LIE THEORY II, SPRING 2020

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Contents

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1. **Introduction**

Classical Harmonic Analysis concerns a decomposition of a function (signal) into a superposition of components corresponding to *simple harmonics*. The analysis of the signal aims at finding these components and the synthesis is the reconstruction of the signal out of them. There is often some "signal processing", dictated by applications, between the analysis and synthesis.

The *simple harmonics* behave well under various symmetries and this is the reason for the decomposition. The fundamental results are Parseval's Theorem (1806) for the Fourier series, [Par06], and Plancherel's Theorem (1910) for the Fourier transform, [Pla10]. Among the best known applications is the Magnetic Resonance Imaging, for which Peter Mansfield and Paul Lauterbur were awarded a Nobel prize in 2003.

If the function is defined on a commutative group, such as the additive group of the real numbers or the multiplicative group of the complex numbers of absolute value one, then the *simple harmonics* are the eigenvectors under the translations. This is the ultimate symmetry one could expect.

Problems arising in Physics and Number Theory motivated a rapid growth of Harmonic Analysis on non-commutative groups. The earliest examples were the Heisenberg group, necessary for a formulation of the principles of Quantum Mechanics (J. Von Neumann 1926, [vN26]), and the compact Lie groups (Peter-Weyl 1927, [PW27]). Here the *simple harmonics* are replaced by *irreducible unitary representations.* All of them may be found by analyzing the square integrable functions on these groups, so that an analog of Plancherel's Theorem may be viewed as the top achievement of the theory.

However, there are plenty of other groups of interest which have irreducible unitary representations occurring outside the space of the square integrable functions on the group. The main class are the non-compact semisimple Lie groups, such as $\mathrm{SL}_2(\mathbb{R})$. Though the irreducible unitary representations of most of them are not understood yet, the representations that can be found in the space of the square integrable functions on the group are known and the decomposition of an arbitrary such function in terms of these representations is known as the *Plancherel formula*. For the group $\mathrm{SL}_n(\mathbb{C})$ this formula was first found by Gelfand and Naimark in 1950, , and for $\mathrm{SL}_n(\mathbb{R})$ by Gelfand and Graev in 1953, . The Plancherel formula on an arbitrary Real Reductive Group was published by Harish-Chandra in 1976, [Har76], and is considered as one of the greatest achievements of Mathematics of the 20th century.

A goal of these lectures is to explain the ingredients of Harish-Chandra's Plancherel formula, explain how they fit together, study particular cases and go through all the details for the group of the real unimodular matrices of size two.

All the necessary information in its original nearly perfect form is contained in Harish-Chadra's articles [HC14a], [HC14b], [HC14d], [HC14c], [HC18]. The example $\mathrm{SL}_2(\mathbb{R})$ is explained in classical books such as [Lan75]. For all of that, a good understanding of the Fourier Transform and Distribution theory on an Euclidean space is needed. Here Hörmender's "The Analysis of Linear Partial Differential Operators I" is one of the best references, [Hör83].

2. The Fourier Transform on the Schwartz space $\mathcal{S}(\mathbb{R})$

Here we follow [Hör83, section 7.1]. Recall that the Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all infinitely many times differentiable functions $f : \mathbb{R} \to \mathbb{C}$ such that for any two integers $n, k \geq 0$

$$\sup_{x\in\mathbb{R}} |x^n \partial_x^k f(x)| < \infty\,.$$

In particular $\mathcal{S}(\mathbb{R}) \subseteq \mathrm{L}^1(\mathbb{R})$ and we have the well defined

Theorem 1. *The Fourier transform*

$$\hat{f}(y) = \mathcal{F}f(y) = \int_{\mathbb{R}} e^{-2\pi ixy} f(x)\,dx \qquad (y \in \mathbb{R},\ f \in \mathcal{S}(\mathbb{R})) \tag{1}$$

maps the Schwartz space $\mathcal{S}(\mathbb{R})$ into itself, is invertible, and the inverse is given by

$$f(x) = \int_{\mathbb{R}} e^{2\pi ixy} \mathcal{F}f(y)\,dy \qquad (x \in \mathbb{R},\ f \in \mathcal{S}(\mathbb{R}))\,. \tag{2}$$

$$f(0) = \int_{\mathbb{R}} \mathcal{F}f(y)\,dy \qquad (f \in \mathcal{S}(\mathbb{R}))\,.$$

Since

$$\frac{d}{dy}\mathcal{F}f(y) = \int_{\mathbb{R}} e^{-2\pi ixy}(-2\pi ix) f(x)\,dx \tag{3}$$

and

$$\int_{\mathbb{R}} e^{-2\pi ixy} f'(x)\,dx = 2\pi iy\mathcal{F}f(y)\,, \tag{4}$$

the inclusion $\mathcal{F}\mathcal{S}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ is easy to check. For the rest we need a few lemmas.

Lemma 2. *Let $f \in \mathcal{S}(\mathbb{R})$ be such that $f(0) = 0$. Set $g(x) = \int_0^1 f'(tx)\,dt$. Then $f(x) = xg(x)$ and $g \in \mathcal{S}(\mathbb{R})$.*

Proof. The equality $f(x) = xg(x)$ is immediate from the Fundamental Theorem of Calculus, via a change of variables $y = tx$.

Fix two non-negative integers n and k. Suppose $|x| \leq 1$. Then

$$\left|x^n g^{(k)}(x)\right| = \left|x^n \int_0^1 t^k f^{(k+1)}(tx)\,dt\right| \leq \int_0^1 |f^{(k+1)}(tx)|\,dt \leq \max_{y\in\mathbb{R}} |f(y)| < \infty.$$

Notice that

$$\begin{aligned} x^n g^{(k)}(x) &= x^n \left(\frac{d}{dx}\right)^k (x^{-1} f(x)) \\ &= x^n \sum_{p=0}^{k} \frac{k!}{p!(k-p)!} (-1)(-2)\dots(-p) x^{-p-1} f^{(k-p)}(x) \end{aligned}$$

and that

$$\max_{|x|\geq 1} \left| x^{n-p-1} f^{(k-p)}(x) \right| < \infty.$$

Hence

$$\max_{x\in\mathbb{R}} |x^n g^{(k)}(x)| < \infty.$$

□

Corollary 3. *Fix $y \in \mathbb{R}$. Let $\phi \in \mathcal{S}(\mathbb{R})$ be such that $\phi(y) = 0$. Then that there is $\psi \in \mathcal{S}(\mathbb{R})$ such that*

$$\phi(x) = (x-y)\psi(x).$$

Lemma 4. *Let $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be a linear map with the property that if $\phi(y) = 0$ for some $y \in \mathbb{R}$ then $T\phi(y) = 0$ for the same y. Then there is a function $c(x)$ such that*

$$T\phi(x) = c(x)\phi(x) \qquad (\phi \in \mathcal{S}(\mathbb{R}).$$

(In other words, T is the multiplication by the function c.)

Proof. Let $\phi_1(x) = e^{-x^2}$. As we know this function belongs to $\mathcal{S}(\mathbb{R})$. Fix $x \in \mathbb{R}$. Then for any $\phi_2 \in \mathcal{S}(\mathbb{R})$

$$(\phi_2(x)\phi_1 - \phi_1(x)\phi_2)(x) = \phi_2(x)\phi_1(x) - \phi_1(x)\phi_2(x) = 0.$$

Hence, by the assumption on T,

$$0 = T(\phi_2(x)\phi_1 - \phi_1(x)\phi_2)(x). \tag{5}$$

Since T is linear

$$T(\phi_2(x)\phi_1) = \phi_2(x)T(\phi_1) \ \text{ and } \ T(\phi_1(x)\phi_2) = \phi_1(x)T(\phi_2).$$

Thus evaluation at x and using (5) we see that

$$0 = \phi_2(x)T(\phi_1)(x) - \phi_1(x)T(\phi_2)(x).$$

Therefore

$$T(\phi_2)(x) = \frac{T(\phi_1)(x)}{\phi_1(x)}\phi_2(x).$$

Thus the claim holds with

$$c(x) = \frac{T(\phi_1)(x)}{\phi_1(x)}.$$

□

Lemma 5. *Suppose $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is a linear map which commutes with the multiplication by x:*

$$T(xf(x)) = xT(f)(x) \qquad (f \in \mathcal{S}(\mathbb{R})).$$

Then there is a function $c(x)$ such that

$$T\phi(x) = c(x)\phi(x) \qquad (\phi \in \mathcal{S}(\mathbb{R})).$$

Proof. It'll suffice to show that T satisfies the assumptions of Lemma 4. Fix $y \in \mathbb{R}$. Let $\phi \in \mathcal{S}(\mathbb{R})$ be such that $\phi(y) = 0$. Then, by Corollary 3 there is $\psi \in \mathcal{S}(\mathbb{R})$ such that

$$\phi(x) = (x - y)\psi(x).$$

Hence

$$T(\phi)(x) = T((x - y)\psi(x)) = (x - y)T(\phi)(x),$$

which is zero if $x = y$. □

Proposition 6. *Suppose $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is a linear map which commutes with the multiplication by x:*

$$T(xf(x)) = xT(f)(x) \qquad (f \in \mathcal{S}(\mathbb{R}))$$

and with the derivative

$$T(f') = T(f)' \qquad (f \in \mathcal{S}(\mathbb{R})).$$

Then there is a constant c such that

$$T\phi(x) = c\phi(x) \qquad (\phi \in \mathcal{S}(\mathbb{R})).$$

Proof. We know from Lemma 5. that T coincides with the multiplication by a function $c(x)$. Since T commutes with the derivative we see that for any $f \in \mathcal{S}(\mathbb{R})$

$$c(x)f'(x) = (c(x)f(x))'.$$

Since the multiplication by $c(x)$ prserves the Schwartz space, $c(x)$ is differentiable and

$$(c(x)f(x))' = c'(x)f(x) + c(x)f'(x)\,.$$

Hence $c'(x) = 0$. therefore $c(x)$ is a constant. □

Lemma 7. *Let $Rf(x) = f(-x)$. The map $T = R\mathcal{F}^2 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ commutes with the multiplication by x and with the derivative.*

Proof. Since

$$R(xf(x)) = -xf(-x) = -xR(f)(x) \;\text{ and }\; R(f') = -R(f)\,,$$

it'll suffice to check that

$$\mathcal{F}^2(xf(x)) = -x\mathcal{F}^2(f)(x) \;\text{ and }\; \mathcal{F}^2(f') = -\left(\mathcal{F}^2(f)\right)'\,,$$

which follows from the relations (3) and (4). □

Lemma 8. *Let $f(x) = e^{-\pi x^2}$. Then $\mathcal{F}f = f$. (Fourier transform of the normalized Gaussian is the same Gaussian.)*

Proof. Since

$$\frac{d}{dx}f(x) = -2\pi x f(x)$$

the formulas (3) and (4) show that

$$\frac{d}{dx}\left(\mathcal{F}f(x)\cdot f(x)\right) = 0\,.$$

Hence

$$\mathcal{F}f(x) = const f(x)\,.$$

Evaluating at $x = 0$ gives

$$\int_{\mathbb{R}} f(y)\,dy = const\,.$$

By squaring the integral and using polar coordinates we show that $const = 1$. □

Now we are ready to prove the inversion formula in Theorem 1. We see from that the map $R\mathcal{F}^2 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is the identity,

$$R\mathcal{F}^2 f = f \qquad (f \in \mathcal{S}(\mathbb{R})).$$

We know from Lemma 7 and Proposition 6 that the map $R\mathcal{F}^2$ is a constant multiple of the identity: $cI = R\mathcal{F}^2$. Now Lemma 8 shows that with $f(x) = e^{-\pi x^2}$

$$cf = R\mathcal{F}^2 f = Rf = f.$$

Thus $c = 1$. Hence

$$R\mathcal{F} = \mathcal{F}^{-1}$$

and the formula2 follows. This completes the proof of Theorem 1.

Problem 1. *Let $P(\mathbb{R})$ denote the space of complex valued polynomials $f(x)$, on the real line. Suppose $T : P(\mathbb{R}) \to P(\mathbb{R})$ is a linear map that commutes with multiplication by x and with the derivative. Is it true that T is a constant multiple of the identity? If yes, prove it. If no, give a counterexample.*

The answer is YES and the proof goes along the same lines we followed for the case of $\mathcal{S}(\mathbb{R})$. Another approach is to define $c(x) = T(1)$ and from the commutation with multiplication by x deduce that

$$T(x^k) = c(x)x^k \qquad (k = 0, 1, 2, \dots.$$

Hence

$$T(f(x)) = c(x)f(x) \qquad (f(x) \in P(\mathbb{R}))\,.$$

Since T commutes with the derivative, we conclude that $c'(x) = 0$. Therefore $c(x)$ is a constant.

Problem 2. *Show that*

$$\mathcal{F}^4 = I\,.$$

Since

$$\mathcal{F}^2 = R^{-1} = R\,.$$

we see that

$$\mathcal{F}^4 = R^2 = I\,.$$

Problem 3. *Let* $\gamma(x) = e^{-\pi x^2}$ *and let* $A = 2\pi x - \partial_x$. *We already know that* $\mathcal{F}\gamma = \gamma$. *Check that*

$$\mathcal{F}(A^k\gamma) = e^{-\frac{\pi}{2}ik}A^k\gamma \qquad (k = 0, 1, 2, 3)\,.$$

Thus we found an eigenvector for each of the four possible eigenvalues.

This may be done by an explicit computation using the formulas (3) *and* (4). *For a systematic treatment see* [HT92, section 2.1].

Integration by parts shows that for any $f \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}(Af)(y) = -iA\mathcal{F}(f)(y)\,.$$

By taking $f = A^k\gamma$ *we obtain the following recurrence equation*

$$\mathcal{F}(A^{k+1}\gamma)(y) = -iA\mathcal{F}(A^k\gamma) \qquad (k = 0, 1, ...)\,.$$

Since $\mathcal{F}\gamma = \gamma$, *we see that*

$$\mathcal{F}(A^k\gamma)(y)) = (-i)^k A^k\gamma(y) \qquad (k = 0, 1, ...)\,.$$

3. Magnetic Resonance Imaging

Suppose a source at $s \in \mathbb{R}$ is emitting a signal with frequency $ks \in \mathbb{R}$ and amplitude $A(s)$. The collective signal received is

$$B(x) = \int_{\mathbb{R}} A(s)e^{2\pi ixks}\,ds\,.$$

By Fourier inversion,

$$A(s) = k\int_{\mathbb{R}} B(x)e^{-2\pi ixks}\,dx\,.$$

Hence we can recover $A(s)$ from $B(x)$. In particular, if we the function $A(s)$ is linear (in some large interval contained in $[0, \infty)$) and if we know $A(s)$ then we know s, i.e. the location of the source. For the related physics see
youtube.com/watch?v = pGcZvSG805Y *youtube.com/watch?v = djAxjtN_7VE.*

4. The Fourier Transform on the space $C^\infty(\mathrm{U}_1)$

Here $\mathrm{U}_1 = \{u \in \mathbb{C};\ |u| = 1\}$ is the group of the unitary matrices of size 1. We shall use the following identification of groups

$$\mathbb{R}/2\pi\mathbb{Z} \ni \theta + 2\pi\mathbb{Z} \to e^{i\theta} \in \mathrm{U}_1 .$$

Here we follow [SS03, section 2.2]. Recall that a function $f : \mathbb{Z} \to \mathbb{C}$ is called rapidly decreasing if

$$\sup_{n\in\mathbb{Z}} |(1+|n|)^k |f(n)| < \infty \qquad (n \in \mathbb{Z}) .$$

Theorem 9. *The Fourier transform*

$$\hat{f}(n) = \mathcal{F}f(n) = \int_0^1 e^{-2\pi i x n} f(x)\, dx \qquad (n \in \mathbb{Z},\ f \in C^\infty(\mathrm{U}_1)) \tag{6}$$

maps the space $C^\infty(\mathrm{U}_1)$ into the space of the rapidly decreasing functions on $\mathbb{Z}$, is invertible, and the inverse given by

$$f(x) = \sum_{n\in\mathbb{Z}} e^{2\pi i x n} \mathcal{F}f(n) \qquad (x \in \mathbb{R},\ f \in C^\infty(\mathrm{U}_1)) . \tag{7}$$

$$f(0) = \sum_{n\in\mathbb{Z}} \mathcal{F}f(n) .$$

Notice that the integral (6) converges as long as the function f is absolutely integrable. Hence we have the Fourier transform $\mathcal{F}f$ for any $f \in \mathrm{L}^1(\mathrm{U}_1)$. Since

$$\int_0^1 e^{-2\pi i x n} f'(x)\, dx = 2\pi i n \mathcal{F}f(y) , \tag{8}$$

the rapid decrease of $\mathcal{F}f$ is easy to check. The inversion formula is immediate from the following Lemma.

Lemma 10. *If $f : \mathrm{U}_1 \to \mathbb{C}$ is continuous and $\mathcal{F}f = 0$, then $f = 0$.*

Proof. Since

$$\int_0^1 e^{-2\pi i x n} f(x+y)\, dx = e^{2\pi i y n} \int_0^1 e^{-2\pi i x n} f(x)\, dx$$

it'll suffice to show that if $\mathcal{F}f = 0$, then $f(0) = 0$.

Suppose the claim is false. We may assume that f is real valued and that $f(0) > 0$. We shall arrive at a contradiction. Choose $0 < \delta \leq \frac{1}{4}$ so that

$$f(x) > \frac{f(0)}{2} \qquad (|x| < \delta) .$$

Let $\epsilon > 0$ be so small that the function

$$p(x) = \epsilon + \cos(2\pi x)$$

satisfies

$$|p(x)| < 1 - \frac{\epsilon}{2} \qquad (\delta \leq |x| \leq \frac{1}{2}) .$$

Choose $0 < \eta < \delta$ so that

$$|p(x)| \geq 1 + \frac{\epsilon}{2} \qquad (|x| \leq \eta)\,.$$

Then for $k = 0, 1, 2, ...,$

$$\left| \int_{\delta \leq |x| \leq \frac{1}{2}} f(x)p(x)^k\, dx \right| \leq 2\pi \parallel f \parallel_\infty \left(1 - \frac{\epsilon}{2}\right)^k ,$$

$$\int_{\eta \leq |x| < \delta} f(x)p(x)^k\, dx \geq 0\,,$$

$$\int_{|x| < \eta} f(x)p(x)^k\, dx \geq 2\eta \frac{f(0)}{2} \left(1 + \frac{\epsilon}{2}\right)^k .$$

Therefore

$$\lim_{k \to \infty} \int_{|x| \leq \frac{1}{2}} f(x)p(x)^k\, dx = +\infty\,.$$

However $p(x)^k$ is a trigonometric polynomial (linear combination of powers of exponentials). Hence the assumption $\mathcal{F}f = 0$ implies

$$\int_{|x| \leq \frac{1}{2}} f(x)p(x)^k\, dx = 0\,.$$

This is a contradiction we have been looking for. □

Set

$$e_n(x) = e^{2\pi i n x} \qquad (n \in \mathbb{Z}, x \in \mathbb{R})\,.$$

Problem 4. *Prove the inclusion*

$$\mathrm{L}^2(\mathbb{R}/\mathbb{Z}) \subseteq \mathrm{L}^1(\mathbb{R}/\mathbb{Z})\,.$$

This follows from Cauchy's inequality. Let $f \in \mathrm{L}^2(\mathbb{R}/\mathbb{Z})$. Then

$$\int_0^1 |f(x)|\, dx = \int_0^1 1 \cdot |f(x)|\, dx \leq \sqrt{\int_0^1 1^2 \cdot\, dx} \sqrt{\int_0^1 |f(x)|^2 \cdot\, dx}$$

$$= \sqrt{\int_0^1 |f(x)|^2 \cdot\, dx} < \infty\,.$$

For $f \in \mathrm{L}^2(\mathbb{R}/\mathbb{Z})$ define the partial Fourier sums

$$S_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e_n \qquad (N = 0, 1, 2, ...)\,.$$

The crown jewel of the L^2 theory for Fourier series is the following theorem.

Theorem 11. *For any* $f \in \mathrm{L}^2(\mathbb{R}/\mathbb{Z})$

$$\lim_{N\to\infty} \parallel f - S_N(f) \parallel_2 = 0 \tag{9}$$

and

$$\parallel f \parallel_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \,. \tag{10}$$

Problem 5. *Find a prove of Theorem 11 and copy it by hand.*

My favorite is [SS03, section 1.2], *but there are plenty of other references.*

Problem 6. *Compute* $\hat{f}(n)$ *for* $f(x) = x$, *restricted to the interval* $[-\frac{1}{2}, \frac{1}{2})$ *(and repeated periodically with period* 1*).*

Integration by parts shows that

$$\hat{f}(n) = \begin{cases} \frac{i(-1)^n}{2\pi n} & \text{if} \quad n \neq 0\,, \\ 0 & \text{if} \quad n = 0\,. \end{cases}$$

Problem 7. *Compute*

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

From Problem 6 and Theorem 11,

$$\left(\frac{1}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n\in\mathbb{Z}\setminus\{0\}} |\hat{f}(n)|^2 = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\, dx = \frac{1}{24}\,.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

5. Symmetries of platonic solids

The group S_4 acts on $\mathbb{R}^4$ by permuting the coordinates and multiplication by the sign of the permutation:

$$\begin{aligned} &\mathbb{R}^4 \ni x = (x_1, x_2, x_3, x_4) \to \sigma\, x = ((\sigma\, x)_1, (\sigma\, x)_2, (\sigma\, x)_3, (\sigma\, x)_4) \in \mathbb{R}^4, \\ &(\sigma\, x)_j = sgn(\sigma)\, x_{\sigma^{-1}(j)} \qquad (\sigma \in S_4,\ 1 \leq j \leq 4). \end{aligned}$$

Recall the scalar product

$$(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$

It is easy to check that

$$(\sigma(x), \sigma(y)) = x \cdot y \qquad (x, y \in \mathbb{R}^4).$$

Let

$$V = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1 + x_2 + x_3 + x_4 = 0\}.$$

The action of S_4 on $\mathbb{R}^4$ preserves the subspace V. The vectors

$$e_1 = (-1, 1, 1, -1),\ e_2 = (1, -1, 1, -1),\ e_3 = (1, 1, -1, -1)$$

form an orthogonal basis of V. Let C be the cube whose faces are centered at point $\pm e_j$, $1 \le j \le 3$.

Here is a description of how each of the 23 nontrivial elements of S_4 acts on the cube C.

Rotations by 180 degrees about the axis going though midpoints of opposite edges:

$$\begin{aligned}
(12):&\ e_1 \to -e_2,\ e_2 \to -e_1,\ e_3 \to -e_3,\\
(13):&\ e_1 \to -e_3,\ e_2 \to -e_2,\ e_3 \to -e_1,\\
(14):&\ e_1 \to -e_1,\ e_2 \to e_3,\ e_3 \to e_2,\\
(23):&\ e_1 \to -e_1,\ e_2 \to -e_3,\ e_3 \to -e_2,\\
(24):&\ e_1 \to e_3,\ e_2 \to -e_2,\ e_3 \to e_1,\\
(34):&\ e_1 \to e_2,\ e_2 \to e_1,\ e_3 \to -e_3.
\end{aligned}$$

Rotations by 120 degrees about the axis going through opposite verticies:

$$\begin{aligned}
(123):&\ e_1 \to e_2,\ e_2 \to e_3,\ e_3 \to e_1,\\
(132):&\ e_1 \to e_3,\ e_2 \to e_1,\ e_3 \to e_2,\\
(134):&\ e_1 \to -e_2,\ e_2 \to -e_3,\ e_3 \to e_1,\\
(143):&\ e_1 \to e_3,\ e_2 \to -e_1,\ e_3 \to -e_2,\\
(243):&\ e_1 \to -e_2,\ e_2 \to e_3,\ e_3 \to -e_1,\\
(234):&\ e_1 \to -e_3,\ e_2 \to -e_1,\ e_3 \to e_2\\
(124):&\ e_1 \to -e_3,\ e_2 \to e_1,\ e_3 \to -e_2,\\
(142):&\ e_1 \to e_2,\ e_2 \to -e_3,\ e_3 \to -e_1.
\end{aligned}$$

Rotations by 180 degrees about the axis going through the centers of the opposite faces:

$$\begin{aligned}
(12)(34):&\ e_1 \to -e_1,\ e_2 \to -e_2,\ e_3 \to e_3,\\
(13)(24):&\ e_1 \to -e_1,\ e_2 \to e_2,\ e_3 \to -e_3,\\
(14)(23):&\ e_1 \to e_1,\ e_2 \to -e_2,\ e_3 \to -e_3.
\end{aligned}$$

Rotations by 90 degrees about the axis going through the centers of the opposite faces:

$$\begin{aligned}
(1234) &:\ e_1 \to e_3,\ e_2 \to e_2,\ e_3 \to -e_1,\\
(1324) &:\ e_1 \to e_3,\ e_2 \to -e_1,\ e_3 \to e_3,\\
(1432) &:\ e_1 \to -e_3,\ e_2 \to e_2,\ e_3 \to e_1\\
(1243) &:\ e_1 \to e_1,\ e_2 \to -e_3,\ e_3 \to e_2,\\
(1342) &:\ e_1 \to e_2,\ e_2 \to e_3,\ e_3 \to -e_2,\\
(1423) &:\ e_1 \to -e_2,\ e_2 \to e_1,\ e_3 \to e_3.
\end{aligned}$$

Let T be a regular tetrahedron whose edges are centered at $\pm e_j$, $1 \le j \le 3$. Then the subgroup of S_4 that preserves T is $Alt(4)$, the subgroup of the even permutations.

Let $Alt(n) \subseteq S_n$ denote the subgroup consisting of all the even permutations. Then $Alt(5)$ is generated by $Alt(4)$, identified with the subgroup of S_5 fixing the number 5, and the cycle $s = (12345)$.

Indeed, Notice that s is an even permutation, i.e. $s \in Alt(5)$. We need to show that for every $g \in Alt(5)$ there is an element $h \in Alt(4)$ and an integer k such that

$$g = hs^k. \tag{11}$$

There is a number $x \in \{1, 2, 3, 4, 5\}$ such that $g(x) = 5$. Since s^{-1} is the full cycle, there is k such that s^{-k} maps 5 to x. Hence the composition $h = gs^{-k}$ maps 5 to 5. Thus h fixes 5 and is an even permutation. In other words, $h \in Alt(4)$. Hence (11) follows.

Recall the golden mean $\mu = \frac{1+\sqrt{5}}{2}$ and the conjugate $\overline{\mu} = \frac{1-\sqrt{5}}{2}$. In the three-dimensional space $V = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$, let D be the regular dodecahedron whose faces are centered on the vectors

$$-\mu e_2 + e_3, \quad e_1 + \mu, \quad e_3 - e_1 + \mu e_3, \quad -\mu e_1 - e_2, \quad -\mu e_2 - e_3, \quad \mu e_1 - e_2 \tag{12}$$

and on the antipodal vectors. The action of $Alt(4)$ preserves the dodecahedron (i.e. it preserves the set of vectors (12) multiplied by ± 1).

Indeed, let

$$\begin{aligned}
a = (12)(34),\ ,b = (14)(23),\ c = (13)(24),\\
x = (123),\ y = (142),\ z = (243),\ w = (134).
\end{aligned}$$

The group $Alt(4)$ consists of

$$1,\ a,\ b,\ c,\ x,\ x^2,\ y,\ y^2,\ z,\ z^2,\ w,\ w^2.$$

Since,

$$xax^{-1} = b,\ x^2ax^{-2} = c,\ axa = y,\ bxb = z,\ cxc = w,$$

we see that $Alt(4)$ is generated by a and x. Hence, it'll suffice to check that a and x preserve the dodecahedron.

Let

$$F_1 = -\mu e_2 + e_3, \quad F_2 = e_1 + \mu e_3, \quad F_3 = -e_1 + \mu e_3,$$
$$F_4 = -\mu e_1 - e_2, \quad F_5 = -\mu e_2 - e_3, \quad F_6 = \mu e_1 - e_2$$

These are the centers of the 6 faces of the dodecahedron. The centers of the antipodal faces are

$$-F_1,\ -F_2,\ -F_3,\ -F_4,\ -F_5,\ -F_6.$$

We see from the explicit action of the group that

$$a : e_1 \to -e_1,\ e_2 \to -e_2,\ e_3 \to e_3$$

and

$$x : e_1 \to e_2,\ e_2 \to e_3,\ e_3 \to e_1.$$

Hence,

$$a : F_1 \to -F_5,\ F_2 \to F_3,\ F_3 \to F_2,\ F_4 \to -F_4,\ F_5 \to -F_1,\ F_6 \to -F_6,$$

and

$$x : F_1 \to -F_3,\ F_2 \to -F_4,\ F_3 \to F_6,\ F_4 \to F_5,\ F_5 \to -F_2,\ F_6 \to -F_1.$$

By computing the angles between the vectors we see that the face F_1 shares one edge with each of the faces F_2, F_3, F_4, F_5, F_6. Furthermore, F_2 shares one edge with F_3, F_3 shares one edge with F_4, F_4 shares one edge with F_5, F_5 shares one edge with F_6 and F_6 shares one edge with F_2. This determines the model of the dodecahedron and we check that both a and x preserve it.

We extend the action of $Alt(4)$ to an action of $Alt(5)$ on V by letting s by

$$s : -\mu e_2 + e_3 \to -\mu e_2 + e_3$$
$$s : e_1 + \mu e_3 \to -e_1 + \mu e_3 \to -\mu e_1 - e_2 \to -\mu e_2 - e_3 \to \mu e_1 - e_2 \to e_1 + \mu e_3.$$

Then the action of s is the rotation by $2\pi/5$ about the axis passing through midpoints $-\mu e_2 + e_3$ and $\mu e_2 - e_3$ of the two opposite faces. (This way $Alt(5)$ is identified with the group of the rotations of a regular dodecahedron.)

Indeed, in terms of the notation used above, the action of s looks as follows:

$$s : F_1 \to F_1,$$
$$s : F_2 \to F_3 \to F_4 \to F_5 \to F_6 \to F_2.$$

We may construct a model of the dodecahedron using, for example a suitable link on the web. Furthermore, since we may compute all the angles between the vectors we know where to place the faces $F_1, F_2, F_3, F_4, F_5, F_6, F_2$. After all that it is clear that s has the desired property.

The matrix of s with respect to the basis $\{e_1, e_2, e_3\}$ is

$$\frac{1}{2}\left(\begin{array}{|c|c|c|} -\overline{\mu} & -1 & -\mu \\ 1 & \mu & \overline{\mu} \\ \mu & \overline{\mu} & 1 \end{array}\right).$$

Indeed, We see from the definition of s that

$$\begin{aligned} s : e_1 &= \frac{1}{2}(F_2 - F_3) \to \frac{1}{2}(F_3 - F_4) = \frac{1}{2}((-1+\mu)e_1 + e_2 + \mu e_3) \\ s : e_2 &= -\frac{1}{2}(F_4 + F_6) \to -\frac{1}{2}(F_5 + F_2) = \frac{1}{2}(-e_1 + \mu e_2 - (\mu - 1)e_3) \\ s : e_3 &= \frac{1}{2}(F_1 - F_5) \to \frac{1}{2}(F_1 - F_6) = \frac{1}{2}(-\mu e_1 + (1-\mu)e_2 + e_3). \end{aligned}$$

Since, $1 - \mu = \overline{\mu}$ the formula for the matrix follows.

6. Basic representation theory of finite groups

A good reference for this section is [FH91]. A representation of a finite group G on a finite dimensional complex vector space V is a group homomorphism

$$\rho : \mathrm{G} \to \mathrm{GL}(\mathsf{V}). \tag{13}$$

We shall denote it by (ρ, V) or just ρ. Also, when convenient we shall write gv instead of $\rho(g)v$. A morphism, or a G-morphism, or a G-intertwinig map, between two representations (ρ, V) and (ρ', V') is a linear map $T : \mathsf{V} \to \mathsf{V}'$ such that

$$T\rho(g)v = \rho'(g)Tv \qquad (g \in \mathrm{G}, v \in \mathsf{V}).$$

Two representations (ρ, V) and (ρ', V') are called isomorphic if there is an invertible morphism between them. In that case we shall write $(\rho, \mathsf{V}) \simeq (\rho', \mathsf{V}')$. If the representations are not isomorphic we shall write $(\rho, \mathsf{V}) \not\simeq (\rho', \mathsf{V}')$. The relation of an isomorphism of representations is an equivalence relation.

Let V^c denote the vector space dual to V. The contragredient representation (ρ^c, V^c) is defined by

$$\rho^c(g)v^c(v) = v^c(\rho(g^{-1})v) \qquad (v \in \mathsf{V},\ v^c \in \mathsf{V}^c, g \in \mathrm{G}).$$

Given two representations (ρ, V) and (ρ', V') define their direct sum $(\rho \oplus \rho', \mathsf{V} \oplus \mathsf{V}')$ by

$$(\rho \oplus \rho')(g)(v, v') = (\rho(g)(v), \rho'(g)(v')) \quad (g \in \mathrm{G}, v \in \mathsf{V}, v' \in \mathsf{V}')$$

and the tensor product $(\rho \otimes \rho', \mathsf{V} \otimes \mathsf{V}')$ by

$$(\rho \otimes \rho')(g)[v \otimes v'] = [\rho(g)(v)] \otimes [\rho'(g)(v')] \quad (g \in \mathrm{G}, v \in \mathsf{V}, v' \in \mathsf{V}').$$

Given a representation (ρ, V), one says that a subspace $\mathrm{U} \subseteq \mathsf{V}$ is invariant if $\rho(g)u \in \mathrm{U}$ for all $u \in \mathrm{U}$. A representation (ρ, V) is called irreducible if the only G-invariant subspaces of V are 0 and V.

Theorem 12. *(Schur's lemma) If (ρ, V) and (ρ', V') are two irreducible representations of* G *and $T : \mathsf{V} \to \mathsf{V}'$ is a morphism, then either T is an isomorphism or $T = 0$. If $(\rho, \mathsf{V}) = (\rho', \mathsf{V}')$ then T is the multiplication by a complex number.*

Proof. Notice thet $Ker(T) \subseteq \mathsf{V}$ is a G-invariant subspace. Hence, either $Ker(T) = 0$ or $Ker(T) = \mathsf{V}$. Similarly, the image of T, $Im(T) \subseteq \mathsf{V}'$ is a G-invariant subspace. Hence, either $Im(T) = 0$ or $Im(T) = \mathsf{V}$. Thus the first statement follows.

The second statement follows from the fact tht $\mathbb{C}$ is algebraically closed. The point is that T has an eigenvalue $\lambda \in \mathbb{C}$ and the corresponding (non-zero) eigen-space in V. By irreducibility this eigenspace is equal to V. □

Let $\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}')$ denote the space of all the morphisms fro V to V'. Theorem 12 implies the following Corollary

Corollary 13. *For two irreducible representations (ρ, V) and (ρ', V') of* G*,*

$$dim\ \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}') = \begin{cases} 1 \ \textit{if}\ (\rho, \mathsf{V}) \simeq (\rho', \mathsf{V}') \\ 0 \ \textit{if}\ (\rho, \mathsf{V}) \not\simeq (\rho', \mathsf{V}') \end{cases}$$

Lemma 14. *For any finite dimensional representation (ρ, V) there is a* G*-invariant positive definite hermitian form $(\ ,\ )$ on V.*

Proof. Let $\langle\ ,\ \rangle$ be any positive definite hermitian form on V. Define

$$(u, v) = \sum_{g \in \mathrm{G}} \langle \rho(g)u, \rho(g)v \rangle \qquad (u, v \in \mathsf{V}).$$

It is easy to check that this form has the required properties. □

If V is equipped with an invariant positive definite hermitian form then (ρ, V) is called a unitary representation of G.

Theorem 15. *Any finite dimensional representation of a finite group decomposes into the direct sum of irreducible representations.*

Proof. This follows from Lemma 14 and the fact that the orthogonal complement of a G-invariant subspace is G-invariant. □

Two unitary representations (ρ, V), (ρ', V') are called unitarily equivalent iff there is a morphism $T : \mathsf{V} \to \mathsf{V}'$ which is also an isometry.

Proposition 16. *If two unitary representations (ρ, V), (ρ', V') are equivalent then they are unitarily equivalent.*

Problem 8. *Proof Proposition 40.*

Proof. Let $(\ ,\ )$ be the invariant scallar product on V and let $(\ ,\ )'$ be the invariant scallar product on V'. Pick an isomorphism $T : \mathsf{V} \to \mathsf{V}'$. Define $T^* : \mathsf{V}' \to \mathsf{V}$ by

$$(Tu, v)' = (u, T^*v) \qquad (u \in \mathsf{V}, v \in \mathsf{V}).$$

Then $T^*\mathsf{V}' \to \mathsf{V}$ is also a morphism. Hence, $T^*T : \mathsf{V} \to \mathsf{V}$ commutes with the action of G. Hence, there is $\lambda \in \mathbb{C}$ such that $T^*T = \lambda I$. Thus, for any $u, v \in \mathsf{V}$,

$$(Tu, Tv)' = (u, T^*Tv) = \overline{\lambda}(u, v).$$

In particular, by taking $u = v \neq 0$ we see that $\lambda > 0$. Hence,

$$\frac{1}{\sqrt{\lambda}} T : \mathsf{V} \to \mathsf{V}'$$

is an isometry and a morphism. $\square$

It is easy to see from the above theorem that any irreducible representation of a finite abelian group is one-dimensional. Indeed, let G be such a group and let (ρ, V) be an irreducible unitary representation of G. For a fixed $g \in \mathrm{G}$, the map $\rho(g) : \mathsf{V} \to \mathsf{V}$ intertwines the action of G. Theorem 12 implies that there is $\mathrm{ch}(g) \in \mathbb{C}$ such that $\rho(g) = \chi(g) I_{\mathsf{V}}$. Since $\rho(g)$ is invertible, we see that $\chi(g) \neq 0$. Moreover the map $\mathrm{G} \ni g \to \chi(g) \in \mathbb{C}^\times$ is a group homomorphism, i.e. a character. Therefore every subspace $\mathrm{U} \subseteq \mathsf{V}$ is G-invariant. Since (ρ, V) is irreducible we see that $\dim \mathsf{V} = 1$.

Any finite abelian group is isomorphic to the direct product of cyclic groups and the characters of cyclic groups are easy to describe. Hence to make things interesting we need a non-abelian group. The simplest one is S_3 the group of all the permutations of 3 elements.

As our first example we shall describe all the irreducible representations of S_3. There are two obvious one-dimensional representations: the trivial representation $(triv, \mathbb{C})$ and the sign representation $(sgn, \mathbb{C})$:

$$triv(g)v = v,\ sgn(g)v = \text{the sign of the permutation } g \text{ times } v \qquad (g \in S_3, v \in \mathbb{C}).$$

There is also a two-dimensional irreducible representation (ρ, V) constructed as follows. Define a representation $(\pi, \mathbb{C}^3)$ by

$$\pi(g)(z_1, z_2, z_3) = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)}) \qquad (g \in S_3, (z_1, z_2, z_3) \in \mathbb{C}^3).$$

This is a unitary representation with respect to the scalar product

$$((z_1, z_2, z_3), (z_1', z_2', z_3')) = z_1\overline{z_1'} + z_2\overline{z_2'} + z_3\overline{z_3'}.$$

The orthogonal complement $\mathrm{U} \subseteq \mathbb{C}^3$ of $\mathbb{C}(1, 1, 1)$ is S_3-invariant and two dimensional. Let ρ denote the restriction of π to U. Then (ρ, U) is a representation of S_3.

Let $c = (123) \in S_3$, $t = (12) \in S_3$, $z = e^{2\pi i/3} \in \mathbb{C}$, $u = (z, 1, z^2) \in \mathbb{C}^3$ and let $v = (1, z, z^2) \in \mathbb{C}^3$. Then the vectors u and v form a basis of the vector space U and

$$\rho(c)u = zu,\ \rho(c)v = z^2v,\ \rho(t)u = v,\ \rho(t)v = u. \tag{14}$$

In particular we see thet U does not have any S_3-invariant one-dimensional subspace. Thus the representation (ρ, U) is irreducible. We claim that up to equivalence there are no other irreducible representations of S_3.

Indeed, consider a finite dimensional irreducible representation (ρ, V) of S_3. Suppose $dim\,\mathsf{V} = 1$. Then $\rho : S_3 \to \mathrm{GL}(\mathsf{V}) = \mathbb{C}^\times$ is a group homomorphism. Hence, the kernel $Ker(\rho) \subseteq S_3$ is a normal subgroup. Since the group $\mathbb{C}^\times$ is abelian, $Ker(\rho) \neq \{1\}$. If $Ker(\rho) = S_3$, then ρ is the trivial representation. If $Ker(\rho) = \{1, c, c^2\}$, the subgroup of cycles of length 3, then ρ is the sign representation. Since there are no other normal subgroups in S_3, we see that there are no other one-dimensional representations of S_3.

Suppose $dim\,\mathsf{V} \geq 2$. Then the subgroup $\{1, c, c^2\} \subseteq S_3$, of index 2, cannot act trivially on V, because the quotient $S_3/\{1, c, c^2\}$ is abelian. Hence there is a vector $u \in \mathsf{V}$ such that $\rho(c)u = \lambda u$, where $\lambda = e^{2\pi i/3}$ or $e^{-2\pi i/3}$. Notice that $ct = tc^2$. Hence

$$\rho(c)\rho(t)u = tc^2 u = \rho(t)(\lambda^2 u) = \lambda^2 \rho(t)u.$$

Therefore we may assume that $\lambda = e^{2\pi i/3} = z$ in our previous notation (14). Let $v = \rho(t)u$. Then, from the above computation, $tv = \lambda^2 v$. We see from the above and from (14) that the vectors u and v span a two dimensional subspace isomorphic to the representation on the space U constructed before. Since V is assumed irreducible, that subspace coincides with V. Thus (ρ, V) is isomorphic to (ρ, U).

Problem 9. *Read the newly added section 5 and correct errors if you find any.*

There was $\overline{\mu}$ in the center of the matrix of s. In the first line of "Rotations by 180 degrees about the axis going though midpoints of opposite edges", in the first line we should have $e_3 \to -e_3$.

Now we come back to general representation theory of a finite group G. By definition the character $\chi_{\mathsf{V}} = \chi_\rho$ of a representation (ρ, V) is the following complex valued function on the group:

$$\chi_{\mathsf{V}}(g) = tr(\rho(g)) \qquad (g \in \mathrm{G}).$$

This function is invariant under conjugation

$$\chi(hgh^{-1}) = \chi(g) \qquad (h, g \in \mathrm{G}).$$

Also, we have

$$\chi_{\mathsf{V}\oplus\mathsf{V}'} = \chi_{\mathsf{V}} + \chi_{\mathsf{V}'},\ \chi_{\mathsf{V}\otimes\mathsf{V}'} = \chi_{\mathsf{V}}\chi_{\mathsf{V}'} \text{ and } \chi_{\rho^c} = \overline{\chi_\rho}. \tag{15}$$

(The last formula follows from the fact that the eigenvalues of $\rho(g)$ are nth roots of unity, where $n = |\mathrm{G}|$, and therefore the inverse of such an eigenvalue is the same as its complex conjugate.)

Theorem 17. *Let X be a set and let V be the space of all the complex valued functions defined on X. Suppose* G *acts on X by permutations. Define a representation (ρ, V) by*

$$\rho(g)v(x) = v(g^{-1}x) \qquad (g \in \mathrm{G}, v \in \mathsf{V}, x \in X).$$

Then

$$\chi_\rho(g) = \text{the number of points in } X \text{ which are fixed by } g.$$

Proof. Let δ_y denote the Dirac delta at $y \in X$. The set of all these Dirac deltas is a basis of the vector space V. Notice that

$$\rho(g)\delta_y = \delta_{gy} \qquad (g \in \mathrm{G}, y \in X).$$

Let us order the set X as $X = \{y_1, y_2, ..., y_n\}$. The matrix $[A(g)_{j,k}]$ of $\rho(g)$ with respect to the order basis $\{\delta_{y_1}, \delta_{y_2}, ..., \delta_{y_n}\}$ are computed from the formula

$$\rho(g)\delta_{y_k} = \sum_{j=1}^{n} A(g)_{j,k}\delta_{y_j}.$$

Explicitly

$$\delta_{gy_k} = \sum_{j=1}^{n} A(g)_{j,k}\delta_{y_j}.$$

Thus $A(g)_{j,k}$ is either 0 or 1, and later happens if and only if $\delta_{gy_k} = \delta_{y_j}$. In particular $A(g)_{j,j} = 1$ if and only if $\delta_{gy_j} = \delta_{y_j}$. Hence the formula for the character follows. $\square$

As an application we compute the characters of the irreducible representations of S_3 in the following table. The characters for the trivial and the sign representation are obvious. We see from the definition of the two-dimensional irreducible representation and from (24) that its character is equal to the difference between the character $\chi_{\mathbb{C}^3}$ of the permutation representation of S_3 on $\mathbb{C}^3$ and the trivial representation. Theorem 17 that

$$\chi_{\mathbb{C}^3}(1) = 3,\ \chi_{\mathbb{C}^3}((12)) = 1,\ \chi_{\mathbb{C}^3}((123)) = 0.$$

Hence the last line of the table follows.

conjugacy class in S_3	[1]	[(12)]	[(123)]
number of elements in the conjugacy class	1	3	2
value of the character on the conjugacy class for			
trivial representation	1	1	1
sign representation	1	-1	1
the two-dimensional representation	2	0	-1

We introduce the following scalar (hermitian) product on the space of all the complex valued functions on G

$$(\phi, \psi) = \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \phi(g)\overline{\psi(g)}. \tag{16}$$

In other words we view the counting measure divided by the cardinality of the group as a fixed Haar measure on G and we consider the corresponding space $\mathrm{L}^2(\mathrm{G})$. Denote by $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} \subseteq \mathrm{L}^2(\mathrm{G})$ the subspace of the functions invariant by the conjugation by all the elements of G. Our characters live in $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}}$.

Lemma 18. *Suppose* (ρ, V) *is a non-trivial irreducible representation of* G*. Then*

$$\sum_{g\in\mathrm{G}} \rho(g)v = 0 \qquad (v \in \mathsf{V}).$$

Proof. Notice that

$$\overline{v} = \sum_{g\in\mathrm{G}} \rho(g)v$$

is a G-invariant vector in V. Since ρ is irreducible, either $\overline{v} = 0$ or $\mathsf{V} = \mathbb{C}\overline{v}$. Since ρ is non-trivial, the second option is impossible. □

Corollary 19. *Suppose* (ρ, V) *is a non-trivial irreducible representation of* G*. Then*

$$\sum_{g\in\mathrm{G}} \chi_\rho(g) = 0.$$

Corollary 20. *Suppose* (ρ, V) *is a representation of* G*. Then*

$$\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \chi_\rho(g) = dim\, \mathsf{V}^{\mathrm{G}},$$

where $\mathsf{V}^{\mathrm{G}} \subseteq \mathsf{V}$ *is the space of the* G*-invariant vectors.*

Lemma 21. *The characters of irreducible representations form an orthonormal set in* $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}}$.

Proof. Consider two such representations (ρ, V) and (ρ', V'). Let $\mathrm{Hom}(\mathsf{V}, \mathsf{V}')$ denote the space of all the linear maps from V to V'. The group G acts on this vector space by

$$gT(v) = \rho'(g)T\rho(g^{-1})v \qquad (g \in \mathrm{G}, T \in \mathrm{Hom}(\mathsf{V}, \mathsf{V}'), v \in \mathsf{V}).$$

This way $\mathrm{Hom}(\mathsf{V}, \mathsf{V}')$ becomes a representation of G. It is easy to check that as such it is isomorphic to $(\rho^c \otimes \rho', \mathsf{V}^c \otimes \mathsf{V}')$. Hence, by (24),

$$\chi_{\mathrm{Hom}(\mathsf{V},\mathsf{V}')}(g) = \overline{\chi_\rho(g)}\chi_{\rho'}(g) \qquad (g \in \mathrm{G}).$$

Therefore,

$$\begin{aligned}(\chi'_\rho, \chi_\rho) &= \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \chi_{\mathrm{Hom}(\mathsf{V},\mathsf{V}')}(g)\\ &= \text{dimension of the space of the G-invariants in } \mathrm{Hom}(\mathsf{V}, \mathsf{V}').\end{aligned}$$

Thus the formula follows from Corollary 39. □

Let (ρ, V) be a representation of G and let

$$\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus ... \oplus \mathsf{V}_n$$

be the decomposition into irreducibles. The number of the V_j which are isomorphic to V_1 is called the multiplicity of V_1 in V, denoted m_1. We may collect the isomorphic representations in the above formula and (after changing the indecies appropriately) get the following decomposition

$$\mathsf{V} = \mathsf{V}_1^{\oplus m_1} \oplus \mathsf{V}_2^{\oplus m_2} \oplus ... \oplus \mathsf{V}_k^{\oplus m_k},$$

where the V_j are irreducible and mutually non-isomorphic and

$$\mathsf{V}_j^{\oplus m_j} = \mathsf{V}_j \oplus \mathsf{V}_j \oplus ... \oplus \mathsf{V}_j \ (m_j \text{ summands}).$$

This is the primary decomposition of V, which is also denoted by

$$\mathsf{V} = m_1\mathsf{V}_1 \oplus m_2\mathsf{V}_2 \oplus ... \oplus m_k\mathsf{V}_k = m_1 \cdot \mathsf{V}_1 \oplus m_2 \cdot \mathsf{V}_2 \oplus ... \oplus m_k \cdot \mathsf{V}_k,$$

for brevity.

Proposition 22. *Let*

$$\mathsf{V} = m_1\mathsf{V}_1 \oplus m_2\mathsf{V}_2 \oplus ... \oplus m_k\mathsf{V}_k,$$

be the primary decomposition of V. *Then*

$$\chi_{\mathsf{V}} = m_1\chi_{\mathsf{V}_1} + m_2\chi_{\mathsf{V}_2} + ..m_k\chi_{\mathsf{V}_k}$$

and

$$m_j = (\chi_{\mathsf{V}}, \chi_{\mathsf{V}_j}).$$

In particular V *is irreducible if and only if* $(\chi_{\mathsf{V}}, \chi_{\mathsf{V}}) = 1$.

Proof. This follows from (24) and from Lemma 49. □

Let $(L, \mathrm{L}^2(\mathrm{G}))$ be the left regular representation of G:

$$L(g)\phi(h) = \phi(g^{-1}h) \qquad (g, h \in \mathrm{G}, \phi \in \mathrm{L}^2(\mathrm{G})).$$

Theorem 23. *The character of the left regular representation is given by*

$$\chi_L(g) = \begin{cases} |\mathrm{G}| \ \textit{if } g = 1, \\ 0 \ \textit{if } g \neq 1. \end{cases} \tag{17}$$

Let (ρ_1, V_1), (ρ_2, V_2), ..., (ρ_k, V_k) *be the set of all (up to equivalence) irreducible representations of* G. *(Lemma 49 shows that this set is finite.) Then*

$$\chi_L = dim\,\mathsf{V}_1\,\chi_{\mathsf{V}_1} + dim\,\mathsf{V}_2\,\chi_{\mathsf{V}_2} + ...dim\,\mathsf{V}_k\,\chi_{\mathsf{V}_k},$$

or equivalently,

$$\mathrm{L}^2(\mathrm{G}) = dim\,\mathsf{V}_1 \cdot \mathsf{V}_1 \oplus dim\,\mathsf{V}_2 \cdot \mathsf{V}_2 \oplus ... \oplus dim\,\mathsf{V}_k \cdot \mathsf{V}_k.$$

In particular every irreducible representation of G *occurs in* $\mathrm{L}^2(\mathrm{G})$. *Also,*

$$|\mathrm{G}| = (dim\,\mathsf{V}_1)^2 + (dim\,\mathsf{V}_2)^2 + ... + (dim\,\mathsf{V}_k)^2.$$

Proof. The formula for χ_L follows from Theorem 17. Suppose a representation (ρ_j, V_j) occurs in the left regular representation. Then the multiplicity of V_j in $\mathrm{L}^2(\mathrm{G})$ is equal to

$$(\chi_{\rho_j}, \chi_L) = \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \chi_{\rho_j}(g)\overline{\chi_L(g)} = \chi_{\rho_j}(1) = dim\,\mathsf{V}_j, ,$$

where the second equality follows from (17). Since we already have Lemma 49, it remains to check that every irreducible representation (ρ_j, V_j) occurs in $(L, \mathrm{L}^2(\mathrm{G}))$. For two a non-zero vector $u, v \in \mathsf{V}_j$ define the corresponding matrix coefficient of the irreducible representation

$$M_{u,v}(g) = (u, \rho_j(g)v) \qquad (g \in \mathrm{G}).$$

It is easy to check that the map

$$\mathsf{V}_j \ni u \to M_{u,v} \in \mathrm{L}^2(\mathrm{G})$$

is a morphism. Since this map is non-zero, we see that (ρ_j, V_j) occurs in $(L, \mathrm{L}^2(\mathrm{G}))$. □

Problem 10. *Describe the irreducible representations of the group S_4 and their characters.*

There are the trivial and the sign representations, as in the case of S_3. The group S_4 acts on $\mathbb{C}^4$ and the orthogonal complement of the trivial representation $(\mathrm{U} = \mathbb{C}(1,1,1,1)^\perp \subseteq \mathbb{C}^3)$ provides a 3 dimensional representation. Its character may be computed from Theorem 17. Then one checks that $(\chi_\mathrm{U}, \chi_\mathrm{U}) = 1$. Hence Proposition 51 shows that U is irreducible. Let U′ stand for the representation obtained from U via tensoring with the sign representation. Then U′ is also irreducible (for the same reason). The squares of the dimensions of the representations we constructed so far add up to $1^2+1^2+3^2+3^2 = 20$. Hence we know from Theorem 23 that we'll be done if we find an irreducible two-dimensional representation, call it W. In fact, since the characters of the irreducible representations form an orthonormal basis of $\mathrm{L}^2(\mathrm{G})^\mathrm{G}$ it suffices to find a function $\chi_\mathsf{W} \in \mathrm{L}^2(\mathrm{G})^\mathrm{G}$ which is orthogonal to all the characters we found so far, has norm 1 $(\chi_\mathsf{W}, \chi_\mathsf{W}) = 1)$ and $\chi_\mathsf{W}(1) = 2$. This can be done, of course. We summarize the results in the following table.

conjugacy class in S_4	[1]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
number of elements in the conjugacy class	1	6	8	6	3
value of the character on the conjugacy class for					
trivial representation	1	1	1	1	1
sign representation	1	−1	1	−1	1
the three-dimensional representation U	3	1	0	−1	−1
U tensored with sign	3	−1	0	1	−1
the two-dimensional representation W	2	0	−1	0	2

(Notice that $\{1, (12)(34), (13)(24), (14)(23)\} \subseteq S_4$ is a normal subgroup. Then one checks that W is the pullback of the irreducible two-dimensional representation of S_3 to S_4 via the quotient map

$$S_4 \to S_4/\{1, (12)(34), (13)(24), (14)(23)\} = S_3.)$$

The alternating group $A_4 \subseteq S_4$ contains $\{1, (12)(34), (13)(24), (14)(23)\}$ as a normal subgroup and the quotient is a cyclic group of order 3. Hence, A_4 has 3 one-dimensional representations. Here is the charter table for A_4, where $z = e^{2\pi i/3}$.

conjugacy class in A_4	[1]	[(123)]	[(132)]	[(12)(34)]
number of elements in the conjugacy class	1	4	4	3
value of the character on the conjugacy class for				
trivial representation	1	1	1	1
second one dimensional representation	1	z	z^2	1
third one dimensional representation	1	z^2	z	1
the three-dimensional representation	3	0	0	-1

Lemma 24. *Given an irreducible unitary representation* (ρ, V) *of* G*, consider* $\mathrm{End}(\mathsf{V})$ *as a Hilbert space with the inner product*

$$(S,T) = \mathrm{tr}(ST^*) \qquad (S,T \in \mathrm{End}(\mathsf{V}))\,.$$

Let

$$\pi(g_1,g_2)T = \rho(g_2)T\rho(g_1^{-1}) \qquad (g_1,g_2 \in \mathrm{G},\ T \in \mathrm{End}(\mathsf{V}))\,.$$

Then $(\pi, \mathrm{End}(\mathsf{V}))$ *is an irreducible unitary representation of* $\mathrm{G}\times\mathrm{G}$.

Proof. The non-trivial part is to check the irreducibility. Notice that $(\pi, \mathrm{End}(\mathsf{V}))$ is isomorphic to outer tensor product $(\rho^c \otimes \rho, \mathsf{V}^c \otimes \mathsf{V})$. Hence,

$$\frac{1}{|\mathrm{G}\times\mathrm{G}|}\sum_{(g_1,g_2)\in\mathrm{G}\times\mathrm{G}} |\chi_\pi(g_1,g_2)|^2 = \frac{1}{|\mathrm{G}\times\mathrm{G}|}\sum_{(g_1,g_2)\in\mathrm{G}\times\mathrm{G}} |\overline{\chi_\rho(g_1)}\chi_{\rho'}(g_2)|^2$$

$$= \frac{1}{|\mathrm{G}|}\sum_{g_1\in\mathrm{G}} |\chi_\rho(g_1)|^2 \frac{1}{|\mathrm{G}|}\sum_{g_2\in\mathrm{G}} |\chi_{\rho'}(g_2)|^2 = 1.$$

Hence, by Proposition 51, $(\pi, \mathrm{End}(\mathsf{V}))$ is irreducible. □

Given a representation (ρ, V) define a map

$$M_\rho : \mathrm{End}(\mathsf{V}) \to \mathrm{L}^2(\mathrm{G}),\ M_\rho(T)(g) = tr(\rho(g)T) = tr(T\rho(g)).$$

In particular, if $I \in \mathrm{End}(\mathsf{V})$ is the identity, then

$$M_\rho(I) = \chi_\rho.$$

Also, it is easy to see that the subspace $M_\rho(\mathrm{End}(\mathsf{V})) \subseteq \mathrm{L}^2(\mathrm{G})$ depends only on the equivalence class of ρ.

Recall the right regular representation of G,

$$R(g)f(h) = f(hg) \quad (g,h \in \mathrm{G},\ f \in \mathrm{L}^2(\mathrm{G})). \tag{18}$$

and the outer tensor product $L \otimes R$,

$$L\otimes R(g_1,g_2)f(g) = f(g_1^{-1}gg_2) \qquad (g_1,g_2,g \in \mathrm{G},\ f \in \mathrm{L}^2(\mathrm{G}))\,. \tag{19}$$

It is easy to see that

$$L(g_1)M_\rho(T) = M_\rho(T\rho(g_1^{-1}))\,, \quad R(g_2)M_\rho(T) = M_\rho(\rho(g_2)T) \qquad (g_1,g_2 \in\ G)\,. \tag{20}$$

In particular,

$$M_\rho(T)(g^{-1}hg) = M_\rho(\rho(g)T\rho(g)^{-1})(h) \qquad (g, h \in \mathrm{G})\,. \tag{21}$$

Lemma 25. *Given an irreducible unitary representation* (ρ, V) *of* G*, The map*

$$M_\rho : \mathrm{End}(\mathsf{V}) \to \mathrm{L}^2(\mathrm{G})$$

intertwines the representation π *with the outer tensor product* $L \otimes R$*, is injective, and*

$$\frac{1}{dim\,\mathsf{V}} tr(ST^*) = (M_\rho(S), M_\rho(T)) \qquad (S, T \in \mathrm{End}(\mathsf{V}))\,.$$

Proof. The intertwining property follows from (20). Therefore the irreducibility of π implies the injectivity of M_ρ.

Recall the adjoint map between Hilbert spaces,

$$M_\rho^* : \mathrm{L}^2(\mathrm{G}) \to \mathrm{End}(\mathsf{V}).$$

Notice also that

$$M_\rho^* M_\rho : \mathrm{End}(\mathsf{V}) \to \mathrm{End}(\mathsf{V})$$

is $\mathrm{G} \times \mathrm{G}$-intertwining. Hence there is $\lambda \in \mathbb{C}$ such that $M_\rho^* M_\rho = \lambda I$. Furtheremore,

$$\lambda \cdot dim\,\mathsf{V} = (I, M_\rho^* M_\rho(I)) = (M_\rho(I), M_\rho(I)) = (\chi_\rho, \chi_\rho) = 1$$

and we are done. □

Theorem 26. *Suppose* (ρ, V) *and* (ρ', V') *are two irreducible unitary representations of* G*. Then*

a) *if* (ρ, V) *is not equivalent to* (ρ', V')*, then* $M_\rho(\mathrm{End}(\mathsf{V})) \perp M_{\rho'}(\mathrm{End}(\mathsf{V}'))$*,*
b) *for any* $S, T \in \mathrm{End}(\mathsf{V})$*,* $(M_\rho(S), M_\rho(T)) = \frac{1}{dim\,\mathsf{V}} tr(ST^*)$*.*

Proof. We have the adjoint map

$$M_{\rho'}^* : \mathrm{L}^2(\mathrm{G}) \to \mathrm{End}(\mathsf{V}').$$

Notice also that

$$M_{\rho'}^* M_\rho : \mathrm{End}(\mathsf{V}) \to \mathrm{End}(\mathsf{V}')$$

is a morphism (i.e. $\mathrm{G} \times \mathrm{G}$ intertwining map). If the two representations are not isomorphic then, by Theorem 12, $M_{\rho'}^* M_\rho = 0$. Thus for $S \in \mathrm{End}(\mathsf{V})$ and $T \in \mathrm{End}(\mathsf{V}')$,

$$(M_\rho(S), M_{\rho'}(T)) = (S, M_{\rho'}^* M_\rho(T)) = 0$$

and a) follows. Part b) was verified in Lemma 25. □

Theorem 27. *There are the following direct sum orthogonal decompositions of Hilbert spaces*

a) $\mathrm{L}^2(\mathrm{G}) = \sum_{\rho \in \hat{\mathrm{G}}} M_\rho(\mathrm{End}(\mathsf{V}))$,
b) $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} = \sum_{\rho \in \hat{\mathrm{G}}} \mathbb{C}\chi_\rho$.

Thus the characters χ_ρ *form an orthonormal basis of* $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}}$*.*

Proof. Part a) follows from Theorem 52 by counting dimensions. Part b) follows from part a), because, by Theorem 12, $\mathrm{End}(\mathsf{V})^{\mathrm{G}} = \mathbb{C}I$. Indeed,

$$\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} = \sum_{\rho\in\hat{\mathrm{G}}}(M_\rho(\mathrm{End}(\mathsf{V})))^{\mathrm{G}} = \sum_{\rho\in\hat{\mathrm{G}}}\mathrm{M}_\rho(\mathrm{End}(\mathsf{V})^{\mathrm{G}}) = \sum_{\rho\in\hat{\mathrm{G}}}\mathbb{C}\chi_\rho.$$

where only the middle equation

$$(M_\rho(\mathrm{End}(\mathsf{V})))^{\mathrm{G}} = \mathrm{M}_\rho(\mathrm{End}(\mathsf{V})^{\mathrm{G}})$$

requires an explanation. We need to show that if

$$tr(T\rho(ghg^{-1})) = tr(T\rho(h))$$

for all $h \in \mathrm{G}$, then $\rho(g^{-1})T\rho(g) = T$. Since $tr(T\rho(ghg^{-1}) = tr(\rho(g^{-1})T\rho(g)\rho(h))$, we'll be done as soon as we show that if $tr(S\rho(h)) = 0$ for all $h \in \mathrm{G}$, then $S = 0$. But, since the map M_ρ is injective (because π is irreducible), this is indeed the case. □

Corollary 28.

$$\delta = \sum_{\rho\in\hat{\mathrm{G}}}\frac{\chi_\rho(1)}{|\mathrm{G}|}\cdot\chi_\rho.$$

Proof. We see from part b) of Theorem 53 that there are numbers $m_\rho \in \mathbb{C}$ such that

$$\delta = \sum_{\rho\in\hat{\mathrm{G}}} m_\rho\cdot\chi_\rho.$$

But Lemma 49 shows that $m_\rho = (\delta, \chi_\rho) = \frac{\chi_\rho(1)}{|\mathrm{G}|}$. □

For a function $f \in \mathrm{L}^1(\mathrm{G})$ define the Fourier transform

$$\hat{f}(\rho) = \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} f(g)\rho^c(g) = \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} f(g)\rho(g^{-1}) \qquad (\rho\in\hat{\mathrm{G}}). \tag{22}$$

Thus for a representation (ρ, V), $\hat{f}(\rho) \in \mathrm{End}(\mathsf{V})$.

Theorem 29. *(Fourier inversion for* G*) For any $f \in \mathrm{L}^1(\mathrm{G})$,*

$$f(g) = \sum_{\rho\in\hat{\mathrm{G}}}\chi_\rho(1)\cdot tr(\hat{f}(\rho)\rho(g)) \qquad (g\in\mathrm{G}).$$

In particular,

$$f(1) = \sum_{\rho\in\hat{\mathrm{G}}}\chi_\rho(1)\cdot\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} f(g)\chi_{\rho^c}(g).$$

Proof. This is straightforward:

$$\begin{aligned}
f(g) &= \sum_{h\in \mathrm{G}} f(h)\delta(h^{-1}g) = \sum_{h\in \mathrm{G}} f(h)\left(\sum_{\rho\in\hat{\mathrm{G}}} \frac{\chi_\rho(1)}{|\mathrm{G}|}\cdot \chi_\rho\right)(h^{-1}g)\\
&= \sum_{h\in \mathrm{G}} f(h)\sum_{\rho\in\hat{\mathrm{G}}} \frac{\chi_\rho(1)}{|\mathrm{G}|}\cdot \chi_\rho(h^{-1}g) = \sum_{h\in \mathrm{G}} f(h)\sum_{\rho\in\hat{\mathrm{G}}} \frac{\chi_\rho(1)}{|\mathrm{G}|}\cdot tr\rho(h^{-1}g)\\
&= \sum_{\rho\in\hat{\mathrm{G}}} \frac{\chi_\rho(1)}{|\mathrm{G}|}\cdot tr\left(\sum_{h\in \mathrm{G}} f(h)\rho(h^{-1}g)\right)\\
&= \sum_{\rho\in\hat{\mathrm{G}}} \chi_\rho(1)\cdot tr\left(\frac{1}{|\mathrm{G}|}\sum_{h\in \mathrm{G}} f(h)\rho(h^{-1})\rho(g)\right).
\end{aligned}$$

□

For two functions $f_1, f_2 \in \mathrm{L}^1(\mathrm{G})$ define the convolution

$$f_1 * f_2(g) = L(f_1)f_2(g) = \frac{1}{|\mathrm{G}|}\sum_{h\in \mathrm{G}} f_1(h)L(h)f_2(g) = \frac{1}{|\mathrm{G}|}\sum_{h\in \mathrm{G}} f_1(h)f_2(h^{-1}g) \qquad (g\in \mathrm{G}).$$

Also, for $f \in \mathrm{L}^1(\mathrm{G})$ define

$$f^*(g) = \overline{f(g^{-1})} \qquad (g\in \mathrm{G}).$$

Lemma 30. *The following formulas hold:*

a) *for two functions* $f_1, f_2 \in \mathrm{L}^1(\mathrm{G})$, $(f_1 * f_2)\hat{} = \hat{f}_1\hat{f}_2$,
b) *for a function* $f \in \mathrm{L}^1(\mathrm{G})$, $(f^*)\hat{} = \hat{f}^*$.

Proof. This is straightforward:

$$\begin{aligned}
(f_1 * f_2)\hat{}(\rho) &= \frac{1}{|\mathrm{G}|}\sum_{g\in \mathrm{G}} f_1 * f_2(g)\rho^c(g)\\
&= \frac{1}{|\mathrm{G}|^2}\sum_{h\in \mathrm{G}}\sum_{g\in \mathrm{G}} f_1(h)f_2(h^{-1}g)\rho^c(g)\\
&= \frac{1}{|\mathrm{G}|^2}\sum_{h\in \mathrm{G}}\sum_{g\in \mathrm{G}} f_1(h)\rho^c(h)f_2(h^{-1}g)\rho^c(h^{-1}g)\\
&= \frac{1}{|\mathrm{G}|^2}\sum_{h\in \mathrm{G}}\sum_{g\in \mathrm{G}} f_1(h)\rho^c(h)f_2(g)\rho^c(g) = \hat{f}_1(\rho)\cdot\hat{f}_2(\rho)
\end{aligned}$$

and

$$\begin{aligned}(f^*)\hat{}(\rho) &= \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} f^*(g)\rho^c(g) = \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \overline{f(g^{-1})}\rho^c(g)\\ &= \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \overline{f(g)}\rho^c(g)^*\\ &= \left(\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} f(g)\rho^c(g)\right)^* = \hat{f}(\rho)^*.\end{aligned}$$

□

Theorem 31. *For any $f \in \mathrm{L}^2(\mathrm{G})$,*

$$\| f \|^2 = \sum_{\rho\in\hat{\mathrm{G}}} \chi_\rho(1)\cdot \| \hat{f}(\rho) \|^2 .$$

Proof. We see from Lemma 30 that the left hand side is equal to

$$f * f^*(1) = \sum_{\rho\in\hat{\mathrm{G}}} \chi_\rho(1)\cdot tr(\hat{f}(\rho)\hat{f}(\rho)^*),$$

which coincides with the right hand side. □

Problem 11. *Read the proof of Theorem 56*

7. Some Functional Analysis on a Hilbert space

Here we follow [Lan85, sections 1.2 and 1.3]. Let V be a Hilbert space. A continuous linear map $A : \mathsf{V} \to \mathsf{V}$ is called compact if it maps any bounded sequence $v_n \in \mathsf{V}$ to a sequence Av_n that has a convergent subsequence.

Theorem 32. *Let A be a compact hermitian operator on the Hilbert space V. Then the family of eigenspaces V_λ, where λ ranges over all eigenvalues (including 0), is an orthogonal decomposition of E:*

$$\mathsf{V} = \bigoplus_\lambda \mathsf{V}_\lambda .$$

For $\lambda \neq 0$ the eigenspace V_λ is finite dimensional.

Let S be a set of operators (continuous linear maps) on V. We say that V is S-irreducible if V has no closed S-invariant subspace other than $\{0\}$ and V itself. We say that V is completely reducible for S if V is the orthogonal direct sum of S-irreducible subspaces. Two subspaces $\mathsf{V}_1, \mathsf{V}_2 \subseteq E$ are called S-isomorphic if there is an isometry from V_1 onto V_2 which intertwines the action of S. The number of elements in such an isomorphism class is called the multiplicity of that isomorphism class in V.

A subalgebra $\mathcal{A}$ of operators on V is said to be $*$-closed if whenever $A \in \mathcal{A}$, then $A^* \in \mathcal{A}$. As explained in [Lan85, section 1.2], the following Theorem is a consequence of Theorem 32.

Theorem 33. *Let $\mathcal{A}$ be a $*$-closed subalgebra of compact operators on a Hilbert space V. Then V is completely reducible for $\mathcal{A}$, and each irreducible subspace occurs with finite multiplicity.*

The following theorem (and the corollary below) is known as Schur's Lemma. For a proof see [Lan85, Appendix 1, Theorem 4]

Theorem 34. *Let S be a set of operators acting irreducibely on the Hilbert space V. Let A be a hermitian operator such that $AB = BA$ for all $B \in S$. Then $A = cI$ for some real number c.*

Corollary 35. *Let S be a set of operators acting irreducibely on the Hilbert space V. Let A be an operator such that $AB = BA$ and $A^*B = BA^*$ for all $B \in S$. Then $A = cI$ for some $c \in \mathbb{C}$. number c.*

Proof. We write $A = B + iC$, where

$$B = \frac{1}{2}(A + A^*)\,, \quad C = \frac{1}{2i}(A - A^*)\,.$$

Then B and C commute with S. Hence there are real numbers b and c such that $B = bI$ and $C = cI$. Hence $A = (b + ic)I$. □

We shall also use a few facts concerning integral kernel operators. The proofs may be found in [Lan85, section 1.3]

Theorem 36. *Let (X, M, dx) and (Y, N, dy) be measured spaces, and assume that $L^2(X)$, $L^2(Y)$ have countable orthogonal bases. Let $q \in L^2(X \times Y)$. Then the the formula*

$$Qf(x) = \int_Y q(x, y) f(y) dy$$

defines a bounded, compact operator from $L^2(Y)$ into $L^2(X)$ with $\| Q \| \leq \| q \|_2$.

Let A be a bounded operator on a Hilbert space V. We say that A is a Hilbert-Schmidt operator if there is an orthonormal basis v_1, v_2, ... of V such that

$$\sum_{n=1}^{\infty} \| Av_n \|^2 < \infty\,.$$

One checks that tjhis quantity does not depend on the orthonormal basis and denotes the square root of it by $\| A \|_{H.S}$. This is the Hilbert-Schmidt norm of A.

A bounded operator T on a Hilbert space V is of trace class if it is the product of two Hilbert-Schmidt operators, $T = AB$. Then for any orthonormal basis v_n of V

$$\sum_n |(Tv_n, v_n)| = \sum_n |(Bv_n, A^* v_n)| \leqq \| A \|_{H.S} \| B^* \|_{H.S} = \| A \|_{H.S} \| B \|_{H.S} < \infty\,.$$

Therefore the following series

$$\operatorname{tr} T = \sum_n (Tv_n, v_n)$$

is converges and defines the trace tr of T. For all tht see [Lan85, section VII.1].

Also, we have the following theorem, see [Kna86, Lemma 10.15] or [Lan85, Theorem 1, page 128] for a special X,

Theorem 37. *Let X be a compact smooth manifold and let dx be a measure on X that is a smooth function times the Lebesgue measure in each coordinate neighborhood. Let $q \in C^\infty(X \times X)$. Then the the formula*

$$Qf(x) = \int_X q(x,y)f(y)dy$$

defines a bounded operator of trace class on $L^2(X)$ and

$$\mathrm{tr}\, Q = \int_{\mathsf{X}} q(x,x)\, dx\,.$$

Problem 12. *Get familiar with the notions and theorems of section 7.*

8. Representations of locally compact groups

Let G be a locally compact group and let V be a topological vector space. We shall always assume that $\mathsf{V} \neq \{0\}$. Let $\mathrm{GL}(\mathsf{V})$ denote the group of the invertible continuous endomorphisms of V. A representation of G on V is a pair (π, V), where $\pi : \mathrm{G} \to \mathrm{GL}(\mathsf{V})$ is a group homomorpism such that the map

$$\mathrm{G} \times \mathsf{V} \ni (g,v) \to \pi(g)v \in \mathsf{V}$$

is continuous. A subspace $\mathsf{W} \subseteq \mathsf{V}$ is called invariant if $\rho(\mathrm{G})\mathsf{W} \subseteq \mathsf{W}$. The representation (ρ, V) is called irreducible if V does not contain any closed invariant subspaces other than $\{0\}$ an V.

If (π_1, V_1), (π_2, V_2) are representations of G, then a continuous linear map $T : \mathsf{V}_1 \to \mathsf{V}_2$ such that

$$T\pi_1(g) = \pi_2(g)T \qquad (g \in \mathrm{G})$$

is called an intertwining operator, or a G-homomorphism. The vector space of all the intertwining maps will be denoted $\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}_1, \mathsf{V}_2)$ or more precisely, $\mathrm{Hom}_{\mathrm{G}}((\pi_1, \mathsf{V}_1), (\pi_2, \mathsf{V}_2))$. The representations (π_1, V_1), (π_2, V_2) are equivalent if there exists a bijective $T \in \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}_1, \mathsf{V}_2)$.

A basic example of a representation of G is the right regular representation $(R, C(\mathrm{G})$. Here $C(\mathrm{G})$ is the space of the continuous, complex valued, functions on G (with the topology of uniform convergence on compact sets) and

$$R(g)f(x) = f(xg) \qquad (g, x \in \mathrm{G}, f \in C(\mathrm{G}))\,.$$

Similarly, we have the left regular representation $(L, C(\mathrm{G}))$,

$$L(g)f(x) = f(g^{-1}x) \qquad (g, x \in \mathrm{G}, f \in C(\mathrm{G}))\,.$$

Given an irreducible representation (π, V) and an element $\lambda \in \mathsf{V}'$ (a continuous linear functional on V) the formula

$$T_\lambda v(x) = \lambda(\pi(x)v) \qquad (x \in \mathrm{G})$$

defines a map $T_\lambda \in \mathrm{Hom}_{\mathrm{G}}((\pi, \mathsf{V}), (R, C(\mathrm{G})))$. If V has enough continuous linear functionals, in the sense that for every $0 \neq v \in \mathsf{V}$ there is $\lambda \in \mathsf{V}'$ such that $\lambda(v) \neq 0$, then T_λ is injective, so (π, V) may be viewed as a subrepresentation of $(R, C(\mathrm{G}))$. Recall, [Rud91, section 1.47], that for $\mathsf{V} = \mathrm{L}^p([0,1])$ with $0 < p < 1$ the dual space $\mathsf{V}' = \{0\}$. Thus for this space the above argument wouldn't work.

The function

$$\mathrm{G} \ni x \to \lambda(\pi(x)v) \in \mathbb{C}$$

is called a matrix coefficient of the representation (π, V).

If V is a Hilbert space, then a representation (π, V) is called unitary if every operator $\pi(g)$, $g \in \mathrm{G}$, is unitary. Two unitary representations (π_1, V_1), (π_2, V_2) are called unitarili equivalen if there is a bjiective and isometric G-homomorphism $T : \mathsf{V}_1 \to \mathsf{V}_2$.

Here is the classical version of Schur's Lemma, which is an immediate consequence of Corollary 35.

Theorem 38. *Let (π, V) be an irreducible unitary representation of* G*. Then*

$$\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}) = \mathbb{C}I\,.$$

Corollary 39. *For two irreducible unitary representations (ρ, V) and (ρ', V') of* G*,*

$$\mathit{dim}\ \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}') = \begin{cases} 1 & \mathit{if}\ (\rho, \mathsf{V}) \simeq (\rho', \mathsf{V}') \\ 0 & \mathit{if}\ (\rho, \mathsf{V}) \not\simeq (\rho', \mathsf{V}') \end{cases}$$

(Here $\simeq$ stands for unitary equivalence.)

Proposition 40. *If two unitary representations (π, V), (π', V') are equivalent then they are unitarily equivalent.*

Proof. Let $(\ ,\ )$ be the invariant scallar product on V and let $(\ ,\ )'$ be the invariant scallar product on V'. Pick an isomorphism $T : \mathsf{V} \to \mathsf{V}'$. Define $T^* : \mathsf{V}' \to \mathsf{V}$ by

$$(Tu, v)' = (u, T^*v) \qquad (u \in \mathsf{V}, v \in \mathsf{V}).$$

Then $T^*\mathsf{V}' \to \mathsf{V}$ is also a morphism. Hence, $T^*T : \mathsf{V} \to \mathsf{V}$ commutes with the action of G. Hence, there is $\lambda \in \mathbb{C}$ such that $T^*T = \lambda I$. Thus, for any $u, v \in \mathsf{V}$,

$$(Tu, Tv)' = (u, T^*Tv) = \overline{\lambda}(u, v).$$

In particular, by taking $u = v \neq 0$ we see that $\lambda > 0$. Hence,

$$\frac{1}{\sqrt{\lambda}}T : \mathsf{V} \to \mathsf{V}'$$

is an isometry and a morphism. □

Problem 13. *Define the Heisenberg group*

$$H = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

with the multiplication

$$(x, \xi, t)(x', \xi', t') = (x + x', \xi + \xi', t + t' + \frac{1}{2}(x \cdot \xi' - x' \cdot \xi))$$

The Schwartz space $\mathcal{S}(\mathbb{R})$ *is equipped with a topology,* [Hör83, section 7.1]. *One may check that the formula*

$$(\rho(x,\xi,t)f)(y) = e^{2\pi i t + \pi i \xi\cdot x - 2\pi i \xi \cdot y} f(y-x) \tag{23}$$

defines a continuous function

$$H \times \mathcal{S}(\mathbb{R}) \ni (g,f) \to \rho(g)f \in \mathcal{S}(\mathbb{R})$$

Assuming that, check that $(\rho, \mathcal{S}(\mathbb{R}))$ *is a representation of* H *on the linear topological space* $\mathcal{S}(\mathbb{R})$*, i.e. show that* $\rho(gg') = \rho(g)\rho(g')$*. Also show that*

$$\mathrm{End}_H(\mathcal{S}(\mathbb{R})) = \mathbb{C} I\,.$$

One may check, but it could take too much of your time, that the representation $(\rho, \mathcal{S}(\mathbb{R}))$ *is irreducible and that the same formula* (23) *defines an irreducible unitary representation of* H *on* $\mathrm{L}^2(\mathbb{R})$*.*

Here is one more version of Schur's Lemma. For a proof see [Wal88, 1.2.2].

Theorem 41. *Let* (π, V) *be an irreducible unitary representation of* G*. Let* $\mathsf{V}_0 \subseteq \mathsf{V}$ *be a dense* G*-invariant subspace and let* $A : \mathsf{V}_0 \to \mathsf{V}$ *be a linear* G*-intertwining map.*

Suppose $\mathsf{V}_1 \subseteq \mathsf{V}$ *be a dense subspace and let* $B : \mathsf{V}_1 \to \mathsf{V}$ *a linear map such that*

$$(Av_0, v_1) = (v_0, Bv_1) \qquad (v_0 \in \mathsf{V}_0, v_1 \in \mathsf{V}_1)\,.$$

Then A *is a scallar multiple of the identity restricted to* V_0*.*

9. Haar measures and extension of a representation of G to a representation of $L^1(\mathrm{G})$

A proof of the following theorem may be found in [HR63, (15.5)-(15.11)]

Theorem 42. *There is a positive Borel measure* dx *on* G *such that*

$$\int_{\mathrm{G}} f(gx)\,dx = \int_{\mathrm{G}} f(x)\,dx \qquad (g \in \mathrm{G}, f \in C_c(\mathrm{G}))\,.$$

This measure is unique up to a constant multiple. Furthermore there is a group homomrphism $\Delta : \mathrm{G} \to \mathbb{R}^+$ *such that*

$$\int_{\mathrm{G}} f(xg)\,dx = \Delta(g) \int_{\mathrm{G}} f(x)\,dx \qquad (g \in \mathrm{G}, f \in C_c(\mathrm{G}))\,.$$

Also, the converse of this theorem holds: if a topological group has a left invariant Borel measure the it is locally compact, see[**?**].

The measure dx is called the left invariant Haar measure on G. The group G is called unimodular if $\Delta = 1$. This is certainly the case if G is compact. The Lebesgue measures we used in sections 2 and 4 are Haar measures and the groups are unimodular. An example of a non-unimodular group is

$$P = \left\{ p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} a \in R^\times, b \in \mathbb{R} \right\}\,.$$

Here $|a|^{-2} da\, db$ is the left invariant Haar measure and $\Delta(p) = a^2$.

The convolution $\phi * \psi$ of two functions $\phi, \psi \in \mathrm{L}^1(\mathrm{G})$ is defined by

$$\phi * \psi(y) = \int_{\mathrm{G}} \phi(x)\psi(x^{-1}y)\, dx\,.$$

The Banach space $\mathrm{L}^1(\mathrm{G})$ with the product defined by the convolution is a Banach algebra. This algebras may have no identity, but they always has an approximate identity.

Theorem 43. *There is a sequence $\phi_n \in C_c(\mathrm{G})$ such that*

$$\begin{aligned}
&(1)\ \phi_n \geq 0 \qquad (n = 0, 1, 2, \ldots)\,,\\
&(2)\ \int_{\mathrm{G}} \phi_n(x)\, dx = 1 \qquad (n = 0, 1, 2, \ldots)\,,\\
&(3)\ \textit{Given a neighborhood } U \textit{ of the identity in } \mathrm{G}\,,\\
&\quad \textit{the support of } \phi_n \textit{ is contained in } U \textit{ for all } n \textit{ sufficiently large.}
\end{aligned}$$

(See [Lan85, secion 1.1] for a proof.) Given a representation (π, V) the formula

$$\pi(\phi) = \int_{\mathrm{G}} \phi(x)\pi(x) \qquad (\phi \in \mathrm{L}^1(\mathrm{G}))$$

defines a representation of the Banach algebra $\mathrm{L}^1(\mathrm{G})$ on V, i.e.

$$\pi(\phi * \psi) = \pi(\phi)\pi(\psi) \qquad (\phi, \psi \in \mathrm{L}^1(\mathrm{G}))\,.$$

By restriction, (π, V) is also a representation of the convolution algebra $C_c(\mathrm{G})$. As straightforward consequence of Theorem 43 we see that a subspace $\mathsf{W} \subseteq \mathsf{V}$ is G-invariant if and only if it is $C_c(\mathrm{G})$-invariant. Also, (π, V) is G-irreducible if and only if it is $C_c(\mathrm{G})$-irreducible.

10. **Representations of a compact group**

Let G be a compact group with the Haar measure of total mass equal to 1. If (π, V) is a representation of G on a Hilbert space V, then, by averaging the norm on V over G, we obtain another norm with respect to which the representation is unitary.

Hence, while considering representations of G on Hilbert spaces, we may assume that they are unitary.

Theorem 44. *Any irreducible unitary representation (π, V) of* G *is finite dimensional.*

Proof. Let $v \in \mathsf{V}$ be a unit vector and let P be the orthogonal projection on the one-dimensional space $\mathbb{C}v$. Let Q be the continuous linear map defined by

$$Q = \int_{\mathrm{G}} \pi(x)^{-1} P \pi(x)\, dx\,.$$

Then $Q = Q^*$ commutes with the action of G. Hence Schur's Lemma, Theorem 36, implies that there is a constant $c \in \mathbb{R}$ such that $Q = cI$. Since π is unitary,

$$(Qv, v) = \int_{\mathrm{G}} (P\pi(x)v, \pi(x)v)\, dx = \int_{\mathrm{G}} (\pi(x)v, v)(v, \pi(x)v)\, dx = \int_{\mathrm{G}} |(\pi(x)v, v)|^2\, dx > 0\,.$$

Hence $c > 0$.

Let v_1, v_2, ... be an orthonormal basis of V. Then for each $x \in \mathrm{G}$, $\pi(x)v_1$, $\pi(x)v_2$, ...is also an orthonormal basis of V. Hence

$$\sum_{n=1}^{\infty}(P\pi(x)v_n, \pi(x)v_n) = \sum_{n=1}^{\infty}((\pi(x)v_n, v)v, \pi(x)v_n) = \sum_{n=1}^{\infty}|(\pi(x)v_n, v)|^2 = \| v \| = 1\,.$$

But $(P\pi(x)v_n, \pi(x)v_n) = (\pi(x)^{-1}P\pi(x)v_n, v_n)$. Hence, after integration over G we see that

$$\sum_{n=1}^{\infty}(Qv_n, v_n) = 1\,.$$

Thus c times the number of the elements in the basis is equal to 1. Hence, $\dim \mathsf{V} < \infty$. $\square$

By combining Theorem 44 with the Spectral Theorem from Linear Algebra we deduce the following corollary.

Corollary 45. *An irreducible unitary representation of a compact abelian group is one dimensional.*

If $\dim \mathsf{V} < \infty$, we let V^c denote the vector space dual to V. The contragredient representation (π^c, V^c) is defined by

$$\pi^c(g)v^c(v) = v^c(\pi(g^{-1})v) \qquad (v \in \mathsf{V},\ v^c \in \mathsf{V}^c, g \in \mathrm{G}).$$

Given two finite dimensional representations (π, V) and (π', V') define their direct sum $(\pi \oplus \pi', \mathsf{V} \oplus \mathsf{V}')$ by

$$(\pi \oplus \pi')(g)(v, v') = (\pi(g)(v), \pi'(g)(v')) \quad (g \in \mathrm{G}, v \in \mathsf{V}, v' \in \mathsf{V}')$$

and the tensor product $(\pi \otimes \pi', \mathsf{V} \otimes \mathsf{V}')$ by

$$(\pi \otimes \pi')(g)[v \otimes v'] = [\pi(g)(v)] \otimes [\pi'(g)(v')] \quad (g \in \mathrm{G}, v \in \mathsf{V}, v' \in \mathsf{V}').$$

By definition the character $\Theta_{\mathsf{V}} = \Theta_\pi$ of a finite dimensional representation (π, V) is the following complex valued function on the group:

$$\Theta_{\mathsf{V}}(g) = tr(\pi(g)) \qquad (g \in \mathrm{G}).$$

This function is invariant under conjugation

$$\Theta(hgh^{-1}) = \Theta(g) \qquad (h, g \in \mathrm{G}).$$

Also, we have

$$\Theta_{\mathsf{V}\oplus\mathsf{V}'} = \Theta_{\mathsf{V}} + \Theta_{\mathsf{V}'},\ \Theta_{\mathsf{V}\otimes\mathsf{V}'} = \Theta_{\mathsf{V}}\Theta_{\mathsf{V}'} \text{ and } \Theta_{\pi^c} = \overline{\Theta_\pi}. \tag{24}$$

Denote by $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} \subseteq \mathrm{L}^2(\mathrm{G})$ the subspace of the functions invariant by the conjugation by all the elements of G. Our characters live in $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}}$.

Lemma 46. *Suppose (π, V) is a non-trivial irreducible unitary representation of G. Then*

$$\int_{\mathrm{G}} \pi(x)v\, dx = 0 \qquad (v \in \mathsf{V}).$$

Proof. Notice that the integral defines a G-invariant vector $u \in \mathsf{V}$. Since π is irreducible, either $u = 0$ or $\mathsf{V} = \mathbb{C}u$. Since π is non-trivial, the second option is impossible. □

Corollary 47. *Suppose (π, V) is a non-trivial irreducible unitary representation of* G. *Then*

$$\int_{\mathrm{G}} \pi(x)\, dx = 0\,.$$

Corollary 48. *Suppose (π, V) is a finite dimensional representation of* G. *Then*

$$\int_{\mathrm{G}} \Theta_\pi(x)\, dx = dim\, \mathsf{V}^{\mathrm{G}},$$

where $\mathsf{V}^{\mathrm{G}} \subseteq \mathsf{V}$ is the space of the G*-invariant vectors.*

Proposition 49. *The characters of irreducible representations form an orthonormal set in $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}}$.*

Proof. Consider two such representations (π, V) and (π', V'). The group G acts on this vector space $\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}')$ by

$$gT(v) = \pi'(g)T\pi(g^{-1})v \qquad (g \in \mathrm{G}, T \in \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}'), v \in \mathsf{V}).$$

This way $\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}')$ becomes a representation of G. It is easy to check that as such it is isomorphic to $(\pi^c \otimes \pi', \mathsf{V}^c \otimes \mathsf{V}')$. Hence, by (24),

$$\Theta_{\mathrm{Hom}(\mathsf{V},\mathsf{V}')}(g) = \overline{\Theta_\pi(g)}\Theta_{\pi'}(g) \qquad (g \in \mathrm{G}).$$

Therefore,

$$\begin{aligned} (\Theta_\pi', \Theta_\pi) &= \int_{\mathrm{G}} \Theta_{\mathrm{Hom}(\mathsf{V},\mathsf{V}')}(g)\, dg \\ &= \text{dimension of the space of the G-invariants in } \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}'). \end{aligned}$$

Thus the formula follows from Corollary 39. □

Proposition 50. *Any finite dimensional representation of* G *decomposes into the direct sum of irreducible representations.*

Proof. This follows from the fact that the orthogonal complement of a G-invariant subspace is G-invariant. □

Let (π, V) be a finite dimensional representation of G and let

$$\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus \ldots \oplus \mathsf{V}_n$$

be the decomposition into irreducibles. The number of the V_j which are isomorphic to V_1 is called the multiplicity of V_1 in V, denoted m_1. We may collect the isomorphic representations in the above formula and (after changing the indecies appropriately) get the following decomposition

$$\mathsf{V} = \mathsf{V}_1^{\oplus m_1} \oplus \mathsf{V}_2^{\oplus m_2} \oplus \ldots \oplus \mathsf{V}_k^{\oplus m_k},$$

where the V_j are irreducible and mutually non-isomorphic and

$$\mathsf{V}_j^{\oplus m_j} = \mathsf{V}_j \oplus \mathsf{V}_j \oplus ... \oplus \mathsf{V}_j \ (m_j \text{ summands}).$$

This is the primary decomposition of V, which is also denoted by

$$\mathsf{V} = m_1\mathsf{V}_1 \oplus m_2\mathsf{V}_2 \oplus ... \oplus m_k\mathsf{V}_k = m_1 \cdot \mathsf{V}_1 \oplus m_2 \cdot \mathsf{V}_2 \oplus ... \oplus m_k \cdot \mathsf{V}_k,$$

for brevity.

Proposition 51. *Let*

$$\mathsf{V} = m_1\mathsf{V}_1 \oplus m_2\mathsf{V}_2 \oplus ... \oplus m_k\mathsf{V}_k,$$

be the primary decomposition of V. *Then*

$$\Theta_{\mathsf{V}} = m_1\Theta_{\mathsf{V}_1} + m_2\Theta_{\mathsf{V}_2} + ..m_k\Theta_{\mathsf{V}_k}$$

and

$$m_j = (\Theta_{\mathsf{V}}, \Theta_{\mathsf{V}_j}).$$

In particular V *is irreducible if and only if* $(\Theta_{\mathsf{V}}, \Theta_{\mathsf{V}}) = 1$.

Proof. This follows from (24) and from Lemma 49. □

Given a finite dimensional representation (π, V) define a map

$$M_\pi : \mathrm{End}(\mathsf{V}) \to \mathrm{L}^2(\mathrm{G}),\ M_\pi(T)(g) = tr(\pi(g)T) = tr(T\pi(g)).$$

In particular, if $I \in \mathrm{End}(\mathsf{V})$ is the identity, then

$$M_\pi(I) = \Theta_\pi.$$

Also, it is easy to see that the subspace $M_\pi(\mathrm{End}(\mathsf{V})) \subseteq \mathrm{L}^2(\mathrm{G})$ depends only on the equivalence class of π.

Theorem 52. *Suppose* (π, V) *and* (π', V') *are two irreducible unitary representations of* G. *Then*

a) *if* (π, V) *is not equivalent to* (π', V'), *then* $M_\pi(\mathrm{End}(\mathsf{V})) \perp M_{\pi'}(\mathrm{End}(\mathsf{V}'))$,
b) *for any* $S, T \in \mathrm{End}(\mathsf{V})$, $(M_\pi(S), M_\pi(T)) = \frac{1}{dim\,\mathsf{V}} tr(ST^*)$.

Proof. Define a representation $(\Pi, \mathrm{End}(\mathsf{V}))$ of the group $\mathrm{G}\times\mathrm{G}$ on the vetor space $\mathrm{End}(\mathsf{V})$ by

$$\Pi(g_1, g_2)T = \pi(g_2)T\pi(g_1^{-1}) \qquad (g_1, g_2 \in \mathrm{G},\ T \in \mathrm{End}(\mathsf{V})).$$

Notice that $(\Pi, \mathrm{End}(\mathsf{V}))$ is isomorphic to outer tensor product $(\pi^c \otimes \pi, \mathsf{V}^c \otimes \mathsf{V})$. Hence,

$$\int_{\mathrm{G}\times\mathrm{G}} |\Theta_\Pi(x_1, x_2)|^2\, dx_1\, dx_2 = \int_{\mathrm{G}\times\mathrm{G}} |\overline{\Theta_\pi(x_1)}\Theta_\pi(x_2)|^2\, dx_1\, dx_2$$
$$= \int_{\mathrm{G}} |\Theta_\pi(x_1)|^2\, dx_1 \int_{\mathrm{G}} |\Theta_\pi(x_2)|^2\, dx_2 = 1.$$

Hence, by Proposition 51, $(\Pi, \mathrm{End}(\mathsf{V}))$ is irreducible.

We view $\mathrm{End}(\mathsf{V})$ as a Hilbert space with the following scalar product

$$(S_1, S_2) = tr(S_1 S_2^*) \qquad (S_1, S_2 \in \mathrm{End}(\mathsf{V})),$$

and similarly for V'. In particular, we have the adjoint map

$$M_{\pi'}^* : \mathrm{L}^2(\mathrm{G}) \to \mathrm{End}(\mathsf{V}).$$

Notice also that

$$M_{\pi'}^* M_\pi : \mathrm{End}(\mathsf{V}) \to \mathrm{End}(\mathsf{V}')$$

is a $\mathrm{G} \times \mathrm{G}$-intertwining map. If the two representations (π, V) and (π', V') are not isomorphic then The $\mathrm{G} \times \mathrm{G}$-modules $\mathrm{End}(\mathsf{V})$ and $\mathrm{End}(\mathsf{V}')$ are not isomorphic. Hence Corollary 39 shows that $M_{\pi'}^* M_\pi = 0$. Thus for $S \in \mathrm{End}(\mathsf{V})$ and $T \in \mathrm{End}(\mathsf{V}')$,

$$(M_\pi(S), M_{\pi'}(T)) = (S, M_{\pi'}^* M_\pi(T)) = 0$$

and a) follows.

Similarly, there is $\lambda \in \mathbb{C}$ such that $M_\pi^* M_\pi = \lambda I$. Furtheremore,

$$\lambda \cdot dim\, \mathsf{V} = (I, M_\pi^* M_\pi(I)) = (M_\pi(I), M_\pi(I)) = (\Theta_\pi, \Theta_\pi) = 1$$

and b) follows. □

The following statement is known as the Peter-Weyl Theorem, see [PW27].

Theorem 53. *There are the following direct sum orthogonal decompositions of Hilbert spaces*

a) $\mathrm{L}^2(\mathrm{G}) = \sum_{\pi \in \hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$,
b) $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} = \sum_{\pi \in \hat{\mathrm{G}}} \mathbb{C}\Theta_\pi$.

Proof. Part b) follows from part a), because, by Theorem 38, $\mathrm{End}(\mathsf{V})^{\mathrm{G}} = \mathbb{C}I$. Indeed,

$$\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} = \sum_{\pi \in \hat{\mathrm{G}}} (M_\pi(\mathrm{End}(\mathsf{V})))^{\mathrm{G}} = \sum_{\pi \in \hat{\mathrm{G}}} \mathrm{M}_\pi(\mathrm{End}(\mathsf{V})^{\mathrm{G}}), = \sum_{\pi \in \hat{\mathrm{G}}} \mathbb{C}\Theta_\pi.$$

where only the middle equation

$$(M_\pi(\mathrm{End}(\mathsf{V})))^{\mathrm{G}} = \mathrm{M}_\pi(\mathrm{End}(\mathsf{V})^{\mathrm{G}})$$

requires an explanation. We need to show that if

$$tr(T\pi(ghg^{-1})) = tr(T\pi(h))$$

for all $h \in \mathrm{G}$, then $\pi(g^{-1})T\pi(g) = T$. Since $tr(T\pi(ghg^{-1}) = tr(\pi(g^{-1})T\pi(g)\pi(h))$, we'll be done as soon as we show that if $tr(S\pi(h)) = 0$ for all $h \in \mathrm{G}$, then $S = 0$. But, since the map M_π is injective (because π is irreducible), this is indeed the case.

Part a) requires some work. We follow the argument in [Kna86, Theorem 1.12]. Let $\mathrm{U} = \sum_{\pi \in \hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$. This is a subspace of $\mathrm{L}^2(\mathrm{G})$ closed under the left and right translations and under the $*$ operation, $\phi \to \phi^*$, $\phi^*(x) = \overline{\phi(x^{-1})}$. Hence so is the orthogonal complement $\mathrm{U}^\perp \subseteq \mathrm{L}^2(\mathrm{G})$.

Suppose $\mathrm{U}^\perp \neq \{0\}$. We shall arrive at a contradiction. Let $\phi \in \mathrm{U}^\perp$ be non-zero. Let $\phi_n \in C_c(\mathrm{G})$ be an approximate identity, as in Theorem 43. Then a straightforward argument shows that

$$\lim_{n \to \infty} \parallel \phi_n * \phi - \phi \parallel_2 = 0\,.$$

Hence there is n such that $\phi_n * \phi \neq 0$. But this function is continuous. Thus we may assume that ϕ is continuous. Furthermore, applying the translations and the $*$ operation we may assume that $\phi(1) > 0$ and $\phi = \phi^*$. Replacing ϕ by the integral

$$\int_{\mathrm{G}} L(x)R(x)\phi\, dx$$

we may assume that ϕ is invariant under conjugation. Let

$$T\psi(x) = \int_{\mathrm{G}} \phi(x^{-1}y)\psi(y)\, dy\,.$$

Since the integral kernel $\phi(x^{-1}y)$ is continuous on $\mathrm{G} \times \mathrm{G}$, T is a compact operator on $\mathrm{L}^2(\mathrm{G})$. Furthermore, $T = T^* \neq 0$, because $\phi = \phi^*$. Hence, by Theorem 32, T has a non-zero finite dimensional eigenspace $\mathsf{V}_\lambda \subseteq \mathrm{L}^2(\mathrm{G})$. Since T commutes with $L(\mathrm{G})$, V_λ is closed under $L(\mathrm{G})$. Hence Theorem 50 shows that there is a G-irreducible subspace $\mathsf{W}_\lambda \subseteq \mathsf{V}_\lambda$. Let f_j be an orthonormal basis of W_λ. Set

$$h_{i,j}(x) = (L(x)f_i, f_j) = \int_{\mathrm{G}} f_i(x^{-1}y)\overline{f_j(y)}\, dy\,.$$

This is a matrix coefficient of an irreducible representation of G, thus it belongs to U. Therefore,

$$\begin{aligned}
0 = (\phi, h_{i,i}) &= \int_{\mathrm{G}}\int_{\mathrm{G}} \phi(x)\overline{f_i(x^{-1}y)}f_i(y)\, dy\, dx \\
&= \int_{\mathrm{G}}\int_{\mathrm{G}} \phi(yx^{-1})\overline{f_i(x)}f_i(y)\, dy\, dx = \int_{\mathrm{G}}\int_{\mathrm{G}} \phi(x^{-1}y)f_i(y)\, dy\, \overline{f_i(x)}\, dx \\
&= \int_{\mathrm{G}} Tf_i(x)\, \overline{f_i(x)}\, dx = \lambda \int_{\mathrm{G}} f_i(x)\overline{f_i(y)}\, dx\,,
\end{aligned}$$

which is a contradiction. □

Theorem 54. *Any unitary representation (ρ, W) of* G *is completely reducible, i.e. the Hilbert space* W *is the orthogonal sum of irreducible finite dimensional representations of* G.

Proof. We follow [Kna86, Theorem 1.12]. Suppose (ρ, W) is not completely reducible. By Zorn's Lemma we may choose a maximal orthogonal set of finite dimensional irreducible invariant subspaces. Let U denote the closure of their sum. Suppose $0 \neq v \in \mathrm{U}^\perp$. We'll arrive at a contradiction.

Let $\phi_n \in C_c(\mathrm{G})$ be an approximate identity, as in Theorem 43. Then, as we have seen before, there is n such that $\rho(\phi_n)v \neq 0$. We fix this n.

Theorem 53 a) implies that there is a finite set $F \subseteq \hat{\mathrm{G}}$ and $\phi \in \bigoplus_{\pi\in F} M_\pi(\mathrm{End}(\mathsf{V}))$, such that

$$\| \phi_n - \phi \|_2 \leq \frac{1}{2 \| v \|} \| \rho(\phi_n)v \| \ .$$

Since the total mass of G is 1, we have

$$\| \phi_n - \phi \|_1 \leq \| \phi_n - \phi \|_2 \ .$$

Hence

$$\| \rho(\phi_n)v - \rho(\phi)v \| \leq \| \phi_n - \phi \|_1 \| v \| \leq \frac{1}{2} \| \rho(\phi_n)v \| \ .$$

Therefore

$$\| \rho(\phi)v \| \geq \| \rho(\phi_n)v \| - \| \rho(\phi_n)v - \rho(\phi)v \| \geq \frac{1}{2} \| \rho(\phi_n)v \| > 0 \, .$$

This is a contradiction because $\rho(\phi)v$ lies in a finite dimensional invariant subspace of W. □

For a function $\phi \in \mathrm{L}^1(\mathrm{G})$ define the Fourier transform

$$\mathcal{F}\phi(\pi) = \int_{\mathrm{G}} \phi(x)\pi(x)\, dx = \pi(\phi) \, . \tag{25}$$

Notice that in order to be consistent with the theory of Fourier series, section 4, we should replace π by π^c in this definition. However we shall follow the tradition and not do that.

Thus for a representation (π, V), $\mathcal{F}\phi(\pi) \in \mathrm{End}(\mathsf{V})$. Notice that

$$\pi(g)\mathcal{F}\phi(\pi) = \mathcal{F}(L(g)\phi) \qquad (g \in \mathrm{G}) \, . \tag{26}$$

Set $d(\pi) = \dim \mathsf{V}) = \Theta_\pi(1)$. This is the degree of the representation (π, V).

Theorem 55. *(Fourier inversion for* G*) For any $\phi \in \bigoplus_{\pi \in \hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$ (algebraic sum),*

$$\phi(g) = \sum_{\pi \in \hat{\mathrm{G}}} d(\pi) \cdot tr(\mathcal{F}\phi(\pi)\pi(g^{-1})) \qquad (g \in \mathrm{G}) \, , \tag{27}$$

or equivalently

$$\phi(1) = \sum_{\pi \in \hat{\mathrm{G}}} d(\pi) \cdot \Theta_\pi(\phi) \qquad (g \in \mathrm{G}) \, , \tag{28}$$

Notice that in the case of the classical Fourier series, section 4, the above formula (27) refers only to trigonometric polynomials, ϕ.

Proof. Clearly (28) is a particular case of (27). Also, (27) follows from (28) and (26).

Let us write (π, V_π) for (π, V) in order to indicate the dependence of the vector space on π. By definition, there is a finite set $F \subseteq \hat{\mathrm{G}}$ and operators $T_\pi \in \mathrm{End}(\mathsf{V}_\pi)$ such that

$$\phi(g) = \sum_{\rho \in F} \mathrm{tr}(T_\rho \rho(g)) \qquad (g \in \mathrm{G}) \, .$$

Hence

$$\mathcal{F}\phi(\pi) = \sum_{\rho \in F} \int_{\mathrm{G}} \mathrm{tr}(T_\rho \rho(g))\pi(g)\, dg \, .$$

Therefore, by Theorem 52

$$\operatorname{tr}\mathcal{F}\phi(\pi) = \sum_{\rho\in F}\int_{\mathrm{G}} \operatorname{tr}(T_\rho\rho(g))\operatorname{tr}\pi(g)\,dg = \int_{\mathrm{G}} \operatorname{tr}(T_\pi\pi(g))\operatorname{tr}\pi(g)\,dg$$

$$= \quad (T_\pi\pi(g)),\pi(g)) = \frac{1}{d_\pi}\operatorname{tr}(T_\pi\pi(g)\pi(g)^*) = \frac{1}{d_\pi}\operatorname{tr}(T_\pi)\,.$$

Hence

$$\sum_{\pi\in\hat{\mathrm{G}}} d(\pi)\cdot\Theta_\pi(\phi) = \sum_{\pi\in\hat{\mathrm{G}}} d(\pi)\cdot\operatorname{tr}\mathcal{F}\phi(\pi) = \sum_{\pi\in\hat{\mathrm{G}}} tr(T_\pi) = \phi(1).$$

□

Theorem 56. *(Parseval formula) For any* $\phi\in\bigoplus_{\pi\in\hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$,

$$\|\,\phi\,\|_2^2 = \sum_{\pi\in\hat{\mathrm{G}}} d_\pi\ \|\,\mathcal{F}\phi(\pi)\,\|^2\ .$$

Proof. That the left hand side is equal to

$$\phi*\phi^*(1) = \sum_{\pi\in\hat{\mathrm{G}}} d_{\pi^c}\,tr(\mathcal{F}\phi(\pi)\mathcal{F}\phi(\pi)^*),$$

which coincides with the right hand side. □

10.1. **Unique factorization domains and Hilbert's Nulstellensatz.** This section is included to explain Hilbert's Nullstellensatz, to be used in the following section.

A ring R, with an identity, is called a Unique Factorization Domain (UFD) if and only if R has no zero divisors and every element of R decomposes into a product of indecomposible elements uniquelly up a permutation of factors and multiplication by units. In particular, any finite subset of an UFD has the greatest common divisor, and any indecomposable element of an UFD is prime.

Proposition 57. *Let F be a field. Then the ring $F[X]$ is a UFD.*

Proof. Euclid's algorithm shows that $R = F[X]$ is a Principal Ideal Domain. Hence, if $a, b\in R$, then the ideal generated by a and b,

$$(a, b) = (c)$$

for some $c\in R$.

In fact c is a greatest common divisor of a and b, denoted $gcd(a, b)$. Indeed, since $a\in(c)$ and $b\in(c)$, we see that c divides a and b. Hence c divides $gcd(a, b)$. On the other hand,

$$c = a'a + b'b$$

for some $a', b'\in R$. Thus $gcd(a, b)$ divides c.

If $p\in R$ is indecomposable, then p is prime. Indeed, suppose $a, b\in R$ are such that p divides ab but does not divide a. Since p is indecomposable we see that $gcd(p, a) = 1$. Hence, there are $a'\in R$ and $p'\in R$ such that

$$1 = a'a + p'p.$$

Therefore

$$b = a'(ab) + p'pb.$$

Thus p divides b.

By looking at the degrees of the polynomials we see that any element $a \in R$ decomposes into a product of indecomposables. Suppose we have two such decompositions

$$a = p_1 p_2 ... p_r = q_1 q_2 ... q_s.$$

Since p_1 is prime, it divides one of the q_i. We may assume that it divides q_1. Hence there is a unit u_1 such that $q_1 = p_1 u_1$. Since R has no zero divisors we see that

$$p_2 ... p_r = u_1 q_2 ... q_s.$$

Repeating this process we see that the decomposition is unique, as stated. □

Theorem 58. *If R is a UFD then $R[x]$ is a UFD.*

Proof. A polynomial in $R[x]$ is called primitive if the greatest common divisor of its coefficients is 1. In particular any primitive element of $R[x]$ is indecomposable if and only if it cannot be written as the product of two elements of positive degree. Also, any element of $R[x]$ is the product of an element of R and a primitive polynomial.

Also, it is clear that any indecomposable element of positive degree in $R[x]$ is the product of a prime in R and an indecomposable primitive element of the same degree.

Since any element of $R[x]$ is the product of indecomposables we'll be done as soon as we show that the decomposition is unique (up to a permutation and multiplication by units). Let F be the field of fractions of R. Then

$$R[x] \subseteq F[x].$$

Hence, by Proposition 57, the only fact we need to verify is that a primitive indecomposable element $f \in R[x]$ of positive degree is also indecomposable in $F[x]$.

Suppose, there are $g_F, h_F \in F[x]$, of positive degree, such that

$$f = g_F h_F.$$

There is $b \in R$ such that $bg_F \in R[x]$. Also, there is $a \in R$ and a primitive $g \in R[x]$, of the same degree as g_F, such that

$$bg_F = ag.$$

Hence,

$$f = g\left(\frac{a}{b} h_F\right).$$

There is $c \in R$ such that

$$c\left(\frac{a}{b} h_F\right) \in R[x].$$

If c is a unit in R then we see that f decomposes in $R[x]$, a contradiction.

Suppose c is not a unit. We see that g divides cf in $R[x]$. Take c with minimal possible number of primes in it such that g divides cf in $R[x]$. Thus there is $h \in R[x]$ such that

$$cf = gh.$$

Let $p \in R$ be a prime which divides c.

Since p divides the left hand side, it divides the right hand side. Since g is primitive, p divides h in $R[x]$. Thus

$$\frac{c}{p} f = g\left(\frac{1}{p} h\right),$$

where $\left(\frac{1}{p}h\right) \in R[x]$, a contradiction. □

Now we turn to Hilbert's Nullstellensatz. We present an argument made by Daniel Allcock, which according to him goes back to Zariski.

Theorem 59. *Let k be a field and K a field extension which is finitely generated as a k-algebra. Then K is algebraic over k.*

Proof. We will assume throughout that K is transcendental over k and finitely generated as a k-algebra, and deduce that K is not finitely generated as a k-algebra, a contradiction.

Suppose first that K has transcendence degree one; this means that it contains a subfield $k(x)$ which is a copy of the one-variable rational function field, and that K is algebraic over $k(x)$. This, together with the finite generation of K, shows that K has finite dimension as a k(x)-vector space. Choose a basis $e_1, ..., e_l$ and write down the multiplication table for K:

$$e_i e_j = \sum_k \frac{a_{ijk}(x)}{b_{ijk}(x)} e_k,$$

with the a's and b's in $k[x]$. We will show that for any $f_1, ..., f_m \in K$, the k-algebra A they generate is smaller than K.

It is convenient to adjoin $f_0 = 1$ as a generator. Express $f_0, ..., f_m$ in terms of our basis:

$$f_i = \sum_j \frac{c_{ij}(x)}{d_{ij}(x)} e_j,$$

with the c's and d's in $k[x]$. Now, an element a of A is a k-linear combination of $f_0 = 1$ and products of $f_1, ..., f_m$. Expanding in terms of our basis, we see that a is a $k(x)$-linear combination of products of the e_i, with the special property that the denominators of the coefficients involve only the d's. Using the multiplication table repeatedly, we see that a is a $k(x)$-linear combination of the e_i, whose coefficients' denominators involve only the b's and d's.

A precise way to state the result of this argument is: when a is expressed as a $k(x)$-linear combination of the e_i, with every coefficient in lowest terms, then all its coefficients' denominators' irreducible factors are among the irreducible factors of the b's and d's. Therefore

$$\frac{1}{\text{some other irreducible polynomial}}$$

cannot lie in A, and A is smaller than K.

This argument requires the existence of infinitely many irreducible polynomials in $k[x]$; to prove this one can mimic Euclid's proof of the infinitude of primes in $\mathbb{Z}$. (If k is infinite then one can just take the infinitely many linear polynomials $x - c$, $c \in k$.)

Now suppose K has transcendence degree > 1 over k, and choose a subextension F over which K has transcendence degree 1. By the above, K is not finitely generated as a F-algebra, so it isn't as a k-algebra either. (To build F explicitly, choose k-algebra generators $x_1, ..., x_n$ for K over k and set $F = k(x_1, ..., x_{l-1})$, where x_l is the last of the x's which is transcendental over the field generated by its predecessors.) □

Theorem 60. *(The 'Weak' Nullstellensatz'.)*

Let k be an algebraically closed field. Then every maximal ideal in the polynomial ring $R = k[x_1, ..., x_n]$ has the form $(x_1 - a_1, ..., x_n - a_n)$ for some $a_1, ..., a_n \in k$. As a consequence, a family of polynomial functions on k_n with no common zeros generates the unit ideal of R, i.e. R.

Proof. If m is a maximal ideal of R then R/m is a field which is finitely generated as a k-algebra. By the previous theorem it is an algebraic extension of k, hence equal to k. Therefore each x_i maps to some $a_i \in k$ under the natural map $R \to R/m = k$, so m contains the ideal $(x_1 - a_1, ..., x_n - a_n)$. This is a maximal ideal, so it equals m.

To see the second statement, consider the ideal generated by some given polynomial functions with no common zeros. If it lay in some maximal ideal, say $(x_1 - a_1, ..., x_n - a_n)$, then all the functions would vanish at $(a_1, ..., a_n) \in k_n$, contrary to hypothesis. Since it doesn't lie in any maximal ideal, it must be all of R. □

Theorem 61. *(Nullstellensatz.)*

Suppose k is an algebraically closed field and g and $f_1, ..., f_m$ are members of $R = k[x_1, ..., x_n]$, regarded as polynomial functions on k^n. If g vanishes on the common zero-locus of the f_i, then some power of g lies in the ideal they generate.

Proof. The polynomials $f_1, ..., f_m$ and $x_{n+1}g - 1$ have no common zeros in k^{n+1}, so by the weak Nullstellensatz we can write $1 = p_1 f_1 + + p_m f_m + p_{m+1} \cdot (x_{n+1} g - 1)$,where the p's are polynomials in $x_1, ..., x_{n+1}$. Taking the image of this equation under the homomorphism $k[x_1, ..., x_{n+1}] \to k(x_1, ..., x_n)$ given by $x_{n+1} \to 1/g$, we find $1 = p_1(x_1, ..., x_n, 1/g) f_1 + + p_m(x_1, ..., x_n, 1/g) f_m$. After multiplying through by a power of g to clear denominators, we have Hilbert's theorem. □

10.2. **Spherical harmonics in $\mathbb{R}^n$.** The Laplacian in $\mathbb{R}^n$, $n \geq 2$, may be written as

$$\Delta_n = \partial_{x_1}^2 + \partial_{x_2}^2 + ... + \partial_{x_n}^2 = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2} L, \tag{29}$$

where $r^2 = x_1^2 + x_2^2 + ... + x_n^2$ and L is the Laplacian on the sphere S^{n-1}.

Lemma 62. *If ϕ is a smooth harmonic function on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree k, then $L\phi = -k(k + n - 2)\phi$.*

Proof. The assumption that ϕ is harmonic means that $\Delta_n \phi = 0$. Thus (29) shows that

$$0 = (\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2} L)\phi.$$

Therefore

$$\begin{aligned} L\phi &= -r^2\partial_r^2\phi-(n-1)r\partial_r\phi=-((r\partial_r)^2-r\partial_r)\phi-(n-1)r\partial_r\phi \\ &= -(r\partial_r)^2\phi-(n-2)r\partial_r\phi=-k^2\phi-(n-2)k\phi. \end{aligned}$$

□

Lemma 63. *Let $\mathcal{P}$ denote the space of the complex valued polynomial functions on $\mathbb{R}^n$. For $\phi,\psi\in\mathcal{P}$ define*

$$(\phi,\psi)=\phi(\partial_{x_1},\partial_{x_2},...,\partial_{x_n})\overline{\psi}(0).$$

Then $(\ ,\ )$ is a positive definite hermitian form on $\mathcal{P}$. Moreover

$$(\phi,\psi\eta)=(\partial(\overline{\psi})\phi,\eta)\qquad(\phi,\psi,\eta\in\mathcal{P}).$$

Proof. In the multi-index notation, let

$$\phi(x)=\sum_\alpha a_\alpha x^\alpha,\psi(x)=\sum_\beta b_\beta x^\beta.$$

Then

$$(\phi,\psi)=\sum_{\alpha,\beta}a_\alpha\overline{b_\beta}\partial^\alpha x^\beta|_{x=0}=\sum_\alpha a_\alpha\overline{b_\alpha}\,\alpha!.$$

Hence the form $(\ ,\ )$ is hermitian and positive definite. Furthermore,

$$\begin{aligned} &(\phi,\psi\eta)=\overline{(\psi\eta,\phi)}=\overline{\partial(\psi)\partial(\eta)\overline{\phi}(0)} \\ &\overline{\partial(\eta)\partial(\psi)\overline{\phi}(0)}=\overline{\partial(\eta)\overline{\partial(\overline{\psi})\phi}(0)}=\overline{(\eta,\partial(\overline{\psi})\phi)}=(\partial(\overline{\psi})\phi,\eta). \end{aligned}$$

□

Lemma 64. *Let $\mathcal{P}_k\subseteq\mathcal{P}$ be the subspace of the polynomials of degree k and let $H_k\subseteq\mathcal{P}_k$ be the subspace of the harmonic polynomials. Then we have the following direct sum orthogonal decomposition:*

$$\mathcal{P}_k=H_k+r^2H_{k-2}+r^4H_{k-4}+...+r^{2m}H_{k-2m},$$

where m is the largest integer such that $2m\leq k$.

Proof. Let $\phi\in\mathcal{P}_k$. Notice that $\phi\in H_k$ iff

$$(\Delta_n\phi,\psi)=0 \text{ for all } \psi\in\mathcal{P}_{k-2},$$

or equivalently, by Lemma 63,

$$(\phi,r^2\psi)=0 \text{ for all } \psi\in\mathcal{P}_{k-2}.$$

Hence

$$H_k=(r^2\mathcal{P}_{k-2})^\perp\cap\mathcal{P}_k.$$

Therefore

$$\mathcal{P}_k=H_k+r^2\mathcal{P}_{k-2}=H_k+r^2H_{k-2}+r^4\mathcal{P}_{k-2}=...$$

□

Theorem 65. *Explicitly, for $k \geq 2$,*

$$H_k = span\{(c_1x_1 + c_2x_2 + ... + c_nx_n)^k;\ c_1^2 + c_2^2 + ... + c_n^2 = 0\}.$$

Proof. It is easy to check that the right hand side of the above equation is contained in the left hand side.

Consider an element $\phi \in H_k$ which is orthogonal to the space on the right hand side. Then

$$\begin{aligned} 0 &= (\phi, (c_1x_1 + ... + c_nx_n)^k) = \partial(\phi)(\overline{c}_1x_1 + ... + \overline{c}_nx_n)^k(0) \\ &= \partial(\phi)\sum_\beta \frac{k!}{\beta_1!...\beta_n!}\overline{c}_\beta x^\beta|_{x=0} = \sum_\alpha \phi_\alpha \overline{c}_\alpha k! = k!\,\phi(\overline{c}). \end{aligned}$$

Thus ϕ vanishes on the variety

$$\mathcal{V} = \{c \in \mathbb{C}^n;\ c_1^2 + ... + c_n^2 = 0\}. \tag{30}$$

Let $I = r^2\mathcal{P}$. This is an ideal in $\mathcal{P}$. Recall Hilbert's Nullstellenstatz (Theorem 61):

$$Ideal(Variety(Ideal)) = \sqrt{Ideal}.$$

Since for our ideal I and our variety $\mathcal{V}$,

$$Variety(I) = \mathcal{V},$$

we see that $\phi \in \sqrt{I}$. Thus there is $p = 1, 2, 3...$ such that

$$\phi^p \in r^2\mathcal{P}.$$

Suppose $n \geq 3$. Then a simple computation using the homogeneity of r^2 shows that r^2 is an irreducible polynomial. Since $\mathcal{P}$ is a Unique Factorization Domain (see Theorem **??**), r^2 is prime and therefore

$$\phi \in r^2\mathcal{P}.$$

Hence,

$$\phi \in r^2\mathcal{P}_{k-2}.$$

Lemma 64 shows that $\phi = 0$. Thus the equality follows.

Suppose $n = 2$. Then $Re\,\phi$ is a harmonic polynomial. Hence there is a holomorphic function $\psi(x_1 + ix_2)$, in fact a polynomial in $x_1 + ix_2$, such that $Re\,\phi(x_1, x_2) = Re\,\psi(x_1 + ix_2)$. Thus

$$Re\,\phi(x_1, x_2) = \frac{1}{2}(\psi(x_1 + ix_2) + \overline{\psi(x_1 + ix_2)}),$$

which is in the desired space. Similarly we deal with the imaginary part. □

Lemma 66. *Let* $\mathrm{U}_k = H_k|_{S^{n-1}}$*. Then* $L^2(S^{n-1}) = \sum_{k=0}^\infty \mathrm{U}_k$*.*

Proof. The algebra $\mathcal{P}|_{S^{n-1}}$ is closed under conjugation and separates points of the sphere. Hence, the algebraic direct sum of the U_k is dense in $C(S^{n-1})$, and therefore also in $\mathrm{L}^2(S^{n-1})$. Furthermore, by Lemma 62, U_k is the $-k(n + k - 2)$-eigensubspace for the action of the selfadjoint operator L. Since all these eigenvalues are different the subspaces are orthogonal. □

(Notice that the hermitian form on the space of the polynomials defined in Lemma 63 is not the same as the one given by the restriction to the sphere and integration there. Indeed, the homogeneous polynomials of different homogeneity degrees are orthogonal with respect to the hermitian form defined in Lemma 63, but they may have the same restriction to the unit sphere ($r^2|_{S^{n-1}} = 1$).)

Define a representation ρ of the group $\mathrm{O}(n)$ on the Hilbert space $\mathrm{L}^2(S^{n-1})$ by the formula

$$\rho(g)\phi(\sigma) = \phi(g^{-1}\sigma) \qquad (g \in \mathrm{O}(n),\ \sigma \in S^{n-1},\ \phi \in \mathrm{L}^2(S^{n-1})). \tag{31}$$

Theorem 67. *For any $k = 0, 1, 2, \ldots$, the restriction of ρ to U_k is an irreducible representation of $\mathrm{O}(n)$.*

Proof. Let $e_n = (0, 0, \ldots, 1) \in S^{n-1}$ be the north pole. Since the evaluation at e_n is a continuous linear functional on the finite dimensional space U_k, there is $\phi \in \mathrm{U}_k$ such that

$$\psi(e_n) = \int_{S^{n-1}} \psi(\sigma)\phi(\sigma)\, d\mu(\sigma) \qquad (\psi \in \mathrm{U}_k).$$

Let $\mathrm{O}(n-1) \subseteq \mathrm{O}(n)$ be the stabilizer of e_n. Clearly ϕ is $\mathrm{O}(n-1)$-invariant. But ϕ, or rather its k-homogeneous extension to $\mathbb{R}^n$, is harmonic.

Consider an arbitrary $\mathrm{O}(n-1)$-invariant function $\eta \in H_k$. Since an O_{n-1}-invariant polynomial on $\mathbb{R}^{n-1}$ is a polynomial in the variable $r_{n-1}^2 = x_1^2 + x_2^2 + \ldots + x_{n-1}^2$,

$$\eta(x) = \sum_{j=0}^{m} c_j x_n^{k-2j} r_{n-1}^{2j},$$

where m is the greatest integer $\leq \frac{k}{2}$. Furthermore,

$$\begin{aligned}
0 &= \sum_{0 \leq j \leq \frac{k}{2}} c_j(\partial_{x_n}^2 (x_n^{k-2j} r_{n-1}^{2j}) + x_n^{k-2j}\Delta_{n-1} r_{n-1}^{2j}) \\
&= \sum_{0 \leq j \leq \frac{k-2}{2}} c_j(k-2j)(k-2j+1) x_n^{k-2j-2} r_{n-1}^{2j} + \sum_{1 \leq j \leq \frac{k}{2}} c_j 2j(n+2j-2) x_n^{2k-2j} r_{n-1}^{2j-2} \\
&= \sum_{1 \leq j \leq \frac{k}{2}} (c_{j-1}(k-2j+2)(k-2j+1) + c_j 2j(n+2j-2)) x_n^{k-2j} r_{n-1}^{2j}.
\end{aligned}$$

Therefore,

$$c_j = -c_{j-1}\frac{(k-2j+2)(k-2j+1)}{2j(n+2j-2)} \qquad (1 \leq j \leq \frac{k}{2}).$$

We see that the space of the $\mathrm{O}(n-1)$-invariants in U_k is one-dimensional.

Let $\mathsf{V} \subseteq \mathrm{U}_k$ be a non-zero $\mathrm{O}(n)$-invariant subspace. Then there is $\psi \in \mathsf{V}$ such that $\psi(e_n) \neq 0$. Let

$$\tilde{\psi}(\sigma) = \int_{\mathrm{O}(n-1)} \psi(g\sigma)\, dg.$$

Then

$$\tilde{\psi}(e_n) = \int_{\mathrm{O}(n-1)} \psi(ge_n)\, dg = \int_{\mathrm{O}(n-1)} \psi(e_n)\, dg \neq 0.$$

Thus $\tilde{\psi}$ is a non-zero $\mathrm{O}(n-1)$-invariant in U_k. Since $\tilde{\psi} \in \mathsf{V}$ we see that $\phi \in \mathsf{V}$. Therefore ϕ is in every non-zero invariant subspace of U_k. This shows that U_k is irreducible. □

Lemma 68.

$$dim\, \mathcal{P}_k = \left(\begin{array}{c} n+k-1 \\ n-1 \end{array} \right) \tag{32}$$

Proof. The dimension is equal to the number of ways one may put k balls in n boxes. Imagine that these boxes are lined up in a row (along the x-axis). In front of each box line up all the balls that go into it (in the direction of the y-axis). Suppose these balls are blue. Insert a red ball between two consecutive boxes. We end up with $k + (n-1)$ balls, $n-1$ of them red. The number of such arrangements is (32). □

Corollary 69. *The space of the* $\mathrm{O}(n-1)$*-invariants in* U_k *is one-dimensional. In particular there is a unique function* $\phi_k \in \mathrm{U}_k$ *such that* $\phi_k(e_n) = 1$*. Furthermore,*

$$dim\, \mathrm{U}_k = \left(\begin{array}{c} n+k-1 \\ n-1 \end{array} \right) - \left(\begin{array}{c} n+k-3 \\ n-1 \end{array} \right).$$

11. Square integrable representations of a locally compact group

In this section G is a locally compact group. We follow [Wal88, section 1.3].

Lemma 70. *Let* (τ, V) *be an irreducible unitary representation of* G*. Suppose that for some non-zero vectors* $u_0, v_0 \in \mathsf{V}$

$$\int_{\mathrm{G}} |(\tau(x)u_0, v_0)|^2\, dx < \infty\,. \tag{33}$$

Then for arbitrary $u, v \in \mathsf{V}$*,*

$$\int_{\mathrm{G}} |(\tau(x)u, v)|^2\, dx < \infty\,. \tag{34}$$

Moreover, the map $T : \mathsf{V} \to \mathrm{L}^2(\mathrm{G})$ *defined by*

$$Tu(x) = (\tau(x)u, v_0) \qquad (u \in \mathsf{V},\ x \in \mathrm{G})$$

is G*-intertwining and has the property that there is* $t > 0$ *such that*

$$(Tu, Tv) = t(u, v) \qquad (u, v \in \mathsf{V})\,. \tag{35}$$

Proof. We may assume that $u_0, v_0 \in \mathsf{V}$ are unit vectors. Let W_0 be the linear span of all the vectors $\pi(g)u_0$, $g \in \mathrm{G}$. This is a G-invariant subspace of V. Since V is irreducible, W_0 is dense in V. Let

$$\mathsf{W} = \{u \in \mathsf{V};\ \int_{\mathrm{G}} |(\tau(x)u, v_0)|^2\, dx < \infty\}\,.$$

Then W is also a G-invariant subspace of V and W contains W_0. Hence W is dense in V.

Clearly

$$T\tau(g)u = R(g)Tu \qquad (g \in \mathrm{G})\,.$$

Define an inner product on W by

$$\langle w_1, w_2\rangle = (w_1, w_2) + (Tw_1, Tw_2) \qquad (w_1, w_2 \in \mathsf{W})\,.$$

One checks that W with this inner product is complete. (Every Cauchy sequence in W has a limit in W.) Thus (R, W) is a unitary representation of G.

The inclusion

$$I_{\mathsf{W}} : \mathsf{W} \ni w \to w \in \mathsf{V}$$

is a bounded linear map which intertwines the actions of G. The adjoint

$$I_{\mathsf{W}}^* : \mathsf{V} \to \mathsf{W}$$

is also a bounded linear map which intertwines the actions of G. Now we apply Theorem 41 with $A = I_{\mathsf{W}}^*$, $\mathsf{V}_0 = \mathsf{V}$, $B = I_{\mathsf{W}}$ and $\mathsf{V}_1 = \mathsf{W}$. The conclusion is that I_{W}^* is a scalar multiple of the identity. Hence $\mathsf{W} = \mathsf{V}$. Since any two elements $u, v \in \mathsf{W}$ satisfy (34), the claim (34) follows.

Now V is equipped with two inner products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ preserved by G. Define a map $S : \mathsf{V} \to \mathsf{V}$ by

$$\langle u, v\rangle = (Su, v)\,.$$

Then S commutes with the action of G. Hence $S = sI$ for some $s \in \mathbb{C}$. Taking $u = v \neq 0$ we see that $s > 0$. This implies (35). $\square$

An irreducible unitary representation of G is called square integrable if one, or equivalently all, its matrix coefficients are square integrable.

Proposition 71. *Let (σ, U) and (τ, V) be irreducible unitary square integrable representations of* G. *Then*

$$\int_{\mathrm{G}} (\sigma(x)u_1, u_2)(v_2, \tau(x)v_1)\, dx = 0 \quad (u_1, u_2 \in \mathrm{U}\,,\ v_1, v_2 \in \mathsf{V}) \ \ \textit{if} \ \ \sigma \not\simeq \tau \tag{36}$$

There is a positive number $d(\tau)$ such that

$$\int_{\mathrm{G}} (\tau(x)u_1, u_2)(v_2, \tau(x)v_1)\, dx = \frac{1}{d(\tau)}(u_1, v_1)(v_2, u_2) \qquad (u_1, u_2, v_1, v_2 \in \mathsf{V})\,. \tag{37}$$

Proof. Define

$$Su(x) = (\sigma(x)u, u_2) \qquad (u \in \mathrm{U},\ x \in \mathrm{G})$$

and

$$Tv(x) = (\tau(x)v, v_2) \qquad (v \in \mathsf{V},\ x \in \mathrm{G})\,.$$

Then (35) shows that there is a positive constant C such that for $u \in \mathrm{U}$ and $v \in \mathsf{V}$

$$|(Su, Tv)| \leq \| Su \|_2 \| Tv \|_2 \leq C \| u \| \| v \|\ .$$

Hence there is a bounded G-intertwinig linear map $A : \mathsf{V} \to \mathrm{U}$ such that

$$(Su, Tv) = (u, Av) \qquad (u \in \mathrm{U}, v \in \mathsf{V})\,.$$

This map is zero if the representations are not equivalent. Since

$$\int_{\mathrm{G}} (\sigma(x)u_1, u_2)(v_2, \tau(x)v_1)\, dx = (Su_1, Tv_1) = (u_1, Av_1)\,,$$

(36) follows. Suppose $\sigma = \tau$. In this case let us write T_{v_2} for T and T_{u_2} for S. Then

$$(T_{u_2}u, T_{v_2}v) = (Su, Tv) = (u, Av) \qquad (u, v \in \mathsf{V})\,,$$

where A is a constant because the representation is irreducible. Thus $A = aI$ for some $a \in \mathbb{C}$. This constant $a = a_{u_2,v_2}$ depends continuously and anti-linearly on u_2 and linearly v_2. Thus there is a constant b such that

$$a_{u_2,v_2} = b(v_2, u_2) \qquad (u_2, v_2 \in \mathsf{V})\,.$$

By taking $u_2 = v_2$ we see that $b > 0$.

$$\int_{\mathrm{G}} (\tau(x)u_1, u_2)(v_2, \tau(x)v_1)\, dx = \overline{a_{u_2,v_2}}(u_1, v_1) = b(u_1, v_1)(v_2, u_2)\,.$$

□

The constant $d(\tau)$ is called the formal degree of τ. It depends on the choice of the Haar measure on G. If G is compact with the Haar measure of total mass 1 then, as we have seen in Theorem 52, $d(\tau) = \dim \mathsf{V}$. Hence the name.

12. $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$

12.1. The finite dimensional unitary representations.

Theorem 72. *The only finite dimensional unitary representation of the group* $\mathrm{SL}_2(\mathbb{R})$ *is the trivial representation.*

Proof. Let (ρ, V) be a finite dimensional representation of the group $\mathrm{SL}_2(\mathbb{R})$. We shall work under the additional assumption that the group homomorphism

$$\mathrm{SL}_2(\mathbb{R}) \ni g \to \rho(g) \in \mathrm{GL}(\mathsf{V})$$

is a polynomial map. We shall see later in these notes that this assumption is not necessary. Then the above map extends to a group homomorphism

$$\mathrm{SL}_2(\mathbb{C}) \ni g \to \rho(g) \in \mathrm{GL}(\mathsf{V})\,.$$

In particular for any $X \in \mathfrak{sl}_2(\mathbb{C})$ we have

$$d\rho(X)v = \frac{d}{dt}\rho(\exp(tX))v|_{t=0}\,.$$

This way we obtain a $\mathbb{C}$-linear map

$$d\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(\mathsf{V})\,.$$

Since the subgroup $\mathrm{SU}_2 \subseteq \mathrm{SL}_2(\mathbb{C})$ is compact, there exists a positive definite hermitian form $(\cdot, \cdot)$ on V preserved by SU_2. Hence all the operators

$$d\rho(X)\,, \quad X \in \mathfrak{su}_2$$

are skew-hermitian and therefore have imaginary eigenvalues.

On the other hand, suppose there exists an $\mathrm{SL}_2(\mathbb{R})$-invariant positive definite hermitian form $\langle\cdot,\cdot\rangle$ on V. Then all the operators

$$d\rho(X)\,,\quad X\in\mathfrak{sl}_2(\mathbb{R})$$

are skew-hermitian and therefore have imaginary eigenvalues. Define

$$\mathfrak{p}=i\mathfrak{su}_2\cap\mathfrak{sl}_2(\mathbb{R})\,.$$

Then the operators

$$d\rho(X)\,,\quad X\in\mathfrak{p}$$

are skew-hermitian and therefore have imaginary eigenvalues. But $d\rho(iX)=id\rho(X)$. Hence the operators

$$d\rho(X)\,,\quad X\in\mathfrak{p}$$

are both hermitian and skew-hermitian. Thus

$$d\rho(\mathfrak{p})=\{0\}\,.$$

It is easy to check that the commutator

$$[\mathfrak{p},\mathfrak{p}]=\mathfrak{so}_2\,.$$

But

$$\mathfrak{sl}_2(\mathbb{R})=\mathfrak{so}_2+\mathfrak{p}\,.$$

Therefore

$$d\rho(\mathfrak{sl}_2(\mathbb{R}))=\{0\}\,.$$

Hence

$$\rho(\exp(\mathfrak{sl}_2(\mathbb{R})))=\{I\}\,.$$

But since the group $\mathrm{SL}_2(\mathbb{R})$ is connected it is generated by $\exp(\mathfrak{sl}_2(\mathbb{R}))$. Thus the representation is trivial. □

This is a particular case of a general theorem of Segal and von Neumann, [SvN50]. Define the following subgroups of G:

$$\begin{aligned}
\mathrm{K} &= \left\{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix};\ \theta\in\mathbb{R}\right\},\\
\mathrm{A} &= \left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}, a>0\right\},\\
\mathrm{N} &= \left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}, n\in\mathbb{R}\right\}.
\end{aligned}$$

The following map, is a bijective diffeomorphism

$$\mathrm{A}\times\mathrm{N}\times\mathrm{K}\ni(a,n,k)\to ank\in\mathrm{G}\,. \tag{38}$$

In particular $\mathrm{G}=\mathrm{ANK}$. This is called the Iwasawa decomposition of G.

Problem 14. *Prove that the map* (38) *is bijective.*

This is straightforward.

Let S be the subset of G consisting of the symmetric and positive definite matrices. Then

$$\mathrm{G} = \mathrm{KS}\,. \tag{39}$$

This is the polar decomposition of G.

Problem 15. *Prove that the map*

$$\mathrm{K} \times \mathrm{S} \ni (k, s) \to ks \in \mathrm{G} \tag{40}$$

is a surjective.

This follows from Spectral Theorem.

Problem 16. *Prove that the map*

$$\mathrm{K} \times \mathrm{A} \times \mathrm{K} \ni (k, n, k) \to kak \in \mathrm{G} \tag{41}$$

is a surjective. In particular $\mathrm{G} = \mathrm{KAK}$. *This is called the Cartan decomposition of* G.

This is immediate from (40) and Spectral Theorem.

12.2. **The maximal compact subgroup** $\mathrm{K} = \mathrm{SO}_2(\mathbb{R})$**.**

We let

$$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and define

$$\mathrm{K} = \mathrm{SO}_2(\mathbb{R}) = \{k(\theta);\ \theta \in \mathbb{R}\}\,.$$

This is a maximal compact subgroup of G, which is unique up to conjugation. The group K is commutative. Define the characters

$$\chi_n(k(\theta)) = e^{in\theta} \qquad (\theta \in \mathbb{R}, n \in \mathbb{Z})\,.$$

For two integers n and m define

$$S_{n,m} = \{f \in C_c(\mathrm{G});\ f(k(\theta_1)gk(\theta_2) = e^{-in\theta_1} f(g) e^{-im\theta_2}\}\,.$$

Lemma 73. *The algebraic sum* $\bigoplus_{n,m} S_{n,m}$ *is* L^1*-dense in* $C_c(\mathrm{G})$*. In fact, given* $\epsilon > 0$ *and* $f \in C_c(\mathrm{G})$*, there exists a function* $g \in \bigoplus_{n,m} S_{n,m}$ *such that the support of* g *is contained in* $\mathrm{K}(\operatorname{supp} f)\mathrm{K}$*, and such that* $\| f - g \|_\infty < \epsilon$*..*

Proof. This follows from the properties of the Fourier series. See [Lan85, page 20]. □

Lemma 74. *The following formulas hold.*

$$\begin{aligned} S_{n,m} * S_{l,q} &= \{0\} \ \ \textit{if} \ \ m \neq l\,, \\ S_{n,m}^* &= S_{m,n}\,, \\ S_{n,m} * S_{m,q} &\subseteq S_{n,q}\,. \end{aligned}$$

Proof. This is straightforward [Lan85, page 22]. □

Lemma 75. *Let* $\mathrm{S} = \{x \in \mathrm{G};\ x = x^t\}$*. Then the map*

$$\mathrm{K} \times \mathrm{S} \ni (k, s) \to ks \in \mathrm{G}$$

is a bijective diffeomrophism.

Proof. This is well known from Linear Algebra. See straightforward [Lan85, page 22]. □

Lemma 76. *The algebra* $S_{0,0}$ *is commutative.*

Proof. The argument is due to I. M. Gelfand. Consider an element $x \in \mathrm{G}$. Then there is $k \in \mathrm{K}$ and $s \in \mathrm{S}$ such that $x = ks$. Hence the transpose

$$x^t = k^t s^t = k^{-1} s = k^{-1} x k^{-1} \,. \tag{42}$$

Also, the group G is unimodular. Hence the Haar measure is invariant under the change of variables $x \to x^t$. Let $f, g \in S_{0,0}$. Set $f^t(x) = f(x^t)$. It is easy to check that

$$(f * g)^t = g^t * f^t \,.$$

Notice that $f^t = f$. Indeed,

$$f^t(x) = f(x^t) = f(k^{-1} x k^{-1}) = f(x) \,,$$

because elements of $S_{0,0}$ are K-bi-invariant. Hence

$$g^t * f^t = g * f \,.$$

By Lemma 74, $f * g \in\in S_{0,0}$. Hence $(f * g)^t = f * g$. The conclusion follows. □

Lemma 77. *For any* $n \in \mathbb{Z}$*, the algebra* $S_{n,n}$ *is commutative.*

Proof. The argument is due to S. Lang, see [Lan85]. Let

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Clearly $\gamma = \gamma^{-1}$. Let $s = s^t \in \mathrm{G}$. Then there is $k \in \mathrm{K}$ such that $ksk^{-1} = d$ is diagonal. Therefore

$$\gamma s \gamma = \gamma k^{-1} d k \gamma = \gamma k^{-1} \gamma \gamma d \gamma \gamma k \gamma = \gamma k^{-1} \gamma d \gamma k \gamma \,.$$

Thus there is $k_s \in \mathrm{K}$ such that

$$\gamma s \gamma = k_s^{-1} s k_s \,.$$

Notice that

$$\gamma k \gamma = k^{-1} \qquad (k \in \mathrm{K}) \,.$$

Recall that $f^t(x) = f(x^t)$ and let $f^\gamma(x) = f(\gamma x \gamma)$. Then, for $f \in S_{n,n}$

$$f^\gamma(x) = f^t(x)$$

Indeed, write $x = ks$. Then

$$\begin{aligned} f^\gamma(ks) &= f(\gamma k s \gamma) = f(k^{-1} \gamma s \gamma) = \chi_n(k) f(\gamma s \gamma) \\ &= \chi_n(k) f(k_s^{-1} s k_s) = \chi_n(k) \chi_n(k_s) f(s) \chi_n(k_s)^{-1} \\ &= \chi_n(k) f(s) = f(s) \chi_n(k) = f(s k^{-1}) = f((ks)^t) \\ &= f^t(ks) \,. \end{aligned}$$

Therefore for any $f, g \in S_{n,n}$,

$$f^\gamma * g^\gamma = f^t * g^t = (g * f)^t = (g * f)^\gamma = g^\gamma * f^\gamma$$

and we are done. □

For a representation (π, V) of G on a Banach space V define

$$\mathsf{V}_n = \{v \in \mathsf{V};\ \pi(k(\theta))v = e^{in\theta}v, \theta \in \mathbb{R}\}.$$

This is the χ_n isotypic component of V.

Lemma 78. *Let (π, V) be a representation of* G *on a Banach space V. Then*

$$\pi(S_{n,m})\mathsf{V} \subseteq \mathsf{V}_n$$

and for $q \neq m$,

$$\pi(S_{n,m})\mathsf{V}_q = 0.$$

Proof. See [Lan85, page 23]. □

Lemma 79. *Let (π, V) be an irreducible representation of* G *on a Banach space V. Then for any q such that $\mathsf{V}_q \neq 0$, the space V_q is $S_{q,q}$-irreducible. Also, $\pi(S_{q,q})\mathsf{V}_q \neq 0$.*

Proof. This follows from the fact that the $*$ algebra

$$\bigoplus_{n,m} S_{n,m}$$

is L^1-dense in $C_c(\mathrm{G})$ see [Lan85, page 24]. □

Theorem 80. *Let (π, V) be an irreducible unitary representation of* G *on a Hilbert space V. Fix an integer n. Then $\dim \mathsf{V}_n = 1$ or 0.*

Proof. In this case $\pi(S_{n,n})$ is a commutative $*$ algebra. Hence the claim follows from Theorem 34. □

From now on we consider only the representations for which $\dim \mathsf{V}_n \leq 1$ for all $n \in \mathbb{Z}$.

Theorem 81. *Let (π, V) be an irreducible representation of* G *on a Banach space V. Then the space of finite sums*

$$\bigoplus_{n \in \mathbb{Z}} \mathsf{V}_n \subseteq \mathsf{V}$$

is dense.

If (π, V) be an irreducible unitary representation of G *on a Hilbert space space V, then*

$$\mathsf{V} = \sum_{n \in \mathbb{Z}} \mathsf{V}_n$$

is a Hilbert space direct sum orthogonal decomposition.

Proof. This follows from the fact that the $*$ algebra

$$\bigoplus_{n,m} S_{n,m}$$

is dense in $C_c(\mathrm{G})$ see [Lan85, page 25]. □

12.3. **Induced representations.** Let us fix Haar measures on these groups

$$d\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{2\pi}d\theta\,, \quad d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{da}{a}\,, \quad d\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = dn\,.$$

Then, in terms of the Iwasawa decomposition, the formula

$$dx = da\,dn\,dk \qquad (x = ank,\ a \in \mathrm{A}, \mathfrak{n} \in \mathrm{N}, k \in \mathrm{K})$$

defines a Haar measure on G. Furthermore

$$\alpha\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^2\,, \quad \rho\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a\,.$$

Let P = AN. This is a subgroup of G with the modular function

$$\Delta\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) = a^2\,.$$

Theorem 82. *Let (σ, V) be a representation of* P *on a Hilbert space* V. *Denote by* $\mathsf{V}(\sigma)$ *the space of functions* $f : \mathrm{G} \to \mathsf{V}$ *such that*

$$f(px) = \Delta(p)^{\frac{1}{2}}\sigma(p)f(x) \qquad (x \in \mathrm{G})$$

whose restriction to K *is square integrable:*

$$\int_{\mathrm{K}} \parallel f(k) \parallel^2\ dk < \infty\,.$$

Set

$$\pi(g)f(x) = f(xg) \qquad (g, x \in \mathrm{G})$$

Then $(\pi, \mathsf{V}(\sigma))$ *is a representation of* G. *If* σ *is unitary then so is* π.

Proof. See [Lan85, Theorem 2, page 44]. □

Our main example is going to be the case when $\mathsf{V} = \mathbb{C}$ and for some fixed complex number s,

$$\sigma(an) = \rho(a)^s \qquad (a \in \mathrm{A}, n \in \mathrm{N})\,.$$

Then we shall write $\mathsf{V}(s) = \mathsf{V}(\sigma)$ and $\pi_s = \pi$. Then the transformation property of functions in $\mathsf{V}(s)$ looks as follows

$$f(anx) = \rho(a)^{1+s}f(x) \qquad (a \in \mathrm{A}, n \in \mathrm{N}, x \in \mathrm{G})\,. \tag{43}$$

The resulting representation$(\pi_s, \mathsf{V}(s))$ is called the principal series representation.

Lemma 83. *Let*

$$\rho_s(ank) = \rho(a)^{1+s} \qquad (a \in \mathrm{A}, n \in \mathrm{N}, k \in \mathrm{K})\,.$$

Then $\rho_s \in \mathsf{V}(s)$ *is* K*-ivariant, has norm* 1*, and*

$$(\pi_s(x)\rho_s, \rho_s) = \int_{\mathrm{K}} \rho_s(kx)\,dk = \int_{\mathrm{K}} \rho(kx)^{1+s}\,dk\,.$$

Proof. See [Lan85, page 47]. □

The function

$$\phi_s = \int_{\mathrm{K}} \rho(kx)^{1+s}\, dk \tag{44}$$

is known as a spherical function.

Lemma 84. *For a function $\psi \in C_c(\mathrm{G})$, $\pi_s(\psi)$ is an integral kernel operator with the integral kernel*

$$q_\psi(k,k') = \int_{\mathrm{A}} \int_{\mathrm{N}} \psi(k'^{-1}ank)\rho(a)^{1+s}\, da\, dn\, dn\,.$$

Furthermore the operator $\pi_s(\psi)$ is of trace class and

$$\operatorname{tr} \pi_s(\psi) = \int_{\mathrm{K}} q_\psi(k,k)\, dk = \int_{\mathrm{A}} \int_{\mathrm{N}} \int_{\mathrm{K}} \psi(kank^{-1})\, dk\, \rho(a)^{1+s}\, dk\, dn\, da\,.$$

Proof. The formula for the integral kernel is obtained via a straightforward computation in [Lan85, page 48]. It is a continuous function. Hence the of trace class. □

By computing a few Jacobians, as in [Lan85, page 68], we obtain the following lemma.

Lemma 85. *For $\phi \in C_c(\mathrm{G})$*

$$\int_{\mathrm{N}} \phi(ana^{-1}n^{-1})\, dn = \frac{1}{|\alpha(a)-1|} \int_{\mathrm{N}} \phi(n)\, dn\,, \tag{45}$$

and therefore

$$\int_{\mathrm{A}\backslash\mathrm{G}} \phi(x^{-1}ax))\, d\dot{x} = \frac{\rho(a)}{|D(a)|} \int_{\mathrm{N}} \int_{\mathrm{K}} \phi(kank^{-1})\, dk\, dn\,, \tag{46}$$

where

$$D(a) = \rho(a) - \rho(a)^{-1} \qquad (a \in \mathrm{A})\,. \tag{47}$$

Theorem 86. *Let Θ_{π_s} be a function on G defined as follows.*

$$\Theta_{\pi_s}(x) = \begin{cases} 2\frac{\rho(a)^s + \rho(a)^{-s}}{|D(a)|} & \text{if } x \text{ is conjugate to } a \in \mathrm{A}\,, \\ 0 & \text{if } x \text{ is not conjugate to any element of } \mathrm{A}\,. \end{cases} \tag{48}$$

Then

$$\operatorname{tr} \pi_s(\psi) = \int_{\mathrm{G}} \Theta_{\pi_s}(x)\psi(x)\, dx \qquad (\psi \in C_c(\mathrm{G}))\,. \tag{49}$$

Proof. By combining Lemmas 84 and 85 we see that

$$\operatorname{tr} \pi_s(\psi) = \int_{\mathrm{A}} |D(a)| \int_{\mathrm{A}\backslash\mathrm{G}} \psi(x^{-1}ax)\, d\dot{x}\, \rho(a)^s\, da\,. \tag{50}$$

Changing the variable a to a^{-1} gives

$$\operatorname{tr} \pi_s(\psi) = \int_{\mathrm{A}} |D(a^{-1})| \int_{\mathrm{A}\backslash\mathrm{G}} \psi(x^{-1}a^{-1}x)\, d\dot{x}\, \rho(a)^{-s}\, da\,.$$

However $|D(a^{-1})| = |D(a)|$ and

$$\int_{\mathrm{A}\backslash\mathrm{G}} \psi(x^{-1}a^{-1}x)\,d\dot{x} = \int_{\mathrm{A}\backslash\mathrm{G}} \psi(x^{-1}ax)\,d\dot{x}\,.$$

Therefore

$$\operatorname{tr}\pi_s(\psi) = \int_{\mathrm{A}} |D(a)| \int_{\mathrm{A}\backslash\mathrm{G}} \psi(x^{-1}ax)\,d\dot{x}\,\frac{\rho(a)^s + \rho(a)^{-s}}{2}\,da\,.$$

On the other hand,

$$\begin{aligned}\int_{\mathrm{G}} \Theta_{\pi_s}(x)\psi(x)\,dx &= \frac{1}{2}\int_{\mathrm{A}} |D(a)|^2 \int_{\mathrm{A}\backslash\mathrm{G}} \Theta_{\pi_s}(x^{-1}ax)\psi(x^{-1}ax)\,d\dot{x}\,da\\ &= \frac{1}{2}\int_{\mathrm{A}} |D(a)| \int_{\mathrm{A}\backslash\mathrm{G}} \psi(x^{-1}ax)\,d\dot{x}\,\Theta_{\pi_s}(a)|D(a)|\,da\end{aligned}$$

and (48) follows. □

Notice that the trace is given via integration against a G-invariant locally integrable function. Let ${}^0\mathrm{M} = \{1, -1\} \subseteq \mathrm{G}$. This is the centralizer A in K. If δ is a character of the group ${}^0\mathrm{M}$, we define $\mathsf{V}(\delta, s) \subseteq \mathsf{V}(s)$ to be the subspace of functions ϕ such that

$$\phi(mx) = \delta(m)\psi(x) \qquad (m \in {}^0\mathrm{M}, x \in \mathrm{G})\,.$$

Denote by $(\pi_{\delta,s}, \mathsf{V}(\delta, s))$ the resulting representation of G. Clearly $(\pi_s, \mathsf{V}(s))$ is the direct sum of the two subrepresentations $(\pi_{\delta,s}, \mathsf{V}(\delta, s))$.

Theorem 87. *Let $\Theta_{\pi_{\delta,s}}$ be a function on* G *defined as follows.*

$$\Theta_{\pi_s}(mx) = \begin{cases} \delta(m)\frac{\rho(a)^s+\rho(a)^{-s}}{D(a)} & \textit{if } x \textit{ is conjugate to } ma, \textit{ where } m \in {}^0\mathrm{M} \textit{ and } a \in \mathrm{A}\,,\\ 0 & \textit{if } x \textit{ is not conjugate to any element of } \pm\mathrm{A}\,. \end{cases}$$

Then

$$\operatorname{tr}\pi_{\delta,s}(\psi) = \int_{\mathrm{G}} \Theta_{\pi_{\delta,s}}(x)\psi(x)\,dx \qquad (\psi \in C_c(\mathrm{G}))\,. \tag{51}$$

Problem 17. *Deduce Theorem 87 from Theorem 86.*

12.4. **Finite dimensional representations.** Define

$$D(k(\theta)) = e^{i\theta} - e^{-i\theta} \qquad (k(\theta) \in \mathrm{K})\,. \tag{52}$$

Lemma 88. *Fix an integer $n \geq 1$. For non-negative integers p,q with $p + q = n - 1$ let $f_{p,q}$ be a function of*

$$x = \begin{pmatrix} a & b\\ c & d \end{pmatrix}$$

such that $f_{p,q}(x) = c^p d^q$. Then $f_{p,q} \in \mathsf{V}(\delta, -n)$, where $\delta(-1) = (-1)^{p+q}$. The functions $f_{p,q}$ span a G*-invariant finite dimensional subspace*

$$\mathrm{U}(\delta, -n) = \bigoplus_{p+q=n-1} \mathbb{C} f_{p,q} \subseteq \mathsf{V}(\delta, -n)$$

of dimension n. Denote the resulting representation by $(\sigma_{\delta,-n}, \mathrm{U}(\delta,-n))$. Then

$$\operatorname{tr}\sigma_{\delta,-n}(ma) = \delta(m)\frac{\rho(a)^n - \rho(a)^{-n}}{D(a)} \qquad (m \in {}^0M, a \in \mathrm{A})\,. \tag{53}$$

Also,

$$\operatorname{tr}\sigma_{\delta,-n}(k(\theta)) = \frac{e^{in\theta} - e^{-in\theta}}{D(k(\theta))} \qquad (k(\theta) \in \mathrm{K})\,. \tag{54}$$

Proof. Everything till (53) is straightforward. See [Lan85, page 151]. The formula (54) follows from (53). Indeed, let $\mathrm{G}_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$. Then $\mathrm{G} \subseteq \mathrm{G}_{\mathbb{C}}$ is a subgroup and the representation $(\sigma_{\delta,-n}, \mathrm{U}(\delta,-n))$ extends to a representation of $\mathrm{G}_{\mathbb{C}}$ so that $\sigma_{\delta,-n} : \mathrm{G}_{\mathbb{C}} \to \mathrm{GL}(\mathrm{U}(\delta,-n))$ is a polynomial map. Notice that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & cos\theta \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Thus $k(\theta)$ is conjugate to

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

within $\mathrm{G}_{\mathbb{C}}$. Hence Lemma 88 shows that

$$\operatorname{tr}\sigma_{\delta,-n}(k(\theta)) = \frac{e^{in\theta} - e^{-in\theta}}{D(k(\theta))} \qquad (k(\theta) \in \mathrm{K})\,. \tag{55}$$

□

Problem 18. *Prove that for $n = 2$ the representation $(\sigma_{\delta,-n}, \mathrm{U}(\delta,-n))$ is irreducible. (We'll see soon that they are all irreducible.)*

In this case this representation is isomorphic to the representation of G *on $\mathbb{C}^2 = M_{1,2}(\mathbb{C})$ via right multiplication. Since no non-zero line in $\mathbb{C}^2$ is preserved by G, the claim follows.*

Problem 19. *Prove that for $n \geq 2$ the representation $(\sigma_{\delta,-n}, \mathrm{U}(\delta,-n))$ are not unitarizable. (We know from Theorem 72 that this is the case. Try to find an independent argument.)*

Each $f_{p,q}$ is an eigenfunction for the action of the subgroup A *and the eigenvalues may go to infinity. Hence, no matter what the inner product, the diagonal matrix coefficient defined by $f_{p,q}$ is not bounded. This is impossible for a unitary representation*

12.5. **Smooth vectors and analytic vectors.** Let (π, V) be a representation of G on a Banach space V. Denote by $\mathsf{V}^\infty \subseteq \mathsf{V}$ the subspace of all vectors v such that the map

$$\mathrm{G} \ni x \to \pi(x)v \in \mathsf{V} \tag{56}$$

is smooth (infinitely many times differentiable). This subspace is dense because, as is easy to check, $\pi(C_c^\infty(\mathrm{G}))\mathsf{V} \subseteq \mathsf{V}^\infty$. For $X \in \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, the Lie algebra of G, define

$$d\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v|_{t=0} \qquad (v \in \mathsf{V}^\infty)\,. \tag{57}$$

One checks without difficulties that the map

$$d\pi : \mathfrak{g} \to \mathrm{End}(\mathsf{V}^\infty) \tag{58}$$

is a Lie algebra homomorphism. By linearity we extend it to the complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$$d\pi : \mathfrak{g}_\mathbb{C} \to \mathrm{End}(\mathsf{V}^\infty) \tag{59}$$

and obtain a representation $(d\pi, \mathsf{V}^\infty)$ of the Lie algebra $\mathfrak{g}_\mathbb{C}$.

One drawback of this representation is that the closure of a $\mathfrak{g}$-invariant subspace $\mathrm{U} \subseteq \mathsf{V}^\infty$ in V does not need to be G-invariant. Indeed, consider the right regular representation $(R, \mathrm{L}^2(\mathrm{G}))$ of G. Let $U \subseteq \mathrm{G}$, $U \neq \mathrm{G}$, be a non-empty open set. Then the space $\mathrm{U} = C_c^\infty(U)$ is closed under the action of $dR(\mathfrak{g})$, but the closure $\mathrm{L}^2(U)$ is not $R(\mathrm{G})$ invariant. For this reason one is lead to study the space $\mathsf{V}^{an} \subseteq \mathsf{V}$ of all vectors v such that the map.

$$\mathrm{G} \ni x \to \pi(x)v \in \mathsf{V} \tag{60}$$

is analytic.

Theorem 89. *Let $\mathsf{X} \subseteq \mathsf{V}^{an}$ be a $\mathfrak{g}$-invariant vector subspace. Then the closure of X in V is* G*-invarant*

Proof. We follow [Var89, Theorem 2, page 108]. It is enough to prove that for any $v \in \mathrm{U}$ and for any $x \in \mathrm{G}$, $\pi(x)u$ belonges to the closure of U. Suppose not. Then by Haan-Banach Theorem there is λ in the dual of V such that λ is equal to zero on the closure of U, but for some $x_0 \in \mathrm{G}$, $\lambda(\pi(x_0))v \neq 0$.

By assumption the function

$$\mathrm{G} \ni x \to \lambda(\pi(x)v) \in \mathbb{C}$$

is analytic. By the choice of λ, its Taylor series at $x = 1$ is zero. Indeed, for any $X_1, X_2, ..., X_n \in \mathfrak{g}$

$$d\pi(X_1)d\pi(X_2)...d\pi(X_n)v \in \mathrm{U}$$

Hence

$$\lambda(d\pi(X_1)d\pi(X_2)...d\pi(X_n)v) = 0$$

Also, $\lambda(v) = 0$. Thus the function is zero because G is connected. This is a contradiction. □

12.6. **The derivative of the right regular representation.** Every element $g \in \mathrm{GL}^+(\mathbb{R})$ has a unique decomposition as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{61}$$

where $u > 0$, $y > 0$, $x \in \mathbb{R}$ and $\theta \in [-\pi, \pi)$. We extend any function defined on $\mathrm{SL}_2(\mathbb{R})$ to a function on $\mathrm{GL}^+(\mathbb{R})$ by making it independent of the variable u. Also, we extend the right regular representation of $\mathrm{SL}_2(\mathbb{R})$ to act on such defined functions on $\mathrm{GL}^+(\mathbb{R})$. Then a straightforward computation, see [Lan85, 113-116] verifies the following formulas,

$$\begin{aligned}
dR\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= y\cos 2\theta\,\partial_x + y\sin 2\theta\,\partial_y + \sin^2\theta\,\partial_\theta\,, \qquad (62)\\
dR\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \partial_\theta\,,\\
dR\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= 2y\cos 2\theta\,\partial_x + 2y\sin 2\theta\,\partial_y - \cos 2\theta\,\partial_\theta\,,\\
dR\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= y\cos 2\theta\,\partial_x + y\sin 2\theta\,\partial_y - \cos^2\theta\,\partial_\theta\,,\\
dR\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= -2y\sin 2\theta\,\partial_x + 2y\cos 2\theta\,\partial_y + \sin 2\theta\,\partial_\theta\,.
\end{aligned}$$

Problem 20. *Verify the first two formulas in* (62).

12.7. **The universal enveloping algebra.** Here $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the complex tensor algebra of $\mathfrak{g}$ divided by the ideal generated by element $AB - BA - [A, B]$, $A, B \in \mathfrak{g}$. The Poincare-Birkhoff-Witt Theorem says that if A, B, C form a basis of the vector space $\mathfrak{g}_{\mathbb{C}}$, then the elements

$$A^a B^b C^c \qquad (0 \le a, b, c \in \mathbb{Z})$$

form a basis of the vector space $\mathcal{U}(\mathfrak{g})$. (Here $A^0 = B^0 = C^0 = 1$.) Furthermore, for any $A, B, C \in \mathfrak{g}_{\mathbb{C}}$ satisfying the commutation relations

$$[A, B] = 2B\,,\ [A, C] = -2C\,,\ [B, C] = A$$

The element, called the Casimir element,

$$\mathcal{C} = A^2 + 2(BC + CB) \tag{63}$$

generates the the center of $\mathcal{U}(\mathfrak{g})$ and does not depend on the choice of the A, B, C.

Problem 21. *Check that* $\mathfrak{g}$ *acts trivially on* $\mathcal{C}$.

Thus the center of $\mathcal{U}(\mathfrak{g})$ is equal to $\mathbb{C}[\mathcal{C}]$. Also, any representation of the Lie algebra $\mathfrak{g}$ extends to a representation of the algebra $\mathcal{U}(\mathfrak{g})$. In terms of the coordinates used in section 12.6,

$$dR(\mathcal{C}) = 4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x\partial_\theta \tag{64}$$

Proofs may be found in [Lan85, pages 191-198].

12.8. K**-multiplicity** 1 **representations.** Here $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$ and we consider only representations (π, V) of G on Banach spaces V such that for each integer n the isotypic component $\mathsf{V}_n \subseteq \mathsf{V}$ has dimension at most 1.

Lemma 90. *For any* $n \in \mathbb{Z}$, *the space* $S_{n,n}^\infty = S_{n,n} \cap C^\infty(\mathrm{G})$ *is dense in* $S_{n,n}$

Proof. See [Lan85, page 101] . □

Theorem 91. *Let* (π, V) *be a representation* G *on a Banach space* V *such that for each integer* n *the isotypic component* $\mathsf{V}_n \subseteq \mathsf{V}$ *has dimension at most* 1*. Then*

$$\mathsf{V}_{\mathrm{K}} = \bigoplus_{n\in\mathbb{Z}} \mathsf{V}_n \subseteq \mathsf{V}^{an} .$$

In particular V_{K} *is a* $(\mathfrak{g}, \mathrm{K})$ *module.*

Proof. Since, by assumption, $\dim \mathsf{V}_n = 1$, we see from Lemmas 90, 78 and 79 that

$$\mathsf{V}_n = \pi(S^\infty_{n,n})\mathsf{V}_n .$$

In particular, $\mathsf{V}_n \subseteq \mathsf{V}^\infty$. It'll suffice to show that for a fixed n and a non-zero vector $v \in \mathsf{V}_n$ the function

$$f_v(g) = \pi(g)v \qquad (g \in \mathrm{G})$$

is analytic. Notice that

$$f_v(gk(\theta)) = e^{in\theta} f_v(g) .$$

Hence (64) shows that

$$dR(\mathcal{C})f_v(g) = (4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x in)f_v(g) .$$

On the other hand $d\pi(\mathcal{C})$ commutes with $\pi(\mathrm{K})$, hence preserves V_n. Since, by assumption, $\dim \mathsf{V}_n = 1$, $d\pi(\mathcal{C})$ acts on V_n via multiplication by a scalar, call it c_n. Thus

$$d\pi(\mathcal{C})v = c_n v .$$

Since

$$R(h)f_v(g) = f_{\pi(h)v}(g) ,$$

this implies

$$dR(\mathcal{C})f_v v = c_n f_v .$$

Since the characteristic variety of the system of differential equations

$$(4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x in)f_v = c_n f_v , \quad \partial_\theta f_v = in$$

is zero the function f_v is analytic. □

Theorem 92. *Let* (π, V) *be a representation* G *on a Banach space* V *such that for each integer* n *the isotypic component* $\mathsf{V}_n \subseteq \mathsf{V}$ *has dimension at most* 1*. Then the map*

$$\mathsf{V} \supseteq \mathrm{U} \to \mathrm{U}_{\mathrm{K}} \subseteq \mathsf{V}_{\mathrm{K}}$$

is a bijection between closed G*-invariant subspaces of* V *and* $(\mathfrak{g}, \mathrm{K})$*-submodules of* V_{K}*. The inverse is given by*

$$\mathsf{V}_{\mathrm{K}} \supseteq \mathsf{X} \to Cl(\mathsf{X}) \subseteq \mathsf{V} .$$

where $Cl(\mathsf{X})$ *denotes the closure of* X *in* V*. In particular* (π, V) *is irreducible if and only if the* $(\mathfrak{g}, \mathrm{K})$*-module* V_{K} *is irreducible.*

Proof. This is clear from Theorems 89 and 91. □

Lemma 93. *For any irreducible* $(\mathfrak{g}, \mathrm{K})$*-module* X *and any integer* n *such that* $\mathsf{X}_n \neq 0$*, any non-zero vector* $v \in \mathsf{X}_n$ *is cyclic for the action of* $\mathcal{U}(\mathfrak{g})$ *on* X*.*

Proof. Since $\mathcal{U}(\mathfrak{g})v$ is a submodule of X, the claim is obvious. □

Let $\mathsf{X} = \bigoplus_n \mathsf{X}_n$ ba a $(\mathfrak{g}, \mathrm{K})$-module with $\dim \mathsf{X}_n \leq 1$. A hermitian form $(\cdot, \cdot)$ on X is called $\mathfrak{g}$-invariant if

$$(Xu, v) = -(u, Xv) \qquad (u, v \in \mathsf{X}, X \in \mathfrak{g})\,. \tag{65}$$

A $(\mathfrak{g}, \mathrm{K})$ module is called unitarizable if it admits an invariant positive definite hemitian form.

Theorem 94. *Let* $\mathsf{X} = \bigoplus_n \mathsf{X}_n$ *ba an irreducible* $(\mathfrak{g}, \mathrm{K})$*-module with* $\dim \mathsf{X}_n \leq 1$*. Then any two positive definite* $\mathfrak{g}$*-invariant hermitian products on* X *are positive multiples of each other (assuming they exist).*

Proof. Denote the two $\mathfrak{g}$-invariant positive hermitian products on X by $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$. Then the spaces V_n are mutually orthogonal with respect to $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$. Let $P_n : \mathsf{V} \to \mathsf{V}_n$ denote the orthogonal projection. Fix $m \in \mathbb{Z}$ such that $\mathsf{V}_m \neq 0$ and a vector $0 \neq v_m \in \mathsf{V}_m$. Lemma 93 implies that for any n there is $a \in \mathcal{U}(\mathfrak{g})$ such that $P_n a v_m \neq 0$. Then

$$\begin{aligned}(av_m, av_m) = (v_m, a^* a v_m) = (v_m, P_m a^* a v_m) = c\langle v_m, P_m a^* a v_m \rangle \\ = c\langle av_m, av_m \rangle\,,\end{aligned}$$

where

$$c = \frac{(v_m, v_m)}{\langle v_m, v_m \rangle}\,.$$

□

Theorem 95. *Two irreducible unitary representations of* G*, of* K*-multiplicity at most* 1*, are unitarily isomorphic if and only if their* $(\mathfrak{g}, \mathrm{K})$*modules are isomorphic.*

Proof. Let (π, V), (σ, U) be the two representations. If they are isomorphic, then clearly so are the $(\mathfrak{g}, \mathrm{K})$-modules. Conversely, suppose

$$L : \mathsf{V}_{\mathrm{K}} \to \mathrm{U}_{\mathrm{K}}$$

is a $\mathfrak{g}$-intertwinig map. Then $(\cdot, \cdot)_{\mathsf{V}}$ and the pull-back of $(\cdot, \cdot)_{\mathrm{U}}$ via L are positive definite $\mathfrak{g}$-invariant hermitian products on V_{K}. Theorem 94 shows that there is a constant $c > 0$ such that

$$(v, v')_{\mathsf{V}} = c(Lv, Lv')_{\mathrm{U}} \qquad (v, v' \in \mathsf{V})\,.$$

Hence

$$T = \sqrt{c}L : \mathsf{V}_{\mathrm{K}} \to \mathrm{U}_{\mathrm{K}}$$

is a $\mathfrak{g}$-intertwining isometry. We need to check that T is also G-intertwining.

Since K-finite vectors are analytic, we have

$$\pi(\exp(X))v = \sum_{n=0}^{\infty} \frac{1}{n!} \left(d\pi(X)\right)^n v \qquad (v \in \mathsf{V}_{\mathrm{K}}) .$$

For $X \in \mathfrak{g}$ in some small neighborhood of zero. Hence, in this neighborhood,

$$T\pi(\exp(X))v = \sum_{n=0}^{\infty} \frac{1}{n!} \left(d\sigma(X)\right)^n Tv = \sigma(\exp(X))Tv .$$

Thus there is an open neighborhood U of $1 \in \mathrm{G}$ such that

$$T\pi(g)v = \sigma(g)Tv \qquad (g \in U) .$$

Since G is connected U generates G, so the proof is complete. □

12.9. **The character of a $(\mathfrak{g}, \mathrm{K})$-module.**

Theorem 96. *Let (π, V) be a representation of G on a Hilbert space V, with $\dim \mathsf{V}_n \leq 1$ for all $n \in \mathbb{Z}$. Then for any $\phi \in C_c^\infty(\mathrm{G})$, the operator $\pi(\phi)$ is of trace class and the map*

$$C_c^\infty(\mathrm{G}) \ni \phi \to \mathrm{tr}\, \pi(\phi) \in \mathbb{C}$$

is a distribution on G.

Proof. Recall that

$$\int_{\mathrm{G}} \phi(g)\pi(g)\, dg = \int_{\mathrm{G}} \phi(g)\pi(gk^{-1}k)\, dg = \int_{\mathrm{G}} R(k)\phi(g)\pi(g)\, dg\, \pi(k) .$$

Hence by taking derivatives at $k = 1$,

$$0 = \int_{\mathrm{G}} dR(J)\phi(g)\pi(g)\, dg + \int_{\mathrm{G}} \phi(g)\pi(g)\, dg\, d\pi(J) .$$

Let $v_n \in \mathsf{V}_n$ be a unit vector. Then

$$0 = \int_{\mathrm{G}} dR(J)\phi(g)\pi(g)\, dg\, v_n + \int_{\mathrm{G}} \phi(g)\pi(g)\, dg\, in\, v_n .$$

By iterating we see that for $m = 0, 1, 2, ...$

$$(-in)^m \pi(\phi)v_n = \pi(dR(J)^m\phi) .$$

Thus

$$|(\pi(\phi)v_n, v_n)| \leq (1 + |n|)^{-m} \parallel \pi(dR(J)^m\phi) \parallel \qquad (n \in \mathbb{Z}) .$$

Since for $m \geq 2$,

$$\sum_{n\in\mathbb{Z}} (1 + |n|)^{-m} < \infty ,$$

the claim follows. □

Theorem 97. *Let* (π, V), (σ, U) *be representations of* G *on Hilbert spaces* V *and* U*, with* $\dim \mathsf{V}_n \leq 1$ *and* $\dim \mathrm{U}_n \leq 1$ *for all* $n \in \mathbb{Z}$*. Assume that the* $(\mathfrak{g}, \mathrm{K})$*-modules* V_K *and* U_K *are isomorphic. Then,*

$$\operatorname{tr} \pi(\phi) = \operatorname{tr} \sigma(\phi) \qquad (\phi \in C_c^\infty(\mathrm{G}))\,.$$

Proof. Given a linear map $A : \mathrm{U} \to \mathsf{V}$, recall the conjugate map $T^* : \mathsf{V} \to \mathrm{U}$,

$$(Au, v)_\mathsf{V} = (u, A^* v)_\mathrm{U}\,.$$

Let

$$T : \mathsf{V}_\mathrm{K} \to \mathrm{U}_\mathrm{K}$$

be a $(\mathfrak{g}, \mathrm{K})$-intertwining isomorphism. Pick unit vectors $v_n \in \mathsf{V}_n$. Notice that Tv_n, $n \in \mathbb{Z}$, form a basis of U and $(T^*)^{-1} v_n$ form a dual basis. Then

$$\operatorname{tr} \pi(\phi) = \sum_{n \in \mathbb{Z}} (\pi(\phi) v_n, v_n)_\mathsf{V} \quad \text{and} \quad \operatorname{tr} \sigma(\phi) = \sum_{n \in \mathbb{Z}} (\sigma(\phi) T v_n, (T^{-1})^* v_n)_\mathrm{U}\,.$$

By definition

$$(\sigma(\phi) T v_n, (T^{-1})^* v_n)_\mathrm{U} = (T^{-1} \sigma(\phi) T v_n, v_n)_\mathrm{U}$$

Since v_n is an analytic vector,

$$T^{-1} \sigma(\exp(X)) T v_n = \pi(\exp(X))$$

in a neighborhood of $0 \in \mathfrak{g}$. Hence

$$(\pi(g) v_n, v_n)_\mathsf{V} = (\sigma(g) T v_n, (T^{-1})^* v_n)_\mathrm{U}$$

in a neighborhood of $1 \in \mathrm{G}$. Since both functions are analytic, they are equal everywhere. Thus

$$(\pi(\phi) v_n, v_n)_\mathsf{V} = (\sigma(\phi) T v_n, (T^{-1})^* v_n)_\mathrm{U}\,.$$

□

In general one says that two representations (π, V) and (σ, U) are infinitesimally equivalent if the $(\mathfrak{g}, \mathrm{K})$-modules V_K and U_K are isomorphic.

Given a representation of G (with K-multiplicities at most 1) we define the character of π as

$$\Theta_\pi(\phi) = \operatorname{tr} \pi(\phi) \qquad (\phi \in C_c^\infty(\mathrm{G})\,.$$

This is a distribution on G which, as shown in Theorem 97, does not depend on the infinitesimal equivalence class of (π, V). Therefore we shall also write $\Theta_\pi = \Theta_{\mathsf{V}_\mathrm{K}}$.

12.10. **The unitary dual.**

Lemma 98. *Let*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Then

$$[E^+, E^-] = -4iJ\,, \quad [J, E^+] = 2iE^+\,, \quad [J, E^-] = -2iE^-\,.$$

Proof. This is straightforward. See [Lan85, page 102]. □

Corollary 99. *For any* $(\mathfrak{g}, \mathrm{K})$ *module* X*, with the* K*-isotypic components* X_n*,*

$$J\mathsf{X}_n \subseteq \mathsf{X}_n\,, \quad E^+\mathsf{X}_n \subseteq \mathsf{X}_{n+2}\,, \quad E^-\mathsf{X}_n \subseteq \mathsf{X}_{n-2}\,.$$

Proof. This is clear from Corollary 98 □

Recall the principal series representation $(\pi_s, \mathsf{V}(s))$.

Lemma 100. *In terms of* (61)*, for* $n \in \mathbb{Z}$ *define*

$$v_n\left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = y^{1+s}e^{in\theta}\,.$$

Then $v_n \in \mathsf{V}(s)$ *and*

$$\begin{array}{rcl} d\pi_s(J)v_n & = & inv_n\,, \\ d\pi_s(E^-)v_n & = & (s+1-n)v_{n-2}\,, \\ d\pi_s(E^+)v_n & = & (s+1+n)v_{n+2}\,. \end{array}$$

Proof. We see from (62) that

$$\begin{array}{rcl} dR(J)v_n & = & inv_n\,, \\ dR(E^-)v_n & = & (s+1-n)v_{n-2}\,, \\ dR(E^+)v_n & = & (s+1+n)v_{n+2}\,. \end{array}$$

But the right regular action coincides with π_s, hence the formulas follow. □

By combining Lemma 100 with Theorem 91 we deduce the following Corollary.

Corollary 101. *The* $(\mathfrak{g}, \mathrm{K})$ *module of the principal series* $(\pi_s, \mathsf{V}(s))$ *is equal to*

$$\mathsf{V}(s)_{\mathrm{K}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}v_n\,.$$

(Here $\mathsf{V}(s) = \mathrm{L}^2(\mathrm{K})$*, by restriction to* K*, independently of* s *and the restriction of* v_n *to* K *is equal* χ_n*.)*

Lemma 100 implies the following Proposition, see [Lan85, pages 119-121].

Proposition 102. *If* s *is not an integer then*

$$\mathsf{V}(s)_{\mathrm{K}}^+ = \bigoplus_{n\in 2\mathbb{Z}} \mathbb{C}v_n \quad \textit{znd} \quad \mathsf{V}(s)_{\mathrm{K}}^- = \bigoplus_{n\in 2\mathbb{Z}+1} \mathbb{C}v_n$$

are irreducible submodules of $\mathsf{V}(s)_{\mathrm{K}}$ *and* $\mathsf{V}(s)_{\mathrm{K}}$ *is the direct sum of them.*

If $s = 0$*, then* $\mathsf{V}(0)_{\mathrm{K}}$ *is the dirct sum of three irreducible submodules*

$$\mathsf{V}(0)_{\mathrm{K}} = \bigoplus_{n\in 2\mathbb{Z}} \mathbb{C}v_n \oplus \bigoplus_{1\le n\in 2\mathbb{Z}+1} \mathbb{C}v_n \oplus \bigoplus_{-1\ge n\in 2\mathbb{Z}+1} \mathbb{C}v_n\,.$$

If $m \geq 2$ *is an integer and* $s = m-1$*, then* $\mathsf{V}(m-1)_{\mathrm{K}}$ *contains three irreducible submodules*

$$\mathsf{X}^m = \bigoplus_{m\le n,\ n-m\in 2\mathbb{Z}} \mathbb{C}v_n\,, \quad \mathsf{X}^{-m} = \bigoplus_{-m\ge n,\ n-m\in 2\mathbb{Z}} \mathbb{C}v_n\,, \quad \bigoplus_{n-m\in 2\mathbb{Z}+1} \mathbb{C}v_n\,.$$

The quotient module, $\mathsf{V}(m-1)_{\mathrm{K}}$ divided by the three submodules is irreducible, finite dimensional of dimension $m-1$. It has a basis represented by the elements

$$v_{-m+2}\,, \quad v_{-m+4}\,, \quad \dots\,, \quad v_{m-2}\,.$$

If $m \geq 2$ is an integer and $s = -m+1$, then $\mathsf{V}(m-1)_{\mathrm{K}}$ contains the finite dimensional submodule

$$\mathbb{C}v_{-m+2} \oplus \mathbb{C}v_{-m+4} \oplus \dots \oplus \mathbb{C}v_{m-2}\,.$$

The quotient module, $\mathsf{V}(m-1)_{\mathrm{K}}$ divided by this module is isomorphic to the direct sum of modules X^m and X^{-m} plus the sum of all K*-types of parity opposite to m.*

The $(\mathfrak{g}, \mathrm{K})$-modules $\mathsf{V}(m-1)_{\mathrm{K}}$ and $\mathsf{V}(-m+1)_{\mathrm{K}}$ are dual to each other.

Thus we have the highest weight modules, lowest weight modules, finite dimensional modules and modules with unbounded K-types on both side.

Problem 22. *Write down a proof of Theorem 102. Imagine studing the structure of the principal series without the notion of a $(\mathfrak{g}, \mathrm{K})$-module and without Harish-Chandra's reduction.*
This is written in Lang's book.

Proposition 103. *The Casimir element $\mathcal{C}$,* (63)*, acts on the principal series representation $(\pi_s, \mathsf{V}(s))$ via multiplication by s^2-1:*

$$d\pi_s(\mathcal{C}) = (s^2-1)\mathrm{I}\,.$$

Proof. As checked in [Lan85, page 195],

$$\mathcal{C} = -1 - (J-i)^2 + E^+E^-\,.$$

Hence a straightforward computation using Lemma 100 implies the formula. □

In particular, since the discrete series modules X^{n+1}, X^{-n-1} are contained in a principal series $\mathsf{V}(n)_{\mathrm{K}}$ we see that the Casimir element acts on them via multiplication by n^2-1. Similarly the Casimir element acts on an irreducible finite dimensional representation of dimension n via multiplication by n^2-1.

The commutation relations Lemma 98 and the formulas Lemma 100 with some work imply the following theorem, due to Bargmann, [Bar47]. See [Lan85, page 123].

Theorem 104. *Here is a complete list of the irreducible $(\mathfrak{g}, \mathrm{K})$ modules, up to equivalence.*

1. *Lowest weight module X^m with lowest weight $m \geq 1$ and the highest weight module X^m with highest weight $m \leq -1$*
2. *Principal series $\mathsf{V}(i\tau)^+_{\mathrm{K}}$ and $\mathsf{V}(i\tau)^-_{\mathrm{K}}$, $\tau \in \mathbb{R}\setminus\{0\}$;*
3. *Principal series $\mathsf{V}(0)^+_{\mathrm{K}}$;*
4. *Complementary series $\mathsf{V}(s)^+_{\mathrm{K}}$, $-1 < s < 1$;*
5. *Trivial module.*

Problem 23. *Based on Lang, write down a proof of this theorem. Also, deduce from it that the only finite dimensional irreducible unitary representation of* G *is the trivial representation.*

A similar result was obtained by Dan Barbasch, [Bar89] for the complex classical groups.

The closures of these modules in the corresponding principal series representation are representations of G on Hilbert spaces. They are unitary representations except the highest and lowest weight representations. In these cases the inner product inherited from the principal series is not $\mathfrak{g}$ invariant. Therefore there is a problem of constructing the unitary representations of G whose $(\mathfrak{g}, \mathrm{K})$-modules are the highest and lowest weight modules. This is explained in the theorem below. See [Lan85, page 181].

Theorem 105. *Define*

$$\tilde{\alpha}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(a+d-ic+ib)\,, \quad \tilde{\beta}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(c+b-ia+id)\,.$$

Let $m \geq 2$. Then the closure $\mathsf{V}^{(m)}$ in $\mathrm{L}^2(\mathrm{G})$ of the space

$$\bigoplus_{r=0}^{\infty} \mathbb{C}\tilde{\alpha}^{-m-r}\tilde{\beta}^{r}$$

is invariant under the left action

$$L(g)\phi(x) = \phi(g^{-1}x)\,.$$

The resulting representation $(L, \mathsf{V}^{(m)})$ of G is irreducible and unitary. Its $(\mathfrak{g}, \mathrm{K})$-module $\mathsf{V}^{(m)}_{\mathrm{K}}$ is isomorphis to the lowest weight module X^m with the lowest weight m. In order to realize the lowest weight representations we take the complex conjugate of $\mathsf{V}^{(m)}$.

We shall skip the construction of the unitary representations whose $(\mathfrak{g}, \mathrm{K})$-modules are X^1 and X^{-1}. They are not square integrable. One may find them in [Kna86, page 36].

12.11. **The character of the sum of discrete series.** For an integer $m \geq 2$ let $\Theta_{\mathsf{X}^{(m)}}$ denote the the character of the $(\mathfrak{g}, \mathrm{K})$-module $\mathsf{X}^{(m)}$, and $\Theta_{\mathsf{X}^{(-m)}}$ denote the the character of the $(\mathfrak{g}, \mathrm{K})$-module $\mathsf{X}^{(-m)}$. Let

$$h_t = \begin{pmatrix} e^t & 0 \\ 0 & e{-}t \end{pmatrix} \qquad t \in \mathbb{R})\,.$$

Proposition 106. *The character of the sum of the two discrete series $(\mathfrak{g}, \mathrm{K})$-modules $\mathsf{X}^{(m)}$ and $\mathsf{X}^{(-m)}$ is represented by the G-conjugation invariant function $\Theta_{\mathsf{X}^{(m)} \oplus \mathsf{X}^{(-m)}}$ given by*

$$\begin{aligned} \Theta_{\mathsf{X}^{(m)} \oplus \mathsf{X}^{(-m)}}(k(\theta)) &= -\frac{e^{i(m-1)\theta} - e^{-i(m-1)\theta}}{e^{i\theta} - e^{-i\theta}} \qquad (\theta \in \mathbb{R})\,, \\ \Theta_{\mathsf{X}^{(m)} + \mathsf{X}^{(-m)}}(zh_t) &= z^m \frac{e^{-t|m-1|}}{|e^t - e^{-t}|} \qquad (z = \pm 1, t \in \mathbb{R})\,. \end{aligned}$$

Proof. From Proposition 102 we know the structure of the principal series $\mathsf{V}(m-1)_{\mathrm{K}}$, from Theorem 86, the character of it, from Lemma 88 the character of the finite dimensional representation which is in the principal series. Now Theorem 97 justifies the formula

$$\begin{aligned}&\text{character of principal series}\\ &\quad = \text{character of the sum of discrete series} + \text{character of finite dimensional module}\,.\end{aligned}$$

This completes the proof □

For convenience define The following G-invariant functions on G

$$\begin{aligned}\Theta_{\mathsf{X}^{(m)}}(k(\theta)) &= -\frac{e^{i(m-1)\theta}}{e^{i\theta}-e^{-i\theta}} \qquad (\theta\in\mathbb{R})\,,\\ \Theta_{\mathsf{X}^{(m)}}(zh_t) &= z^m\frac{e^{-|t||m-1|}}{|e^t-e^{-t}|} \qquad (z=\pm 1, t\in\mathbb{R})\,.\end{aligned}$$

and

$$\begin{aligned}\Theta_{\mathsf{X}^{(-m)}}(k(\theta)) &= \frac{e^{i(m-1)\theta}}{e^{i\theta}-e^{-i\theta}} \qquad (\theta\in\mathbb{R})\,,\\ \Theta_{\mathsf{X}^{(-m)}}(zh_t) &= z^m\frac{e^{-|t||m-1|}}{|e^t-e^{-t}|} \qquad (z=\pm 1, t\in\mathbb{R})\,.\end{aligned}$$

These are the characters of the individual discrete series representations, but we don't need to know it to explain Harish-Chandra's Plancharel formula in the $\mathrm{SL}_2(\mathbb{R})$ case.

12.12. **Harish-Chandra's Plancherel formula for $\mathrm{L}^2(\mathrm{G})$.** It is not difficult to check that K and $\mathrm{H} = \mathrm{A}\cup(-A)$ are the only Cartan subgroups of G up to conjugacy. Let $\mathrm{A}' = \{a\in\mathrm{A};\ a\neq 1\}$ and let $\mathrm{K}' = \{t\in\mathrm{K};\ t\neq\pm 1\}$. We begin by recalling Harish-Chandra's orbital integrals of a function $\phi\in C_c^\infty(\mathrm{G})$, using the notation of [Lan85],

$$H_\phi^{\mathrm{A}}(zh_t) = |D(h_t)|\int_{\mathrm{A}\backslash\mathrm{G}}\phi(x^{-1}zh_tx)\,d\dot{x} \qquad (z=\pm1, t\in\mathbb{R}, t\neq 0)\,, \tag{66}$$

$$H_\phi^{\mathrm{K}}(k) = D(k)\int_{\mathrm{K}\backslash\mathrm{G}}\phi(x^{-1}kx)\,d\dot{x} \qquad (k\in\mathrm{K}')\,,$$

The Haar measure on G may be expressed in terms of these integrals by

$$\int_{\mathrm{G}}\phi(x)\,dx = \int_{\mathrm{K}}H_\phi^{\mathrm{K}}(k)\overline{D(k)}\,dk + \frac{1}{2}\int_{\mathrm{A}'}H_\phi^{\mathrm{A}}(a)|D(a)|\,da + \frac{1}{2}\int_{\mathrm{A}'}H_\phi^{\mathrm{A}}(-a)|D(a)|\,da\,. \tag{67}$$

Theorem 107. *The function H_ϕ^{A} extends to a smooth function on* H. *The function H_ϕ^{K} is smooth on* H′ *and its derivatives have one sided limits on the boundary. In these terms,*

$$H_\phi^{\mathrm{K}}(k(0+)) - H_\phi^{\mathrm{K}}(k(0-)) = \frac{i}{2}H_\phi^{\mathrm{A}}(1)\,, \tag{68}$$

$$H_\phi^{\mathrm{K}}(k(\pi+)) - H_\phi^{\mathrm{K}}(k(\pi-)) = \frac{i}{2}H_\phi^{\mathrm{A}}(-1))\,,$$

Furthermore,

$$\partial_\theta H_\phi^{\mathrm{K}}(k(\theta))|_{\theta=0} = -i\phi(1)\,. \tag{69}$$

Proof. This is a problem concerning integrals on a 3 dimensional manifold, $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$. Notice that we may replace ϕ by a K conjugation invarian function $\int_{\mathrm{K}} \phi(kxk^{-1})\, dk$. This leads to analysis on the two dimensional manifold G/K. The computations are done in [Lan85, page 164 -167]. $\square$

Problem 24. *Take a look at Lemma on page 164 in Lang's book to understand the kind of elementary calculus we are using here. Harish-Chandra proved an analogous statement for an arbitrary semisimple group. Imagine the difficulties.*

Recall the character $\chi_n(k(\theta)) = e^{in\theta}$.

Theorem 108. *For any integer $m \geq 2$ and $\phi \in C_c^\infty(\mathrm{G})$*

$$\begin{aligned}\int_{\mathrm{G}} \Theta_{\mathsf{X}^{(m)}}(x)\phi(x)\, dx &= \int_{\mathrm{K}} H_\phi^{\mathrm{K}}(k)\chi_{m-1}(k)\, dk \\ &+ \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t||m-1|}\, dt + \frac{1}{2}\int_{\mathbb{R}} (-1)^m H_\phi^{\mathrm{A}}(-h_t)e^{-|t||m-1|}\, dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathrm{G}} \Theta_{\mathsf{X}^{(-m)}}(x)\phi(x)\, dx &= -\int_{\mathrm{K}} F_\phi^{\mathrm{K}}(t)\chi_{-m+1}(k)\, dk \\ &+ \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t||m-1|}\, dt + \frac{1}{2}\int_{\mathbb{R}} (-1)^m H_\phi^{\mathrm{A}}(-h_t)e^{-|t||m-1|}\, dt\,.\end{aligned}$$

Proof. This follows from (67) and the formulas for $\Theta_{\mathsf{X}^{(\pm m)}}$ in previous section. $\square$

Theorem 108 gives formulas for the Fourier coefficients of H_ϕ^{K}:

$$\hat{H}_\phi^{\mathrm{K}}(n) = \int_{\mathrm{K}} H_\phi^{\mathrm{K}}(k)\chi_n(k)\, dk \qquad (n \neq 0)\,.$$

Indeed, for $n \geq 1$,

$$\hat{H}_\phi^{K}(n) = \Theta_{\mathsf{X}^{(n+1)}}(\phi) - \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t|n}\, dt - \frac{1}{2}\int_{\mathbb{R}} (-1)^m H_\phi^{\mathrm{A}}(-h_t)e^{-|t|n}\, dt$$

and

$$\hat{H}_\phi^{K}(-n) = \Theta_{\mathsf{X}^{(-n-1)}}(\phi) - \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t||n|}\, dt - \frac{1}{2}\int_{\mathbb{R}} (-1)^m H_\phi^{\mathrm{A}}(-h_t)e^{-|t||n|}\, dt$$

In particular,

$$\begin{aligned}\hat{H}_\phi^{K}(n) - \hat{H}_\phi^{\mathrm{K}}(-n) &= \Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi) \\ &- \int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t|n}\, dt - \int_{\mathbb{R}} (-1)^m H_\phi^{\mathrm{A}}(-h_t)e^{-|t|n}\, dt\,. \qquad (70)\end{aligned}$$

On the other hand, for $k \neq \pm 1$

$$H_\phi^{\mathrm{K}}(k) = \sum_{n\in\mathbb{Z}} \hat{H}_\phi^{\mathrm{K}}(n)\chi_{-n}(k) = \sum_{n\in\mathbb{Z}} \hat{H}_\phi^{\mathrm{K}}(n)\chi_n(k^{-1}) = \sum_{n\in\mathbb{Z}} \hat{H}_\phi^{\mathrm{K}}(-n)\chi_{-n}(k^{-1})$$

and therefore

$$H^{\mathrm{K}}_\phi(k) - H^{\mathrm{K}}_\phi(k^{-1}) = \sum_{0\neq n\in\mathbb{Z}} (H^{\mathrm{K}}_\phi(n) - H^{\mathrm{K}}_\phi(-n))\chi_{-n}(k)\,.$$

Thus,

$$H^{\mathrm{K}}_\phi(k(\theta)) - H^{\mathrm{K}}_\phi(k(-\theta)) = \sum_{n=1}^{\infty}(H^{\mathrm{K}}_\phi(n) - H^{\mathrm{K}}_\phi(-n))(-i)\sin(n\theta)\,. \tag{71}$$

Continuing this way (and correcting the constants, if necessary) one obtains the following lemma, see [Lan85, page 174],

Lemma 109.

$$\frac{\pi}{i}(H^{\mathrm{K}}_\phi(k(\theta)) - H^{\mathrm{K}}_\phi(k(-\theta))) = -\sum_{n=1}^{\infty}\Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi)\sin(n\theta)$$
$$+\sum_{n=1}^{\infty}\int_{\mathbb{R}}\frac{1}{2}\left(H^{\mathrm{A}}_\phi(h_t) + (-1)^{n+1}H^{\mathrm{A}}_\phi(-h_t)\right)\sin(n\theta)e^{-|t|n}\,dt\,. \tag{72}$$

Let

$$\phi^+(x) = \frac{\phi(x)+\phi(-x)}{2} \quad \text{and} \quad \phi^-(x) = \frac{\phi(x)-\phi(-x)}{2}\,.$$

Then a straightforward argument shows that

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}}\frac{1}{2}\left(H^{\mathrm{A}}_\phi(h_t) + (-1)^{n+1}H^{\mathrm{A}}_\phi(-h_t)\right)\sin(n\theta)e^{-|t|n}\,dt$$
$$= \int_{\mathbb{R}} H^{\mathrm{A}}_{\phi^+}(h_t)\frac{\sin(\theta)\cosh(t)}{\cosh(2t)-\cos(2\theta)}\,dt + \int_{\mathbb{R}} H^{\mathrm{A}}_{\phi^+}(h_t)\frac{\sin(2\theta)}{\cosh(2t)-\cos(2\theta)}\,dt$$

For $\psi \in C_c^\infty(\mathrm{G})$ we have the Fourier transform

$$\hat{H}^{\mathrm{A}}_\psi(\lambda) = \int_{\mathbb{R}} H^{\mathrm{A}}_\psi(h_t)\,e^{it\lambda}\,dt = \Theta_{\pi_{i\lambda}}(\phi)\,, \tag{73}$$

where the second equality follows from (50). One computes the Fourier transforms of the functions

$$\frac{\sin(\theta)\cosh(t)}{\cosh(2t)-\cos(2\theta)} \quad \text{and} \quad \frac{\sin(2\theta)}{\cosh(2t)-\cos(2\theta)}$$

to deduce from Lemma (109) and (73) the following theorem. See [Lan85, page 173]

Theorem 110.

$$\frac{\pi}{i}(H^{\mathrm{K}}_\phi(k(\theta)) - H^{\mathrm{K}}_\phi(k(-\theta))) = -\sum_{n=1}^{\infty}\Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi)\sin(n\theta)$$
$$+\frac{1}{2}\int_0^\infty \Theta_{\pi_{+,i\lambda}}(\phi)\frac{\cosh((\frac{\pi}{2}-\theta)\lambda)}{\cosh(\frac{\pi\lambda}{2})}\,d\lambda + \frac{1}{2}\int_0^\infty \Theta_{\pi_{-,i\lambda}}(\phi)\frac{\sinh((\frac{\pi}{2}-\theta)\lambda)}{\sinh(\frac{\pi\lambda}{2})}\,d\lambda$$

Now we take the derivative with respect to θ of both sides, go to limit with $\theta \to 0$ and apply (69) to deduce the Harish-Chandra's Plancherel formula.

Theorem 111. *For any $\phi \in C_c^\infty(\mathrm{G})$,*

$$2\pi\phi(1) = \sum_{n=1}^{\infty} n\Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi) \\ + \frac{1}{2}\int_0^\infty \Theta_{\pi_{+,i\lambda}}(\phi)\,\lambda\tanh(\frac{\pi\lambda}{2})\,d\lambda + \frac{1}{2}\int_0^\infty \Theta_{\pi_{-,i\lambda}}(\phi)\,\lambda\coth(\frac{\pi\lambda}{2})\,d\lambda\,.$$

This theorem was published by Harish-Chandra in 1952, [HC52]. Notice that knowing the composition series of the principal series was crucial for this proof.

Since $\Theta_\pi(\phi) = \mathrm{tr}\,\pi(\phi)$ the formula of Theorem 111 may be rewritten as

$$\phi(1) = \frac{1}{2\pi}\sum_{n=1}^{\infty} n\left(\mathrm{tr}(\pi_{n+1}(\phi)) + \mathrm{tr}(\pi_{-n-1}(\phi))\right) \\ + \frac{1}{4\pi}\int_0^\infty \mathrm{tr}\,\pi_{+,i\lambda}(\phi)\,\lambda\tanh(\frac{\pi\lambda}{2})\,d\lambda + \frac{1}{4\pi}\int_0^\infty \mathrm{tr}\,\pi_{-,i\lambda}(\phi)\,\lambda\coth(\frac{\pi\lambda}{2})\,d\lambda\,, \tag{74}$$

where π_m is the discrete series representation of G with the $(\mathfrak{g}, \mathrm{K})$-module $\mathsf{X}^{(m)}$. Let us fix $x \in \mathrm{G}$ and replace ϕ by $R(x)\phi$. Then

$$R(x)\phi(1) = \phi(x)$$

and

$$\pi(R(x)\phi) = \int_{\mathrm{G}} \phi(gx)\pi(g)\,dg = \int_{\mathrm{G}} \phi(g)\pi(gx^{-1})\,dg = \int_{\mathrm{G}} \phi(g)\pi(g)\,dg\pi(x^{-1}) = \pi(\phi)\pi(x^{-1})\,.$$

Hence,

$$\phi(x) = \frac{1}{2\pi}\sum_{n=1}^{\infty} n\left(\mathrm{tr}(\pi_{n+1}(\phi)\pi(x^{-1})) + \mathrm{tr}(\pi_{-n-1}(\phi)\pi(x^{-1}))\right) \\ + \frac{1}{4\pi}\int_0^\infty \mathrm{tr}(\pi_{+,i\lambda}(\phi)\pi(x^{-1}))\,\lambda\tanh(\frac{\pi\lambda}{2})\,d\lambda + \frac{1}{4\pi}\int_0^\infty \mathrm{tr}(\pi_{-,i\lambda}(\phi)\pi(x^{-1}))\,\lambda\coth(\frac{\pi\lambda}{2})\,d\lambda\,.$$

Therefore one defines the Fourier transform of ϕ to be an operator valued function

$$\mathcal{F}(\phi)(\pi) = \pi(\phi) = \int_{\mathrm{G}} \phi(g)\pi(g)\,dg \in \mathrm{End}(\mathsf{V}_\pi) \qquad (\phi \in \mathbb{C}_c^\infty(\mathrm{G}))\,, \tag{75}$$

where π is one of the irreducible unitary representations of G which occur in Theorem (111), realized on a Hilbert space V_π. Furthermore we have the Hilbert-Schmidt inner product and norm on the subspace $\mathrm{End}(\mathsf{V}_\pi)_{HS} \subseteq \mathrm{End}(\mathsf{V}_\pi)$,

$$(S,T) = \mathrm{tr}(ST^*)\,, \quad \| T \|_2^2 = (T,T) \qquad (S,T \in \mathrm{End}(\mathsf{V}_\pi))\,.$$

With this notation, we have the following Fourier Inversion Formula

Theorem 112. *For any $\phi \in C_c^\infty(\mathrm{G})$,*

$$\phi(x) = \frac{1}{2\pi}\sum_{n=1}^{\infty} n\left(\mathrm{tr}(\mathcal{F}(\phi)(\pi_{n+1})\pi(x^{-1})) + \mathrm{tr}(\mathcal{F}(\phi)(\pi_{-n-1})\pi(x^{-1}))\right)$$
$$+\frac{1}{4\pi}\int_0^\infty \mathrm{tr}(\mathcal{F}(\phi)(\pi_{+,i\lambda})\pi(x^{-1}))\,\lambda\tanh(\frac{\pi\lambda}{2})\,d\lambda+\frac{1}{4\pi}\int_0^\infty \mathrm{tr}(\mathcal{F}(\phi)(\pi_{-,i\lambda})\pi(x^{-1}))\,\lambda\coth(\frac{\pi\lambda}{2})\,d\lambda\,.$$

Recall also that $\phi^*(x) = \overline{\phi(x^{-1})}$ so that

$$\| \phi \|_2^2 = \int_{\mathrm{G}} |\phi(x)|^2\,dx = \int_{\mathrm{G}} \phi(x)\phi^*(x^{-1})\,dx = \int_{\mathrm{G}} \phi(x)\phi^*(x^{-1}1)\,dx = \phi * \phi^*(1)\,.$$

Also, for a unitary representation π,

$$\pi(\phi)^* = \left(\int_{\mathrm{G}} \pi(g)\phi(g)\,dg\right)^* = \int_{\mathrm{G}} \pi(g)^*\overline{\phi(g)}\,dg = \int_{\mathrm{G}} \pi(g^{-1})\overline{\phi(g)}\,dg = \int_{\mathrm{G}} \pi(g)\overline{\phi(g^{-1})}\,dg$$
$$= \pi(\phi^*)\,.$$

Hence (74) implies the following theorem

Theorem 113. *For any $\phi \in C_c^\infty(\mathrm{G})$,*

$$\| \phi \|_2^2 = \frac{1}{2\pi}\sum_{n=1}^{\infty} n\left(\| \mathcal{F}(\phi)(\pi_{n+1}) \|_2^2 + \| \mathcal{F}(\phi)(\pi_{-n-1}) \|_2^2\right)$$
$$+\frac{1}{4\pi}\int_0^\infty \| \mathcal{F}(\phi)(\pi_{+,i\lambda}) \|_2^2\ \lambda\tanh(\frac{\pi\lambda}{2})\,d\lambda + \frac{1}{4\pi}\int_0^\infty \| \mathcal{F}(\phi)(\pi_{-,i\lambda}) \|_2^2\ \lambda\coth(\frac{\pi\lambda}{2})\,d\lambda\,.$$

Also, for $\psi \in C_c^\infty(\mathrm{G})$,

$$(\phi,\psi) = \frac{1}{2\pi}\sum_{n=1}^{\infty} n\left((\mathcal{F}(\phi)(\pi_{n+1}),\mathcal{F}(\psi)(\pi_{n+1})) + (\mathcal{F}(\phi)(\pi_{-n-1}),\mathcal{F}(\psi)(\pi_{-n-1}))\right)$$
$$+\frac{1}{4\pi}\int_0^\infty (\mathcal{F}(\phi)(\pi_{+,i\lambda}),\mathcal{F}(\psi)(\pi_{+,i\lambda}))\,\lambda\tanh(\frac{\pi\lambda}{2})\,d\lambda$$
$$+\frac{1}{4\pi}\int_0^\infty (\mathcal{F}(\phi)(\pi_{-,i\lambda}),\mathcal{F}(\psi)(\pi_{-,i\lambda}))\,\lambda\coth(\frac{\pi\lambda}{2})\,d\lambda\,.$$

Recall, [Kir78, page 68] the notion of the direct integral of Hilbert spaces $\mathcal{H}_x$, indexed by x in some set X equipped with a measure μ,

$$\int_X \mathcal{H}_x\,d\mu(x)\,.$$

This is a Hilbert space of functions f such that for each $x \in X$, $f(x) \in \mathcal{H}_x$ and

$$(f,h)_{\int_X \mathcal{H}_x\,d\mu(x)} = \int_X (f(x),h(x))_{\mathcal{H}_x}\,d\mu(x)\,.$$

We take X to be the set $\hat{\mathrm{G}}_{temp}$ of the equivalence classes of the irreducible unitary representations of G which occur in Theorem 111 and for each $\pi \in \hat{\mathrm{G}}_{temp}$, $\mathcal{H}_\pi = \mathrm{End}(\mathsf{V}_\pi)$. We may identify

$$\hat{\mathrm{G}}_{temp} = (\mathbb{Z} \setminus \{0\}) \cup (\{\pm\} \times \mathbb{R}^+)$$

and define the measure

$$\begin{aligned} d\mu(\pi_n) &= d\mu(\pi_{-n}) = d\mu(n) = \frac{1}{2\pi}|n| \qquad (n \in \mathbb{Z} \setminus \{0\})\,, \\ d\mu(\pi_{+,\lambda}) &= d\mu(+,\lambda) = \frac{1}{4\pi}\lambda \tanh(\frac{\pi\lambda}{2})\, d\lambda \qquad (\lambda \in \mathbb{R}^+)\,, \\ d\mu(\pi_{-,\lambda}) &= d\mu(-,\lambda) = \frac{1}{4\pi}\lambda \coth(\frac{\pi\lambda}{2})\, d\lambda \qquad (\lambda \in \mathbb{R}^+)\,. \end{aligned}$$

The subspace $C_c^\infty(\mathrm{G}) \subseteq \mathrm{L}^2(\mathrm{G})$ inherits the inner product and Theorem 113 implies that the Fourier transform is an injective isometry of Hilbert spaces

$$\mathcal{F} : C_c^\infty(\mathrm{G}) \to \int_{\hat{\mathrm{G}}_{temp}} \mathrm{End}(\mathsf{V}_\pi)_{HS}\, d\mu(\pi)\,. \tag{76}$$

The group $\mathrm{G} \times \mathrm{G}$ acts on $C_c^\infty(\mathrm{G})$ via the left and right regular representation:

$$\phi(x) \to \phi(g^{-1}xh) \qquad (g, h, x \in \mathrm{G})\,.$$

This groups also acts on the direct integral of Hilbert spaces, (76),

$$\mathrm{End}(\mathsf{V}_\pi)_{HS} \ni T \to \pi(g)T\pi(h)^{-1} \in \mathrm{End}(\mathsf{V}_\pi)_{HS} \qquad (\pi \in \hat{\mathrm{G}}_{temp}) \tag{77}$$

and it is easy to see that the Fourier transform (76) intertwines these two actions. Clearly both actions are by unitary representations. With some more effort one deduces the following theorem.

Theorem 114. *The Fourier transform* (76) *extends to a* $\mathrm{G} \times \mathrm{G}$ *intertwining bijective isometry of Hilbert spaces*

$$\mathcal{F} : \mathrm{L}^2(\mathrm{G}) \to \int_{\hat{\mathrm{G}}_{temp}} \mathrm{End}(\mathsf{V}_\pi)_{HS}\, d\mu(\pi)\,. \tag{78}$$

Recall that the map

$$\begin{aligned} &\mathsf{V}_\pi \otimes \mathsf{V}_\pi \ni u \otimes v \to E_{u\otimes v} \in \mathrm{End}(\mathsf{V}_\pi)\,, \\ &E_{u\otimes v}(w) = (w, u)_{\mathsf{V}_\pi} v \end{aligned}$$

extends to a bijective isometry. It is linear with respect to v and anti-linear with respect to u. Moreover,

$$\pi(g)E_{u\otimes v}\pi(h)^{-1}(w) = (\pi(h)^{-1}w, u)_{\mathsf{V}_\pi}\pi(g)v = (w, \pi(h)u)_{\mathsf{V}_\pi}\pi(g)v = E_{\pi(h)u\otimes\pi(g)v}\,.$$

If we introduce a new multiplication by complex numbers on the "left" copy of V_π by

$$z \cdot u = \overline{z}u \qquad (z \in \mathbb{C})$$

then we still have $z \cdot \pi(h)u = \pi(h)z \cdot u$. Thus the new V_π, call it $\overline{\mathsf{V}}_\pi$ still carries a unitary representation of G, call it $\overline{\pi}$. The result is a representation $(\overline{\pi}, \overline{\mathsf{V}}_\pi)$. Altogether we have a $\mathrm{G} \times \mathrm{G}$ intertwining bijective isometry which extends the map

$$\overline{\mathsf{V}}_\pi \otimes \mathsf{V}_\pi \ni u \otimes v \to E_{u\otimes v} \in \mathrm{End}(\mathsf{V}_\pi)\,.$$

Theorem 115. *The Fourier transform* (76) *extends to a* $\mathrm{G} \times \mathrm{G}$ *intertwining bijective isometry of Hilbert spaces*

$$\mathcal{F} : \mathrm{L}^2(\mathrm{G}) \to \int_{\hat{\mathrm{G}}_{temp}} \overline{\mathsf{V}}_\pi \otimes \mathsf{V}_\pi \, d\mu(\pi)\,. \tag{79}$$

Symbolically

$$\mathcal{F} : (L \otimes R, \mathrm{L}^2(\mathrm{G})) \to \int_{\hat{\mathrm{G}}_{temp}} (\overline{\pi} \otimes \pi, \overline{\mathsf{V}}_\pi \otimes \mathsf{V}_\pi)\, d\mu(\pi)\,, \tag{80}$$

or

$$L \otimes R = \int_{\hat{\mathrm{G}}_{temp}} \overline{\pi} \otimes \pi \, d\mu(\pi)\,. \tag{81}$$

12.13. **An irreducible unitary non-tempered representation of** $\mathrm{SL}_2(\mathbb{R})$**.**

12.13.1. **An abstract** $(\mathfrak{g}, \mathrm{K})$**-module.** We begin by constructing an irreducible unitarizable $(\mathfrak{g}, \mathrm{K})$-modules which is infinite dimensional and does not occur on the list of $(\mathfrak{g}, \mathrm{K})$-modules which are present in the Harish-Chandra Plancherel formula, Theorem 111.

Let

$$e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$[e^+, e^-] = h\,, \quad [h, e^+] = 2e^+\,, \quad [h, e^-] = -2e^-$$

and

$$\mathfrak{g} = \mathbb{R}e^+ \oplus \mathbb{R}e^- \oplus \mathbb{R}h\,.$$

Let

$$\hbar = i(e^- - e^+)\,, \quad n^+ = \frac{1}{2}(h + i(e^+ + e^-))\,, \quad n^- = \frac{1}{2}(h - i(e^+ + e^-))\,.$$

Then

$$[n^+, n^-] = \hbar\,, \quad [\hbar, n^+] = 2n^+\,, \quad [\hbar, n^-] = -2n^-$$

and

$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C}n^+ \oplus \mathbb{C}n^- \oplus \mathbb{C}\hbar\,.$$

Moreover

$$\overline{\hbar} = -\hbar\,, \quad \overline{n^+} = n^-\,.$$

As a vector space

$$\mathrm{X} = \bigoplus_{j=0}^{\infty} \mathbb{C}v_j\,,$$

where v_0, v_1, ... is a basis of X and the action of $\mathfrak{g}_{\mathbb{C}}$ is given by

$$\begin{aligned} v_j &= (n^+)^j v_0 \qquad (j = 0, 1, 2, ...)\,, \\ n^- v_j &= -j^2 v_{j-1} \qquad (j = 1, 2, ...)\,, \\ n^- v_0 &= 0\,, \\ \hbar v_j &= (1 + 2j) v_j \qquad (j = 0, 1, 2, ...)\,. \end{aligned}$$

The group $\mathrm{K} = \mathrm{SO}_2$ acts by taking the exponential of the last formula

$$k(\theta) v_j = e^{i(1+2j)\theta} v_j \qquad (j = 0, 1, 2, ...)\,.$$

Indeed,

$$k(\theta) v_j = \exp(\theta((e^+ - e^-)) v_j$$

and

$$\theta((e^+ - e^-) v_j = i\theta(i(e^- - e^+)) v_j = i\theta(1 + 2j) v_j\,.$$

Hence the formula follows. Thus X is a $(\mathfrak{g}, \mathrm{K})$-module and it is not difficult to check that it is irreducible.

We need to define an invariant positive definite Hermitian product $(\cdot, \cdot)$ on X. Thus we must have

$$((X + iY)u, v) = (u, -\overline{(X + iY)} v) \qquad (X, Y \in \mathfrak{g};\ u, v \in \mathrm{X})\,.$$

We do it as follows. We declare the v_j to be orthogonal and set

$$(v_j, v_j) = (j!)^2 \qquad (j = 0, 1, 2, ...)\,.$$

Then

$$(v_{j+1}, v_{j+1}) = (n^+ v_j, v_{j+1}) = (v_j, -n^- v_{j+1}) = (v_j, (j + 1)^2 v_j) = (j + 1)^2 (v_j, v_j)\,.$$

This implies that the inner product is invariant indeed.

12.13.2. **A realization of the abstract $(\mathfrak{g}, \mathrm{K})$-module.** Next we realize the $(\mathfrak{g}, \mathrm{K})$-module explicitly.

Define te following operators on the space $\mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned} \omega(h) &= x_1 \partial_{x_1} + x_2 \partial_{x_2} + 1\,, \\ \omega(e^+) &= \frac{i}{2}(x_1^2 + x_2^2)\,, \\ \omega(e^-) &= \frac{i}{2}(\partial_{x_1}^2 + \partial_{x_2}^2)\,. \end{aligned}$$

The resulting map $\omega : \mathfrak{g} \to \mathrm{End}(\mathcal{S}(\mathbb{R}^2))$ defines a representation of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $a_j = x_j + \partial_{x_j}$, $b_j = x_j - \partial_{x_j}$, $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ and let $r^2 = x_1^2 + x_2^2$ A straightforward computation shows that

$$\begin{aligned} \omega(\hbar) &= \frac{1}{4}\sum_{j=1}^{2}(a_jb_j+b_ja_j)\,,\\ \omega(n^+) &= \frac{i}{4}e^{\frac{r^2}{2}}\Delta e^{-\frac{r^2}{2}}\,,\\ \omega(n^-) &= \frac{i}{4}e^{-\frac{r^2}{2}}\Delta e^{\frac{r^2}{2}}\,. \end{aligned}$$

Let $v_0=\frac{1}{\sqrt{\pi}}e^{-\frac{r^2}{2}}$ and let $(\cdot,\cdot)$ denote the usual L^2 inner product on $\mathcal{S}(\mathbb{R})$. Then

$$(v_0,v_0)=1\,.$$

We can realize the $(\mathfrak{g},\mathrm{K})$-module constructed previously as

$$\mathrm{X}=\bigoplus_{j=0}^{\infty}\mathbb{C}v_j\subseteq\mathcal{S}(\mathbb{R}^2)\,.$$

where $v_j=\omega(n^+)^jv_0$, $j=0,1,2,\ldots.$

12.13.3. **The irreducible unitary representation of the group.** Finally we describe the representation of the group G whose $(\mathfrak{g},\mathrm{K})$-module is equal to X. The orthogonal group O_2 acts on $\mathcal{S}(\mathbb{R}^2)$ in the usual fashion

$$\omega(g)v(x)=v(xg)\qquad(v\in\mathcal{S}(\mathbb{R}^2),x\in\mathbb{R}^2,g\in\mathrm{O}_2)\,.$$

Let $\mathcal{S}(\mathbb{R}^2)^{\mathrm{O}_2}\subseteq\mathcal{S}(\mathbb{R}^2)$ denote the subspace of the invariants. Recall the Iwasawa decomposition $\mathrm{G}=\mathrm{KAN}$, (38). For $v\in\mathcal{S}(\mathbb{R}^2)$ set

$$\begin{aligned} \omega(\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix})v(x) &= av(ax)\,,\\ \omega(\begin{pmatrix} 1 & n\\ 0 & 1\end{pmatrix})v(x) &= e^{in\frac{r^2}{2}}v(x)\,,\\ \omega(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix})v(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(x_1y_1+x_2y_2)}v(y_1,y_2)\,dy_1\,dy_2\,. \end{aligned}$$

Since he space $\mathcal{S}(\mathbb{R}^2)^{\mathrm{O}_2}$ is dense in $\mathrm{L}^2(\mathbb{R}^2)^{\mathrm{O}_2}$ and since the subgroups A, N together with the element $\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$ generate G we conclude that $(\omega,\mathrm{L}^2(\mathbb{R}^2)^{\mathrm{O}_2})$ is an irreducible unitary representation of G with the $(\mathfrak{g},\mathrm{K})$-module X desribed above.

13. The Heisenberg group.

13.1. **Structure of the group.** Let W be a finite dimensional vector space over $\mathbb{R}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$. This means that $\langle\ ,\ \rangle$ is a bilinear form such that

$$\langle w,w'\rangle=-\langle w',w\rangle$$

and if

$$\langle w, w' \rangle = 0$$

for all $w' \in \mathsf{W}$ then $w = 0$. As is well known, the dimension of W is even and W has a basis

$$e_1, e_2, ..., e_n, f_1, f_2, ..., f_n$$

such that

$$\langle e_j, f_k \rangle = \delta_{j,k}\,, \quad \langle e_j, e_k \rangle = \langle f_j, f_k \rangle = 0\,.$$

By fixing such a basis we get linear isomorphism

$$\mathbb{R}^{2n} \ni (w_1, ..., w_n, w_{n+1}, .., w_{2n}) \to w = \sum_{j=1}^{n} w_j e_j + \sum_{k=1}^{n} w_{n+k} f_k \in \mathsf{W}\,. \tag{82}$$

Moreover the formula

$$J(e_j) = -f_j\,, \quad J(f_k) = e_k$$

defines a linear map $J \in \mathrm{End}(\mathsf{W})$ such that $J^2 = -I$ and the bilinear form

$$\langle J\cdot, \cdot \rangle \tag{83}$$

is symmetric and positive definite. Any map with these properies is calle a positive definite compatible complex structure on W.

Using the identification (82) we define a Lebesgue measure dw on W by

$$dw = dw_1...dw_{2n}$$

It is characterized by the property that the volume of the unit cube with respect to the form (83) is 1.

Furthermore, the two subspaces

$$\mathrm{X} = \sum_{j=1}^{n} \mathbb{R} e_j\,, \quad \mathrm{Y} = \sum_{j=1}^{n} \mathbb{R} f_j$$

are maximal isotropic and

$$\mathsf{W} = \mathrm{X} \oplus \mathrm{Y}\,. \tag{84}$$

By the Heisenberg group we understand the direct product $\mathrm{H} = \mathrm{H}(\mathsf{W}) = \mathsf{W} \times \mathbb{R}$ with the multiplication given by

$$(w, r)(w', r') = (w + w', r + r' + \frac{1}{2}\langle w, w' \rangle) \qquad ((w, r), (w', r') \in \mathrm{H}(\mathsf{W})).$$

Then $\mathrm{Z} = \{0\} \times \mathbb{R}$ is the center of H.

13.2. **Decomposition of the right regular representation.** For $f_1, f_2 \in \mathcal{S}(\mathrm{H})$ we have the convolution

$$f_1 * f_2(w,r) = \int_{\mathrm{H}} f_1(w',r') f_2((w,r)(w',r')^{-1})\, dw'\, dr'$$
$$= \int_{\mathrm{H}} f_1(w',r') f_2((w,r)(-w',-r'))\, dw'\, dr'$$
$$= \int_{\mathrm{H}} f_1(w',r') f_2((w-w', r-r' - \frac{1}{2}\langle w, w'\rangle)\, dw'\, dr'$$

Define the central Fourier transform

$$Z\mathcal{F} : \mathcal{S}(\mathrm{H}) \to \mathcal{S}(\mathrm{H})$$

by the formula

$$Z\mathcal{F} f(w,\chi) = \int_{\mathbb{R}} f(w,r)\chi(r)\, dr\,, \tag{85}$$

where χ is the unitary character of $\mathbb{R}$ identified with a real number χ by $\chi(r) = e^{2\pi i \chi r}$, $r \in \mathbb{R}$. A straightforward computation shows that

$$Z\mathcal{F}(f_1 * f_2)(w,\chi) = \int_{\mathsf{W}} Z\mathcal{F} f_1(w',\chi) Z\mathcal{F} f_2(w-w',\chi)\chi(\frac{1}{2}\langle w, w'\rangle)\, dw'\,. \tag{86}$$

Thus if we introduce the following twisted convolution in $\mathcal{S}(\mathsf{W})$

$$\psi \natural \phi(w) = \int_{\mathsf{W}} \psi(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, du \qquad (w \in \mathsf{W}) \tag{87}$$

then the central Fourier transform followed by evaluation at χ reults in the twisted convolution of the central Fourier transforms evaluated at χ. Also,

$$Z\mathcal{F} f^*(w,\chi) = \overline{Z\mathcal{F} f(w,\chi)}\,. \tag{88}$$

Plancherel formula for $\mathbb{R}$ shows that

$$\| f \|_2^2 = \| Z\mathcal{F} f \|_2^2\ . \tag{89}$$

Define the Weyl transform

$$\mathcal{K}_\chi : \mathcal{S}(\mathsf{W}) \to \mathcal{S}(\mathrm{X} \times \mathrm{X})\,, \tag{90}$$
$$\mathcal{K}_\chi(\phi)(x,x') = \int_{\mathrm{Y}} \phi(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, dy\,.$$

A straightforward computation shows that

$$\mathcal{K}_\chi(\phi \natural \psi)(x,x'') = \int_{\mathrm{X}} \mathcal{K}_\chi(\phi)(x,x')\mathcal{K}_\chi(\psi)(x',x'')\, dx' \tag{91}$$

and

$$\| \mathcal{K}_\chi(\phi) \|_2^2 = |\chi|^{-n} \| \phi \|_2^2\ . \tag{92}$$

Problem 25. *Verify* (91) *and* (92).

To check (92) *we compute,*

$$\int_{\mathsf{X}}\int_{\mathsf{X}} |\mathcal{K}_\chi(\phi)(x,x')|^2\,dx\,dx' = 2^{-n}\int_{\mathsf{X}}\int_{\mathsf{X}} |\mathcal{K}_\chi(\phi)(\frac{u+v}{2},\frac{u-v}{2})|^2\,du\,dv$$

$$2^{-n}\int_{\mathsf{X}}\int_{\mathsf{X}} |\int_{\mathrm{Y}} \phi(v+y)\chi(\frac{1}{2}\langle y,u\rangle)\,dy|^2\,du\,dv = 2^{-n}\int_{\mathsf{X}}\int_{\mathsf{X}} |\int_{\mathrm{Y}} \phi(v+y)e^{i\pi\chi\langle y,u\rangle}\,dy|^2\,du\,dv$$

$$= \int_{\mathsf{X}}\int_{\mathsf{X}} |\int_{\mathrm{Y}} \phi(v+y)e^{2\pi i\chi\langle y,u\rangle}\,dy|^2\,du\,dv = \int_{\mathsf{X}}\int_{\mathsf{X}} |\int_{\mathrm{Y}} \phi(v+y)e^{2\pi i\langle y,\chi u\rangle}\,dy|^2\,du\,dv$$

$$= |\chi|^{-n}\int_{\mathsf{X}}\int_{\mathsf{X}} |\int_{\mathrm{Y}} \phi(v+y)e^{2\pi i\langle y,u\rangle}\,dy|^2\,du\,dv = |\chi|^{-n}\int_{\mathsf{X}}\int_{\mathsf{X}} |\phi(v+u)|^2\,du\,dv = |\chi|^{-n} \parallel \phi \parallel_2^2,$$

where in the second to last equation we use the fact that the Fourier transform on $\mathbb{R}^n$ is an isometry.

Set

$$Z\mathcal{F}_\chi(f)(w) = Z\mathcal{F}(f)(w,\chi)$$

Then our computations show that

$$\int_{\mathbb{R}} \parallel \mathcal{K}_\chi(Z\mathcal{F}_\chi(f)) \parallel_2^2\ |\chi|^n\,d\chi = \parallel f \parallel_2^2\ . \tag{93}$$

Each element $K\in\mathcal{S}(\mathsf{X}\times\mathsf{X})$ defines an operator $\mathrm{Op}(K)\in\mathrm{Hom}(\mathcal{S}(\mathsf{X}),\mathcal{S}(\mathsf{X}))$ by

$$\mathrm{Op}(K)v(x) = \int_{\mathsf{X}} K(x,x')v(x')\,dx'\,. \tag{94}$$

Recall the Hilbert-Schmidt norm for operators on $\mathrm{L}^2(\mathrm{X})$,

$$\parallel \mathrm{Op}(K) \parallel_{\mathrm{H.S.}}^2 = \int_{\mathsf{X}}\int_{\mathsf{X}} |K(x,x')|^2\,dx\,dx'$$

Then (93) may be rewritten as

$$\int_{\mathbb{R}} \parallel \mathrm{Op}\circ\mathcal{K}_\chi\circ Z\mathcal{F}_\chi(f)) \parallel_{\mathrm{H.S.}}^2\ |\chi|^n\,d\chi = \parallel f \parallel_2^2 \qquad (f\in\mathcal{S}(\mathrm{H}))\,. \tag{95}$$

Set

$$\omega_\chi = \mathrm{Op}\circ\mathcal{K}_\chi\circ Z\mathcal{F}_\chi\,.$$

Then, from what we just computed

$$\omega_\chi : \mathcal{S}(\mathrm{H}(\mathsf{W})) \to \mathrm{End}(\mathcal{S}(\mathrm{X}))$$

is an injective $*$-algebra homomorphism. By using approximate identity sequences on $\mathcal{S}(\mathrm{H})$ we extend ω_χ to a map

$$\omega_\chi : \mathrm{H}(\mathsf{W}) \to \mathrm{End}(\mathcal{S}(\mathrm{X}))\,.$$

Let

$$\mathrm{U}(\mathrm{L}^2(\mathsf{X}))$$

denote the group of the unitary operators on the Hilbert space $\mathrm{L}^2(\mathsf{X})$.

Theorem 116. *The map*

$$\omega_\chi \colon \mathrm{H}(\mathsf{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X})) \cap \mathrm{End}(\mathcal{S}(\mathrm{X}))$$

is an injective group homomorphism. For each $v \in \mathrm{L}^2(\mathsf{X})$, the map

$$\mathrm{H}(\mathsf{W}) \ni g \to \omega_\chi(g)v \in \mathrm{L}^2(\mathsf{X})$$

is continuous, so that $(\omega_\chi, \mathrm{L}^2(\mathsf{X}))$ is a unitary representation of the group. Explicitly, for $v \in \mathrm{L}^2(\mathsf{X})$,

$$\begin{aligned} \omega_\chi(x_0, r)v(x) &= \chi(r)v(x - x_0) \quad (x, x_0 \in \mathsf{X}, r \in \mathbb{R}), \\ \omega_\chi(y_0, r)v(x) &= \chi(r)\chi(\langle y_0, x\rangle)v(x) \quad (y_0 \in \mathsf{Y}, r \in \mathbb{R}), \end{aligned} \tag{96}$$

Hence, the representation $(\omega_\chi, \mathrm{L}^2(\mathsf{X}))$ of $\mathrm{H}(\mathsf{W})$ is irreducible. Moreover

$$\| f \|_2^2 = \int_{\mathbb{R}} \| \omega_\chi(f)) \|_{\mathrm{H.S.}}^2 \ |\chi|^n \, d\chi \,. \tag{97}$$

Hence, $|\chi|^n \, d\chi$ is the Plancherel measure and

$$(L, \mathrm{L}^2(\mathrm{H}(\mathsf{W}))) = \int_{\mathbb{R}\setminus\{0\}} (\omega_\chi, \mathrm{L}^2(\mathrm{X}))|\chi|^n \, d\chi \,. \tag{98}$$

Proof. We'll check that the representation $(\omega_\chi, \mathrm{L}^2(\mathsf{X}))$ given by the formula (96) is irreducible. Suppose $T : \mathrm{L}^2(\mathsf{X}) \to \mathrm{L}^2(\mathsf{X})$ is a bounded operator that commutes with $\omega_\chi(g)$ for all $g \in \mathrm{H}(\mathsf{W})$. In particular

$$T : \chi(\langle y_0, x\rangle)v(x) \to \chi(\langle y_0, x\rangle)Tv(x) \quad (y_0 \in \mathsf{Y}).$$

Hence, for any $f \in \mathcal{S}(\mathsf{Y})$,

$$T : \int_{\mathsf{Y}} f(y_0)\chi(\langle y_0, x\rangle)\, dy_0 \, v(x) \to \int_{\mathsf{Y}} f(y_0)\chi(\langle y_0, x\rangle)\, dy_0 \, Tv(x) \quad (y_0 \in \mathsf{Y}).$$

Set

$$\int_{\mathsf{Y}} f(y_0)\chi(\langle y_0, x\rangle)\, dy = \tilde{f}(x).$$

Then $\tilde{f}$ is a Fourier transform of f. Thus we checked that T commutes with the multiplication by $\tilde{f}$. Since the Fourier transform is bijective on the space of the Schwartz functions, we see that T commutes with the multiplication by any Schwartz function,

$$v(x) \to \tilde{f}(x)v(x) \qquad (x \in \mathsf{X}, \tilde{f} \in \mathcal{S}(\mathsf{X}), v \in \mathrm{L}^2(\mathsf{X})).$$

Let v_R be the indicator function of the ball B_R of radius R centered at the origin. Then for any $v \in \mathbb{C}_c^\infty(\mathsf{X})$ supported in that ball,

$$T(v) = T(v_R v) = vT(v_R) = T(v_R)v \,.$$

Hence $T(v_R)$ is a bounded measurable function and

$$T(v) = T(v_R)v \qquad (v \in C_c^\infty(B_R)).$$

This implies that
$$T(v_{R'})|_{B_R} = T(v_R) \qquad (R < R')\,.$$
Hence there is a function F such that
$$T(v)(x) = F(x)v(x) \qquad (v \in C_c^\infty(\mathsf{X}))\,.$$
But we also know that T commutes with the translations. Hence F is a constant function. Therefore the representation is irreducible. □

13.3. **Structure of the Lie algebra.** The Lie algebra $\mathfrak{h}(\mathsf{W})$ of the Heisenberg group is the direct sum $\mathfrak{h}(\mathsf{W}) = \mathsf{W} \oplus \mathbb{R}$ with the obvious structure of the vector space over $\mathbb{R}$ and the Lie bracket given by
$$[(w,r),(w',r')] = (0, \langle w, w'\rangle) \qquad ((w,r),(w',r') \in \mathfrak{h}(\mathsf{W})).$$
Then, under the identification of manifolds, $\mathfrak{h}(\mathsf{W}) = \mathrm{H}(\mathsf{W})$, the commutator in the group and the Lie bracket in the Lie algebra are related by the formula
$$(w,r)(w',r')(w,r^{-1})(w',r')^{-1} = [(w,r),(w',r')]\,.$$
The commutators of higher order are trivial. Hence $\mathrm{H}(\mathsf{W})$ is a step two nilpotent Lie group. Similarly, $\mathfrak{h}(\mathsf{W})$ is a step two nilpotent Lie algebra.

13.4. **A generic representation of the Lie algebra.** Let us fix a nontrivial character χ as in Theorem 116. Then by taking the derivatives we obtain the following action of the Lie algebra
$$\begin{aligned} &\omega_\chi(x_0,0)v(x) = -\partial_{x_0}v(x) \quad (x, x_0 \in \mathsf{X}, v \in \mathcal{S}(\mathsf{X}))\,, \\ &\omega_\chi(y_0,0)v(x) = 2\pi i\chi\langle y_0, x\rangle v(x) \quad (x \in \mathsf{X}, y_0 \in \mathsf{Y}, v \in \mathcal{S}(\mathsf{X}))\,, \\ &\omega_\chi(0,r)v(x) = 2\pi i\chi r v(x) \quad (r \in \mathbb{R}, v \in \mathcal{S}(\mathsf{X}))\,. \end{aligned} \tag{99}$$
where ∂_{x_0} is the directional derivative in the direction of x_0
$$\partial_{x_0}v(x) = \lim_{t\to 0}(v(x+tx_0) - v(x))t^{-1}\,.$$
Let us identify X with $\mathbb{R}^n$, by
$$\mathrm{X} \ni \sum_{j=1}^n n_j e_j = (x_1, x_2, ..., x_n) \in \mathbb{R}^n\,.$$
Also, lets choose χ so that $2\pi\chi = 1$. Set $\omega = \omega_\chi$. Then
$$\omega_\chi(e_j) = -\partial_j\,, \quad \omega_\chi(f_j) = ix_j \quad (1 \le j, k \le n)\,. \tag{100}$$
This is the "easiest on the eyes" version of the Heisenberg representation of the Heisenberg Lie algebra with its Canonical Commutation Relations
$$[\partial_j, x_k] = \delta_{j,k}\,.$$

14. **An example of a wild group.**

Here we follow [Kir78, $19]. This is the last section of this book. It contains some "removable singularities". Please check comparing with the text below.

14.1. **The Mautner group.** Fix an irrational number $\alpha \in \mathbb{R}$. Let G be the group of matrices

$$\begin{pmatrix} e^{it} & 0 & z \\ 0 & e^{i\alpha t} & w \\ 0 & 0 & 1 \end{pmatrix},$$

where $t \in \mathbb{R}$, $z, w \in \mathbb{C}$. Let us identity the above matrix with the string of numbers (t, z, w). Then t

$$\begin{aligned} (t,z,w)(\tau,\xi,\eta) &= (t+\tau, z+e^{it}\xi, w+e^{i\alpha t}\eta)\,, \\ (t,z,w)(0,0,0) &= (t,z,w)\,, \\ (\tau,\xi,\eta)^{-1} &= (-\tau, -e^{-i\tau}\xi, -e^{-i\alpha\tau}\eta)\,. \end{aligned}$$

If we identify $\mathbb{R} = \{(t,0,0);\ t \in \mathbb{R}\}$ and $\mathbb{C}^2 = \{(0,z,w);\ z, w \in \mathbb{C}\}$, then we see that G is the semidirect product of $\mathbb{R}$ and $\mathbb{C}^2$. Thus G is a solvable group. It is easy to check that the formula

$$\int f(t,z,w)\,dt\,dz\,dw$$

defines a left and right invariant Haar measure on G. Here dz is the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

14.2. **The right regular representation 1.** Recall the right regular representation $(R, \mathrm{L}^2(\mathrm{G}))$,

$$R(\tau,\xi,\eta)f(t,z,w) = f((t,z,w)(\tau,\xi,\eta)) = f((t+\tau, z+e^{it}\xi, w+e^{i\alpha t}\eta))\,.$$

Let

$$\chi(z) = e^{2\pi i \mathrm{Re}(z)} \qquad (z \in \mathbb{C})\,.$$

Then the formula

$$Ff(t,a,b) = \tilde{f}(t,a,b) = \int_{\mathbb{C}\times\mathbb{C}} \chi(a\overline{z} + b\overline{w}) f(t,z,w)\,dz\,dw$$

defines a bijective isometry

$$F : \mathrm{L}^2(\mathrm{G}) \to \mathrm{L}^2(\mathrm{G})\,.$$

Lemma 117. *Set*

$$\tilde{R}(g) = FR(g)F^{-1} \qquad (g \in \mathrm{G})\,.$$

Then

$$F : (R, \mathrm{L}^2(\mathrm{G})) \to (\tilde{R}, \mathrm{L}^2(\mathrm{G}))$$

is an isomorphism of unitary representations of G *and explicitly,*

$$\tilde{R}(\tau,\xi,\eta)\tilde{f}(t,a,b) = \chi(-(ae^{-it}\overline{\xi} + be^{-i\alpha t}\overline{\eta}))\tilde{f}(t+\tau,a,b)\,.$$

Proof. The first part is obvious. Since

$$\int \chi(a\overline{z}+b\overline{w})f(t+\tau, z+e^{it}\xi, w+e^{i\alpha t}\eta))\,dz\,dw$$
$$= \chi(-(ae^{-it}\overline{\xi}+be^{-i\alpha t}\overline{\eta}))\int \chi(a\overline{z}+b\overline{w})f(t,z,w)\,dz\,dw$$

the claim follows. □

For $a, b \in \mathbb{C}$ set

$$U_{a,b}(\tau,\xi,\eta)v(t) = \chi(-(ae^{-it}\overline{\xi}+be^{-i\alpha t}\overline{\eta}))v(t+\tau) \qquad (v \in \mathrm{L}^2(\mathbb{R}))\,.$$

Then a straightforward computation shows that $(U_{a,b}, \mathrm{L}^2(\mathbb{R}))$ is a unitary representation of G. Also, by fixing the $a, b \in C$ we may view a function $\tilde{f}(t,a,b)$ as a function of t and write

$$\tilde{f}_{a,b}(t) = \tilde{f}(t,a,b)\,.$$

This way the map

$$\mathrm{L}^2(\mathrm{G}) \ni \tilde{f} \to \tilde{f}_{a,b} \in \mathrm{L}^2(\mathbb{R})_{a,b} = \mathrm{L}^2(\mathbb{R})$$

gives an isomorphism of Hilbert spaces

$$\mathrm{L}^2(\mathrm{G}) = \int_{\mathbb{C}\times\mathbb{C}} \mathrm{L}^2(\mathbb{R})_{a,b}\,da\,db\,.$$

Furthermore, Lemma 117 shows that the above map intertwines $\tilde{R}$ with the direct integral of the representations $U_{a,b}$. Thus

$$(\tilde{R}, \mathrm{L}^2(\mathrm{G})) \quad \text{is unitarily equivalent to} \quad \int_{\mathbb{C}\times\mathbb{C}} (U_{a,b}, \mathrm{L}^2(\mathbb{R})_{a,b})\,da\,db\,. \tag{101}$$

Thus

$$(R, \mathrm{L}^2(\mathrm{G})) \quad \text{is unitarily equivalent to} \quad \int_{\mathbb{C}\times\mathbb{C}} (U_{a,b}, \mathrm{L}^2(\mathbb{R})_{a,b})\,da\,db\,. \tag{102}$$

Lemma 118. *Let* $\mathbb{T} = \{(z_1,z_2) \in \mathbb{C}\times\mathbb{C};\ |z_1| = |z_2| = 1\}$. *Then the path*

$$P = \{(e^{it}, e^{i\alpha t});\ t \in \mathbb{R}\}$$

is not self intersecting.

Proof. Indeed, suppose

$$(e^{it}, e^{i\alpha t}) = (e^{is}, e^{i\alpha s})\,.$$

Then there are integers m, n such that

$$t = s+m \quad \text{and} \quad \alpha t = \alpha s + n\,.$$

But the first equation implies that $\alpha t = \alpha s + \alpha m$. Hence the second equations shows that α is rational - contradiction. □

Lemma 119. *For any* $\phi \in C_c^\infty(\mathbb{R})$, *there is* $\tilde{\phi} \in C_c^\infty(\mathbb{T})$ *such that*

$$\phi(t) = \tilde{\phi}(e^{it}, e^{i\alpha t}) \qquad (t \in \mathbb{R})\,.$$

Proof. This is immediate from Lemma 118. □

Lemma 120. *Fix two non-zero complex numbers a and b. For any $\phi \in C_c^\infty(\mathbb{R})$, there is $\tilde{f} \in C_c^\infty(\mathbb{C} \times \mathbb{C})$ such that*

$$\phi(t) = \tilde{f}(e^{-it}a, e^{-i\alpha t}b) \qquad (t \in \mathbb{R})\,.$$

Proof. This is immediate from Lemma 119. □

Lemma 121. *Fix two non-zero complex numbers a and b. For any $\phi \in C_c^\infty(\mathbb{R})$, there is $f \in \mathcal{S}(\mathbb{C} \times \mathbb{C})$ such that*

$$\phi(t) = \int_{\mathbb{C}\times\mathbb{C}} \chi(-(ae^{-it}\overline{\xi} + be^{-i\alpha t}\overline{\eta}))f(\xi,\eta)\,d\xi\,d\eta \qquad (t \in \mathbb{R})\,.$$

Proof. This is immediate from Lemma 120 and the fact that the Fourier transform on $\mathbb{C} \times \mathbb{C}$ is a bijection from $\mathcal{S}(\mathbb{C} \times \mathbb{C})$ onto itself. □

Lemma 122. *Fix two non-zero complex numbers a and b. Suppose*

$$T : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$$

is a bounded linear map sch that

$$U_{a,b}(0,\xi,\eta)T = TU_{a,b}(0,\xi,\eta) \qquad (\xi, \eta \in \mathbb{C})\,.$$

Then there is a function F such that

$$Tv(t) = F(t)v(t) \qquad (v \in \mathrm{L}^2(\mathbb{R})\,,\ t \in \mathbb{R})\,.$$

Proof. We see from Lemma 121 and the definition of the representation $U_{a,b}$ that T commutes with the multiplication by functions from $C_c^\infty(\mathbb{R})$:

$$T(\phi v) = \phi T v \qquad (v \in \mathrm{L}^2(\mathbb{R})\,,\ \phi \in C_c^\infty(\mathbb{R}))\,.$$

Fix $N > 0$ and let $v_N \in C_c^\infty(\mathbb{R})$ be equal to 1 in the interval $[-N, N]$. Then for any $v \in C_c^\infty(-N, N)$

$$Tv = T(v_N v) = T(v_N)v\,.$$

Thus on this interval, T coincides with the multiplication by the function $T(v_N)$. Hence there is F whose restriction to $(-N, N)$ is equal to $T(v_N)$ and

$$Tv(t) = F(t)v(t) \qquad (v \in \mathrm{L}^2(\mathbb{R})\,,\ t \in \mathbb{R})\,.$$

□

Corollary 123. *For any two non-zero complex numbers a and b the representation $U_{a,b}$ is irreducible.*

Proof. Indeed, suppose a bounded operator $T : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ commutes with $U_{a,b}(g)$ for all $g \in \mathrm{G}$. Then Lemma 122 shows that T cincides with the multiplication by a function F. Since

$$U_{a,b}(\tau, 0, 0)v(t) = v(t + \tau)$$

and since T commutes with all $U_{a,b}(\tau, 0, 0)$, the function F is constant. Therefore the only orthogonal projection that commutes with T is a constant multiple of the identity. This there are no proper closed invariant subspaces in $\mathrm{L}^2(\mathbb{R})$. $\square$

Lemma 124. *For $a, a', b, b' \in \mathbb{C}^\times$, the representations $U_{a,b}$, $U_{a',b'}$ are equivalent if and only if there is $s \in \mathbb{R}$ such that $a' = ae^{is}$ and $b' = be^{i\alpha s}$.*

Proof. Notice that

$$(s, 0, 0)(\tau, \xi, \eta)(s, 0, 0)^{-1} = (\tau, e^{is}\xi, e^{i\alpha s}\eta)\,.$$

Hence,

$$U_{a,b}(s, 0, 0)U_{a,b}(\tau, \xi, \eta)U_{a,b}(s, 0, 0)^{-1} = U_{a,b}(\tau, e^{is}\xi, e^{i\alpha s}\eta) = U_{ae^{-is},be^{-i\alpha s}}(\tau, \xi, \eta)\,.$$

Therefore the representations $U_{a,b}$ and $U_{ae^{is},be^{i\alpha s}}$ are unitarily equivalent for any $s \in \mathbb{R}$.

Conversely, if the representations $U_{a,b}$, $U_{a',b'}$ are equivalent, then their restrictions to the subgroup $\mathbb{C} \times \mathbb{C} \subseteq \mathrm{G}$ are equivalent. But the restriction of $U_{a,b}$ to $\mathbb{C} \times \mathbb{C} \subseteq \mathrm{G}$ acts via multiplication as follows

$$U_{a,b}(0, \xi, \eta)v(t) = \chi(-(ae^{-it}\overline{\xi} + be^{-i\alpha t}\overline{\eta}))v(t)$$

Hence for any $f \in \mathcal{S}(\mathbb{C} \times \mathbb{C})$

$$T : \int_{\mathbb{C}\times\mathbb{C}} f(\xi, \eta)\chi(-(ae^{-it}\overline{\xi} + be^{-i\alpha t}\overline{\eta}))v(t)\, d\xi\, d\eta$$

$$\to \int_{\mathbb{C}\times\mathbb{C}} f(\xi, \eta)\chi(-(a'e^{-it}\overline{\xi} + b'e^{-i\alpha t}\overline{\eta}))Tv(t)\, d\xi\, d\eta\,.$$

Therefore for any $\tilde{f} \in \mathcal{S}(\mathbb{C} \times \mathbb{C})$

$$T : \tilde{f}(ae^{-it} + be^{-i\alpha t})v(t) \to \tilde{f}(a'e^{-it} + b'e^{-i\alpha t})Tv(t)\,.$$

Choose $\tilde{f} \in C_c^\infty)$ to be concentrated very close to the path

$$P_{a,b} = \{(ae^{-it}, be^{-i\alpha t});\ t \in \mathbb{R}\}\,.$$

Then $\tilde{f}(a'e^{-it} + b'e^{-i\alpha t})v(t)$ is zero unless $\tilde{f}$ is concentrated very close to the the same path. Which implies

$$\{(ae^{-it}, be^{-i\alpha t});\ t \in \mathbb{R}\} = \{(a'e^{-it}, b'e^{-i\alpha t});\ t \in \mathbb{R}\}\,.$$

Hence there is $s \in \mathbb{R}$ such that $a' = ae^{is}$ and $b' = be^{i\alpha s}$. $\square$

14.3. **The right regular representation 2.** The following formula defines a bijection $T : \mathrm{G} \to \mathrm{G}$,

$$T(t, z, w) = (t, e^{it}z, e^{i\alpha t}w)\,.$$

We shall use T to define a different multiplication on G,

$$g \circ h = T^{-1}(T(g)T(h)) \qquad (g, h \in G)\,.$$

Obviously G with the original multiplication is isomorphic to G with the new multiplication. Explicitly

$$(t, z, w) \circ (\tau, \xi, \eta) = (t + \tau, e^{-i\tau}z + \xi, e^{-i\alpha\tau}w + \eta)\,.$$

It is easy to see from the above formula that the original Haar measure on G is left and right invariant with respect to the new multiplication. Let r denote the right regular representation with respect to the new multiplication. Explicitly

$$r(\tau,\xi,\eta)\phi(t,z,w)=\phi((t,z,w)\circ(\tau,\xi,\eta))=\phi(t+\tau,e^{-i\tau}z+\xi,e^{-i\alpha\tau}w+\eta)\,.$$

The formula

$$\Phi\phi(s,a,b)=\tilde{\phi}(s,a,b)=\int_{\mathbb{R}\times\mathbb{C}\times\mathbb{C}}\chi(ts+a\overline{z}+b\overline{w})\phi(t,z,w)\,dt\,dz\,dw$$

defines a bijective isometry

$$\Phi:\mathrm{L}^2(\mathrm{G})\to\mathrm{L}^2(\mathrm{G})\,.$$

Lemma 125. *Set*

$$\tilde{r}(g)=\Phi r(g)\Phi^{-1}\qquad(g\in\mathrm{G})\,.$$

Then

$$\Phi:(r,\mathrm{L}^2(\mathrm{G}))\to(\tilde{r},\mathrm{L}^2(\mathrm{G}))$$

is an isomorphism of unitary representations of G *and explicitly,*

$$\tilde{r}(\tau,\xi,\eta)\tilde{\phi}(s,a,b)=\chi(-(\tau s+ae^{-i\tau}\overline{\xi}+be^{-i\alpha\tau}\overline{\eta}))\tilde{\phi}(s,e^{-i\tau}a,e^{-i\alpha\tau}b)\,.$$

Proof. The first part is obvious. Since

$$\begin{aligned}&\int\chi(ts+a\overline{z}+b\overline{w})\phi((t+\tau,e^{-i\tau}z+\xi,e^{-i\alpha\tau}w+\eta)\,dt\,dz\,dw\\&\qquad=\chi(-(\tau s+ae^{-i\tau}\overline{\xi}+be^{-i\alpha\tau}\overline{\eta}))\int\chi(ts+ae^{-i\tau}\overline{z}+be^{-i\alpha\tau}\overline{w})\phi(t,z,w)\,dt\,dz\,dw\end{aligned}$$

the claim follows. □

For $r,\rho>0$ let

$$X_{r,\rho}=\{(a,b)\in\mathbb{C}\times\mathbb{C};\ |a|=r\,,\ |b|=\rho\}\,.$$

This is the direct product of to circles. We equip it with a measure

$$\int_{X_{r,\rho}}\psi(a,b)\,d\mu(a,b)=\int_{X_{r,\rho}}\psi(re^{i\theta_1},\rho e^{i\theta_2})\,d\theta_1\,d\theta_2\,.$$

Then

$$\mathrm{L}^2(\mathbb{C}\times\mathbb{C})=\int_0^\infty\int_0^\infty\mathrm{L}^2(X_{r,\rho})\,r\,dr\,\rho\,d\rho\,.$$

For $s\in\mathbb{R}$ and $r,\rho>0$ let

$$V_{r,\rho,s}(\tau,\xi,\eta)\psi(a,b)=\chi(-(\tau s+ae^{-i\tau}\overline{\xi}+be^{-i\alpha\tau}\overline{\eta}))\psi(e^{-i\tau}a,e^{-i\alpha\tau}b)\qquad(\psi\in\mathrm{L}^2(X_{r,\rho}))\,.$$

A straightforward computation shows that $(V_{r,\rho,s},\mathrm{L}^2(X_{r,\rho}))$ is a unitary representation of G (with the new multiplication).

For $\phi\in\mathcal{S}(\mathrm{G})$,

$$\int_{\mathrm{G}}|\phi(g)|^2\,dg=\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_{X_{r,\rho,s}}|\phi|_{\{s\}\times X_{r,\rho,s}}|^2\,d\mu\,rdrd\rho\,ds\,.$$

Let

$$\phi_{s,r,\rho}(a,b) = \phi|_{\{s\}\times X_{r,\rho,s}}(s,a,b) = \psi(s,a,b)\,.$$

Then the map

$$\mathcal{S}(\mathrm{G}) \ni \phi \to \phi_{s,r,\rho} \in \mathrm{L}^2(X_{r,\rho})$$

intertwines $\tilde{r}$ with $V_{r,\rho,s}$. Thus

$$(\tilde{r}, \mathrm{L}^2(\mathrm{G})) \quad \text{is unitarily equivalent to} \quad \int_{\mathbb{R}} \int_0^\infty \int_0^\infty (V_{r,\rho,s}, \mathrm{L}^2(X_{r,\rho}))\, rdr\, \rho d\rho\, ds\,. \tag{103}$$

Hence,

$$(r, \mathrm{L}^2(\mathrm{G})) \quad \text{is unitarily equivalent to} \quad \int_{\mathbb{R}} \int_0^\infty \int_0^\infty (V_{r,\rho,s}, \mathrm{L}^2(X_{r,\rho}))\, rdr\, \rho d\rho\, ds\,. \tag{104}$$

Thus

$$(R, \mathrm{L}^2(\mathrm{G})) \quad \text{is unitarily equivalent to} \quad \int_{\mathbb{R}} \int_0^\infty \int_0^\infty (V_{r,\rho,s}, \mathrm{L}^2(X_{r,\rho}))\, rdr\, \rho d\rho\, ds\,. \tag{105}$$

Problem 26. *Prove that for all r, ρ and s the representations $V_{r,\rho,s}$ are irreducible.*

Let T be abounded operator on the Hilbert space $\mathrm{L}^2(X_{r,\rho})$ which commutes with the action of the group. As in the (partial) proof of Theorem (116) we check that since T commutes with all the operators

$$V_{r,\rho,s}(0,\xi,\eta) \qquad (\xi,\eta \in \mathbb{C})\,,$$

T coincides with the multiplication by a function $F \in C^\infty(X_{r,\rho})$. Since T commutes with all the operators

$$V_{r,\rho,s}(\tau,0,0) \qquad (\tau \in \mathbb{R})\,,$$

the function F is constant any orbit

$$\{(ae^{it}, be^{i\alpha t});\ t \in \mathbb{R}\} \qquad ((a,b) \in X_{r,\rho})\,.$$

Since α is irrational any such orbit is dense in $X_{r,\rho}$. Hence F is constant. Thus T coincides with a constant multiple of the identity.

14.4. **A surprise.**

None of the representations $(U_{a,b}, \mathrm{L}^2(\mathbb{R}))$ is equivalent to any $(V_{r,\rho,s}, \mathrm{L}^2(X_{r,\rho}))$. Indeed, consider the restriction of both to the abelian subgroup

$$N = \{(0,z,w);\ z,w \in \mathbb{C}\}\,.$$

Thus

$$U_{a,b}(0,\xi,\eta)v(t) = \chi(-(e^{-it}a\overline{\xi} + e^{-i\alpha t}b\overline{\eta}))v(t)$$

and

$$V_{r,\rho,s}(0,\xi,\eta)\psi(a,b) = \chi(-(a\overline{\xi} + b\overline{\eta}))\psi(a,b)\,.$$

If the two representation would be equivalent then the two characters

$$\mathbb{C}\times\mathbb{C} \ni (\xi,\eta) \to \chi(-(e^{-it}a\overline{\xi} + e^{-i\alpha t}b\overline{\eta})) \in \mathbb{C}^\times\,,$$
$$\mathbb{C}\times\mathbb{C} \ni (\xi,\eta) \to \chi(-(a\overline{\xi} + b\overline{\eta})) \in \mathbb{C}^\times$$

would be equal. But they are not (for almost all t)!

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