & \{\mathcal{O}(2N-1,1), \mathcal{O}(2N-3,1^3), \underline{\mathcal{O}(N-5,1^2)}\} \end{array}.$$

If $N \geq 3$, let $\vartheta^{2,N}_{2} : \mathbf{S}(\mathrm{Sp}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{O}_{2N}(\bar{\mathbb{F}}_q))$ be defined by

$$\vartheta^{2,N}_{2} : \begin{array}{ccc} \Lambda^{\mathrm{u,sp}}_{(2),\emptyset} & \mapsto & \{\mathcal{O}(2N-1,1), \underline{\mathcal{O}(2N-5,5)}, \mathcal{O}(2N-3,3)\} \\ \Lambda^{\mathrm{u,sp}}_{(1),(1)} & \mapsto & \{\mathcal{O}(2N-3,3), \underline{\mathcal{O}(2N-5,3,1^2)}, \mathcal{O}(2N-1,1), \mathcal{O}(2N-3,1^3)\} \\ \Lambda^{\mathrm{u,sp}}_{(1^2),\emptyset} & \mapsto & \{\underline{\mathcal{O}(2N-5,3,1^2)}, \mathcal{O}(2N-3,3)\} \\ \Lambda^{\mathrm{u,sp}}_{\emptyset,(2)} & \mapsto & \{\mathcal{O}(2N-1,1), \mathcal{O}(2N-3,1^3), \underline{\mathcal{O}(2N-5,2^2,1)}\} \\ \Lambda^{\mathrm{u,sp}}_{\emptyset,(1^2)} & \mapsto & \mathcal{O}(2N-3,1^3), \underline{\mathcal{O}(2N-5,1^5)}\} \end{array}.$$

If $N \geq 3$, let $\vartheta^{2,N}_{3} : \mathbf{S}(\mathrm{O}_4(\bar{\mathbb{F}}_q)) \to \mathcal{N}(\mathrm{Sp}_{2N}(\bar{\mathbb{F}}_q))$ be defined by

$$\vartheta^{2,N}_{3} : \begin{array}{ccc} \Lambda^{\mathrm{u,or}}_{(2),\emptyset} & \mapsto & \{\mathcal{O}(2N), \mathcal{O}(2N-2,2), \underline{\mathcal{O}(2N-4,4)}\} \\ \Lambda^{\mathrm{u,or}}_{(1),(1)} & \mapsto & \{\mathcal{O}(2N), \mathcal{O}(2N-2,1^2), \mathcal{O}(2N-2,2), \underline{\mathcal{O}(2N-4,2^2)}\} \\ \Lambda^{\mathrm{u,or}}_{(1^2),\emptyset} & \mapsto & \{\mathcal{O}(2N-2,2), \underline{\mathcal{O}(2N-4,2,1^2)}\} \\ \Lambda^{\mathrm{u,or}}_{\emptyset,(2)} & \mapsto & \{\mathcal{O}(2N), \mathcal{O}(2N-2,1^2), \underline{\mathcal{O}(2N-4,2^2)}\} \\ \Lambda^{\mathrm{u,or}}_{\emptyset,(1^2)} & \mapsto & \{\mathcal{O}(2N), \underline{\mathcal{O}(2N-4,1^4)}\} \end{array}.$$

Notice that in each case, for $N \geq 3$, the underlined orbit is in the closure of any orbit in the set.

Howe correspondence over finite fields is not bijective on the level of the irreducible characters. Nevertheless, Theorem 10 below shows that it should be possible to *extract from* $\Theta^{\mathrm{G},\mathrm{G}'}_k$ *a bijective correspondence* at least when $N' = 2$, given by the map $\underline{\theta}^{2,N}$ introduced in Definition 5.

THEOREM 10. *We assume that* $N' = 2$, $N \geq 2$ *and that Conjecture* 1 *holds. Then Howe correspondence for the dual pair* $(\mathrm{G}', \mathrm{G})$ *induces the map*

$$\begin{cases} \vartheta^{2,N}_{1,a} & \textit{if } (\mathrm{G}', \mathrm{G}) = (\mathrm{O}^{\varepsilon_k}_{2(k^2+2)}(q), \mathrm{Sp}_{2(k^2+k+N)}(q)), \\ \vartheta^{2,N}_{1,b} & \textit{if } (\mathrm{G}', \mathrm{G}) = (\mathrm{Sp}_{2(k^2+k+2)}(q), \mathrm{O}^{\varepsilon_k}_{2(k^2+N)}(q)), \\ \vartheta^{2,N}_{2} & \textit{if } (\mathrm{G}', \mathrm{G}) = (\mathrm{Sp}_{2(k^2+k+2)}(q), \mathrm{O}^{\varepsilon_{k+1}}_{2((k+1)^2+N)}(q)), \\ \vartheta^{2,N}_{3} & \textit{if } (\mathrm{G}', \mathrm{G}) = (\mathrm{O}^{\varepsilon_{k+1}}_{2((k+1)^2+2)}(q), \mathrm{Sp}_{2(k^2+k+N)}(q)). \end{cases}$$

Moreover, if $N \geq 3$*, then the following holds:*

Let $\rho_{\xi',\eta'} \in \mathrm{Irr}(\mathsf{W}_2)$, and let $\rho_{\xi_0,\eta_0} = \underline{\theta}^{2,N}(\rho_{\xi',\eta'})$. Then every irreducible representation $\rho_{\xi,\eta}$ of W_N which corresponds to $\rho_{\xi',\eta'}$ by $\Theta_k^{\mathrm{G},\mathrm{G}'}$ satisfies

$$\mathcal{O}_{\xi_0,\eta_0} \leq \mathcal{O}_{\xi,\eta}. \tag{12}$$

REMARK 1. Proposition 8 and Theorem 10 are unconditional (since Conjecture 1 is known to be true, see [**AMR96**, § 6]) for the following triples $(\mathrm{G}', \mathrm{G}, k)$:

- in Case 1. (a):
  - $(\mathrm{O}_6^-(q), \mathrm{Sp}_{2(N+2)}(q), 1)$ where $2 \leq N \leq 9$;
  - $(\mathrm{O}_{12}^+(q), \mathrm{Sp}_{2(N+6)}(q), 2)$ where $2 \leq N \leq 5$;
- in Case 1. (b):
  - $(\mathrm{Sp}_8(q), \mathrm{O}_{2(N+1)}^-(q), 1)$ where $2 \leq N \leq 10$;
  - $(\mathrm{Sp}_{16}(q), \mathrm{O}_{2(N+4)}^+(q), 2)$ where $2 \leq N \leq 7$;
- in Case 2:
  - $(\mathrm{Sp}_8(q), \mathrm{O}_{2(N+4)}^+(q), 1)$ where $2 \leq N \leq 7$;
  - $(\mathrm{Sp}_{16}(q), \mathrm{O}_{22}^-(q), 2)$;
- in Case 3:
  - $(\mathrm{O}_{12}^+(q), \mathrm{Sp}_{2(N+2)}(q), 1)$ where $2 \leq N \leq 9$;
  - $(\mathrm{O}_{22}^-(q), \mathrm{Sp}_{2(N+6)}(q), 2)$ where $2 \leq N \leq 5$.

PROOF. We will use the examples studied in Section 4.2 and we will also need the following additional computations:

- $\Lambda^{\mathrm{u,sp}}_{(N),\emptyset} = \begin{pmatrix} N \\ - \end{pmatrix}$. We have $\lambda^{\mathrm{u,sp}}_{(N),\emptyset} = (2N)$.
- $\Lambda^{\mathrm{u,sp}}_{\emptyset,(N)} = \begin{pmatrix} 0 & & 2 \\ & N+1 & \end{pmatrix} \sim_{\mathrm{sim}} \begin{pmatrix} 0 & & N+1 \\ & 2 & \end{pmatrix} = \Lambda^{\mathrm{u,sp}}_{(N-1),(1)}$. Example 7 gives $\lambda^{\mathrm{u,sp}}_{\emptyset,(N)} = (2N-2, 2)$.
- $\Lambda^{\mathrm{u,sp}}_{(2),(N-2)} = \begin{pmatrix} 0 & & 4 \\ & N-1 & \end{pmatrix} \sim_{\mathrm{sim}} \begin{pmatrix} 0 & & N-1 \\ & 4 & \end{pmatrix} = \Lambda^{\mathrm{u,sp}}_{(N-2),(2)}$. Example 8 gives $\lambda^{\mathrm{sp}}_{(2),(N-2)} = (2N-6, 6)$.
- $\Lambda^{\mathrm{u,sp}}_{\emptyset,(N-1,1)} = \begin{pmatrix} 0 & & 2 & & 4 \\ & 2 & & N+2 & \end{pmatrix} \sim_{\mathrm{sim}} \begin{pmatrix} 0 & & 2 & & N+2 \\ & 2 & & 4 & \end{pmatrix}$ is distinguished if $N \geq 2$. Example 5 gives $\lambda^{\mathrm{sp}}_{\emptyset,(N-1,1)} = (2N-4, 2, 1^2)$.
- $\Lambda^{\mathrm{u,sp}}_{(1),(N-1)} = \begin{pmatrix} 0 & & 3 \\ & N & \end{pmatrix} \sim_{\mathrm{sim}} \begin{pmatrix} 0 & & N \\ & 3 & \end{pmatrix} = \Lambda^{\mathrm{u,sp}}_{(N-2),(2)}$. Example 6 gives $\lambda^{\mathrm{u,sp}}_{(1),(N-1)} = (2N-4, 4)$.
- $\Lambda^{\mathrm{u,or}}_{(N),\emptyset} = \begin{pmatrix} N \\ 0 \end{pmatrix} \sim_{\mathrm{sim}} \begin{pmatrix} 0 \\ N \end{pmatrix}$. Hence $\lambda^{\mathrm{or}}_{N,\emptyset} = (2N-1, 1)$.
- $\Lambda^{\mathrm{u,or}}_{\emptyset,(N)} = \begin{pmatrix} 0 \\ N \end{pmatrix}$. Hence $\lambda^{\mathrm{or}}_{\emptyset,N} = (2N-1, 1)$.
- $\Lambda^{\mathrm{u,or}}_{(2),(N-2)} = \begin{pmatrix} 2 \\ N-2 \end{pmatrix} \sim_{\mathrm{sim}} \Lambda^{\mathrm{u,or}}_{(N-2),(2)}$. Example 6 gives $\lambda^{\mathrm{or}}_{(2),(N-2)} = (2N-5, 5)$.
- $\Lambda^{\mathrm{u,or}}_{(1),(N-1)} = \begin{pmatrix} 1 \\ N-1 \end{pmatrix} \sim_{\mathrm{sim}} \Lambda^{\mathrm{u,or}}_{(N-1),(1)}$. Example 7 gives $\lambda^{\mathrm{or}}_{(1),(N-1)} = (2N-3, 3)$.
- $\Lambda^{\mathrm{u,or}}_{(1),(N-2,1)} = \begin{pmatrix} 0 & 3 \\ 1 & N \end{pmatrix}$. Example 9 gives $\lambda^{\mathrm{or}}_{(1),(N-2,1)} = (2N-5, 3, 1^2)$.

- $\Lambda^{\mathrm{u,or}}_{\emptyset,(N-1,1)} = \begin{pmatrix} 0 & 2 \\ 1 & N+1 \end{pmatrix}$. Example 3 gives $\lambda^{\mathrm{or}}_{\emptyset,(N-1,1)} = (2N-3, 1^3)$.
- $\Lambda^{\mathrm{u,or}}_{\emptyset,(N-2,2)} = \begin{pmatrix} 0 & 2 \\ 2 & N \end{pmatrix}$. Example 3 gives $\lambda^{\mathrm{or}}_{\emptyset,(N-2,2)} = (2N-5, 2^2, 1)$.
- $\Lambda^{\mathrm{u,or}}_{(1^2),(N-2)} = \begin{pmatrix} 1 & 3 \\ 0 & N \end{pmatrix} = \Lambda^{\mathrm{u,or}}_{(n-2),(1^2)}$. Example 5 gives $\lambda^{\mathrm{or}}_{(1^2),(N-2)} = (2N-5, 3, 1^2)$.
- $\Lambda^{\mathrm{u,or}}_{\emptyset,(N-2,1^2)} = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & N+2 \end{pmatrix}$. Example 4 gives $\lambda^{\mathrm{or}}_{\emptyset,(N-2,1^2)} = (2N-5, 1^5)$.

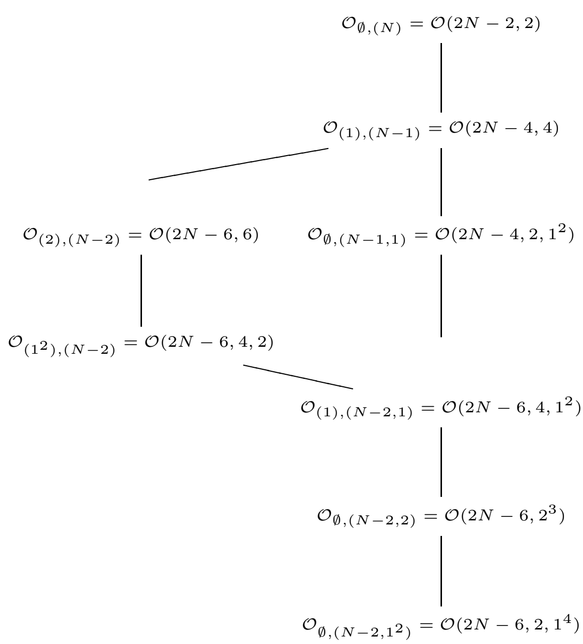
We will consider the four cases above separately.

Case 1 (a): The map $\theta_1^{2,N}$ induces the following correspondence between u-symbols:

$$\begin{array}{ccc} \Lambda^{\mathrm{u,or}}_{(2),\emptyset} & \mapsto & \underline{\Lambda^{\mathrm{u,sp}}_{(2),(N-2)}} \\ \Lambda^{\mathrm{u,or}}_{(1),(1)} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{(1),(N-1)}, \underline{\Lambda^{\mathrm{u,sp}}_{(1),(N-2,1)}}\} \\ \Lambda^{\mathrm{u,or}}_{(1^2),\emptyset} & \mapsto & \underline{\Lambda^{\mathrm{u,sp}}_{(1^2),(N-2)}} \\ \Lambda^{\mathrm{u,or}}_{\emptyset,(2)} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{\emptyset,(N)}, \Lambda^{\mathrm{u,sp}}_{\emptyset,(N-1,1)}, \underline{\Lambda^{\mathrm{u,sp}}_{\emptyset,(N-2,2)}}\} \\ \Lambda^{\mathrm{u,or}}_{\emptyset,(1^2)} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{\emptyset,(N)}, \Lambda^{\mathrm{u,sp}}_{\emptyset,(N-1,1)}, \underline{\Lambda^{\mathrm{u,sp}}_{\emptyset,(N-2,1^2)}}\} \end{array}, \quad \text{if } N \geq 3. \tag{13}$$

Here and in the rest of this proof the underlined symbols correspond to the representations $\rho_{\xi_0,\eta_0} = \underline{\theta}^{2,N}(\rho_{\xi',\eta'})$.

Combining the above computations with Example 10, Example 11, Example 12, and Example 13, we obtain the following closure order on the unipotent classes of $\mathrm{Sp}_{2N}(\bar{\mathbb{F}}_q)$ occurring in the above correspondence:



Hence (13) induces $\vartheta_{1,a}^{2,N}$ and the second assertion of the Theorem follows in the case 1 (a).

Case 1 (b): The map $\theta_1^{2,N}$ induces the following correspondence between u-symbols:

$$
\begin{array}{ccc}
\Lambda^{\mathrm{u,sp}}_{(2),\emptyset} & \mapsto & \underline{\Lambda^{\mathrm{u,or}}_{(N-2),(2)}} \\
\Lambda^{\mathrm{u,sp}}_{(1),(1)} & \mapsto & \{\Lambda^{\mathrm{u,or}}_{(N-1),(1)}, \underline{\Lambda^{\mathrm{u,or}}_{(N-2,1),(1)}}\} \\
\Lambda^{\mathrm{u,sp}}_{(1^2),\emptyset} & \mapsto & \underline{\Lambda^{\mathrm{u,or}}_{(N-2),(1^2)}} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(2)} & \mapsto & \{\Lambda^{\mathrm{u,or}}_{(N),\emptyset}, \Lambda^{\mathrm{u,or}}_{(N-1,1),\emptyset}, \underline{\Lambda^{\mathrm{u,or}}_{(N-2,2),\emptyset}}\} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(1^2)} & \mapsto & \{\Lambda^{\mathrm{u,or}}_{(N),\emptyset}, \Lambda^{\mathrm{u,or}}_{(N-1,1),\emptyset}, \underline{\Lambda^{\mathrm{u,or}}_{(N-2,1^2),\emptyset}}\}
\end{array}
\quad, \quad \text{if } N \geq 3. \tag{14}
$$

We obtain the following closure order on the unipotent classes of $\mathrm{O}_{2N}(\bar{\mathbb{F}}_q)$ occurring in the above correspondence:

$$
\begin{tikzcd}[column sep=tiny]
 & \mathcal{O}_{\emptyset,(N)} = \mathcal{O}(2N-1,1) \arrow[d, dash] & \\
 & \mathcal{O}_{(1),(N-1)} = \mathcal{O}(2N-3,3) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_{\emptyset,(N-1,1)} = \mathcal{O}(2N-3,1^3) \arrow[dr, dash] & & \mathcal{O}_{(2),(N-2)} = \mathcal{O}(2N-5,5) \arrow[dl, dash] \\
 & \mathcal{O}_{(1),(N-2,1)} = \mathcal{O}_{(1^2),(N-2)} = \mathcal{O}(2N-5,3,1^2) \arrow[d, dash] & \\
 & \mathcal{O}_{\emptyset,(N-2,2)} = \mathcal{O}(2N-5,2^2,1) \arrow[d, dash] & \\
 & \mathcal{O}_{\emptyset,(N-2,1^2)} = \mathcal{O}(2N-5,1^5) &
\end{tikzcd}
$$

Hence (14) induces $\vartheta_{1,b}^{2,N}$ and the second assertion of the Theorem follows in the case 1 (b).

Case 2: The map $\theta_2^{2,N}$ induces the following correspondence between u-symbols:

$$
\begin{array}{ccc}
\Lambda^{\mathrm{u,sp}}_{(2),\emptyset} & \mapsto & \{\Lambda^{\mathrm{u,or}}_{(N),\emptyset}, \underline{\Lambda^{\mathrm{u,or}}_{(N-2),(2)}}, \Lambda^{\mathrm{u,or}}_{(N-1),(1)}\} \\
\Lambda^{\mathrm{u,sp}}_{(1),(1)} & \mapsto & \{\Lambda^{\mathrm{u,or}}_{(N-1),(1)}, \underline{\Lambda^{\mathrm{u,or}}_{(N-2,1),(1)}}, \Lambda^{\mathrm{u,or}}_{(N),\emptyset}, \Lambda^{\mathrm{u,or}}_{(N-1,1),\emptyset}\} \\
\Lambda^{\mathrm{u,sp}}_{(1^2),\emptyset} & \mapsto & \{\underline{\Lambda^{\mathrm{u,or}}_{(N-2),(1^2)}}, \Lambda^{\mathrm{u,or}}_{(N-1),(1)}\} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(2)} & \mapsto & \{\Lambda^{\mathrm{u,or}}_{(N),\emptyset}, \Lambda^{\mathrm{u,or}}_{(N-1,1),\emptyset}, \underline{\Lambda^{\mathrm{u,or}}_{(N-2,2),\emptyset}}\} \\
\Lambda^{\mathrm{u,sp}}_{\emptyset,(1^2)} & \mapsto & \Lambda^{\mathrm{u,or}}_{(N-1,1),\emptyset}, \underline{\Lambda^{\mathrm{u,or}}_{(N-2,1^2),\emptyset}}\}
\end{array}
\quad, \quad \text{if } N \geq 3. \tag{15}
$$

We obtain the following closure order on the unipotent classes of $\mathrm{O}_{2N}(\bar{\mathbb{F}}_q)$ occurring in the above correspondence:

$$\begin{tikzcd}[column sep=tiny]
 & \mathcal{O}_{\emptyset,(N)} = \mathcal{O}(2N-1,1) \arrow[d, dash] & \\
 & \mathcal{O}_{(1),(N-1)} = \mathcal{O}(2N-3,3) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_{\emptyset,(N-1,1)} = \mathcal{O}(2N-3,1^3) \arrow[dr, dash] & & \mathcal{O}_{(2),(N-2)} = \mathcal{O}(2N-5,5) \arrow[dl, dash] \\
 & \mathcal{O}_{(1^2),(N-2)} = \mathcal{O}_{(1),(N-2,1)} = \mathcal{O}(2N-5,3,1^2) \arrow[d, dash] & \\
 & \mathcal{O}_{\emptyset,(N-2,2)} = \mathcal{O}(2N-5,2^2,1) \arrow[d, dash] & \\
 & \mathcal{O}_{\emptyset,(N-2,1^2)} = \mathcal{O}(2N-5,1^5) &
\end{tikzcd}.$$

Hence (15) induces $\vartheta_2^{2,N}$ and the second assertion of the Theorem follows in the case 2.

Case 3: The map $\theta_3^{2,N}$ induces the following correspondence between u-symbols:

$$\begin{array}{ccc}
\Lambda^{\mathrm{u,or}}_{(2),\emptyset} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{(N),\emptyset}, \Lambda^{\mathrm{u,sp}}_{(N-1,1),\emptyset}, \underline{\Lambda^{\mathrm{u,sp}}_{(N-2,2),\emptyset}}\} \\
\Lambda^{\mathrm{u,or}}_{(1),(1)} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{(N),\emptyset}, \Lambda^{\mathrm{u,sp}}_{(N-1,1),\emptyset}, \Lambda^{\mathrm{u,sp}}_{(N-1),(1)}, \underline{\Lambda^{\mathrm{u,sp}}_{(N-2,1),(1)}}\} \\
\Lambda^{\mathrm{u,or}}_{(1^2),\emptyset} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{(N-1,1),\emptyset}, \underline{\Lambda^{\mathrm{u,sp}}_{(N-2,1^2),\emptyset}}\} \\
\Lambda^{\mathrm{u,or}}_{\emptyset,(2)} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{(N),\emptyset}, \Lambda^{\mathrm{u,sp}}_{(N-1),(1)}, \underline{\Lambda^{\mathrm{u,sp}}_{(N-2),(2)}}\} \\
\Lambda^{\mathrm{u,or}}_{\emptyset,(1^2)} & \mapsto & \{\Lambda^{\mathrm{u,sp}}_{(N-1),(1)}, \underline{\Lambda^{\mathrm{u,sp}}_{(N-2),(1^2)}}\}
\end{array}, \quad \text{if } N \geq 3. \tag{16}$$

We obtain the following closure order on the unipotent classes of $\mathrm{Sp}_{2N}(\overline{\mathbb{F}}_q)$ occurring in the correspondence above.

$$\begin{tikzcd}[column sep=tiny]
& \mathcal{O}_{(N),\emptyset} = \mathcal{O}(2N) \arrow[d, dash] & \\
& \mathcal{O}_{(N-1),(1)} = \mathcal{O}(2N-2,2) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{O}_{(N-1,1),\emptyset} = \mathcal{O}(2N-2,1^2) \arrow[dr, dash] & & \mathcal{O}_{(N-2),(2)} = \mathcal{O}(2N-4,4) \arrow[dl, dash] \\
& \mathcal{O}_{(N-2,2),\emptyset} = \mathcal{O}_{(N-2,1),(1)} = \mathcal{O}(2N-4,2^2) \arrow[d, dash] & \\
& \mathcal{O}_{(N-2),(1^2)} = \mathcal{O}(2N-4,2,1^2) \arrow[d, dash] & \\
& \mathcal{O}_{(N-2,1^2),\emptyset} = \mathcal{O}(2N-4,1^4) &
\end{tikzcd}$$

Hence (16) induces $\vartheta_3^{2,N}$ and the second assertion of the Theorem follows in the case 3. $\square$

Property (12) shows that the map $\underline{\theta}^{2,N}$ plays a special role in Howe correspondence. We will now restrict our attention to it and see how it relates to [**AKP13**, (19)].

Let $\widetilde{\mathrm{sgn}} := \mathrm{sgn} \otimes \mathrm{sgn}_{\mathrm{CD}}$ denote the product of the sign character $\mathrm{sgn} = \rho_{\emptyset,(1^N)}$ of the group W_N by the character $\mathrm{sgn}_{\mathrm{CD}} = \rho_{\emptyset,(N)}$, *i.e.*:

$$\widetilde{\mathrm{sgn}} \otimes \rho_{\xi,\eta} = \rho_{{}^t\xi,{}^t\eta}.$$

Then, when $k_2 = k$, let $\underline{\theta}^{2,N}_{\mathrm{twist}}$ be the map defined by

$$\underline{\theta}^{2,N}_{\mathrm{twist}} := \begin{cases} \mathrm{sgn} \circ \underline{\theta}^{2,N} \circ \mathrm{sgn} & \text{if } \mathrm{G}' \text{ symplectic,} \\ \widetilde{\mathrm{sgn}} \circ \underline{\theta}^{2,N} \circ \widetilde{\mathrm{sgn}} & \text{if } \mathrm{G}' \text{ orthogonal.} \end{cases} \tag{17}$$

We obtain

$$\underline{\theta}^{2,N}_{\mathrm{twist}}(\rho_{\xi',\eta'}) = \begin{cases} \rho_{(1^2)\oplus\xi',\eta'} & \text{if } \mathrm{G}' \text{ symplectic,} \\ \rho_{\xi',(1^2)\oplus\eta'} & \text{if } \mathrm{G}' \text{ orthogonal.} \end{cases} \tag{18}$$

In the case where $k = 0$ and $\varepsilon = +$, we have $\tilde{n}_1(k) = n_1$, $\tilde{n}_2(k) = n_2$, $N = \max(n_1, n_2)$ and $N' = \min(n_1, n_2)$. Hence Howe correspondence between the irreducible components of the unipotent principal series of the groups G and G' (where $(\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_{2n}(q), \mathrm{O}^+_{2n'}(q))$ or $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}^+_{2n'}(q), \mathrm{Sp}_{2n}(q))$ is given by the correspondence $\Theta_0^{\mathrm{G},\mathrm{G}'}$ between irreducible characters of the groups W_N and $W_{N'}$, and (18) coincides with [**AKP13**, (19)].

5. Howe correspondence and wave front set

Recall (see [**Lus92**], [**GM99**], [**AA07**]) that for every irreducible character χ_Π of the $\mathbb{F}_q$-points $G = \mathbf{G}^F$ of a split connected reductive group $\mathbf{G}$ defined over $\mathbb{F}_q$ (assuming that the characteristic p of $\mathbb{F}_q$ is "good for G": for instance, if G a symplectic group or a split special orthogonal group, then p must be odd) there is a unique rational unipotent class $\mathcal{O}_\Pi$ in $\mathbf{G}$ which has the property that there exists $u \in \mathcal{O}_\Pi(q)$ such that $\chi_\Pi(u) \neq 0$ and $\mathcal{O}_\Pi$ has maximal dimension among classes with that property. The class $\mathcal{O}_\Pi$ is called the *unipotent support* of χ_Π. It coincides with the class defined in [**Lus84**, §13.3].

More precisely, suppose Π is unipotent. Then there exists an irreducible representation ρ of the Weyl group W of $\mathbf{G}$ such that the scalar product between χ_Π and the *almost character* R_ρ (which is defined as a certain linear combination of Deligne-Lusztig generalized characters in [**Lus84**, page 347 and (4.24.1)], and coincides with the virtual character in Eqn. (3) when $\mathrm{G} = \mathrm{GL}_n(q)$) is non-zero. Moreover, if ρ' is another irreducible representation of W such that χ_Π has non-scalar product with $R_{\rho'}$, then ρ and ρ' belong to the same family of characters of W (see [**Lus84**, Theorems 5.25 and 6.17]). Thus, we can associate with χ_Π a unique family of characters of W, or equivalently, a unique two-sided cell in W. Let ρ_{spe} be the unique *special character* in this family (for $\mathbf{G}$ of classical type a family of characters of W corresponds to a similarity class of u-symbols, and the symbol corresponding to ρ_{spe} is the unique distinguished u-symbol in that family, [**Lus84**, (4.5.6)]). Then the class $\mathcal{O}_\Pi$ coincides with the unipotent class corresponding to ρ_{spe} by the Springer correspondence for the group W. In particular the unipotent class $\mathcal{O}_\Pi$ is always special.

Moreover, every rational unipotent class $\mathcal{O}$ on which χ_Π does not vanish (i.e., such that there exists $u \in \mathcal{O}(q)$ with $\chi_\Pi(u) \neq 0$) satisfies

$$\mathcal{O} \leq \mathcal{O}_\Pi, \tag{19}$$

see [**AA07**, Theorem 6.1].

Suppose Π is an irreducible unipotent representation of a split group G, such as $\mathrm{O}^+_{2n}(q)$ or $\mathrm{Sp}_{2n}(q)$, which belongs to the principal series. The algebra of the endomorphisms of the principal series which commute with the action of G is the Iwahori-Hecke algebra, whose irreducible representations coincide with the irreducible representations of the Weyl group W. Hence, as we remarked previously, there is a one to one correspondence between the irreducible representations of W and the irreducible representations of G which occur in the principal series. Given an irreducible representation ρ of W we denoted by Π_ρ the corresponding representation of G. Furthermore, the almost character R_ρ has a non-zero scalar product with the character of Π_ρ. (This follows from [**Lus84**, Theorem 4.23]. For an explicit argument see pages 297 and 298 in [**Lus84**].)

If the group $\mathbf{G}$ is disconnected and Π be an irreducible representation of $\mathrm{G} = \mathbf{G}^F$, we define the unipotent support of Π, denoted $\mathcal{O}_\Pi$, to be the union of the rational unipotent classes $\mathcal{O} \subseteq \mathbf{G}$ of maximal dimension, such that $\mathcal{O} \cap \mathrm{G}$ has a non-empty intersection with the support of the character χ_Π.

Let $\Pi = \Pi_{\rho_{\xi,\eta}}$ be an irreducible unipotent representation of $\mathrm{O}^+_{2n}(q)$ which belongs to the principal series, where (ξ, η) is a pair of partitions of n. Two cases can occur: the restriction Π_{SO} of Π to $\mathrm{SO}^+_{2n}(q)$ is either irreducible, or is the direct sum of two nonequivalent irreducible representations $\Pi^{\mathrm{I}}_{\mathrm{SO}}$ and $\Pi^{\mathrm{II}}_{\mathrm{SO}}$. The latter case arises if and only if the restriction of the representation $\rho_{\xi,\eta}$ to the Weyl group of the special orthogonal group splits into the sum of two inequivalent representations, i.e. the partition (ξ, η) is such that $\xi = \eta$, see [**Car93**, Prop. 11.4.4].

$(*)$ We recall (see for instance [**Spa82**]) that any rational unipotent class $\mathcal{O}$ in O_{2n}^+ is either a rational unipotent class in SO_{2n}^+ or is the disjoint union of a rational unipotent class $\mathcal{O}(u)$ in SO_{2n}^+ and a rational unipotent class $\mathcal{O}(sus^{-1})$ in SO_{2n}^+, with $u \in \mathrm{SO}_{2n}^+(q)$ unipotent and some $s \in \mathrm{O}_{2n}(q) \setminus \mathrm{SO}_{2n}(q)$. Both these classes have the same dimension.

LEMMA 11. *If Π_{SO} is irreducible then*

$$\mathcal{O}_{\Pi} = \mathcal{O}_{\Pi_{\mathrm{SO}}}. \tag{20}$$

If $\Pi_{\mathrm{SO}} = \Pi_{\mathrm{SO}}^{\mathrm{I}} \oplus \Pi_{\mathrm{SO}}^{\mathrm{II}}$, then

$$\mathcal{O}_{\Pi} = \mathcal{O}_{\Pi_{\mathrm{SO}}^{\mathrm{I}}} \cup \mathcal{O}_{\Pi_{\mathrm{SO}}^{\mathrm{II}}}. \tag{21}$$

In both cases, $\mathcal{O}_{\Pi}$ is a single unipotent class in O_{2n}^+.

PROOF. Let us assume first that Π_{SO} is irreducible. Then the restrictions of the two characters χ_{Π} and $\chi_{\Pi_{\mathrm{SO}}}$ to $\mathrm{SO}_{2n}(q)$ are equal. In particular, $\chi_{\Pi_{\mathrm{SO}}}(g) = \chi_{\Pi}(g) = \chi_{\Pi}(sgs^{-1}) = \chi_{\Pi_{\mathrm{SO}}}(sgs^{-1})$ for any $g \in \mathrm{SO}_{2n}(q)$ and s as in $(*)$. Let $\mathcal{O}(u)$ be a rational unipotent class in O_{2n}^+, as in $(*)$. We see that, with the notation of $(*)$, the restriction of χ_{Π} to $\mathcal{O}(u) \cap \mathrm{SO}_{2n}^+(q)$ is non-zero if and only if the restriction of χ_{Π} to $\mathcal{O}(sus^{-1}) \cap \mathrm{SO}_{2n}^+(q)$ is non-zero. But the classes $\mathcal{O}(sus^{-1})$ and $\mathcal{O}(u)$ have the same dimension. Therefore, if that dimension is maximal among the unipotent classes which have a non-empty intersection with the support of $\chi_{\Pi_{\mathrm{SO}}}$, we get a contradiction. Thus the unipotent support of Π_{SO} is the unipotent class in O_{2n}^+ which is also a single unipotent class in SO_{2n}^+. Hence, (20) follows.

Assume now that $\Pi_{\mathrm{SO}} = \Pi_{\mathrm{SO}}^{\mathrm{I}} \oplus \Pi_{\mathrm{SO}}^{\mathrm{II}}$. In this case the representations $\Pi_{\mathrm{SO}}^{\mathrm{I}}$ and $\Pi_{\mathrm{SO}}^{\mathrm{II}}$ are permuted via the action of the group element s, as in $(*)$, and so are their unipotent supports. More precisely, $\chi_{\Pi_{\mathrm{SO}}^{\mathrm{II}}}(u) = \chi_{\Pi_{\mathrm{SO}}^{\mathrm{I}}}(sus^{-1})$, $\mathcal{O}_{\Pi_{\mathrm{SO}}^{\mathrm{I}}} = \mathcal{O}(u)$, $\mathcal{O}_{\Pi_{\mathrm{SO}}^{\mathrm{II}}} = \mathcal{O}(sus^{-1})$ and the right hand side of (21) is a single unipotent rational class in O_{2n}. (Since, as we noticed before, $\xi = \eta$, these classes have the same set of elementary divisors and hence the same dimension, see [**Car93**, page 399]. They are described explicitly in [**Car93**, § 13.3, Type D_l]).

Let $u \in \mathrm{SO}_{2n}^+(q) \cap \mathcal{O}_{\Pi}$ be such that $\chi_{\Pi}(u) \neq 0$. Since $\chi_{\Pi}(u) = \chi_{\Pi_{\mathrm{SO}}^{\mathrm{I}}}(u) + \chi_{\Pi_{\mathrm{SO}}^{\mathrm{II}}}(u)$, we see that that $\chi_{\Pi_{\mathrm{SO}}^{\mathrm{I}}}(u) \neq 0$ or $\chi_{\Pi_{\mathrm{SO}}^{\mathrm{II}}}(u) \neq 0$. Hence, $\mathcal{O}_{\Pi} = \mathcal{O}_{\Pi_{\mathrm{SO}}^{\mathrm{I}}} \cup \mathcal{O}_{\Pi_{\mathrm{SO}}^{\mathrm{II}}}$. □

Since (19) holds for the representations of $\mathrm{SO}_{2n}(q)$, we see from Lemma 11 that it also holds for the representations of $\mathrm{O}_{2n}^+(q)$.

COROLLARY 12. *Let $\Pi'_{\rho_{\xi',\eta'}}$ be an irreducible unipotent representation of $\mathrm{G}' = \mathrm{Sp}_4(q)$ (resp. $\mathrm{G}' = \mathrm{O}_4^+(q)$) which belongs to the principal series of G'. Let $n \geq 3$, and let $(\xi_0, \eta_0) := (\xi', (n-2) \cup \eta')$.*

Assume that Conjecture 1 holds. Then every representation of $\mathrm{G} = \mathrm{O}_{2n}^+(q)$ (resp. $\mathrm{G}' = \mathrm{Sp}_{2n}(q)$) which occurs in the image of Π' by Howe correspondence for the dual pair $(\mathrm{G}', \mathrm{G})$ is such that the closure of its unipotent support contains the closure of the unipotent support of $\Pi_{\xi_0 \cdot \eta_0}$.

PROOF. This follows directly from (12) and from the fact that, for the map $\vartheta_1^{2,n}$ with $n \geq 3$, the underlined orbit in the output set is contained in the closure of each orbit in that set. $\square$

Recall Alvis-Curtis Duality $D_{\mathrm{G}}\colon \mathfrak{R}(\mathrm{G}) \to \mathfrak{R}(\mathrm{G})$ ([**Alv79**], [**Cur80**], [**Aub92**]), which is defined for representations of $\mathrm{G} = \mathbf{G}^F$, when $\mathbf{G}$ is connected.

Let Π be an irreducible unipotent representation of $\mathrm{O}_{2n} := \mathrm{O}_{2n}^+(q)$, as in Lemma 11. If Π_{SO} is irreducible, define $D_{\mathrm{O}_{2n}}(\Pi)$ to be the unique irreducible representation $\tilde{\Pi}$ of O_{2n} such that $\tilde{\Pi}_{\mathrm{SO}} = D_{\mathrm{SO}_{2n}}(\tilde{\Pi}_{\mathrm{SO}})$. If $\Pi_{\mathrm{SO}} = \Pi_{\mathrm{SO}}^I \oplus \Pi_{\mathrm{SO}}^{II}$, let $D_{\mathrm{O}_{2n}}(\Pi)$ to be the only irreducible representation $\tilde{\Pi}$ of $\mathrm{O}_{2n}^+(q)$ such that $\tilde{\Pi}_{\mathrm{SO}} = D_{\mathrm{SO}_{2n}}(\Pi_{\mathrm{SO}}^I) \oplus D_{\mathrm{SO}_{2n}}(\Pi_{\mathrm{SO}}^{II})$. Then in both cases, $D_{\mathrm{O}_{2n}}(\Pi_{\rho_{\xi,\eta}}) = \Pi_{\rho_{t_\eta, t_\xi}}$. In other words, $D_{\mathrm{O}_{2n}}(\Pi_\rho) = \Pi_{\rho\otimes\mathrm{sgn}}$, see [**Lus84**, (6.8.6)]. Hence, $\mathcal{O}_\Pi = \mathcal{O}_{\xi,\eta}$ if and only if $\mathcal{O}_{D_{\mathrm{O}_{2n}}(\Pi)} = \mathcal{O}_{t_\xi, t_\eta}$. Also, tensoring with the sign representation of the Weyl group translates via Springer correspondence to an order reversing involution on the special unipotent orbits, see [**Car93**, pages 389, 390]. By combining this with Corollary 12, (18), (17) and [**AKP13**, Proposition 5], we deduce the following theorem.

THEOREM 13. *Consider the dual pair* $(\mathrm{G}' = \mathrm{Sp}_4(q), \mathrm{G} = \mathrm{O}_{2n}^+(q))$ *with* $n \geq 4$. *(This is a dual pair in the stable range with* G' *the smaller member.) Assume that Conjecture* 1 *holds.*

Let π' *be an irreducible representation of* G' *such that* $D_{\mathrm{G}'}(\pi')$ *is unipotent and belongs to the principal series of* G'. *Then there is a unique irreducible representation* π_{pref} *of* G *such that* $D_{\mathrm{G}}(\pi_{\mathrm{pref}})$ *corresponds to* $D_{\mathrm{G}'}(\pi')$ *via Howe Correspondence for the pair* $(\mathrm{G}', \mathrm{G})$ *and the unipotent support* $\mathcal{O}_{\pi_{\mathrm{pref}}}$ *of* π_{pref} *contains in its closure the unipotent support of any irreducible representation* π *of* G *such that* $D_{\mathrm{G}}(\pi)$ *corresponds to* $D_{\mathrm{G}'}(\pi')$.

Let λ', λ' *be the partitions describing the rational unipotent class* $\mathcal{O}_{\pi'}$ *and* $\mathcal{O}_{\pi_{\mathrm{pref}}}$, *respectively. Then* λ *is obtained from* λ' *by adding a column of length* $2N-4$ *to* λ', *as in* [**AKP13**, Theorem 1].

Lusztig has proved in [**Lus92**, Theorem 11.2], under the assumption that p is large enough, that the closure of the unipotent support of Π coincides with the wave front set (as defined by Kawanaka in [**Kaw87**]) of its dual. Recall that $D_{\mathrm{GL}_n(q)}$ maps the unipotent character R_{ρ_μ} to the unipotent character $R_{\rho_{t_\mu}}$. Hence Theorem 3 and Theorem 13 imply the following result.

COROLLARY 14. *Let* $(\mathrm{G}', \mathrm{G})$ *be one of the dual pairs* $(\mathrm{GL}_{n'}(q), \mathrm{GL}_n(q))$ *or* $(\mathrm{Sp}_4(q), \mathrm{O}_{2n}^+(q))$ *with* $n \geq 4$. *In the latter case, we assume that Conjecture* 1 *holds.*

Let Π' *be a unipotent irreducible representation of* G' *that belongs to the principal series of* G'. *Then there is a unique irreducible representation* Π_{pref} *of* G *such that* Π_{pref} *corresponds to* Π' *via Howe Correspondence for the pair* $(\mathrm{G}', \mathrm{G})$ *and the wave front set of* Π_{pref} *contains the wave front set of any irreducible representation* Π *of* G *such that* Π *corresponds to* Π'.

PROOF. Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_4(q), \mathrm{O}_{2n}^+(q))$. We put $\pi' := D_{\mathrm{G}'}(\Pi')$. Since $D_{\mathrm{G}'}$ is an involution, we have $\Pi' = D_{\mathrm{G}'}(\pi')$. Then we apply Theorem 13 to the representation π', and we set

$$\Pi_{\mathrm{pref}} := D_G(\pi_{\mathrm{pref}}).$$

From Theorem 13, it follows that Π_{pref} corresponds to Π' by Howe correspondence, and that the unipotent support $\mathcal{O}_{\pi_{\mathrm{pref}}}$ of π_{pref} contains in its closure the unipotent support $\mathcal{O}_\pi$ of any irreducible representation π of G such that $D_{\mathrm{G}}(\pi)$ corresponds to $D_{\mathrm{G}'}(\pi') = \Pi'$. Setting $\Pi := D_{\mathrm{G}}(\pi)$, and using the fact that the closure of $\mathcal{O}_\pi$ coincides with the wave front set of $D_G(\pi) = \Pi$, we get that the wave front set of Π_{pref} contains the wave front set of any irreducible representation Π of G such that Π corresponds to Π'.

A similar argument using Theorem 3 instead of 13 gives the result when $(\mathrm{G}', \mathrm{G})$ is of type II. □

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