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Characters, Dual Pairs, and Unipotent Representations

TOMASZ PRZEBINDA*

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803-4913

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We lift distribution characters of irreducible unitary representations of classical groups from the group to the Lie algebra via the Cayley Transform. Then a specific class of these characters admits Fourier transform supported on the closure of a single nilpotent coadjoint orbit. We calculate also the wave front set of the most singular low rank representations. © 1991 Academic Press, Inc.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to exhibit some irreducible unitary representations of real reductive groups which are attached to nilpotent coadjoint orbits in a very explicit fashion. For some abstract conjectures, see [V1, V3]. We work in the formalism of real reductive dual pairs [H7]. Thus there is a real symplectic vector space W, with a symplectic form $\langle\ ,\ \rangle$, the corresponding symplectic group $Sp = Sp(W)$, and a pair of subgroups $G, G' \subseteq Sp$. We consider (mainly) the pairs of type I . Thus there is a

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division algebra $\mathbb{D}$ over $\mathbb{R}$ (the reals) with an involution $\#$ and there are two right $\mathbb{D}$-vectors spaces V, V' with forms $(\ ,\ )$, $(\ ,\ )'$ (one $\#$-Hermitian and the other $\#$-skew-Hermitian) so that G is isomorphic to the group of isometries of $(\ ,\ )$ and G' to the group of isometries of $(\ ,\ )'$, (see Section 2). Denote by $\tilde{G}, \tilde{G}'$ the preimages of G, G' in the metaplectic group $\widetilde{Sp}$. Let (as in [H7, Sect. 6] or [H6]) $R(\tilde{G} \cdot \tilde{G}', \omega)$ denote the set of infinitesimal equivalence classes of representations of $\tilde{G} \cdot \tilde{G}'$ which occur in Howe's duality correspondence. Here ω is the oscillator representation of $\widetilde{Sp}$ [H2]. Each such representation $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$ determines (and is determined by) a temperated distribution $f \in S^*(W)$. We call it an initertwining distribution (5.1). We prove (see (6.15), (6.17)) the following

(1.1) THEOREM. *Suppose that the pair G, G' is in the stable range with G, the smaller member* [H8, (2.14)] *compact. Assume that the form $(\ ,\ )'$ is split and that the representation Π is trivial on the identity component of $\tilde{G}$. Then the pullback of the distribution character $\Theta_{\Pi'}$ of Π' to the Lie algebra $\underline{g}'$ of $\tilde{G}'$ via the Cayley Transform $\tilde{c}_{-}$* (3.16), *when divided by a real analytic function $ch_{\underline{g}'}$* (5.6), *coincides with a finite sum of homogeneous distributions on $\underline{g}'$. The Fourier Transform* (4.14) *of this sum is supported on the closure of a single nilpotent coadjoint orbit $O'_{\max} \subseteq \underline{g}'^*$* (2.19).

This clearly resembles Kirillov's character formula for irreducible unitary representations of nilpotent groups [K], and Harish-Chandra's formula for p-adic groups [Ha, P4]. We conjecture that (under some additional assumptions) (1.1) should hold even if G is not compact (see (6.16)). Theorem (1.1) was discovered in an attempt to prove the following

(1.2) *Conjecture* (Howe). Suppose that the pair G, G' is in the stable range with G the smaller member. Assume that $\Pi \in R(\tilde{G}, \omega)$ is unitary and finite dimensional. Then $WF(\Pi') = \tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0))$. (This is the closure of a single nilpotent coadjoint orbit in $\underline{g}'^*$ (2.19).)

Here $WF(\Pi')$ stands for the wave front set of the representation Π' [H1] and $\tau_{\underline{g}'}, \tau_{\underline{g}}$ are the "orbit parameter maps" (2.1). By [Li1], Π' is unitary (with some exceptions (see (1.3.1)) not covered by his proof, for which Π' should be unitary too). We reduce (1.2) to a manageable conjecture (8.1) and prove (see (8.2))

(1.3) THEOREM. *The statement* (1.2) *is true if*

(1.3.1) *(G, G') is not one of the pairs $(Sp(n, \mathbb{R}),\ O(2n, 2n))$ or $(Sp(n, \mathbb{C}),\ O(4n, \mathbb{C}))$,*

(1.3.2) *the form $(\ ,\ )'$ is split (then the covering $\tilde{G} \to G$ splits over the Zariski identity component G_0 of G and Π defines a representation π of G_0* (5.26.6)), (1.3.3) *the representation* π *is trivial, and*

(1.3.4) *if* $\mathbb{D}=\mathbb{R}$ *and if* G *is not compact then* $(1/2)\dim V' - 2\dim \underline{g}/\dim V$ *is an even integer.*

The reason for the assumptions (1.3.1)–(1.3.4) is to ensure that the corresponding intertwining distribution $f \in S^*(W)$ (5.1) is a finite sum of homogeneous distributions ((5.9), (5.26)) so that the wave front set of f is easily computable (5.51). It is possible to determine the Langlands–Vogan parameters of Π' (1.3) and we'll report on it elsewhere.

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2. The Orbit Parameter Maps

Here we collect some simple technical results about the structure of $G \cdot G'$-orbits in W to be used in Section 5. For a subspace $\underline{h} \subseteq \underline{sp}$ define a quadratic map

$$\tau_{\underline{h}}: W \to \underline{h}^*, \tau_{\underline{h}}(w)(x) = \tfrac{1}{4}\langle x(w), w\rangle \qquad (w \in W, x \in \underline{h}). \tag{2.1}$$

Assume for the rest of this section that G, G' is an irreducible pair of type I in $Sp(W)$ [H7, Sect. 6]. This means that there is a division algebra $\mathbb{D}(=\mathbb{R}, \mathbb{C}, \mathbb{H})$ with involution $\#$ and two right $\mathbb{D}$-vector spaces V, V' with non-degenerate forms $(\ ,\ )$, $(\ ,\ )'$ one $\#$-Hermitian and the other $\#$-skew-Hermitian such that G is the group of isometries of $(\ ,\ )$ and G' is the group of isometries of $(\ ,\ )'$. The symplectic space is defined by

$$\begin{gathered} W = \mathrm{Hom}_{\mathbb{D}}(V', V),\ \langle w', w\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}\, w^*w', \\ (w(v'), v) = (v', w^*(v))' \qquad (v' \in V'; v \in V; w', w \in W). \end{gathered} \tag{2.2}$$

Here $\mathrm{tr}_{\mathbb{D}/\mathbb{R}}$ stands for the reduced trace. The embeddings of G and G' into $Sp(W) = Sp$ are defined via the following action of these groups on W.

$$g(w) = gw, \qquad g'(w) = wg'^{-1}\ (w \in W, g \in G, g' \in G'). \tag{2.3}$$

We shall denote by $\underline{g} \subseteq \mathrm{End}_{\mathbb{D}}(V)$ and by $\underline{g} \subseteq \mathrm{End}_{\mathbb{D}}(V')$ the Lie algebras of G and G', respectively. There are maps [H5, Chap. I, (7.5)]

$$\begin{gathered} \tilde{\tau}: W \to \underline{g}, \qquad \tilde{\tau}': W \to \underline{g}', \\ \tilde{\tau}(w) = ww^*, \qquad \tilde{\tau}'(w) = w^*w \quad (w \in W). \end{gathered} \tag{2.4}$$

Clearly if we identify the real vector space $\underset{\sim}{g}$ with its algebraic dual $\underset{\sim}{g}^*$ via the bilinear form

$$\underset{\sim}{g} \times \underset{\sim}{g} \ni (x, y) \to \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) \in \mathbb{R} \tag{2.5}$$

then $\tau_{\underset{\sim}{g}}$ (2.1) will coincide with $(1/4)\tilde{\tau}$. Similarly $\tau_{\underset{\sim}{g}'}$ will coincide with $(1/4)\tilde{\tau}'$.

(2.6) LEMMA. *Let $d\tau_{\underset{\sim}{g}}(w)$ denote the derivative of $\tau_{\underset{\sim}{g}}$ at $w \in W$. Then the annihilator of the image of $d\tau_{\underset{\sim}{g}}(w)$ in $\underset{\sim}{g}$ $(\mathrm{im}\, d\tau_{\underset{\sim}{g}}(w))^{\perp} = \{x \in \underset{\sim}{g}; xw = 0\}$.*

Proof. Since

$$d\tilde{\tau}(w)(w') = ww'^* + w'w^* \qquad (w, w' \in W)$$

we see that for $x \in \underset{\sim}{g}$

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x\, d\tilde{\tau}(w)(w')) = \langle xw, w' \rangle + \langle xw', w \rangle = 2\langle xw, w' \rangle.$$

This clearly implies the lemma. Q.E.D.

(2.7) LEMMA. *Let V_1' be a maximal isotropic subspace of V' and let $W_1 = \{w \in W; V_1'^{\perp} \subseteq \ker w\}$. Then $\tau_{\underset{\sim}{g}}^{-1}(0) = G'W_1$. Here $V_1^{\perp}$ is the annihilator of V_1' in V' and $G'W_1$ is the union of G' orbits of elements of W_1.*

Proof. Some elementary linear algebra implies

$$\mathrm{im}\, w^* = (\ker w)^{\perp} \qquad (w \in W). \tag{2.8}$$

Consider a $w \in W$. Clearly

$$\tau_{\underset{\sim}{g}}(w) = 0 \qquad \text{iff} \quad ww^* = 0. \tag{2.9}$$

Combining (2.8) and (2.9) we see that

$$(\ker w)^{\perp} \text{ is an isotropic subspace of } V'. \tag{2.10}$$

Since, by Witt's theorem, G' acts transitively on the set of maximal isotropic subspaces of V' the lemma follows from (2.10). Q.E.D.

Similarly one can show that if V_1 is a maximal isotropic subspace of V and if $W_{11} = \{w \in W_1; \mathrm{im}\, w \subseteq V_1\}$ then

$$\tau_{\underset{\sim}{g}}^{-1}(0) \cap \tau_{\underset{\sim}{g}'}^{-1}(0) = G \cdot G'W_{11}. \tag{2.11}$$

Define

$$(2.12)\qquad W^0 = \{w \in W;\, d\tau_{\underline{g}}(w) \text{ is surjective}\},$$
$$W^{00} = W^0 \cap \tau_{\underline{g}}^{-1}(0) \qquad \text{and} \qquad W_1^0 = W^0 \cap W_1.$$

Here $d\tau_{\underline{g}}$ is as in (2.6) and W_1 as in (2.7). Lemma (2.7) implies that

$$(2.13)\qquad W^{00} = G' W_1^0.$$

It is obvious that

$$(2.14)\qquad W^{00} \text{ is not empty if } \dim_{\mathbb{D}} V \leqslant \dim_{\mathbb{D}} V_1'.$$

The condition (2.14) means that the pair G, G' is in the stable range with G the smaller member [H8].

(2.15) *Remark.* If G is compact then W^{00} is non-empty iff

(2.15.1) $n \leqslant m+1$ for $G \cong O(n)$, $G' \cong Sp(m, \mathbb{R})$,

(2.15.2) $n \leqslant q$ for $G \cong U(n)$, $G' \cong U(p, q)$, $q \leqslant p$,

(2.15.3) $n \leqslant m/2$ for $G \cong Sp(n)$. $G' \cong O^*(2m)$.

Indeed, by the assumption, the form (,) is anisotropic. Therefore for each $w \in W$ we have a direct sum decomposition

$$V = \operatorname{im} w \oplus (\operatorname{im} w)^{\perp}.$$

The restriction of (,) to $(\operatorname{im} w)^{\perp}$ is nondegenerate and the corresponding Lie algebra of isometries is isomorphic to the Lie algebra $(\operatorname{im} \tau_{\underline{g}}(w))^{\perp}$ (2.6). In particular $(\operatorname{im} \tau_{\underline{g}}(w))^{\perp}$ depends only on the rank of w. An easy case by case verification using (2.13) completes the proof of (2.15).

(2.16) Lemma. *Suppose that G is compact or that the pair G, G' is in the stable range with G the smaller member. Then the set $\tau_{\underline{g}}^{-1}(0)$ is a finite union of $G \cdot G'$ orbits and containes a unique open dense orbit $O_{\max}$. Moreover if W^{00} is not empty then $O_{\max} \subseteq W^{00}$.*

Proof. By the definition of the space W_1 (2.7) we have an identification

$$W_1 \cong \operatorname{Hom}_{\mathbb{D}}(V'/V_1'^{\perp}, V).$$

The orbit decomposition of this space under the obvious action of $GL(V'/V_1'^{\perp}) \times G$ is well known. In particular [H5, Chap. I, Proposition 8.1] implies that

(2.17) any two elements of W_1 of maximal rank are in one $GL(V'/V_1'^{\perp}) \times G$ orbit.

Pick $w_{\max}$ in W_1 of maximal rank and define

$$O_{\max} = G \cdot G' w_{\max}. \tag{2.18}$$

By (2.17), $O_{\max}$ does not depend on the choice of $w_{\max}$. The lemma follows from (2.7) and [H5, Chap. I, Proposition 8.1]. Q.E.D.

Finally we calculate the dimension of $O_{\max}$ and the dimension of the image of $O_{\max}$ under the map $\tau_{\underline{g}}$ (2.1).

(2.19) LEMMA. *Suppose that the pair G, G' is in the stable range with G the smaller member. Let $O'_{\max} = \tau_{\underline{g}'}(O_{\max})$. Then*

(2.19.1) $O_{\max} = G' w_{\max}$ ($w_{\max}$ *as in* (2.18)),

(2.19.2) $\dim O_{\max} = \dim W - \dim \underline{g}$, *and*

(2.19.3) $\dim O'_{\max} = \dim W - 2 \dim \underline{g}$.

Here, and in the rest of this paper, $\dim = \dim_{\mathbb{R}}$.

Proof. The statement (2.19.1) follows by the argument used in the proof of (2.16). Since by (2.16), $O_{\max}$ is dense in W^{00} and since $\tau_{\underline{g}}$, when restricted to W^0, is a submersion a well known fact [D, 16.8.8.1] implies (2.19.2). Define

$$\underline{g}'_0 = \{x \in \underline{g}'; [x, \tilde{\tau}'(w_{\max})] = 0\}, \qquad \underline{g}'_1 = \{x \in \underline{g}'; x(w_{\max}) = 0\}.$$

Clearly $\underline{g}'_1$ is a Lie subalgebra of $\underline{g}'_0$ and

$$\dim O_{\max} - \dim O'_{\max} = \dim(\underline{g}'_0/\underline{g}'_1). \tag{2.20}$$

Consider the pullback of the form $(\ ,\ )$ to V' by $w_{\max}$:

$$(u, v)_{\max} = (w_{\max}(u), w_{\max}(v)) \qquad (u, v \in V').$$

One checks easily that each element $x \in \underline{g}'_0$ is skew-symmetric with respect to $(\ ,\ )_{\max}$. Let $(\ ,\ )'_{\max}$ be the corresponding form on $V'/\ker w_{\max}$. This form $(\ ,\ )'_{\max}$ is nondegenerate and of the same type as $(\ ,\ )$. Therefore there is an injection

$$\underline{g}'_0/\underline{g}'_1 \to \underline{g}. \tag{2.21}$$

Since $\underline{g}'/\underline{g}'_1$ map surjectively onto $\mathrm{End}_{\mathbb{D}}(V'/\ker w_{\max})$, (2.21) is a surjection. Consequently

$$\dim(\underline{g}'_0/\underline{g}'_1) = \dim \underline{g}$$

and (2.19.3) follows from (2.19.2) and (2.20). Q.E.D.

3. The Cayley Transform

Let $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and let $\#$ be an involution of $\mathbb{D}$ as in Section 2. Let V be a finite dimensional right $\mathbb{D}$-vector space. For $x \in \mathrm{End}_{\mathbb{D}} V$ such that $x-1$ is invertible define the Cayley Transform

$$c(x) = (x+1)(x-1)^{-1}. \tag{3.1}$$

Then c is a rational map on $\mathrm{End}_{\mathbb{D}} V$ and

$$c(c(x)) = x,\ c(0) = -1,\ c(gxg^{-1}) = gc(x)\,g^{-1} \qquad (g \in GL_{\mathbb{D}}(V)). \tag{3.2}$$

Let $(\ ,\ )$ be a nondegenerate $\#$-Hermitian or $\#$-skew-Hermitian form on V and let $G \subseteq \mathrm{End}_{\mathbb{D}} V$ be the group of isometries of $(\ ,\ )$ with the Lie algebra $\underline{g} \subseteq \mathrm{End}_{\mathbb{D}} V$. One checks easily [H5, Chap. I, Proposition 7.3] that

$$c(\underline{g}) \subseteq G \qquad \text{and} \qquad c(G) \subseteq \underline{g}. \tag{3.3}$$

Suppose now that G is a member of a reductive dual pair G, G' as in Section 2. Let $\widetilde{Sp}$ denote the metaplectic group covering $Sp = Sp(W)$. Denote by $\tilde{G}$ the preimage of G in $\widetilde{Sp}$. Let us fix once and for all an element

$$(-1)^{\sim} \in \widetilde{Sp} \text{ in the preimage of } -1 \in Sp. \tag{3.4}$$

(3.5) Lemma. *Assume that the group G is compact. Then the domain of c* (3.1) *contains the Lie algebra $\underline{g}$. Moreover there is a unique smooth map $\tilde{c}\colon \underline{g} \to \tilde{G}$ such that $\tilde{c}(0) = (-1)^{\sim}$* (3.4) *and the following diagram commutes*

$$\begin{tikzcd} & \tilde{G} \arrow[d] \\ \underline{g} \arrow[ur, "\tilde{c}"] \arrow[r, "c"'] & G. \end{tikzcd}$$

Here the vertical arrow indicates the covering map.

Proof. Since G is compact the spectrum of $x \in \underline{g}$ is imaginary and therefore $x-1$ is invertible. The second part of the lemma follows from the monodromy principle [D, 16.28.8] because $\underline{g}$ is simple connected and c is smooth. Q.E.D.

(3.6) *Remark.* In general, with G not necessarily compact, the diagram (3.5) exists with $\underline{g}$ replaced by a Zariski open neighborhood of zero. We shall refer to this neighborhood as to the domain of $\tilde{c}$. We shall always assume that $\tilde{c}(0) = (-1)^{\sim}$. For more explanation see (4.8).

There is an involution $x \to x^+$ on $\mathrm{End}_{\mathbb{D}} V$ defined by

$$(x(u), v) = (u, x^+(v)) \qquad (u, v \in V). \tag{3.7}$$

Let $\mathbb{F}$ be the field of $\#$-fixed points in $\mathbb{D}$. Then $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and the algebra $\underline{g}$ is a vector space over $\mathbb{F}$. The group $GL_{\mathbb{D}}(V)$ acts on $\mathrm{End}_{\mathbb{D}}(V)$ by

$$x \to gxg^+ \qquad (x \in \mathrm{End}_{\mathbb{D}}(V),\ g \in GL_{\mathbb{D}}(V)). \tag{3.8}$$

This action preserves the $\mathbb{F}$-subspace $\underline{g} \in \mathrm{End}_{\mathbb{D}}(V)$.

(3.9) LEMMA. *The determinant of the linear transformation* (3.8) *of the vector space $\underline{g}$ over $\mathbb{F}$ is equal to* $(\det_{\mathbb{F}} g)^r$, *where* $r = 2\,\mathrm{sim}_{\mathbb{F}}\,\underline{g}/\dim_{\mathbb{F}} V$. *Here we view V as a vector space over $\mathbb{F}$ by restricting scalars and* $\det_{\mathbb{F}} g$ *indicates the determinant of g viewed as an element of* $\mathrm{End}_{\mathbb{F}}(V)$.

Proof. The determinant in question is an $\mathbb{F}^\times$ valued character of the group $GL_{\mathbb{D}}(V)$. Therefore it has to be of the form $g \to (\det_{\mathbb{F}} g)^r$ for some r. In order to find r we take $a \in \mathbb{F}^\times$ and $g = a \cdot$ (identity on V). Then for $x \in \underline{g}$,

$$gxg^+ = a^2 x \qquad \text{and} \qquad \det_{\mathbb{F}} g = a^m, \qquad \text{where} \quad m = \dim_{\mathbb{F}} V.$$

This clearly implies the lemma. Q.E.D.

Now we shall calculate the pullback of the Haar measure on $\tilde{G}$ to $\underline{g}$ via $\tilde{c}$ (3.6). As is well known there is a rational function whose absolute value $j(x)$ is defined for x in the domain of $\tilde{c}$ and satisfies

$$\int_{\tilde{G}} \Psi(g)\, dg = \int_{\underline{g}} \Psi \circ \tilde{c}(x)\, j(x)\, dx \tag{3.10}$$

for any continuous function Ψ with compact support contained in the image of $\tilde{c}$.

(3.11) LEMMA. *One can normalize the Lebesgue measure on $\underline{g}$ so that* $j(x) = |\det_{\mathbb{R}}(1-x)|^{-r}$, *where r is as in* (3.9).

Proof. For x and y in the domain of $\tilde{c}$ with y sufficiently close to zero and $x + y$ invertible we have the formula

$$\tilde{c}^{-1}(\tilde{c}(x)\,\tilde{c}(y)) = c(c(x)\,c(y)) = (y-1)(x+y)^{-1}(x-1) + 1, \tag{3.12}$$

where the last equality is taken from [H2, (10.2.3)]. By fixing y in (3.12) we obtain a function of x. Let h denote the inverse of this function. A straightforward calculation using (3.2) shows that

$$h(x) = -y - (y+1)(x-y)^{-1}(y-1).$$

Let Ψ be as in (3.10) and let $\psi = \Psi \circ \tilde{c}$.

The invariance of the Haar measure on $\tilde{G}$ implies

$$\int_{\underline{g}} \psi(x)\, j(x)\, dx = \int_{\underline{g}} \psi \circ h^{-1}(x)\, j(x)\, dx.$$

In particular if $\mathscr{J}(x)$ denotes the Jacobian of h at x then

$$j(x) = j(h(x))\, |\mathscr{J}(x)|. \tag{3.13}$$

We may normalize the Lebesgue measure on $\underline{g}$ so that $j(0) = 1$. Since $h(0) = -y^{-1}$ (assuming that y is invertible) (3.13) implies that

$$j(-y^{-1}) = |\mathscr{J}(0)|^{-1}.$$

Thus our problem is to calculate $\mathscr{J}(0)$. The derivative of h at zero coincides with the map

$$\underline{g} \ni x \to (1 + y^{-1})\, x(1 - y^{-1}) \in \underline{g}. \tag{3.14}$$

Put $g = 1 + y^{-1}$. Then $g \in \mathrm{End}_{\mathbb{D}}\, V$ and $g^{+} = 1 - y^{-1}$, (3.7). We may assume that g is invertible. Then (3.14) coincides with (3.8) and the lemma follows from (3.9). Q.E.D.

We shall also need another version of the Cayley Transform, namely

$$c_{-}(x) = -c(x) \qquad (x \in \underline{g} \text{ in the domaini of } c). \tag{3.15}$$

The point is that, by (3.2), $c_{-}(0) = 1$. For x in the domain of $\tilde{c}$ (3.6) define

$$\tilde{c}_{-}(x) = \tilde{c}(x)((-1)^{\sim})^{-1}. \tag{3.16}$$

Here $(-1)^{\sim}$ is as in (3.4) to that $\tilde{c}_{-}(0)$ is the identity of the group $\tilde{G}$. The invariance of the Haar measure on $\tilde{G}$ implies that (3.10)–(3.11) hold with $\tilde{c}$ replaced by $\tilde{c}_{-}$ and the same function $j(x)$.

4. The Stone–von Neumann Theorem

For the reader's convenience we recall some well known results here. Our main references are [H2, H4]. The Schwartz space of W, $S(W)$ has a structure of associative algebra with multiplication

$$\phi_1 \natural \phi_2(w') = \int_W \phi_1(w)\, \phi_1(w' - w)\, \chi(\tfrac{1}{2}\langle w, w' \rangle)\, dw, \tag{4.1}$$

where $\phi_1, \phi_2 \in S(W)$, $w' \in W$, and $\chi(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$.

We embed $S(W)$ into the space $S^*(W)$ of tempered distributions on W by

$$f(\phi\,) = \int_W f(w)\,\phi(w)\,dw \qquad (f, \phi \in S(W)). \tag{4.2}$$

Here dw stands for a (convenient choice of) a Lebesgue measure on W.

The symplectic group $Sp(W)$ acts on $S(W)$ by algebra automorphisms as follows.

$$\omega_{1,1}(g)\,\phi(w) = \phi(g^{-1}(w)) \qquad (w \in W, g \in Sp(W), \phi \in S(W)). \tag{4.3}$$

By dualizing (4.3) we obtain an $Sp(W)$ action on $S^*(W)$

$$\omega_{1,1}(g)\,f(\phi) = f(\omega_{1,1}(g^{-1})\phi) \qquad (g \in Sp(W), \phi \in S(W), f \in S^*(W)). \tag{4.4}$$

The formula (4.2) implies that the action (4.4) is an extension of the action (4.3) from $S(W)$ to $S^*(W)$. Let ω be the oscillator representation of the group $Sp(W)$ attached to the character χ (4.1). Let us choose realization of ω on a Hilbert space $\mathscr{H}$. Denote by $\mathscr{H}^\infty$ the space of smooth vectors in $\mathscr{H}$ and by $\mathscr{H}^{\infty *}$ the linear topological dual of $\mathscr{H}^\infty$. The symbols $B(\mathscr{H})$, $H.S.(\mathscr{H})$, $\mathrm{Hom}(\mathscr{H}^\infty, \mathscr{H}^{\infty *})$ will stand for the spaces of bounded operators on $\mathscr{H}$, Hilbert–Schmidt operators on $\mathscr{H}$, and continuous linear maps from $\mathscr{H}^\infty$ to $\mathscr{H}^{\infty *}$. We combine the Stone–von Neumann theorem [H4] with a result of Howe [H2, 16.3] in the following

(4.5) THEOREM. *There is an algebra homomorphism*

$$\rho: S(W) \to B(\mathscr{H})$$

which extends to a surjective isometry

$$\rho: L^2(W) \to H.S.(\mathscr{H})$$

and even further to a liner bijection

$$\rho: S^*(W) \to \mathrm{Hom}(\mathscr{H}^\infty, \mathscr{H}^{\infty *})$$

which has the intertwining property

$$\omega(\tilde{g})\,\rho(f)\,\omega(\tilde{g})^{-1} = \rho(\omega_{1,1}(g)\,f), \tag{4.6}$$

where $f \in S^*(W)$ and $\tilde{g} \in \widetilde{Sp}(W)$ is in the preimage of $g \in Sp(W)$. Moreover, *for* $x \in \underline{sp}(W)$ *in the domain of the Cayley Transform* $\tilde{c}$, $\rho^{-1}\omega(\tilde{c}(x))$ *is a function on* W. *It is possible to chose* $\tilde{c}$ *so that*

$$(4.7) \qquad \rho^{-1}\omega \circ \tilde{c}(x)(w) = \mathrm{ch}(x)\, \chi(\tfrac{1}{4}\langle x(w), w\rangle) \qquad (w \in W),$$

where $\mathrm{ch}(x) = z\, |\det(1-x)|^{1/2}$ *and* $z \in \mathbb{C}$ *is a constant.*

(4.8) *Remark.* The choice of the function ch in (4.7) determines a real analytic lifting

$$(4.8.1) \qquad \tilde{c}: \underline{sp} \to \widetilde{Sp}$$

of the Cayley Transform c: $\underline{sp} \to Sp$ (3.1). Conversely given a $\tilde{c}$ (4.8.1) the function ch is the pullback of the distribution character of the oscillator representation ω to $\underline{sp}$ via $\tilde{c}$. Let G be a member of reductive dual pair in Sp as in (2.2), (2.3). Then $\tilde{G}$ injects into $\widetilde{Sp}$ and (4.8.1) determines the lifting $\tilde{c}: \underline{g} \to G$ (3.6). The constant z (4.7) will play no significant role in our calculations. A choice of z is equivalent to a choice of $(-1)^{\sim}$ (3.4).

Let $\delta \in S^*(W)$ be the Dirac delta at the origin. Then $\rho(\delta)$ is the identity operator on $\mathscr{H}$. Fix a positive constant s such that in terms of the oscillatory integrals [Hö, (7.8.5)]

$$(4.9) \qquad \delta = s \natural s.$$

In [H2], $s = 2^{-n}$ with $2n = \dim W$. Define the symplectic Fourier Transform on $S(W)$ [H2, Sect. 2]

$$(4.10) \qquad \widehat{\phi} = \phi \natural s \qquad (\phi \in S(W))$$

and its extension to $S^*(W)$ by dualization

$$(4.11) \qquad \widehat{f}(\phi) = f(\widehat{\phi}) \qquad (f \in S^*(W)).$$

Since $((-1)^{\sim})^4) = 1$ (see (3.4)), the formula (4.7) implies that the constant z (4.7) satisfies

$$z \natural z \natural z \natural z = \delta.$$

Therefore $z^4 = s^4$. Put $\zeta = sz^{-1}$. Then $\zeta^4 = 1$ and

$$(4.12) \qquad f \natural \rho^{-1}\omega((-1)^{\sim}) = \zeta \widehat{f} \qquad (f \in S^*(W)).$$

For future reference we recall here the definition of the Fourier Transform of a tempered distribution. Let U be a real vector space of finite dimension. Denote by U^* the algebraic dual of U. Let

$$(4.13) \qquad \mathscr{F}(\psi) = \int_U \psi(\xi)\, \chi(\xi(x))\, d\xi \qquad (\psi \in S(U^*), x \in U).$$

Here $d\xi$ stands for a Lebesgue measure on U. The choice of this measure will play no role in our calculations. The formula (4.13) defines a continuous map

$$\mathscr{F}: S(U^*) \to S(U).$$

Denote by

$$\mathscr{F}^*: S^*(U) \to S^*(U^*) \tag{4.14}$$

the adjoint map.

5. Intertwining Distributions

Let G, G' be a real reductive dual pair in $Sp(W)$ and let $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$. Here ω is as in (4.5). A smooth version of the representation $\Pi \check{\otimes} \Pi'$ may be realized on a subspace of $\mathscr{H}^{\infty *}$ [P1, Proposition 1.2.19]. Therefore (4.5) implies that there is $f \in S^*(W)$ such that

$$\rho(f) \text{ intertwines } \omega^{\infty}|_{\tilde{G} \cdot \tilde{G}'} \text{ and } \Pi \check{\otimes} \Pi'. \tag{5.1}$$

Moreover by [H6, Theorem 1] this f (5.1) is determined up to a non-zero scalar multiple (which we shall ignore). In particular since $(-1)^{\sim}$ (3.4) is in the center of $\tilde{G} \cdot \tilde{G}'$ the formulas (4.12) and (4.9) imply that

$$\hat{f} = \pm f. \tag{5.2}$$

The title of this section refers to the distributions f (5.1).

(5.3) Example. Let $G' = Sp(W)$. Then $G \cong 0(1)$ and, as is well known, ω decomposes into a direct sum of two irreducible representations of $\tilde{G}'$. Call them ω_+ and ω_-. We may normalize the corresponding intertwining distributions f_+ and f_- so that

$$f_+ + f_- = \delta. \tag{5.3.1}$$

This two distributions can't satisfy (5.2) with the same sign because they correspond to two different representations of $\tilde{G}$. Chose the notation so that $\hat{f}_+ = f_+$ and $\hat{f}_- - f_-$. Then (5.3.1) and (4.9) imply

$$f_+ - f_- = \hat{\delta} = s. \tag{5.3.2}$$

From (5.3.1) and (5.3.2) we find

$$f_+ = \tfrac{1}{2}(\delta + s) \qquad \text{and} \qquad f_- = \tfrac{1}{2}(\delta - s). \tag{5.3.3}$$

The simplicity of the formulas (5.3.3) is remarkable. In particular f_+ and f_- are finite sums of homogeneous distributions ([Hö, 3.2], (B.3)) of distinct degrees. The intention of this section is to find out for which intertwining distributions this phenomenon persists. Surprisingly it does if one of the corresponding representations (Π or Π') is most likely to be unipotent ini the sense of Barbasch and Vogan [V1].

(5.4) LEMMA. *Let G, G' be an irreducible pair of type I and let $\Pi \otimes \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$. Suppose that*

(5.4.1) *G is compact, or*

(5.4.2) *the pair G, G' is in the stable range with G the smaller member, the representation Π is unitary and finite dimensional. We exclude the pairs $G \cong Sp(n, \mathbb{R})$, $G' \cong O(2n, 2n)$, and $G \cong Sp(n, \mathbb{C})$, $G' \cong O(4n, \mathbb{C})$.*

Then the intertwining distribution corresponding to $\Pi \check{\otimes} \Pi'$ (5.1) is given by the formula

$$f = \int_{\tilde{G}} \bar{\Theta}_{\Pi}(g)\, \rho^{-1}\omega(g)\, dg, \tag{5.4.3}$$

where Θ_{Π} is the distribution character of Π.

Proof. Assume that G is compact. Then, as is well known [W, 1.4.6], the projection onto the Π-isotypic component of $\mathscr{H}$ (4.5)

$$P_{\Pi} = \int_{\tilde{G}} \dim \Pi \cdot \bar{\Theta}_{\Pi}(g)\, \omega(g)\, dg.$$

Applying ρ^{-1} (4.5) to both sides of the above equation and dividing by $\dim \Pi$ we get (5.4.3). Suppose that (5.4.2) holds. Then [Li1, Corollary 3.3] (see also (A.1)) implies that the integral

$$\int_{\tilde{G}} \bar{\Theta}_{\Pi}(g)\, \omega(g)\, dg$$

is a well defined operator in $\mathrm{Hom}(\mathscr{H}^{\infty}, \mathscr{H}^{\infty *})$ which intertwines $\omega^{\infty}|_{\tilde{G} \cdot \tilde{G}'}$ with $\Pi \check{\otimes} \Pi'$. Again by applying ρ^{-1} to this integral we get (5.4.3)Q.E.D.

The main result of [Li1] implies that the representation Π' (5.4) is unitary.

We shall study the integral (5.4.3) via a change of variables provided by the Cayley Transform (3.1). Therefore we define

$$\tilde{f} = \int_{im\tilde{c}} \bar{\Theta}_{\Pi}(g)\, \rho^{-1}\omega(g)\, dg. \tag{5.5}$$

It follows from (A.1) (and from [H4, Theorem 3.5.4]) that (5.5) does indeed define a tempered distribution on W. We would like to replace g by $\tilde{c}(x)$ in (5.5). In order to do this we need some additional notation. The formula (2.3) induces an embedding of the Lie algebra $\underline{g}$ into $\underline{sp}(W)$. Using this embedding we can pull back the function ch (4.7) to $\underline{g}$. We denote this new function by $\mathrm{ch}_{\underline{g}}$. If follows directly from (2.2) and (4.7) that

$$(5.6) \qquad \mathrm{ch}_{\underline{g}}(x) = z\,|\det_{\mathbb{R}}(1-x)|^{d'(1/2)} \qquad (x \in \underline{g},\ d' = \dim_{\mathbb{D}} V'),$$

where z is the same constant as in (4.7), and $\det_{\mathbb{R}}(1-x)$ stands for the determinant of $1-x$ viewed as an element of $\mathrm{End}_{\mathbb{R}}(V)$. Next for $\Pi \in R(\tilde{G}, \omega)$, define

$$(5.7) \qquad a(x) = \mathrm{ch}_{\underline{g}}(x)\, \bar{\Theta}_{\Pi}(\tilde{c}(x))\, j(x) \qquad (x \in \underline{g} \text{ in the domain of } \tilde{c}).$$

Here $j(x)$ is as in (3.11). The formula (4.7) implies that the distribution $\mathcal{J}$ (5.5) is formally given by the integral

$$(5.8) \qquad \mathcal{J}(w) = \int_{\underline{g}} a(x)\, \chi(\tau_{\underline{g}}(w)(x))\, dx \qquad (w \in W).$$

The support of the function $a(x)$ is too large for this to be an oscillatory integral [Hö, Theorem 7.8.2]. However, using (A.1) we'll show that (5.8) is a limit of oscillatory integrals. We are most interested in the cases where $a(x)$ (5.7) is a polynomial function. Then (5.8) indicates that $\mathcal{J}$ should be a finite sum of homogeneous distributions. We prove this in the following

(5.9) THEOREM. *Let G, G' be an irreducible dual pair of type I in the stable range with G the smaller member. Suppose that the assumptions of* (5.4) *are satisfied and that the function* (5.7) *is a polynomial with homogeneous decomposition*

$$(5.9.1) \qquad a = \sum_i a_i, \qquad a_i\text{—homogeneous of degree } i,$$
$$0 \leqslant i < \tfrac{1}{2}\dim W - \dim \underline{g}.$$

Then the distribution (5.5) *is a finite sum of homogeneous distributions*

$$(5.9.2) \qquad \mathcal{J} = \sum_i \mathcal{J}_i,$$

where the summation is over the i's with $a_i \neq 0$ and

$$(5.9.3) \qquad \mathcal{J}_i \text{ is homogeneous of degree } d_i = -2i - 2\dim \underline{g}.$$

Moreover each $\tilde{f}_i$ (5.9.2) *is* $\omega_{1,1}(\tilde{G}\cdot\tilde{G}')$*-invariant and*

$$\operatorname{supp}\tilde{f}=\tau_{\underline{g}}^{-1}(0). \tag{5.9.4}$$

Proof. Clearly $\Gamma_{\underline{g}}=\{x\in \underline{g};\ \ker x=\{0\}\}$ is an open cone in $\underline{g}$. Choose an open cone Γ in $\underline{g}$ with the closure $\bar{\Gamma}\subseteq\Gamma_{\underline{g}}\cup\{0\}$. Let

$$\gamma(x)=\begin{cases}1 & \text{if } x\in\Gamma\\ 0 & \text{if } x\in\underline{g}\backslash\Gamma.\end{cases} \tag{5.10}$$

By (A.1) the following integral defines a tempered distribution on W.

$$f_\gamma=\int_{\tilde{G}}\gamma(\tilde{c}^{-1}(g))\,\bar{\Theta}_\Pi(g)\,\rho^{-1}\omega(g)\,dg. \tag{5.11}$$

On the other hand the function $\gamma\cdot a$ (5.7) defines a tempered distribution on $\underline{g}$ (by integration). By [Hö, Lemma 8.1.7] the Fourier Transform of this distribution (4.14) satisfies

$$WF(\mathscr{F}^*(\gamma\cdot a))\subseteq\underline{g}^*\times\bar{\Gamma}. \tag{5.12}$$

Here $WF(v)$ stands for the wave front set of a distribution v [Hö, Definition 8.1.2]. We shall denote by $WF_\xi(v)$ the fiber of $WF(v)$ over $\xi\in\underline{g}^*$. It follows from (5.12) and (2.6) that

$$(\operatorname{im} d\tau_{\underline{g}}(w))^\perp\cap WF_\xi(\mathscr{F}^*(\gamma a))=\varnothing \qquad \text{for}\quad w\in W\backslash\{0\}\ \text{and}\ \xi\in\underline{g}^*. \tag{5.13}$$

Therefore [Hö, Theorem 8.2.4] implies that there is a well defined pullback of the distribution $\mathscr{F}^*(\gamma a)$ to W by $\tau_{\underline{g}}$. This pullback is denoted by $\tau_{\underline{g}}^*(\mathscr{F}^*(\gamma a))$. In fact the distribution (5.11)

$$f_\gamma=\tau_{\underline{g}}^*(\mathscr{F}^*(\gamma a)). \tag{5.14}$$

Indeed, we may chose a sequence α_n of continuous compactly supported functions on $\underline{g}$ so that

$$0\leqslant\alpha_n(x)\leqslant\gamma(x)\qquad(x\in\underline{g}),$$

and

$$\lim_{n\to\infty}\alpha_n(x)=\gamma(x)\qquad\text{almost everywhere on } \underline{g}.$$

Then (A.1) implies that

$$f_\gamma=\lim_{n\to\infty}\int_{\tilde{G}}\alpha_n(\tilde{c}^{-1}(g))\,\bar{\Theta}_\Pi(g)\,\rho^{-1}\omega(g)\,dg\qquad\text{in}\quad S^*(W). \tag{5.15}$$

Each distribution on the right hand side of (5.15) is a smooth function

$$W \ni w \to \int_{\underline{g}} \alpha_n(x)\, a(x)\, \chi(\tau_{\underline{g}}(w)(x))\, dx \in \mathbb{C}. \tag{5.16}$$

The function (5.16) is the pullback of $\mathscr{F}^*(\alpha_n a)$ (4.14) by $\tau_{\underline{g}}$. A straightforward calculation shows that

$$\mathscr{F}^*(\gamma a) = \lim_{n \to \infty} \mathscr{F}^*(\alpha_n a) \quad \text{in} \quad D'_{\underline{g}^* \times \Gamma}(\underline{g}^*) \tag{5.17}$$

(for notation see [Hö, Definition 8.2.2]).

Combining [Hö, Theorem 8.2.4] with (5.15)–(5.17) we get (5.14).

Sinice (5.13) holds with a replaced by a_i (5.9.1) we may define a distribution on $W\setminus\{0\}$

$$f_{\gamma,i} = \tau_{\underline{g}}^*(\mathscr{F}^*(\gamma a_i)). \tag{5.18}$$

The map $\tau_{\underline{g}}$ is quadratic ($\tau_{\underline{g}}(tw) = t^2 \tau_{\underline{g}}(w)$, $t \in \mathbb{R}$, $w \in W$). Therefore a straightforward calculation implies that $f_{\gamma,i}$ (5.18) is homogeneous of degree d_i (5.9.3). Since by (5.9.1), $d_i > -\dim W$, each $f_{\gamma,i}$ extends uniquely to a homogeneous distribution on W of the same degree [Hö, Theorem 3.2.3]. Clearly

$$f_\gamma = \sum_i f_{\gamma,i}. \tag{5.19}$$

We may choose a sequence of open cones $\Gamma_n \subseteq \underline{g}$ with $\bar{\Gamma}_n \subseteq \Gamma_{\underline{g}} \cup \{0\}$ and such that $\bigcup_n \Gamma_n = \Gamma_{\underline{g}}$. Let γ_n be the characteristic function of Γ_n as in (5.10). Then (A.1) implies that the distribution (5.5)

$$\tilde{f} = \lim_{n \to \infty} f_{\gamma_n} \quad \text{in} \quad S^*(W). \tag{5.20}$$

If follows from (5.20), (B.4), and from the decomposition (5.19) with γ replaced by γ_n that

$$\tilde{f}_i = \lim_{n \to \infty} f_{\gamma_n, i} \tag{5.21}$$

defines a distribution on $W\setminus\{0\}$, homogeneous of degree d_i (5.9.3). By an argument used previously, this distribution extends uniquely to a homogeneous distribution on W of the same degree. Moreover it is clear from the above construction that $\tilde{f}_i$ is $\omega_{1,1}(G \cdot G')$-invariant.

The restriction of $\tilde{f}_0$ to W^0 (2.12) coincides with the pullback of the Dirac delta at $0 \in \underline{g}^*$ to W^0 via $\tau_{\underline{g}}$ [Hö, Theorem 6.1.2]. In particular this restriction is a $\omega_{1,1}(G \cdot G')$-invariant measure with support equal to $\overline{W^{00}}$ (2.12). It follows from (2.16) that $\tau_{\underset{=}{g}}^{-1}(0) = \overline{W^{00}}$. Therefore

$$(5.22) \qquad \operatorname{supp} \tilde{f}_0 = \tau_{\underset{=}{g}}^{-1}(0).$$

The statement (5.9.4) follows from (5.22) and (B.4). Q.E.D.

(5.23) *Conjecture.* The distribution $\tilde{f}_0$ is a measure.

We'll provide some additional evidence for this conjecture in the next section. Now we address two questions.

(5.24) When is the assumption (5.9.1) satisfied?

(5.25) How to recover the intertwining distribution f (5.1) from $\tilde{f}$ (5.5)?

We do not answer any of them completely. For the first one we'll satisfy ourselves with the following

(5.26) LEMMA. *The condition* (5.9.1) *holds under assumption* (5.4.2) *if*

(5.26.1) $\Theta_{\Pi} \circ \tilde{c}$ *is a constant function,*

(5.26.2) *the form* $(\ ,\ )'$ *is split, and*

(5.26.3) *if* $\mathbb{D} = \mathbb{R}$ *and if* G *is not compact then* $d'/2 - r$ *is even* (*r as in* (3.9)).

Proof. Let $x \in \underset{=}{g}$. Then the subspace W_1 (2.7) is isotropic for the symmetric bilinear form

$$(5.26.4) \qquad W \times W \ni (\mathbf{w}, \mathbf{w}') \to \langle x(\mathbf{w}), \mathbf{w}' \rangle \to \langle x(\mathbf{w}), \mathbf{w}' \rangle \in \mathbb{R}.$$

The assumption (5.26.2) implies that $\dim W = 2 \dim W_1$. Therefore if the form (5.26.4) is nondegenerate then it has signature zero. Thus it follows from [H2, (8.1)–(8.2)] (see also [Hö, (3.4.6) and Theorem 7.6.1]) and form [H2, Sects. 16 and 17] that one may choose the constant z (4.7) so that the map

$$(5.26.5) \qquad G \ni c(x) \to \tilde{c}(x) \in \tilde{G} \qquad (x \in \underset{=}{g} \text{ in the domain of } c)$$

extends to a group isomorphism onto a subgroup of $\tilde{G}$ of index 2. By (5.26.1), Θ_{Π} is constant on this subgroup. Therefore the function a (5.7) is a constant multiple of $[\det_{\mathbb{R}}(1 - x)]^k$, where $k = d'/2 - r$, $d' = \dim_{\mathbb{D}} V'$, and r is as in (3.9). This is a polynomial function by (5.6) and (5.26.3). It remains to check that

$$0 \leqslant k \cdot \dim_{\mathbb{R}} V \leqslant \tfrac{1}{2} \dim W - \dim \underset{=}{g}.$$

The first inequality may be verified by a case by case analysis and the second one as follows (for notation see (3.9)):

$$(d'/2-r)\dim_{\mathbb{R}} V=\tfrac{1}{2}\dim W-2\dim_{\mathbb{F}} \underline{g}\cdot\dim_{\mathbb{R}}\mathbb{D}/\dim_{\mathbb{F}}\mathbb{D}$$
$$\leqslant\tfrac{1}{2}\dim W-\dim \underline{g}. \qquad \text{Q.E.D.}$$

(5.26.6) *Remark.* The above proof shows that there is an isomorphism $\tilde{G}\cong\{\pm 1\}\times G$. The representation Π coincides with the nontrivial character of $\{\pm 1\}$ tensored with a representation π of G. The condition (5.26.1) means that π is trivial on the Zariski identity component of G. In particular if $G\cong O(p,q)$ then π is either the trivial or the "determinant" representation of G. There are two other representations of $O(p,q)$ $(pq\neq 0)$ trivial on the identity component (with respect to the usual topology) which we do not consider here.

Now we discuss the question (5.25). Since $\operatorname{im}\tilde{c}=\operatorname{im}\tilde{c}_-\cdot(-1)^{\sim}$ (3.16), the symplectic Fourier Transform of the distribution (5.5)

$$\widehat{f}=\text{const}\int_{\operatorname{im}\tilde{c}}\bar{\Theta}_{\Pi}(g)\,\rho^{-1}\omega(g)\,dg. \tag{5.27}$$

Let $\tilde{1}\in\widetilde{Sp}$ be the element different than the identity of $\widetilde{Sp}$ in the preimage of $1\in Sp$. Then obviously

$$\widehat{f}=\text{const}\int_{\operatorname{im}\tilde{c}_-}\bar{\Theta}_{\Pi}(g)\,\rho^{-1}\omega(g)\,dg. \tag{5.28}$$

Let $\tilde{1}\in\widetilde{Sp}$ be the element different than the identity of $\widetilde{Sp}$ in the preimage of $1\in Sp$. Then obviously

$$\operatorname{im}\tilde{c}_-\cap\tilde{1}\cdot\operatorname{im}\tilde{c}_-=\varnothing. \tag{5.28}$$

Denote by G_1 the Zariski component of the identity of G. Let $\tilde{G}_1$ be the preimage of G_1 in $\tilde{G}$. Since $\operatorname{im} c_-$ is of full measure in G_1, (5.28) implies

$$\operatorname{im}\tilde{c}_-\cup\tilde{1}\cdot\operatorname{im}\tilde{c}_-\ \text{is of full measure in}\ \tilde{G}_1. \tag{5.29}$$

Since $\omega(\tilde{1})=-$identity,

$$\bar{\Theta}_{\Pi}(g\cdot\tilde{1})\,\omega(g\tilde{1})=\bar{\Theta}_{\Pi}(g)\,\omega(g)\qquad(g\in\tilde{G}).$$

Combining (5.27)–(5.30) we conclude that

$$\widehat{f}=\text{const}\int_{\tilde{G}_1}\bar{\Theta}_{\Pi}(g)\,\rho^{-1}\omega(g)\,dg. \tag{5.31}$$

In particular (5.2) and (5.31) imply

(5.32) LEMMA. *Under the assumptions of* (5.4) *the intertwining distribution* $f = \text{const}\,\tilde{f}$ *if* $G = G_1$.

It follows from the classification of real reductive dual pairs [H9] that (if G, G' is of type I) $G \neq G_1$, implies that G is an orthogonal group (over $\mathbb{R}$ or $\mathbb{C}$). Then

$$(5.33) \qquad G \cong G_1 \times \{\pm 1\} \qquad \text{if} \quad \dim_{\mathbb{D}} V \text{ is odd } (\mathbb{D} = \mathbb{R} \text{ or } \mathbb{C}).$$

In particular (4.12), (5.31), and (5.33) imply

(5.34) LEMMA. *Under the assumptions of* (5.4) *the intertwining distribution* $f = \text{const}(\tilde{f} \pm \widehat{\tilde{f}}\,)$, *if* $G, \neq G_1$, *and* $\dim_{\mathbb{D}} V$ *is odd.*

he remaining case, when G is orthogonal with $\dim_{\mathbb{D}} V$ even, is more complicated.

Let us close an orthogonal direct sum decomposition of the formed space

$$(5.35) \qquad V = V_1 \oplus V_2, (\ ,\ ) = (\ ,\ )_1 \oplus (\ ,\ )_2 \qquad \text{with} \quad \dim_{\mathbb{D}} V_1 = 1.$$

Thus the symplectic space (2.2)

$$(5.36) \qquad W = W_1 \oplus W_2, \langle\ ,\ \rangle = \langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_2,$$

where $w \in W$ belongs to W_i iff $\operatorname{im} w \subseteq V_i$ and the symplectic form $\langle\ ,\ \rangle_i$ is defined as in (2.2) with respect to the pair of forms $(\ ,\ )'$, $(\ ,\ )_i$ $(i = 1, 2)$. Corresponding to $\langle\ ,\ \rangle_1$ we have a symplectic Fourier Transform on $S^*(W_1)$ defined as in (4.11). By tensoring it with the identity map on $S^*(W_2)$ we obtain a partial symplectic Fourier Transform on $S^*(W)$. We shall denote it by $\breve{F}$ $(F \in S^*(W))$. Explicitly if $F \in L^1(W)$ then for $w_1' \in W_1$ and $w_2' \in W_2$

$$(5.37) \qquad \breve{F}(w_1' + w_2') = s_1 \int_{W_1} F(w_1 + w_2')\, \chi(\tfrac{1}{2}\langle w_1, w_1' \rangle_1)\, dw_1.$$

Here s_1 is a constant defined by (4.9) for W_1. As in (5.34) we obtain the following

(5.38) LEMMA. *Under the assumptions of* (5.4) *if* $G \neq G_1$ *and* $\dim_{\mathbb{D}} V$ *is even the intertwining distribution* $f = \text{const}(\tilde{f} \pm \breve{\tilde{f}})$.

We turn now to a heuristic investigation of the distribution $\tilde{f}$ (5.38). We'll work under the assumptions of (5.26). Then $a(x)$ (5.8) is a polynomial.

Let $\underline{g}_2$ be the Lie subalgebra of $\underline{g}$ consisting of elements preserving V_2 (5.35). Denote by $\underline{g}_1$ the orthogonal complement (with respect to the Killing form—see (2.5)) of $\underline{g}_2$ in $\underline{g}$. Thus $\underline{g} = \underline{g}_{12} \oplus \underline{g}_2$.

We can decompose the polynomial function $a(x)$ (5.7) into a sum of terms homogeneous with respect to the dilatations on $\underline{g}_1$ and on $\underline{g}_2$:

(5.39) $$a(x) = \sum_{j,k} a_{jk}(x), \qquad \text{where} \quad x = x_1 + x_2,\, x_1 \in \underline{g}_1,\, x_2 \in \underline{g}_2,$$
$$a_{jk}(t_1 x_1 + t_2 x_2) = t_1^j t_2^k a_{jk}(x_1 + x_2) \qquad (t_1, t_2 > 0).$$

For a_{jk} as ini (5.39) define (formally)

(5.40) $$\mathcal{J}_{jk}(w) = \int_{\underline{g}} a_{jk}(w)\, \chi(\tau_{\underline{g}}(w)(x))\, dx \qquad (w \in W).$$

Then

(5.41) $$\mathcal{J} = \sum_{jki} \mathcal{J}_{jk}, \qquad \text{where the summation is over } a_{jk} \neq 0.$$

As in (2.2) we have the map

(5.42) $$\mathrm{Hom}_{\mathbb{D}}(V', V_i) \ni w \to w_i^{*j} \in \mathrm{Hom}_{\mathbb{D}}(V_i, V') \qquad (i = 1, 2).$$

Since $\dim_{\mathbb{D}} V_1 = 1$, $w_1 w_1^{*1} = O$ for any $w_1 \in \mathrm{Hom}_{\mathbb{D}}(V', V_1)$. In terms of (5.36) and (5.42)

(5.43) $$(w_1 + w_2)(w_1 + w_2)^* = (w_1 w_2^{*2} + w_2 w_1^{*1}) + w_2 w_2^{*2}$$
$$(w_i \in W_i,\ i = 1, 2).$$

The first term on the right hand side of the equation (5.43) belongs to $\underline{g}_1$ and the second one to $\underline{g}_2$. If we identify $\underline{g}^*, \underline{g}_1^*, \underline{g}_2^*$ with $\underline{g}, \underline{g}_1, \underline{g}_2$ via (2.5) respectively then

(5.44) $$\tau_{\underline{g}_1}(w_1 + w_2) = \tfrac{1}{4}(w_1 w_2^{*2} + w_2 w_1^{*1}),$$
$$\tau_{\underline{g}_2}(w_1 + w_2) = \tfrac{1}{4} w_2 w_2^{*2}$$
$$\tau_{\underline{g}}(w_1 + w_2) = \tau_{\underline{g}_1}(w_1 + w_2) + \tau_{\underline{g}_2}(w_2) \qquad (w_i \in W_i,\ i = 1, 2).$$

The definition (5.40) and the relations (5.44) indicate that for $w_i \in W_i$ and for $t_i > 0$ $(i = 1, 2)$,

(5.45) $$\mathcal{J}_{jk}(t_1 w_1 + t_2 w_2) = t_1^{d_j'} t_2^{d_{jk}'} \mathcal{J}_{jk}(w_1 + w_2),$$
$$d_j' = -\dim \underline{g}_1 - j,\ d_{jk}' = -\dim \underline{g}_1 - 2 \dim \underline{g}_2 - j - 2k.$$

Notice that $\ker(x_1+x_2)=\{0\}$ ($x_1\in \underline{g}_1$, $x_2\in \underline{g}_2$) iff for any $t_1, t_2>0$, ker $(t_1x_1+t_2x_2)=0$. Thus the cone $\Gamma_{\underline{g}}$ used in the proof of (5.9) is invariant under the double dilatations

(5.46) $$\underline{g}_1\oplus \underline{g}\ni (x_1, x_2)\to (t_1x_1, t_2x_2)\in \underline{g}_1\oplus \underline{g}_2 \qquad (t_1, t_2>0).$$

One may approximate the cone of $\Gamma_{\underline{g}}$ by cones Γ (5.10) which are also invariant under (5.46). Then (5.14) implies that the decomposition (5.41) really does hold on $W\setminus\{0\}$ and that the distributions $\tilde{f}_{jk}$ have the homogenity properties indicated in (5.45). Since, by (5.26), $d_j'+d_{jk}'>-\dim W$ each $\tilde{f}_{jk}$ extends to a distribution on W, homogeneous of degree $d_j'+d_{dk}'$. In fact, the distribution (5.9.3)

(5.47) $\tilde{f}_i=\sum \tilde{f}_{jk}$, where the summation is over these indecies j, for which $d_i=d_j'+d_{jk}'$.

Therefore the intertwining distribution (5.38)

(5.48) $$f=\sum_{j,k} (\tilde{f}_{jk}\pm \breve{\tilde{f}}_{jk}),$$

where the summation is over $a_{jk}\neq 0$ (5.39) and the choice of sign does not depend on j, k.

By chasing through the proof of (5.26) we see that (with the notation (5.45))

(5.49) $$d_j'>-\dim W_1,\ d_{jk}'>\dim W_2,$$

(so that $d_j'+d_{jk}'>1-\dim W$) and

$$-\dim W_1-d_j'+d_{jk}'<-2\dim \underline{g}.$$

Thus [Hö, Theorem 3.2.3] implies that each $\tilde{f}_{jk}\in S^*(W)$ has the homogenity properties indicated in (5.45). Consequently the distribution

(5.50) f (5.48) is a finite sum of homogeneous distributions where the only homogeneous term of degree $d_0=2\dim \underline{g}$ (5.9.3) is $\tilde{f}_0$.

Since for any Lie algebra $\underline{g}$ under consideration (3.3)

$$\det_{\mathbb{R}}(1-x)=\det_{\mathbb{R}}(1+x) \qquad (x\in \underline{g})$$

a straightforward argument (extending the proof of (5.26)) shows that the statement (5.50) remains valid for f as in (5.34) under the assumptions of (5.26).

Thus (5.50), (5.34), (5.32), (5.26), (5.9), and (B.4) imply the following

(5.51) COROLLARY. *Suppose that* G, G', *and* $\Pi \in R(\tilde{G}, \omega)$ *satisfy the assumptions of* (5.4.2) *and* (5.26). *Let* f *be the corresponding intertwining distribution* (5.1). *Then* f *is a finite sum of homogeneous distributions and* $\bigcup_{w \in W} WF_w f \supseteq \tau_{\underline{g}}^{-1}(0)$. *Here we identify* W^* *with* W *via* $\langle\ ,\ \rangle$.

6. HOLOMORPHIC REPRESENTATIONS

In this section G, G' is an irreducible dual pair of type I (as in (2.1)–(2.3)) with G compact. We'll derive an explicit formula for the distribution character $\Theta_{\Pi'}$ of the representation Π' (5.1) in terms of the corresponding intertwining distribution f.

(6.1) LEMMA. *The pullback by* $\tau_{\underline{g}'}$

$$S(\underline{g}'^*) \ni \psi \to \psi \circ \tau_{\underline{g}'} \in S(W)$$

is a well defined continuous map. In particular, by dualizing we obtain a push-forward of tempered distributions

$$\tau_{\underline{g}'*}: S^*(W) \to S^*(\underline{g}'^*),$$
$$\tau_{\underline{g}'*}(f)(\psi) = f(\psi \circ \tau_{\underline{g}}) \qquad (f \in S^*(W), \psi \in S(\underline{g}'^*)).$$

Proof. Since G is compact it centralizes a positive compatible complex structure $\mathscr{J}$ on W. Thus $\mathscr{J} \in \underline{g}'$ and the quadratic form

$$W \ni w \to \langle \mathscr{J}(w), w \rangle \in \mathbb{R}$$

is positive definite. Let $|\ |$ be a norm on the real vector space $\underline{g}'$ such that $|\mathscr{J}| = 1$. Define a norm $|\ |$ on W by

$$|w|^2 = \tau_{\underline{g}'}(w)(\mathscr{J})(= \tfrac{1}{4}\langle \mathscr{J}(w), w \rangle) \qquad (w \in W).$$

For a fixed $w \in W$

$$|\tau_{\underline{g}'}(w)| = \sup\{|\tau_{\underline{g}'}(w)(x)|\,;\, x \in \underline{g}', |x| = 1)$$

is the norm of the functional $\tau_{\underline{g}'}(w)$. Clearly

$$|\tau_{\underline{g}'}(w)| \geqslant |w|^2 \qquad (w \in W). \tag{6.2}$$

Let $\psi \in S(\underline{g}'^*)$ and let n be a positive integer. Then (6.2) implies

$$|w|^{2n} |\psi \circ \tau_{\underline{g}'}(w)| \leqslant |\tau_{\underline{g}'}(w)|^n\, |\psi(\tau_{\underline{g}'}(w))| \tag{6.3}$$

and the right hand side of (6.3) is bounded independently of $w \in W$. The chain rule combined with (6.3) completed the proof. Q.E.D.

We may compose the push-forward $\tau_{\underline{g}'*}$ (6.1) with the Fourier Transform $\mathscr{F}^*\colon S^*(\underline{g}'^*) \to S^*(\underline{g}')$ (4.14) to obtain a continuous linear map

$$\mathscr{F}^* \circ \tau_{\underline{g}'*}\colon S^*(W) \to S^*(\underline{g}'). \tag{6.4}$$

Explicitly, for an absolutely integrable function f on W,

$$\mathscr{F}^* \circ \tau_{\underline{g}'*}(f)(x) = \int_W f(w)\,\chi(\tau_{\underline{g}'}(w)(x))\,dw \qquad (x \in \underline{g}). \tag{6.5}$$

Let $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$. Denote by $\tilde{c}_-^*\,\Theta_{\Pi'}$ the pullback of $\Theta_{\Pi'}$ by $\tilde{c}_-$ (3.16). This means [Hö, Theorem 6.1.2] that

$$\tilde{c}_-^*\,\Theta_{\Pi'}(\psi) = \Theta_{\Pi'}(\Psi), \qquad \text{where} \quad \Psi \in C_c^\infty(\tilde{G}') \tag{6.6}$$

is supported in the image of $\tilde{c}_-$ and

$$\psi(x) = \Psi(\tilde{c}_-(x))\,j(x) \qquad (\text{see (3.11) for } j(x)).$$

(6.7) Theorem. *Suppose G, G' is an irreducible dual pair with G compact. Let $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$ and let $f \in S^*(W)$ be the corresponding intertwining distribution* (5.1). *Then*

$$\frac{1}{\mathrm{ch}_{\underline{g}'}}\,\tilde{c}_-^*\,\Theta_{\Pi'} = \mathrm{const}_\Pi\,\mathscr{F}^* \circ \tau_{\underline{g}'*}(\hat{f}) \tag{6.7.1}$$

in the sense that the left hand side, originally defined on the domain of $\tilde{c}_-$, extends to a tempered distribution on $\underline{g}'$ equal to the one on the right hand side. Notice that, by (5.2), *we can replace $\hat{f}$ by f in* (6.7.1).

The characters (6.7.1) have been studied by Hecht from a different view point [He].

Proof. Let us normalize f so that $\rho(f)$ is a projection on the Π-isotypic component of the Hilbert space $\mathscr{H}$ (4.5). We calculate using the formulas (6.6), (3.16), and (4.12), respectively,

$$\begin{aligned}
\dim \Pi \cdot \tilde{c}_-^*\,\Theta_{\Pi'}(\psi) &= \mathrm{tr}\left(\int_{\tilde{G}'} \Psi(g)\,\omega(g)\,\rho(f)\,dg\right) \\
&= \mathrm{tr}\left(\int_{\underline{g}'} \psi(x)\,\omega(\tilde{c}(x))\,\omega((-1)^\sim)^{-1}\,\rho(f)\,dx\right) \\
&= \mathrm{const}\;\mathrm{tr}\int_{\underline{g}'} \psi(x)\,\omega(\tilde{c}(x))\,\rho(\hat{f})\,dx.
\end{aligned}$$

This combined with (4.7) and [H4, Theorem 3.5.4] shows that

$$\tilde{c}^*_- \Theta_{\Pi'}(\psi) = \text{const}_\Pi \int_{\underset{\sim}{g}'} \psi(x)\rho^{-1} \circ \omega(\tilde{c}(x)) \natural \hat{f}(0)\,dx$$
$$= \text{const}_\Pi\, \hat{f}(\mathcal{F}(\text{ch}_{\underset{\sim}{g}'}\psi) \circ \tau_{\underset{\sim}{g}}). \qquad \text{Q.E.D.}$$

A straightforward calculation verifies the following

(6.8) LEMMA. *If* $f \in S^*(W)$ *is homogeneous of degree* $d \in \mathbb{C}$ (B.3), *then* $\tau_{\underset{\sim}{g}'*}(f)$ *is homogeneous of degree* $(1/2)d + (1/2)\dim W - \dim \underset{\sim}{g}'$.

In particular the distribution $\tau_{\underset{\sim}{g}'*}(\tilde{f}_0)$ (where $\tilde{f}_0$ is as in (5.9.2)) is homogeneous of degree $(1/2)\,\text{dimm}\, O'_{\max} - \dim \underset{\sim}{g}'$. Therefore the proof of [B-V1, Corollary 3.9] implies that

(6.9) $\tau_{\underset{\sim}{g}*}(\tilde{f}_0)$ is a constant multiple of the orbital integral on $\underset{\sim}{g}'$ [R, Theorem 1] defined by the orbit $O'_{\max}$ (2.19).

Clearly (6.9) provides some evidence for the conjecture (5.23).

Recall [H1] that there is a notion of the wave front set of a unitary representation of a Lie group. In particular [H1, Theorem 1.8] shows that for Π' unitary

$$(6.10) \qquad WF(\Pi') = WF_1(\Theta_{\Pi'}) \qquad (= WF_0(\tilde{c}^*_- \Theta_{\Pi'})).$$

Here $WF_1(\Theta_{\Pi'})$ is the fiber of $WF(\Theta_{\Pi'})$ over the identity $1 \in \tilde{G}'$.

(6.11) THEOREM. *Let* G, G' *be an irreducible dual pair with* G *compact and let* $\Pi \tilde{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$. *Then* $WF(\Pi') = \tau_{\underset{\sim}{g}'}(\tau_{\underset{\sim}{g}}^{-1}(0))$.

Proof. Theorem (6.7) and [Hö, Lemma 8.1.7] imply that

$$(6.12) \qquad WF_0(\tilde{c}^*_- \Theta_{\Pi'}) \subseteq \tau_{\underset{\sim}{g}'}(W).$$

Denote by $\mathcal{N}'$ the nilpotent cone in $\underset{\sim}{g}'^*$. As is well known [H1, Proposition 2.4] $WF(\Pi') \subseteq \mathcal{N}'$, and (2.4) implies that

$$(6.13) \qquad \tilde{\tau}'(\omega) \text{ is nilpotent iff } \tilde{\tau}(w) \text{ is nilpotent } (w \in W).$$

Combining (6.12), (6.13) with the fact that the only nilpotent element of $\underset{\sim}{g}$ is zero we see that

$$(6.14) \qquad WF(\Pi) \subseteq \tau_{\underset{\sim}{g}'}(\tau_{\underset{\sim}{g}}^{-1}(0)).$$

On the other hand the Gelfand–Kirillov dimension [V1] of Π is known ([H10], (C.1)) and is equal to the dimension of the variety on the right hand side of the inclusion (6.14). Therefore [B-V1, Theorem 4.8] and [B-V2, Proposition 4] imply that (6.14) is an equality. Q.E.D.

(6.15) COROLLARY. *Under the assumptions of* (6.11) *if*

(6.15.1) *the intertwining distribution f is a finite sum of homogeneous distributions, then*

(6.15.2) $(1/\mathrm{ch}_{\underline{g}'}) \cdot \tilde{c}^*_{\underline{-}}\, \Theta_{\Pi'}$ *extends to a finite sum of homogeneous distributions on $\underline{g}'$ and the support of the Fourier Transform of this sum coincides with $\tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0))$—which is the closure of one nilpotent orbit $O'_{\max}$* (2.9).

(6.16) *Conjecture.* The statement (6.15.2) holds under the assumptions of (5.4.2) and (5.26)—with G not necessarily compact.

(6.17) *Remark.* It follows from (5.51) and (6.15) that (6.16) is true for G compact. In this case the assumptions of (6.16) are satisfied if the representation Π is trivial on the identity component of $\tilde{G}$. The main obstacle for a rapid proof of (6.16) is that the set (2.11) is not empty if G is not compact.

7. THE WEYL ALGEBRA AND ASSOCIATED VARIETIES

Consider a real reductive dual pair $G, G' \subseteq Sp(W)$. The enveloping algebra $\mathscr{U}(\mathbf{g})$ of the complexification $\mathbf{g}$ of $\underline{g}$ carries a natural filtration by vector-subspaces [B]

$$\mathscr{U}_{-1}(\mathbf{g}) = 0, \qquad \mathscr{U}_0(\mathbf{g}) = \mathbb{C}, \qquad \mathscr{U}_1(\mathbf{g}) = \mathbb{C} + \mathbf{g},$$

The corresponding graded algebra

$$\operatorname{gr} \mathscr{U}(\mathbf{g}) = \bigoplus_{n \geqslant 0} \mathscr{U}_n(\mathbf{g})/\mathscr{U}_{n-1}(\mathbf{g})$$

is isomorphic to the ring $\mathscr{P}(\mathbf{g}^*)$ of polynomial functions on the dual vector space $\mathbf{g}^*$ of $\mathbf{g}$. Given an ideal $I \subseteq \mathscr{U}(\mathbf{g})$, the graded ideal

$$\operatorname{gr} I = \bigoplus_{n \geqslant 0} (I \cap \mathscr{U}_n(\mathbf{g}))/(I \cap \mathscr{U}_{n-1}(\mathbf{g}))$$

in $\operatorname{gr} \mathscr{U}(\mathbf{g})$, defines a set of common zeros in $\mathbf{g}^*$ which is called the associated variety $\mathscr{V}(\operatorname{gr} I)$ of I [B, 2.1]. The goal of this section is to prove the following

(7.1) THEOREM. *Suppose that $\Pi \otimes \Pi \in R(\tilde{G} \cdot \tilde{G}', \omega)$. Let $I_\Pi (I_{\Pi'})$ denote the annihilator of the Harish-Chandra module of Π (Π') in $\mathscr{U}(\mathbf{g})$ $(\mathscr{U}(\mathbf{g}'))$.*

Let $f \in S^*(W)$ be the corresponding intertwining distribution as in (5.1). *Then*

$$\tau_{\mathbf{g}'}\left(\bigcup_{w \in W} WF_w f\right) \subseteq \mathscr{V}(\mathrm{gr}\, I_{\Pi'}) \subseteq \tau_{\mathbf{g}'}(\tau_{\mathbf{g}}^{-1}(\mathscr{V}(\mathrm{gr}\, I_{\Pi}))). \tag{7.2}$$

Here we denote by $\tau_{\mathbf{g}}$ the extension of the polynomial map $\tau_g \colon W \to g^*$ (2.1) *to the complexification $\mathbf{W}$ of W, i.e., $\tau_{\mathbf{g}} \colon \mathbf{W} \to \mathbf{g}^*$ (the same refers to $\tau_{g'}$).*

Let $\mathscr{W}$ denote the subspace of $S^*(W)$ consisting of all distributions with support contained in $\{0\}$. This subspace is an algebra with twisted convolution $\natural$ [H4, (2.2.5)] so that

$$\begin{aligned} &(w' \natural w) \natural \phi = w' \natural (w \natural \phi) \qquad (w', w \in \mathscr{W}, \phi \in S(W)), \\ &w \natural \phi(\mathbf{w}') = w(\phi_{\mathbf{w}'}), \\ &\phi_{\mathbf{w}'}(\mathbf{w}) = \phi(\mathbf{w}' - \mathbf{w})\, \chi(\tfrac{1}{2}\langle \mathbf{w}, \mathbf{w}' \rangle) \qquad (\mathbf{w}, \mathbf{w}' \in W). \end{aligned} \tag{7.3}$$

There is an embedding

$$\begin{aligned} &\partial \colon W \to \mathscr{W}, \\ &\partial_{\mathbf{w}}(\phi) = \lim_{t \to 0} t^{-1}(\phi(tw) - \phi(0)) \qquad (w \in W, \phi \in S(W)) \end{aligned} \tag{7.4}$$

which satisfies the Cannonical Commutation Relations [H2, (22.1.1)]

$$[\partial_{\mathbf{w}}, \partial_{\mathbf{w}'}] = 2\pi i \langle \mathbf{w}, \mathbf{w}' \rangle \delta \qquad (\mathbf{w}, \mathbf{w}' \in W). \tag{7.5}$$

The symplectic form $\langle\ ,\ \rangle$ and the map ∂ (7.4) extend to $\mathbf{W}$ so that (7.5) holds for $\mathbf{w}, \mathbf{w}' \in \mathbf{W}$. As is well known, the map ∂ extends to an isomorphism from the quotient of the tensor algebra of $\mathbf{W}$ by the ideal generated by elements

$$\mathbf{w} \otimes \mathbf{w}' - \mathbf{w}' \otimes \mathbf{w} - 2\pi i \langle \mathbf{w}, \mathbf{w}' \rangle \qquad (\mathbf{w}, \mathbf{w}' \in \mathbf{W})$$

onto $\mathscr{W}$. In other words, $\mathscr{W}$ is the Weyl algebra associated to the form $2\pi i\langle\ ,\ \rangle$ on $\mathbf{W}$. Let $\mathscr{W}_0 = \mathbb{C}\delta$ and let for $n \geqslant 1$

(7.6) $\mathscr{W}_n$ be the subspace of $\mathscr{W}$ spanned by δ and the monomials $\partial_{\mathbf{w}_1} \natural \partial_{\mathbf{w}_2} \natural \cdots \natural \partial_{\mathbf{w}_m}$, with $1 \leqslant m \leqslant n$, $\mathbf{w}_i \in \mathbf{W}$, $i = 1, 2, ..., m$.

Since $\partial(\mathbf{W})$ generates $\mathscr{W}$ we have an exhaustive filtration

$$\mathscr{W}_0 \subseteq \mathscr{W}_1 \subseteq \mathscr{W}_2 \subseteq \cdots \subseteq \mathscr{W}. \tag{7.7}$$

Using the obvious identification

$$(7.8)\qquad \mathbb{C} \cong \mathbb{C}\delta$$

we get an isomorphism from the graded algebra

$$(7.9)\qquad \operatorname{gr} \mathscr{W} = \bigoplus_{n=0}^{\infty} \mathscr{W}_n/\mathscr{W}_{n-1} \qquad (\mathscr{W}_{-1} = 0)$$

onto $\mathscr{P}(\mathbf{W})$, polynomial functions on $\mathbf{W}$, by

$$(7.10)\quad \operatorname{gr}(w)(\mathbf{w}) = [\cdots[[w, \underbrace{\partial_{\mathbf{w}}]\partial_{\mathbf{w}}] \cdots \partial_{\mathbf{w}}]}_{n \text{ times}} \qquad (w \in \mathscr{W}_n \backslash \mathscr{W}_{n-1}, \mathbf{w} \in \mathbf{W}).$$

The oscillator representation ω when composed with ρ^{-1} (4.5) maps $\mathscr{U}_n(\mathbf{sp})$ into $\mathscr{W}_{2n}$, $n \geqslant 0$. In particular there is a homomorphism of graded algebras

$$(7.11)\qquad \operatorname{gr} \rho^{-1}\omega \colon \operatorname{gr} \mathscr{U}(\mathbf{sp}) \to \operatorname{gr} \mathscr{W}.$$

(7.12) Lemma. *Under the identification* (7.10)

$$\operatorname{gr} \rho^{-1}\omega(x)(\mathbf{w}) = 2\pi i\langle x(\mathbf{w}), \mathbf{w}\rangle \qquad (x \in \mathbf{sp}, \mathbf{w} \in \mathbf{W}).$$

Proof. The formulas (4.6) and (7.5) imply

$$\begin{aligned} \operatorname{gr} \rho^{-1}\omega(x)(\mathbf{w}) &= [[\rho^{-1}\omega(x), \partial_{\mathbf{w}}]\partial_{\mathbf{w}}] \\ &= [\partial_{x(\mathbf{w})}, \partial_{\mathbf{w}}] = 2\pi i\langle x(\mathbf{w}), \mathbf{w}\rangle\delta. \end{aligned}$$

Q.E.D.

The map (2.1) extends to

$$\tau_{\mathbf{h}} \colon \mathbf{W} \to \mathbf{h}^* \qquad (\mathbf{h} = \text{the complexification of } h)$$

and defines the pullback

$$(7.13)\qquad \mathscr{P}(\mathbf{h}^*) \ni a \to a \circ (8\pi i \tau_{\mathbf{h}}) \in \mathscr{P}(\mathbf{W}).$$

(7.14) Lemma. *For any Lie subalgebra $\underline{h} \subseteq \underline{sp}$ the following diagram commutes*

$$\begin{array}{ccc} \operatorname{gr} \mathscr{W} & \xrightarrow{(7.10)} & \mathscr{P}(\mathbf{W}) \\ \uparrow{\scriptstyle (7.11)} & & \uparrow{\scriptstyle (7.13)} \\ \operatorname{gr} \mathscr{U}(\mathbf{h}) & \xrightarrow{\cong} & \mathscr{P}(\mathbf{h}^*). \end{array}$$

Proof. Since all the maps in this diagram are algebra homomorphisms it will suffice to check the commutativity on the generators of gr $\mathscr{U}(\mathbf{h})$. Let $x \in \mathbf{h}$ and let $\mathbf{w} \in \mathbf{W}$. Then by (7.12)

$$\operatorname{gr} \rho^{-1}\omega(x)(\mathbf{w}) = 2\pi i\langle x(\mathbf{w}), \mathbf{w}\rangle = 8\pi i\tau_{\mathbf{h}}(\mathbf{w})(x). \qquad \text{Q.E.D.}$$

Denote by

(7.15) Diff($\mathbf{W}$) the algebra of polynomial coefficient differential operators on $\mathbf{W}$.

This algebra has the usual filtration by the degree of the differential operator and if we identify $\mathbf{W}$ with its dual $\mathbf{W}^*$ by

$$(7.16) \qquad \mathbf{W} \ni \mathbf{w} \to (\mathbf{W} \ni \mathbf{w}' \to 2\pi\langle \mathbf{w}, \mathbf{w}'\rangle \in \mathbb{C})$$

then there is an isomorphism [Hö, (8.3.2)′]

$$(7.17) \qquad \operatorname{gr} \operatorname{Diff}(\mathbf{W}) \to \mathscr{P}(\mathbf{W} \times \mathbf{W})$$

$$\operatorname{gr} P(\omega_1, \omega_2) = \lim_{t \to \infty} t^{-m}\chi(-t\langle w_{2,} \_\rangle)\, P\chi(t\langle w_{2,} \_\rangle)|_{w_1}.$$

Here m is the degree of the differential operator P. For example, if $P = \partial_{\mathbf{w}} \natural$, $\mathbf{w} \in W$, then a straightforward calculation shows that

$$(7.18) \qquad \operatorname{gr}(\partial_{\mathbf{w}} \natural)(\mathbf{w}, \mathbf{w}_2) = 2\pi i\langle \mathbf{w}, \mathbf{w}_2\rangle.$$

There is an injection

$$(7.19) \qquad \mathscr{W} \to \operatorname{Diff}(\mathbf{W}), \qquad \text{defined by } \partial_{\mathbf{w}} \to \partial_{\mathbf{w}} \natural \quad (\mathbf{w} \in \mathbf{W}).$$

Define an injection

$$(7.20) \qquad \mathscr{P}(\mathbf{W}) \ni p \to q \in \mathscr{P}(\mathbf{W} \times \mathbf{W}),$$

$$q(\mathbf{w}, \mathbf{w}_2) = p(\mathbf{w}_2) \qquad (\mathbf{w}, \mathbf{w}_2 \in \mathbf{W}).$$

It follows easily from (7.16)–(7.20) and (7.9) that the following diagram commutes

$$(7.21) \qquad \begin{array}{ccccc} \mathscr{W} & \xrightarrow{(7.19)} & \operatorname{Diff}(\mathbf{W}) & \longrightarrow & \operatorname{gr} \operatorname{Diff}(\mathbf{W}) \\ \downarrow & & & & \downarrow{\scriptstyle (7.17)} \\ \operatorname{gr} \mathscr{W} & \xrightarrow{(7.10)} & P(\mathbf{W}) & \xrightarrow{(7.20)} & \mathscr{P}(\mathbf{W} \times \mathbf{W}). \end{array}$$

In fact if $\underline{h}$ is any Lie subalgebra of $\underline{sp}(W)$, then a simple calculation shows that the symbol (7.17) of the differential operator $\rho^{-1}\omega(x)\natural$ $(x \in \mathscr{U}(\mathbf{h}))$ coincides with the polynomial function

$$\text{(7.22)} \qquad \mathbf{W} \times \mathbf{W} \ni (w, w_2) \to \operatorname{gr} x(8\pi i \tau_{\mathbf{h}}(w_2)) \in \mathbb{C}.$$

Here $\operatorname{gr} x \in \mathscr{P}(\mathbf{h}^*)$ as in (7.14). After these general preliminaries about the Weyl algebra we come back to our reductive dual pair G, G' (7.1).

(7.23) LEMMA. *Let $\underline{h} = \underline{g} + \underline{g}'$ or $\underline{h} = \underline{g}'$. Denote by $I \subseteq \mathscr{U}(\mathbf{h})$ the kernel of $\omega|_{\mathscr{U}(\mathbf{h})}$. Then $\mathscr{V}(\operatorname{gr} I) = \tau_{\mathbf{h}}(\mathbf{W})$.*

Proof. Consider the case $\underline{h} = \underline{g}'$. The other one is analogous. By [H11, Theorem 7] we have the following short exact sequence

$$\text{(7.24)} \qquad O \to I \to \mathscr{U}(\mathbf{g}') \xrightarrow{\rho^{-1}\omega} \mathscr{W}^G \to O.$$

Here $\mathscr{W}^G$ is the algebra of G-invariants in $\mathscr{W}$ and G acts of $\mathscr{W}$ by conjugation [H2, (13.1.3)]. Since the maps $\operatorname{gr}: \mathscr{U}(\mathbf{g}') \to \operatorname{gr} \mathscr{U}(\mathbf{g}')$ and $\operatorname{gr}: \mathscr{W}^G \to \operatorname{gr}(\mathscr{W})^G = (\operatorname{gr} \mathscr{W})^G$ are isomorphisms of vector spaces we obtain from (7.24) the following short exact sequence

$$\text{(7.25)} \qquad O \to \operatorname{gr} I \to \operatorname{gr} \mathscr{U}(\mathbf{g}') \to \operatorname{gr} \mathscr{W}^G \to O.$$

Thus, by (7.14), $\mathscr{V}(\operatorname{gr} I)$ coincides with the Zariski closure of $\tau_{\mathbf{g}}(\mathbf{W})$. But this set is Zariski closed (see (D.3)). Q.E.D.

(7.26) LEMMA. *Suppose that $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$. Let $f \in S^*(W)$ be the corresponding intertwining distribution* (5.1). *Denote by $\underline{h}$ either $\underline{g} + \underline{g}'$ or $\underline{g}$. Let $A(\mathbf{h}, f) = \{w \in \rho^{-1}\omega(\mathscr{U}(\mathbf{h})) \mid w \natural f = 0\} \subseteq \mathscr{W}$. Then $\mathscr{V}(\operatorname{gr} I_\Pi) = \tau_{\mathbf{g}}(\mathscr{V}(\operatorname{gr} A(\mathbf{g}, f)))$ and $\mathscr{V}(\operatorname{gr} I_{\Pi \check{\otimes} \Pi'}) = \tau_{\mathbf{g}+\mathbf{g}'}(\mathscr{V}(\operatorname{gr} A(\mathbf{g}+\mathbf{g}', f)))$.*

Here $I_{\Pi \check{\otimes} \Pi'}$ is the annihilator of $\Pi \check{\otimes} \Pi'$ in $\mathscr{U}(\mathbf{g}+\mathbf{g}')$.

Proof. Consider $x \in \mathscr{U}(\mathbf{g}+\mathbf{g}')$ $(x \in \mathscr{U}(\mathbf{g}))$. Then by (5.1), $x \in I_{\Pi \check{\otimes} \Pi'}$ $(x \in I_\Pi)$ iff $\omega(x)\,\rho(f) = 0$. This condition means that $\rho^{-1}\omega(x) \natural f = 0$. It follows from (7.23) that $\mathscr{V}(\operatorname{gr} I_{\Pi \check{\otimes} \Pi'}) \subseteq \tau_{\mathbf{g}+\mathbf{g}'}(\mathbf{W})$ $(\mathscr{V}(\operatorname{gr} I_\Pi) \subseteq \tau_{\mathbf{g}}(\mathbf{W}))$. Now it is clear that (7.14) implies the lemma. Q.E.D.

Proof of (7.1). The first inclusion in (7.2) follows immediately from (7.26), (7.22), and [Hö, Theorem 8.1.8]. Since, with the notation of (7.26), $A(\mathbf{g}+\mathbf{g}', f) \supseteq A(\mathbf{g}, f)$ we have

$$\mathscr{V}(\operatorname{gr} A(\mathbf{g}+\mathbf{g}', f)) \subseteq \mathscr{V}(\operatorname{gr} A(\mathbf{g}, f)).$$

Consequently

$$\text{(7.27)} \qquad \mathscr{V}(\operatorname{gr} I_{\Pi \check{\otimes} \Pi'}) \subseteq \tau_{\mathbf{g}+\mathbf{g}'}(\tau_{\mathbf{g}}^{-1}(\mathscr{V}(\operatorname{gr} I_\Pi))).$$

If $\mathbf{g}+\mathbf{g}' \cong \mathbf{g} \oplus \mathbf{g}'$ then $\mathscr{V}(\mathrm{gr}\, I_{\Pi \hat{\otimes} \Pi'}) \cong \mathscr{V}(\mathrm{gr}\, I_{\Pi}) \times \mathscr{V}(\mathrm{gr}\, I_{\Pi'})$ and $\tau_{\mathbf{g}+\mathbf{g}} \cong \tau_{\mathbf{g}} \times \tau'_{\mathbf{g}}$. Thus (7.27) implies the second inclusion in (7.2). If $\mathbf{g}+\mathbf{g}' \not\cong \mathbf{g} \oplus \mathbf{g}'$ then $\mathbf{g} \cap \mathbf{g}'$ is the center of $\mathbf{g}$ and of $\mathbf{g}'$ so that similarly one gets the same conclusion. Q.E.D.

8. The Wave Front Set of a Unipotent Representation

(8.1) *Conjecture.* Under the assumption (5.4.2)

$$\bigcup_{w \in W} WF_w f \supseteq \tau_{\underline{g}}^{-1}(0).$$

Corollary (5.51) is a partial solution of this conjecture and the complete proof does not seem to be that far out to reach. Writing it down, however, could take some space-time and some case by case analysis which we would like to avoid here.

In this section we consider the pairs G, G' and the representations $\Pi \hat{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$ for which (8.1) (and (5.4.2)) is valid.

(8.2) Theorem. *Under the above assumption* $WF(\Pi') = \tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0))$.

Proof. By (8.1) and (7.1) we have the inclusions

$$\tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0)) \subseteq \mathscr{V}(\mathrm{gr}\, I_{\Pi'}) \subseteq \tau_{\mathbf{g}'}(\tau_{\mathbf{g}}^{-1}(0)).$$

By (2.19) the dimension of the dense G'-orbit in $\tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0))$ is $\dim W - 2\dim \underline{g}$. A straightforward calculation using (D.2) and the argument of the proof of (2.19) implies that $\tau_{\mathbf{g}'}(\tau_{\mathbf{g}}^{-1}(0))$ is the closure of one $\mathbf{G}'$-orbit ($\mathbf{G}'$ = the complexification of the algebraic group G'), whose dimension (over $\mathbb{C}$) is $\dim_{\mathbb{C}} \mathbf{W} - 2\dim_{\mathbb{C}} \mathbf{g}$. This clearly shows that

$$\mathscr{V}(\mathrm{gr}\, I_{\Pi'}) = \tau_{\mathbf{g}'}(\tau_{\mathbf{g}}^{-1}(0)). \tag{8.3}$$

Let $\xi \in WF(\Pi')$. The $\xi \in \underline{g}'^*$ corresponds to an element $x \in \underline{g}'$ via (2.5), and by (8.3), $x^2 = 0$. Therefore there is a maximal isotropic subspace $V'_1 \subseteq V'$ such that $\mathrm{im}\, x \subseteq V'_1$. Let P'_1 be the maximal parabolic subgroup of G' preserving V'_1. Then x belongs to the center $\underline{n}'_1$ of the Lie algebra of the unipotent radical of P'_1. Choose another maximal isotropic subspace $V'_2 \subseteq V'$ such that $V' = V_1'^{\perp} \oplus V'_2$. Let $\underline{n}'_2$ be the Lie algebra of the center N'_2 of the unipotent radical of the maximal parabolic subgroup of G' preserving V'_2. Let $r\colon \underline{g}'^* \to \underline{n}_2'^*$ be the restriction map. Then by [H1, Proposition 1.5]

$$WF(\Pi'|_{N'_2}) \supseteq (WF(\Pi')).$$

Howe's theory of rank [Li2, Theorem 4.7] implies that

$$WF(\Pi'|_{N_2'}) = \tau_{\underline{n}_2'}(W_1) \qquad (W_1 \text{ as in } (2.7)).$$

Thus $r(\xi) \in \tau_{\underline{n}_2'}(W_1)$. Since $\underline{n}_1'$ and $\underline{n}_2'$ are paired nondegenerately via the form (2.5) this implies that $x \in \tilde{\tau}'(W_1)$ ($\tilde{\tau}'$ as in (2.4)). Consequently (by 2.7))

$$\xi \in \mathrm{Ad}^* G'(\tau_{\underline{g}'}(W_1)) = \tau_{\underline{g}'}(G' W_1) = \tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0)),$$

so that

$$WF(\Pi') \subseteq \tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0)). \tag{8.4}$$

Since $\dim WF(\Pi') = \dim_{\mathbb{C}} \mathscr{V}(\mathrm{gr}\, I_{\Pi'})$ [B-V1, Theorem 4.1], we see that (8.3) and (8.4) imply the theorem. Q.E.D.

APPENDIX A: An Estimate of Jian Shu Li

Here we show that the proof of Theorem 3.2 in [Li1] verifies the following

(A.1) Theorem. *Suppose that G, G' is as in (5.4.2). Then for any $\phi \in S(W)$ the function*

$$\tilde{G} \ni g \to \mathrm{tr}(\omega(g)\, \rho(\phi)) \in \mathbb{C}$$

belongs to $L^1(\tilde{G})$, and the integral of this function defines a tempered distribution on W.

Proof. Let V_1 be a maximal isotropic subspace of V (2.2). Define $X_1 = \{w \in W \mid \mathrm{im}\, w \subseteq V_1\}$. Then X_1 is an isotropic subspace of W. Pick a maximal isotropic subspace X of W and a complement X_2 of X_1 in X so that $X = X_1 + X_2$ is a direct sum. We realize the oscillator representation ω (4.5) on the Hilbert space $\mathscr{H} = L^2(X)$ as in [H4]. Then by [H4, Theorem 1.4.1], for each $\phi \in S(W)$, $\rho(\phi)$ is an integral operator with kernel K_ϕ in the Schwartz space $S(X \times X)$. Moreover the map

$$S(W) \ni \phi \to K_\phi \in S(X \times X) \tag{A.2}$$

is a linear topological isomorphism and

$$\mathrm{tr}\, \rho(\phi) = \int_X K_\phi(x, x)\, dx.$$

Let A be a maximal split torus in G which preserves the subspace V_1. Choose a Cartan decomposition of $\tilde{G}$

$$\tilde{G} = \tilde{K}A^+\tilde{K}$$

and the corresponding decomposition of the Haar measure on $\tilde{G}$

$$dg = \gamma(a)\, dk_1\, da\, dk_2 \tag{A.3}$$

as in, for example, [W, 2.4.2]. The formula (17) in [H8] together with (A.2) imply

$$\operatorname{tr}(\omega(a)\,\rho(\phi)) = \alpha(a) \int_{X_1} \int_{X_2} K_\phi(a^{-1}x_1 + x_2, x_1 + x_2)\, dx_1\, dx_2, \tag{A.4}$$

where $a \in A$ and $\phi \in S(W)$.

Here α is a function on A which, by the proof of Theorem 3.2 in [Li] satisfies

$$\int_{A^+} \gamma(a)\, |\alpha(a)|\, da < \infty. \tag{A.5}$$

Let us introduce a scalar product on the real vector space X such that X_1 is orthogonal to X_2. Denote by $|\ |$ the corresponding norm on X. Choose positive constants N_1, N_2 such that

$$\int_{Xj} (1 + |x_j|)^{-Nj}\, dx_j < \infty \qquad (j = 1, 2).$$

Define a seminorm q on $S(W)$ by

$$q(\phi) = \sup_{x, x' \in X} (1 + |x'| + |x|)^{N_1 + N_2}\, |K_\phi(x', x)| \qquad (\phi \in S(W)).$$

Then for any $x_1', x_1 \in X_1$ and any $\phi \in S(W)$

$$\left| \int_{X_2} K_\phi(x_1' + x_2, x_1 + x_2)\, dx_2 \right|$$

$$\leqslant q(\phi) \int_{X_2} (1 + |x_2|)^{-N_2}\, dx_2 \cdot (1 + |x_1|)^{-N_1}.$$

Therefore by (A.4) and (A.5)

$$\int_{A^+} |\operatorname{tr}(\omega(a)\,\rho(\phi))|\, \gamma(a)\, da \leqslant \text{const}\ q(\phi) \qquad (\phi \in S(W)). \tag{A.6}$$

It follows from [H7, (11.4)] and [Wa, 4.1.1] that the function

$$\tilde{K} \times S(W) \times \tilde{K} \ni (k_1, \phi, k_2) \to \rho^{-1}(\omega(k_1)\, \rho(\phi)\, \omega(k_2)) \in S(W)$$

is continuous. Since $\tilde{K}$ is compact there is a seminorm q' on $S(W)$ such that

$$q(\rho^{-1}(\omega(k_1)\, \rho(\phi)\, \omega(k_1))) \leqslant \text{const}\; q'(\phi).$$

Therefore (A.3) and (A.6) imply the estimate

$$\begin{aligned}
&\int_{\tilde{G}} |\mathrm{tr}(\omega(g)\, \rho(\phi))|\, dg \\
&\quad = \int_{\tilde{K}} \int_{A^+} \int_{\tilde{K}} |\mathrm{tr}(\omega(a)\, \omega(k_1)\, \rho(\phi)\, \omega(k_2))|\, \gamma(a)\, dk_1\, da\, dk_2 \\
&\quad \leqslant \text{const}\; q'(\phi).
\end{aligned}$$

Q.E.D.

APPENDIX B: HOMOGENEOUS DISTRIBUTIONS

Let U be an open conical subset of a real vector space of dimension $n < \infty$. For $t > 0$ and $\phi \in C_c^\infty(U)$. Put

(B.1) $$\phi_t(x) = t^{-n} \phi(t^{-1}x) \qquad (x \in U).$$

Dualizing (B.1) define

(B.2) $$u_t(\phi) = u(\phi_t) \qquad (u \in D'(U),\ \phi \in C_c^\infty(U)).$$

A distribution $u \in D'(U)$ is called homogeneous of degree $d \in \mathbb{C}$ iff

(B.3) $$u_t = t^d u \qquad (t > 0).$$

(B.4) LEMMA. *Let $d_1, d_2, ..., d_r$ be distinct complex numbers. Denote by E_i the space of all distributions on U homogeneous of degree d_i, $1 \leqslant i \leqslant r$. Let*

(B.4.1) $$E = E_1 + E_2 + \cdots + E_r \subseteq D'(U)$$

be equipped with the relative topology. Then the sum (B.4.1) *is direct and the corresponding projections*

$$P_i \colon E \to E_i \qquad (l \leqslant i \leqslant r)$$

are continuous.

Proof. Since the functions

$$[1,2] \ni r \to t^{d_i} \in \mathbb{C} \qquad (1 \leqslant i \leqslant r)$$

are lineary independent we may choose some linear combinations of them $p_1(t), p_2(t), ..., p_2(t)$ so that

$$\int_1^2 p_j(t)\, t^{d_i}\, dt = \delta_{ij} \qquad \text{(Kronecker delta)}\ (1 \leqslant i, j \leqslant r).$$

Therefore it follows from (B.3) that

$$P_j(u) = \int_1^2 p_j(t) u_t\, dt \qquad (u \in F,\ 1 \leqslant j \leqslant r). \qquad \text{Q.E.D.}$$

APPENDIX C: THE GELFAND–KIRYLLOV DIMENSION OF A HOLOMORPHIC REPRESENTATION

Here we assume that G, G' is an irreducible dual pair with G compact. Our goal is to prove the following known

(C.1) THEOREM. *Let* $\Pi' \in R(\tilde{G}', \omega)$. *Then* $2\,\mathrm{Dim}\,\Pi' = \dim \tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0))$. (*For the notation* "Dim Π'" *see* [V2, *Proposition* 5.5].)

In the view of the inclusion (6.14) and [B-V1, Theorem 4.8] it will suffice to show the inequality

(C.2) $$2\,\mathrm{Dim}\,\Pi' \geqslant \dim \tau_{\underline{g}'}(\tau_{\underline{g}}^{-1}(0)).$$

Let $\mathscr{J}$ be a compatible positive complex structure on W centralized by G. Denote K' the centralizer of $\mathscr{J}$ in G'. Then K' is a maximal compact subgroup of G' and we get a Cartan decomposition

(C.3) $$\underline{g}' = \underline{k}' + \underline{p}'.$$

We shall work in the Harish-Chandra module of the Fock model of ω (4.5) adapted to $\mathscr{J}$ [P1, (1.4.5)]. This module coincides with the space

(C.4) $\mathscr{P} = \mathscr{P}(W)$ of polynomial functions on W where W is viewed as a complex vector space ($iw = \mathscr{J}(W)$, $w \in W$). The complexification $\mathbf{p}'$ of $\underline{p}'$ (C.3) has a direct sum decomposition

(C.5) $$\mathbf{p}' = \mathbf{p}'_+ + \mathbf{p}'_-$$

with the property that $\omega(\mathbf{p}'_+)$ is spaned by certain quadratic polynomials (viewed as multiplication operators). In the notation of [H6], $\omega(\mathbf{p}'_+) = \mathbf{g}'^{(2,0)}$. The $\mathbf{p}'_+$ (C.5) is a commutative Lie subalgebra of $\mathbf{g}'$. Denote by A the image of the universal enveloping algebra $\mathscr{U}(\mathbf{p}'_+)$ under ω and by I the kernel of $\omega|_{\mathscr{U}(\mathbf{p}'_+)}$. Thus we have a short exact sequence.

$$O \to I \to \mathscr{U}(\mathbf{p}'_+) \xrightarrow{\omega} A \to O. \tag{C.6}$$

Denote by $\mathscr{P}_\Pi$ the Π-isotypic component of $\mathscr{P}$ (C.4) and by $H(G)_\Pi$ the subspace of $\mathscr{P}_\Pi$ spanned by the non-zero polynomials of lowest possible degree in $\mathscr{P}_\Pi$ (see [H6; P1, (5.18)]). Then, as is well known [H6, (3.9)]

$$\mathscr{P}_\Pi = AH(G)_\Pi. \tag{C.7}$$

Since $\omega(\mathbf{k} + \mathbf{p}'_-)$ normalizes $H(G)_\Pi$ the P-B-W theorem implies that

$$\omega(\mathscr{U}_n(\mathbf{g}'))\, H(G)_\Pi = \omega(\mathscr{U}_n(\mathbf{p}'_+))\, H(G)_\Pi \tag{C.8}$$

(see (7.9) for $\mathscr{U}_n$). By combining (C.7), (C.8) with [V2, Theorem 1.1] we see that

$$\operatorname{Dim} \Pi' = \operatorname{Dim} A, \tag{C.9}$$

where Dim A is the Gelfand–Kiryllov dimension of the algebra A [B-K]. This dimension coincides with the dimension of the variety

(C.10) $\mathscr{V}(\operatorname{gr} I)$ of the homogeneous ideal gr I (C.6) in the dual $\mathbf{p}'^*_+$.

Lema (7.14) implies that

$$\mathscr{V}(\operatorname{gr} I) \supseteq \tau_{\mathbf{p}'_+}(\mathbf{W}). \tag{C.11}$$

It remains to show that

$$2 \dim_{\mathbb{C}} \tau_{\mathbf{p}'_+}(\mathbf{W}) = \dim_{\mathbb{C}} \tau_{\mathbf{g}'}(\tau_{\mathbf{g}}^{-1}(0)) \tag{C.12}$$

which may be verified by a calculation similar to the one used in the proof of (2.9).

APPENDIX D: Complex Pairs

Let G, G' be a real reductive dual pair in $Sp(W)$. Then the complexified Lie algebras $\mathbf{g}, \mathbf{g}'$ form a complex dual pair of Lie algebras in $\mathbf{sp}$. Suppose that $\mathbf{h}, \mathbf{h}'$ is another complex dual pair in $\mathbf{sp}$ and that $\mathbf{h}$ is isomorphic to $\mathbf{g}$ and $\mathbf{h}'$ to $\mathbf{g}'$. Then (as follows from the classification of such pairs [H7]) there is $g \in Sp(\mathbf{W})$ such that $\mathrm{Ad}\, g(\mathbf{g}) = \mathbf{h}$ and $\mathrm{Ad}\, g(\mathbf{g}') = \mathbf{h}'$. As in (2.1) we have a quadratic map

$$\tau_{\mathbf{g}}: \mathbf{W} \to \mathbf{g}^*, \ \tau_{\mathbf{g}}(\mathbf{w})(x) = \tfrac{1}{4}\langle x(\mathbf{w}), \mathbf{w}\rangle, \ x \in \mathbf{g}, \ \mathbf{w} \in \mathbf{W}. \tag{D.1}$$

Clearly this is the extension of the map (2.1) $\tau_{\underline{g}}: W \to \underline{g}^*$. All together we have the following commuting diagram

(D.2)
$$\begin{tikzcd} \mathbf{g}'^* \arrow[d, "\mathrm{Ad}^* g"] & \mathbf{W} \arrow[l, "\tau_{\mathbf{g}'}"'] \arrow[r, "\tau_{\mathbf{g}}"] \arrow[d] & \mathbf{g}^* \arrow[d, "\mathrm{Ad}^* g"] \\ \mathbf{h}'^* & \mathbf{W} \arrow[l, "\tau_{\mathbf{h}'}"] \arrow[r, "\tau_{\mathbf{h}}"'] & \mathbf{h}^* \end{tikzcd}$$

where the unmarked arrow maps $\mathbf{w} \in \mathbf{W}$ to $g(\mathbf{w}) \in \mathbf{W}$.

(D.3) LEMMA. *The set $\tau_{\mathbf{g}}(\mathbf{W}) \subseteq \mathbf{g}^*$ is an affine algebraic variety.*

Proof. The pair $\mathbf{g}, \mathbf{g}'$ may be reducible [H7]. Then we have the orthogonal direct sum decompositions

$$\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \cdots, \ \mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2 \oplus \cdots, \ \mathbf{g}' = \mathbf{g}_1' \oplus \mathbf{g}_2' \oplus \cdots$$

such that $\mathbf{g}_i, \mathbf{g}_i'$ is an irreducible pair in $sp(\mathbf{W})$. Notice that

$$\tau_{\mathbf{g}} = \tau_{\mathbf{g}_1} \times \tau_{\mathbf{g}_2} \times \cdots,$$

where $\tau_{\mathbf{g}_i}: \mathbf{W}_i \to \mathbf{g}_i^*$ is defined by (D.1) on $\mathbf{W}_i$ $(i = 1, 2, \ldots)$.

Thus we may assume that our pair $\mathbf{g}, \mathbf{g}'$ is irreducible. By (D.2) there are two cases to consider:

(D.3.1) $\mathbf{g}(\mathbf{g}')$ is the Lie algebra of isometries of a complex vector space $\mathbf{V}(\mathbf{V}')$ with a nondegenerate symmetric form $(\ ,\ )$ (antisymmetric form $(\ ,\ )'$), and $\mathbf{W} = \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}', \mathbf{V})$; or

(D.3.2) $\mathbf{g} = \mathrm{End}_{\mathbb{C}}(\mathbf{U})$ $(\mathbf{g}' = \mathrm{End}_{\mathbb{C}}(\mathbf{U}'))$ where $\mathbf{U}(\mathbf{U}')$ is a complex vector space and

$$\mathbf{W} = \mathrm{Hom}_{\mathbb{C}}(\mathbf{U}, \mathbf{U}') \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbf{U}', \mathbf{U}).$$

In the situation (D.3.1) there is a linear isomorphism

(D.3.3) $\mathbf{g} \ni x \to \beta(x) \in B(\mathbf{V})$, $\beta(x)(u, v) = (x(u), v)$ $(u, v \in \mathbf{V})$ onto the space $B(\mathbf{V})$ of antisymmetric forms on $\mathbf{V}$. If we identify $\mathbf{g}^*$ with $\mathbf{g}$ as in (2.5) then $4\tau_{\mathbf{g}}$ will coincide with the map

$$\tilde{\tau}(\mathbf{w}) = \mathbf{w}\mathbf{w}^* \qquad (\mathbf{w} \in \mathbf{W}),$$

where the "*" is defined as in (2.2). Thus $\tau_{\mathfrak{g}}(\mathbf{W})$ as a subset of $\mathrm{End}_{\mathbb{C}}(\mathbf{V})$ is mapped, via a linear isomorphism, onto $\beta \circ \tilde{\tau}(\mathbf{W}) \subseteq B(\mathbf{W})$. This set $\beta \circ \tilde{\tau}(\mathbf{W})$ coincides with the variety of all antisymmetric forms on $\mathbf{V}$ of rank at most $\dim_{\mathbb{C}} \mathbf{V}'$. Thus $\tau_{\mathfrak{g}}(\mathbf{W})$ (and similarly $\tau_{\mathfrak{g}'}(\mathbf{W})$) is Zariski closed and irreducible.

The case (D.3.4) is simpler. Hence, by similar procedure, $\tau_{\mathfrak{g}}$ may be identified with a map

(D.3.4) $\quad \mathrm{Hom}_{\mathbb{C}}(\mathbf{U}, \mathbf{U}') \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbf{U}', \mathbf{U}) \ni \mathbf{w}_1 \oplus \mathbf{w}_2 \to \mathbf{w}_2 \mathbf{w}_1 \in \mathfrak{g}.$

Thus $\tau_{\mathfrak{g}}(\mathbf{W})$ is the variety of linear endomorphisms of $\mathbf{V}$ of rank at most $\dim_{\mathbb{C}} \mathbf{V}'$. Q.E.D.

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