An entropy-based uncertainty principle for a locally compact abelian group [☆]

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Abstract

We classify all functions on a locally compact, abelian group giving equality in an entropy inequality generalizing the Heisenberg Uncertainty Principle. In particular, for functions on a real line, we proof a conjecture of Hirschman published in 1957.
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0. Introduction and background

The Heisenberg inequality essentially states that a function of a real variable and its Fourier transform cannot both be arbitrarily concentrated. However there is no straightforward analog of the Heisenberg inequality even for functions defined on a finite cyclic group (one problem is that the "position operator" does not make sense in this case). One way around this problem is to consider the L^p norms of a function, for $1 < p < \infty$ (see [B1,B2,L] and their references for a sample of the extensive literature in this direction). More to the point is via information theory, where the definitive measure of the concentration of a probability density function is entropy [S].

The entropy approach in the continuous case goes back to Hirschman, who in 1957 proved that the sum of entropies of a function f of a real variable, with $\|f\|_2 = 1$, and its Fourier transform is nonnegative [Hi]. He also observed that Weyl's formulation of the Heisenberg Uncertainty Principle is a consequence of a stronger version of this inequality, namely

$$-\int_{\mathbb{R}} |f(x)|^2 \log(|f(x)|^2) \, dx - \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \log(|\hat{f}(\xi)|^2) \, d\xi \geqslant \log\left(\frac{e}{2}\right), \qquad (0.1)$$

where the log stands for the natural logarithm with base e, $f \in S(\mathbb{R})$, the Schwartz space (see [H1]) and $\|f\|_2 = 1$. This was proven by Beckner [B1] in 1975.

In [Hi], Hirschman also conjectured that the minimizers for the sharp inequality (0.1) were Gaussians, as is the case for the Heisenberg Uncertainty Principle. The special case of our Main Theorem 1.5 below, where the group A is chosen to be the additive group of real numbers, verifies this conjecture. The related problem of finding all the maximizers realizing the norm of the Fourier transform viewed as an operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ was solved by Lieb (see [L] and the references there).

There is an analog of (0.1) for functions f defined on a finite cyclic group, proved in [D-C-T], with applications in Signal Processing. The minimizers for this finite case were determined in [P-D-O] (verifying the conjecture in [D-O-P]). This corresponds to choosing a finite cyclic group as A in our Main Theorem 1.5. These minimizers depend on the factorization of the order of the finite cyclic group and are not "Gaussians" or discretized Gaussians. This discrepancy between the finite and the continuous case seems to be unexpected in the Signal Processing community. In fact the main motivation for our result was to certify this fundamental difference.

Another Uncertainty Principle in the finite case, due to Matolcsi and Szücs [M-S] states that the product of the cardinalities of the support of a function, defined on a finite cyclic group, and the support of its Fourier transform is no less than the order of the group (see also [D-S]). This inequality follows from the Heisenberg–Weyl (entropy) version [D-C-T]. It turns out that the minimizers for both versions are exactly the same functions [P-D-O]. The main theorem below classifies the minimizers of the entropy inequality in the multidimensional case where the finite cyclic group may be replaced by an arbitrary locally compact abelian group. The minimizers for the corresponding Donoho–Stark inequality for an arbitrary finite abelian group are the same. The first proof of this is in [Sm], an elementary proof is given in [M-Ö-P]. Another elementary proof of a generalization as well as related results are in [G-G-I], (see also [F-S,Te,Ta]).

1. The main theorem

Let A be a locally compact abelian group. As was explained to the second author by Michael Cowling, a result of Ahern and Jewett [A-J], together with [H-R, 9.8], imply that A is isomorphic to the direct product of a finite number of copies of $\mathbb{R}$ (the reals) and an abelian locally compact group B which contains an open compact

subgroup:

$$A = \mathbb{R}^n \times B. \tag{1.1}$$

Let $\hat{A}$ be the Pontriagin dual of A. Then $\hat{A} = R^n \times \hat{B}$, where $\hat{B}$ also contains an open compact subgroup. Let α be a Haar measure on A and let $\hat{\alpha}$ be the Haar measure on $\hat{A}$, dual to α, so that the Fourier transform and the inverse Fourier transform are given by the following formulas:

$$\hat{f}(\hat{a}) = \int_A f(a)\hat{a}(-a)\,d\alpha(a) \quad (\hat{a} \in \hat{A}),$$

$$f(a) = \int_A \hat{f}(\hat{a})\hat{a}(a)\,d\hat{\alpha}(\hat{a}) \quad (a \in A), \tag{1.2}$$

whenever both integrals make sense.

   Let $G \subseteq U(L^2(A,\alpha))$ be the Heisenberg group generated by the translations and the modulations:

$$T_{a_0} f(a) = f(a + a_0), \quad M_{\hat{a}_0} f(a) = \hat{a}_0(a)f(a)$$

$$(f \in L^2(A,\alpha);\ a_0, a \in A;\ \hat{a}_0 \in \hat{A}), \tag{1.3}$$

and by multiplications by complex numbers of absolute value 1. Recall the notion of entropy of a probability density ϕ on a measure space (M, μ):

$$H(\phi) = -\int_M \phi(m)\log(\phi(m))\,d\mu(m), \tag{1.4}$$

which is well defined whenever integral (1.4) is absolutely convergent (see [S]). Here the log stands for the natural logarithm with base e.

Theorem 1.5. *For any function $f \in L^2(A,\alpha)$, with $\|f\|_2 = 1$, satisfying*

$$f \in L^1(A,\alpha) \ \text{and} \ \hat{f} \in L^1(\hat{A}, \hat{\alpha}) \tag{$*$}$$

the following inequality holds:

$$H(|f|^2) + H(|\hat{f}|^2) \geqslant n \log\!\left(\frac{e}{2}\right), \tag{a}$$

where 'n' is defined in (1.1). The set of minimizers for (a) coincides with the union of orbits

$$G \cdot f, \tag{b}$$

where, according to decomposition (1.1), $f = g \otimes h$, g is a normalized Gaussian on $\mathbb{R}^n$ and h is the normalized indicator function of a subgroup of B.

Here by a Gaussian on $\mathbb{R}^n$ we understand a function of the form $g(x) = const\, e^{-Q(x)}$, $x \in \mathbb{R}^n$, where Q is a positive definite quadratic form on $\mathbb{R}^n$.

We proof Theorem 1.5(b) in the next three sections. The first assertion of Theorem 1.5(a), is essentially known. When $A = \mathbb{R}^n$ it is due to Beckner, [B1]. When A is a finite abelian group it can be found in [M,M-U,D-C-T, p. 1513]. The general case can be obtained by taking the left derivative of both sides of inequality (4.1) below, at $p = 2$ (since (4.1) is an equality for $p = 2$). Assumption 1.5(∗) assures that all the integrals we are going to consider converge, and that, in an appropriate context, we shall be able to reverse the order of differentiation and integration. These details are left to the reader.

2. The case $A = B$

Let f be a minimizer for 1.5(b). Consider the following function:

$$F(z) = \int_{\hat{A}} (|f|^{2z} \frac{f}{|f|})\hat{}(\hat{a})\, |\hat{f}(\hat{a})|^{2z} \frac{\overline{\hat{f}(\hat{a})}}{|\hat{f}(\hat{a})|}\, d\hat{\alpha}(\hat{a}) \quad (z \in \mathbb{C},\ \tfrac{1}{2} \leqslant Re(z) \leqslant 1), \qquad (2.1)$$

where $\frac{f}{|f|} = 0$ outside the support of f, and similarly for $\frac{\hat{f}}{|\hat{f}|}$. Integral (2.1) is absolutely convergent. Indeed, a straightforward application of Hölder's inequality and the Riesz–Thorin Theorem shows that for $\frac{1}{2} \leqslant x \leqslant 1$, $y \in \mathbb{R}$, $p = \frac{1}{x}$ and q defined by the equation $\frac{1}{p} + \frac{1}{q} = 1$ (with $q = \infty$ if $p = 1$), we have

$$|F(x + iy)| \leqslant ||(|f|^{2x+i2y} \frac{f}{|f|})\hat{}||_q \cdot ||\,|\hat{f}|^{2x+i2y}||_p \leqslant ||\,|f|^{2x+i2y}||_p \cdot ||\,|\hat{f}|^{2x+i2y}||_p$$

$$= ||f||_2 \cdot ||\hat{f}||_2 = 1. \qquad (2.2)$$

The function F is analytic in the open strip (2.1) and continuous in the closed strip. A straightforward calculation shows that

$$F'(z) = \int_{\hat{A}} (|f|^{2z} \frac{f}{|f|} \log(|f|^2))\hat{}(\hat{a}) |\hat{f}(\hat{a})|^{2z} \frac{\overline{\hat{f}(\hat{a})}}{|\hat{f}(\hat{a})|}\, d\hat{\alpha}(\hat{a})$$

$$+ \int_{\hat{A}} (|f|^{2z} \frac{f}{|f|})\hat{}(\hat{a})\, |\hat{f}(\hat{a})|^{2z} \frac{\overline{\hat{f}(\hat{a})}}{|\hat{f}(\hat{a})|} \log(|\hat{f}(\hat{a})|^2)\, d\hat{\alpha}(\hat{a}).$$

Hence, by the Plancherel formula,

$$F'(\tfrac{1}{2}) = -H(|f|^2) - H(|\hat{f}|^2). \qquad (2.3)$$

Since f is a minimizer, the right-hand side of Eq. (2.3) is zero. In particular $Re\, F(z)$ is a real valued harmonic function, on the interior of the disc of radius $\frac{1}{4}$ centered at $z = \frac{3}{4}$, which achieves the maximum at $z = \frac{1}{2}$ and has derivative equal to zero at this

point. Hence the Hopf's Maximum Principle, [H2, Theorem 3.1.6'], implies that $Re\, F(z) = 1$ on the disc. Hence, $F(z) = 1$ on the disc. In particular,

$$1 = F(1) = \int_{\hat{A}} (|f|f)\hat{\,}(\hat{a}) |\hat{f}(\hat{a})| \overline{\hat{f}(\hat{a})} \, d\hat{\alpha}(\hat{a}). \tag{2.4}$$

Formula (2.4) may be rewritten as

$$1 = \int_{\hat{A}} \int_{A} |f(a)|^2 |\hat{f}(\hat{a})|^2 \hat{a}(-a) \frac{f(a)}{|f(a)|} \frac{\overline{\hat{f}(\hat{a})}}{|\hat{f}(\hat{a})|} \, d\alpha(a) \, d\hat{\alpha}(\hat{a}). \tag{2.5}$$

Since,

$$1 = \int_{\hat{A}} \int_{A} |f(a)|^2 |\hat{f}(\hat{a})|^2 \, d\alpha(a) \, d\hat{\alpha}(\hat{a})$$

Eq. (2.5) implies that (almost everywhere, with respect to the measure $\alpha \times \hat{\alpha}$), we have

$$1 = \hat{a}(-a) \frac{f(a)}{|f(a)|} \frac{\overline{\hat{f}(\hat{a})}}{|\hat{f}(\hat{a})|} \quad (a \in supp\, f, \hat{a} \in supp\, \hat{f}). \tag{2.6}$$

Hence,

$$\hat{a}(-a) = \frac{\overline{f(a)}}{|f(a)|} \frac{\hat{f}(\hat{a})}{|\hat{f}(\hat{a})|}.$$

Thus for $\hat{a} \in supp\, \hat{f}$

$$\hat{f}(\hat{a}) = \int_{A} f(a)\hat{a}(-a) \, d\alpha(a) = \int_{A} f(a) \frac{\overline{f(a)}}{|f(a)|} \, d\alpha(a) \frac{\hat{f}(\hat{a})}{|\hat{f}(\hat{a})|} = ||f||_1 \frac{\hat{f}(\hat{a})}{|\hat{f}(\hat{a})|}.$$

Therefore

$$|\hat{f}(\hat{a})| = ||f||_1 \quad (\hat{a} \in supp\, \hat{f}), \tag{2.7}$$

and similarly

$$|f(a)| = ||\hat{f}||_1 \quad (a \in supp\, f). \tag{2.8}$$

Statement (2.7) implies that the function $|\hat{f}|$ is constant on its support. Since $||\hat{f}||_2 = 1$, the constant is equal to $\hat{\alpha}(supp\, \hat{f})^{-1/2}$. Hence,

$$H(|\hat{f}|^2) = \log(\hat{\alpha}(supp\, \hat{f})).$$

Similarly

$$H(|f|^2) = \log(\alpha(supp\, f)).$$

Since f is a minimizer,

$$\log(\alpha(supp\, f) \cdot \hat{\alpha}(supp\, \hat{f})) = 2(H(|f|^2) + H(|\hat{f}|^2)) = 0.$$

Therefore

$$\alpha(supp\, f) \cdot \hat{\alpha}(supp\, \hat{f}) = 1. \tag{2.9}$$

We may assume that $0 \in supp\, \hat{f}$ and $0 \in supp\, f$. Then (2.7) implies

$$\left| \int_A f(a)\, d\alpha(a) \right| = \int_A |f(a)|\, d\alpha(a).$$

Therefore there is $\lambda \in \mathbb{C}$ such that $f = \lambda|f|$. Hence (2.7) may be rewritten as

$$\left| \int_A \lambda|f(a)|\hat{a}(-a)\, d\alpha(a) \right| = \int_A |f(a)|\, d\alpha(a) \quad (\hat{a} \in supp\, \hat{f}). \tag{2.10}$$

Therefore

$$supp\, \hat{f} \subseteq (-supp\, f)^\perp, \tag{2.11}$$

where for a subset $S \subseteq A$, $S^\perp = \{\hat{a} \in \hat{A};\ \hat{a}|_S = 1\}$. Similarly (2.8) implies

$$supp\, f \subseteq (supp\, \hat{f})^\perp. \tag{2.12}$$

By dualizing (2.11) and (2.12) we deduce

$$-supp\, f \subseteq (-supp\, f)^{\perp\perp} \subseteq (supp\, \hat{f})^\perp \tag{2.13}$$

and

$$supp\, \hat{f} \subseteq (supp\, \hat{f})^{\perp\perp} \subseteq (supp\, f)^\perp$$

But, as is well known (see (1.2)),

$$\hat{\alpha}((supp\, \hat{f})^\perp) \cdot \alpha((supp\, \hat{f})^{\perp\perp}) = 1. \tag{2.14}$$

By combining (2.9), (2.13) and (2.14) we see that inclusions (2.13) are equalities (almost everywhere). In particular $supp\, f$ is a subgroup of A and f is invariant under the translations by this subgroup. Thus f is a constant multiple of the indicator function of a subgroup of A, as claimed.

3. The case $A = \mathbb{R}^n$

In this section we consider $A = \mathbb{R}^n$ and we identify $\hat{A}$ with A via the formula

$$\hat{a}(b) = e^{2\pi i a \cdot b} \quad (a, b \in \mathbb{R}^n),$$

where $a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ is the usual dot product in $\mathbb{R}^n$. Then the Lebesgue measure dx serves as the Haar measure $d\alpha(x)$ and as the dual Haar measure $d\hat{\alpha}(x)$.

Our proof follows closely some arguments of Lieb [L]. Let $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^n \times \mathbb{R}^n)$, and let

$$F(\hat{\xi}, y) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} F(x, y) \, dx \quad (x, y \in \mathbb{R}^n)$$

denote the partial Fourier transform with respect to the first variable. Also, for convenience, let

$$F(\hat{\xi}, \hat{\eta}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-2\pi i (\xi \cdot x + \eta \cdot y)} F(x, y) \, dx \, dy \quad (\xi, \ \eta \in \mathbb{R}^n)$$

denote the Fourier transform of F. From now on we assume that $||F||_2 = 1$. For $1 < p < 2$ and for q, defined by the equation $\frac{1}{p} + \frac{1}{q} = 1$, we deduce the following inequalities from [B1], as in [L, p. 193]:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(\hat{\xi}, \hat{\eta})|^q \, d\xi \, d\eta$$

$$\leqslant (A_p)^{nq} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(\hat{\xi}, y)|^p \, dy \right)^{\frac{q}{p}} d\xi$$

$$\leqslant (A_p)^{nq} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(\hat{\xi}, y)|^q \, d\xi \right)^{\frac{p}{q}} dy \right)^{\frac{q}{p}}$$

$$\leqslant (A_p)^{2nq} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x, y)|^p \, dx \, dy \right)^{\frac{q}{p}}, \tag{3.1}$$

where $A_p = \left(p^{\frac{1}{p}} / q^{\frac{1}{q}} \right)^{\frac{1}{2}}$ is the Babenko–Beckner constant, see [Ba,B1, p. 162]. If $p = 2$, all the inequalities in (3.1) are equalities. Hence, by taking the left derivative with

respect to p, at $p = 2$, we deduce the following inequalities:

$$-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|F(\hat\xi,\hat\eta)|^2\log(|F(\hat\xi,\hat\eta)|)\,d\xi\,d\eta$$

$$\geqslant\frac{n}{2}\log\left(\frac{e}{2}\right)-\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,dy\right)\log\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,dy\right)d\xi$$

$$+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\log(|F(\hat\xi,y)|)\,dy\,d\xi$$

$$\geqslant\frac{n}{2}\log\left(\frac{e}{2}\right)+\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,d\xi\right)\log\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,d\xi\right)dy$$

$$-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\log(|F(\hat\xi,y)|)\,dy\,d\xi$$

$$\geqslant n\log\left(\frac{e}{2}\right)+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|F(x,y)|^2\log(|F(x,y)|)\,dx\,dy. \tag{3.2}$$

Suppose the function F is a minimizer for $\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$. Then (3.2) implies that the following equation holds:

$$-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\log(|F(\hat\xi,y)|^2)\,d\xi\,d\eta$$

$$=-\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,dy\right)\log\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,dy\right)d\xi$$

$$-\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,d\xi\right)\log\left(\int_{\mathbb{R}^n}|F(\hat\xi,y)|^2\,d\xi\right)dy. \tag{3.3}$$

Hence, by Shannon [S] the following equation holds almost everywhere on $\mathbb{R}^{2n}$:

$$|F(\hat\xi,y)|^2=\int_{\mathbb{R}^n}|F(\hat\xi,s)|^2ds\int_{\mathbb{R}^n}|F(\hat t,y)|^2dt. \tag{3.4}$$

The rest of the proof is straightforward. We reproduce an argument of Lieb in a concise form [L, pp. 202, 203], for the readers convenience.

Let f be a minimizer for inequality 1.5(a) for $\mathbb{R}^n$, and let $g(y)=2^{\frac{n}{4}}e^{-\pi y\cdot y}$, $y\in\mathbb{R}^n$. Then the tensor product, $f(x)g(y)$ is a minimizer for $\mathbb{R}^{2n}$. Since the rotation

$$\mathbb{R}^n\times\mathbb{R}^n\ni(x,y)\rightarrow\left(\frac{x+y}{\sqrt2},\frac{x-y}{\sqrt2}\right)\in\mathbb{R}^n\times\mathbb{R}^n$$

leaves the Lebesgue measure invariant and commutes with the Fourier transform, the function

$$F(x,y) = f\left(\frac{x+y}{\sqrt{2}}\right)g\left(\frac{x-y}{\sqrt{2}}\right) \tag{3.5}$$

is also a minimizer for $\mathbb{R}^{2n}$.

A straightforward calculation shows that

$$F(\hat{\xi},y) = 2^{\frac{n}{4}}e^{-\pi(\xi\cdot\xi+y\cdot y)}Q(\pi(y-i\xi)),$$

where

$$Q(w) = e^{-\frac{1}{\pi}w\cdot w}\int_{\mathbb{R}^n} e^{-\frac{\pi}{2}t\cdot t+2w\cdot t}f\left(\frac{t}{\sqrt{2}}\right)dt \quad (\xi,y,w\in\mathbb{R}^n). \tag{3.6}$$

The function Q extends to an analytic function on $\mathbb{C}^n$. Clearly, there are finite constants c_1 and c_2 such that

$$|Q(w)| \leqslant c_1 e^{c_2|w|^2} \quad (w\in\mathbb{C}^n). \tag{3.7}$$

Let $M(\xi,y) = Q(\pi(y-i\xi))$, and let $M^*(\xi,y) = \overline{M(\bar{\xi},\bar{y})}$, $\xi,y\in\mathbb{C}^n$. Then M and M^* are analytic functions on $\mathbb{C}^n \times \mathbb{C}^n$ and

$$M(\xi,y)M^*(\xi,y) = |F(\hat{\xi},y)|^2 2^{-n/2}e^{2\pi(\xi\cdot\xi+y\cdot y)} \quad (\xi,y\in\mathbb{R}^n).$$

Hence, by (3.4), there are functions M_1, M_2 such that

$$M(\xi,y)M^*(\xi,y) = M_1(\xi)M_2(y) \quad (\xi,y\in\mathbb{R}^n). \tag{3.8}$$

It is easy to see that the functions M_1 and M_2 extend to analytic functions on $\mathbb{C}^n$ and that Eq. (3.8) holds for all $\xi,y\in\mathbb{C}^n$. The zero set of the left-hand side of (3.8) is the union of sets of the form

$$\{(\xi,y)\in\mathbb{C}^n \times \mathbb{C}^n; \ y-i\xi=z\}, \ \{(\xi,y)\in\mathbb{C}^n \times \mathbb{C}^n; \ y+i\xi=z\} \quad (z\in\mathbb{C}^n).$$

On the other hand, the zero set of the right-hand side of (3.8) is the union of sets of the form

$$\{\xi\} \times \mathbb{C}^n, \ \mathbb{C}^n \times \{y\} \quad (\xi,y\in\mathbb{C}^n).$$

Since function (3.8) is not identically equal to zero we see from the above, that it has no zeros. Thus the function $F(\hat{\xi},y)$ has no zeros. Therefore the function

$$\mathbb{C}^n \ni \xi \rightarrow \log(F(\hat{\xi},0))\in\mathbb{C}$$

is well defined, analytic, and satisfies the following estimate:

$$|\log(F(\hat{\xi},0))| \leqslant \log(c_1) + c_2|\xi|^2 \quad (\xi \in \mathbb{C}^n).$$

Hence, by the Cauchy estimate, there is a symmetric matrix $\mathscr{A}$ with complex entries, a vector $\mathscr{B} \in \mathbb{C}^n$ and a number $\mathscr{C} \in \mathbb{C}$ such that

$$\log(F(\hat{\xi},0)) = \xi^t \mathscr{A} \xi + \mathscr{B} \cdot \xi + \mathscr{C} \quad (\xi \in \mathbb{C}^n).$$

Therefore

$$F(\hat{\xi},0) = e^{\xi^t \mathscr{A} \xi + \mathscr{B} \cdot \xi + \mathscr{C}} \quad (\xi \in \mathbb{R}^n).$$

Since the above function is integrable, the matrix $\mathscr{A}$ is real and positive definite. Hence, by Fourier inversion, the function $F(x,0)$ is a translation and a modulation of a Gaussian. Since $F(x,0) = f\left(\frac{x}{\sqrt{2}}\right)g\left(\frac{x}{\sqrt{2}}\right)$, we see that f is also a translation and a modulation of a Gaussian.

4. The general case

In this section $A = \mathbb{R}^n \times B$, where B contains an open compact subgroup. Let β be a Haar measure on the group B, and let $\hat{\beta}$ be the dual Haar measure on the dual group $\hat{B}$. Then $dx\,d\beta(b)$ is a Haar measure on A and $dx\,d\hat{\beta}(\hat{b})$ is the dual Haar measure on $\hat{A} = \mathbb{R}^n \times \hat{B}$.

Notice that by Riesz–Thorin, Beckner and Minkowski we have for a suitable function f (with $1 < p \leqslant 2$ and q defined by $\frac{1}{p} + \frac{1}{q} = 1$),

$$\int_{\mathbb{R}^n} \int_{\hat{B}} |\hat{f}(\hat{b},\xi)|^q \, d\hat{\beta}(\hat{b}) \, d\xi \leqslant \int_{\mathbb{R}^n} \left(\int_B |f(b,\hat{\xi})|^p d\beta(b) \right)^{q/p} d\xi$$

$$\leqslant \left(\int_B \left(\int_{\mathbb{R}^n} |f(b,\hat{\xi})|^q \, d\xi \right)^{p/q} d\beta(b) \right)^{q/p}$$

$$\leqslant (A_p)^{nq} \left(\int_B \int_{\mathbb{R}^n} |f(b,x)|^p \, dx \, d\beta(b) \right)^{q/p}.$$

Therefore,

$$\int_{\mathbb{R}^n} \int_{\hat{B}} |\hat{f}(\hat{b},\xi)|^q \, d\hat{\beta}(\hat{b}) \, d\xi \leqslant (A_p)^{nq} \left(\int_B \int_{\mathbb{R}^n} |f(b,x)|^p \, dx \, d\beta(b) \right)^{q/p}. \qquad (4.1)$$

Hence, the argument used to prove (3.4) shows that if F is a minimizer for $\mathbb{R}^n \times \mathbb{R}^n \times B$ then

$$|F(\hat{\xi}, y, b)|^2 = \left(\int_{\mathbb{R}^n} \int_B |F(\hat{\xi}, y_1, b_1)|^2 \, d\beta(b_1) dy_1 \right)$$
$$\times \left(\int_{\mathbb{R}^n} |F(\hat{\xi}_1, y, b)|^2 \, d\xi_1 \right) \quad (b \in B; \ \xi, y \in \mathbb{R}^n), \qquad (4.2)$$

and if f is a minimizer for A, then for $x \in \mathbb{R}^n, b \in B$ and $\hat{b} \in \hat{B}$,

$$|f(\hat{\xi}, b)|^2 = \left(\int_{\mathbb{R}^n} |f(\hat{\xi}_1, b)|^2 \, d\xi_1 \right) \left(\int_B |f(\hat{\xi}, b_1)|^2 \, d\beta(b_1) \right),$$
$$|f(x, \hat{b})|^2 = \left(\int_{\mathbb{R}^n} |f(x_1, \hat{b})|^2 \, dx_1 \right) \left(\int_{\hat{B}} |f(x, \hat{b_1})|^2 \, d\hat{\beta}(b_1) \right), \qquad (4.3)$$

where $f(x, \hat{b})$ stands for the Fourier transform of f with respect to the second variable. Let f be a minimizer for A and let $g(y) = 2^{\frac{n}{4}} e^{-\pi y \cdot y}$, $y \in \mathbb{R}^n$. Then $F(x, y, b) = f\left(\frac{x+y}{\sqrt{2}}, b \right) g\left(\frac{x-y}{\sqrt{2}} \right)$ is a minimizer for A. As in Section 3, we deduce from (4.2) that there are matrix valued function $\mathscr{A}$ and $\mathscr{B}$ such that

$$f(\hat{\xi}, b) = e^{-\xi^t \cdot \mathscr{A}(b)\xi - \mathscr{B}(b) \cdot \xi} h(b) \quad (\xi \in \mathbb{R}^n, b \in B) \qquad (4.4)$$

But then (4.3) implies that the functions $\mathscr{A}$ and $\mathscr{B}$ are constant. Hence f is a tensor product of a translation and a modulation of a Gaussian and a function h on B. By normalizing the Gaussian we may assume that $\|h\|_2 = 1$. Then it is easy to see that h is a minimizer for the group B, and therefore has the desired form, by the results of Section 2.

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