# SYMMETRY BREAKING OPERATORS FOR DUAL PAIRS WITH ONE MEMBER COMPACT 

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#### Abstract

We consider a dual pair ( $\mathrm{G}, \mathrm{G}^{\prime}$ ), in the sense of Howe, with G compact acting on $L^{2}\left(\mathbb{R}^{n}\right)$, for an appropriate $n$, via the Weil representation $\omega$. Let $\widetilde{\mathrm{G}}$ be the preimage of $G$ in the metaplectic group. Given a genuine irreducible unitary representation $\Pi$ of $\widetilde{G}$, let $\Pi^{\prime}$ be the corresponding irreducible unitary representation of $\widetilde{\mathrm{G}}^{\prime}$ in Howe's correspondence. The orthogonal projection onto the $\Pi$-isotypic component $L^{2}\left(\mathbb{R}^{n}\right)_{\Pi}$ is,  $\left.\mathcal{H}_{\Pi^{\prime}}^{\infty}\right)$. We study this operator by computing its Weyl symbol. Our results allow us to recover the known list of highest weights of irreducible representations of $\widetilde{G}$ occurring in Howe's correspondence when the rank of G is strictly bigger than the rank of G'. They also allow us to compute the wavefront set of $\Pi^{\prime}$ by elementary means.


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## Introduction

Let W be a finite dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate symplectic form $\langle\cdot, \cdot\rangle$ and let $\mathrm{Sp}(\mathrm{W})$ denote the corresponding symplectic group. Write $\widetilde{\mathrm{Sp}}(\mathrm{W})$ for the metaplectic group. Let us fix the character $\chi$ of $\mathbb{R}$ given by $\chi(r)=e^{2 \pi i r}$, $r \in \mathbb{R}$. Then the Weil representation of $\widetilde{\operatorname{Sp}}(\mathrm{W})$ associated to $\chi$ is denoted by $\left(\omega, \mathcal{H}_{\omega}\right)$.

For $\mathrm{G}, \mathrm{G}^{\prime} \subseteq \mathrm{Sp}(\mathrm{W})$ forming a reductive dual pair in the sense of Howe, let $\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}^{\prime}$ denote their preimages in $\widetilde{\mathrm{Sp}}(\mathrm{W})$. Howe's correspondence (or local $\theta$-correspondence) for $\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}^{\prime}$ is a bijection $\Pi \leftrightarrow \Pi^{\prime}$ between the irreducible admissible representations of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}^{\prime}}$ which occur as smooth quotients of $\omega$, [How89b]. It can be formulated as follows. Assume that $\operatorname{Hom}_{\widetilde{\mathrm{G}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty}\right) \neq 0$. Then $\operatorname{Hom}_{\widetilde{\mathrm{G}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty}\right)$ is a $\widetilde{\mathrm{G}^{\prime}}$-module under the action via $\omega$. Howe proved that it has a unique irreducible quotient, which is an irreducible admissible representation $\left(\Pi^{\prime}, \mathcal{H}_{\Pi^{\prime}}\right)$ of $\widetilde{\mathrm{G}^{\prime}}$. Conversely, $\operatorname{Hom}_{\widetilde{G^{\prime}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi^{\prime}}^{\infty}\right)$ is a $\widetilde{\mathrm{G}}$-module which has a unique irreducible admissible quotient, infinitesimally equivalent to $\left(\Pi, \mathcal{H}_{\Pi}\right)$. Furthermore, $\Pi \otimes \Pi^{\prime}$ occurs as a quotient of $\omega^{\infty}$ in a unique way, i.e.

$$
\begin{equation*}
\operatorname{dim} \operatorname{Hom}_{\widetilde{\mathrm{G}} \widetilde{\mathbb{G}^{\prime}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty} \otimes \mathcal{H}_{\Pi^{\prime}}^{\infty}\right)=1 \tag{1}
\end{equation*}
$$

In [Kob15], the elements of

$$
\operatorname{Hom}_{\widetilde{\mathrm{G}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty}\right), \quad \operatorname{Hom}_{\widetilde{G^{\prime}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi^{\prime}}^{\infty}\right) \quad \text { and } \quad \operatorname{Hom}_{\widetilde{\mathrm{G}} \widetilde{G^{\prime}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty} \otimes \mathcal{H}_{\Pi^{\prime}}^{\infty}\right)
$$

are called symmetry breaking operators. Their construction is part of Stage C of Kobayashi's program for branching problems in the representation theory of real reductive groups.

Since the last space is one dimensional, it deserves a closer look. The explicit contruction of the (essentially unique) symmetry breaking operator in $\operatorname{Hom}_{\widetilde{\mathrm{G}} \widetilde{G}^{\prime}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty} \otimes \mathcal{H}_{\Pi^{\prime}}^{\infty}\right)$ provides an alternative and direct approach to Howe's correspondence. To do this is the aim of the present paper.

Our basic assumption is that $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ is an irreducible dual pair with G compact. As shown by Howe [How79], up to an isomorphism, $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ is one of the pairs

$$
\begin{equation*}
\left(\mathrm{O}_{d}, \mathrm{Sp}_{2 m}(\mathbb{R})\right), \quad\left(\mathrm{U}_{d}, \mathrm{U}_{p, q}\right), \quad\left(\mathrm{Sp}_{d}, \mathrm{O}_{2 m}^{*}\right) \tag{2}
\end{equation*}
$$

Then the representations $\Pi$ and $\Pi^{\prime}$ together with their contragredients are arbitrary irreducible unitary highest weight representations. They have been defined by HarishChandra in [Har55], were classified in [EHW83] and have been studied in terms of Zuckerman functors in [Wal84], [Ada83] and [Ada87].

The crucial fact for constructing the symmetry breaking operator in $\operatorname{Hom}_{\widetilde{\mathrm{G}}{\widetilde{G^{\prime}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty} \otimes\right)}$ $\left.\mathcal{H}_{\Pi^{\prime}}^{\infty}\right)$ is that, up to a non-zero constant multiple, there is a unique $\mathrm{GG}^{\prime}$-invariant tempered distribution $f_{\Pi \otimes \Pi^{\prime}}$ on W such that
where Op and $\mathcal{K}$ are classical transformations which we shall review in section 1 . In [Prz93], $f_{\Pi \otimes \Pi^{\prime}}$ is called the intertwining distribution associated to $\Pi \otimes \Pi^{\prime}$. In fact, if we work in a Schrödinger model of $\omega$, then $f_{\Pi \otimes \Pi^{\prime}}$ happens to be the Weyl symbol, [Hör83], of the operator $(\mathrm{Op} \circ \mathcal{K})\left(f_{\Pi \otimes \Pi^{\prime}}\right)$.

The last paragraph does not require G to be compact. Suppose that the group G is compact. Let $\Theta_{\Pi}$ and $d_{\Pi}$ respectively denote the character and the degree of $\Pi$. Then the projection onto the $\Pi$-isotypic component of $\omega$ is equal to $d_{\Pi} / 2$ times

$$
\begin{equation*}
\int_{\widetilde{\mathrm{G}}} \omega(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d \tilde{g}=\omega\left(\check{\Theta}_{\Pi}\right) \tag{4}
\end{equation*}
$$

where $\Theta_{\Pi}(\tilde{g})=\Theta_{\Pi}\left(\tilde{g}^{-1}\right)$ and we normalize the Haar measure $d \tilde{g}$ of $\tilde{G}$ to have the total mass 2. (This explains the constant multiple $\frac{1}{2}$ needed for the projection. In this way, the mass of G is equal to 1.) By Howe's correspondence with G compact, the projection onto the $\Pi$-isotypic component of $\omega$ is a symmetry breaking operator for $\Pi \otimes \Pi^{\prime}$. The intertwining distribution for $\Pi \otimes \Pi^{\prime}$ is therefore determined by the equation

$$
\begin{equation*}
(\mathrm{Op} \circ \mathcal{K})\left(f_{\Pi \otimes \Pi^{\prime}}\right)=\frac{1}{2} \omega\left(\check{\Theta}_{\Pi}\right) . \tag{5}
\end{equation*}
$$

There are more cases when $f_{\Pi \otimes \Pi^{\prime}}$ may be computed via the formula (5), see [Prz93]. However, if the group $G$ is compact then the distribution character $\Theta_{\Pi^{\prime}}$ may also be recovered from $f_{\Pi \otimes \Pi^{\prime}}$ via an explicit formula, see [Prz91]. Thus, in this case, we have a diagram

$$
\begin{equation*}
\Theta_{\Pi} \longrightarrow f_{\Pi \otimes \Pi^{\prime}} \longrightarrow \Theta_{\Pi^{\prime}} \tag{6}
\end{equation*}
$$

In general, the asymptotic properties of $f_{\Pi \otimes \Pi^{\prime}}$ relate the associated varieties of the primitive ideals of $\Pi$ and $\Pi^{\prime}$ and, under some more assumptions, the wave front sets of these representations, see [Prz93] and [Prz91].

The usual, often very successful, approach to Howe's correspondence avoids any work with distributions on the symplectic space. Instead, one finds Langlands parameters (see [Moe89], [AB95], [Pau98], [Pau00], [Pau05], [LPTZ03]), character formulas (see [Ada98], [Ren98], [DP96], [Prz00], [BP14] ), or candidates for character formulas (as in [BP14]), or one establishes preservation of unitarity (as in [Li89], [He03], [Prz93], [ABP $\left.{ }^{+} 07\right]$ or [MSZ17] ). However, in the background (explicit or not), there is the orbit correspondence induced
by the unnormalized moment maps

$$
\mathfrak{g}^{*} \longleftarrow \mathrm{~W} \longrightarrow \mathfrak{g}^{\prime *}
$$

where $\mathfrak{g}$ and $\mathfrak{g}^{\prime}$ denote the Lie algebras of $G$ and $G^{\prime}$, respectively, and $\mathfrak{g}^{*}$ and $\mathfrak{g}^{* *}$ are their duals. This correspondence of orbits has been studied in [DKP97], [DKP05] and [Pan10]. Furthermore, in their recent work, [LM15], H. Y. Loke and J. J. Ma computed the associated variety of the representations for the dual pairs in the stable range in terms of the orbit correspondence. The $p$-adic case was studied in detail in [Moe98].

Working with the $\mathrm{GG}^{\prime}$-invariant distributions on W is a more direct approach than relying on the orbit correspondence and provides different insights and results. As a complementary contribution to all work mentioned above, we compute the intertwining distributions $f_{\Pi \otimes \Pi^{\prime}}$ explicitly, see section 5 . As an application, we obtain the wave front set of $\Pi^{\prime}$ by elementary means. The computation will be sketched in section 15 , but the detailed proofs will appear in a forthcoming paper, [MPP23b]. Another application of the methods presented in this paper leads to the explicit formula for the character of the corresponding irreducible unitary representation $\Pi^{\prime}$ of $\widetilde{\mathrm{G}}^{\prime}$. This can be found in [Mer17, Mer20].

The explicit formulas for the intertwining distribution provide important information on the nature of the symmetry breaking operators. Namely, they show that none of the symmetry breaking operators of the form $(\mathrm{Opo} \mathcal{K})\left(f_{\Pi \otimes \Pi^{\prime}}\right)$ is a differential operator. For the present situation, this answers in the negative the question on the existence of differential symmetry breaking operators, addressed in different contexts by several authors (see for instance [KP16a, KP16b, KS15] and the references given there). This property is the content of Corollary 13.

Finally, observe that our computations leading to the intertwining distributions apply to any genuine irreducible representation $\Pi$ of the compact member $\widetilde{G}$ of a dual pair. They provide an explicit formula for the Weyl symbol of the projection of $\left.\omega\right|_{\widetilde{G}}$ onto the $\Pi$-isotypic component. According to Howe's duality theorem, this projection is non-zero if and only if there is a unitary highest weight representation $\Pi^{\prime}$ of $\widetilde{\mathrm{G}}^{\prime}$ such that $\Pi \otimes \Pi^{\prime}$ occurs in $\left.\omega\right|_{\widetilde{G} \widetilde{G}^{\prime}}$, i.e. $\Pi$ occurs in Howe's correspondence. When the rank of G is strictly bigger than that of $\mathrm{G}^{\prime}$, we recover the known necessary and sufficient conditions on the highest weights of $\Pi$ so that it occurs in Howe's correspondence. See Corollary 11.

The paper is organized as follows. In section 1, we introduce some notation and review the construction of the intertwining distributions. Section 2 computes the intertwining distribution for the dual pair $(\mathrm{Z}, \mathrm{Sp}(\mathrm{W}))$, where $\mathrm{Z}=\mathrm{O}_{1}$ is the center of of the symplectic group $\mathrm{Sp}(\mathrm{W})$, and introduces some properties needed in the sequel. Section 3 recalls how to realize the dual pairs with one member compact as Lie supergroups, and section 4 collects some definitions and properties of Weyl-Harish-Chandra integration formulas on W that we will need to compute the intertwining distributions. Section 5 states the main results of this paper. The dual pairs $\left(\mathrm{O}_{2}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ are particular because the group $\mathrm{SO}_{2}$ is abelian. The intertwining distributions corresponing to these pairs are computed in section 6 . The smallest example of $\left(\mathrm{O}_{2}, \mathrm{Sp}_{2}(\mathbb{R})=\mathrm{SL}_{2}(\mathbb{R})\right)$ is presented with more details. An additional example in given in section 7 , where we illustrate the main two theorems when $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{l}, \mathrm{U}_{p, p}\right)$ and $\Pi$ is the trivial representation of $\mathrm{U}_{l}$. The proofs of the main results are in sections 8,9 and 10 . We treat the special cases concerning the non-identity connected components of the orthogonal groups in sections 11, 12 and 13. Section 14
contains the proof of a necessary condition of a representation of $\widetilde{\mathrm{U}}_{l}$ to occur in Howe's correspondence for ( $\mathrm{U}_{l}, \mathrm{U}_{p, q}$ ) when $p \leq q<l \leq l^{\prime}=p+q$. Finally, in section 15, we outline how the results of this paper lead, for each representation $\Pi$ of $\widetilde{G}$ occurring in Howe's duality, to the computation of the wave front set of the representation $\Pi^{\prime}$ dual to $\Pi$. The nine appendices collect and prove some auxiliary results.

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## 1. Notation and preliminaries

Let us first recall the construction of the metaplectic group $\widetilde{\mathrm{Sp}_{\mathrm{p}}}(\mathrm{W})$ and the Weil representation $\omega$. We are using the approach of [AP14, Section 4], to which we refer the reader for more details.

Let $\mathfrak{s p}$ denote the Lie algebra of $\operatorname{Sp}(\mathrm{W})$, both contained in $\operatorname{End}(\mathrm{W})$. Fix a positive definite compatible complex structure $J$ on W , that is an element $J \in \mathfrak{s p}$ such that $J^{2}=-1$ (minus the identity on W ) and the symmetric bilinear form $\langle J \cdot, \cdot\rangle$ is positive definite on W . For an element $g \in \operatorname{Sp}(\mathrm{~W})$, let $J_{g}=J^{-1}(g-1)$. The adjoint of $J_{g}$ with respect to the form $\langle J \cdot, \cdot\rangle$ is $J_{g}^{*}=J g^{-1}(1-g)$. In particular, $J_{g}$ and $J_{g}^{*}$ have the same kernel. Hence the image of $J_{g}$ is

$$
J_{g} \mathrm{~W}=\left(\operatorname{Ker} J_{g}^{*}\right)^{\perp}=\left(\operatorname{Ker} J_{g}\right)^{\perp},
$$

where $\perp$ denotes the orthogonal complement with respect to $\langle J \cdot, \cdot\rangle$. Therefore, the restriction of $J_{g}$ to $J_{g} \mathrm{~W}$ defines an invertible element. Thus for every $g \neq 1$, it makes sense to talk about $\operatorname{det}\left(J_{g}\right)_{J_{g} \mathrm{~W}}^{-1}$, the reciprocal of the determinant of the restriction of $J_{g}$ to $J_{g} \mathrm{~W}$. With this notation, we have

$$
\begin{equation*}
\widetilde{\mathrm{Sp}}(\mathrm{~W})=\left\{\tilde{g}=(g ; \xi) \in \mathrm{Sp}(\mathrm{~W}) \times \mathbb{C}, \quad \xi^{2}=i^{\operatorname{dim}(g-1) \mathrm{W}} \operatorname{det}\left(J_{g}\right)_{J_{g} \mathrm{~W}}^{-1}\right\} \tag{7}
\end{equation*}
$$

with the convention that $\operatorname{det}\left(J_{g}\right)_{J_{g} \mathrm{~W}}^{-1}=1$ if $g=1$. There exists a 2 -cocycle $C: \operatorname{Sp}(\mathrm{W}) \times$ $\mathrm{Sp}(\mathrm{W}) \rightarrow \mathbb{C}$, explicitly described in [AP14, Proposition 4.13], such that $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a group with respect to the multiplication

$$
\begin{equation*}
\left(g_{1} ; \xi_{1}\right)\left(g_{2} ; \xi_{2}\right)=\left(g_{1} g_{2} ; \xi_{1} \xi_{2} C\left(g_{1}, g_{2}\right)\right) \tag{8}
\end{equation*}
$$

and the homomorphism

$$
\begin{equation*}
\widetilde{\mathrm{Sp}}(\mathrm{~W}) \ni(g ; \xi) \rightarrow g \in \mathrm{Sp}(\mathrm{~W}) \tag{9}
\end{equation*}
$$

does not split.
Let $\mu_{\mathrm{W}}$ (or simply $d w$ ) be the Lebesgue measure on W normalized by the condition that the volume of the unit cube with respect to the form $\langle J \cdot, \cdot\rangle$ is 1 . (Since all positive complex structures are conjugate by elements of Sp , this normalization does not depend on the particular choice of $J$.) Let $\mathrm{W}=\mathrm{X} \oplus \mathrm{Y}$ be a complete polarization. Similar normalizations are fixed for the Lebesgue measures on every vector subspace of W , for instance on X and on Y . Furthermore, for every vector space V , we write $\mathcal{S}(\mathrm{V})$ for the Schwartz space on V and $\mathcal{S}^{\prime}(\mathrm{V})$ for the space of tempered distributions on V . We use the notation $\mathrm{G}^{\prime}$ for the second member of a dual pair because it is the centralizer of G in
$\operatorname{Sp}(\mathrm{W})$. We also use the notation $\cdot^{\prime}$ for all the objects associated with $\mathrm{G}^{\prime}$, such as $\mathfrak{g}^{\prime}, \Pi^{\prime}$, ... Unfortunately, this collides with the usual notation for the dual of a linear topological space in functional analysis, also used in this paper, such as $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right), \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right), \ldots$ We hope the reader will guess from the context the correct meaning of the notation.

Each element $K \in \mathcal{S}^{\prime}(\mathrm{X} \times \mathrm{X})$ defines an operator $\operatorname{Op}(K) \in \operatorname{Hom}\left(\mathcal{S}(\mathrm{X}), \mathcal{S}^{\prime}(\mathrm{X})\right)$ by

$$
\begin{equation*}
\operatorname{Op}(K) v(x)=\int_{\mathrm{X}} K\left(x, x^{\prime}\right) v\left(x^{\prime}\right) d x^{\prime} \tag{10}
\end{equation*}
$$

The map

$$
\begin{equation*}
\text { Op : } \mathcal{S}^{\prime}(\mathrm{X} \times \mathrm{X}) \rightarrow \operatorname{Hom}\left(\mathcal{S}(\mathrm{X}), \mathcal{S}^{\prime}(\mathrm{X})\right) \tag{11}
\end{equation*}
$$

is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [Trè67, Theorem 51.7].

The Weyl transform is the linear isomorphism $\mathcal{K}: \mathcal{S}^{\prime}(\mathrm{W}) \rightarrow \mathcal{S}^{\prime}(\mathrm{X} \times \mathrm{X})$ defined for $f \in \mathcal{S}(\mathrm{~W})$ by

$$
\begin{equation*}
\mathcal{K}(f)\left(x, x^{\prime}\right)=\int_{\mathrm{Y}} f\left(x-x^{\prime}+y\right) \chi\left(\frac{1}{2}\left\langle y, x+x^{\prime}\right\rangle\right) d y \tag{12}
\end{equation*}
$$

(Recall that $\chi$ is the character of $\mathbb{R}$ we fixed at the beginning of the introduction.)
For $g \in \mathrm{Sp}(\mathrm{W})$, let

$$
\begin{equation*}
\chi_{c(g)}(u)=\chi\left(\frac{1}{4}\left\langle(g+1)(g-1)^{-1} u, u\right\rangle\right) \quad(u=(g-1) w, w \in \mathrm{~W}) \tag{13}
\end{equation*}
$$

Notice that, if $g-1$ is invertible on W , then

$$
\chi_{c(g)}(u)=\chi\left(\frac{1}{4}\langle c(g) u, u\rangle\right),
$$

where $c(g)=(g+1)(g-1)^{-1}$ is the usual Cayley transform.
Following [AP14, Definition 4.23 and (114)], we define

$$
\begin{equation*}
T: \widetilde{\mathrm{Sp}}(\mathrm{~W}) \ni \tilde{g}=(g ; \xi) \longrightarrow \xi \chi_{c(g)} \mu_{(g-1) \mathrm{W}} \in \mathcal{S}^{\prime}(\mathrm{W}), \tag{14}
\end{equation*}
$$

where $\mu_{(g-1) \mathrm{W}}$ is the Lebesgue measure on the subspace $(g-1) \mathrm{W}$ normalized as above. Set

$$
\begin{equation*}
\omega=\mathrm{Op} \circ \mathcal{K} \circ T \tag{15}
\end{equation*}
$$

As proved in [AP14, Theorem 4.27], $\omega$ is a unitary representation of $\widetilde{\mathrm{Sp}}$ on $\mathrm{L}^{2}(\mathrm{X})$. In fact, $\left(\omega, \mathrm{L}^{2}(\mathrm{X})\right)$ is the Schrödinger model of Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character $\chi$ and the polarization $\mathrm{W}=\mathrm{X} \oplus \mathrm{Y}$. In this realization, $\mathcal{H}_{\omega}=\mathrm{L}^{2}(\mathrm{X})$ and $\mathcal{H}_{\omega}^{\infty}=\mathcal{S}(\mathrm{X})$.

The distribution character of the Weil representation turns out to be the function

$$
\begin{equation*}
\Theta: \widetilde{\mathrm{Sp}}(\mathrm{~W}) \ni(g ; \xi) \rightarrow \xi \in \mathbb{C}^{\times} \tag{16}
\end{equation*}
$$

[AP14, Proposition 4.27]. Hence for $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ in the preimage of $g \in \operatorname{Sp}(\mathrm{~W})$ under the double covering map (9), we have

$$
\begin{equation*}
T(\tilde{g})=\Theta(\tilde{g}) \chi_{c(g)} \mu_{(g-1) \mathrm{W}} \quad(\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{~W})) \tag{17}
\end{equation*}
$$

Suppose now that $\mathrm{G}, \mathrm{G}^{\prime} \subseteq \mathrm{Sp}(\mathrm{W})$ is a dual pair. Every irreducible admissible representation $\Pi \otimes \Pi^{\prime}$ of $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}}^{\prime}$ occurring in Howe's correspondence may be realized, up to infinitesimal equivalence, as a subspace of $\mathcal{H}_{\omega}^{\infty \prime}=\mathcal{S}^{\prime}(\mathrm{X})$. Hence

The existence of the interwining distribution $f_{\Pi \otimes \Pi^{\prime}} \in \mathcal{S}^{\prime}(\mathrm{W})$ defined (up to a multiplicative constant) by (3) is thus a consequence of (1), (11) and (12).

Finally, because of (15), equation (4) and (5) lead to the equality

$$
\begin{equation*}
f_{\Pi \otimes \Pi^{\prime}}=\frac{1}{2} T\left(\check{\Theta}_{\Pi}\right)=\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g . \tag{18}
\end{equation*}
$$

The problem of finding an explicit expression for $f_{\Pi \otimes \Pi^{\prime}}$ is hence transformed into the task of computing the right-hand side of (18).

## 2. The center of the metaplectic group

Let $\mathrm{Z}=\{1,-1\}$ be the center of the symplectic group $\operatorname{Sp}(\mathrm{W})$. Then $(\mathrm{Z}, \mathrm{Sp}(\mathrm{W}))$ is a dual pair in $\mathrm{Sp}(\mathrm{W})$ with compact member Z . Let $(\widetilde{\mathrm{Z}}, \widetilde{\mathrm{Sp}}(\mathrm{W}))$ be the corresponding dual pair in the metaplectic group $\widetilde{\mathrm{Sp}^{\prime}}(\mathrm{W})$. Then $\widetilde{\mathrm{Z}}$ coincides with the center of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ and is equal to

$$
\begin{equation*}
\widetilde{Z}=\{(1 ; 1),(1 ;-1),(-1 ; \zeta),(-1 ;-\zeta)\}, \tag{19}
\end{equation*}
$$

where $\zeta=\left(\frac{i}{2}\right)^{\frac{1}{2} \operatorname{dim} \mathrm{~W}}$.
In this section we illustrate how to compute the intertwining distributions for the pair $(\mathrm{Z}, \mathrm{Sp}(\mathrm{W}))$. At the same time, we introduce some facts that will be needed in the rest of the paper.

The formula for the cocycle in (8) is particularly simple over Z :

$$
C(1, \pm 1)=C(-1,1)=1 \text { and } C(-1,-1)=2^{\operatorname{dim} \mathrm{W}}
$$

Also, $C(g, 1)=C(1, g)=1$ for all $g \in \mathrm{Sp}(\mathrm{W})$ by [AP14, Proposition 4.13]. Notice that

$$
\begin{equation*}
(-1 ; \pm \zeta)^{2}=\left(1 ; \zeta^{2} C(-1,-1)\right)=\left(1 ;(-1)^{\frac{1}{2} \operatorname{dim} \mathrm{~W}}\right) \tag{20}
\end{equation*}
$$

Hence the covering (9) restricted to $\widetilde{Z}$,

$$
\begin{equation*}
\widetilde{\mathrm{Z}} \ni \tilde{z} \rightarrow z \in \mathrm{Z} \tag{21}
\end{equation*}
$$

splits if and only if $\frac{1}{2} \operatorname{dim} W$ is even.
By (14) we have

$$
\begin{array}{ll}
T(1 ; 1)=\delta, & T(1 ;-1)=-\delta \\
T(-1 ; \zeta)=\zeta \mu_{\mathrm{W}}, & T(-1 ;-\zeta)=-\zeta \mu_{\mathrm{W}}
\end{array}
$$

Moreover, [AP14, Proposition 4.28] shows that for $v \in \mathrm{~L}^{2}(\mathrm{X})$ and $x \in \mathrm{X}$,

$$
\begin{array}{ll}
\omega(1 ; 1) v(x)=v(x), & \omega(1 ;-1) v(x)=-v(x), \\
\omega(-1 ; \zeta) v(x)=\frac{\zeta}{|\zeta|} v(-x), & \omega(-1 ;-\zeta) v(x)=-\frac{\zeta}{|\zeta|} v(-x) .
\end{array}
$$

Since $T(\tilde{z})=\Theta(\tilde{z}) \chi_{c(z)} \mu_{(z-1) \mathrm{W}}$ for $\tilde{z} \in \tilde{Z}$, it follows that

$$
\begin{equation*}
\omega(\tilde{z}) v(x)=\frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} v(z x) \quad(\tilde{z} \in \widetilde{Z}) . \tag{22}
\end{equation*}
$$

The fraction

$$
\begin{equation*}
\chi_{+}(\tilde{z})=\frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \quad(\tilde{z} \in \widetilde{Z}) \tag{23}
\end{equation*}
$$

defines an irreducible character $\chi_{+}$of the group $\widetilde{Z}$. Let $\varepsilon$ be the unique non-trivial irreducible character of the two element group Z . Then

$$
\begin{equation*}
\chi_{-}(\tilde{z})=\varepsilon(z) \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \quad(\tilde{z} \in \widetilde{Z}) \tag{24}
\end{equation*}
$$

is also an irreducible character of $\widetilde{Z}$.
Let $\mathrm{L}^{2}(\mathrm{X})_{+} \subseteq \mathrm{L}^{2}(\mathrm{X})$ denote the subspace of the even functions and let $\mathrm{L}^{2}(\mathrm{X})_{-} \subseteq \mathrm{L}^{2}(\mathrm{X})$ denote the subspace of the odd functions. Then, as is well known, [KV78, (6.9)], the restriction $\omega_{ \pm}$of $\omega$ to $\mathrm{L}^{2}(\mathrm{X})_{ \pm}$is irreducible. As we have seen above, the center $\widetilde{\mathrm{Z}}$ acts on $\mathrm{L}^{2}(\mathrm{X})_{ \pm}$via the character $\chi_{ \pm}$. Thus $\chi_{ \pm}$is the central character of $\omega_{ \pm}$.

Hence, in the case of the dual pair ( $\mathrm{Z}, \mathrm{Sp}(\mathrm{W})$ ), Howe's correspondence looks as follows

$$
\begin{equation*}
\left(\chi_{+}, \mathbb{C}\right) \leftrightarrow\left(\omega_{+}, \mathrm{L}^{2}(\mathrm{X})_{+}\right) \quad \text { and } \quad\left(\chi_{-}, \mathbb{C}\right) \leftrightarrow\left(\omega_{-}, \mathrm{L}^{2}(\mathrm{X})_{-}\right) \tag{25}
\end{equation*}
$$

The projections

$$
\mathrm{L}^{2}(\mathrm{X}) \rightarrow \mathrm{L}^{2}(\mathrm{X})_{+} \quad \text { and } \quad \mathrm{L}^{2}(\mathrm{X}) \rightarrow \mathrm{L}^{2}(\mathrm{X})_{-}
$$

are respectively given by

$$
\frac{1}{2} \omega\left(\check{\chi}_{+}\right)=\frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \tilde{\chi}_{+}(\tilde{z}) \omega(\tilde{z}) \quad \text { and } \quad \frac{1}{2} \omega\left(\check{\chi}_{-}\right)=\frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_{-}(\tilde{z}) \omega(\tilde{z})
$$

The corresponding intertwining distributions are

$$
\begin{align*}
& f_{\chi_{+} \otimes \omega_{+}}=\frac{1}{4} \sum_{\tilde{z} \in \tilde{\mathrm{Z}}} \check{\chi}_{+}(\tilde{z}) T(\tilde{z})=\frac{1}{2}\left(\delta+2^{-\frac{1}{2} \operatorname{dim} \mathrm{~W}} \mu_{\mathrm{W}}\right), \\
& f_{\chi-\otimes \omega_{-}}=\frac{1}{4} \sum_{\tilde{z} \in \tilde{\mathrm{Z}}} \check{\chi}_{-}(\tilde{z}) T(\tilde{z})=\frac{1}{2}\left(\delta-2^{-\frac{1}{2} \operatorname{dim} \mathrm{~W}} \mu_{\mathrm{W}}\right), \tag{26}
\end{align*}
$$

where we normalize the total mass of Z to be 1 , as we did for a general dual pair ( $\mathrm{G}, \mathrm{G}^{\prime}$ ) with G compact.

The right-hand side of (26) is a sum of two homogenous distributions of different homogenity degrees. So, asymptotically, they can be isolated. This allows us to recover $\mu_{\mathrm{W}}$, and hence $\tau_{\mathfrak{s p}(\mathrm{W})}(\mathrm{W})$, the wave front of $\omega_{ \pm}$, out of the intertwining distribution.

## 3. Dual pairs as Lie supergroups

To present the main results of this paper, we need the realization of dual pairs with one member compact as Lie supergroups. The content of this section is taken from [Prz06] and [MPP15]. We recall the relevant material for making our exposition self-contained.

For a dual pair $\left(G, G^{\prime}\right)$ as in (2), there is a division algebra $\mathbb{D}=\mathbb{R}, \mathbb{C}, \mathbb{H}$ with an involution over $\mathbb{R}$, a finite dimensional right $\mathbb{D}$-vector space V with a positive definite hermitian form $(\cdot, \cdot)$ and a finite dimensional right $\mathbb{D}$-vector space $\mathrm{V}^{\prime}$ with a non-degenerate skew-hermitian form $(\cdot, \cdot)^{\prime}$ such that G coincides with the isometry group of $(\cdot, \cdot)$ and $\mathrm{G}^{\prime}$ coincides with the isometry group of $(\cdot, \cdot)^{\prime}$. We assume that G centralizes the complex structure $J$ and that $J$ normalizes $\mathrm{G}^{\prime}$. Then the conjugation by $J$ is a Cartan involution on $\mathrm{G}^{\prime}$, which we denote by $\theta$.

Let $\mathrm{V}_{\overline{0}}=\mathrm{V}, d=\operatorname{dim}_{\mathbb{D}} \bigvee_{\overline{0}}, \mathrm{~V}_{\overline{1}}=\mathrm{V}^{\prime}$ and $d^{\prime}=\operatorname{dim}_{\mathbb{D}} \mathrm{V}_{\overline{1}}$. We assume that both $\mathrm{V}_{\overline{0}}$ and $\mathrm{V}_{\overline{1}}$ are right vector spaces over $\mathbb{D}$. Set $\mathrm{V}=\mathrm{V}_{\overline{0}} \oplus \mathrm{~V}_{\overline{1}}$ and define an element $\mathrm{S} \in \operatorname{End}(\mathrm{V})$ by

$$
\mathrm{S}\left(v_{0}+v_{1}\right)=v_{0}-v_{1} \quad\left(v_{0} \in \mathrm{~V}_{\overline{0}}, v_{1} \in \mathrm{~V}_{\overline{1}}\right)
$$

Let

$$
\begin{aligned}
& \operatorname{End}(\mathrm{V})_{\overline{0}}=\{x \in \operatorname{End}(\mathrm{~V}) ; \mathrm{S} x=x \mathrm{~S}\} \\
& \operatorname{End}(\mathrm{V})_{\overline{1}}=\{x \in \operatorname{End}(\mathrm{~V}) ; \mathrm{S} x=-x \mathrm{~S}\} \\
& \operatorname{GL}(\mathrm{V})_{\overline{0}}=\operatorname{End}(\mathrm{V})_{\overline{0}} \cap \mathrm{GL}(\mathrm{~V})
\end{aligned}
$$

Denote by $(\cdot, \cdot)^{\prime \prime}$ the direct sum of the two forms $(\cdot, \cdot)$ and $(\cdot, \cdot)^{\prime}$. Let

$$
\begin{align*}
& \mathfrak{s}_{\overline{0}}=\left\{x \in \operatorname{End}(\mathrm{~V})_{\overline{0}} ; \quad(x u, v)^{\prime \prime}=-(u, x v)^{\prime \prime}, u, v \in \mathrm{~V}\right\},  \tag{27}\\
& \mathfrak{s}_{\overline{1}}=\left\{x \in \operatorname{End}(\mathrm{~V})_{\overline{1}} ; \quad(x u, v)^{\prime \prime}=(u, \mathrm{~S} x v)^{\prime \prime}, u, v \in \mathrm{~V}\right\}, \\
& \mathfrak{s}^{\prime}=\mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}, \\
& \mathrm{~S}=\left\{s \in \operatorname{GL}(\mathrm{~V})_{\overline{0}} ;(s u, s v)^{\prime \prime}=(u, v)^{\prime \prime}, u, v \in \mathrm{~V}\right\}, \\
& \langle x, y\rangle=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}(\mathrm{S} x y) . \tag{28}
\end{align*}
$$

(Here $\operatorname{tr}_{\mathbb{D} / \mathbb{R}}(x)$ denotes the trace of $x$ considered as a real endomorphism of V .) Then $(S, \mathfrak{s})$ is a real Lie supergroup, i.e. a real Lie group $S$ together with a real Lie superalgebra $\mathfrak{s}=\mathfrak{s}_{0} \oplus \mathfrak{s}_{\overline{1}}$, whose even component $\mathfrak{s}_{0}$ is the Lie algebra of S. (In terms of [DM99, §3.8], $(\mathrm{S}, \mathfrak{s})$ is a Harish-Chandra pair.) We shall write $\mathfrak{s}(\mathrm{V})$ instead of $\mathfrak{s}$ whenever we want to specify the Lie superalgebra $\mathfrak{s}$ constructed as above from V and $(\cdot, \cdot)^{\prime \prime}$. By restriction, we have the identification

$$
\begin{equation*}
\mathfrak{s}_{\overline{1}}=\operatorname{Hom}_{\mathbb{D}}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}}\right) \tag{29}
\end{equation*}
$$

The group S acts on $\mathfrak{s}$ by conjugation and $\langle\cdot, \cdot\rangle$ is a non-degenerate S -invariant form on the real vector space $\mathfrak{s}$, whose restriction to $\mathfrak{s}_{0}$ is symmetric and restriction to $\mathfrak{s}_{1}$ is skew-symmetric. We shall employ the notation

$$
\begin{align*}
& s . x=\operatorname{Ad}(s) x=s x s^{-1} \quad(s \in \mathrm{~S}, x \in \mathfrak{s})  \tag{30}\\
& x(w)=\operatorname{ad}(x)(w)=x w-w x \quad\left(x \in \mathfrak{s}_{0}, w \in \mathfrak{s}_{\overline{1}}\right) . \tag{31}
\end{align*}
$$

In terms of the notation introduced at the beginning of this section,

$$
\mathfrak{g}=\mathfrak{s}_{\overline{0}}\left|v_{\overline{0}}, \quad \mathfrak{g}^{\prime}=\mathfrak{s}_{0}\right| \mathrm{v}_{\overline{\mathrm{T}}}, \quad \mathrm{G}=\left.\mathrm{S}\right|_{\mathrm{v}_{\overline{0}}}, \quad \mathrm{G}^{\prime}=\left.\mathrm{S}\right|_{\mathrm{v}_{\overline{\mathrm{T}}}}
$$

Define $\mathrm{W}=\operatorname{Hom}_{\mathbb{D}}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}}\right)$. Then, by restriction, we have the identification

$$
\begin{equation*}
\mathrm{W}=\mathfrak{s}_{\overline{1}} \tag{32}
\end{equation*}
$$

Under this identification, the adjoint action of G on $\mathfrak{s}_{\overline{1}}$ becomes the action on W by the left (post)multiplication. Similarly, the adjoint action of $\mathrm{G}^{\prime}$ on $\mathfrak{s}_{\overline{1}}$ becomes the action of $\mathrm{G}^{\prime}$ on W via the right (pre)multiplication by the inverse. Also, we have the unnormalized moment maps

$$
\begin{equation*}
\tau: \mathrm{W} \ni w \rightarrow w^{2}\left|\mathrm{v}_{\overline{0}} \in \mathfrak{g}, \quad \tau^{\prime}: \mathrm{W} \ni w \rightarrow w^{2}\right| \mathrm{v}_{\bar{T}} \in \mathfrak{g}^{\prime} \tag{33}
\end{equation*}
$$

An element $x \in \mathfrak{s}$ is called semisimple (resp., nilpotent) if $x$ is semisimple (resp., nilpotent) as an endomorphism of V . We say that a semisimple element $x \in \mathfrak{s}_{1}$ is regular if it is nonzero and $\operatorname{dim}(S . x) \geq \operatorname{dim}(S . y)$ for all semisimple $y \in \mathfrak{s}_{1}$. Let $x \in \mathfrak{s}_{\mathrm{T}}$ be fixed. For $x, y \in \mathfrak{s}_{1}$ let $\{x, y\}=x y+y x \in \mathfrak{s}_{0}$ be their anticommutator.

The anticommutant and the double anticommutant of $x$ in $\mathfrak{s}_{\overline{1}}$ are

$$
\begin{aligned}
x_{\mathfrak{S}_{\overline{1}}} & =\left\{y \in \mathfrak{s}_{\overline{1}}:\{x, y\}=0\right\}, \\
{ }_{\mathfrak{s}_{\overline{1}}} \mathfrak{s}_{\overline{1}} & =\bigcap_{y \in{ }^{x_{\mathfrak{s}_{\overline{1}}}}} y_{\mathfrak{S}_{\overline{1}}},
\end{aligned}
$$

respectively. A Cartan subspace $\mathfrak{h}_{\overline{1}}$ of $\mathfrak{s}_{\overline{1}}$ is defined as the double anticommutant of a regular semisimple element $x \in \mathfrak{s}_{\overline{1}}$. We denote by $\mathfrak{h}_{\overline{1}}^{\text {reg }}$ the set of regular elements in $\mathfrak{h}_{\overline{1}}$.

Next we describe the Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathfrak{s}_{\overline{1}}$. We refer to [Prz06, §6] and [MPP15, $\S 4]$ for the proofs omitted here. Let $l$ be the rank of $\mathfrak{g}, l^{\prime}$ the rank of $\mathfrak{g}^{\prime}$, and set

$$
\begin{equation*}
l^{\prime \prime}=\min \left\{l, l^{\prime}\right\} \tag{34}
\end{equation*}
$$

Given a Cartan subspace $\mathfrak{h}_{\overline{1}}$, there are $\mathbb{Z} / 2 \mathbb{Z}$-graded subspaces $\mathrm{V}^{j} \subseteq \mathrm{~V}$ such that the restriction of the form $(\cdot, \cdot)^{\prime \prime}$ to each $\mathrm{V}^{j}$ is non-degenerate, $\mathrm{V}^{j}$ is orthogonal to $\mathrm{V}^{k}$ for $j \neq k$ and

$$
\begin{equation*}
\mathrm{V}=\mathrm{V}^{0} \oplus \mathrm{~V}^{1} \oplus \mathrm{~V}^{2} \oplus \cdots \oplus \mathrm{~V}^{l^{\prime \prime}} \tag{35}
\end{equation*}
$$

The subspace $\mathrm{V}^{0}$ coincides with the intersection of the kernels of the elements of $\mathfrak{h}_{\overline{1}}$ (equivalently, $\mathrm{V}^{0}=\operatorname{Ker}(x)$ if $\mathfrak{h}_{\overline{1}}={ }^{x_{\mathfrak{S}_{\bar{T}}}} \mathfrak{s}_{\overline{1}}$ ). For $1 \leq j \leq l^{\prime \prime}$, the subspaces $\mathrm{V}^{j}=\mathrm{V}_{\overline{0}}^{j} \oplus \mathrm{~V}_{\bar{T}}^{j}$ are described as follows. Suppose $\mathbb{D}=\mathbb{R}$. Then there is a basis $v_{0}, v_{0}^{\prime}$ of $\mathrm{V}_{\overline{0}}^{j}$ and basis $v_{1}, v_{1}^{\prime}$ of $\mathrm{V}_{\frac{1}{1}}^{j}$ such that

$$
\begin{array}{ll}
\left(v_{0}, v_{0}\right)^{\prime \prime}=\left(v_{0}^{\prime}, v_{0}^{\prime}\right)^{\prime \prime}=1, & \left(v_{0}, v_{0}^{\prime}\right)^{\prime \prime}=0  \tag{36}\\
\left(v_{1}, v_{1}\right)^{\prime \prime}=\left(v_{1}^{\prime}, v_{1}^{\prime}\right)^{\prime \prime}=0, & \left(v_{1}, v_{1}^{\prime}\right)^{\prime \prime}=1
\end{array}
$$

The following formulas define an element $u_{j} \in \mathfrak{s}_{1}\left(\mathrm{~V}^{j}\right)$,

$$
\begin{array}{ll}
u_{j}\left(v_{0}\right)=\frac{1}{\sqrt{2}}\left(v_{1}-v_{1}^{\prime}\right), & u_{j}\left(v_{1}\right)=\frac{1}{\sqrt{2}}\left(v_{0}-v_{0}^{\prime}\right), \\
u_{j}\left(v_{0}^{\prime}\right)=\frac{1}{\sqrt{2}}\left(v_{1}+v_{1}^{\prime}\right), & u_{j}\left(v_{1}^{\prime}\right)=\frac{1}{\sqrt{2}}\left(v_{0}+v_{0}^{\prime}\right)
\end{array}
$$

Suppose $\mathbb{D}=\mathbb{C}$. Then $\mathrm{V}_{\overline{0}}^{j}=\mathbb{C} v_{0}, \mathrm{~V}_{\overline{1}}^{j}=\mathbb{C} v_{1}$, where $\left(v_{0}, v_{0}\right)^{\prime \prime}=1$ and $\left(v_{1}, v_{1}\right)^{\prime \prime}=\delta_{j} i$, with $\delta_{j}= \pm 1$. The following formulas define an element $u_{j} \in \mathfrak{s}_{\overline{1}}\left(\mathrm{~V}^{j}\right)$,

$$
\begin{equation*}
u_{j}\left(v_{0}\right)=e^{-i \delta_{j} \frac{\pi}{4}} v_{1}, \quad u_{j}\left(v_{1}\right)=e^{-i \delta_{j} \frac{\pi}{4}} v_{0} . \tag{37}
\end{equation*}
$$

Suppose $\mathbb{D}=\mathbb{H}$. Then $\bigvee_{\overline{0}}^{j}=\mathbb{H} v_{0}, \mathrm{~V}_{\overline{1}}^{j}=\mathbb{H} v_{1}$, where $\left(v_{0}, v_{0}\right)^{\prime \prime}=1$ and $\left(v_{1}, v_{1}\right)^{\prime \prime}=i$. The following formulas define an element $u_{j} \in \mathfrak{s}_{\overline{1}}\left(\mathrm{~V}^{j}\right)$,

$$
u_{j}\left(v_{0}\right)=e^{-i \frac{\pi}{4}} v_{1}, \quad u_{j}\left(v_{1}\right)=e^{-i \frac{\pi}{4}} v_{0} .
$$

In any case, by extending each $u_{j}$ by zero outside $\mathrm{V}^{j}$, we have

$$
\begin{equation*}
\mathfrak{h}_{\bar{T}}=\sum_{j=1}^{l^{\prime \prime}} \mathbb{R} u_{j} . \tag{38}
\end{equation*}
$$

The formula (38) describes a maximal family of mutually non-conjugate Cartan subspaces of $\mathfrak{s}_{\mathrm{I}}$. Notice that there is only one such subspace unless the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ is isomorphic to $\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right)$ with $l^{\prime \prime}=l<p+q$. In the last case there are $\min (l, p)-\max (l-q, 0)+$ 1 such subspaces, assuming $p \leq q$. For each $m$ such that $\max (l-q, 0) \leq m \leq \min (p, l)$ there is a Cartan subspace $\mathfrak{h}_{\overline{1}, m}$ determined by the condition that $m$ is the number of
positive $\delta_{j}$ 's in (37). We may assume that $\delta_{1}=\cdots=\delta_{m}=1$ and $\delta_{m+1}=\cdots=\delta_{l}=-1$. The choice of the spaces $\mathrm{V}_{\overline{0}}^{j}$ may be done independently of $m$. The spaces $\mathrm{V}_{\mathrm{T}}^{j}$ depend on $m$. If ( $\mathrm{G}, \mathrm{G}^{\prime}$ ) is isomorphic to $\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right)$ with $l \geq l^{\prime \prime}=p+q$, then there is a unique Cartan subspace of $\mathfrak{s}_{\overline{1}}$ up to conjugation. It is determined by the condition that in (37) there are $p$ positive and $q$ negative $\delta_{j}$ 's. We may assume that the positive $\delta_{j}$ 's are the first $p$.

The Weyl group $W\left(\mathrm{~S}, \mathfrak{h}_{\overline{1}}\right)$ is the quotient of the stabilizer of $\mathfrak{h}_{\overline{1}}$ in S by the subgroup $S^{\mathfrak{h}_{\overline{1}}}$ fixing each element of $\mathfrak{h}_{\overline{1}}$. If $\mathbb{D} \neq \mathbb{C}$, then $W\left(\mathrm{~S}, \mathfrak{h}_{\overline{1}}\right)$ acts by all sign changes and all permutations of the $u_{j}$ 's. If $\mathbb{D}=\mathbb{C}$, the Weyl group acts by all sign changes and all permutations of the $u_{j}$ 's which preserve $\left(\delta_{1}, \ldots, \delta_{l^{\prime \prime}}\right)$, see [Prz06, (6.3)].

Set $\delta_{j}=1$ for all $1 \leq j \leq l^{\prime \prime}$, if $\mathbb{D} \neq \mathbb{C}$, and define

$$
\begin{equation*}
J_{j}=\delta_{j} \tau\left(u_{j}\right), \quad J_{j}^{\prime}=\delta_{j} \tau^{\prime}\left(u_{j}\right) \quad\left(1 \leq j \leq l^{\prime \prime}\right) \tag{39}
\end{equation*}
$$

Then $J_{j}, J_{j}^{\prime}$ are complex structures on $\mathrm{V}_{\overline{0}}^{j}$ and $\mathrm{V}_{\overline{1}}^{j}$ respectively. Explicitly,

$$
\begin{align*}
& J_{j}\left(v_{0}\right)=-v_{0}^{\prime}, \quad J_{j}\left(v_{0}^{\prime}\right)=v_{0}, \quad J_{j}^{\prime}\left(v_{1}\right)=-v_{1}^{\prime}, \quad J_{j}^{\prime}\left(v_{1}^{\prime}\right)=v_{1}, \quad \text { if } \quad \mathbb{D}=\mathbb{R},  \tag{40}\\
& J_{j}\left(v_{0}\right)=-i v_{0}, \quad J_{j}^{\prime}\left(v_{1}\right)=-i v_{1}, \quad \text { if } \mathbb{D}=\mathbb{C} \text { or } \mathbb{D}=\mathbb{H} .
\end{align*}
$$

(The point of the multiplication by the $\delta_{j}$ in (39) is that the complex structures $J_{j}, J_{j}^{\prime}$ do not depend on the Cartan subspace $\mathfrak{h}_{\overline{1}}$.) In particular, if $w=\sum_{j=1}^{l^{\prime \prime}} w_{j} u_{j} \in \mathfrak{h}_{\overline{1}}$, then

$$
\begin{equation*}
\tau(w)=\sum_{j=1}^{l^{\prime \prime}} w_{j}^{2} \delta_{j} J_{j} \text { and } \tau^{\prime}(w)=\sum_{j=1}^{l^{\prime \prime}} w_{j}^{2} \delta_{j} J_{j}^{\prime} \tag{41}
\end{equation*}
$$

Let $\mathfrak{h}_{\overline{1}}^{2} \subseteq \mathfrak{s}_{\overline{0}}$ be the subspace spanned by all the squares $w^{2}, w \in \mathfrak{h}_{\overline{1}}$. Then

$$
\begin{equation*}
\mathfrak{h} \frac{2}{1}=\sum_{j=1}^{l^{\prime \prime}} \mathbb{R}\left(J_{j}+J_{j}^{\prime}\right) . \tag{42}
\end{equation*}
$$

We shall use the following identification

$$
\begin{equation*}
\mathfrak{h}_{\overline{1}}^{2}\left|\mathrm{v}_{\overline{0}} \ni \sum_{j=1}^{l^{\prime \prime}} y_{j} J_{j}=\sum_{j=1}^{l^{\prime \prime}} y_{j} J_{j}^{\prime} \in \mathfrak{h}_{\overline{\mathrm{I}}}^{2}\right| \mathrm{v}_{\overline{\mathrm{T}}} . \tag{43}
\end{equation*}
$$

Recall from (34) that $l^{\prime \prime}=\min \left\{l, l^{\prime}\right\}$. If $l^{\prime \prime}=l$, then $\mathfrak{h}_{\overline{1}}^{2} \mid v_{\overline{0}}$ is an elliptic Cartan subalgebra of $\mathfrak{g}$ which we denote by $\mathfrak{h}$. (This means that all the roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ are purely imaginary.) The identification (43) embeds $\mathfrak{h}$ diagonally in $\mathfrak{g}$ and in $\mathfrak{g}^{\prime}$. It is contained in an elliptic Cartan subalgebra of $\mathfrak{g}^{\prime}$, say $\mathfrak{h}^{\prime}$. Similarly, if $l^{\prime \prime}=l^{\prime}$, then $\left.\mathfrak{h}_{\frac{1}{1}}^{2} \right\rvert\, v_{\overline{1}}$ is an elliptic Cartan subalgebra of $\mathfrak{g}^{\prime}$ which we denote by $\mathfrak{h}^{\prime}$. If $l \leq l^{\prime}$ we denote by $\mathfrak{z}^{\prime} \subseteq \mathfrak{g}^{\prime}$ the centralizer of $\mathfrak{h}$. Similarly, if $l^{\prime} \leq l$ we denote by $\mathfrak{z} \subseteq \mathfrak{g}$ the centralizer of $\mathfrak{h}^{\prime}$. In particular, if $l^{\prime}=l$, then $\mathfrak{z}^{\prime}=\mathfrak{h}^{\prime}=\mathfrak{h}=\mathfrak{z}$, where the first equality is in $\mathfrak{g}$, the second is (43) and the last is in $\mathfrak{g}^{\prime}$.

Let $\mathfrak{s}_{\overline{0} \mathbb{C}}=\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{\prime}$ be the complexification of $\mathfrak{s}_{0}$. Fix a system of positive roots for the adjoint action of $\mathfrak{h}_{\frac{2}{1}}^{2}$ on $\mathfrak{s}_{0} \mathbb{C}$. Suppose first that $l \leq l^{\prime}$. By the identification (43), $\mathfrak{h}$ preserves both $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}^{\prime}$. So our choice of positive roots for $\left(\mathfrak{h}_{\overline{1}}^{2}, \mathfrak{s}_{\overline{0} \mathbb{C}}\right)$ fixes a positive root system of $\left(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}\right)$ and extends to a compatible positive root system for $\left(\mathfrak{h}_{\mathbb{C}}^{\prime}, \mathfrak{g}_{\mathbb{C}}^{\prime}\right)$. Let $\pi_{\mathfrak{g} / \mathfrak{h}}$ be the product of positive roots of $\left(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}\right)$ and let $\pi_{\mathfrak{g}^{\prime} / \mathfrak{s}^{\prime}}$ be the product of positive roots of $\left(\mathfrak{h}_{\mathbb{C}}^{\prime}, \mathfrak{g}_{\mathbb{C}}^{\prime}\right)$ such that the corresponding root spaces do not occur in $\mathfrak{z}_{\mathbb{C}}^{\prime}$. If $l>l^{\prime}$,
then $\pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}$ and $\pi_{\mathfrak{g} / \mathfrak{z}}$ can be similarly defined. See Appendix A for the explicit expressions of these root products restricted to the elements in (43).

Suppose $l^{\prime}<l$. Then $\mathrm{V}_{\overline{1}}^{0}=0, \mathrm{~V}_{\overline{0}}^{0} \neq 0$ and

$$
\begin{equation*}
\mathrm{V}=\mathrm{V}_{\overline{0}}=\mathrm{V}_{\overline{0}}^{0} \oplus \mathrm{~V}_{\overline{0}}^{1} \oplus \mathrm{~V}_{\overline{0}}^{2} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l^{\prime \prime}} \tag{44}
\end{equation*}
$$

is a direct sum orthogonal decomposition with respect to the positive definite hermitian form $(\cdot, \cdot)$. We extend $\mathfrak{h} \subseteq \mathfrak{g}$ to a Cartan subalgebra $\mathfrak{h}(\mathfrak{g}) \subseteq \mathfrak{g}$ as follows. The restriction of $\mathfrak{h}(\mathfrak{g})$ to $\mathrm{V}_{\overline{0}}^{1} \oplus \mathrm{~V}_{\overline{0}}^{2} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{{l^{\prime \prime}}^{\prime}}$ coincides with $\mathfrak{h}$. Pick an orthogonal direct sum decomposition

$$
\begin{equation*}
\mathrm{V}_{\overline{0}}^{0}=\mathrm{V}_{\overline{0}}^{0,0} \oplus \mathrm{~V}_{\overline{0}}^{0, l^{\prime \prime}+1} \oplus \mathrm{~V}_{\overline{0}}^{0, l^{\prime \prime}+2} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{0}, l \tag{45}
\end{equation*}
$$

where for $j>l^{\prime \prime}, \operatorname{dim}_{\mathbb{D}} \bigvee_{\overline{0}}^{0, j}=2$ if $\mathbb{D}=\mathbb{R}$ and $\operatorname{dim}_{\mathbb{D}} \bigvee_{\overline{0}}^{0, j}=1$ if $\mathbb{D} \neq \mathbb{R}$. Also $\bigvee_{\overline{0}}^{0,0}=0$ unless $\mathrm{G}=\mathrm{O}_{2 l+1}$, in which case $\operatorname{dim}_{\mathbb{D}} \mathrm{V}_{\overline{0}}^{0,0}=1$. In each space $\mathrm{V}_{\overline{0}}^{0, j}$, with $j>l^{\prime \prime}$, we pick an orthonormal basis and define $J_{j}$ as in (40). Then

$$
\begin{equation*}
\mathfrak{h}(\mathfrak{g})=\sum_{j=1}^{l} \mathbb{R} J_{j} . \tag{46}
\end{equation*}
$$

If $l \leq l^{\prime}$, then we set $\mathfrak{h}(\mathfrak{g})=\mathfrak{h}$.
Let $J_{j}^{*}, 1 \leq j \leq l$, be the basis of the space $\mathfrak{h}(\mathfrak{g})^{*}$ which is dual to $J_{1}, \ldots, J_{l}$, and set

$$
\begin{equation*}
e_{j}=-i J_{j}^{*}, \quad 1 \leq j \leq l . \tag{47}
\end{equation*}
$$

If $\mu \in i \mathfrak{h}(\mathfrak{g})^{*}$, then $\mu=\sum_{j=1}^{l} \mu_{j} e_{j}$ with $\mu_{j} \in \mathbb{R}$. We say that $\mu$ is strictly dominant if $\mu_{1}>\mu_{2}>\cdots>\mu_{l}$.

## 4. Orbital integrals on W

In this section we recall from [MPP15] and [MPP20] some definitions and results concerning the orbital integrals on W that we will need in the following. Moreover, using the surjectivity properties of Harish-Chandra's orbital integrals, we prove that the image of the so-called Harish-Chandra regular almost-elliptic orbital integral on W is "large enough" in the sense of Corollary 2 when $l \geq l^{\prime}$.

Let $\mathcal{S}^{\prime}(\mathrm{W})^{\mathrm{S}}$ denote the space of S -invariant tempered distributions on W , where the S-action is induced by (30). Let $\mathfrak{h}_{\overline{1}}$ be a Cartan subspace of W. Suppose first that G is different from $\mathrm{O}_{2 l+1}$ with $l<l^{\prime}$. For $w \in \mathfrak{h}_{\overline{1}}^{\text {reg }}$, the orbital integral attached to the orbit $\mathcal{O}(w)=$ S. $w$ is the element $\mu_{\mathcal{O}(w), \mathfrak{h}_{\bar{T}}}$ of $\mathcal{S}^{\prime}(\mathrm{W})^{\mathrm{S}}$ defined for $\phi \in \mathcal{S}(\mathrm{W})$ by

$$
\begin{equation*}
\mu_{\mathcal{O}(w), \mathfrak{h}_{\bar{T}}}(\phi)=\int_{\mathrm{S} / \mathrm{S}^{\mathfrak{b}_{\overline{\mathrm{T}}}}} \phi(s . w) d\left(s S^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right)=\frac{1}{\operatorname{vol}\left(\mathrm{~S}^{\mathfrak{h}_{\mathrm{T}}}\right)} \int_{\mathrm{S}} \phi(s . w) d s . \tag{48}
\end{equation*}
$$

Suppose now that $\mathrm{G}=\mathrm{O}_{2 l+1}$ with $l<l^{\prime}$. Then one needs to modify (48) because the union of the orbits S. $w$ over all $w \in \mathfrak{h}_{\overline{1}}^{\text {reg }}$ would not be dense in W; see [MPP15, Theorem 20]. Let $w_{0} \in \mathfrak{s}_{\overline{1}}\left(\mathrm{~V}^{0}\right)$ be a non-zero element, $w \in \mathfrak{h}_{\overline{1}}^{\text {reg }}$ and $\mathrm{S}^{\mathfrak{h}+w_{0}}$ the centralizer of $w+w_{0}$ in S . Set $\mathcal{O}(w)=\mathrm{S} .\left(w+w_{0}\right)$ and define

$$
\begin{equation*}
\mu_{\mathcal{O}(w), \mathfrak{h}_{\overline{1}}}(\phi)=\int_{\mathrm{S}^{\mathbf{S}} \mathrm{S}_{\overline{\mathrm{T}}}+w_{0}} \phi\left(s .\left(w+w_{0}\right)\right) d\left(s \mathrm{~S}^{\mathfrak{h}_{\bar{T}}+w_{0}}\right)=\frac{1}{\operatorname{vol}\left(\mathrm{~S}^{\mathfrak{h}_{\overline{1}}+w_{0}}\right)} \int_{\mathrm{S}} \phi\left(s .\left(w+w_{0}\right)\right) d s . \tag{49}
\end{equation*}
$$

(This is independent of the choice of $w_{0} \in \mathfrak{s}_{1}\left(\mathrm{~V}^{0}\right)$.) The orbital integrals (48) and (49) are well defined, tempered distribution on W , which depend only on $\tau(w)$, or equivalently $\tau^{\prime}(w)$ via the identification (43).

For $w \in \mathfrak{h}_{\overline{1}}$, set

$$
\pi_{\mathfrak{s}_{0} / \mathfrak{h}_{1}^{2}}\left(w^{2}\right)= \begin{cases}\pi_{\mathfrak{g} / \mathfrak{h}}(\tau(w)) \pi_{\mathfrak{g}^{\prime} / \mathfrak{s}^{\prime}}\left(\tau^{\prime}(w)\right) & \text { if } l \leq l^{\prime}  \tag{50}\\ \pi_{\mathfrak{g} / \mathfrak{z}}(\tau(w)) \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(\tau^{\prime}(w)\right) & \text { if } l \geq l^{\prime}\end{cases}
$$

As shown in [MPP20, Lemma 1.2], there is a constant $C\left(\mathfrak{h}_{\overline{1}}\right)$, depending on $\mathfrak{h}_{\overline{1}}$ and with $\left|C\left(\mathfrak{h}_{\overline{1}}\right)\right|=1$, such that

$$
\begin{equation*}
\pi_{\mathfrak{s}_{\overline{0}} / \mathfrak{h}}^{\frac{2}{1}}\left(w^{2}\right)=C\left(\mathfrak{h}_{\overline{1}}\right)\left|\pi_{\mathfrak{s}_{\overline{0}} / \mathfrak{b} \frac{2}{1}}\left(w^{2}\right)\right| . \tag{51}
\end{equation*}
$$

The set $\mathfrak{h}_{\overline{1}}{ }^{\text {reg }}$ of regular elements of $\mathfrak{h}_{\overline{1}}$ is explicitly given by

$$
\begin{equation*}
\mathfrak{h}_{\overline{1}}^{r e g}=\left\{w \in \mathfrak{h}_{\overline{1}} ; \pi_{\mathfrak{s}_{\sigma} / \mathfrak{h}_{\frac{2}{1}}}\left(w^{2}\right) \neq 0\right\} . \tag{52}
\end{equation*}
$$

Choose a positive Weyl chamber $\mathfrak{h}_{\overline{1}}^{+} \subseteq \mathfrak{h}_{\overline{1}}^{\text {reg }}$, i.e. an open fundamental domain for the action of the Weyl group, $W\left(\mathrm{~S}, \mathfrak{h}_{\overline{1}}\right)$. There is a normalization $d \tau(w)$ of the Lebsegue measure on $\mathfrak{h}$, respectively a normalization $d \tau^{\prime}(w)$ of the Lebsegue measure on $\mathfrak{h}^{\prime}$, such that the following equalities hold for all $\phi \in \mathcal{S}(\mathrm{W})$ :

$$
\begin{array}{ll}
\mu_{\mathrm{W}}(\phi)=\sum_{\mathfrak{h}_{\overline{1}}} \int_{\tau\left(\mathfrak{h}_{\frac{1}{1}}^{+}\right)}\left|\pi_{\mathfrak{s}_{\overline{0}} / \mathfrak{h}_{\overline{1}}^{2}}\left(w^{2}\right)\right| \mu_{\mathcal{O}(w), \mathfrak{h}_{\overline{\mathrm{T}}}}(\phi) d \tau(w) & \text { if } l \leq l^{\prime}, \\
\mu_{\mathrm{W}}(\phi)=\int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{+}\right)}\left|\pi_{\mathfrak{s}_{\bar{\delta}} / \mathfrak{h}_{\overline{1}}^{2}}\left(w^{2}\right)\right| \mu_{\mathcal{O}(w), \mathfrak{h}_{\bar{T}}}(\phi) d \tau^{\prime}(w) & \text { if } l \geq l^{\prime} . \tag{54}
\end{array}
$$

Formulas (53) and (54) are the Weyl-Harish-Chandra integration formulas on W, [MPP15, Theorem 21]. The sum in (53) is over the family of mutually non-conjugate Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathrm{~W}$. (It therefore reduces to a single term for (G, $\mathrm{G}^{\prime}$ ) different from $\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right)$ with $l<l^{\prime}=p+q$.) The formulas agree for $l=l^{\prime}$ once we identify $\tau(w)$ and $\tau^{\prime}(w)$ via (43).

Let $C_{\mathfrak{h}_{\overline{1}}}=C\left(\mathfrak{h}_{\overline{1}}\right) \cdot i^{\operatorname{dim} \mathfrak{g} / \mathfrak{h}}$, where $C\left(\mathfrak{h}_{\overline{1}}\right)$ is as in (51). Let $\bigcup_{\mathfrak{h}_{\overline{1}}} \tau\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)$ be as in [MPP20, Lemma 3.1] if $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right)$ with $l<l^{\prime}=p+q$, and equal to $\tau\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)$, where $\mathfrak{h}_{\overline{1}}$ is the fixed Cartan subspace, otherwise. The Harish-Chandra regular almost-elliptic orbital integral on W is the function $F: \bigcup_{\mathfrak{h}_{\bar{T}}} \tau\left(\mathfrak{h}_{\overline{1}}^{r e g}\right) \rightarrow \mathcal{S}^{\prime}(\mathrm{W})^{\mathrm{S}}$ defined as follows for every $y \in \bigcup_{\mathfrak{h}_{\bar{T}}} \tau\left(\mathfrak{h}_{\overline{1}}^{r e g}\right), y=\tau(w)=\tau^{\prime}(w):$

$$
F(y)= \begin{cases}\sum_{\mathfrak{h}_{\bar{T}}} C_{\mathfrak{h}_{\bar{T}}} \pi_{\mathfrak{g}^{\prime} / \mathfrak{z}^{\prime}}(y) \mu_{\mathcal{O}(w), \mathfrak{h}_{\overline{1}}} & \text { if } l \leq l^{\prime}  \tag{55}\\ C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}^{\prime}} / \mathfrak{h}^{\prime}(y) \mu_{\mathcal{O}(w), \mathfrak{h}_{\overline{1}}} & \text { if } l>l^{\prime}\end{cases}
$$

Following Harish-Chandra's notation, we shall write $F_{\phi}(y)$ for $F(y)(\phi)$.
Suppose first that $l \leq l^{\prime}$. According to [MPP20, Theorem 3.6], $F$ uniquely extends to a function $F: \mathfrak{h} \rightarrow \mathcal{S}^{\prime}(\mathrm{W})^{\mathrm{S}}$ satisfying

$$
\begin{equation*}
F(s y)=\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(s) F(y) \quad(s \in W(\mathrm{G}, \mathfrak{h}), y \in \mathfrak{h}) \tag{56}
\end{equation*}
$$

This extension is supported in $\mathfrak{h} \cap \tau(\mathrm{W})$. The extended map $F$ is smooth on the subset of $y=\sum_{j=1}^{l} y_{j} J_{j}$ where each $y_{j} \neq 0$ and, for any multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{l}\right)$ with

$$
\max \left(\alpha_{1}, \ldots, \alpha_{l}\right) \leq \begin{cases}d^{\prime}-r-1 & \text { if } \mathbb{D}=\mathbb{R} \text { or } \mathbb{C}  \tag{57}\\ 2\left(d^{\prime}-r\right) & \text { if } \mathbb{D}=\mathbb{H}\end{cases}
$$

the function $\partial\left(J_{1}^{\alpha_{1}} J_{2}^{\alpha_{2}} \ldots J_{l}^{\alpha_{l}}\right) F(y)$ extends to a continuous function on $\mathfrak{h} \cap \tau(\mathrm{W})$ vanishing on the boundary of $\mathfrak{h} \cap \tau(\mathrm{W})$.

Before considering the case $l>l^{\prime}$, let us look at the pullback of the unnormalized moment map $\tau^{\prime}: \mathrm{W} \rightarrow \mathfrak{g}^{\prime}$ (for any values of $l$ and $l^{\prime}$ ). The pullback of $\tau^{\prime}$ is

$$
\tau^{\prime *}: \mathcal{S}\left(\mathfrak{g}^{\prime}\right) \ni \psi \rightarrow \psi \circ \tau^{\prime} \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}
$$

According to [MPP20, (25)] (a special case of a theorem of Astengo, Di Blasio and Ricci [ABR09, Theorem 6.1]), there is a surjective map $\tau_{*}^{\prime}: \mathcal{S}(\mathrm{W})^{\mathrm{G}} \rightarrow \mathcal{S}\left(\mathfrak{g}^{\prime}\right)$ such that

$$
\tau^{\prime *} \circ \tau_{*}^{\prime}(\phi)=\phi \quad\left(\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}\right)
$$

Suppose now that $l>l^{\prime}$. Then by [MPP20, (39)],

$$
\begin{equation*}
F_{\phi}(y)=C_{\mathfrak{h}_{\overline{1}}}^{\prime} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y) \int_{\mathrm{G}^{\prime} / \mathrm{H}^{\prime}} \psi\left(g^{\prime} \cdot y\right) d\left(g^{\prime} \mathrm{H}^{\prime}\right) \quad\left(\phi \in \mathcal{S}(\mathrm{W}), y \in \mathfrak{h}_{\overline{1}}^{r e g}\right), \tag{58}
\end{equation*}
$$

where $\mathrm{H}^{\prime} \subseteq \mathrm{G}^{\prime}$ is the Cartan subgroup corresponding to $\mathfrak{h}^{\prime}$,

$$
\begin{equation*}
\psi=\tau_{*}^{\prime}\left(\phi^{\mathrm{G}}\right) \in \mathcal{S}\left(\mathfrak{g}^{\prime}\right), \tag{59}
\end{equation*}
$$

and $C_{\mathfrak{h}_{\boldsymbol{T}}}^{\prime}$ is a suitable non-zero constant. The right-hand side of (58) is Harish-Chandra's orbital integral of $\psi$. It provides a $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-skew-invariant extension of $F_{\phi}$ to $\mathfrak{h}^{\prime \text { In } n-\text { reg }}$, where $\mathfrak{h}^{\prime \text { In-reg }} \subseteq \mathfrak{h}^{\prime}$ is the subset where no non-compact roots vanish. Furthermore, $F(y)$ is G-invariant for $y \in \mathfrak{h}^{\prime \text { In-reg }}$ and $\mathrm{G}^{\prime}$-invariant for $y \in \mathfrak{h}^{\text {In-reg }} \cap \tau^{\prime}(\mathrm{W})$; see [MPP20, Theorem 3.4].

Notice that, by [MPP20, (69)-(72)], formulas (58) and (59) also hold when $l=l^{\prime}$ because $\mathrm{Z}^{\prime}=\mathrm{H}^{\prime}$ in this case.

Remark 1. The Cartan subalgebra $\mathfrak{h}^{\prime}$ is $\theta$-stable, where $\theta$ is the fixed Cartan involution of $\mathfrak{g}^{\prime}$. Let $\mathrm{H}^{\prime} \subseteq \mathrm{G}^{\prime}$ be the Cartan subgroup which is the centralizer of $\mathfrak{h}^{\prime}$ in $G^{\prime}$, and let $K^{\prime}$ be the maximal compact subgroup of $\mathrm{G}^{\prime}$ which is fixed by $\theta$. Then, by [Har56, Lemma 10], the Weyl group $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$ coincides with $W\left(\mathrm{~K}^{\prime}, \mathfrak{h}^{\prime}\right)$, i.e. the normalizer of $\mathrm{H}^{\prime}$ in $\mathrm{K}^{\prime}$ modulo the centralizer of $\mathrm{H}^{\prime}$ in $\mathrm{K}^{\prime}$. Explicitly, $\mathrm{K}^{\prime}$ is $\mathrm{U}_{l^{\prime}}$ if $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, and $\mathrm{U}_{p} \times \mathrm{U}_{q}$ if $\mathbb{D}=\mathbb{C}$. Hence $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$ acts on $\mathfrak{h}^{\prime}$ by permuting the $J_{j}^{\prime},(39)$, if $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, and by separately permuting the first $p$ and the last $q$ elements $J_{j}^{\prime}$ if $\mathbb{D}=\mathbb{C}$. Since $\delta_{j}=1$ for all $j=1, \ldots, l^{\prime}$ if $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, and $\delta_{j}=1$ for $j=1, \ldots, p$ and $\delta_{j}=-1$ for $j=p+1, \ldots, p+q$ if $\mathbb{D}=\mathbb{C}$, it follows from (41) that the domain of integration $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)$ appearing in (54) is $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant. This property will be relevant in Corollary 2 below.

The following results show that the image of $F$ is "large enough" when $l \geq l^{\prime}$.
Proposition 1. Suppose that $l \geq l^{\prime}$. Given a $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-skew-invariant function $\psi_{0} \in$ $C_{c}^{\infty}\left(\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)\right)$, there is a function $\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}$ such that

$$
F_{\phi}(y)=\psi_{0}(y) \quad\left(y \in \tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)=\tau\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)\right) .
$$

Proof. This is a consequence of [Bou94, Théorème, p. 164]. We keep the notation of [Bou94] with $\mathfrak{g}^{\prime}$ instead of $\mathfrak{g}$ and $\mathcal{U}=\mathfrak{g}^{\prime}$ : for $j=0, k=1$ and $\mathfrak{h}_{1}=\mathfrak{h}^{\prime}$, the function $\psi_{0}$ belongs to $\mathcal{D}\left(\mathfrak{h}^{\prime}\right)^{\varepsilon_{I}}$. Hence part (ii) of the cited theorem proves that there is $\Psi_{0} \in \mathcal{J} \mathcal{D}_{0}\left(\mathfrak{g}^{\prime}\right)$ such that

$$
\psi_{0}(y)=\left(\prod_{\alpha>0} \frac{\alpha(y)}{|\alpha(y)|}\right) \Psi_{0}(y) \quad\left(y \in \mathfrak{h}^{\prime r e g}\right)
$$

where the product is over the positive roots of $\left(\mathfrak{h}_{\mathbb{C}}^{\prime}, \mathfrak{g}_{\mathbb{C}}^{\prime}\right)$ and $\mathfrak{h}^{\prime \text { reg }}$ is the set where none of the root vanishes. Notice that $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}{ }^{\text {reg }}\right) \subseteq \mathfrak{h}^{\prime \text { reg }}$. Part (i) of the same theorem then shows that there is $\psi \in C_{c}^{\infty}\left(\mathfrak{g}^{\prime}\right)$ such that

$$
\psi_{0}(y)=C_{\mathfrak{h}_{\overline{1}}}^{\prime} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y) \int_{\mathrm{G}^{\prime} / \mathrm{H}^{\prime}} \psi\left(g^{\prime} \cdot y\right) d\left(g^{\prime} \mathrm{H}^{\prime}\right) \quad\left(y \in \mathfrak{h}^{\prime r e g}\right)
$$

The claim then follows by the surjectivity of $\tau_{*}^{\prime}$ and (59).
Corollary 2. Suppose that $l \geq l^{\prime}$. Let $\Phi$ be a $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant real-valued continuous function on $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)$. Suppose that for every $\phi \in \mathcal{S}(\mathrm{W})$ the integral

$$
\Phi^{\sharp}(\phi)=\int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}} \text { reg }\right)} \Phi(y) \pi_{\mathfrak{g} / \mathfrak{s}}(y) F_{\phi}(y) d y
$$

converges absolutely and hence defines a linear map $\Phi^{\sharp}: \mathcal{S}(\mathrm{W}) \rightarrow \mathbb{C}$. (In the integral, the function $\pi_{\mathfrak{g} / \mathfrak{s}}(y)$ is computed using the identification $y=\tau^{\prime}(w)=\tau(w)$ from (43).) If $\Phi \neq 0$, then $\Phi^{\sharp} \neq 0$.

Proof. Choose a $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant open subset $U_{0}$ of $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)$ on which $\Phi$ is non-zero and of constant sign. Notice that $\pi_{\mathfrak{g} / \mathfrak{z}}(y) F_{\phi}(y)$ is $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant for every $\phi \in \mathcal{S}(\mathrm{W})$; see (A.4) for the explicit expression of $\pi_{\mathfrak{g} / \mathfrak{s}}(y)$. Choose $\psi_{0} \in C_{c}^{\infty}\left(\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}{ }^{\text {reg }}\right)\right)$ supported in $U_{0}$, $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-skew invariant and positive on a fundamental domain of the $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-action on $U_{0}$. Let $\phi \in \mathcal{S}(\mathrm{W})$ be chosen for $\psi_{0}$ as in Proposition 1. Then the integral defining $\Phi^{\sharp}(\phi)$ is nonzero.

## 5. Main results

Suppose an irreducible representation $\Pi$ of $\widetilde{G}$ occurs in Howe's correspondence. This means that there is a subspace $\mathcal{H}_{\Pi} \subseteq \mathrm{L}^{2}(\mathrm{X})$ on which the restriction of $\omega$ coincides with П. Since $\widetilde{\mathrm{Z}} \subseteq \widetilde{\mathrm{G}} \cap \widetilde{\mathrm{G}}^{\prime}$, then either $\mathcal{H}_{\Pi} \subseteq \mathrm{L}^{2}(\mathrm{X})_{+}$or $\mathcal{H}_{\Pi} \subseteq \mathrm{L}^{2}(\mathrm{X})_{-}$. In the first case the restriction of the central character $\chi_{\Pi}$ of $\Pi$ to $\widetilde{\mathrm{Z}}$ is equal to $\chi_{+}$and in the second case to $\chi_{-}$. Thus for $\tilde{z} \in \widetilde{\mathrm{Z}}$ and $\tilde{g} \in \widetilde{\mathrm{G}}$,

$$
\begin{array}{lll}
\Theta_{\Pi}(\tilde{z} \tilde{g})=\chi_{+}(\tilde{z}) \Theta_{\Pi}(\tilde{g}) & \text { if } & \mathcal{H}_{\Pi} \subseteq \mathrm{L}^{2}(\mathrm{X})_{+}  \tag{60}\\
\Theta_{\Pi}(\tilde{z} \tilde{g})=\chi_{-}(\tilde{z}) \Theta_{\Pi}(\tilde{g}) & \text { if } & \mathcal{H}_{\Pi} \subseteq \mathrm{L}^{2}(\mathrm{X})_{-}
\end{array}
$$

We see from equations (17), (23), (24) and (60) that the function

$$
\widetilde{\mathrm{G}} \ni \tilde{g} \rightarrow T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) \in \mathcal{S}^{\prime}(\mathrm{W})
$$

is constant on the fibers of the covering map (9). The following lemma is a restatement of (18). Our main results will be the explicit expressions of the various integrals appearing on the right-hand sides of the equations below.

Lemma 3. Let $\mathrm{G}^{0} \subseteq \mathrm{G}$ denote the connected identity component. Suppose $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=$ $\left(\mathrm{U}_{d}, \mathrm{U}_{p, q}\right)$ or $\left(\mathrm{Sp}_{d}, \mathrm{O}_{2 m}^{*}\right)$. Then $\mathrm{G}=\mathrm{G}^{0}=-\mathrm{G}^{0}$ and

$$
\begin{equation*}
f_{\Pi \otimes \Pi^{\prime}}=\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g . \tag{61}
\end{equation*}
$$

Formula (61) holds also if $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{d}, \mathrm{Sp}_{2 m}(\mathbb{R})\right)$ with d even and $\Theta_{\Pi}$ supported in $\widetilde{\mathrm{G}^{0}}$, because $\mathrm{G}^{0}=\mathrm{SO}_{d}=-\mathrm{SO}_{d}=-\mathrm{G}^{0}$. In the remaining cases

$$
\begin{equation*}
f_{\Pi \otimes \Pi^{\prime}}=\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g+\int_{\mathrm{G} \backslash\left(-\mathrm{G}^{0}\right)} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g . \tag{62}
\end{equation*}
$$

The integrals over $-\mathrm{G}^{0}$ in (61) and (62) are given in Theorems 4 and 5 below, proved in section 10. The integrals over the other connected component, in the cases where (62) holds, are computed in Theorems 7, 8 and 9, respectively, and proved in sections 11, 12 , and 13. Theorem 6 , proved in this section, will furthermore show that the second integral on the right-hand side of (62) coincides with the first integral when ( $\mathrm{G}, \mathrm{G}^{\prime}$ ) $=$ $\left(\mathrm{O}_{d}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$, where $d=2 l$ or $d=2 l+1$, provided $l>l^{\prime}$.

Remark 2. Notice that, since the character $\Theta_{\Pi}$ is conjugation invariant,

$$
\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})\left(\phi^{\mathrm{G}}\right) d g
$$

where $\phi^{\mathrm{G}}$ is the projection of $\phi$ onto the space of the G-invariants in $\mathcal{S}(\mathrm{W})$,

$$
\begin{equation*}
\phi^{\mathrm{G}}(w)=\int_{\mathrm{G}} \phi(g \cdot w) d g \quad(w \in \mathrm{~W}) \tag{63}
\end{equation*}
$$

(Recall that we have normalized the Haar measure on $G$ so that its mass is 1.)
Let

$$
\iota= \begin{cases}1 & \text { if } \mathbb{D}=\mathbb{R} \text { or } \mathbb{C}  \tag{64}\\ \frac{1}{2} & \text { if } \mathbb{D}=\mathbb{H}\end{cases}
$$

and let

$$
\begin{equation*}
r=\frac{2 \operatorname{dim} \mathfrak{g}}{\operatorname{dim} \mathrm{~V}_{\mathbb{R}}} \tag{65}
\end{equation*}
$$

where the subscript $\mathbb{R}$ indicates that we are viewing $V$ as a vector space over $\mathbb{R}$. Explicitly,

$$
r= \begin{cases}2 l-1 & \text { if } \mathrm{G}=\mathrm{O}_{2 l}  \tag{66}\\ 2 l & \text { if } \mathrm{G}=\mathrm{O}_{2 l+1} \\ l & \text { if } \mathrm{G}=\mathrm{U}_{l} \\ l+\frac{1}{2} & \text { if } \mathrm{G}=\mathrm{Sp}_{l}\end{cases}
$$

Let

$$
\begin{equation*}
\delta=\frac{1}{2 \iota}\left(d^{\prime}-r+\iota\right) \quad \text { and } \quad \beta=\frac{2 \pi}{\iota} \tag{67}
\end{equation*}
$$

Fix an irreducible representation $\Pi$ of $\widetilde{\mathrm{G}}$ that occurs in the restriction of the Weil representation $\omega$ to $\widetilde{\mathrm{G}}$. Let $\mu \in i \mathfrak{h}(\mathfrak{g})^{*}$ be the Harish-Chandra parameter of $\Pi$ with $\mu_{1}>\mu_{2}>\cdots$. This means that $\mu=\lambda+\rho$, where $\lambda$ is the highest weight of $\Pi$ and $\rho$ is one half times the sum of the positive roots of $\left(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}\right)$. See Appendix H.

Let $P_{a, b}$ and $Q_{a, b}$ be the piecewise polynomial functions defined in (D.4) and (D.5). Define

$$
\begin{align*}
& a_{j}=-\mu_{j}-\delta+1, \quad b_{j}=\mu_{j}-\delta+1  \tag{68}\\
& p_{j}(\xi)=P_{a_{j}, b_{j}}(\beta \xi) e^{-\beta|\xi|}, \quad q_{j}(\xi)=\beta^{-1} Q_{a_{j}, b_{j}}\left(\beta^{-1} \xi\right) \quad\left(1 \leq j \leq l^{\prime \prime}, \xi \in \mathbb{R}\right), \tag{69}
\end{align*}
$$

where $\delta$ and $\beta$ are as in (67). Notice that $a_{j}$ and $b_{j}$ are integers (see Lemma 19). Furthermore, set

$$
\kappa_{0}= \begin{cases}1 / 2 & \text { if } \mathrm{G}=\mathrm{O}_{2 l} \text { and } \lambda_{l}=\mu_{l}=0  \tag{70}\\ 1 & \text { otherwise }\end{cases}
$$

Theorem 4. Let $l \leq l^{\prime}$. Then there is a non-zero constant $C$ which depends only of the dual pair ( $\mathrm{G}, \mathrm{G}^{\prime}$ ) such that for all $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h} \cap \tau(\mathrm{W})}\left(\prod_{j=1}^{l}\left(p_{j}\left(y_{j}\right)+q_{j}\left(-\partial_{y_{j}}\right) \delta_{0}\left(y_{j}\right)\right)\right) \cdot F_{\phi}(y) d y \tag{71}
\end{equation*}
$$

where $\chi_{\Pi}$ is the central character of $\Pi$ (see (60)), $\tilde{c}$ is a real analytic lift of the Cayley transform (see (123)), $\delta_{0}$ is the Dirac delta at 0 , and $F_{\phi}(y)$ is the image of $\phi$ under the Harish-Chandra regular almost-elliptic orbital integral on W (see [MPP20, Definition 3.2] and (55)).

The term

$$
\begin{equation*}
\prod_{j=1}^{l}\left(p_{j}\left(y_{j}\right)+q_{j}\left(-\partial_{y_{j}}\right) \delta_{0}\left(y_{j}\right)\right) \tag{72}
\end{equation*}
$$

is:
(1) a function of $y$ if and only if all the $q_{j}$ 's are zero, and this happens if and only if $l=l^{\prime}$ and $\left(\mathrm{G}, \mathrm{G}^{\prime}\right) \neq\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$;
(2) a linear combination of products of functions and Dirac delta's at 0 in some coordinates $y_{j}$ if and only if all the $q_{j}$ 's are of degree zero. This happens if and only if either $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l}(\mathbb{R})\right)$, or $l^{\prime}=l+1$ and $\mathbb{D}=\mathbb{C}$ or $\mathbb{H}$.
In the remaining cases, (72) is a distribution, but not a measure.
Remark 3. The integration domain $\mathfrak{h} \cap \tau(\mathrm{W})$ appearing in Theorem 4 was explicitly determined in [MPP20, Lemma 3.4]. It is equal to $\mathfrak{h}$ if $\mathbb{D} \neq \mathbb{C}$ or if $\mathbb{D}=\mathbb{C}$ and $l \leq$ $\min \{p, q\}$. By (167), (168) and Appendix H, we see that $a_{j} \leq 0$ for all $1 \leq j \leq l$ when $l \leq l^{\prime}$. Hence each $P_{a_{j}, b_{j}}\left(\beta y_{j}\right)$ vanishes for $y_{j}<0$. In cases (1) and (2) of Theorem 4 with $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, we can therefore replace the domain of integration $\mathfrak{h}$ with the smaller domain $\tau\left(\mathfrak{h}_{\overline{1}}\right)$.

In the case $l>l^{\prime}$, up to conjugation, there is a unique Cartan subspace $\mathfrak{h}_{\overline{1}}$ in W. Recall that for $\mathbb{D}=\mathbb{C}$ we are supposing that $p \leq q$.

Define $s_{0} \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ by

$$
\begin{align*}
& s_{0}\left(J_{j}\right)=J_{j} \quad(1 \leq j \leq l)  \tag{73}\\
& s_{0}\left(J_{j}\right)=\left\{\begin{array}{ll}
J_{j} & (1 \leq j \leq p) \\
J_{q+j} & (p+1 \leq j \leq l-q) \\
J_{j-l+l^{\prime}} & (l-q+1 \leq j \leq l)
\end{array} \quad \text { if } \mathbb{D}=\mathbb{R} \text { or } \mathbb{H},\right. \tag{74}
\end{align*}
$$

Theorem 5. Let $l>l^{\prime}$. Then

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g \neq 0 \tag{75}
\end{equation*}
$$

if and only if the highest weight $\lambda=\sum_{j=1}^{l} \lambda_{j} e_{j}$ of $\Pi$ is of the form
(a) $\quad \lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{l^{\prime}} \geq 0$ and $\lambda_{j}=0$ for $l^{\prime}+1 \leq j \leq l \quad$ if $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$,
(b) $\quad \lambda_{j}=\frac{q-p}{2}+\nu_{j} \quad$ where

$$
\nu_{1} \geq \cdots \geq \nu_{p} \geq 0, \nu_{j}=0 \text { for } p+1 \leq j \leq l-q, 0 \geq \nu_{l-q+1} \geq \cdots \geq \nu_{l} \quad \text { if } \mathbb{D}=\mathbb{C}
$$

Suppose that (a) and (b) are satisfied. Then there is a non-zero constant $C$ which depends only of the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ such that for all $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)}\left(\prod_{j=1}^{l^{\prime}} p_{j}\left(\left(s_{0}^{-1} y\right)_{j}\right)\right) \cdot F_{\phi}(y) d y \tag{76}
\end{equation*}
$$

where $\kappa_{0}$ is as in (70) and, explicitly,

$$
\prod_{j=1}^{l^{\prime}} p_{j}\left(\left(s_{0}^{-1} y\right)_{j}\right)= \begin{cases}\prod_{j=1}^{l^{\prime}} p_{j}\left(y_{j}\right) & \text { if } \mathbb{D}=\mathbb{R} \text { or } \mathbb{H} \\ \left(\prod_{j=1}^{p} p_{j}\left(y_{j}\right)\right)\left(\prod_{j=l-q+1}^{l} p_{j}\left(y_{j}\right)\right) & \text { if } \mathbb{D}=\mathbb{C}\end{cases}
$$

Moreover,

$$
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C \int_{\mathrm{W}} \Phi(w) \phi(w) d w
$$

where $\Phi$ is the locally integrable S-invariant function on W whose restriction to $\mathfrak{h}_{\overline{1}}^{\text {reg }}$ is a non-zero constant multiple of

$$
\begin{align*}
& \frac{\sum_{s^{\prime} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta\left(s^{\prime} y\right)_{j}\right)}{\pi_{\mathfrak{g} / \mathfrak{z}}(y)} e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} \\
& \quad\left(y_{j}=J_{j}^{\prime *} y, y=\tau(w)=\tau^{\prime}(w), w \in \mathfrak{h}_{\overline{1}}{ }^{\text {reg }}\right) . \tag{77}
\end{align*}
$$

In (77), $\mu$ is the Harish-Chandra parameter of $\Pi$,

$$
\begin{equation*}
a_{s, j}=-(s \mu)_{j}-\delta+1, \quad b_{s, j}=(s \mu)_{j}-\delta+1 \quad(s \in W(\mathrm{G}, \mathfrak{h}), 1 \leq j \leq l) \tag{78}
\end{equation*}
$$

$P_{a, b, \pm 2}$ is the polynomial defined in (D.1) or (D.2), and the $\delta_{j}$ 's are as in (37). (See (43) for the identifications $y=\tau(w)=\tau^{\prime}(w)$ in (77).) The fraction in (77) is the pullback by $\tau^{\prime}$ of a $\mathrm{G}^{\prime}$-invariant polynomial on $\mathfrak{g}^{\prime}$. The exponential in (77) extends to a $\mathrm{GG}^{\prime}$-invariant function. It is a Gaussian on W if and only if $\mathrm{G}^{\prime}$ is compact.

Remark 4. Recall from Remark 1 that the domain of integration $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)$ appearing in Theorem 5 is $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant.

Remark 5. Let $\Pi$ be a genuine irreducible representation of $\widetilde{\mathrm{G}}$ of highest weight $\lambda$. Conditions (a) and (b) in Theorem 5 are precisely those ensuring that $\Pi$ occurs in Howe's correspondence. See [Prz96, Appendix] and Corollary 11 below.

Before considering the integrals over $G \backslash\left(-\mathrm{G}^{0}\right)$ in the cases where (62) holds, let us introduce some notation concerning the irreducible representations of the orthogonal groups.

Suppose that $\mathrm{G}=\mathrm{O}_{d}$. Then, for each highest weight $\lambda$ of an irreducible representation of $\mathrm{G}^{0}$ there are one or two unitary genuine representations of $\widetilde{\mathrm{G}}$ having highest weight $\lambda$. There are two if and only if either $d=2 l$ and $\lambda_{l}=0$, or $d=2 l+1$. See e.g. [GW98, §5.5.5]. Let $\Pi_{\lambda,+}$ and $\Pi_{\lambda,-}$ be these representations. Set

$$
\begin{equation*}
\chi_{+}(\widetilde{g})=\frac{\Theta(\widetilde{g})}{|\Theta(\widetilde{g})|} \quad\left(g \in \mathrm{O}_{d}\right) \tag{79}
\end{equation*}
$$

where $\Theta$ is defined in (16). Then $\chi_{+}$is a character of $\widetilde{\mathrm{G}}$. Notice that (79) is an extension of (23) from $\widetilde{\mathrm{Z}}$ to $\widetilde{\mathrm{G}}$. Let $\mathrm{X}=\mathrm{M}_{d, l^{\prime}}(\mathbb{R})$ denote the space of $d \times l^{\prime}$ matrices with real coefficients. Then, in the Schrödinger model for the Weil representation $\omega$, for which the space of smooth vectors is $\mathcal{S}(\mathrm{X})$,

$$
\begin{equation*}
\omega \otimes \chi_{+}^{-1}(\widetilde{g}) f(x)=f\left(g^{-1} x\right) \quad(g \in \mathrm{G}, f \in \mathcal{S}(\mathrm{X}), x \in \mathrm{X}) \tag{80}
\end{equation*}
$$

Hence $\omega \otimes \chi_{+}$descends to a representation $\omega_{0}$ of G given by

$$
\begin{equation*}
\omega_{0}(g) f(x)=f\left(g^{-1} x\right) \quad(g \in \mathrm{G}, f \in \mathcal{S}(\mathrm{X}), x \in \mathrm{X}) \tag{81}
\end{equation*}
$$

Theorem 6. Suppose that $l>l^{\prime}$. Let $\Pi$ be an irreducible representation of $\widetilde{\mathrm{O}}_{d}$ occurring in the restriction of the Weil representation to $\widetilde{\mathrm{O}}_{d}$. If $d=2 l$, then $\lambda_{l}=0$. In both cases $d=2 l$ or $d=2 l+1$, the second irreducible genuine representation of $\widetilde{\mathrm{O}}_{d}$ having the same highest weight as $\Pi$ does not occur in the restriction of the Weil representation to $\widetilde{\mathrm{O}}_{d}$. Moreover,

$$
\begin{equation*}
\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=2 \int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=2 \int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g \tag{82}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\int_{\mathrm{G} \backslash\left(-\mathrm{G}^{0}\right)} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g \tag{83}
\end{equation*}
$$

The integral on the very right-hand side of (82) was computed in Theorem 5.
Proof. Let $\lambda$ be the highest weight of $\Pi$. Suppose first that $d=2 l$. The assumption that the representation $\Pi$ occurs in the Weil representation means that, up to multiplication by a character, it occur in the $l^{\prime}$ fold tensor product of the defining representation $\mathbb{C}^{2 l}$. Hence the highest weight of $\Pi$ is equal to $l^{\prime} e_{1}$ minus a sum of positive roots. Since $l>l^{\prime}$, a straightforward argument shows that $\lambda_{l}=0$. Therefore $\Pi$ is one of the two inequivalent irreducible representations of $\widetilde{\mathrm{O}}_{2 l}$ having highest weight $\lambda$.

From now on, let $d=2 l$ or $2 l+1$. Keeping the notation introduced before (79), we prove that only one of the representations $\Pi_{\lambda,+}$ and $\Pi_{\lambda,-}$ (i.e. $\Pi$ ) can occur in the Weil representation.

Suppose on the contrary that they both occur. Then $\Pi_{\lambda, \pm} \otimes \chi_{+}^{-1}$ descends to a representation $\left.\left(\Pi_{\lambda, \pm} \otimes \chi_{+}^{-1}\right)\right|_{\mathrm{G}}$ of G occurring in $\omega_{0}$. By (81),

$$
\begin{equation*}
\left.\left(\Pi_{\lambda, \pm} \otimes \chi_{+}^{-1}\right)\right|_{\mathrm{G}}(g) f(x)=f\left(g^{-1} x\right) \quad(g \in \mathrm{G}, f \in \mathcal{S}(\mathrm{X}), x \in \mathrm{X}) \tag{84}
\end{equation*}
$$

Let $\Pi_{\lambda, 0}$ denote the irreducible representation of $\mathrm{G}^{0}$ of highest weight $\lambda$. As one can see from [GW98, §5.5.5],

$$
\begin{equation*}
\text { if }\left.\left(\Pi_{\lambda,+} \otimes \chi_{+}^{-1}\right)\right|_{\mathrm{G}}=\Pi_{\lambda, 0}, \text { then }\left.\left(\Pi_{\lambda,-} \otimes \chi_{+}^{-1}\right)\right|_{\mathrm{G}}=\Pi_{\lambda, 0} \otimes \text { det. } \tag{85}
\end{equation*}
$$

Hence $\Pi_{\lambda, 0} \otimes \Pi_{\lambda, 0} \otimes$ det occurs in $\omega_{0} \otimes \omega_{0}$, acting on $\mathcal{S}(\mathrm{X} \oplus \mathrm{X})$. Recall that $\Pi_{\lambda, 0}=\Pi_{\lambda, 0}^{c}$ is self-contragredient. Since $\Pi_{\lambda, 0}^{c} \otimes \Pi_{\lambda, 0}$ contains the trivial representation, we conclude that det occurs in $\omega_{0} \otimes \omega_{0}$. Observe that $\omega_{0} \otimes \omega_{0}$ acts on $\mathcal{S}(\mathrm{X} \oplus \mathrm{X})$ by

$$
\omega_{0} \otimes \omega_{0}(g) f(x)=f\left(g^{-1} x\right) \quad(g \in \mathrm{G}, f \in \mathcal{S}(\mathrm{X} \oplus \mathbf{X}), x \in \mathbf{X})
$$

It is therefore the "representation $\omega_{0}$ " corresponding to a dual pair $\left(\mathrm{O}_{d}, \mathrm{Sp}_{4 l^{\prime}}\right)$. By Proposition F.1, it follows that $d \leq 2 l^{\prime}$, contrary to our assumption.

The above shows that the second representation of $\widetilde{\mathrm{O}}_{d}$ which has the same restriction as $\Pi$ to $\mathrm{G}^{0}=\mathrm{SO}_{d}$, does not occur. Hence the $\left.\Pi\right|_{\widetilde{\mathrm{SO}}_{d}}$-isotypic component of $\omega$ coincides with the $\Pi$-isotypic component of $\omega$. Therefore

$$
\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=2 \int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g
$$

(The factor 2 is a consequence of the normalization of the measures.) In particular, $\int_{\mathrm{G} \backslash \mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g$. If $\mathrm{G}=\mathrm{O}_{2 l}$, then $\mathrm{G}^{0}=-\mathrm{G}^{0}$ and if $\mathrm{G}=\mathrm{O}_{2 l+1}$, then $\mathrm{G} \backslash \mathrm{G}^{0}=-\mathrm{G}^{0}$. This explains the second equality in (82).

Remark 6. It should be pointed out that the proof of Theorem 6 does not use the known classification of the highest weights of the genuine irreducible representations occurring in Howe's correspondence.

Consider now the case $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ and the character $\Theta_{\Pi}$ not supported on the preimage $\widetilde{\mathrm{G}^{0}}$ of the connected identity component $\mathrm{G}^{0} \subseteq \mathrm{G}$.

Suppose that $l \leq l^{\prime}$ and $l \neq 1$. Then the graded vector space (35) is equal to

$$
\mathrm{V}=\mathrm{V}_{\mathrm{I}}^{0} \oplus \mathrm{~V}^{1} \oplus \mathrm{~V}^{2} \oplus \cdots \oplus \mathrm{~V}^{l}
$$

Recall from (36) that in each $\mathrm{V}_{\overline{0}}^{j}$ we selected an orthonormal basis $v_{0}, v_{0}^{\prime}$. For convenience, we introduce the index $j$ in the notation and we write $v_{2 j-1}=v_{0}$ and $v_{2 j}=v_{0}^{\prime}$, for $1 \leq j \leq l$. Then $v_{1}, v_{2}, \ldots, v_{2 l}$ is an orthonormal basis of $\mathrm{V}_{\overline{0}}$ and

$$
J_{j} v_{2 j-1}=-v_{2 j}, \quad J_{j} v_{2 j}=v_{2 j-1} \quad(1 \leq j \leq l)
$$

In terms of the dual basis (47) of $\mathfrak{h}_{\mathbb{C}}^{*}$, the positive roots are

$$
e_{j} \pm e_{k} \quad(1 \leq j<k \leq l)
$$

Define an element $s \in \mathrm{G}$ by

$$
\begin{equation*}
s v_{1}=v_{1}, s v_{2}=v_{2}, \ldots, s v_{2 l-1}=v_{2 l-1}, s v_{2 l}=-v_{2 l} \tag{86}
\end{equation*}
$$

Then $G=G^{0} \cup G^{0} s$ is the disjoint union of two connected components. Set

$$
\mathrm{V}_{\overline{0}, s}=\mathrm{V}_{\overline{0}}^{1} \oplus \mathrm{~V}_{\overline{0}}^{2} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l-1} \oplus \mathbb{R} v_{2 l}, \quad \text { and } \quad \mathrm{V}_{s}=\mathrm{V}_{\overline{0}, s} \oplus \mathrm{~V}_{\overline{1}}
$$

The dual pair corresponding to $\left(\mathrm{V}_{\overline{0}, s}, \mathrm{~V}_{\overline{1}}\right)$ is $\left(\mathrm{G}_{s}, \mathrm{G}_{s}^{\prime}\right)=\left(\mathrm{O}_{2 l-1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ acting on the symplectic space $\mathrm{W}_{s}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}, s}\right)$. Since $\mathrm{V}_{\overline{0}}=\mathrm{V}_{\overline{0}, s} \oplus \mathbb{R} v_{2 l-1}$, we have two natural maps
$\kappa:\left\{\right.$ endomorphisms of $\mathrm{V}_{\overline{0}}$ preserving $\mathrm{V}_{\overline{0}, s}$ and $\left.\mathbb{R} v_{2 l-1}\right\} \longrightarrow\left\{\right.$ endomorphisms of $\left.\mathrm{V}_{\overline{0}, s}\right\}$, $\kappa^{-1}:\left\{\right.$ endomorphisms of $\left.\bigvee_{\overline{0}, s}\right\} \longrightarrow\left\{\right.$ endomorphisms of $\bigvee_{\overline{0}}$ preserving $\bigvee_{\overline{0}, s}$ and $\left.\mathbb{R} v_{2 l-1}\right\}$ defined by

$$
\begin{equation*}
\kappa(a)=\left.a\right|_{v_{0, s}} \quad \text { and } \quad \kappa^{-1}(b)=b \times \operatorname{id}_{\mathbb{R} v_{2 l-1}} . \tag{87}
\end{equation*}
$$

If we identify $a$ with its $(2 l) \times(2 l)$-matrix with respect to the basis $\left\{v_{j}\right\}_{1 \leq j \leq 2 l}$ of $\mathrm{V}_{\overline{0}}$, then $\kappa$ removes the $(2 l-1)^{\text {th }}$ row and column of $a$.

Let $\mathfrak{h}_{s}=\sum_{j=1}^{l-1} \mathbb{R} J_{j}$ be the centralizer of $s$ in $\mathfrak{h}$. Denote by

$$
\begin{equation*}
\lambda=\lambda_{1} e_{1}+\lambda_{2} e_{2}+\cdots+\lambda_{l-1} e_{l-1} \tag{88}
\end{equation*}
$$

the highest weight of $\Pi$. (Here $\lambda_{l}=0$ because the character $\Theta_{\Pi}$ is not supported on $\widetilde{\mathrm{G}^{0}}$.) See [GW09, p. 277] and [Wen01, Theorem 2.6].

Let $\mathrm{H}_{s}^{0} \subseteq \mathrm{H}$ be the identity connected component of the centralizer of $s$. The map

$$
\begin{equation*}
\kappa: \mathrm{H}_{s}^{0} \cup \mathrm{H}_{s}^{0} s \rightarrow \mathrm{G}_{s}, \tag{89}
\end{equation*}
$$

is a bijection onto the Cartan subgroup of $\mathrm{G}_{s}$. Notice that $\kappa: \mathrm{H}_{s}^{0} \rightarrow \kappa\left(\mathrm{H}_{s}^{0}\right)$ is an isomorphism. Moreover,

$$
\begin{equation*}
\kappa(h s)=\kappa(h) \kappa(s) \quad\left(h \in \mathrm{H}_{s}^{0}\right) . \tag{90}
\end{equation*}
$$

Let $\mathfrak{g}_{s}$ be the Lie algebra of $\mathrm{G}_{s}$. Then $\kappa\left(\mathfrak{h}_{s}\right)$ is a Cartan subalgebra of $\mathfrak{g}_{s}$.
Let $\sigma$ be the spin representation of the spin double cover $\operatorname{Spin}_{2 l-1}$ of $\mathrm{G}_{s}^{0}=\mathrm{SO}(2 l-1)$. Given a representation $\Pi$ of $\widetilde{G}$, of highest weight $\lambda$, then

$$
\begin{equation*}
\lambda_{s}=\lambda+\frac{1}{2} \sum_{j=1}^{l-1} e_{j} \tag{91}
\end{equation*}
$$

is the highest weight of an irreducible representation $\Pi_{\lambda_{s}}$ of $\operatorname{Spin}_{2 l-1}$. The function

$$
-\kappa\left(\mathrm{H}_{s}^{0}\right) \ni h_{1} \rightarrow \Theta_{\Pi_{\lambda_{s}} \otimes \sigma^{c}}\left(\widehat{\kappa(s) h_{1}}\right)
$$

is $W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)$ invariant. Hence there a $\mathrm{G}_{s}^{0}$-conjugation invariant function $\Phi_{\Pi}$ such that

$$
\Phi_{\Pi}\left(\tilde{h}_{1}\right)=\Theta_{\Pi_{\lambda_{s}} \otimes \sigma^{c}}\left(\widehat{\kappa(s) h_{1}}\right) \quad\left(h_{1} \in-\kappa\left(\mathrm{H}_{s}^{0}\right)\right)
$$

Theorem 7. Let $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$. Assume that the character $\Theta_{\Pi}$ is not supported on $\widetilde{\mathrm{G}^{0}}$. Suppose that $l \leq l^{\prime}$ and $l \neq 1$. Then for all $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{equation*}
\int_{\mathrm{G}^{0} s} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C(\Pi) \int_{-\mathrm{G}_{s}^{0}} \check{\Phi}_{\Pi}(\tilde{g}) T_{s}(\tilde{g})\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d g \tag{92}
\end{equation*}
$$

where $C(\Pi)$ is a constant equal to $\pm 1$ and $T_{s}$ is the map $T$, see (14), corresponding to the symplectic space $\mathrm{W}_{s}$. Once the function $\Phi_{\Pi}$ is decomposed as a linear combination of irreducible characters and Theorem 4 is applied to each summand, one obtains explicit formulas for (92).

Theorem 7 excludes the dual pairs $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ because its proof relies on an analogue of the Weyl's character formula for $G \backslash G^{0}$ proved by [Wen01] for nonconnected compact semisimple Lie groups. These excluded cases will be treated in section 6.

Now we consider the case $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $1 \leq l \leq l^{\prime}$. Recall from (35) the graded vector space V . In the case we consider, $\operatorname{dim} \mathrm{V}_{\overline{0}}^{0}=1$, $\operatorname{dim} \mathrm{V}_{\overline{1}}^{0}=2\left(l^{\prime}-l\right)$ and for $1 \leq j \leq l$, $\operatorname{dim} \bigvee_{\overline{0}}^{j}=\operatorname{dim} \bigvee_{\overline{1}}^{j}=2$. Let

$$
\mathrm{W}_{s}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}}^{1} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l}\right) \quad \text { and } \quad \mathrm{W}_{s}^{\perp}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}}^{0}\right) .
$$

Then

$$
\begin{equation*}
\mathrm{W}=\mathrm{W}_{s} \oplus \mathrm{~W}_{s}^{\perp} \tag{93}
\end{equation*}
$$

is a direct sum orthogonal decomposition. Let $\mathrm{G}_{s} \subseteq \mathrm{G}$ be the subgroup acting trivially on the space $\mathrm{V}_{\overline{0}}^{0}$. Then the Lie algebra $\mathfrak{g}_{s}$ of $\mathfrak{g}$ embeds as those elements acting as zero on $\mathrm{V}_{\overline{0}}^{0}$. Let $\mathrm{G}_{s}^{\prime}=\mathrm{G}^{\prime}$. Then the dual pair corresponding to $\mathrm{W}_{s}$ is $\left(\mathrm{G}_{s}, \mathrm{G}_{s}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ and dual pair corresponding to $\mathrm{W}_{s}^{\perp}$ is $\left(\mathrm{O}_{1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$. If H is a Cartan subgroup of G , then $H^{0}=H_{s}^{0}$ is a Cartan subgroup of $\mathrm{G}_{s}$, and the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_{s}$ share the same Cartan subalgebra $\mathfrak{h}=\mathfrak{h}_{s}$.

Theorem 8. Let $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $1 \leq l \leq l^{\prime}$. Then for all $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{equation*}
\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=\int_{\mathrm{G}_{s}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) \operatorname{det}(1-g) T_{s}(\tilde{g})\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d g \tag{94}
\end{equation*}
$$

where $T_{s}$ is the operator $T$, see (14), corresponding to the symplectic space $\mathrm{W}_{s}$. In fact, there is a nonzero constant $C$ such that

$$
\begin{equation*}
\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C \int_{\mathfrak{h}} \prod_{j=1}^{l}\left(p_{j}\left(y_{j}\right)+q_{j}\left(-\partial_{y_{j}}\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi}(y) d y \tag{95}
\end{equation*}
$$

where $p_{j}, q_{j}$ are defined as in (69). In particular,

$$
\begin{equation*}
\int_{\mathrm{G} \backslash \mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=C_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g \tag{96}
\end{equation*}
$$

where $C_{0}$ is a nonzero constant which depends only on the pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$.
Remark 7. As in Theorem 4, the term

$$
\prod_{j=1}^{l}\left(p_{j}\left(y_{j}\right)+q_{j}\left(-\partial_{y_{j}}\right) \delta_{0}\left(y_{j}\right)\right)
$$

is a function of $y$ (i.e. all the $q_{j}$ 's are zero) if and only if $l=l^{\prime}$. In the remaining cases, it is a distribution, but not a measure. Furthermore, if $l=l^{\prime}$, we can replace the domain of integration $\mathfrak{h}$ with the smaller domain $\tau\left(\mathfrak{h}_{\overline{1}}\right)$.

Remark 8. The term $\operatorname{det}(1-g)$ appearing in (94) admits a representation theoretical interpretation. Indeed, let $\sigma$ be the spin representation of the spin cover of $\mathrm{G}_{s}^{0}$. Then the tensor product $\sigma \otimes \sigma^{c}$ is a representation of $\mathrm{G}_{s}^{0}$ and, by [Lit06, Ch. XI, III., p. 254]

$$
\begin{equation*}
\Theta_{\sigma \otimes \sigma^{c}}(g)=\left|\Theta_{\sigma}(g)\right|^{2}=\operatorname{det}(1+g) \quad\left(g \in \mathrm{G}_{s}^{0}\right) . \tag{97}
\end{equation*}
$$

So $\operatorname{det}(1-g)=\Theta_{\sigma \otimes \sigma^{c}}(-g)$.

Remark 9. The pair $\left(\mathrm{O}_{1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ was studied in detail in section 2 .
Suppose $l>l^{\prime}$. Theorem 6 reduces the computation of $\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g$ to that of $\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g$, done in Theorem 5. One could still try to compute the integral on $\mathrm{G} \backslash-\mathrm{G}^{0}$ directly using the methods developed in this paper (Weyl's integrations on $\mathfrak{g}$ and W). We will do it for $\mathrm{O}_{2 l+1}$ in Theorem 9 below. Nevertheless, the result is less precise than the one obtained in Theorem 6 since we are only able to obtain that the integral over $G \backslash-G^{0}$ is a nonzero constant multiple of the one over $-G^{0}$, but determining the constant is a serious issue even in the much easier situation of $\left(\mathrm{U}_{l}, \mathrm{U}_{l^{\prime}}\right)$; see [MPP23a]. An additional result of Theorem 6 below is that formula (94), proved for the pair (G, $\mathrm{G}^{\prime}$ ) $=$ $\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $1 \leq l \leq l^{\prime}$, holds for $l>l^{\prime}$ as well. This formula is needed in [MPP23b].

To consider the case $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $l>l^{\prime}$, recall the graded vector space V , (35) and (44),

$$
\mathrm{V}=\mathrm{V}^{0} \oplus \mathrm{~V}^{1} \oplus \cdots \oplus \mathrm{~V}^{l^{\prime}}
$$

where, as in (45),

$$
\begin{aligned}
& \mathrm{V}_{\overline{0}}^{0}=\mathrm{V}_{\overline{0}}^{0,0} \oplus\left(\mathrm{~V}_{\overline{0}}^{0,0}\right)^{\perp} \\
& \mathrm{V}_{\overline{1}}^{0}=0
\end{aligned}
$$

with $\operatorname{dim} \bigvee_{\overline{0}}^{0,0}=1$ and $\operatorname{dim}\left(\bigvee_{\overline{0}}^{0,0}\right)^{\perp}=2\left(l-l^{\prime}\right)$. Let

$$
\mathrm{W}_{s}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}^{1} \oplus \cdots \oplus \mathrm{~V}_{\overline{1}}^{l^{\prime}},\left(\mathrm{V}_{\overline{0}}^{0,0}\right)^{\perp} \oplus \mathrm{V}_{\overline{0}}^{1} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l^{\prime}}\right), \quad \mathrm{W}_{s}^{\perp}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}}^{0,0}\right)
$$

Then

$$
\begin{equation*}
\mathrm{W}=\mathrm{W}_{s} \oplus \mathrm{~W}_{s}^{\perp} \tag{98}
\end{equation*}
$$

is a direct sum orthogonal decomposition. Let $\mathrm{G}_{s} \subseteq \mathrm{G}$ be the subgroup acting trivially on the space $\mathrm{V}_{\overline{0}}^{0,0}$ and let $\mathrm{G}_{s}^{\prime}=\mathrm{G}^{\prime}$. The dual pair corresponding to $\mathrm{W}_{s}$ is $\left(\mathrm{G}_{s}, \mathrm{G}_{s}^{\prime}\right)=$ $\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ and dual pair corresponding to $\mathrm{W}_{s}^{\perp}$ is $\left(\mathrm{O}_{1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$.
Theorem 9. Let $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $l>l^{\prime}$. Then

$$
\begin{equation*}
\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g \neq 0 \tag{99}
\end{equation*}
$$

if and only if the highest weight $\lambda=\sum_{j=1}^{l} \lambda_{j} e_{j}$ of $\Pi$ satisfies condition (a) of Theorem 5 for $\mathbb{D}=\mathbb{R}$. Suppose that this condition is satisfied. Then there is a non-zero constant $C$ which depends only of the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ such that for all $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{align*}
\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g & =\int_{\mathrm{G}_{s}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) \operatorname{det}(1-g) T_{s}(\tilde{g})\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d g  \tag{100}\\
& =C \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}{ }^{\text {reg }}\right)}\left(\prod_{j=1}^{l^{\prime}} p_{j}\left(y_{j}\right)\right) F_{\phi}(y) d y \tag{101}
\end{align*}
$$

Theorems 6 and 8 prove, in particular, that the integrals of $\check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g})$ over $\mathrm{G}^{0}$ and over $\mathrm{G} \backslash \mathrm{G}^{0}$ are nonzero multiples of each other when either $l>l^{\prime}$ or $\mathrm{G}=\mathrm{O}_{2 l+1}$ and $1 \leq l \leq l^{\prime}$. The formula from Theorem 7 does not allow us to compare the integrals of $\Theta_{\Pi}(\widetilde{g}) T(\widetilde{g})$ over $\mathrm{G}^{0}$ and over $\mathrm{G} \backslash \mathrm{G}^{0}$ if $\mathrm{G}=\mathrm{O}_{2 l}$ and $l \leq l^{\prime}$. Still, the fact that the two
integrals are either both zero or both nonzero can be directly inferred from the properties of the distribution $T$ with respect to the twisted convolution, as shown in the following proposition.

Proposition 10. Let $\mathrm{G}=\mathrm{O}_{d}$ and let $\Pi$ be a genuine irreducible representation of $\widetilde{\mathrm{G}}$. Then

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g \neq 0 \quad \text { if and only if } \quad \int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g \neq 0 . \tag{102}
\end{equation*}
$$

Proof. For every $g \in-\mathrm{G}^{0}$ there is $\varepsilon_{g} \in\{(1, \pm 1)\}$ such that $\widetilde{-g}=\varepsilon_{g}(\widetilde{-1}) \widetilde{g}$. Hence

$$
\left.\check{\Theta}_{\Pi}(\widetilde{(-1}) \widetilde{g}\right) T((\widetilde{-1}) \widetilde{g})=\check{\Theta}_{\Pi}\left(\varepsilon_{g}(\widetilde{(-g})\right) T\left(\varepsilon_{g}(\widetilde{-g})\right)=\check{\chi}_{\Pi}\left(\varepsilon_{g}\right) \check{\chi}_{\omega}\left(\varepsilon_{g}\right) \check{\Theta}_{\Pi}(\widetilde{-g}) T(\widetilde{-g})
$$

because $\Pi$ is genuine. Notice that $T((\widetilde{-1}) \widetilde{g})=T((\widetilde{-1})) দ T(\widetilde{g})$, where $\downarrow$ denotes the twisted convolution (see e.g. [How80]). Hence, by replacing $g$ with $-g$ in the integral,

$$
\begin{aligned}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g & =\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}((\widetilde{-1}) \widetilde{g}) T((\widetilde{-1}) \widetilde{g}) d g \\
& =T(\widetilde{-1}) দ\left(\check{\chi}_{\Pi}(\widetilde{c}(0)) \int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g\right) .
\end{aligned}
$$

The result follows because the map $T\left(\widetilde{(-1)} \mathfrak{t}\right.$. is a bijection of $\mathcal{S}^{\prime}(\mathrm{W})$ onto itself. In fact, by $(20)$ and since $\frac{1}{2} \operatorname{dim} \mathrm{~W}=(2 l+1) l^{\prime}$,

$$
(T(\widetilde{-1}) \mathfrak{\square} \cdot)^{2}=T\left((\widetilde{-1})^{2}\right) \mathfrak{দ} \cdot=(-1)^{l^{\prime}} T((1,1)) \mathfrak{দ} \cdot=(-1)^{l^{\prime}} \delta_{0} \mathfrak{দ} \cdot=(-1)^{l^{\prime}} \mathrm{id}_{\mathcal{S}^{\prime}(\mathrm{W})}
$$

where $\operatorname{id}_{\mathcal{S}^{\prime}(\mathrm{W})}$ is the identity operator on $\mathcal{S}^{\prime}(\mathrm{W})$.
Remark 10. Comparing (96) and (102) and using the fact that $(T \widetilde{(-1)}) \underline{\square} \cdot)^{2}=(-1)^{l^{\prime}} \mathrm{id}_{\mathcal{S}^{\prime}(\mathrm{W})}$, we see that the constant $C_{0}$ in (96) satisfies $C_{0}^{2}=(-1)^{l^{\prime}}$.

As a byproduct of our calculations of the intertwining distributions, we obtain the list of highest weights of the genuine irreducible representations $\Pi$ of $\widetilde{G}$ that occur in Howe's correspondence when $l>l^{\prime}$. This list was first determined (without any restrictions on the ranks $l$ and $l^{\prime}$ ) in [KV78]. As it will be seen in the proof of Theorems 5 and 9 , our list is obtained by comparing the support of the function $\prod_{j=1}^{l^{\prime}} p_{j}\left(y_{j}\right)$ and the domain of integration, $\tau^{\prime}\left(\mathfrak{h}_{1}^{\text {reg }}\right)$. Unfortunately, this method is not refined enough to give the result when $l \leq l^{\prime}$.
Corollary 11. Suppose that $l>l^{\prime}$. A genuine representation $\Pi \in \widetilde{\mathrm{G}}^{\wedge}$ occurs in Howe's correspondence if and only if its highest weight satisfies conditions (a) or (b) of Theorem 5.

Proof. Our computations of the intertwining distribution $\int_{\mathrm{G}} \widetilde{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g$ can be applied to any genuine irreducible representation $\Pi \in \widetilde{\mathrm{G}}^{\wedge}$ (not necessarily occurring in Howe's correspondence). This distribution is nonzero if and only if $\left.\omega\right|_{\widetilde{G}}$ has a nonzero $\Pi$ isotypic component. This is equivalent to the fact that there is a unitary highest weight representation $\Pi^{\prime}$ of $\widetilde{\mathrm{G}^{\prime}}$ such that $\Pi \otimes \Pi^{\prime}$ occurs in $\left.\omega\right|_{\widetilde{\mathrm{G}} \widetilde{G}^{\prime}}$. The nonvanishing of the intertwining distributions leads to conditions (a) or (b) of Theorem 5 when $\mathrm{G}=\mathrm{U}_{l}$ or $\mathrm{Sp}_{l}$. In the case of orthogonal groups, we can further use Theorem 6 and conclude
that the nonvanishing of the intertwining distributions is equivalent to the nonvanishing of the integral of $\widetilde{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g})$ over $-\mathrm{G}^{0}$. The claim then follows again from Theorem 5 .

A more refined analysis of the intertwining distribution still allows us to recover the necessary condition for a representation $\Pi \in \mathrm{G}^{\wedge}$ to occur in Howe's correspondence when $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right)$ and $p=\min \{p, q\}<l \leq l^{\prime}=p+q$. We shall prove the following corollary in section 14.
Corollary 12. Suppose that $\mathbb{D}=\mathbb{C}$ and $p=\min \{p, q\}<l \leq l^{\prime}=p+q$. Let $\Pi \in \widetilde{\mathrm{G}}^{\wedge}$ be a genuine irreducible representation of highest weight $\lambda$. If either $\lambda_{q+1}>\frac{q-p}{2}$ or $\lambda_{l-p}<\frac{q-p}{2}$, then $\Pi$ does not occur in Howe's correspondence.

We conclude this section with a result on the non-differential operator nature of the symmetry breaking operators in $\left.\operatorname{Hom}_{\widetilde{\mathrm{G}}{\widetilde{G^{\prime}}}^{\prime}} \mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty} \otimes \mathcal{H}_{\Pi^{\prime}}^{\infty}\right)$.
Corollary 13. Let $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ be a real reductive dual pair with one member compact. Then the essentially unique non-zero symmetry breaking operator in

$$
\operatorname{Hom}_{\widetilde{\mathrm{G}}{\widetilde{G^{\prime}}}\left(\mathcal{H}_{\omega}^{\infty}, \mathcal{H}_{\Pi}^{\infty} \otimes \mathcal{H}_{\Pi^{\prime}}^{\infty}\right)}
$$

is not a differential operator.
Proof. We are going to show that $(\mathrm{Op} \circ \mathcal{K})\left(f_{\Pi \otimes \Pi^{\prime}}\right)$ is not a differential operator.
Let $f \in \mathcal{S}^{\prime}(\mathrm{W})$ and recall the definition of $\mathcal{K}(f)$ in (12). According to [Hör83, Theorems 5.2.1 (the Schwartz kernel theorem) and 5.2.3], the continuous linear map Op $\circ \mathcal{K}(f)$ is a distribution-valued differential operator if and only if $\mathcal{K}(f) \in \mathcal{S}^{\prime}(\mathrm{X} \times \mathrm{X})$ is supported by the diagonal $\Delta=\{(x, x) ; x \in \mathrm{X}\}$. This implies that $f$ is supported in Y. Indeed, given $\varphi \in \mathcal{S}(\mathrm{X} \times \mathrm{X})$, let $\psi \in \mathcal{S}(\mathrm{X} \times \mathrm{X})$ be defined by $\varphi\left(x, x^{\prime}\right)=\psi\left(x-x^{\prime}, x+x^{\prime}\right)$ for all $x, x^{\prime} \in \mathrm{X}$. Furthermore, let $\psi(\cdot, \cdot) \in \mathcal{S}(\mathrm{X} \times \mathrm{Y})$ denote the partial Fourier transform of $\psi$ with respect to its second variable, defined by

$$
\psi(a, \widehat{y})=\int_{\mathrm{X}} \chi\left(\frac{1}{2}\langle y, b\rangle\right) \psi(a, b) d b \quad((a, y) \in \mathrm{X} \times \mathrm{Y})
$$

Then

$$
\operatorname{supp} \varphi \cap \Delta=\emptyset \quad \text { if and only if } \quad \operatorname{supp} \psi(\cdot, \cdot) \cap(\{0\} \times \mathrm{Y})=\emptyset
$$

Since $\mathcal{K}(f)(\varphi)=f(\psi(\cdot, \cdot))$ by (12), we obtain the claim.
Notice that this cannot happen in our case. Indeed, the support of $f_{\Pi \otimes \Pi^{\prime}}$ is $G^{\prime}$ invariant. Since the pair is of type I, GG' acts irreducibly on W. Therefore the inclusion supp $f_{\Pi \otimes \Pi^{\prime}} \subseteq$ Y would imply supp $f_{\Pi \otimes \Pi^{\prime}}=\{0\}$. This would mean that the wavefront set of $\Pi^{\prime}$ is 0 , i.e. $\Pi^{\prime}$ is finite dimensional. By classification, see e.g. [Prz96, Appendix] all highest weight representations occurring in Howe's correspondence are infinite dimensional unless $\mathrm{G}^{\prime}=\mathrm{U}_{l^{\prime}}$, which is compact. In this case, the intertwining distribution is a smooth function; see [MPP23a]. In particular, its support is not 0 . Hence the intertwining operator is not a differential operator.

$$
\text { 6. The pair }\left(\mathrm{O}_{2}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)
$$

We consider here the case $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$. By (E.6) and Proposition E.1, we can identify

$$
\widetilde{\mathrm{O}}_{2}=\left\{(g ; \zeta) \in \mathrm{O}_{2} \times \mathbb{C}^{\times} ; \zeta^{2}=\operatorname{det} g\right\}
$$

and the $\operatorname{det}^{1 / 2}$-covering $\widetilde{\mathrm{O}}_{2} \ni(g ; \zeta) \rightarrow g \in \mathrm{O}_{2}$ does not split. Let $\Pi \in \widetilde{\mathrm{O}}_{2}$ occur in Howe's correspondence and let $\chi_{+}: \widetilde{\mathrm{O}}_{2} \rightarrow \mathbb{C}^{\times}$be the character of $\widetilde{\mathrm{O}}_{2}$ defined by (79).

Since $\Pi$ is genuine, there is $\Pi_{0} \in \widehat{O_{2}}$ such that $\Pi_{0}(g)=\left(\Pi \otimes \chi_{+}^{-1}\right)(\widetilde{g})$. Accordingly,

$$
\int_{\mathrm{O}_{2}} \check{\Theta}_{\Pi}(\widetilde{g}) \omega(\widetilde{g}) d g=\int_{\mathrm{O}_{2}} \check{\Theta}_{\Pi_{0}}(g) \omega_{0}(g) d g
$$

where $\omega_{0}$ is as in (81).
Observe that the image under the metaplectic cover of $\operatorname{supp}\left(\Theta_{\Pi}\right)$ is equal to $\operatorname{supp}\left(\Theta_{\Pi_{0}}\right)$. Since ${\widetilde{\mathrm{SO}_{2}}}_{2} \rightarrow \mathrm{SO}_{2}$ splits by (E.10), we conclude that $\Theta_{\Pi}$ is supported on $\widetilde{\mathrm{G}^{0}}=\widetilde{\mathrm{SO}_{2}}$ if and only if $\Theta_{\Pi_{0}}$ is supported on $\mathrm{SO}_{2}$.

Proposition 14. Let $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ and let $\Pi$ be a genuine irreducible representation of $\widetilde{\mathrm{G}}$ with character $\Theta_{\Pi}$ not supported on $\widetilde{\mathrm{G}^{0}}$. Then either $\Pi=\chi_{+}^{-1}$, or $\Pi=\widetilde{\operatorname{det}}$ is the character of $\widetilde{\mathrm{G}}$ such that $\left(\widetilde{\operatorname{det}} \otimes \chi_{+}^{-1}\right)(\tilde{g})=\operatorname{det}(g)$ for all $\tilde{g} \in \widetilde{\mathrm{G}}$.

If $\Pi=\chi_{+}^{-1}$, then

$$
\begin{equation*}
\int_{\mathrm{O}_{2}} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g}) d g=2 \int_{\mathrm{SO}_{2}} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g}) d g \tag{103}
\end{equation*}
$$

which is computed by Theorem 4; see also subsection 6.1 when $l^{\prime}=1$.
Suppose $\Pi=\widetilde{\operatorname{det}}$. Then $\widetilde{\operatorname{det}}$ does not occur in Howe correspondence if $l^{\prime}=1$ and hence

$$
\int_{\mathrm{O}_{2}} \Theta_{\widetilde{\mathrm{det}}}(\tilde{g}) T(\tilde{g}) d g=0
$$

for $\left(\mathrm{O}_{2}, \mathrm{Sp}_{2}(\mathbb{R})\right)$. Let $l^{\prime}>1$. Decompose $\mathrm{W}=\mathrm{M}_{2,2 l^{\prime}}(\mathbb{R})$ as $\mathrm{W}=\mathrm{W}_{1} \oplus \mathrm{~W}_{2}$, where $\mathrm{W}_{1}$ is subspace of the $w \in \mathrm{~W}$ for which all coefficients of the second row are 0 and $\mathrm{W}_{2}$ is subsapce of the $w \in \mathrm{~W}$ for which all coefficients of the first row are 0 . Then

$$
\begin{equation*}
\int_{\mathrm{O}_{2}} \Theta_{\widetilde{\mathrm{det}}}(\widetilde{g}) T(\widetilde{g})(\phi) d g=\int_{\mathrm{SO}_{2}} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g})(\phi) d g-\mu_{\mathcal{O}}(\phi) \quad(\phi \in \mathcal{S}(\mathrm{W})) \tag{104}
\end{equation*}
$$

where $\mathcal{O}$ is the $\mathrm{O}_{2} \times \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})$-orbit of $n_{0}=\left(\begin{array}{cccc}0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0\end{array}\right) \in \mathrm{W}$ and $\mu_{\mathcal{O}} \in \mathcal{S}^{\prime}(\mathrm{W})$ is the invariant measure on $\mathcal{O}$ defined by

$$
\begin{equation*}
\mu_{\mathcal{O}}(\phi)=2^{\operatorname{dim} \mathrm{W}_{2} / 2} \int_{\mathrm{W}_{2}} \int_{\mathrm{O}_{2}} \phi(g w) d g d \mu_{\mathrm{W}_{2}}(w) \quad(\phi \in \mathcal{S}(\mathrm{W})) \tag{105}
\end{equation*}
$$

The integral over $\mathrm{SO}_{2}$ is computed by Theorem 4.
Proof. For $n \in \mathbb{Z}$, let $\rho_{n}$ be the character of $\mathrm{SO}_{2}$ defined by

$$
\rho_{n}\left(\left(\begin{array}{ll}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{array}\right)\right)=e^{i n \theta} .
$$

Up to equivalence, the irreducible representations of $\mathrm{O}_{2}$ are of the form $\Pi_{0, n}=\operatorname{Ind}_{\mathrm{SO}_{2}}^{\mathrm{O}_{2}}\left(\rho_{n}\right)$ with $n>0$, together with the trivial representation triv and det. (Moreover, $\Pi_{0, n} \simeq \Pi_{0,-n}$ and $\Pi_{0,0}=1 \oplus$ det.) Hence $\Theta_{\Pi_{0}}$ does not have support contained in $\widetilde{\mathrm{SO}_{2}}$ if and only if $\left.\Pi_{0}\right|_{\mathrm{SO}_{2}}=1$. Hence the only possible cases are triv and det.

Suppose first that $\Pi_{0}=$ triv, i.e. $\Pi=\chi_{+} . \quad$ Since $\left.\left(\Theta_{\text {triv }}+\Theta_{\text {det }}\right)\right|_{\mathrm{O}_{2} \backslash \mathrm{SO}_{2}}=0$ and $\left.\Theta_{\text {triv }}\right|_{\mathrm{SO}_{2}}=\left.\Theta_{\text {det }}\right|_{\mathrm{SO}_{2}}=1$, we conclude that

$$
\begin{aligned}
\int_{\mathrm{O}_{2}} \Theta_{\text {triv }}(g) \omega_{0}(g) d g & =\int_{\mathrm{O}_{2}}\left(\Theta_{\text {triv }}+\Theta_{\text {det }}\right)(g) \omega_{0}(g) d g=\int_{\mathrm{SO}_{2}}\left(\Theta_{\text {triv }}+\Theta_{\text {det }}\right)(g) \omega_{0}(g) d g \\
& =2 \int_{\mathrm{SO}_{2}} \Theta_{\text {triv }}(g) \omega_{0}(g) d g=2 \int_{\mathrm{SO}_{2}} \chi_{+}^{-1}(\widetilde{g}) \omega(\widetilde{g}) d g
\end{aligned}
$$

Thus

$$
\int_{\mathrm{O}_{2}} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g}) d g=2 \int_{\mathrm{SO}_{2}} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g}) d g
$$

which proves (103).
Let now $\Pi_{0}=$ det. If $l^{\prime}=1$, then det does not occur in $\omega_{0} \mid{ }_{O_{2}}$; see Proposition F.1. So,

$$
\int_{\mathrm{O}_{2}} \Theta_{\operatorname{det}}(g) \omega_{0}(g) d g=0
$$

for $\left(\mathrm{O}_{2}, \mathrm{Sp}_{2}(\mathbb{R})\right)$.
Suppose then $l^{\prime}>1$. Then

$$
\check{\Theta}_{\widetilde{\operatorname{det}}}(\tilde{g}) T(\tilde{g})=\check{\Theta}_{\widetilde{\operatorname{det}}}(\tilde{g}) \chi_{+}(\tilde{g}) \chi_{+}^{-1}(\tilde{g}) T(\tilde{g})=\check{\Theta}_{\widetilde{\operatorname{det}} \otimes \chi_{+}^{-1}}(\tilde{g}) \chi_{+}^{-1}(\tilde{g}) T(\tilde{g})=\operatorname{det}(g)^{-1} \chi_{+}^{-1}(\tilde{g}) T(\tilde{g}) .
$$

Hence

$$
\int_{\mathrm{O}_{2}} \check{\Theta}_{\widetilde{\operatorname{det}}}(\tilde{g}) T(\tilde{g}) d g=\int_{\mathrm{SO}_{2}} \chi_{+}^{-1}(\tilde{g}) T(\tilde{g}) d g-\int_{\left(\mathrm{SO}_{2}\right) s} \chi_{+}^{-1}(\tilde{g}) T(\tilde{g}) d g
$$

We now compute the integral over $\left(\mathrm{SO}_{2}\right) s$. Let $g_{t}=\left(\begin{array}{cc}\cos (t) & \sin (t) \\ -\sin (t) & \cos (t)\end{array}\right) \in \mathrm{SO}_{2}$. Then $g_{t} s g_{-t}=g_{2 t} s$. If $f$ is any function on $\mathrm{SO}_{2}$, then

$$
\begin{aligned}
\int_{\left(\mathrm{SO}_{2}\right) s} f\left(g_{t}\right) d g_{t} & =\int_{\mathrm{SO}_{2}} f\left(g_{t} s\right) d g_{t}=\int_{\mathrm{SO}_{2}} f\left(g_{t / 2} s g_{-t / 2}\right) d g_{t} \\
& =2 \int_{\mathrm{SO}_{2}} f\left(g_{t} s g_{-t}\right) d g_{t}=2 \int_{\mathrm{SO}_{2}} f\left(g_{-t} s g_{t}\right) d g_{t}
\end{aligned}
$$

Applying this to $\mathrm{SO}_{2} \ni g \rightarrow \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g}) \in \mathcal{S}^{\prime}(\mathbb{R})$, we get

$$
\begin{equation*}
\int_{\left(\mathrm{SO}_{2}\right) s} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g}) d g=2 \int_{\mathrm{SO}_{2}} \chi_{+}^{-1}\left(\widetilde{\left.g^{-1} s g\right)} T\left(\widetilde{g^{-1} s g}\right) d g_{t}\right. \tag{106}
\end{equation*}
$$

Decompose $\mathrm{W}=\mathrm{M}_{2,2 l^{\prime}}(\mathbb{R})$ as in the statement of the theorem and let $g \in \mathrm{O}_{2}$. Then $\mathrm{W}=g^{-1} \mathrm{~W}_{1} \oplus g^{-1} \mathrm{~W}_{2}$ is an orthogonal decomposition such that $g^{-1} s g$ preserves both $g^{-1} \mathrm{~W}_{1}$ and $g^{-1} \mathrm{~W}_{2}$. Notice that

$$
\begin{aligned}
& \left.g^{-1} s g\right|_{g^{-1} W v_{1}}=1_{g^{-1} \mathrm{~W}_{1}} \quad \text { because }\left.s\right|_{\mathrm{W}_{1}}=1, \\
& \left.g^{-1} s g\right|_{g^{-1} W v_{2}}=-1_{g^{-1} \mathrm{~W}_{2}} \quad \text { because }\left.s\right|_{\mathrm{W}_{2}}=-1 .
\end{aligned}
$$

By Lemma G.1,

$$
\begin{equation*}
\chi_{+}^{-1}\left(\widetilde{g^{-1} s g}\right) T_{\mathrm{W}}\left(\widetilde{g^{-1} s g}\right)=\chi_{+}^{-1}\left(\widetilde{1_{g^{-1} \mathrm{~W}_{1}}}\right) T_{\mathrm{W}}\left(\widetilde{\left(\widetilde{1_{g^{-1} \mathrm{~W}_{1}}}\right) \otimes \chi_{+}^{-1}\left(\widetilde { ( - \widetilde { 1 _ { g ^ { - 1 } \mathrm { W } _ { 2 } } } ) } T _ { \mathrm { W } } \left(\widetilde{\left(-{1_{g^{-1} \mathrm{~W}_{2}}}\right)}, ., ~ . ~\right.\right.}\right. \tag{107}
\end{equation*}
$$

independently of the choices of the preimages of $g^{-1} s g, 1_{g^{-1} \mathrm{~W}_{1}}$ and $-1_{g^{-1} \mathrm{~W}_{2}}$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$, $\widetilde{\mathrm{Sp}}\left(g^{-1} \mathrm{~W}_{1}\right)$ and $\widetilde{\mathrm{Sp}}\left(g^{-1} \mathrm{~W}_{2}\right)$. We can therefore fix $\widetilde{g_{g^{-1} \mathrm{~W}_{1}}}$ to be the identity element of $\widetilde{\mathrm{Sp}}\left(g^{-1} \mathrm{~W}_{1}\right)$, which gives $\chi_{+}^{-1}\left(\widetilde{1_{g^{-1} \mathrm{~W}_{1}}}\right)=1$. Hence

$$
\chi_{+}^{-1}\left(\widetilde{1_{g^{-1} \mathrm{~W}_{1}}}\right) T_{\mathrm{W}}\left(\widetilde{1_{g^{-1} \mathrm{~W}_{1}}}\right)=\delta_{0, g^{-1} \mathrm{~W}_{1}},
$$

where $\delta_{0, g^{-1} \mathrm{~W}_{1}}$ indicates Dirac's delta at 0 in the space $g^{-1} \mathrm{~W}_{1}$.
By [AP14, Definition 4.16 and Remark 4.5], $\Theta_{\mathrm{W}}^{2}(-1)=(-2 i)^{\operatorname{dim} \mathrm{W}}$. Hence $\left|\Theta_{\mathrm{W}}(\widetilde{-1})\right|=$ $2^{\operatorname{dim} \mathrm{W} / 2}$ only depends on the dimension of W . In particular,

$$
\left|\Theta_{g^{-1} \mathrm{~W}_{2}}(\widetilde{(-1})\right|=\left|\Theta_{\mathrm{W}_{2}}(\widetilde{(-1})\right|=2^{\operatorname{dim} \mathrm{W}_{2} / 2}
$$

So

Thus (107) becomes

$$
\begin{equation*}
\chi_{+}^{-1}\left(\widetilde{g^{-1} s g}\right) T_{\mathrm{W}}\left(\widetilde{g^{-1} s g}\right)=2^{\operatorname{dim} \mathrm{W}_{2} / 2} \delta_{0, g^{-1} \mathrm{~W}_{1}} \otimes \mu_{g^{-1} \mathrm{~W}_{2}} \tag{108}
\end{equation*}
$$

By (106), for all $\phi \in \mathcal{S}(\mathrm{W})$,

$$
\begin{aligned}
\int_{\left(\mathrm{SO}_{2}\right) s} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g})(\phi) d g & =2^{1+\operatorname{dim} \mathrm{W}_{2} / 2} \int_{\mathrm{SO}_{2}}\left(\delta_{0, g^{-1} \mathrm{~W}_{1}} \otimes \mu_{g^{-1} \mathrm{~W}_{2}}\right)(\phi) d g \\
& =2^{1+\operatorname{dim} \mathrm{W}_{2} / 2} \int_{\mathrm{SO}_{2}} \int_{g^{-1} \mathrm{~W}_{2}} \phi(w) d \mu_{g^{-1} \mathrm{~W}_{2}}(w) d g \\
& =2^{1+\operatorname{dim} \mathrm{W}_{2} / 2} \int_{\mathrm{W}_{2}} \int_{\mathrm{SO}_{2}} \phi(g w) d g d \mu_{\mathrm{W}_{2}}(w)
\end{aligned}
$$

Notice that, since $s w=-w$ for $w \in \mathrm{~W}_{2}$,

$$
\begin{aligned}
\int_{\mathrm{W}_{2}} \int_{\mathrm{SO}_{2}} \phi(g w) d g d \mu_{\mathrm{W}_{2}}(w) & =\int_{\mathrm{W}_{2}} \int_{\mathrm{SO}_{2}} \phi(-g w) d g d \mu_{\mathrm{W}_{2}}(w) \\
& =\int_{\mathrm{W}_{2}} \int_{\mathrm{SO}_{2}} \phi(g s w) d g d \mu_{\mathrm{W}_{2}}(w) \\
& =\int_{\mathrm{W}_{2}} \int_{\left(\mathrm{SO}_{2}\right) s} \phi(g w) d g d \mu_{\mathrm{W}_{2}}(w) .
\end{aligned}
$$

In conclusion,

$$
\int_{\left(\mathrm{SO}_{2}\right) s} \chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g})(\phi) d g=\mu_{\mathcal{O}}(\phi) \quad(\phi \in \mathcal{S}(\mathrm{W}))
$$

where $\mu_{\mathcal{O}}$ is as in (105).
It remains to show that $\mu_{\mathcal{O}}$ is a $\mathrm{O}_{2} \times \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})$-invariant measure on the orbit $\mathcal{O}$. Notice first that $\mathrm{W}_{2} \backslash\{0\}=\mathrm{Sp}_{2 l^{\prime}}(\mathbb{R}) . n_{0}$. Indeed, $n_{0} \in \mathrm{~W}_{2}$ and $\mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})$ preserves $\mathrm{W}_{2}$. Conversely, let $w_{2}=\left(\begin{array}{cc}0 & 0 \\ u & v\end{array}\right) \in \mathrm{W}_{2} \backslash\{0\}$, where $u, v \in \mathrm{M}_{l^{\prime}, 1}(\mathbb{R})$. Since $J=\left(\begin{array}{cc}0 & I_{l^{\prime}} \\ I_{l^{\prime}} & 0\end{array}\right) \in \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})$ and $w_{2} J=\left(\begin{array}{cc}0 & 0 \\ -v & u\end{array}\right)$, we can suppose that $u \neq 0$. If $a \in \mathrm{GL}_{l^{\prime}}(\mathbb{R})$ has $u$ as its first row and $b$ is a symmetric matrix having $v$ as its first row, then $\left(\begin{array}{cc}a & b \\ 0 & \left(a^{t}\right)^{-1}\end{array}\right) \in \operatorname{Sp}_{2 l^{\prime}}(\mathbb{R})$ and
$n_{0}\left(\begin{array}{cc}a & b \\ 0 & \left(a^{t}\right)^{-1}\end{array}\right)=w_{2}$. It follows from this that $\left\{g w_{2} ; g \in \mathrm{O}_{2}, w_{2} \in \mathrm{~W}_{2}\right\}=\mathcal{O} \cup\{0\}$. The right-hand side of (105) is clearly $\mathrm{O}_{2}$-invariant, and we see that it is $\mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})$-invariant by linear changes of variables in the integral over $\mathrm{W}_{2}$ because the elements of $\mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})$ have determinant 1 .
6.1. The case $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2}, \mathrm{Sp}_{2}(\mathbb{R})=\mathrm{SL}_{2}(\mathbb{R})\right)$. In this case, $\mathrm{H}=\mathrm{SO}_{2}$ and $\mathfrak{g}=\mathfrak{h}=\mathbb{R} J_{1}$, where $J_{1}=\mathbb{R}\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$. Moreover, $\tau\left(\mathfrak{h}_{\overline{1}}\right)=\mathbb{R}^{+} J_{1}$ and $\mathfrak{h} \cap \tau(\mathrm{W})=\mathfrak{h}$. The Harish-Chandra parameter of $\Pi \in \widetilde{\mathrm{O}}_{2}^{\wedge}$ (which coincides with its highest weight since $\rho=0$ ) is of the form $\mu e_{1}$, where $\mu \geq 0$ is an integer. Hence, in the notation of Theorem 4, $a=-b=-\mu$ and $\beta=2 \pi$.

If $\mu=0$, then $P_{-\mu, \mu}=0$. If $\mu>0$, then the function $P_{-\mu, \mu}$ is supported in $[0,+\infty)$ and, by (D.4) and Remark 13,

$$
\begin{equation*}
P_{-\mu, \mu, 2}\left(2 \pi y_{1}\right)=2(-1)^{\mu-1} L_{\mu-1}^{1}\left(4 \pi y_{1}\right)=2(-1)^{\mu-1} \sum_{h=0}^{\mu-1}\binom{\mu}{\mu-1-h} \frac{\left(-4 \pi y_{1}\right)^{h}}{h!} \tag{109}
\end{equation*}
$$

where $L_{\mu-1}^{1}$ is a Laguerre polynomial. Moreover, by (D.5), $Q_{-\mu, \mu}(y)=2 \pi(-1)^{\mu}$ for all $\mu \geq 0$.

Suppose first $\mu>0$. Then, by Lemma 3, Theorem 4 and Remark ??, for every $\phi \in$ $\mathcal{S}(\mathrm{W})$,

$$
\begin{align*}
f_{\Pi \otimes \Pi^{\prime}}(\phi) & =\int_{\mathrm{SO}_{2}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g})(\phi) d g \\
& =2 \pi C(-1)^{\mu} \int_{0}^{+\infty} P_{-\mu, \mu, 2}\left(2 \pi y_{1}\right) e^{-2 \pi y_{1}} F_{\phi}\left(y_{1} J_{1}\right) d y_{1}+C \int_{\mathfrak{h}} \delta_{0}(y) F_{\phi}(y) d y \tag{110}
\end{align*}
$$

where $C$ is the constant appearing in Theorem 4. To make formula (110) explicit, we need to calculate terms involving $F(y)$, the Harish-Chandra regular almost-elliptic orbital integral on W.

By [MPP20, Definition 3.1, (39) and (27)] and (I.2) with $\mathrm{Z}^{\prime}=\mathrm{H}^{\prime}$, there are constants $C_{\mathfrak{h}_{\overline{1}}}$ and $C_{\mathfrak{h}_{\overline{1}}}^{\prime}$ such that, for all $y=y_{1} J_{1}=\tau(w) \in \tau\left(\mathfrak{h}_{\overline{1}}\right)$,

$$
\begin{equation*}
F_{\phi}(y)=C_{\mathfrak{h}_{\bar{T}}} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(y^{\prime}\right) \int_{\mathrm{S} / \mathrm{S}^{\mathfrak{b}} \overline{\mathrm{T}}} \phi(s . w) d\left(s S_{\overline{\mathrm{h}}}^{\mathfrak{h}}\right)=C_{\mathfrak{h}_{\bar{T}}}^{\prime} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(y^{\prime}\right) \int_{\mathrm{G}^{\prime} / \mathrm{H}^{\prime}} \psi\left(g^{\prime} \cdot y^{\prime}\right) d\left(g^{\prime} \mathrm{H}^{\prime}\right) \tag{111}
\end{equation*}
$$

where $y^{\prime}=y_{1} J_{1}^{\prime}=y_{1}\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)=\tau^{\prime}(w)$, and $\psi=\tau_{*}^{\prime}\left(\phi^{\mathrm{G}}\right) \in \mathcal{S}\left(\mathfrak{g}^{\prime}\right)$. The right-hand side of (111) is Harish-Chandra's orbital integral for the orbit $\mathrm{G}^{\prime} . y^{\prime}$.

Notice that, for $\mathrm{G}=\mathrm{O}_{2}$ and $l=1 \leq l^{\prime}$, the extension of $F(y)$ from $y \in \mathfrak{h}^{+}=\tau\left(\mathfrak{h}_{\overline{1}}\right)$ to $-\tau\left(\mathfrak{h}_{\overline{1}}\right)$ is even in $y$; see [MPP20, Theorem 3.6]. Hence,

$$
\int_{\mathfrak{h}} \delta_{0}(y) F_{\phi}(y) d y=\lim _{y_{1} \rightarrow 0+} F_{\phi}\left(y_{1} J_{1}\right) \quad(\phi \in \mathcal{S}(\mathrm{W})) .
$$

Write $x \in \mathfrak{g}^{\prime}$ as

$$
x=x_{1}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)+x_{2}\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)+x_{3} J_{1}^{\prime}=\left(\begin{array}{cc}
x_{1} & x_{2}+x_{3} \\
x_{2}-x_{3} & -x_{1}
\end{array}\right)=A\left(x_{1}, x_{2}, x_{3}\right),
$$

where $\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$. Then the map $A: \mathbb{R}^{3} \rightarrow \mathfrak{g}^{\prime}$ is a linear isomorphism. It transfers the adjoint action of $\mathrm{G}^{\prime}$ on $\mathfrak{g}^{\prime}$ to the natural action on $\mathbb{R}^{3}$ by $\operatorname{SO}(2,1)^{0}$, the identity component of $\mathrm{SO}(2,1)$, i.e. the group of isometries of $x_{1}^{2}+x_{2}^{2}-x_{3}^{2}=-\operatorname{det}\left(A\left(x_{1}, x_{2}, x_{3}\right)\right)$ preserving the positive light cone

$$
X^{0+}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3} ; x_{1}^{2}+x_{2}^{2}=x_{3}^{2}, x_{3}>0\right\}
$$

See [HT92, Chapter IV, §5.1]. Under the map $A$, the orbit $\mathrm{G}^{\prime} . y^{\prime}$ with $y^{\prime}=y_{1} J_{1}^{\prime}$ and $y_{1}>0$ is the image of the hyperboloid's upper sheet

$$
O_{y_{1}}^{-}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3} ; x_{1}^{2}+x_{2}^{2}-x_{3}^{2}=-y_{1}^{2}, x_{3}>0\right\} .
$$

Under $A$, the positive light cone $X^{0+}$ corresponds to the $\mathrm{G}^{\prime}$-orbit of $x_{0}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$. Moreover $\mathrm{G}^{\prime} \cdot x_{0} \simeq \mathrm{G}^{\prime} / \mathrm{MN}$, where $\mathrm{M}=\{ \pm 1\}$ and $\mathrm{N}=\exp \left(\mathbb{R} x_{0}\right)=\left\{\left(\begin{array}{cc}1 & t \\ 0 & 1\end{array}\right) ; t \in \mathbb{R}\right\}$. As the geometry suggests, for suitable normalizations of the $\mathrm{SO}(2,1)^{0}$-invariant orbital measures,

$$
\lim _{y_{1} \rightarrow 0^{+}} \int_{O_{y_{1}}^{-}} f d \mu_{O_{\overline{y_{1}}}}=\int_{X^{0+}} f d \mu_{X^{0+}} \quad\left(f \in \mathcal{S}\left(\mathbb{R}^{3}\right)\right)
$$

Thus, for a suitable positive constant $C_{\mathfrak{h}_{\bar{T}}}^{\prime \prime}$

$$
\begin{equation*}
\int_{\mathfrak{h}} \delta_{0}(y) F_{\phi}(y) d y=C_{\mathfrak{h}_{\bar{T}}}^{\prime \prime} \int_{\mathrm{G}^{\prime} / \mathrm{MN}} \psi\left(g^{\prime} \cdot x_{0}\right) d\left(g^{\prime} \mathrm{MN}\right) \quad\left(\phi \in \mathcal{S}(\mathrm{W}), \psi \in \mathcal{S}\left(\mathfrak{g}^{\prime}\right)^{\mathrm{G}}, \psi \circ \tau^{\prime}=\phi^{\mathrm{G}}\right) \tag{112}
\end{equation*}
$$

Suppose now that $\mu=0$. By Remark ??, $\Pi=\nu^{-1}$ where $\nu((g, \xi))=\operatorname{det}(g)^{1 / 2}$ for all $\widetilde{g}=(g, \xi) \in \widetilde{\mathrm{O}}_{2}$. Moreover, for all $\phi \in \mathcal{S}(\mathrm{W})$,

$$
f_{\nu^{-1} \otimes\left(\nu^{-1}\right)^{\prime}}(\phi)=2 \int_{\mathrm{SO}_{2}} \nu(\widetilde{g}) T(\widetilde{g})(\phi) d g=2 \int_{\mathrm{SO}_{2}} \nu(\widetilde{g}) T(\widetilde{g})(\phi) d g=2 C \int_{\mathfrak{h}} \delta_{0}(y) F_{\phi}(y) d y
$$

where $C$ is the constant appearing in Theorem 4 and the last integral is computed in (112).
7. Another example: $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{l}, \mathrm{U}_{p, p}\right)$ and $\Pi=\widetilde{\text { triv }}$

Let $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{l}, \mathrm{U}_{p, p}\right)$. Hence $l^{\prime}=2 p$. Consider the trivial representation triv of $\mathrm{U}_{l}$. In the Schrödinger model, with a polarization $\mathrm{W}=\mathrm{X} \oplus \mathrm{Y}$ preserved by G , we have

$$
\begin{equation*}
\omega(\widetilde{g}) v(x)=\chi_{+}(\widetilde{g}) v\left(g^{-1} x\right) \quad(\widetilde{g} \in \widetilde{\mathrm{G}}, v \in \mathcal{S}(\mathrm{X}), x \in \mathrm{X}) \tag{113}
\end{equation*}
$$

where $\chi_{+}: \widetilde{\mathrm{Sp}}(\mathrm{W}) \rightarrow \mathrm{U}_{1}$ is a function whose restriction to $\widetilde{\mathrm{G}}$ is a character. See [AP14, Proposition 4.28]. Let $\widetilde{\text { triv }}$ denote this restriction. Then $\widetilde{\text { triv }}$ is the lift to $\widetilde{\mathrm{U}}_{l}$ of triv that occurs in Howe's correspondence. Moreover, (113) implies that

$$
\omega\left(\check{\Theta}_{\text {triv }}\right) v(x)=\int_{\mathrm{G}} v\left(g^{-1} x\right) d g \quad(v \in \mathcal{S}(\mathrm{X}), x \in \mathrm{X}) .
$$

Let $\widetilde{\text { triv }}^{\prime}$ be the representation of $\widetilde{\mathrm{U}}_{p, p}$ which corresponds to $\widetilde{\operatorname{triv}}$. If $l=1$, then $\widetilde{\operatorname{triv}}^{\prime}$ is a minimal representation of $\mathrm{U}_{p, p}$, called the Wallach representation.

In this section we are computing $f_{\widetilde{\text { triv }} \otimes \widetilde{\text { rriv }}}$, which is the Weyl symbol of the operator $\omega\left(\check{\Theta}_{\text {triv }}\right)$. As in our main theorems, we distinguish the cases $l \leq l^{\prime}$ and $l>l^{\prime}$. Notice first that the parameters appearing in (67) are

$$
\beta=2 \pi \quad \text { and } \quad \delta=p+\frac{1-l}{2}=\frac{1+l^{\prime}-l}{2} .
$$

Moreover, $\rho=\sum_{j=1}^{l}\left(\frac{l+1}{2}-j\right) e_{j}$ for $\mathrm{G}=\mathrm{U}_{l}$.
7.0.1. The case $l \leq l^{\prime}$. The parameters (68) corresponding to $\Pi=\widetilde{\text { triv }}$ are

$$
\begin{equation*}
a_{j}=-\frac{l^{\prime}}{2}+j \quad \text { and } \quad b_{j}=-\frac{l^{\prime}}{2}+l+1-j \tag{114}
\end{equation*}
$$

where $1 \leq j \leq l$. Observe that the $a_{j}$ 's and the $b_{j}$ 's describe the same set

$$
\left\{-l^{\prime} / 2+1, \ldots,-l^{\prime} / 2+l-1,-l^{\prime} / 2+l\right\}
$$

and $b_{l+1-j}=a_{j}$ for all $1 \leq j \leq l$. Hence, by (D.6),

$$
\begin{align*}
P_{a_{l+1-j}, b_{l+1-j}}(\xi) & =P_{b_{j}, a_{j}}(\xi)=P_{a_{j}, b_{j}}(-\xi),  \tag{115}\\
Q_{a_{l+1-j}, b_{l+1-j}}(\xi) & =Q_{b_{j}, a_{j}}(\xi)=Q_{a_{j}, b_{j}}(-\xi) . \tag{116}
\end{align*}
$$

Since $a_{j}=b_{l+1-j} \leq 0$ for all $1 \leq j \leq \min \left\{l, l^{\prime} / 2\right\}$, by (D.3),

$$
\begin{equation*}
P_{a_{j}, b_{j},-2}(\xi)=P_{a_{l+1-j}, b_{l+1-j}, 2}(\xi)=0 \quad\left(1 \leq j \leq \min \left\{l, l^{\prime} / 2\right\}\right) \tag{117}
\end{equation*}
$$

In particular, $a_{j} \leq 0$ for all $j$ (and hence $b_{j} \leq 0$ for all $j$ ) if and only if $l \leq l^{\prime} / 2$. Furthermore, $a_{j}+b_{j}=l-l^{\prime}+1$, which is independent of $j$, is $\geq 1$ if and only if $l=l^{\prime}=2 p$. As a consequence,

$$
\begin{array}{ll}
P_{a_{j}, b_{j}}=0 \text { for all } 1 \leq j \leq l & \text { if and only if } \quad l \leq \frac{l^{\prime}}{2} \\
Q_{a_{j}, b_{j}} \neq 0 \text { for all } 1 \leq j \leq l & \text { if } l<l^{\prime} \\
Q_{a_{j}, b_{j}}=0 \text { for all } 1 \leq j \leq l & \text { if } l=l^{\prime}=2 p,
\end{array}
$$

We now examine more precisely the formula for $f_{\text {trive } \otimes \text { triv }}$, when $l \leq l^{\prime} / 2$. This is the stable range case. As remarked above, $P_{a_{j}, b_{j}}=0$ for all $1 \leq j \leq l$, whereas

$$
Q_{a_{j}, b_{j}}\left(y_{j}\right)=2 \pi\left(1+y_{j}\right)^{-a_{j}}\left(1-y_{j}\right)^{-b_{j}} .
$$

Hence $p_{j}=0$ for all $1 \leq j \leq l$, whereas

$$
q_{j}\left(-\partial_{y_{j}}\right)^{*}=q_{j}\left(\partial_{y_{j}}\right)=\left(1+\frac{1}{2 \pi} \partial_{y_{j}}\right)^{\frac{l^{\prime}}{2}-j}\left(1-\frac{1}{2 \pi} \partial_{y_{j}}\right)^{\frac{l^{\prime}}{2}-(l-j+1)}
$$

where * denotes the formal adjoint. Theorem 4 yields for $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{align*}
f_{\stackrel{\text { triv } \otimes \operatorname{triv}}{ }}(\phi) & =\int_{\mathrm{U}_{l}} \check{\Theta}_{\not{\text { triv }}}(\widetilde{g}) T(\widetilde{g})(\phi) d g \\
& =C \int_{\mathfrak{h}}\left[\prod_{l=1}^{l} q_{j}\left(-\partial_{y_{j}}\right) \delta_{0}\left(y_{j}\right)\right] F_{\phi}(y) d y \\
& =C\left[\left(\prod_{l=1}^{l} q_{j}\left(\partial_{y_{j}}\right)\right) F_{\phi}\right](0) \tag{118}
\end{align*}
$$

where $C$ is a nonzero constant. Hence $f_{\text {triv } \otimes \text { triv }}$, has support inside the nilpotent cone in W.

Another case where the formula for $f_{\not{\text { triv }} \otimes \text { triv }}{ }^{\prime}$ simplifies is when $l=l^{\prime}=2 p$ because $Q_{a_{j}, b_{j}}=0$ for all $j$. Since $a_{j}=b_{2 p+1-j} \leq 0$ for $1 \leq j \leq p$, we have

$$
P_{a_{j}, b_{j}}(\xi)= \begin{cases}2 \pi P_{a_{j}, b_{j}, 2}(\xi) \mathbb{I}_{\mathbb{R}^{+}}(\xi) & \text { if } 1 \leq j \leq p \\ 2 \pi P_{a_{j}, b_{j},-2}(\xi) \mathbb{I}_{\mathbb{R}^{-}}(\xi) & \text { if } p+1 \leq j \leq 2 p\end{cases}
$$

In particular, in this case, we can replace in (71) the domain of integration $\mathfrak{h} \cap \tau(\mathrm{W})$ with $\tau\left(\mathfrak{h}_{\overline{1}}\right)$, where $\mathfrak{h}_{\overline{1}}$ is the unique Cartan subspace of W and $\tau\left(\mathfrak{h}_{\overline{1}}\right)$ determined by the condition that the first $p$ values $\delta_{j}$ in (37) are equal to 1 and the last $p$ are equal to -1 . The explicit expression for $f_{\underset{\text { triv } \otimes \text { triv }}{ }}$ can be easily computed using (71), (D.1) and (D.2). For instance, if $p=1$, i.e. $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{2}, \mathrm{U}_{1,1}\right)$, then

$$
f_{\text {triv } \otimes \text { triv }}(\phi)=C \int_{0}^{\infty} \int_{-\infty}^{0} e^{2 \pi\left(y_{2}-y_{1}\right)} F_{\phi}\left(y_{1}, y_{2}\right) d y_{1} d y_{2} \quad(\phi \in \mathcal{S}(\mathrm{~W}))
$$

where $C$ is a nonzero constant.
7.0.2. The case $l>l^{\prime}$. In this case, $Q_{a_{j}, b_{j}}=0$ for all $j$ and, according to Theorem 5, the distribution $f_{\overparen{\text { triv }} \otimes \text { triv }}$, is the locally integrable $\mathrm{U}_{l} \times \mathrm{U}_{p, p}$-invariant function on W whose restriction to $\mathfrak{h}_{\overline{1}}{ }^{\text {reg }}$ is equal to the function given by (77). The Weyl group $W\left(\mathrm{U}_{p, p}, \mathfrak{h}^{\prime}\right)$ acts on $\mathfrak{h}^{\prime}$ by permuting the first $p$ coordinates and the last $p$ coordinates (see Remark 1). The parameter $a_{s, j}$ and $b_{s, j}$ appearing in (77) are therefore obtained by separately permuting the first $p=l^{\prime} / 2$ and the last $p$ terms appearing in (114), but the index $j$ now ranges between 1 and $l^{\prime}$. Notice that

$$
\begin{array}{ll}
a_{j} \leq 0 & \text { if and only if } \quad 1 \leq j \leq \frac{l^{\prime}}{2} \\
b_{j} \leq 0 & \text { if and only if } l \leq \frac{3 l^{\prime}}{2}-1 \text { and } l+1-\frac{l^{\prime}}{2} \leq j \leq l^{\prime}
\end{array}
$$

In particular, since $l>l^{\prime}$, for each $j$, at most one between $a_{j}$ and $b_{j}$ can be $\leq 0$. Moreover, there is at least one index $j$ for which both $a_{j}$ and $b_{j}$ are positive, namely $j=\frac{l^{\prime}}{2}+1$.

When $\mathrm{G}=\mathrm{U}_{1,1}$ (and hence $l^{\prime}=2$ ), then $W\left(\mathrm{U}_{1,1}, \mathfrak{h}^{\prime}\right)$ is trivial, $s_{0}=1$ and (77) simplifies to a nonzero constant multiple of

$$
\frac{P_{a_{1}, b_{1}, 2}\left(2 \pi y_{1}\right) P_{a_{2}, b_{2},-2}\left(2 \pi y_{2}\right)}{\left(y_{2}-y_{1}\right)\left(y_{1} y_{2}\right)^{l-2}} e^{-2 \pi\left(y_{1}-y_{2}\right)} \quad\left(y=\tau^{\prime}(w), w \in \mathfrak{h}_{\overline{1}}^{r e g}\right),
$$

where $a_{j}, b_{j}$ are as in (114) and the denominator is the root product (A.4).

## 8. The integral over $-G^{0}$ as an integral over $\mathfrak{g}$

Let $\mathfrak{s p}(\mathrm{W})$ be the Lie algebra of $\operatorname{Sp}(\mathrm{W})$. Set

$$
\begin{align*}
& \mathfrak{s p}(\mathrm{W})^{c}=\{x \in \mathfrak{s p}(\mathrm{~W}) ; x-1 \text { is invertible }\}  \tag{119}\\
& \mathrm{Sp}(\mathrm{~W})^{c}=\{g \in \mathrm{Sp}(\mathrm{~W}) ; g-1 \text { is invertible }\} \tag{120}
\end{align*}
$$

The Cayley transform $c: \mathfrak{s p}(\mathrm{W})^{c} \rightarrow \mathrm{Sp}(\mathrm{W})^{c}$ is the bijective rational map defined by $c(x)=(x+1)(x-1)^{-1}$. Its inverse $c^{-1}: \mathrm{Sp}(\mathrm{W})^{c} \rightarrow \mathfrak{s p}(\mathrm{~W})^{c}$ is given by the same formula, $c^{-1}(g)=(g+1)(g-1)^{-1}$.

Since all eigenvalues of $x \in \mathfrak{g} \subseteq \operatorname{End}(W)$ are purely imaginary, $x-1$ is invertible. Therefore $\mathfrak{g} \subseteq \mathfrak{s p}(W)^{c}$. Moreover, $c(\mathfrak{g}) \subseteq G$. Since the map $c$ is continuous, the range $c(\mathfrak{g})$ is connected. Also, $-1=c(0)$ is in $c(\mathfrak{g})$. Furthermore, for $x \in \mathfrak{g}$,

$$
c(x)-1=(x+1)(x-1)^{-1}-(x-1)(x-1)^{-1}=2(x-1)^{-1}
$$

is invertible. Hence $c(\mathfrak{g}) \subseteq \mathrm{G} \cap \mathrm{Sp}(\mathrm{W})^{c}$. This is an equality since $c(c(y))=y$ and $c(\mathrm{G}) \subseteq \mathfrak{g}$. Thus

$$
c(\mathfrak{g})=\{g \in \mathrm{G} ; \operatorname{det}(g-1) \neq 0\}
$$

This is an open dense subset of G if $\mathrm{G}=\mathrm{U}_{d}$ or $\mathrm{G}=\mathrm{Sp}_{d}$. If $\mathrm{G}=\mathrm{O}_{2 l}$ then $c(\mathfrak{g})$ is dense in $\mathrm{SO}_{2 l}$. Therefore, if $\mathrm{G} \neq \mathrm{O}_{2 l+1}$, then $\mathrm{G}^{0}=-\mathrm{G}^{0}$. Hence

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d g=\int_{c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d g \tag{121}
\end{equation*}
$$

If $\mathrm{G}=\mathrm{O}_{2 l+1}$, then $c(\mathfrak{g})$ is dense in the connected component of -1 , which coincides with $-\mathrm{SO}_{2 l+1}$. Therefore

$$
\begin{equation*}
\int_{\mathrm{G}^{0}} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d g=\int_{-c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d g \tag{122}
\end{equation*}
$$

and

$$
\int_{-\mathrm{G}^{0}} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d g=\int_{c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d g
$$

Let

$$
\begin{equation*}
\tilde{c}: \mathfrak{g} \rightarrow \widetilde{\mathrm{G}} \tag{123}
\end{equation*}
$$

be a real analytic lift of $c$. Set $\tilde{c}_{-}(x)=\tilde{c}(x) \tilde{c}(0)^{-1}$. Then $\tilde{c}_{-}(0)$ is the identity of the group $\widetilde{\mathrm{Sp}}(\mathrm{W})$.

By (14), we have

$$
\begin{equation*}
T(\tilde{c}(x))=\Theta(\tilde{c}(x)) \chi_{x} \mu_{\mathrm{W}} \tag{124}
\end{equation*}
$$

Therefore, for a suitable normalization of the Lebesgue measure on $\mathfrak{g}$,

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=\int_{\mathfrak{g}} \check{\Theta}_{\Pi}(\tilde{c}(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \chi_{x} \mu_{\mathrm{W}} d x \tag{125}
\end{equation*}
$$

where $j_{\mathfrak{g}}(x)$ is the Jacobian of the map $c: \mathfrak{g} \rightarrow c(\mathfrak{g})$ (see Appendix B for its explicit expression). Also, since $\tilde{c}(0)$ is in the center of the metaplectic group,

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=\check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_{\Pi}\left(\tilde{c}_{-}(x)\right) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \chi_{x} \mu_{\mathrm{W}} d x \tag{126}
\end{equation*}
$$

where $\chi_{\Pi}$ is the central character of $\Pi$; see (60). In the rest of this paper we shall write $d w=d \mu_{\mathrm{W}}(w)$, when convenient.

## 9. The invariant integral over $\mathfrak{g}$ as an integral over $\mathfrak{h}$

We now apply the Weyl integration formula to reduce the integral on $\mathfrak{g}$ in (126) to an integral on a Cartan subalgebra of $\mathfrak{g}$. In section 3, this Cartan subalgebra was denoted by $\mathfrak{h}(\mathfrak{g})$, see (46). To make our notation lighter, in this section we will write $\mathfrak{h}$ instead of $\mathfrak{h}(\mathfrak{g})$. Let $H \subseteq G$ be the corresponding Cartan subgroup. Fix a system of positive roots of $\left(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}\right)$. For any positive root $\alpha$ let $\mathfrak{g}_{\mathbb{C}, \alpha} \subseteq \mathfrak{g}_{\mathbb{C}}$ be the corresponding ad $\left(\mathfrak{h}_{\mathbb{C}}\right)$-eigenspace and let $X_{\alpha} \in \mathfrak{g}_{\mathbb{C}, \alpha}$ be a non-zero vector. Then there is a character (group homomorphism) $\xi_{\alpha}: \mathrm{H} \rightarrow \mathbb{C}^{\times}$such that

$$
\operatorname{Ad}(h) X_{\alpha}=\xi_{\alpha}(h) X_{\alpha} \quad(h \in \mathrm{H})
$$

The derivative of $\xi_{\alpha}$ at the identity coincides with $\alpha$. Let $\rho \in \mathfrak{h}_{\mathbb{C}}^{*}$ denote one half times the sum of all the positive roots. Then in all cases except when $\mathrm{G}=\mathrm{O}_{2 l+1}$ or $\mathrm{G}=\mathrm{U}_{l}$ with $l$ even, there is a character $\xi_{\rho}: \mathrm{H} \rightarrow \mathbb{C}^{\times}$whose derivative at the identity is equal to $\rho$, see [GW09, (2.21) and p. 145]. When $\mathrm{G}=\mathrm{O}_{2 l+1}$ or $\mathrm{G}=\mathrm{U}_{l}$ with $l$ even, the character $\xi_{\rho}$ exists as a map defined on a non-trivial double cover

$$
\begin{equation*}
\widehat{\mathrm{H}} \ni \widehat{h} \rightarrow h \in \mathrm{H} . \tag{127}
\end{equation*}
$$

of H. In particular the Weyl denominator

$$
\begin{equation*}
\Delta(h)=\xi_{\rho}(h) \prod_{\alpha>0}\left(1-\xi_{-\alpha}(h)\right) \tag{128}
\end{equation*}
$$

is defined for $h \in \mathrm{H}$ or $h \in \widehat{\mathrm{H}}$ according to the cases described above. The sign representation $\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}$ of the Weyl group $W(\mathrm{G}, \mathfrak{h})$ is defined by

$$
\begin{equation*}
\Delta(s h)=\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(s) \Delta(h) \quad(s \in W(\mathrm{G}, \mathfrak{h})), \tag{129}
\end{equation*}
$$

where either $h \in \mathrm{H}$ or $h \in \widehat{\mathrm{H}}$. Notice that $W(\mathrm{G}, \mathfrak{h})=W(\widetilde{\mathrm{G}}, \mathfrak{h})$ because $\widetilde{\mathrm{G}}$ is a central extension of G .

We now determine the preimage $\widetilde{H}$ of H in $\widetilde{\mathrm{G}}$ and, where needed, a double cover $\widehat{\widetilde{\mathrm{H}}}$ of $\widetilde{\mathrm{H}}$ on which all functions $\Delta, \xi_{\mu}$ and $\check{\Theta}_{\Pi}$ can be defined. At the same time, we define a lift to $\widetilde{\mathrm{H}}$, or to $\widehat{\widetilde{\mathrm{H}}}$, of the modified Cayley transform defined on $\mathfrak{h}$ by

$$
\begin{equation*}
c_{-}(x)=(1+x)(1-x)^{-1}=-c(x) . \tag{130}
\end{equation*}
$$

Suppose first that $\mathrm{G}=\mathrm{O}_{2 l+1}$. Then $\mathrm{H}=\mathrm{H}^{0} \cdot \mathrm{Z}$ is the direct product of the connected identity component $\mathrm{H}^{0}$ of H and the center Z of $\mathrm{Sp}(\mathrm{W})$. According to Proposition E.1, $\widetilde{\mathrm{G}} \rightarrow \mathrm{G}$ is isomorphic to the $\operatorname{det}^{1 / 2}$-covering (E.12). Hence $\widetilde{\mathrm{H}^{0}} \rightarrow \mathrm{H}^{0}$ is isomorphic to the $\operatorname{det}^{1 / 2}$-covering $\sqrt{\mathrm{H}^{0}}=\left\{(h ; \zeta) \in \mathrm{H}^{0} \times \mathbb{C}^{\times}: \zeta^{2}=1\right\}=\left\{(h ; \pm 1): h \in \mathrm{H}^{0}\right\}$, whose identity component is isomorphic to $\mathrm{H}^{0}$. This connected component will be again denoted by $\mathrm{H}^{0}$. In this notation, $\widetilde{H}=\mathrm{H}^{0} \cdot \widetilde{\mathrm{Z}}$.

We have a chain of double covering homomorphisms

$$
\begin{gather*}
\widehat{\mathrm{H}^{0}} \times \widetilde{\mathrm{Z}} \longrightarrow \mathrm{H}^{0} \times \widetilde{\mathrm{Z}} \longrightarrow \mathrm{H}^{0} \times \mathrm{Z} \longrightarrow \mathrm{H}^{0} \\
(\widehat{h}, \tilde{z}) \longrightarrow(h, \tilde{z}) \longrightarrow(h, z) \longrightarrow h, \tag{131}
\end{gather*}
$$

where $\widehat{\mathrm{H}^{0}}$ is the preimage of $\mathrm{H}^{0}$ in $\widehat{\mathrm{H}}$. We set $\widehat{\widetilde{\mathrm{H}}}=\widehat{\mathrm{H}^{0}} \times \widetilde{\mathrm{Z}}$. We lift $\Delta$ and $\xi_{\mu}$ to functions on $\widehat{\widetilde{H}}$ constant on the fibers of composition of all the maps (131) and the character $\check{\Theta}_{\Pi}$ to a function on $\widehat{\mathrm{H}^{0}} \times \widetilde{\mathrm{Z}}$ constant on the fibers of the leftmost map in (131).

The subset $c_{-}(\mathfrak{h}) \subseteq \mathrm{H}^{0}$ consists of all $h \in \mathrm{H}^{0}$ such that $h+1$ is invertible. In particular, it is dense in $\mathrm{H}^{0}$. We choose a real analytic section

$$
\sigma: c_{-}(\mathfrak{h}) \rightarrow \widehat{\mathrm{H}^{0}}
$$

to the covering map (127) and set

$$
\begin{equation*}
\widehat{c}_{-}: \mathfrak{h} \ni x \rightarrow\left(\sigma\left(c_{-}(x)\right), 1\right) \in \widehat{\widetilde{\mathrm{H}}}=\widehat{\mathrm{H}^{0}} \times \widetilde{\mathrm{Z}} \tag{132}
\end{equation*}
$$

Suppose now that $\mathrm{G}=\mathrm{U}_{l}$ with $l$ even. If $\mathrm{G}^{\prime}=\mathrm{U}_{p, q}$ with $p+q$ odd, then the covering $\widetilde{\mathrm{H}} \rightarrow \mathrm{H}$ does not split (see the proof of Proposition E.1). Hence $\widehat{\mathrm{H}}=\widetilde{\mathrm{H}}$. Therefore $\Delta$, $\xi_{\mu}$ and $\check{\Theta}_{\Pi}$ are defined on $\widehat{H}$. We have the Cayley transform $c_{-}: \mathfrak{h} \rightarrow H$, an analytic section $\sigma: c_{-}(\mathfrak{h}) \rightarrow \widehat{\mathrm{H}}$ and the map

$$
\begin{equation*}
\widehat{c}_{-}: \mathfrak{h} \ni x \rightarrow \sigma\left(c_{-}(x)\right) \in \widehat{\mathrm{H}} . \tag{133}
\end{equation*}
$$

If $\mathrm{G}^{\prime}=\mathrm{U}_{p, q}$ with $p+q$ even, then $\widetilde{\mathrm{H}}=\mathrm{H} \times\{1, \tilde{1}\}$ splits by Proposition E.1, and we have maps

$$
\begin{align*}
& \widehat{\mathrm{H}} \longrightarrow \mathrm{H} \longrightarrow \widetilde{\mathrm{H}} \longrightarrow \mathrm{H}  \tag{134}\\
& \widehat{h} \rightarrow h \rightarrow(h ; 1) \rightarrow h
\end{align*}
$$

Again $\Delta, \xi_{\mu}$ and $\check{\Theta}_{\Pi}$ are defined on $\widehat{\mathrm{H}}$ and (133) defines the lift of the Cayley transform we shall use. In this case, we set then $\widehat{\widetilde{H}}=\widehat{\mathrm{H}}$.

For the remaining dual pairs, $\widehat{\mathrm{H}}=\mathrm{H}$ and we lift $\Delta$ and $\xi_{\mu}$ to functions on $\widetilde{\mathrm{H}}$ constant on the fibers of the covering map $\widetilde{\mathrm{H}} \rightarrow \mathrm{H}$ and write $\widehat{c}_{-}$for $\tilde{c}_{-}$, which was defined under the equation (123).
Lemma 15. Let $\mu \in i \mathfrak{h}^{*}$. Then

$$
\begin{equation*}
\xi_{-\mu}\left(\widehat{c}_{-}(x)\right)=\prod_{j=1}^{l}\left(\frac{1+i x_{j}}{1-i x_{j}}\right)^{\mu_{j}}=\prod_{j=1}^{l}\left(1+i x_{j}\right)^{\mu_{j}}\left(1-i x_{j}\right)^{-\mu_{j}} \quad(x \in \mathfrak{h}) \tag{135}
\end{equation*}
$$

Proof. By (35), it is enough to verify this formula when $l=1$. In this case, $x=x_{1} J_{1}$ and $\mu=\mu_{1} e_{1}=-i \mu_{1} J_{1}^{*}$. Let log denote the local inverse of the exponential map near 1. Then, for $x$ sufficiently close to 0 ,

$$
\log \left(c_{-}(x)\right)=\log \left((1+x)(1-x)^{-1}\right)=\log (1+x)-\log (1-x)
$$

is a real analytic odd function of $x$. Hence it admits a Taylor series expansion

$$
\sum_{n \geq 0} a_{n} x^{2 n+1}=\sum_{n \geq 0} a_{n}(-1)^{n} x_{1}^{2 n+1} J_{1}
$$

Thus

$$
\mu\left(\log \left(c_{-}(x)\right)\right)=-\sum_{n \geq 0} a_{n}(-1)^{n} x_{1}^{2 n+1} i \mu_{1}=-\sum_{n \geq 0} a_{n}\left(i x_{1}\right)^{2 n+1} \mu_{1}=\ln \left(\frac{1-i x_{1}}{1+i x_{1}}\right) \mu_{1}
$$

By taking exponentials, we obtain

$$
\xi_{-\mu}\left(\widehat{c}_{-}(x)\right)=e^{-\mu\left(\log \left(c_{-}(x)\right)\right)}=\left(\frac{1+i x_{1}}{1-i x_{1}}\right)^{\mu_{1}}
$$

and the result extends to all $x \in \mathfrak{h}$ by real analyticity.
Let $\Pi$ be an irreducible representation of $\widetilde{\mathrm{G}}$, and let $\mu \in i \mathfrak{h}^{*}$ represent the infinitesimal character of $\Pi$. When $\mu$ is dominant, then we will refer to it as the Harish-Chandra parameter of $\Pi$. This is consistent with the usual terminology; see e.g. [Kna86, Theorem 9.20]. Then the corresponding character $\xi_{\mu}$ is defined as $\xi_{\mu}=\xi_{\rho} \xi_{\mu-\rho}$, where $\xi_{\mu-\rho}$ is one of the extremal H-weights of $\Pi$. In these terms, Weyl's character formula looks as follows,

$$
\begin{equation*}
\Theta_{\Pi}(h) \Delta(h)=\kappa_{0} \sum_{s \in W(\mathbf{G}, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(s) \xi_{s \mu}(h), \tag{136}
\end{equation*}
$$

where $h \in \widetilde{\mathrm{H}^{0}}$ or $h \in \widehat{\widetilde{\mathrm{H}^{0}}}$, according to the cases above, and $\kappa_{0}$ is as in (70).
Lemma 16. Let $\pi_{\mathfrak{g} / \mathfrak{h}}$ be the product of the positive roots of $\left(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}\right)$ and let

$$
\kappa(x)=\kappa_{0} \frac{\pi_{\mathfrak{g} / \mathfrak{h}}(x)}{\Delta\left(\widehat{c}_{-}(x)\right)} \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \quad(x \in \mathfrak{h}) .
$$

Then, for a suitable normalization of the Lebesgue measure on $\mathfrak{h}$ and any $\phi \in \mathcal{S}(\mathrm{W})$,

$$
\begin{aligned}
\int_{-\mathrm{G}^{0}} & \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g \\
& =\frac{\check{\chi}_{\Pi}(\tilde{c}(0))}{|W(\mathrm{G}, \mathfrak{h})|} \int_{\mathfrak{h}}\left(\check{\Theta}_{\Pi} \Delta\right)\left(\widehat{c}_{-}(x)^{-1}\right) \frac{\kappa(x)}{\kappa_{0}} \pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w d x \\
& =\check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}\left(\widehat{c}_{-}(x)\right) \kappa(x) \pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w d x,
\end{aligned}
$$

where $\phi^{\mathrm{G}}$ is as in (63) and each consecutive integral is absolutely convergent.
Proof. Applied to a test function $\phi \in \mathcal{S}(\mathrm{W})$, the integral (122) over $c(\mathfrak{g})$ is absolutely convergent because both, the character and the function $T(\widetilde{g})(\phi)$ are continuous and bounded (see for example [Prz93, Proposition 1.13]) and the group G is compact. Hence, each consecutive integral in the formula (126) applied to $\phi$,

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=\check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_{\Pi}\left(\tilde{c}_{-}(x)\right) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi(w) d w d x \tag{137}
\end{equation*}
$$

is absolutely convergent. Since

$$
\chi_{g . x}(w)=\chi_{x}\left(g^{-1} \cdot w\right)
$$

and the Lebesgue measure $d w$ is G-invariant,

$$
\int_{\mathrm{G}} \int_{\mathrm{W}} \chi_{g . x}(w) \phi(w) d w d g=\int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w
$$

Observe also that $\widetilde{\operatorname{Ad}}(\tilde{g})=\operatorname{Ad}(g)$ and the characters $\check{\Theta}_{\Pi}$ and $\Theta$ are $\widetilde{\mathrm{G}}$-invariant. Moreover, by (128) and (135),

$$
\overline{\Delta\left(\widehat{c}_{-}(x)\right)}=\Delta\left(\widehat{c}_{-}(x)^{-1}\right)=(-1)^{m} \Delta\left(\widehat{c}_{-}(x)\right) \quad(x \in \mathfrak{h})
$$

where $m$ is the number of positive roots, and

$$
\overline{\pi_{\mathfrak{g} / \mathfrak{h}}(x)}=(-1)^{m} \pi_{\mathfrak{g} / \mathfrak{h}}(x) \quad(x \in \mathfrak{h}) .
$$

Therefore the Weyl integration formula on $\mathfrak{g}$ shows that (137) is equal to $\frac{\check{\chi}_{\Pi}(\tilde{c}(0))}{|W(G, \mathfrak{h})|}$ times

$$
\begin{aligned}
\int_{\mathfrak{h}} \mid & \left.\pi_{\mathfrak{g} / \mathfrak{h}}(x)\right|^{2} \check{\Theta}_{\Pi}\left(\tilde{c}_{-}(x)\right) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w d x \\
& =\int_{\mathfrak{h}} \check{\Theta}_{\Pi}\left(\widehat{c}_{-}(x)\right) \overline{\Delta\left(\widehat{c}_{-}(x)\right)}\left(\frac{\pi_{\mathfrak{g} / \mathfrak{h}}(x)}{\Delta\left(\widehat{c}_{-}(x)\right)} \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x)\right) \pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w d x \\
& =\int_{\mathfrak{h}} \Theta_{\Pi}\left(\widehat{c}_{-}(x)^{-1}\right) \Delta\left(\widehat{c}_{-}(x)^{-1}\right) \frac{\kappa(x)}{\kappa_{0}} \pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w d x .
\end{aligned}
$$

(Here, we suppose that the Haar measure on H is normalized to have total mass 1.) This verifies the first equality and the absolute convergence. Since

$$
\begin{aligned}
\Theta_{\Pi}\left(\widehat{c}_{-}(x)^{-1}\right) \Delta\left(\widehat{c}_{-}(x)^{-1}\right) & =\kappa_{0} \sum_{s \in W(\mathrm{G}, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(s) \xi_{s \mu}\left(\widehat{c}_{-}(x)^{-1}\right) \\
& =\kappa_{0} \sum_{s \in W(\mathrm{G}, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(s) \xi_{-s \mu}\left(\widehat{c}_{-}(x)\right),
\end{aligned}
$$

the second equality follows via a straightforward computation.
Since any element $x \in \mathfrak{g}$, viewed as an endomorphism of V over $\mathbb{R}$, has imaginary eigenvalues which come in complex conjugate pairs, we have $\operatorname{det}(1-x)_{\mathrm{V}_{\mathbb{R}}} \geq 1$. Define

$$
\begin{equation*}
\operatorname{ch}(x)=\operatorname{det}(1-x)_{V_{\mathbb{R}}}^{1 / 2} \quad(x \in \mathfrak{g}) . \tag{138}
\end{equation*}
$$

Recall the symbols $r$ and $\iota$ from (65) and (64).
Lemma 17. There is a constant $C$ which depends only on the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ such that

$$
\frac{\kappa(x)}{\kappa_{0}}=C \operatorname{ch}^{d^{\prime}-r-\iota}(x) \quad(x \in \mathfrak{h}) .
$$

Proof. Recall [Prz93, Lemma 5.7] that $\pi_{\mathfrak{g} / \mathfrak{h}}(x)$ is a constant multiple of $\Delta\left(\widehat{c}_{-}(x)\right) \operatorname{ch}^{r-\iota}(x)$,

$$
\begin{equation*}
\pi_{\mathfrak{g} / \mathfrak{h}}(x)=C \Delta\left(\widehat{c}_{-}(x)\right) \operatorname{ch}^{r-\iota}(x) . \tag{139}
\end{equation*}
$$

(For the reader's convenience, this is verified in Appendix C.) As is well known, [AP14, Definition 4.16],

$$
\begin{equation*}
\Theta(\tilde{c}(x))^{2}=i^{\operatorname{dim} \mathrm{W}} \operatorname{det}\left(2^{-1}(x-1)\right)_{\mathrm{W}} \quad(x \in \mathfrak{s p}(\mathrm{~W}), \operatorname{det}(x-1) \neq 0) \tag{140}
\end{equation*}
$$

Hence there is a choice of $\tilde{c}$ so that

$$
\begin{equation*}
\Theta(\tilde{c}(x))=\left(\frac{i}{2}\right)^{\frac{1}{2} \operatorname{dim} \mathrm{~W}} \operatorname{det}(1-x)_{\mathrm{W}}^{\frac{1}{2}} \quad(x \in \mathfrak{g}) \tag{141}
\end{equation*}
$$

Furthermore, since the symplectic space may be realized as $W=\operatorname{Hom}_{\mathbb{D}}\left(\mathrm{V}^{\prime}, \mathrm{V}\right)$, see (32), we obtain that

$$
\begin{equation*}
\operatorname{det}(1-x)_{\mathrm{W}}=\operatorname{det}(1-x)_{\mathrm{V}_{\mathbb{R}}}^{d^{\prime}} \quad(x \in \mathfrak{g}) \tag{142}
\end{equation*}
$$

Also, as checked in [Prz91, (3.11)], the Jacobian of $\tilde{c}_{-}: \mathfrak{g} \rightarrow \mathrm{G}$ is a constant multiple of $\operatorname{ch}^{-2 r}(x)$. (Again, for reader's convenience a -slightly different- proof is included in Appendix B.) Hence the claim follows.
Corollary 18. For any $\phi \in \mathcal{S}(\mathrm{W})$
$\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}\left(\widehat{c}_{-}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w d x$,
where $C$ is a constant which depends only on the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right), \phi^{\mathrm{G}}$ is as in (63), and each consecutive integral is absolutely convergent.

## 10. An intertwining distribution in terms of orbital integrals on the symplectic space

We keep the notation introduced in section 3. Let

$$
W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))= \begin{cases}\Sigma_{l} & \text { if } \mathbb{D}=\mathbb{C}  \tag{143}\\ \Sigma_{l} \ltimes\{ \pm 1\}^{l} & \text { otherwise }\end{cases}
$$

Denote the elements of $\Sigma_{l}$ by $\eta$ and the elements of $\{ \pm 1\}^{l}$ by $\epsilon=\left(\epsilon_{1}, \epsilon_{2}, \ldots, \epsilon_{l}\right)$, so that an arbitrary element of the group (143) is of the form $t=\epsilon \eta$, with $\epsilon=(1,1, \ldots, 1)$, if $\mathbb{D}=\mathbb{C}$. This group acts on $\mathfrak{h}(\mathfrak{g})$, see (46), as follows: for $t=\epsilon \eta$,

$$
\begin{equation*}
t\left(\sum_{j=1}^{l} y_{j} J_{j}\right)=\sum_{j=1}^{l} \epsilon_{j} y_{\eta^{-1}(j)} J_{j} . \tag{144}
\end{equation*}
$$

As indicated by the notation, $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ coincides with the Weyl group, equal to the quotient of the normalizer of $\mathfrak{h}(\mathfrak{g})$ in $G$ by the centralizer of $\mathfrak{h}(\mathfrak{g})$ in $G$.

The action of $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ on $\mathfrak{h}(\mathfrak{g})$ extends by duality to $i \mathfrak{h}(\mathfrak{g})^{*}$. More precisely, let $e_{j}$ be as in (47). If $\mu \in i \mathfrak{h}(\mathfrak{g})^{*}$, then $\mu=\sum_{j=1}^{l} \mu_{j} e_{j}$ with all $\mu_{j} \in \mathbb{R}$. If $t=\epsilon \eta \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$, then

$$
\begin{equation*}
t\left(\sum_{j=1}^{l} \mu_{j} e_{j}\right)=\sum_{j=1}^{l} \epsilon_{j} \mu_{\eta^{-1}(j)} e_{j} . \tag{145}
\end{equation*}
$$

Recall the notation of Lemma 16 and the symbol $\delta$ from (67).
Lemma 19. The following formulas hold for any $y=\sum_{j=1}^{l} y_{j} J_{j} \in \mathfrak{h}(\mathfrak{g})$,

$$
\begin{equation*}
\xi_{-\mu}\left(\widehat{c}_{-}(t y)\right)=\xi_{-t^{-1} \mu}\left(\widehat{c}_{-}(y)\right) \quad(t \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))) \tag{146}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi_{-\mu}\left(\widehat{c}_{-}(y)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(y)=\prod_{j=1}^{l}\left(1+i y_{j}\right)^{\mu_{j}+\delta-1}\left(1-i y_{j}\right)^{-\mu_{j}+\delta-1} \tag{147}
\end{equation*}
$$

where all the exponents are integers:

$$
\begin{equation*}
\pm \mu_{j}+\delta \in \mathbb{Z} \quad(1 \leq j \leq l) \tag{148}
\end{equation*}
$$

In particular, (147) is a rational function in the variables $y_{1}, y_{2}, \ldots, y_{l}$.
Proof. By (135),

$$
\xi_{-\mu}\left(\widehat{c}_{-}(y)\right)=\prod_{j=1}^{l}\left(\frac{1+i y_{j}}{1-i y_{j}}\right)^{\mu_{j}}=\prod_{j=1}^{l}\left(1+i y_{j}\right)^{\mu_{j}}\left(1-i y_{j}\right)^{-\mu_{j}}
$$

Hence (146) and (147) follow from the definition of the action of $W(G, \mathfrak{h}(\mathfrak{g}))$, the definition of ch in (138), and the following easy-to-check formula:

$$
\begin{equation*}
\operatorname{ch}(y)=\prod_{j=1}^{l}\left(1+y_{j}^{2}\right)^{\frac{1}{2 \iota}}=\prod_{j=1}^{l}\left(1+i y_{j}\right)^{\frac{1}{2 \iota}}\left(1-i y_{j}\right)^{\frac{1}{2 \iota}} . \tag{149}
\end{equation*}
$$

Let $\lambda=\sum_{j=1}^{l} \lambda_{j} e_{j}$ be the highest weight of the representation $\Pi$ and let $\rho=\sum_{j=1}^{l} \rho_{j} e_{j}$ be one half times the sum of the positive roots of $\mathfrak{h}(\mathfrak{g})$ in $\mathfrak{g}_{\mathbb{C}}$. If $\mu$ is the Harish-Chandra parameter of $\Pi$, then $\lambda+\rho=\mu=\sum_{j=1}^{l} \mu_{j} e_{j}$; see Appendix H. Hence, the statement (148) is equivalent to

$$
\begin{equation*}
\lambda_{j}+\rho_{j}+\frac{1}{2 \iota}\left(d^{\prime}-r+\iota\right) \in \mathbb{Z} . \tag{150}
\end{equation*}
$$

Indeed, if $\mathrm{G}=\mathrm{O}_{d}$, then with the standard choice of the positive root system, $\rho_{j}=\frac{d}{2}-j$. Also, $\lambda_{j} \in \mathbb{Z}, \iota=1, r=d-1$. Hence, (150) follows. Similarly, if $\mathrm{G}=\mathrm{U}_{d}$, then $\rho_{j}=\frac{d+1}{2}-j$, $\lambda_{j}+\frac{d^{\prime}}{2} \in \mathbb{Z}, \iota=1, r=d$, which implies (150). If $\mathrm{G}=\operatorname{Sp}_{d}$, then $\rho_{j}=d+1-j, \lambda_{j} \in \mathbb{Z}$, $\iota=\frac{1}{2}, r=d+\frac{1}{2}$, and (150) follows.

Our next goal is to understand the integral

$$
\pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w
$$

occurring in the formula for $\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g$ in Lemma 16 and Corollary 18 , in terms of orbital integrals on the symplectic space W . The results depend on whether $l \leq l^{\prime}$ or $l>l^{\prime}$ and will be given in Lemmas 22 and 23 . We first need two other lemmas.

Lemma 20. Fix an element $z \in \mathfrak{h}(\mathfrak{g})$. Let $\mathfrak{z} \subseteq \mathfrak{g}$ and $\mathrm{Z} \subseteq G$ denote the centralizer of $z$. (Then Z is a real reductive group with Lie algebra $\mathfrak{z}$.) Denote by $\mathfrak{c}$ the center of $\mathfrak{z}$ and by $\pi_{\mathfrak{g} / \mathfrak{z}}$ the product of the positive roots for $\left(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}(\mathfrak{g})_{\mathbb{C}}\right)$ which do not vanish on z. Let $B(\cdot, \cdot)$ be any non-degenerate symmetric G-invariant real bilinear form on $\mathfrak{g}$. Then there is a constant $C_{\mathfrak{z}}$ such that for $x \in \mathfrak{h}(\mathfrak{g})$ and $x^{\prime} \in \mathfrak{c}$,

$$
\begin{align*}
\pi_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(x) \pi_{\mathfrak{g} / \mathfrak{z}}\left(x^{\prime}\right) & \int_{\mathrm{G}} e^{i B\left(g . x, x^{\prime}\right)} d g \\
& =C_{\mathfrak{z}} \sum_{t W(\mathbf{Z}, \mathfrak{h}(\mathfrak{g})) \in W(\mathbf{G}, \mathfrak{h}(\mathfrak{g})) / W(\mathbf{Z}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} x\right) e^{i B\left(x, t\left(x^{\prime}\right)\right)} . \tag{151}
\end{align*}
$$

(Here $\pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}=1$ if $\mathfrak{z}=\mathfrak{h}$. Recall also the notation $g \cdot x=g x g^{-1}$.)
Proof. The proof is a straightforward modification of the argument proving Harish-Chandra's formula for the Fourier transform of a regular semisimple orbit, [Har57, Theorem 2, page 104]. A more general, and by now classical, result is [DV90, Proposition 34, p. 49].

The symplectic form $\langle\cdot, \cdot\rangle$ on W according to the Lie superalgebra structure introduced in (28) is

$$
\begin{equation*}
\left\langle w^{\prime}, w\right\rangle=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\mathrm{S} w^{\prime} w\right) \quad\left(w^{\prime}, w \in \mathrm{~W}\right) . \tag{152}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\langle x w, w\rangle=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\mathrm{S} x w^{2}\right) \quad\left(x \in \mathfrak{g} \oplus \mathfrak{g}^{\prime}, w \in \mathrm{~W}\right) . \tag{153}
\end{equation*}
$$

Set

$$
\begin{equation*}
B(x, y)=\pi \operatorname{tr}_{\mathbb{D} / \mathbb{R}}(x y) \quad(x, y \in \mathfrak{g}) \tag{154}
\end{equation*}
$$

Lemma 21. Recall the Gaussian $\chi_{x}$ from (13). Then

$$
\begin{equation*}
\chi_{x}(w)=e^{i B(x, \tau(w))} \quad(x \in \mathfrak{g}, w \in \mathrm{~W}) \tag{155}
\end{equation*}
$$

Proof. Notice that, for $x \in \mathfrak{g} \oplus \mathfrak{g}^{\prime}$ and $w \in \mathrm{~W}$,

$$
\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\mathrm{S} x w^{2}\right)=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(x w^{2} \mid v_{\overline{0}}\right)-\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(x w^{2} \mid v_{\overline{1}}\right)
$$

where

$$
\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(x w^{2} \mid{v_{\overline{0}}}\right)=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\left.\left.\left.x\right|_{v_{\overline{0}}} w\right|_{v_{\overline{1}}} w\right|_{v_{\overline{0}}}\right)=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\left.\left.\left.w\right|_{v_{\overline{0}}} x\right|_{\vee_{\overline{0}}} w\right|_{\left.\left.{v_{\overline{1}}}\right)\right)=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\left.w x\right|_{\mathrm{v}_{\overline{1}}}\right)}\right.
$$

and similarly

$$
\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\left.x w^{2}\right|_{v_{\overline{1}}}\right)=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\left.w x w\right|_{v_{\overline{0}}}\right) .
$$

Hence

$$
\langle x w, w\rangle=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\mathrm{S} x w^{2}\right)=-\operatorname{tr}_{\mathbb{D} / \mathbb{R}}(\mathrm{S} w x w)=-\langle w x, w\rangle .
$$

Therefore

$$
\begin{equation*}
\langle x(w), w\rangle=2 \operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(\mathrm{S} x w^{2}\right) \quad\left(x \in \mathfrak{g} \oplus \mathfrak{g}^{\prime}, w \in \mathrm{~W}\right) . \tag{156}
\end{equation*}
$$

Then (153) and (33) show that

$$
\frac{\pi}{2}\langle x(w), w\rangle=B(x, \tau(w)) \quad(x \in \mathfrak{g}, w \in \mathrm{~W})
$$

which completes the proof.
The Harish-Chandra regular almost semisimple orbital integral $F(y), y \in \mathfrak{h}$, was defined in [MPP20, Definition 3.2 and Theorems 3.4 and 3.6]; see also section 4 above. In particular, [MPP20, Theorem 3.6] implies that, in the statements below, all the integrals over $\mathfrak{h}$ involving $F(y)$ are absolutely convergent. Recall the notation $F_{\phi}(y)$ for $F(y)(\phi)$.

Lemma 22. Suppose $l \leq l^{\prime}$. Then, with the notation of Lemma 16,

$$
\pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w=C \int_{\mathfrak{h} \cap \tau(\mathrm{W})} e^{i B(x, y)} F_{\phi}(y) d y,
$$

where $C$ is a non-zero constant which depends only of the dual pair ( $\mathrm{G}, \mathrm{G}^{\prime}$ ).
Proof. The Weyl integration formula on W, see (53) and (50), shows that

$$
\begin{equation*}
\int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w=\sum_{\mathfrak{h}_{\overline{1}}} \int_{\tau\left(\mathfrak{h}_{\frac{1}{1}}^{ \pm}\right)} \pi_{\mathfrak{g} / \mathfrak{h}}(\tau(w)) \pi_{\mathfrak{g}^{\prime} / \mathfrak{s}^{\prime}}(\tau(w)) C\left(\mathfrak{h}_{\overline{1}}\right) \mu_{\mathcal{O}(w), \mathfrak{h}_{\overline{1}}}\left(\chi_{x} \phi^{\mathrm{G}}\right) d \tau(w), \tag{157}
\end{equation*}
$$

where $\mathfrak{h}_{\overline{1}}^{+} \subseteq \mathfrak{h}_{\overline{1}}^{\text {reg }}$ is an open fundamental domain for the action of the Weyl group $W\left(\mathrm{~S}, \mathfrak{h}_{\overline{1}}\right)$ and $C\left(\mathfrak{h}_{\overline{1}}\right)$ is a constant, determined in [MPP20, Lemma 2.1]. Let us consider first the case of a semisimple orbital integral

$$
\mu_{\mathcal{O}(w), \mathfrak{h}_{\mathrm{I}}}\left(\chi_{x} \phi^{\mathrm{G}}\right)=\int_{\mathrm{S} / \mathrm{S}^{\mathfrak{b}_{\bar{T}}}}\left(\chi_{x} \phi^{\mathrm{G}}\right)(s . w) d\left(s S^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right),
$$

where $\mathrm{S}^{\mathfrak{h}}{ }_{\bar{T}}$ is the centralizer of $\mathfrak{h}_{\bar{\top}}$ in S . Recall the identification $y=\tau(w)=\tau^{\prime}(w)$ and let us write $s=g g^{\prime}$, where $g \in \mathrm{G}$ and $g^{\prime} \in \mathrm{G}^{\prime}$. Then

$$
\begin{equation*}
\chi_{x}(s . w)=e^{i \frac{\pi}{2}\langle x(s . w), s . w\rangle}=e^{i B(x, \tau(s . w))}=e^{i B(x, g \cdot \tau(w))}=e^{i B(x, g \cdot y)} \tag{158}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi^{\mathrm{G}}(s \cdot w)=\phi^{\mathrm{G}}\left(g^{\prime} \cdot w\right) . \tag{159}
\end{equation*}
$$

Since $l \leq l^{\prime}$, equation (I.1) implies that there is a positive constant $C_{1}$ such that

$$
\mu_{\mathcal{O}(w), \mathfrak{h}_{\overline{\mathrm{T}}}}\left(\chi_{x} \phi^{\mathrm{G}}\right)=C_{1} \int_{\mathrm{G}} e^{i B(x, g \cdot y)} d g \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime}} \phi^{\mathrm{G}}\left(g^{\prime} \cdot w\right) d\left(g^{\prime} \mathbf{Z}^{\prime}\right) .
$$

However we know from Harish-Chandra (Lemma 20) that

$$
\pi_{\mathfrak{g} / \mathfrak{h}}(x)\left(\int_{\mathrm{G}} e^{i B(x, g \cdot y)} d g\right) \pi_{\mathfrak{g} / \mathfrak{h}}(y)=C_{2} \sum_{t \in W(\mathbf{G}, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(t) e^{i B(x, t \cdot y)}
$$

Hence, using (157) and [MPP20, Definition 3.2 and Lemma 3.4], we obtain for some suitable positive constants $C_{k}$,

$$
\begin{align*}
& \pi_{\mathfrak{g} / \mathfrak{h}}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w  \tag{160}\\
& =C_{3} \sum_{t \in W(\mathbf{G}, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(t) \sum_{\mathfrak{h}_{\bar{T}}} \int_{\tau\left(\mathfrak{h _ { \overline { 1 } } ^ { + }}\right)} e^{i B(x, t . y)} C\left(\mathfrak{h}_{\overline{1}}\right) \pi_{\mathfrak{g}^{\prime} / \mathfrak{z}^{\prime}}(y) \int_{\mathbf{G}^{\prime} / \mathbf{Z}^{\prime}} \phi^{\mathrm{G}}\left(g^{\prime} \cdot w\right) d\left(g^{\prime} \mathbf{Z}^{\prime}\right) d y \\
& =C_{4} \sum_{t \in W(\mathrm{G}, \mathfrak{\mathfrak { h }})} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(t) \int_{\bigcup_{\mathfrak{b}_{\bar{T}}} \tau\left(\mathfrak{h}_{\overline{\mathrm{T}}}^{ \pm}\right)} e^{i B(x, t . y)} F_{\phi^{\mathrm{G}}}(y) d y \\
& =C_{4} \sum_{t \in W(\mathrm{G}, \mathfrak{h})} \int_{\bigcup_{\mathfrak{b}_{\mathrm{T}}} \tau\left(\mathfrak{h}_{\frac{1}{\mathrm{I}}}^{ \pm}\right)} e^{i B(x, t . y)} F_{\phi^{\mathrm{G}}}(t \cdot y) d y \\
& =C_{4} \int_{W(\mathrm{G}, \mathfrak{h})\left(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau\left(\mathfrak{h}_{\frac{1}{1}}^{+}\right)\right)} e^{i B(x, y)} F_{\phi^{\mathrm{G}}}(y) d y \\
& =C_{4} \int_{\mathfrak{h} \cap \tau(\mathrm{W})} e^{i B(x, y)} F_{\phi^{\mathrm{G}}}(y) d y .
\end{align*}
$$

Since $F_{\phi^{\mathrm{G}}}=\operatorname{vol}(\mathrm{G}) F_{\phi}=F_{\phi}$, the formula follows.
Next we consider the case $\mathrm{G}=\mathrm{O}_{2 l+1}, \mathrm{G}^{\prime}=\mathrm{Sp}_{2 l^{\prime}}(\mathbb{R}), l<l^{\prime}$. Then

$$
\mu_{\mathcal{O}(w)}\left(\chi_{x} \phi^{\mathrm{G}}\right)=\int_{\mathrm{S} / \mathrm{S}^{\mathrm{h}_{\mathrm{T}}+w_{0}}}\left(\chi_{x} \phi^{\mathrm{G}}\right)\left(s .\left(w+w_{0}\right)\right) d\left(s \mathrm{~S}^{\mathfrak{h}_{\mathbb{1}}+w_{0}}\right),
$$

where $w_{0} \in \mathfrak{s}_{1}\left(\mathrm{~V}^{0}\right)$ is a nonzero element. Since the Cartan subspace $\mathfrak{h}_{\overline{1}}$ preserves the decomposition (35), $\left(w+w_{0}\right)^{2}=w^{2}+w_{0}^{2}$. Hence, $\left(s .\left(w+w_{0}\right)\right)^{2}=s .\left(w^{2}+w_{0}^{2}\right)$. The
element $x \in \mathfrak{h}$ acts by zero on $\mathfrak{g}^{\prime}$. Therefore $x\left(s .\left(w+w_{0}\right)\right)^{2}=x\left(s .\left(w+w_{0}\right)\right)^{2} \mid \vee_{\overline{0}}$. Since $\mathrm{S}\left(\mathrm{V}^{0}\right)=\mathrm{O}_{1} \times \mathrm{Sp}_{2\left(l^{\prime}-l\right)}(\mathbb{R})$ we see that $w_{0}^{2} \mid \mathrm{v}_{\overline{0}}=0$. Thus $x s . w_{0}^{2} \mid \mathrm{v}_{\overline{0}}=0$. Therefore, by (27),

$$
\left\langle x\left(s .\left(w+w_{0}\right)\right), s .\left(w+w_{0}\right)\right\rangle=\operatorname{tr}\left(x\left(s .\left(w+w_{0}\right)\right)^{2}\right)=\operatorname{tr}\left(x s . w^{2} \mid \mathrm{v}_{\overline{0}}\right)=\operatorname{tr}(x g . \tau(w)),
$$

because $s=g g^{\prime}$. Hence,

$$
\chi_{x}\left(s .\left(w+w_{0}\right)\right)=e^{i \frac{\pi}{2}\left\langle x\left(s .\left(w+w_{0}\right)\right), s .\left(w+w_{0}\right)\right\rangle}=e^{i B(x, g \cdot \tau(w))}=e^{i B(x, g \cdot y)}
$$

and

$$
\phi^{\mathrm{G}}\left(s \cdot\left(w+w_{0}\right)\right)=\phi^{\mathrm{G}}\left(g^{\prime} \cdot\left(w+w_{0}\right)\right) .
$$

Therefore, with $n=\tau^{\prime}\left(w_{0}\right)$, we obtain from (I.3) that

$$
\mu_{\mathcal{O}(w)}\left(\chi_{x} \phi^{\mathrm{G}}\right)=C_{1} \int_{\mathrm{G}} e^{i B(x, g \cdot g)} d g \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime n}} \phi^{\mathrm{G}}\left(g^{\prime} \cdot w\right) d\left(g^{\prime} \mathbf{Z}^{\prime n}\right),
$$

where $\mathrm{Z}^{\prime n}$ is the centralizer of $n$ in $\mathrm{Z}^{\prime}$. Thus, the computation (160) holds again, and we are done.

Lemma 23. Suppose $l>l^{\prime}$. Let $\mathfrak{z} \subseteq \mathfrak{g}$ and $\mathrm{Z} \subseteq \mathrm{G}$ be the centralizers of $\tau\left(\mathfrak{h}_{\overline{1}}\right)$. Then for $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{aligned}
\pi_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(x) & \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w \\
& =C \sum_{t W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g})) \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g})) / W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} \cdot x\right) \int_{\tau^{\prime}\left(\mathfrak{h} \overline{\mathbf{1}}^{\text {reg }}\right)} e^{i B(x, t \cdot y)} F_{\phi}(y) d y,
\end{aligned}
$$

where $C$ is a non-zero constant which depends only on the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$.
Proof. By the Weyl integration formula with the roles of G and $\mathrm{G}^{\prime}$ reversed, see (54) and (50),

$$
\int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w=C_{1} \int_{\tau^{\prime}\left(\mathfrak{\mathfrak { h } _ { \overline { 1 } } ^ { r e g } )}\right.} \pi_{\mathfrak{g} / \mathfrak{z}}\left(\tau^{\prime}(w)\right) \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(\tau^{\prime}(w)\right) \mu_{\mathcal{O}(w)}\left(\chi_{x} \phi^{\mathrm{G}}\right) d \tau^{\prime}(w)
$$

where

$$
\mu_{\mathcal{O}(w)}\left(\chi_{x} \phi^{\mathrm{G}}\right)=\int_{\mathrm{S} / \mathrm{S}^{\mathrm{h}} \overline{\mathrm{~T}}}\left(\chi_{x} \phi^{\mathrm{G}}\right)(s . w) d\left(s \mathrm{~S}^{\mathrm{h}_{\overline{\mathrm{T}}}}\right) .
$$

Recall the identification $y=\tau(w)=\tau^{\prime}(w)$ and let us write $s=g g^{\prime}$, where $g \in \mathrm{G}$ and $g^{\prime} \in \mathrm{G}^{\prime}$. Then, as in (158) and (159),

$$
\chi_{x}(s . w)=e^{i B(x, g \cdot y)} \quad \text { and } \quad \phi^{\mathrm{G}}(s . w)=\phi^{\mathrm{G}}\left(g^{\prime} \cdot w\right)
$$

Since $l>l^{\prime}$, equation (I.2) implies that there is a constant $C_{2}$ such that

$$
\mu_{\mathcal{O}(w)}\left(\chi_{x} \phi^{\mathrm{G}}\right)=C_{2} \int_{\mathrm{G}} e^{i B(x, g \cdot y)} d g \int_{\mathrm{G}^{\prime} / \mathrm{H}^{\prime}} \phi^{\mathrm{G}}\left(g^{\prime} \cdot w\right) d\left(g^{\prime} \mathrm{H}^{\prime}\right) .
$$ By (151) in Lemma 20 and [MPP20, (34)], we obtain for some constants $C_{k}$

$$
\begin{align*}
& \pi_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w  \tag{161}\\
& =C_{3} \sum_{t W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}) \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g})) / W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} \cdot x\right) \int_{\tau^{\prime}\left(\mathfrak{h} \overline{\mathrm{T}}^{r e g}\right)} e^{i B(x, t \cdot y)} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}}(y) \\
& \quad \times \int_{\mathbf{G}^{\prime} / \mathrm{H}^{\prime}} \phi^{\mathrm{G}}\left(g^{\prime} \cdot w\right) d\left(g^{\prime} \mathrm{H}^{\prime}\right) d y \\
& =C_{4} \sum_{t W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}) \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g})) / W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} \cdot x\right) \int_{\tau^{\prime}\left(\mathfrak{( ) ^ { - }}\right.} e^{i B(x, t \cdot y)} F_{\phi^{\mathrm{G}}}(y) d y .
\end{align*}
$$

Since $F_{\phi^{\mathrm{G}}}=\operatorname{vol}(\mathrm{G}) F_{\phi}=F_{\phi}$, the formula follows.
Lemma 24. Suppose $l \leq l^{\prime}$. Then there is a seminorm $q$ on $\mathcal{S}(\mathrm{W})$ such that

$$
\left|\int_{\mathfrak{h} \cap \tau(\mathrm{W})} F_{\phi}(y) e^{i B(x, y)} d y\right| \leq q(\phi) \operatorname{ch}(x)^{-d^{\prime}+r-1} \quad(x \in \mathfrak{h}, \phi \in \mathcal{S}(\mathrm{~W}))
$$

Proof. The boundedness of the function $T(\widetilde{g})(\phi), \tilde{g} \in \widetilde{\mathrm{G}}$, means that there is a seminorm $q(\phi)$ on $\mathcal{S}(\mathfrak{g})$ such that

$$
\begin{equation*}
\left|\Theta(\tilde{c}(x)) \int_{\mathrm{W}} \chi_{x}(w) \phi(w) d w\right| \leq q(\phi) \quad(x \in \mathfrak{g}) \tag{162}
\end{equation*}
$$

Equivalently, replacing $q(\phi)$ with a constant multiple of $q(\phi)$, and using (138), (141) and (142), we see that

$$
\begin{equation*}
\left|\int_{\mathrm{W}} \chi_{x}(w) \phi(w) d w\right| \leq q(\phi) \operatorname{ch}^{-d^{\prime}}(x) \quad(x \in \mathfrak{g}) \tag{163}
\end{equation*}
$$

Since $l \leq l^{\prime}$, Lemma 22 together with (163) prove that (again up to a multiplicative constant that can be absorbed by $q(\phi))$,

$$
\left|\int_{\mathfrak{h} \cap \tau(\mathrm{W})} F_{\phi}(y) e^{i B(x, y)} d y\right| \leq q(\phi)\left|\pi_{\mathfrak{g} / \mathfrak{h}}(x)\right| \operatorname{ch}(x)^{-d^{\prime}}
$$

Recall the constants $r$ and $\iota$ from (66) and (64). Then, as one can verify from (A.1),

$$
\begin{equation*}
\max \left\{\operatorname{deg}_{y_{j}} \pi_{\mathfrak{g} / \mathfrak{h}} ; 1 \leq j \leq l\right\}=\frac{1}{l}(r-1) \tag{164}
\end{equation*}
$$

where $\operatorname{deg}_{y_{j}} \pi_{\mathfrak{g} / \mathfrak{h}}$ denotes the degree of $\pi_{\mathfrak{g} / \mathfrak{h}}(y)$ with respect to the variable $y_{j}$.
Also, (164) and (149) imply that

$$
\left|\pi_{\mathfrak{g} / \mathfrak{h}}(x)\right| \leq C_{5} \operatorname{ch}^{r-1}(x) \leq C_{5} \operatorname{ch}^{r-\iota}(x) \quad(x \in \mathfrak{h}),
$$

where $C_{5}$ is a constant. Thus, the claim follows.
Lemmas 22 and 23 allow us to restate Corollary 18 in terms of orbital integrals on the symplectic space $W$.

Corollary 25. Suppose $l \leq l^{\prime}$. Then for any $\phi \in \mathcal{S}(\mathrm{W})$

$$
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \int_{\mathfrak{h} \cap \tau(\mathrm{W})} e^{i B(x, y)} F_{\phi}(y) d y d x
$$

where $C$ is a constant that depends only on the dual pair ( $\mathrm{G}, \mathrm{G}^{\prime}$ ) and each consecutive integral is absolutely convergent.

Proof. The equality is immediate from Corollary 18 and Lemma 22. The absolute convergence of the outer integral over $\mathfrak{h}$ follows from Lemma 24 .

Corollary 26. Suppose $l>l^{\prime}$. Then for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$
\begin{array}{r}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \sum_{s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(s) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-s \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \\
\times \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}(x) \int_{\tau^{\prime}(\mathfrak{( \mathfrak { r }}}{ }^{\text {reg })} \\
\end{array} e^{i B(x, y)} F_{\phi}(y) d y d x,
$$

where $C$ is a constant that depends only on the dual pair ( $\mathrm{G}, \mathrm{G}^{\prime}$ ) and each consecutive integral is absolutely convergent.

Proof. The formula is immediate from Corollary 18, Lemma 23 and formula (146):

$$
\begin{aligned}
& \frac{1}{\kappa_{0}} \int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g(\phi) \\
& =C_{1} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x)\left(\pi_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(x) \int_{\mathrm{W}} \chi_{x}(w) \phi^{\mathrm{G}}(w) d w\right) d x \\
& =C_{2} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{d^{-}-r-\iota}(x)} \\
& \times\left(\sum_{t W(\mathbb{Z}, \mathfrak{h}(\mathfrak{g}) \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g})) / W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} \cdot x\right) \int_{\left.\tau^{\prime}(\mathfrak{h})_{\bar{T}}^{r e g}\right)} e^{i B(x, t \cdot y)} F_{\phi}(y) d y\right) d x \\
& =\frac{C_{2} \check{\chi}_{\Pi}(\tilde{c}(0))}{|W(\mathrm{Z}, \mathfrak{h}(\mathfrak{g}))|} \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \\
& \times\left(\sum_{t \in W(\mathbf{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} \cdot x\right) \int_{\tau^{\prime}\left(\mathfrak{h}_{\bar{T}}^{\text {reg }}\right)} e^{i B(x, t . y)} F_{\phi}(y) d y\right) d x \\
& =C_{3} \check{\chi}_{\Pi}(\tilde{c}(0)) \sum_{t \in W(\mathbf{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}\left(\widehat{c_{-}}(t . x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(t . x) \\
& \times\left(\pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}(x) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)} e^{i B(t . x, x . t y)} F_{\phi}(y) d y\right) d x \\
& =C_{3} \check{\chi}_{\Pi}(\tilde{c}(0)) \sum_{t \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-t^{-1} \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \\
& \times\left(\pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}(x) \int_{\tau^{\prime}\left(\mathfrak{h} \overline{\mathbf{T}}^{\text {reg }}\right)} e^{i B(x, y)} F_{\phi}(y) d y\right) d x .
\end{aligned}
$$

Let $G^{\prime \prime}$ be the isometry group of the restriction of the form $(\cdot, \cdot)$ to $\mathrm{V}_{\overline{0}}^{0,0}$ and let $\mathfrak{h}^{\prime \prime}=$ $\sum_{j=l^{\prime}+1}^{l} \mathbb{R} J_{j}$. Then, as in (164), we check that

$$
\max \left\{\operatorname{deg}_{x_{j}} \pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})} ; 1 \leq j \leq l\right\}=\max \left\{\operatorname{deg}_{x_{j}} \pi_{\mathfrak{z}^{\prime \prime} / \mathfrak{h}^{\prime \prime}} ; l^{\prime}+1 \leq j \leq l\right\}=\frac{1}{\iota}\left(r^{\prime \prime}-1\right),
$$

where $r^{\prime \prime}=\frac{2 \operatorname{dim} \mathfrak{g}_{0}^{\prime \prime}}{\operatorname{dim} V_{\overline{0} \mathbb{R}}^{\delta 0}}$ is defined as in (65). Since $r-r^{\prime \prime}=d^{\prime}$, we see that

$$
\operatorname{ch}^{d^{\prime}-r-1}(x)\left|\pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}(x)\right| \leq \text { const } \operatorname{ch}^{d^{\prime}-r-1+r^{\prime \prime}-1}(x)=\text { const } \operatorname{ch}^{-2}(x)
$$

Furthermore, $F_{\phi}$ is absolutely integrable. Therefore, the absolute convergence of the last integral over $\mathfrak{h}(\mathfrak{g})$ follows from the fact that $\mathrm{ch}^{-2 \iota}$ is absolutely integrable.

To prove Theorem 4 (and Theorem 5), we still need the following explicit formula for the form $B(x, y)$. Let $\beta=\frac{2 \pi}{\iota}$, where $\iota$ is as in (64). Then

$$
\begin{equation*}
B(x, y)=-\beta \sum_{j=1}^{l} x_{j} y_{j} \quad\left(x=\sum_{j=1}^{l} x_{j} J_{j}, y=\sum_{j=1}^{l} y_{j} J_{j} \in \mathfrak{h}(\mathfrak{g})\right) . \tag{165}
\end{equation*}
$$

Indeed, the definition of the form $B,(154)$, shows that

$$
\begin{align*}
B(x, y) & =\pi \operatorname{tr}_{\mathbb{D} / \mathbb{R}}(x y)=\pi \sum_{j, k} \operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(J_{j} J_{k}\right) x_{j} y_{k} \\
& =\pi \sum_{j} \operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left(-1_{\mathrm{V} \frac{j}{0}}\right) x_{j} y_{j}=-\frac{2 \pi}{\iota} \sum_{j} x_{j} y_{j} . \tag{166}
\end{align*}
$$

Proof of Theorem 4. Notice that the degree of the polynomial $Q_{a_{j}, b_{j}}$ is $-a_{j}-b_{j}=2 \delta-2$ and is independent of $\mu$ and $j$. Explicitly,

$$
\begin{equation*}
2 \delta-2=\frac{1}{\iota}\left(d^{\prime}-r-\iota\right) \tag{167}
\end{equation*}
$$

(where $\iota=1 / 2$ if $\mathbb{D}=\mathbb{H}$ and 1 otherwise). Hence, by [MPP20, Theorem 3.6], the function $F_{\phi}$ has the required number of continuous derivatives for the formula (71) to make sense. The operators appearing in the integrand of (71) act on different variables and therefore commute. Also, the constants $a_{j}, b_{j}$ are integers by (148). Hence, equation (71) follows from Corollary 25, Lemma 19, formula (165), and Proposition D.5.

For the last statement about (72), we have

$$
d^{\prime}-r-\iota= \begin{cases}2 l^{\prime}-2 l & \text { if }\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right),  \tag{168}\\ 2 l^{\prime}-2 l-1 & \text { if }\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right), \\ l^{\prime}-l-1 & \text { if }\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right), p+q=l^{\prime}, \\ l^{\prime}-l-1 & \text { if }\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{Sp}_{l}, \mathrm{O}_{2 l^{\prime}}^{*}\right)\end{cases}
$$

Thus, since we assume $l \leq l^{\prime}$, the product (72) is a function if and only if $d^{\prime}-r-\iota<0$, i.e. if and only if $l=l^{\prime}$ and $\left(\mathrm{G}, \mathrm{G}^{\prime}\right) \neq\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}\right)$. Furthermore, (72) contains no derivatives (but terms involving $\delta_{0}$ are allowed) if and only if $d^{\prime}-r-\iota=0$, which corresponds to either $l=l^{\prime}$ and $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$, or $l^{\prime}=l+1$ and $\mathbb{D}=\mathbb{C}$ or $\mathbb{H}$. This completes the proof.

Suppose now $l>l^{\prime}$. Let $\mathfrak{h}^{\prime \prime}=\sum_{j=l^{\prime}+1}^{l} \mathbb{R} J_{j}$, so that

$$
\begin{equation*}
\mathfrak{h}(\mathfrak{g})=\mathfrak{h} \oplus \mathfrak{h}^{\prime \prime} \tag{169}
\end{equation*}
$$

Then the centralizer of $\tau\left(\mathfrak{h}_{\overline{1}}\right)$ coincides with the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ and is equal to $\mathfrak{z}=\mathfrak{h} \oplus \mathfrak{g}^{\prime \prime}$, where $\mathfrak{g}^{\prime \prime}$ is the Lie algebra of the group $G^{\prime \prime}$ of the isometries of the restriction of the form $(\cdot, \cdot)$ to $\bigvee \frac{0}{0}$. Furthermore, the derived Lie algebras of $\mathfrak{z}$ and $\mathfrak{g}^{\prime \prime}$ coincide (i.e. $\left.[\mathfrak{z}, \mathfrak{z}]=\left[\mathfrak{g}^{\prime \prime}, \mathfrak{g}^{\prime \prime}\right]\right)$ and $\mathfrak{h}^{\prime \prime}$ is a Cartan subalgebra of $\mathfrak{g}^{\prime \prime}$. We shall identify $\mathfrak{h}$ and $\mathfrak{h}^{\prime}$ by means of (43). This justifies writing $\mathfrak{h}(\mathfrak{g})=\mathfrak{h}^{\prime} \oplus \mathfrak{h}^{\prime \prime}$ when we need to emphasize the role of $\mathfrak{g}^{\prime}$.

Lemma 27. Suppose $l>l^{\prime}$. In terms of Corollary 26 and the decomposition (169)

$$
\begin{align*}
& \xi_{-s \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \pi_{\mathfrak{z}} / \mathfrak{h}(\mathfrak{g}) \\
&=\left(\xi_{-s \mu}\left(\widehat{c_{-}}\left(x^{\prime}\right)\right) \operatorname{ch}^{d^{\prime}-r-\iota}\left(x^{\prime}\right)\right)\left(\xi_{-s \mu}\left(\widehat{c_{-}}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{d^{\prime}-r-\iota}\left(x^{\prime \prime}\right) \pi_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right)\right), \tag{170}
\end{align*}
$$

where $x=x^{\prime}+x^{\prime \prime} \in \mathfrak{h}(\mathfrak{g})$, with $x^{\prime} \in \mathfrak{h}^{\prime}$ and $x^{\prime \prime} \in \mathfrak{h}^{\prime \prime}$. Moreover,

$$
\begin{align*}
& \int_{\mathfrak{h}^{\prime \prime}} \xi_{-s \mu}\left(\widehat{c_{-}}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{d^{\prime}-r-\iota}\left(x^{\prime \prime}\right) \pi_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right) d x^{\prime \prime} \\
&=C \sum_{s^{\prime \prime} \in W\left(\mathfrak{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right) \mathbb{I}_{\{0\}}\left(-\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}+s^{\prime \prime} \rho^{\prime \prime}\right), \tag{171}
\end{align*}
$$

where $C$ is a constant, $\rho^{\prime \prime}$ is one half times the sum of the positive roots for $\left(\mathfrak{g}_{\mathbb{C}}^{\prime \prime}, \mathfrak{h}_{\mathbb{C}}^{\prime \prime}\right)$ and $\mathbb{I}_{\{0\}}$ is the indicator function of zero.
Proof. Formula (170) is obvious, because $\pi_{\mathfrak{z} / \mathfrak{h}(\mathfrak{g})}\left(x^{\prime}+x^{\prime \prime}\right)=\pi_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right)$. We shall verify (171). Let $r^{\prime \prime}$ denote the number defined in (65) for the Lie algebra $\mathfrak{g}^{\prime \prime}$. A straightforward computation verifies the following table:

| $\mathfrak{g}$ | $r$ | $r^{\prime \prime}$ | $d^{\prime}-r+r^{\prime \prime}$ |
| :---: | :---: | :---: | :---: |
| $\mathfrak{u}_{d}$ | $d$ | $d-d^{\prime}$ | 0 |
| $\mathfrak{o}_{d}$ | $d-1$ | $d-d^{\prime}-1$ | 0 |
| $\mathfrak{s p}_{d}$ | $d+\frac{1}{2}$ | $d-d^{\prime}+\frac{1}{2}$ | 0 |

By (139) applied to $\mathrm{G}^{\prime \prime} \supseteq \mathrm{H}^{\prime \prime}$ and $\mathfrak{g}^{\prime \prime} \supseteq \mathfrak{h}^{\prime \prime}$,

$$
\pi_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right)=C_{1}^{\prime \prime} \Delta^{\prime \prime}\left(\widehat{c_{-}}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{r^{\prime \prime}-\iota}\left(x^{\prime \prime}\right) \quad\left(x^{\prime \prime} \in \mathfrak{h}^{\prime \prime}\right)
$$

where $\Delta^{\prime \prime}$ is the Weyl denominator for $\mathrm{G}^{\prime \prime}$,

$$
\begin{equation*}
\Delta^{\prime \prime}=\kappa_{0}^{\prime \prime} \sum_{s^{\prime \prime} \in W\left(\mathrm{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right) \xi_{s^{\prime \prime} \rho^{\prime \prime}} \tag{172}
\end{equation*}
$$

and

$$
\kappa_{0}^{\prime \prime}= \begin{cases}\frac{1}{2} & \text { if } \mathrm{G}^{\prime \prime}=\mathrm{O}_{d^{\prime \prime}} \text { where } d^{\prime \prime} \text { is even }  \tag{173}\\ 1 & \text { otherwise }\end{cases}
$$

Hence, by (149), the integral (171) is a constant multiple of

$$
\begin{equation*}
\int_{\mathfrak{h}^{\prime \prime}} \xi_{-s \mu}\left(\widehat{c_{-}}\left(x^{\prime \prime}\right)\right) \Delta^{\prime \prime}\left(\widehat{c_{-}}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{d^{\prime}-r+r^{\prime \prime}}\left(x^{\prime \prime}\right) \operatorname{ch}^{-2 \iota}\left(x^{\prime \prime}\right) d x^{\prime \prime}=2^{\operatorname{dim} \mathfrak{h}^{\prime \prime}} \int_{\widehat{c_{-}\left(\mathfrak{h}^{\prime \prime}\right)}} \xi_{-s \mu}(h) \Delta^{\prime \prime}(h) d h \tag{174}
\end{equation*}
$$

where $\widehat{c_{-}}\left(\mathfrak{h}^{\prime \prime}\right) \subseteq \widehat{\mathrm{H}^{\prime \prime \prime}}$.
Notice that the function

$$
\widehat{\mathrm{H}^{\prime \prime 0}} \ni h \rightarrow \xi_{-s \mu}(h) \Delta^{\prime \prime}(h) \in \mathbb{C}
$$

is constant on the fibers of the covering map

$$
\begin{equation*}
\widehat{\mathrm{H}^{\prime 0}} \rightarrow \mathrm{H}^{\prime \prime 0} \tag{175}
\end{equation*}
$$

Indeed, the covering (175) is non-trivial only in two cases, namely $\mathrm{G}^{\prime \prime}=\mathrm{O}_{2 l^{\prime \prime+1}}$ and $\mathrm{G}^{\prime \prime}=\mathrm{U}_{l^{\prime \prime}}$ with $l^{\prime \prime}$ even; see (127). In these cases, (172) shows that this claim is true provided that the weight $-s \mu+s^{\prime \prime} \rho^{\prime \prime}$ is integral for the Cartan subgroup $H^{\prime \prime}$ (i.e. it is equal to the derivative of a character of $\mathrm{H}^{\prime \prime}$ ).

Suppose $\mathrm{G}^{\prime \prime}=\mathrm{O}_{2 l^{\prime \prime}+1}$. Then $\mathrm{G}=\mathrm{O}_{2 l+1}, \lambda_{j} \in \mathbb{Z}$ and $\rho_{j} \in \mathbb{Z}+\frac{1}{2}$. Hence, $(-s \mu)_{j} \in \mathbb{Z}+\frac{1}{2}$. Since, $\rho_{j}^{\prime \prime} \in \mathbb{Z}+\frac{1}{2}$, we see that $(-s \mu)_{j}+\rho_{j}^{\prime \prime} \in \mathbb{Z}$.

Suppose now that $\mathrm{G}^{\prime \prime}=\mathrm{U}_{l^{\prime \prime}}$ with $l^{\prime \prime}$ even. Then $\mathrm{G}=\mathrm{U}_{l}$ and $(-s \mu)_{j} \in \mathbb{Z}+\frac{1}{2}$. In fact, if $l^{\prime}$ is even, i.e. $l=l^{\prime}+l^{\prime \prime}$ is even, then $\lambda_{j} \in \mathbb{Z}$ and $\rho_{j} \in \mathbb{Z}+\frac{1}{2}$. If $l^{\prime}$ is odd, i.e. $l=l^{\prime}+l^{\prime \prime}$ is odd, then $\lambda_{j} \in \mathbb{Z}+\frac{1}{2}$ and $\rho_{j} \in \mathbb{Z}$. Since $\rho_{j}^{\prime \prime} \in \mathbb{Z}+\frac{1}{2}$, in both cases, we conclude that $(-s \mu)_{j}+\rho_{j}^{\prime \prime} \in \mathbb{Z}$.

Therefore, (174) is a constant multiple of

$$
\begin{align*}
& \sum_{s^{\prime \prime} \in W\left(\mathfrak{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right) \int_{\mathrm{H}^{\prime \prime 0}} \xi_{-s \mu}(h) \xi_{s^{\prime \prime} \rho^{\prime \prime}}(h) d h  \tag{176}\\
&= \begin{cases}\operatorname{vol}\left(\mathrm{H}^{\prime \prime 0}\right) \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right) & \text { if }\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}=s^{\prime \prime} \rho^{\prime \prime}, \\
0 & \text { if }\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}} \notin W\left(\mathrm{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right) \rho^{\prime \prime},\end{cases} \\
&=\operatorname{vol}\left(\mathrm{H}^{\prime \prime 0}\right) \sum_{s^{\prime \prime} \in W\left(\mathfrak{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right) \mathbb{I}_{\{0\}}\left(-\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}+s^{\prime \prime} \rho^{\prime \prime}\right) .
\end{align*}
$$

Corollary 28. Suppose $l>l^{\prime}$ and keep the notation of Lemma 27. Then

$$
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=0
$$

unless there is $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ such that

$$
\begin{equation*}
\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime} . \tag{177}
\end{equation*}
$$

If $\mathrm{G}=\mathrm{O}_{2 l+1}$ or $\mathrm{Sp}_{l}$, then (177) is equivalent to

$$
\begin{equation*}
\left.\mu\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime} \quad \text { and }\left.\quad s\right|_{\mathfrak{h}^{\prime \prime}}=1 \tag{178}
\end{equation*}
$$

Suppose $\mathrm{G}=\mathrm{O}_{2 l}$ and write $\mathfrak{h}^{\prime \prime}=\mathfrak{h}_{0}^{\prime \prime} \oplus \mathbb{R} J_{l}$, where $\mathfrak{h}_{0}^{\prime \prime}=\sum_{j=l^{\prime}+1}^{l-1} \mathbb{R} J_{j}$. Then (177) is equivalent to

$$
\begin{equation*}
\left.\mu\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime},\left.\quad s\right|_{\mathfrak{h}_{0}^{\prime \prime}}=1, \quad \text { and }\left.\quad s\right|_{\mathbb{R} J_{l}}= \pm 1 \tag{179}
\end{equation*}
$$

Finally, if $\mathrm{G}=\mathrm{U}_{l}$, then (177) holds if and only if there is $j_{0} \in\left\{0,1, \ldots, l^{\prime}\right\}$ such that

$$
\begin{equation*}
\mu_{j_{0}+j}=\rho_{l^{\prime}+j}^{\prime \prime} \quad \text { and } \quad s\left(J_{j_{0}+j}\right)=J_{l^{\prime}+j} \quad\left(1 \leq j \leq l-l^{\prime}\right) \tag{180}
\end{equation*}
$$

Suppose that (177) holds. Then for any $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{align*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g(\phi)= & C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \sum_{s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g})),\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}=\rho^{\prime \prime}}} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(s) \\
& \times \int_{\mathfrak{h}^{\prime}} \xi_{-s \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}{ }^{\text {reg }}\right)} e^{i B(x, y)} F_{\phi}(y) d y d x \tag{181}
\end{align*}
$$

where $C$ is a non-zero constant which depends only on the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$, and each consecutive integral is absolutely convergent.

Proof. Observe that $B\left(x^{\prime}+x^{\prime \prime}, y\right)=B\left(x^{\prime}, y\right)$ for $x^{\prime} \in \mathfrak{h}^{\prime}, x^{\prime \prime} \in \mathfrak{h}^{\prime \prime}$ and $y \in \tau^{\prime}\left(\mathfrak{h}_{1}^{r e g}\right) \subseteq \mathfrak{h}^{\prime}$. We see therefore from Corollary 26 and Lemma 27 that

$$
\begin{align*}
& \int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g(\phi) \\
& =C \kappa_{0} \check{\chi} \Pi(\tilde{c}(0)) \sum_{s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \sum_{s^{\prime \prime} \in W\left(\mathrm{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(s) \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right) \mathbb{I}_{\{0\}}\left(-\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}+s^{\prime \prime} \rho^{\prime \prime}\right) \\
&  \tag{182}\\
& \quad \times \int_{\mathfrak{h}^{\prime}} \xi_{-s \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)} e^{i B(x, y)} F_{\phi}(y) d y d x . \quad 1
\end{align*}
$$

Notice that for $x \in \mathfrak{h}^{\prime}$ and $s^{\prime \prime} \in W\left(\mathrm{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)$, we have $s^{\prime \prime} x=x$. Thus $\xi_{-s \mu}\left(\widehat{c_{-}}(x)\right)=$ $\xi_{-s^{\prime \prime} s \mu}\left(\widehat{c_{-}}(x)\right)$ by (146). Notice also that $W\left(\mathrm{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right) \subseteq W(\mathrm{G}, \mathfrak{h})$ and $\operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right)=$ $\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}\left(s^{\prime \prime}\right)$. Moreover, $\mathbb{I}_{\{0\}}\left(-\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}+s^{\prime \prime} \rho^{\prime \prime}\right)=\mathbb{I}_{\{0\}}\left(-\left.\left(s^{\prime \prime-1} s \mu\right)\right|_{\mathfrak{h}^{\prime \prime}}+\rho^{\prime \prime}\right)$. Hence, replacing $s$ by $s^{\prime \prime} s$ in (182), we see that this expression is equal to

$$
\begin{align*}
C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g}))} & \sum_{s^{\prime \prime} \in W\left(\mathrm{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g}(\mathfrak{h}) / \mathfrak{h}}(s) \mathbb{I}_{\{0\}}\left(-\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}+\rho^{\prime \prime}\right) \\
& \times \int_{\mathfrak{h}^{\prime}} \xi_{-s \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}{ }^{r e g}\right)} e^{i B(x, y)} F_{\phi}(y) d y d x, \tag{183}
\end{align*}
$$

which yields (181), with a new non-zero constant $C$, equal to $C\left|W\left(\mathrm{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)\right|$. Clearly (183) is zero if there is no $s$ such that $\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}$. The absolute convergence of the integrals was checked in the proof of Corollary 26.

Recall that $\mathfrak{h}^{\prime \prime}=\sum_{j=l^{\prime}+1}^{l} \mathbb{R} J_{j}$ and $\mu=\lambda+\rho$ where $\lambda$ is the highest weight of the genuine representation $\Pi$; see Appendix H. If $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, then $\left.\rho\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}$. All coefficients of $\rho$ are positive and strictly decreasing by 1 except when $\mathrm{G}=\mathrm{O}_{2 l}$, where $\rho_{l}=0$. Hence $\left.s\right|_{\mathfrak{h}^{\prime \prime}}$ cannot contain sign changes when $\mathrm{G}=\mathrm{O}_{2 l+1}$ or $\mathrm{Sp}_{l}$, whereas $\left.s\right|_{\mathfrak{h}_{0}^{\prime \prime}}$ cannot contain sign changes when $\mathrm{G}=\mathrm{O}_{2 l}$. Using the form of the coefficients of $\lambda$, one easily sees that (177) is equivalent to (178) or (179). If $\mathrm{G}=\mathrm{U}_{l}$, then $\lambda=\frac{q-p}{2}+\nu$, where $\nu_{1} \geq \nu_{2} \geq \cdots \geq \nu_{l}$ are integers. Moreover,

$$
\begin{equation*}
\frac{q-p}{2}+\rho_{q+j}=\frac{l-p-q+1}{2}-j=\rho_{l^{\prime}+j}^{\prime \prime} \quad\left(1 \leq j \leq l-l^{\prime}\right) . \tag{184}
\end{equation*}
$$

The Weyl group $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ consists of permutations of the $J_{j}$ 's. Hence a genuine HarishChandra parameter $\mu$ satisfies (177) if and only if among its coefficients $\mu_{1}, \ldots, \mu_{l}$ we can find a string of $l-l^{\prime}$ successive coefficients $\mu_{j}$ equal to $\rho_{l^{\prime}+1}^{\prime \prime}, \ldots, \rho_{l}^{\prime \prime}$ and the permutation $s$ translates the corresponding string of $J_{j}$ 's onto $J_{l^{\prime}+1}, \ldots, J_{l}$. This proves (180).

In the next lemmas we study the integrals appearing on the right-hand side of (181).
Lemma 29. For $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ and $y \in \tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$, in the sense of distributions on $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}{ }^{\text {reg }}\right)$,

$$
\begin{equation*}
\int_{\mathfrak{h}^{\prime}} \xi_{-s \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) e^{i B(x, y)} d x=\left(\prod_{j=1}^{l^{\prime}} P_{a_{s, j}, b_{s, j}}\left(\beta y_{j}\right)\right) e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} \tag{185}
\end{equation*}
$$

where $a_{s, j}, b_{s, j}$ and $\beta$ are as in (78) and (67), and $P_{a, b}$ is defined in (D.4).
Proof. This follows immediately from Lemma 19, (D.5), and Proposition D.5, since $a_{s, j}+$ $b_{s, j}=-2 \delta+2 \geq 1$ for $l>l^{\prime}$.

Suppose that $\mu$ satisfies (177) for some $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$. The integral corresponding to $s$ in (181) vanishes when the intersection of the support of the right-hand side of (185) and $\tau^{\prime}\left(\mathfrak{h}_{\bar{T}}^{\text {reg }}\right)$ has an empty interior. We first study this intersection for some specific elements in $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$.

If $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, define $s_{0}=1$ as in (73). Then clearly $\left.s_{0} \mu\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}$ by (178). If $\mathbb{D}=\mathbb{C}$, fix $j_{0} \in\left\{0,1, \ldots, l^{\prime}\right\}$ as in (180) and define $s_{0, j_{0}}$ as the permutation in $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ given by

$$
s_{0, j_{0}}\left(J_{j}\right)= \begin{cases}J_{j} & \left(1 \leq j \leq j_{0}\right)  \tag{186}\\ J_{l^{\prime}-j_{0}+j} & \left(j_{0}+1 \leq j \leq j_{0}+l-l^{\prime}\right) \\ J_{j-l+l^{\prime}} & \left(j_{0}+l-l^{\prime}+1 \leq j \leq l\right)\end{cases}
$$

i.e.


Equivalently,

$$
\left(s_{0, j_{0}} \mu\right)_{j}=\mu_{s_{0, j_{0}}^{-1}(j)}= \begin{cases}\mu_{j} & \left(1 \leq j \leq j_{0}\right)  \tag{187}\\ \mu_{l-l^{\prime}+j} & \left(j_{0}+1 \leq j \leq l^{\prime}\right) \\ \mu_{j_{0}-l^{\prime}+j} & \left(l^{\prime}+1 \leq j \leq l\right)\end{cases}
$$

Hence $\left.\left(s_{0, j_{0}} \mu\right)\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}$. Notice that $s_{0, p}$ is the element $s_{0}$ defined in (74).
Lemma 30. Let $l>l^{\prime}$ and suppose that $\mu$ satisfies (177). Let $s_{0}=1$, as in (73), if $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, and let $s_{0, j_{0}}$ be as in (186) if $\mathbb{D}=\mathbb{C}$.

If $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$, then

$$
\begin{equation*}
\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}}\left(\beta y_{j}\right)=(2 \pi)^{l^{\prime}} \prod_{j=1}^{l^{\prime}} P_{a_{j}, b_{j}, 2}\left(\beta y_{j}\right) \mathbb{I}_{\mathbb{R}^{+}}\left(y_{j}\right) \quad\left(y=\sum_{j=1}^{l^{\prime}} y_{j} J_{j}^{\prime} \in \mathfrak{h}^{\prime}\right) \tag{188}
\end{equation*}
$$

has support equal to $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$.
If $\mathbb{D}=\mathbb{C}$, then

$$
\begin{gather*}
\prod_{j=1}^{l^{\prime}} P_{a_{s_{0, j}, j}, j, b_{s_{0, j_{0}}, j}}\left(\beta y_{j}\right)=(2 \pi)^{l^{\prime}}\left(\prod_{j=1}^{j_{0}} P_{a_{j}, b_{j}, 2}\left(\beta y_{j}\right) \mathbb{I}_{\mathbb{R}^{+}}\left(y_{j}\right)\right)\left(\prod_{j=j_{0}+1}^{l^{\prime}} P_{a_{j+l-l^{\prime}}, b_{j+l-l^{\prime},-2}}\left(\beta y_{j}\right) \mathbb{I}_{\mathbb{R}^{-}}\left(y_{j}\right)\right) \\
\left(y=\sum_{j=1}^{l^{\prime}} y_{j} J_{j}^{\prime} \in \mathfrak{h}^{\prime}\right) \tag{189}
\end{gather*}
$$

has support equal to $\left(\sum_{j=1}^{j_{0}} \mathbb{R}^{+} J_{j}^{\prime}\right) \oplus\left(\sum_{j=j_{0}+1}^{l^{\prime}} \mathbb{R}^{-} J_{j}^{\prime}\right)$. This support is equal to $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ if $j_{0}=p$, whereas its intersection with $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ has empty interior if $j_{0} \neq p$.

Proof. Let $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$. By (67) and (167) and since $\left.\mu\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}=\left.\rho\right|_{\mathfrak{h}^{\prime \prime}}$, we see that

$$
\mu_{1}>\cdots>\mu_{l^{\prime}}>\mu_{l^{\prime}+1}=\rho_{l^{\prime}+1}^{\prime \prime}=-\delta,
$$

These inequalities are equivalent to

$$
\begin{equation*}
a_{1}=-\mu_{1}-\delta+1<a_{2}=-\mu_{2}-\delta+1<\cdots<a_{l^{\prime}}=-\mu_{l^{\prime}}-\delta+1 \leq 0 \tag{190}
\end{equation*}
$$

because the $\mu_{j}$ 's and $\delta$ are either all in $\mathbb{Z}$ or all in $\mathbb{Z}+\frac{1}{2}$. Hence $P_{a_{j}, b_{j},-2}=0$ for all $1 \leq j \leq l^{\prime}$ by (D.2). Moreover, since $a_{j}+b_{j}=-2 \delta+2>2$, the polynomial $P_{a_{j}, b_{j}, 2}$ is nonzero for all $1 \leq j \leq l^{\prime}$. Hence the function on the right-hand side of (188) has support equal $\sum_{j=1}^{l^{\prime}} \mathbb{R}^{+} J_{j}^{\prime}=\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$.

Let now $\mathbb{D}=\mathbb{C}$. By (180), (184), (67) and (167),

$$
\begin{aligned}
& \mu_{1}>\mu_{2}>\cdots>\mu_{j_{0}}>\mu_{j_{0}+1}=\rho_{l^{\prime}+1}^{\prime \prime}=\frac{l-l^{\prime}-1}{2}=-\delta(>0) \\
& (0>) \delta=-\frac{l-l^{\prime}-1}{2}=\rho_{l}^{\prime \prime}=\mu_{j_{0}+l-l^{\prime}}>\mu_{j_{0}+l-l^{\prime}+1}>\cdots>\mu_{l}
\end{aligned}
$$

Since the $\mu_{j}$ 's and $\delta$ are either all in $\mathbb{Z}$ or all in $\mathbb{Z}+\frac{1}{2}$, these inequalities are equivalent to

$$
\begin{align*}
& a_{1}=-\mu_{1}-\delta+1<a_{2}=-\mu_{2}-\delta+1<\cdots<a_{j_{0}}=-\mu_{j_{0}}-\delta+1 \leq 0 \\
& 0 \geq b_{j_{0}+l-l^{\prime}+1}=\mu_{j_{0}+l-l^{\prime}+1}-\delta+1>\cdots>b_{l}=\mu_{l}-\delta+1 \tag{191}
\end{align*}
$$

Hence

$$
\begin{array}{rlll}
P_{a_{j}, b_{j},-2}=0 & \text { i.e. } & P_{a_{j}, b_{j}}\left(y_{j}\right)=2 \pi P_{a_{j}, b_{j}, 2}\left(y_{j}\right) \mathbb{I}_{\mathbb{R}^{+}}\left(y_{j}\right) & \left(1 \leq j \leq j_{0}\right) \\
P_{a_{j}, b_{j}, 2}=0 & \text { i.e. } & P_{a_{j}, b_{j}}\left(y_{j}\right)=2 \pi P_{a_{j}, b_{j},-2}\left(y_{j}\right) \mathbb{I}_{\mathbb{R}^{-}}\left(y_{j}\right) & \left(j_{0}+l-l^{\prime}+1 \leq j \leq l\right)
\end{array}
$$

The polynomials appearing in these expressions of $P_{a_{j}, b_{j}}$ are nonzero because $a_{j}+b_{j}=$ $-2 \delta+2>0$ for all $j$. By (41) and the convention on the symbols $\delta_{j}$ 's for the dual pair $\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right)$ with $l>l^{\prime}=p+q$, the claims on the support of the right-hand side of (189) follow.

Let $\mathbb{D}=\mathbb{C}$. Suppose that there is $s \in W(G, \mathfrak{h}(\mathfrak{g}))$ such that $\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}$ and that the string of coefficients of $\mu$ equal to those of $\rho^{\prime \prime}$, see (180), starts at $j_{0}+1$, where $j_{0} \in\left\{0,1, \ldots, l^{\prime}\right\}$. Then $s=s_{0, j_{0}}$ satisfies $\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}$. Lemma 30 shows that if $j_{0} \neq p$ then the intersection of the support of $\prod_{j=1}^{l^{\prime}} P_{a_{s_{0, j_{0}}, j}, b_{s_{0}, j_{0}}, j}$ with $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ has empty interior. We now prove that if $j_{0} \neq p$ the same holds for the support of $\prod_{j=1}^{l^{\prime}} P_{a_{s, j}, b_{s, j}}$ for every $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ such that $\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}=\rho^{\prime \prime}$.
Lemma 31. Let $\mathbb{D}=\mathbb{C}$. Suppose that $\mu$ and $s \in W(G, \mathfrak{h}(\mathfrak{g}))$ satisfy (180) for $j_{0} \in$ $\left\{0,1, \ldots, l^{\prime}\right\}$. If $j_{0} \neq p$, then the intersection of the support of $\prod_{j=1}^{l^{\prime}} P_{a_{s, j}, b_{s, j}}$ with $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ has empty interior.

Proof. Since

$$
s_{0, j_{0}}\left(J_{j_{0}+j}\right)=J_{l^{\prime}+j}, \quad s\left(J_{j_{0}+j}\right)=J_{l^{\prime}+j} \quad\left(1 \leq j \leq l-l^{\prime}\right)
$$

the composition $s^{-1} s_{0, j_{0}}$ fixes the elements of $\left\{J_{j_{0}+1}, \ldots, J_{j_{0}+l-l^{\prime}}\right\}$ and permutes those of $\left\{J_{1}, \ldots, J_{j_{0}}\right\} \cup\left\{J_{j_{0}+l-l^{\prime}+1}, \ldots, J_{l}\right\}$. Then $s^{-1}=\left(s^{-1} s_{0, j_{0}}\right) s_{0, j_{0}}^{-1}$ maps $\left\{J_{l^{\prime}+1}, \ldots, J_{l}\right\}$ onto $\left\{J_{j_{0}+1}, \ldots, J_{j_{0}+l-l^{\prime}}\right\}$ and hence $\left\{J_{1}, \ldots, J_{l^{\prime}}\right\}$ bijectively onto $\left\{J_{1}, \ldots, J_{j_{0}}\right\} \cup\left\{J_{j_{0}+l-l^{\prime}+1}, \ldots, J_{l}\right\}$. Therefore $\left\{(s \mu)_{j}=\mu_{s^{-1}(j)} ; 1 \leq j \leq l^{\prime}\right\}$ is a permutation of $\left\{\mu_{j} ; 1 \leq j \leq j_{0}\right\} \cup\left\{\mu_{j} ; j_{0}+l-\right.$ $\left.l^{\prime}+1 \leq j \leq l\right\}$. By (191), there are $j_{0}$ negative $a_{j}$ and $l^{\prime}-j_{0}$ negative $b_{j}$ for $1 \leq j \leq l^{\prime}$. The same is then true for the $a_{s, j}$ and the $b_{s, j}$. The support of $\prod_{j=1}^{l^{\prime}} P_{a_{s, j}, b_{s, j}}$ is therefore a
product (in some order) of $j_{0}$ copies of $\mathbb{R}^{+}$and $l^{\prime}-j_{0}$ copies of $\mathbb{R}^{-}$. Its intersection with $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ has therefore empty interior if $j_{0} \neq p$.

When the intersection of the support of $\prod_{j=1}^{l^{\prime}} P_{a_{s, j}, b_{s, j}}$ and $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ has empty interior, the integral on the right-hand side of (181) that corresponds to $s$ vanishes. Lemma 31 shows that every such integral is zero when $j_{0} \neq p$. This yields the following corollary. Recall that we are supposing that $q \geq p$.

Corollary 32. Suppose that $\Pi$ is a genuine representation of $\widetilde{\mathrm{U}}_{l}$ with Harish-Chandra parameter $\mu$ satisfying (180) for $j_{0} \in\left\{0,1, \ldots, l^{\prime}\right\}$. If $j_{0} \neq p$ then

$$
f_{\Pi \otimes \Pi^{\prime}}=\int_{\mathrm{U}_{l}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g=0
$$

Thus, if $\Pi$ is a genuine representation of $\widetilde{\mathrm{U}}_{l}$ which occurs in Howe's correspondence, then its highest weight must be of the form $\lambda=\sum_{j=1}^{l}\left(\frac{q-p}{2}+\nu_{j}\right) e_{j}$ where

$$
\nu_{1} \geq \nu_{2} \geq \cdots \geq \nu_{p} \geq \nu_{p+1}=\cdots=\nu_{l-q}=0 \geq \nu_{l-q+1} \geq \cdots \geq \nu_{l} .
$$

In the proof of Theorem 5 we will see that the condition on the highest weight of $\Pi$ is also sufficient for the nonvanishing of the intertwining distributions.

Because of Corollary 32, we can restrict ourselves to the case $j_{0}=p$ when $\mathrm{G}=\mathrm{U}_{l}$. In this case, to simplify notation, we will write $s_{0}$ instead of $s_{0, p}$. Hence

$$
\begin{equation*}
s_{0}=1 \quad(\text { if } \mathbb{D}=\mathbb{R} \text { or } \mathbb{H}) \quad \text { and } \quad s_{0}=s_{0, p} \quad(\text { if } \mathbb{D}=\mathbb{C}) \tag{192}
\end{equation*}
$$

Observe that this notation allows us to write

$$
\begin{equation*}
\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}}\left(\beta y_{j}\right)=(2 \pi)^{l^{\prime}} \prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta y_{j}\right) \mathbb{I}_{\delta_{j} \mathbb{R}^{+}}\left(y_{j}\right), \tag{193}
\end{equation*}
$$

which unifies (188) and (189).
Suppose that $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ satisfies (177) and $j_{0}=p$ if $\mathbb{D}=\mathbb{C}$. Then

$$
\begin{equation*}
\left.s s_{0}^{-1}\right|_{\mathfrak{h}^{\prime \prime}}=1 \quad \text { and } \quad s s_{0}^{-1}(\mathfrak{h})=\mathfrak{h} . \tag{194}
\end{equation*}
$$

The condition $s s_{0}^{-1}(\mathfrak{h})=\mathfrak{h}$ and the identification (43), allow us to consider $s s_{0}^{-1}$ as isomorphisms of $\mathfrak{h}^{\prime}$. In the following lemma we prove that such a $s$ contributes to the right-hand side of (181) if and only if $s s_{0}^{-1} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$. Moreover, in this case, the contribution from $s$ agree with that of $s_{0}$.

Lemma 33. Let $l>l^{\prime}$ and let $\mu$ and $s \in W(G, \mathfrak{h}(\mathfrak{g}))$ satisfy (177) with $j_{0}=p$ if $\mathbb{D}=\mathbb{C}$. The integral

$$
\begin{equation*}
\int_{\mathfrak{h}^{\prime}} \xi_{-s \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \int_{\tau^{\prime}\left(\mathfrak{h}_{\mathrm{T}} \text { reg }\right)} e^{i B(x, y)} F_{\phi}(y) d y d x \tag{195}
\end{equation*}
$$

is zero:
(a) if $\left.s s_{0}^{-1}\right|_{\mathfrak{h}}$ acts by some sign changes, when $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$,
(b) if $\left.s s_{0}^{-1}\right|_{\mathfrak{h}}$ does not stabilize $\left\{J_{1}, \ldots, J_{p}\right\}$ (and $\left\{J_{p+1}, \ldots, J_{l^{\prime}}\right\}$ ), when $\mathbb{D}=\mathbb{C}$.

Equivalently, by identifying $\mathfrak{h}$ and $\mathfrak{h}^{\prime}$ via (43), the integral (195) is zero unless ss ${ }_{0}^{-1} \in$ $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$. Moreover, (181) becomes: for any $\phi \in \mathcal{S}(\mathrm{W})$

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g(\phi)=C \kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\tau^{\prime}\left(\mathfrak{h}_{\mathrm{T}}{ }^{r e g}\right)}\left(\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}}\left(\beta y_{j}\right)\right) e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} F_{\phi}(y) d y \tag{196}
\end{equation*}
$$

where $C$ is a non-zero constant which depends on the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$.
Proof. Let $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$. Suppose that $s s_{0}^{-1}\left(J_{j}\right)=-J_{j}$ for some $j \in\left\{1, \ldots, l^{\prime}\right\}$. Then $(s \mu)_{j}=-\left(s_{0} \mu\right)_{j}$, which interchanges the $j$-th indices $a$ and $b$ of $s \mu$ and $s_{0} \mu$. Thus $P_{a_{s, j}, b_{s, j}}$ is supported in $\mathbb{R}^{-}$, and the support of (185) has a lower dimensional intersection with $\tau^{\prime}\left(\mathfrak{h}_{\bar{T}}\right)$.

The case $\mathbb{D}=\mathbb{C}$ is similar: if $s s_{0}^{-1}\left(J_{i}\right)=J_{j}$ where $1 \leq i \leq p<j \leq l^{\prime}$, then $(s \mu)_{j}=$ $\left(s_{0} \mu\right)_{i}$, which interchanges the $i$-th and $j$-th indices $a$ and $b$ of $s \mu$ and $s_{0} \mu$. The support of (185) has therefore a lower dimensional intersection with $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$.

By the above and by identifying $\mathfrak{h}$ and $\mathfrak{h}^{\prime}$ via (43), we can restrict the sum on the right-hand side of (181) to the set of $s \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ such that $\left.s s_{0}^{-1}\right|_{\mathfrak{h}} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$ and $\left.s s_{0}^{-1}\right|_{\mathfrak{h}^{\prime \prime}}=1$. Therefore, the sum can be parametrized by $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$. By (185) and since $\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}\left(s s_{0}^{-1}\right)=\operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s s_{0}^{-1}\right)$, we obtain that $\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g(\phi)$ is $\kappa_{0} \check{\chi}_{\Pi}(\tilde{c}(0))$ times a constant multiple of

$$
\begin{aligned}
& \sum_{s^{\prime} \in W\left(\mathfrak{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \int_{\mathfrak{h}^{\prime}} \xi_{-s^{\prime} s_{0} \mu}\left(\widehat{c_{-}}(x)\right) \operatorname{ch}^{d^{\prime}-r-\iota}(x) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{\mathfrak{T}}}^{r e g}\right)} e^{i B(x, y)} F_{\phi}(y) d y d x \\
&=\sum_{s^{\prime} \in W\left(\mathfrak{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{\mathrm{T}}}{ }^{r e g}\right)}\left(\prod_{j=1}^{l^{\prime}} P_{a_{s^{\prime} s_{0}, j}, b_{s^{\prime} s_{0}, j}}\left(\beta y_{j}\right)\right) e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} F_{\phi}(y) d y
\end{aligned}
$$

Observe that

$$
\prod_{j=1}^{l^{\prime}} P_{a_{s^{\prime} s_{0}, j}, b_{s^{\prime} s_{0}, j}}\left(\beta y_{j}\right)=\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}}\left(\beta\left(s^{\prime-1} y\right)_{j}\right)
$$

because $s^{\prime} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$ permutes the indices $1 \leq j \leq l^{\prime}$. Recall also that $F_{\phi}(y)$ transforms as the sign representation with respect to the action of $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$. Formula (196) therefore follows. The new non-zero constant $C$ is the one appearing in (181) times $\left|W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)\right|$ times $\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}\left(s_{0}\right)$, which is equal to 1 if $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$ and $(-1)^{q\left(l-l^{\prime}\right)}$ if $\mathbb{D}=\mathbb{C}$.

Proof of Theorem 5. The conditions on the highest weight $\lambda$ rephrase the condition (177) for the Harish-Chandra parameter $\mu$ of $\Pi$ determined in Corollary 28 and its proof. The case of $\mathbb{D}=\mathbb{C}$ was further considered in Corollary 32. Hence $\int_{-G^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g=0$ unless $\lambda$ satisfies them.

Formula (76) holds because, by Lemma 30, it is a restatement of (196).
By (193), the function $\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}}\left(\beta y_{j}\right)$ has support equal to $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ and we can rewrite the right-hand side of (196) as a constant multiple of

$$
\begin{equation*}
\int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)}\left(\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta y_{j}\right)\right) e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} F_{\phi}(y) d y \tag{197}
\end{equation*}
$$

By the $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-skew-invariance of $F_{\phi}$, we can replace the term

$$
\left(\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta y_{j}\right)\right) e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|}
$$

in the integral (197) by its $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-skew-invariant component

$$
\begin{equation*}
\left(\frac{1}{\left|W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)\right|} \sum_{s^{\prime} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta\left(s^{\prime} y\right)_{j}\right)\right) e^{-\beta \sum_{j=1}^{l^{\prime}\left|y_{j}\right|}} \tag{198}
\end{equation*}
$$

Here we have used that $\sum_{j=1}^{l^{\prime}}\left|\left(s^{\prime} y\right)_{j}\right|=\sum_{j=1}^{l^{\prime}}\left|y_{j}\right|$. Notice that

$$
\prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta\left(s^{\prime} y\right)_{j}\right)=\prod_{j=1}^{l^{\prime}} P_{a_{s^{\prime}-1} s_{s_{0}, j}, b_{s^{\prime}-1}} s_{s_{0}, j} 2 \delta_{j}\left(\beta y_{j}\right)
$$

because $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$ only permutes the $y$-coordinates for which the $\delta_{j}$ 's have equal sign. Moreover, (198) is non-zero because $P_{a_{s^{\prime}-1} s_{0, j}, j^{\prime}, b_{s^{\prime}-1} \mu_{s_{0}, j}, 2 \delta_{j}}\left(\beta y_{j}\right)$ is not $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant when $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right) \neq 1$. Indeed, the condition $\mu_{1}>\mu_{2}>\cdots>\mu_{l^{\prime}}$ implies $b_{1}>b_{2}>\cdots>b_{l^{\prime}}$ and $a_{1}<a_{2}<\cdots<a_{l^{\prime}}$. If $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right) \neq 1$, then there are at least two indices $j \neq j^{\prime}$ such that $\delta_{j}=\delta_{j^{\prime}}$ and the corresponding factors in (198) have different degrees. (If $b \geq 1$ then the degree of $P_{a, b, 2}$ is $b-1$ and if $a \geq 1$ then that of $P_{a, b,-2}$ is $a-1$.) Recall from [MPP20, Definition 3.2] that

$$
\begin{equation*}
F_{\phi}(y)=C_{\mathfrak{h}_{\bar{T}}} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y) \int_{\mathrm{S}^{\prime} \mathrm{S}_{\overline{\overline{1}}}} \phi(s . y) d\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{1}}}\right) \quad\left(\phi \in \mathcal{S}(\mathrm{W}), y \in \tau\left(\mathfrak{h}_{\overline{1}}^{r e g}\right)\right), \tag{199}
\end{equation*}
$$

where $\mathrm{S}^{\mathfrak{h}}{ }^{\boldsymbol{T}}$ is the centralizer of $\mathfrak{h}_{\overline{1}}$ in S and $C_{\mathfrak{h}_{\bar{T}}}$ is an explicit constant. By (199), [MPP20, Lemma 2.1] and the definition of $C_{\mathfrak{h}_{\bar{T}}}$,

$$
\begin{align*}
& C_{\mathfrak{h}_{\bar{T}}} \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)} \frac{\sum_{s^{\prime} \in W\left(\mathfrak{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta\left(s^{\prime} y\right)_{j}\right)}{\pi_{\mathfrak{g} / \mathfrak{z}}(y)} e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} \\
& \times \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y) \pi_{\mathfrak{g} / \mathfrak{\mathfrak { j }}}(y) \int_{\mathrm{S} / \mathrm{S}^{\mathfrak{b}_{\overline{1}}}} \phi(s . y) d\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{1}}}\right) d y \\
& =i^{\operatorname{dim} \mathfrak{g} / \mathfrak{h}} \int_{\tau^{\prime}\left(\mathfrak{h}_{\bar{T}}{ }^{\text {reg }}\right)} \frac{\sum_{s^{\prime} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta\left(s^{\prime} y\right)_{j}\right)}{\pi_{\mathfrak{g} / \mathfrak{s}}(y)} e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} \\
& \times\left|\pi_{\mathfrak{s}_{0} / h^{\frac{2}{1}}}\left(w^{2}\right)\right| \int_{\mathrm{S} / \mathrm{S}^{\mathfrak{h}} \overline{\mathrm{T}}} \phi(s . y) d\left(s S^{\mathfrak{h} \overline{1}}\right) d y \quad\left(w \in \mathfrak{h}_{\overline{1}}^{r e g}, y=\tau^{\prime}(w)\right) . \tag{200}
\end{align*}
$$

We now prove that

$$
\begin{equation*}
\frac{\sum_{s^{\prime} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta\left(s^{\prime} \tau^{\prime}(w)\right)_{j}\right)}{\pi_{\mathfrak{g} / \mathfrak{s}}\left(\tau^{\prime}(w)\right)} e^{-\beta \sum_{j=1}^{l^{\prime}}\left|\tau^{\prime}(w)_{j}\right|} \tag{201}
\end{equation*}
$$

is the restriction to $\mathfrak{h}_{\overline{1}}^{\text {reg }}$ of a locally integrable function, say $\Phi$, on W.
Using (A.3) and (A.4), we see that there is a non-zero constant $C_{\mathfrak{z}}$, depending of (G, G'), such that

$$
\begin{equation*}
\pi_{\mathfrak{g} / \mathfrak{z}}(y)=C_{\mathfrak{z}} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y) \operatorname{det}(y)_{\mathrm{V}^{\prime}}^{\gamma} \quad\left(y=\tau(w)=\tau^{\prime}(w), w \in \mathfrak{h}_{\overline{1}}\right) \tag{202}
\end{equation*}
$$

where

$$
\gamma= \begin{cases}l-l^{\prime} & \text { if } \mathbb{D}=\mathbb{C}  \tag{203}\\ l-l^{\prime}+\frac{1}{2} & \text { if } \mathbb{D}=\mathbb{H} \\ l-l^{\prime}-\frac{1}{2} & \text { if } \mathbb{D}=\mathbb{R} \text { and } g=\mathfrak{s o}_{2 l} \\ l-l^{\prime} & \text { if } \mathbb{D}=\mathbb{R} \text { and } g=\mathfrak{s o}_{2 l+1}\end{cases}
$$

and $\operatorname{det}\left(g^{\prime}\right)_{\mathrm{V}^{\prime}}$ denotes the determinant of $g^{\prime}$ as an element of $\mathrm{G}^{\prime} \subseteq \mathrm{GL}_{\mathbb{D}}\left(\mathrm{V}^{\prime}\right)$. (See the remark after (E.11) in Appendix E for the case $\mathbb{D}=\mathbb{H}$.)

The polynomial in parenthesis in (198) is $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-skew-invariant. Hence it is divisible by $\pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y)$. Therefore the fraction

$$
\begin{equation*}
\frac{\sum_{s^{\prime} \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(s^{\prime}\right) \prod_{j=1}^{l^{\prime}} P_{a_{s_{0}, j}, b_{s_{0}, j}, 2 \delta_{j}}\left(\beta\left(s^{\prime} y\right)_{j}\right)}{\pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y)} \quad\left(y \in \mathfrak{h}^{\prime}\right) \tag{204}
\end{equation*}
$$

is a $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant polynomial. By Chevalley's restriction theorem, see e.g. [Wal88, 3.1.3], it extends to a $\mathrm{G}^{\prime}$-invariant polynomial on $\mathfrak{g}^{\prime}$. Thus its pullback by $\tau^{\prime}$ is a $\mathrm{GG}^{\prime}$ invariant polynomial on W .

We now consider the term $e^{-\beta \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|}$. Notice that for $w=\sum_{j=1}^{l^{\prime}} w_{j} u_{j} \in \mathfrak{h}_{\overline{1}}$,

$$
\sum_{j=1}^{l^{\prime}}\left|y_{j}\right|=\sum_{j=1}^{l^{\prime}}\left|J_{j}^{\prime *}\left(\tau^{\prime}(w)\right)\right|=\sum_{j=1}^{l^{\prime}} w_{j}^{2}=\sum_{j=1}^{l^{\prime}} \delta_{j} J_{j}^{\prime *}\left(\tau^{\prime}(w)\right)=\left(\sum_{j=1}^{l^{\prime}} \delta_{j} J_{j}^{\prime *}\right) \circ \tau^{\prime}(w)
$$

This is a quadratic polynomial on $\mathfrak{h}_{\overline{1}}$, invariant under the Weyl group $W\left(\mathrm{~S}, \mathfrak{h}_{\overline{1}}\right)$. Such a polynomial has no $\mathrm{GG}^{\prime}$-invariant extension to W , unless $\mathrm{G}^{\prime}$ is compact.

Indeed, suppose $P$ is a real-valued $\mathrm{GG}^{\prime}$-invariant polynomial on W such that

$$
P(w)=\left(\sum_{j=1}^{l^{\prime}} \delta_{j} J_{j}^{\prime *}\right) \circ \tau^{\prime}(w) \quad\left(w \in \mathfrak{h}_{\overline{1}}\right)
$$

Then $P$ extends uniquely to a complex-valued $\mathrm{G}_{\mathbb{C}} \mathrm{G}_{\mathbb{C}}^{\prime}$-invariant polynomial on the complexification $\mathrm{W}_{\mathbb{C}}$ of W . Hence, by the Classical Invariant Theory, [How89a, Theorems 1A and 1 B$]$ there is a $\mathrm{G}_{\mathbb{C}}^{\prime}$-invariant polynomial $Q$ on $\mathfrak{g}_{\mathbb{C}}^{\prime}$ such that $P=Q \circ \tau^{\prime}$. Hence,

$$
Q\left(\tau^{\prime}(w)\right)=P(w)=\left(\sum_{j=1}^{l^{\prime}} \delta_{j} J_{j}^{\prime *}\right) \circ \tau^{\prime}(w) \quad\left(w \in \mathfrak{h}_{\overline{1}}\right)
$$

Since $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}\right)$ spans $\mathfrak{h}^{\prime}$, we see that the restriction of $Q$ to $\mathfrak{h}^{\prime}$ is

$$
\left.Q\right|_{\mathfrak{h}^{\prime}}=\sum_{j=1}^{l^{\prime}} \delta_{j} J_{j}^{\prime *} \in \mathfrak{h}_{\mathbb{C}}^{\prime}
$$

Since $Q$ is $\mathrm{G}_{\mathbb{C}^{\prime}}^{\prime}$ invariant, the restriction $\left.Q\right|_{\mathfrak{h}^{\prime}}$ has to be invariant under the corresponding Weyl group. There are no linear invariants if $\mathrm{G}^{\prime}=\mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})$ or $\mathrm{O}_{2 l^{\prime}}^{*}$. Therefore $\mathrm{G}^{\prime}=\mathrm{U}_{p, q}$, $p+q=l^{\prime}$. But in this case the invariance means that all the $\delta_{j}$ are equal. Hence $\mathrm{G}^{\prime}=\mathrm{U}_{l^{\prime}}$ is compact. In the case $\mathrm{G}^{\prime}=\mathrm{U}_{l^{\prime}}$, the sum of squares coincides with $\langle J(w), w\rangle$ for a positive complex structure $J$ on W which commutes with G and $\mathrm{G}^{\prime}$ and therefore

$$
\begin{equation*}
e^{-\beta \sum_{j=1}^{l^{\prime}}\left|\delta_{j} J_{j}^{\prime *}\left(\tau^{\prime}(w)\right)\right|} \tag{205}
\end{equation*}
$$

extends to a Gaussian on $W$. If $\mathrm{G}^{\prime}$ is not compact then (205) extends to a $\mathrm{GG}^{\prime}$-invariant function on W , which is bounded but is not a Gaussian.

Finally, the function $\operatorname{det}(y)_{\mathrm{V}^{\prime}}^{\gamma}$ appearing in (202) is the restriction to $\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)$ of the $\mathrm{G}^{\prime}-$ invariant polynomial $x \rightarrow \operatorname{det}(x)_{\mathrm{V}^{\prime}}^{\gamma}$ on $\mathfrak{g}^{\prime}$. Its pullback by $\tau^{\prime}$ is a $\mathrm{GG}^{\prime}$-invariant polynomial on W.

The above constructions define the $\mathrm{GG}^{\prime}$-invariant function $\Phi$ on W restricting to (201) on $\mathfrak{h}_{\overline{1}}^{\text {reg }}$. It remains to prove that $\Phi$ is locally integrable. For this, it is enough to prove that the integral $\int_{\mathrm{W}} \Phi(w) \phi(w) d w$ is finite for every $\phi \in C_{c}^{\infty}(\mathrm{W})$. Weyl's integration formula on W with $l>l^{\prime}$, see (54), together with the $\mathrm{GG}^{\prime}$-invariance of $\Phi$, yields

$$
\begin{equation*}
\int_{\mathrm{W}} \Phi(w) \phi(w) d w=\int_{\tau^{\prime}\left(\mathfrak{h}_{\bar{T}}^{r e g}\right)} \Phi(y)\left|\pi_{\mathfrak{S}_{\sigma} / \mathfrak{h}_{\overline{\mathrm{T}}}^{2}}\left(w^{2}\right)\right| \int_{\mathrm{S} / \mathrm{S}^{\mathfrak{h}_{\bar{T}}}} \phi(s . y) d\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right) d y \quad\left(y=w^{2}\right) \tag{206}
\end{equation*}
$$

which coincides up to a non-zero constant with the finite integral $\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g(\phi)$ we started with.

It is still left to prove that if the highest weight $\lambda$ of $\Pi$ satisfies the conditions (a) or (b), then the integral (75) is nonzero. Let

$$
\tau^{\prime *}: \mathcal{S}\left(\mathfrak{g}^{\prime}\right) \ni \psi \rightarrow \psi \circ \tau^{\prime} \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}
$$

be the pullback by $\tau^{\prime}$. According to [MPP20, (25)], there is a surjective map $\tau_{*}^{\prime}: \mathcal{S}(\mathrm{W})^{\mathrm{G}} \rightarrow$ $\mathcal{S}\left(\mathfrak{g}^{\prime}\right)$ such that

$$
\tau^{\prime *} \circ \tau_{*}^{\prime}(\phi)=\phi \quad\left(\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}\right) .
$$

By [MPP20, Theorem 3.4 and (39)], since $l>l^{\prime}$,

$$
F_{\phi}(y)=C_{\mathfrak{h}_{\bar{T}}}^{\prime} \pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(y) \int_{\mathrm{G}^{\prime} / \mathrm{H}^{\prime}} \tau_{*}^{\prime}(\phi)\left(g^{\prime} \cdot y\right) d\left(g^{\prime} \mathrm{H}^{\prime}\right)
$$

where $\mathrm{H}^{\prime} \subseteq \mathrm{G}^{\prime}$ is the Cartan subgroup corresponding to $\mathfrak{h}^{\prime}$ and $C_{\mathfrak{h}_{\bar{T}}}^{\prime}$ is a suitable nonzero constant. Observe that $\Phi(y)$ is a not-identically zero $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$-invariant real-valued continuous function. (Recall that $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)=W\left(\mathrm{~K}^{\prime}, \mathfrak{h}^{\prime}\right)$, where $\mathrm{K}^{\prime}$ is maximal compact in $\mathrm{G}^{\prime}$, hence a unitary group.) Therefore (200) can be written as

$$
\begin{equation*}
\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g(\phi)=C^{\prime} \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{1}}^{\text {reg }}\right)} \Phi(y) \pi_{\mathfrak{g} / \mathfrak{\mathfrak { s }}}(y) F_{\phi}(y) d y \quad\left(w \in \mathfrak{h}_{\overline{1}}^{\text {reg }}, y=\tau^{\prime}(w)\right) . \tag{207}
\end{equation*}
$$

Thus Corollary 2 shows that this integral does not vanish for a suitably chosen $\phi \in \mathcal{S}(\mathrm{W})$.
The proof is now complete.

## 11. The special case for the pair $\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $l \leq l^{\prime}$

Here we consider the case $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ and suppose that the character $\Theta_{\Pi}$ is not supported on the preimage of the connected identity component $\widetilde{\mathrm{G}^{0}}$. Since the case $l>l^{\prime}$ was considered in Theorem 6 and the dual pair $\left(\mathrm{O}_{2}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ was treated in section 6 , we will suppose in the following that $2<l \leq l^{\prime}$. Recall the element $s \in \mathrm{G}$, (86), with centralizer in $\mathfrak{h}$ equal to $\mathfrak{h}_{s}=\sum_{j=1}^{l-1} \mathbb{R} J_{j}$, and the spaces

$$
\mathrm{V}_{\overline{0}, s}=\mathrm{V}_{\overline{0}}^{1} \oplus \mathrm{~V}_{\overline{0}}^{2} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l-1} \oplus \mathbb{R} v_{2 l}, \quad \mathrm{~V}_{s}=\mathrm{V}_{\overline{0}, s} \oplus \mathrm{~V}_{\overline{1}}
$$

The corresponding dual pair is $\left(\mathrm{G}_{s}, \mathrm{G}_{s}^{\prime}\right)=\left(\mathrm{O}_{2 l-1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ acting on the symplectic space $\mathrm{W}_{s}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}, s}\right)$.

Let $\mathrm{H}_{s}^{0} \subseteq \mathrm{H}$ be the identity connected component of the centralizer of $s$. The map

$$
\begin{equation*}
\kappa: \mathrm{H}_{s}^{0} \cup \mathrm{H}_{s}^{0} s \rightarrow \mathrm{G}_{s}, \tag{208}
\end{equation*}
$$

obtained by restricting an element of $\mathrm{H}_{s}^{0} \cup \mathrm{H}_{s}^{0} s \subseteq \operatorname{End}\left(\mathrm{~V}_{\overline{0}}\right)$ to $\mathrm{V}_{\overline{0}, s}$, is a bijection onto the Cartan subgroup of $\mathrm{G}_{s}$. Notice that $\kappa: \mathrm{H}_{s}^{0} \rightarrow \kappa\left(\mathrm{H}_{s}^{0}\right)$ is an isomorphism. Moreover,

$$
\begin{equation*}
\kappa(h s)=\kappa(h) \kappa(s) \quad\left(h \in \mathrm{H}_{s}^{0}\right) . \tag{209}
\end{equation*}
$$

If we identify $h \in \mathrm{H}_{s}^{0}$ with its $(2 l) \times(2 l)$-matrix with respect to the basis $\left\{v_{j}\right\}_{1 \leq j \leq 2 l}$ of $\mathrm{V}_{\overline{0}}$, then $\kappa$ removes the $(2 l-1)^{\text {th }}$ row and column of $h$. Let $\mathfrak{g}_{s}$ be the Lie algebra of $\mathrm{G}_{s}$. Then $\kappa$ defined as above extends to $\mathfrak{h}_{s}$, and $\kappa\left(\mathfrak{h}_{s}\right)$ is a Cartan subalgebra of $\mathfrak{g}_{s}$.

Lemma 34. Every element of the connected component $\mathrm{G}^{0}$ s is G -conjugate to an element of $\mathrm{H}_{s}^{0}$.

Proof. Fix an element $g \in \mathrm{G}$. As shown in [Cur84, page 114], $g$ preserves a subspace of V of dimension 1 or 2 . Hence V decomposes into a direct sum of $g$-irreducible subspaces of dimension 1 or 2 , and the claim follows.

Let us consider the $\rho$-function and the Weyl denominator for the root system of type $C_{l-1}$ which is dual to that of $\left(\mathfrak{g}_{s}, \kappa\left(\mathfrak{h}_{s}\right)\right)$ :

$$
\begin{equation*}
\rho_{s}^{\mathrm{D}}=(l-1) e_{1}+(l-2) e_{2}+\cdots+e_{l-1} \tag{210}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta_{s}^{\mathrm{D}}(h)=\xi_{\rho_{s}^{\mathrm{D}}}(h) \prod_{1 \leq j<k \leq l-1}\left(1-\xi_{e_{k}-e_{j}}(h)\right)\left(1-\xi_{-e_{j}-e_{k}}(h)\right) \cdot \prod_{j=1}^{l-1}\left(1-\xi_{-2 e_{j}}(h)\right) \quad\left(h \in \mathrm{H}_{s}^{0}\right) . \tag{211}
\end{equation*}
$$

Here the $e_{j}$ 's are considered both as elements of $i \kappa\left(\mathfrak{h}_{s}\right)^{*}$ and elements of $i \mathfrak{h}^{*}$ vanishing on $\mathbb{R} J_{l}$. Hence, for every root $\alpha$ of $\left(\mathfrak{g}_{s}, \kappa\left(\mathfrak{h}_{s}\right)\right)$, we can think of $\xi_{\alpha}$ as a character of $\mathrm{H}_{s}^{0}$ satisfying

$$
\begin{equation*}
\xi_{\alpha}(h)=\xi_{\alpha}(\kappa(h)) \quad\left(h \in \mathrm{H}_{s}^{0}\right) \tag{212}
\end{equation*}
$$

Observe that the Weyl group of the root system of type $C_{l-1}$ coincides with $W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)$. Its elements act by sign changes and permutations of the $J_{1}, \ldots, J_{l-1}$. The following two lemmas follow respectively from [Wen01, Theorems 2.5 and 2.6].

Lemma 35. For any continuous G -invariant function $f: \mathrm{G}^{0} s \rightarrow \mathbb{C}$,

$$
\int_{\mathrm{G}^{0} s} f(g) d g=\frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{\mathrm{H}_{s}^{0}} f(h s)\left|\Delta_{s}^{\mathrm{D}}(h)\right|^{2} d h .
$$

Notice that the coverings

$$
\widetilde{\mathrm{G}^{0} s} \rightarrow \mathrm{G}^{0} s, \quad \widetilde{\mathrm{G}^{0}} \rightarrow \mathrm{G}^{0}
$$

split (see Appendix E). Hence we may choose continuous sections

$$
\begin{equation*}
\mathrm{H}_{s}^{0} s \ni h s \rightarrow \widetilde{h s} \in \widetilde{\mathrm{H}_{s}^{0} s} \quad \text { and } \quad \mathrm{H}_{s}^{0} \ni h \rightarrow \tilde{h} \in \widetilde{\mathrm{H}_{s}^{0}} . \tag{213}
\end{equation*}
$$

Lemma 36. Consider the map

$$
\mathrm{H}_{s}^{0} \ni h \rightarrow \widetilde{h s} \in \widetilde{\mathrm{H}_{s}^{0} s}
$$

obtained by composing the multiplication by s and the fixed continuous section. Then

$$
\begin{equation*}
\Theta_{\Pi}(\widetilde{h s})=\frac{\Theta_{\Pi}(\tilde{s})}{\Theta_{\Pi_{s}}(1)} \Theta_{\Pi_{s}}(h) \quad\left(h \in \mathrm{H}_{s}^{0}\right), \tag{214}
\end{equation*}
$$

where

$$
\begin{equation*}
\Theta_{\Pi_{s}}(h)=\frac{\sum_{w \in W\left(G_{s}^{0}, k\left(h_{s}\right)\right)} \operatorname{sgn}(w) \xi_{w \cdot\left(\lambda+\rho_{s}^{\mathrm{D}}\right)}(h)}{\Delta_{s}^{\mathrm{D}}(h)} \quad\left(h \in \mathrm{H}_{s}^{0}\right), \tag{215}
\end{equation*}
$$

$\lambda$ is the highest weight of $\Pi$, see (88), and

$$
\begin{equation*}
\frac{\Theta_{\Pi}(\tilde{s})}{\Theta_{\Pi_{s}}(1)}= \pm 1 . \tag{216}
\end{equation*}
$$

Let

$$
\begin{equation*}
\rho_{s}=\rho_{s}^{\mathrm{D}}-\frac{1}{2} \sum_{j=1}^{l-1} e_{j} \tag{217}
\end{equation*}
$$

be the $\rho$-function of $\left(\mathfrak{s o}_{2 l-1}, \kappa\left(\mathfrak{h}_{s}\right)\right)$. Then $\xi_{\rho_{s}}$ is well defined on the double covering

$$
\widehat{\mathrm{H}_{s}^{0}} \ni \widehat{h} \rightarrow h \in \mathrm{H}_{s}^{0}
$$

introduced in (127). Set

$$
\begin{equation*}
\Delta_{s}(\widehat{h})=\xi_{\rho_{s}}(\widehat{h}) \prod_{1 \leq j<k \leq l-1}\left(1-\xi_{e_{k}-e_{j}}(h)\right)\left(1-\xi_{-e_{j}-e_{k}}(h)\right) \cdot \prod_{j=1}^{l-1}\left(1-\xi_{-e_{j}}(h)\right) \quad\left(h \in \mathrm{H}_{s}^{0}\right) \tag{218}
\end{equation*}
$$

Notice that $\Delta_{s}(\widehat{h})$ depends only on $\kappa(h)$. Hence we shall write $\Delta_{s}(\widehat{h})=\Delta_{s}(\widehat{\kappa(h)})$, when needed. Let $\sigma$ denote the spin representation of $\operatorname{Spin}_{2 l-1}$ and let $\Theta_{\sigma}$ be its character. By [Lit06, Ch. XI, III., p. 254],

$$
\begin{equation*}
\Theta_{\sigma}\left(\widehat{h_{1}}\right)=\prod_{j=1}^{l-1} \xi_{e_{j} / 2}\left(\widehat{h_{1}}\right)\left(1+\xi_{-e_{j}}\left(\widehat{h_{1}}\right)\right) \quad\left(h_{1} \in \kappa\left(\mathrm{H}_{s}^{0}\right)\right) \tag{219}
\end{equation*}
$$

Lemma 37. Let $\lambda$ be the highest weight of $\Pi$. Then

$$
\begin{equation*}
\lambda_{s}=\lambda+\frac{1}{2} \sum_{j=1}^{l-1} e_{j} \tag{220}
\end{equation*}
$$

is the highest weight of an irreducible representation $\Pi_{\lambda_{s}}$ of $\operatorname{Spin}_{2 l-1}$. Let $\Theta_{\Pi_{\lambda_{s}}}$ denote its character. Then

$$
\begin{equation*}
\Theta_{\Pi_{s}}(h) \Delta_{s}^{\mathrm{D}}(h) \overline{\Delta_{s}^{\mathrm{D}}(h)}=\Theta_{\Pi_{\lambda_{s}}}(\widehat{\kappa(h)}) \Theta_{\sigma}(\widehat{\kappa(h)}) \Delta_{s}(\widehat{h}) \overline{\Delta_{s}(\widehat{h})} \quad\left(h \in \mathrm{H}_{s}^{0}\right), \tag{221}
\end{equation*}
$$

where $\Theta_{\sigma}$ is the character of the spin representation $\sigma$ of $\operatorname{Spin}_{2 l-1}$.

Proof. By (211), (218) and (219),

$$
\begin{equation*}
\Delta_{s}^{\mathrm{D}}(h)=\Theta_{\sigma}(\kappa(h)) \Delta_{s}(\widehat{h}) \quad\left(\widehat{h} \in \widehat{\mathrm{H}}_{s}\right) . \tag{222}
\end{equation*}
$$

Notice that this character is real valued. Since $\lambda+\rho_{s}^{\mathrm{D}}=\lambda_{s}+\rho_{s}$, formula (221) follows immediately from (215) and (222).

Remark 11. The same computations leading to (221) show that

$$
\Theta_{\Pi_{s}}(h)=\frac{\Theta_{\Pi_{\lambda_{s}}}(\widehat{\kappa(h)})}{\Theta_{\sigma}(\widehat{\kappa(h)})} \quad\left(h \in \mathrm{H}_{s}^{0}\right) .
$$

The right-hand side is a continuous $W\left(\mathrm{G}_{s}^{0}, \mathfrak{h}_{s}\right)$-invariant function on $\widehat{\kappa\left(\mathrm{H}_{s}^{0}\right)}$. It is therefore the restriction to $\widehat{\kappa\left(\mathrm{H}_{s}^{0}\right)}$ of a central function $\Phi$ on $\operatorname{Spin}_{2 l-1}$. Hence $\Phi$ is a finite linear combination of irreducible characters of $\mathrm{Spin}_{2 l-1}$. Let us embed $\mathrm{G}_{s}$ into G by letting it act trivially on $\mathbb{R} v_{2 l-1}$. Thus, despite the fact that the denominator of (215) is the Weyl denominator for a root system of type $C_{l-1}$, the function $\Theta_{\Pi_{s}}$ extends to a finite linear combination of irreducible characters of $\operatorname{Spin}_{2 l-1}$ on $\mathrm{G}_{s}^{0}$, the connected component of the identity in $\mathrm{G}_{s}$.

To compute $\Theta(\widetilde{h s})$, we need the following lemma. Recall from [AP14, Definition 4.16, (35) and (101)] that

$$
\begin{equation*}
\Theta^{2}(g)=i^{\operatorname{dim}(g-1) \mathrm{W}} \operatorname{det}\left(J^{-1}(g-1)\right)_{J^{-1}(g-1) \mathrm{W}}^{-1} \quad(g \in \mathrm{Sp}(\mathrm{~W})) \tag{223}
\end{equation*}
$$

Define

$$
\begin{equation*}
t(c(x))=\chi_{x} \mu_{\mathrm{W}} \quad(x \in \mathfrak{g}) \tag{224}
\end{equation*}
$$

Let $\Theta_{s}, t_{s}, T_{s}$ be defined as in (16) and (14) for the dual pair $\left(\mathrm{G}_{s}, \mathrm{G}_{s}^{\prime}\right)$.
Lemma 38. For $h \in \mathrm{H}_{s}^{0}$ we have

$$
\begin{equation*}
\Theta^{2}(h s)=\Theta_{s}^{2}(\kappa(h s)) \tag{225}
\end{equation*}
$$

and

$$
\begin{equation*}
t(h s)=t_{s}(\kappa(h s)) \otimes \delta_{0} \tag{226}
\end{equation*}
$$

where $\delta_{0}$ is the Dirac delta on $\mathrm{W}_{s}^{\perp}$.
Proof. By definition

$$
\mathrm{W}=\mathrm{W}_{s} \oplus \mathrm{~W}_{s}^{\perp} \quad \text { where } \quad \mathrm{W}_{s}^{\perp}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathbb{R} v_{2 l-1}\right) .
$$

Since

$$
\left.h\right|_{\mathrm{W}_{\frac{1}{s}}}=\left.1\right|_{\mathrm{W}_{\frac{1}{s}}} \quad \text { and }\left.\quad s\right|_{\mathrm{W}_{\stackrel{1}{s}}}=\left.1\right|_{\mathrm{W}_{\frac{1}{s}}}
$$

we have $(h s-1) \mathrm{W}=(h s-1) \mathrm{W}_{s}=(\kappa(h s)-1) \mathrm{W}_{s}$. Furthermore

$$
\mathrm{W}_{s}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}}^{1} \oplus \mathrm{~V}_{\overline{0}}^{2} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l-1}\right) \oplus \operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathbb{R} v_{2 l}\right),
$$

and

$$
\begin{aligned}
\left.(h s-1)\right|_{\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V} \oplus \oplus V_{\overline{0}}^{2} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l-1}\right)} & =\left.(h-1)\right|_{\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}, \frac{1}{\overline{0}} \oplus \mathrm{~V}_{\overline{0}} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l-1}\right)} \\
& =\left.(\kappa(h s)-1)\right|_{\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V} \frac{1}{0} \oplus \mathrm{~V}_{\overline{0}} \oplus \cdots \oplus \mathrm{~V}_{\overline{0}}^{l-1}\right)}
\end{aligned}
$$

whereas

$$
\left.(h s-1)\right|_{\operatorname{Hom}\left(\mathfrak{V}_{\overline{1}}, \mathbb{R} v_{2 l}\right)}=-21_{\operatorname{Hom}\left(\mathbb{V}_{\overline{1}}, \mathbb{R} v_{2 l}\right)}=\left.(\kappa(h s)-1)\right|_{\operatorname{Hom}\left(\mathbb{V}_{\bar{T}}, \mathbb{R} v_{2 l}\right)} .
$$

We may fix the complex structure $J$ that acts as an element of $\mathrm{G}^{\prime}$, see the proof of Lemma 5.4 in [DKP05]. Then $J^{-1}(h s-1)=(h s-1) J^{-1}$ so that $J^{-1}(h s-1) \mathrm{W}=(h s-1) \mathrm{W}=$ $(\kappa(h s)-1) \mathrm{W}_{s}=J^{-1}(\kappa(h s)-1) \mathrm{W}_{s}$. Hence,

$$
\operatorname{det}\left(J^{-1}(h s-1)\right)_{J^{-1}(h s-1) \mathrm{W}}=\operatorname{det}(\kappa(h s)-1)_{J^{-1}(\kappa(h s)-1) \mathrm{W}_{s}} .
$$

By (223), this verifies (225).
Let $w_{s} \in \mathrm{~W}_{s}$ and $w_{s}^{\perp} \in \mathrm{W}_{s}^{\perp}$. Then, writing $\langle\cdot, \cdot\rangle=\langle\cdot, \cdot\rangle_{\mathrm{W}_{s}}+\langle\cdot, \cdot\rangle_{\mathrm{W}_{\frac{\perp}{s}}}$, we have

$$
\begin{aligned}
\left\langle c(h s)(h s-1)\left(w_{s}+w_{s}^{\perp}\right)\right. & \left.,(h s-1)\left(w_{s}+w_{s}^{\perp}\right)\right\rangle \\
& =\left\langle c(h s)(h s-1) w_{s},(h s-1) w_{s}\right\rangle \\
& =\left\langle(h s+1) w_{s},(h s-1) w_{s}\right\rangle \\
& =\left\langle c(\kappa(h s))(\kappa(h s)-1) w_{s},(\kappa(h s)-1) w_{s}\right\rangle_{\mathrm{W}_{s}},
\end{aligned}
$$

which proves equation (226).

Proof of Theorem 7. Let us choose $\tilde{h}=h$ in (213) and define the section $\mathrm{H}_{s}^{0} s \rightarrow \widetilde{\mathrm{H}_{s}^{0} s}$ so that

$$
\begin{equation*}
\Theta(\widetilde{h s})=\Theta_{s}(\widetilde{\kappa(h s)}) . \tag{227}
\end{equation*}
$$

Then

$$
\begin{equation*}
T(\widetilde{h s})=T_{s}(\widetilde{\kappa(h s)}) \otimes \delta_{0} . \tag{228}
\end{equation*}
$$

In these terms, Lemmas 35, 37 and 38 show that

$$
\begin{aligned}
& \int_{\mathrm{G}^{0} s} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g \\
& =\frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{\mathrm{H}_{s}^{0}} \check{\Theta}_{\Pi}(\widetilde{h s})\left|\Delta_{s}^{\mathrm{D}}(h)\right|^{2} T(\widetilde{h s})\left(\phi^{\mathrm{G}}\right) d h \\
& =\frac{1}{\mid W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right) \mid\right.} \int_{\mathrm{H}_{s}^{0}} \check{\Theta}_{\Pi}(\widetilde{h s})\left|\Theta_{\sigma}(\widehat{\kappa(h)})\right|^{2}\left|\Delta_{s}(h)\right|^{2} T(\widetilde{h s})\left(\phi^{\mathrm{G}}\right) d h \\
& =\frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{\mathrm{H}_{s}^{0}} \check{\Theta}_{\Pi}(\widetilde{h s})\left|\Theta_{\sigma}(\widehat{\kappa(h)})\right|^{2}\left|\Delta_{s}(h)\right|^{2} T_{s}(\widetilde{\kappa(h s)})\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h \\
& \left.=\frac{\check{\Theta}_{\Pi( }(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{\mathrm{H}_{s}^{0}} \check{\Theta}_{\Pi_{s}}(h)\left|\Theta_{\sigma}(\widehat{\kappa(h)})\right|^{2}\left|\Delta_{s}(h)\right|^{2} T_{s}(\widehat{\kappa(h) \kappa(s)})\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h \\
& \left.=\frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\mid W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right) \mid\right.} \int_{\mathrm{H}_{s}^{0}} \check{\Theta}_{\Pi_{s}}(\kappa(h))\left|\Theta_{\sigma}(\widehat{\kappa(h)})\right|^{2}\left|\Delta_{s}(\kappa(h))\right|^{2} T_{s}(\widetilde{\kappa(h) \kappa(s)})\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h \\
& \left.=\frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{\kappa\left(\mathrm{H}_{s}^{0}\right)} \check{\Theta}_{\Pi_{s}}\left(h_{1}\right)\left|\Theta_{\sigma}\left(\widehat{h_{1}}\right)\right|^{2}\left|\Delta_{s}\left(h_{1}\right)\right|^{2} T_{s}\left(\widetilde{h_{1} \kappa(s)}\right)\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h_{1} \\
& \left.=\frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{\kappa\left(\mathrm{H}_{s}^{0}\right)} \check{\Theta}_{\Pi_{s}}\left(-\kappa(s) h_{1}\right)\left|\Theta_{\sigma}\left(\widetilde{-\kappa(s) h_{1}}\right)\right|^{2}\left|\Delta_{s}\left(-\kappa(s) h_{1}\right)\right|^{2} T_{s}\left(\widetilde{-h_{1}}\right)\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h_{1} \\
& \left.=\frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{\kappa\left(\mathrm{H}_{s}^{0}\right)} \check{\Theta}_{\Pi_{s}}\left(-\kappa(s) h_{1}\right)\left|\Theta_{\sigma}\left(\widehat{-\kappa(s) h_{1}}\right)\right|^{2}\left|\Delta_{s}\left(-h_{1}\right)\right|^{2} T_{s}\left(\widetilde{-h_{1}}\right)\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h_{1} \\
& \left.=\frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{-\kappa\left(\mathrm{H}_{s}^{0}\right)} \check{\Theta}_{\Pi_{s}}\left(\kappa(s) h_{1}\right)\left|\Theta_{\sigma}\left(\widehat{\kappa(s) h_{1}}\right)\right|^{2}\left|\Delta_{s}\left(h_{1}\right)\right|^{2} T_{s}\left(\widetilde{h}_{1}\right)\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h_{1} \\
& \left.=\frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{-\kappa\left(\mathrm{H}_{s}^{0}\right)} \check{\Theta}_{\Pi_{\lambda_{s}}}\left(\kappa(s) h_{1}\right) \Theta_{\sigma}\left(\widehat{\kappa(s) h_{1}}\right)\left|\Delta_{s}\left(h_{1}\right)\right|^{2} T_{s}\left(\widetilde{h_{1}}\right)\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h_{1} \\
& \left.=\frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_{s}}(1)} \frac{1}{\left|W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)\right|} \int_{-\kappa\left(\mathrm{H}_{s}^{0}\right)} \check{\Theta}_{\Pi_{\lambda_{s}} \otimes \sigma^{c}}\left(\widehat{\kappa(s) h_{1}}\right)\left|\Delta_{s}\left(h_{1}\right)\right|^{2} T_{s}\left(\widetilde{h_{1}}\right)\right)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h_{1} .
\end{aligned}
$$

The function

$$
-\kappa\left(\mathrm{H}_{s}^{0}\right) \ni h_{1} \rightarrow \Theta_{\Pi_{\lambda_{s}} \otimes \sigma^{c}}\left(\widehat{\kappa(s) h_{1}}\right)
$$

is $W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right.$ )-invariant. Hence there a $\mathrm{G}_{s}^{0}$-conjugation invariant function $\Phi_{\Pi}$ such that

$$
\Phi_{\Pi}\left(\tilde{h}_{1}\right)=\Theta_{\Pi_{\lambda_{s}} \otimes \sigma^{c}}\left(\widehat{\kappa(s) h_{1}}\right) \quad\left(h_{1} \in-\kappa\left(\mathrm{H}_{s}^{0}\right)\right)
$$

Therefore

$$
\int_{\mathrm{G}^{0} s} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=C(\Pi) \int_{-\mathrm{G}_{s}^{0}} \check{\Phi}_{\Pi}(g) T_{s}(g) d g
$$

where $C(\Pi)=\frac{\check{\Theta}_{\Pi(\tilde{s})}}{\Theta_{\Pi_{s}}(1)}$.
Clearly $\Phi_{\Pi}(-g)$ is a finite linear combination of some irreducible characters $\Theta_{\pi}(g)$ of $\mathrm{G}_{s}^{0}$ :

$$
\Phi_{\Pi}(-g)=\sum_{\pi} a_{\pi} \Theta_{\pi}(g) \quad\left(g \in \mathrm{G}_{s}^{0}\right)
$$

Thus

$$
\Phi_{\Pi}(g)=\sum_{\pi} a_{\pi} \Theta_{\pi}(-g) \quad\left(g \in-\mathrm{G}_{s}^{0}\right) .
$$

In particular,

$$
\Phi_{\Pi}(c(x))=\sum_{\pi} a_{\pi} \Theta_{\pi}(-c(x)) \quad\left(x \in \mathfrak{g}_{s}\right)
$$

Remark 12. In fact,

$$
\begin{equation*}
C(\Pi) \Phi_{\Pi}\left(\tilde{h}_{1}\right)=\Theta_{\Pi}\left(\widetilde{\kappa^{-1}\left(h_{1}\right)}\right) \operatorname{det}\left(1+s \kappa^{-1}\left(h_{1}\right)\right) \quad\left(h_{1} \in-\kappa\left(\mathrm{H}_{s}^{0}\right)\right) . \tag{229}
\end{equation*}
$$

Indeed, by [Lit06, Ch. XI, III., p. 254], the character of $\sigma$ is real valued and

$$
\begin{equation*}
\Theta_{\sigma \otimes \sigma^{c}}(g)=\Theta_{\sigma}^{2}(g)=\operatorname{det}(1+g) \quad\left(g \in \mathrm{G}_{s}^{0}\right) . \tag{230}
\end{equation*}
$$

By Remark 11,

$$
\begin{equation*}
\Theta_{\Pi_{s}}(h) \Theta_{\sigma}^{2}(\widehat{\kappa(h)})=\Theta_{\Pi_{\lambda_{s}}}(\widehat{\kappa(h)}) \Theta_{\sigma}(\widehat{\kappa(h)}) \quad\left(h \in \mathrm{H}_{s}^{0}\right) . \tag{231}
\end{equation*}
$$

If $h_{1} \in-\kappa\left(\mathrm{H}_{s}^{0}\right)$, then $\kappa(s) h_{1} \in \kappa\left(\mathrm{H}_{s}^{0}\right)$. Hence, by (231), for $h_{1} \in-\kappa\left(\mathrm{H}_{s}^{0}\right)$,

$$
\Phi_{\Pi}\left(\tilde{h}_{1}\right)=\Theta_{\Pi_{\lambda_{s}}}\left(\widehat{\kappa(s) h_{1}}\right) \Theta_{\sigma}\left(\widehat{\kappa(s) h_{1}}\right)=\Theta_{\Pi_{s}}\left(\kappa^{-1}\left(\kappa(s) h_{1}\right)\right) \Theta_{\sigma}^{2}\left(\kappa(s) h_{1}\right) .
$$

Notice that

$$
\begin{equation*}
\kappa^{-1}\left(\kappa(s) h_{1}\right)=s \kappa^{-1}\left(h_{1}\right)=\kappa^{-1}\left(h_{1}\right) s \quad\left(h_{1} \in-\kappa\left(\mathrm{H}_{s}^{0}\right)\right) . \tag{232}
\end{equation*}
$$

Moreover, $\kappa^{-1}\left(h_{1}\right) \in-\mathrm{G}_{s}^{0}$. So, $\Theta_{\Pi}\left(\widetilde{\kappa^{-1}\left(h_{1}\right)}\right)$ is well-defined, and by (214) and (231), we obtain

$$
\begin{equation*}
C(\Pi) \Phi_{\Pi}\left(\tilde{h}_{1}\right)=\Theta_{\Pi}\left(\widetilde{\kappa^{-1}\left(h_{1}\right)}\right) \operatorname{det}\left(1+\kappa(s) h_{1}\right) \quad\left(h_{1} \in-\kappa\left(\mathrm{H}_{s}^{0}\right)\right) \tag{233}
\end{equation*}
$$

By (232), $\operatorname{det}\left(1+\kappa(s) h_{1}\right)=\operatorname{det}\left(1+s \kappa^{-1}\left(h_{1}\right)\right)$, where the first determinant is taken in $\mathrm{O}_{2 l-1}$ and the second in $\mathrm{O}_{2 l}$. This proves (229).

The function $-\kappa\left(\mathrm{H}_{s}^{0}\right) \ni h_{1} \rightarrow \operatorname{det}\left(1+s \kappa^{-1}\left(h_{1}\right)\right) \in \mathbb{R}$ is $W\left(\mathrm{G}_{s}^{0}, \kappa\left(\mathfrak{h}_{s}\right)\right)$-invariant. Hence it extends to a $\mathrm{G}_{s}^{0}$-invariant function on $-\mathrm{G}_{s}^{0}$. However, this extension is not obvious, as one can see for example when $\mathrm{G}_{s}^{0}=\mathrm{SO}_{3}$. In this case

$$
s=\left(\begin{array}{cc|c|c}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\hline 0 & 0 & 1 & 0 \\
\hline 0 & 0 & 0 & -1
\end{array}\right), \quad \kappa^{-1}\left(h_{1}\right)=\left(\begin{array}{cc|c|c}
\cos (t) & \sin (t) & 0 & 0 \\
-\sin (t) & \cos (t) & 0 & 0 \\
\hline 0 & 0 & 1 & 0 \\
\hline 0 & 0 & 0 & -1
\end{array}\right)
$$

and hence

$$
\operatorname{det}\left(1+s \kappa^{-1}\left(h_{1}\right)\right)=4(1+\cos (t))=2\left(2 \operatorname{det}\left(\kappa^{-1}\left(h_{1}\right)\right)+\operatorname{tr}\left(\kappa^{-1}\left(h_{1}\right)\right)\right),
$$

and the right-hand side extends to a $\mathrm{G}_{s}^{0}$-invariant function on $-\mathrm{G}_{s}^{0}$.

## 12. The special case for the pair $\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $1 \leq l \leq l^{\prime}$

Recall the decomposition (93). Let us denote the objects corresponding to $\mathrm{W}_{s}$ by the subscript $s$ : $\Theta_{s}, t_{s}$ and $T_{s}$. If H is a Cartan subgroup of G , then $\mathrm{H}^{0}=\mathrm{H}_{s}^{0}$ is a Cartan subgroup of $\mathrm{G}_{s} \subseteq \mathrm{G}$, and the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_{s}$ share the same Cartan subalgebra $\mathfrak{h}=\mathfrak{h}_{s}$. Since any element $h \in \mathrm{H}$ acts trivially on $\mathrm{W}_{s}^{\perp}$, we see that $(h-1) \mathrm{W}=(h-1) \mathrm{W}_{s}$. Hence,

$$
\operatorname{det}(h-1)_{(h-1) \mathrm{W}}=\operatorname{det}(h-1)_{(h-1) \mathrm{W}_{s}}
$$

and, just as in section 11, we check that

$$
\begin{equation*}
\Theta(\tilde{h})=\Theta_{s}(\tilde{h}), \quad t(h)=t_{s}(h) \otimes \delta_{0} \quad\left(h \in \mathrm{H}^{0}\right), \tag{234}
\end{equation*}
$$

where $\delta_{0}$ is the Dirac delta on $\mathrm{W}_{s}^{\perp}$.
Proof of Theorem 8. Let $\widehat{\mathrm{H}} \ni \widehat{h} \rightarrow h \in \mathrm{H}$ be the double covering of H on which the functions $\xi_{\rho}$ and $\Delta$ are well-defined; see (127), It is easy to check that

$$
\begin{equation*}
|\Delta(\widehat{h})|^{2}=\left|\Delta_{s}(h)\right|^{2} \operatorname{det}(1-h) \quad(h \in \mathrm{H}), \tag{235}
\end{equation*}
$$

where

$$
\Delta_{s}(h)=\xi_{\rho_{s}}(h) \prod_{1 \leq j<k \leq l}\left(1-\xi_{-e_{j}+e_{k}}(h)\right)\left(1-\xi_{-e_{j}-e_{k}}(h)\right) .
$$

Hence, for $\phi \in \mathcal{S}(\mathrm{W})$,

$$
\begin{array}{rl}
\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T & T(\tilde{g})(\phi) d g=\frac{1}{\left|W\left(\mathrm{G}^{0}, \mathfrak{h}\right)\right|} \int_{\mathrm{H}^{0}} \check{\Theta}_{\Pi}(\tilde{h}) \Delta(\widehat{h}) \overline{\Delta(\widehat{h})} T(\tilde{h})\left(\phi^{\mathrm{G}}\right) d h \\
& =\frac{1}{\left|W\left(\mathrm{G}_{s}, \mathfrak{h}_{s}\right)\right|} \int_{\mathrm{H}_{s}^{0}} \check{\Theta}_{\Pi}(\tilde{h}) \operatorname{det}(1-h) \Delta_{s}(h) \overline{\Delta_{s}(h)} \Theta_{s}(\tilde{h}) t_{s}(h)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h \\
& =\int_{\mathrm{G}_{s}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) \operatorname{det}(1-g) T_{s}(\tilde{g})\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d g
\end{array}
$$

This proves (94).
Consider the Cayley transform $c: \mathfrak{h}_{s} \rightarrow \mathrm{H}_{s}^{0}$ where

$$
\begin{equation*}
c\left(i x_{1}, \ldots, i x_{l}, 0\right)=\left(v_{1}, \ldots, v_{l}, 1\right), \quad v_{j}=\frac{i x_{j}+1}{i x_{j}-1} \quad\left(x_{j} \in \mathbb{R}, 1 \leq j \leq l\right) \tag{236}
\end{equation*}
$$

By (234) and since $c\left(\mathfrak{h}_{s}\right)$ is dense in $\mathrm{H}_{s}^{0}=\mathrm{H}^{0}$,

$$
\begin{align*}
\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g= & \frac{1}{\left|W\left(\mathrm{G}_{s}, \mathfrak{h}_{s}\right)\right|} \int_{\mathrm{H}_{s}^{0}} \check{\Theta}_{\Pi}(\tilde{h}) \Delta(\widehat{h}) \overline{\Delta(\widehat{h})} \Theta_{s}(\tilde{h}) t_{s}(h)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d h \\
= & \frac{4}{\left|W\left(\mathrm{G}_{s}, \mathfrak{h}_{s}\right)\right|} \int_{\mathfrak{h}_{s}} \check{\Theta}_{\Pi}(\widehat{c}(x)) \overline{\Delta(\widehat{c}(x))} \Delta(\widehat{c}(x)) \Theta_{s}(\widehat{c}(x)) \frac{1}{\pi_{\mathfrak{g}_{s} / \mathfrak{h}_{s}}(x)} \\
& \times \pi_{\mathfrak{g}_{s} / h_{s}}(x) \int_{\mathrm{W}_{s}} \chi_{x}(w)\left(\phi^{\mathrm{G}} \mid \mathrm{W}_{s}\right)(w) d w \cdot \operatorname{ch}^{-2}(x) d x, \tag{237}
\end{align*}
$$

where the jacobian of the map $c: \mathfrak{h}_{s} \rightarrow \mathrm{H}_{s}^{0}$ is computed using Appendix B for $\mathrm{G}=\mathrm{SO}_{2}$. We now proceed as in Lemma 16:

$$
\check{\Theta}_{\Pi}(\widehat{c}(x)) \overline{\Delta(\widehat{c}(x))}=\Theta_{\Pi}\left(\widehat{c}(x)^{-1}\right) \Delta\left(\widehat{c}(x)^{-1}\right)=\sum_{s \in W(\mathrm{G}, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}}(s) \xi_{-s \mu}(\widehat{c}(x)),
$$

Appendix C, (141) and (149) show that there is a constant $C_{1}$ such that

$$
\begin{align*}
\Delta(\widehat{c}(x)) \Theta_{s}(\widehat{c}(x)) \frac{1}{\pi_{\mathfrak{g}_{s} / \mathfrak{h}_{s}}(x)} \operatorname{ch}^{-2}(x) & =C_{1} \operatorname{ch}^{-2 l+1}(x) \operatorname{ch}^{2 l^{\prime}}(x) \operatorname{ch}^{-2}(x) \\
& =C_{1} \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) \tag{238}
\end{align*}
$$

By Lemma 22, there is a constant $C_{2}$ such that

$$
\begin{equation*}
\pi_{\mathfrak{g}_{s} / \mathfrak{h}_{s}}(x) \int_{\mathrm{W}_{s}} \chi_{x}(w)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right)(w) d w=\left.C_{2} \int_{\mathfrak{h}_{s}} e^{i B(x, y)} F_{\phi^{\mathrm{G}}}\right|_{\mathrm{W}_{s}}(y) d y \tag{239}
\end{equation*}
$$

This, together with Lemma 19, implies that (237) is equal to a constant multiple of

$$
\begin{align*}
& \int_{\mathfrak{h}_{s}} \xi_{-\mu}(\widehat{c}(x)) \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) e^{i B(x, y)} F_{\left.\phi^{\mathrm{G}}\right|_{\mathbf{w}_{s}}}(y) d y d x \\
&=\int_{\mathfrak{h}_{s}} \prod_{j=1}^{l}\left(1-i x_{j}\right)^{\mu_{j}+\delta-1}\left(1+i x_{j}\right)^{-\mu_{j}+\delta-1} \int_{\mathfrak{h}_{s}} e^{i B(x, y)} F_{\phi^{\mathrm{G}} \mid \mathrm{w}_{s}}(y) d y d x \\
&=\int_{\mathfrak{h}_{s}} \prod_{j=1}^{l}\left(1-i x_{j}\right)^{-a_{j}}\left(1+i x_{j}\right)^{-b_{j}} \int_{\mathfrak{h}_{s}} e^{i B(x, y)} F_{\phi^{\mathrm{G}} \mid \mathrm{w}_{s}}(y) d y d x, \tag{240}
\end{align*}
$$

where $\delta=\frac{1}{2}\left(2 l^{\prime}-2 l-1\right)$, see (67), and $a_{j}$ and $b_{j}$ are as in (68).
Since $\tau\left(\mathrm{W}_{s}\right) \cap \mathfrak{h}_{s}=\mathfrak{h}_{s}$ for $\mathbb{D}=\mathbb{R}$, we are in the situation considered by Theorem 4, see also Corollary 25. The same computation as in Theorem 4 (with $a_{j}$ and $b_{j}$ reversed) shows that (240) is equal to

$$
\begin{equation*}
\int_{\mathfrak{h}_{s}} \prod_{j=1}^{l}\left(P_{b_{j}, a_{j}}\left(\beta y_{j}\right) e^{-\beta\left|y_{j}\right|}+\beta^{-1} Q_{b_{j}, a_{j}}\left(-\beta^{-1} y_{j}\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y) d y . \tag{241}
\end{equation*}
$$

Since $-1 \in W\left(\mathrm{G}_{s}, \mathfrak{h}_{s}\right)$ has signature 1 and $F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}$ is $W\left(\mathrm{G}_{s}, \mathfrak{h}_{s}\right)$-skew-invariant, we have $F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(-y)=F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y)$. By (D.6), the change of variable $y \rightarrow-y$ transforms the previous integral into

$$
\begin{align*}
& \int_{\mathfrak{h}_{s}} \prod_{j=1}^{l}\left(P_{a_{j}, b_{j}}\left(\beta y_{j}\right) e^{-\beta\left|y_{j}\right|}+\beta^{-1} Q_{a_{j}, b_{j}}\left(-\beta^{-1} y_{j}\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y) d y= \\
& \int_{\mathfrak{h}_{s}} \prod_{j=1}^{l}\left(p_{j}\left(y_{j}\right)+q_{j}\left(-\partial_{y_{j}}\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y) d y . \tag{242}
\end{align*}
$$

Suppose first that $l<l^{\prime}$ and let $w_{0} \in \mathfrak{s}_{\overline{1}}\left(\mathrm{~V}^{0}\right)$, as in (49). Then by (I.3), there is a constant $C_{3}>0$ such that

$$
\begin{equation*}
\int_{\mathrm{S} / \mathrm{S}^{\mathfrak{h}}+w_{0}} \phi(s . w) d\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{1}}+w_{0}}\right)=C_{3} \int_{\mathrm{G}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime n}} \phi\left(\left(g, g^{\prime}\right) \cdot w\right) d g d\left(g^{\prime} \mathrm{Z}^{\prime n}\right) \quad\left(\phi \in \mathcal{S}(\mathrm{W}), w \in \mathfrak{h}_{\overline{1}}^{r e g}\right), \tag{243}
\end{equation*}
$$

where $\mathrm{Z}^{\prime n}$ is the centralizer of $n=\tau^{\prime}\left(w_{0}\right)$ in $\mathrm{G}^{\prime}$. Because of the embedding $\mathrm{G}_{s} \subseteq \mathrm{G}$ and the normalization $\operatorname{vol}\left(\mathrm{G}_{s}\right)=1$,

$$
\begin{aligned}
& \int_{\mathrm{G}_{s}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime n}} \phi^{\mathrm{G}}\left(\left(g_{s}, g^{\prime}\right) \cdot w\right) d g_{s} d\left(g^{\prime} \mathrm{Z}^{\prime n}\right) \\
&=\int_{\mathrm{G}_{s}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime n}} \int_{\mathrm{G}} \phi\left(\left(g g_{s}, g^{\prime}\right) \cdot w\right) d g d g_{s} d\left(g^{\prime} \mathrm{Z}^{\prime n}\right) \\
&=\int_{\mathrm{G}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime n}} \phi\left(\left(g, g^{\prime}\right) \cdot w\right) d g d\left(g^{\prime} \mathrm{Z}^{\prime n}\right) \quad\left(\phi \in \mathcal{S}(\mathrm{W}), w \in \mathfrak{h}_{\overline{1}}^{r e g}\right) .
\end{aligned}
$$

Hence, by (243), there is a positive constant $C_{3}^{\prime}$ such that

$$
\begin{equation*}
\mu_{\mathcal{O}(w), \mathfrak{h}_{\bar{T}}}\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right)=C_{3}^{\prime} \mu_{\mathcal{O}(w), \mathfrak{h}_{\bar{T}}}(\phi) \quad\left(w \in \mathfrak{h}_{\overline{1}}^{r e g}\right) . \tag{244}
\end{equation*}
$$

Proceeding as above with (I.1) instead of (I.3), we see that (244) holds in the case $l=l^{\prime}$ as well. Observe that $\pi_{\mathfrak{g}^{\prime} / \mathfrak{z}^{\prime}}(y)=\pi_{\mathfrak{g}_{s}^{\prime} / z_{s}^{\prime}}(y)$ by (A.4). Hence

$$
F_{\phi \mathrm{G} \mid \mathrm{W}_{s}}(y)=C_{3}^{\prime \prime} F_{\phi}(y) \quad\left(y=\tau(w), w \in \mathfrak{h}_{\overline{\mathrm{I}}}{ }^{r e g}\right) .
$$

Since $\mathfrak{h}_{s}=\mathfrak{h},(242)$ becomes

$$
C_{3} \int_{\mathfrak{h}} \prod_{j=1}^{l}\left(p_{j}\left(y_{j}\right)+q_{j}\left(-\partial_{y_{j}}\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi}(y) d y d x
$$

which is a positive constant multiple of the integral $\int_{-\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) d g$ from Theorem 4.

## 13. A different look at the pair $\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $l>l^{\prime}$

Recall the decompositions $\mathfrak{h}(\mathfrak{g})=\mathfrak{h} \oplus \mathfrak{h}^{\prime \prime}$ from (169) and $\mathrm{W}=\mathrm{W}_{s} \oplus \mathrm{~W}_{s}^{\perp}$ from (98). Recall also that we often identify $\mathfrak{h}$ and $\mathfrak{h}^{\prime}$ via (43). As before, we denote the objects corresponding to $W_{s}$ by the subscript $s: \mathfrak{g}_{s}, \mathrm{G}_{s}, \Theta_{s}, t_{s}$ and $T_{s}$. In particular, $\mathfrak{h}_{s}=\mathfrak{h}(\mathfrak{g})$, see (46), and $\mathrm{H}_{s}^{0}=\mathrm{H}(\mathfrak{g})^{0}$. Since any element $h \in \mathrm{H}(\mathfrak{g})$ acts trivially on $\mathrm{W}_{s}^{\perp}$, we see that

$$
(h-1) \mathrm{W}=(h-1) \mathrm{W}_{s} .
$$

Hence

$$
\operatorname{det}(h-1)_{(h-1) \mathrm{W}}=\operatorname{det}(h-1)_{(h-1) \mathrm{W}_{s}} .
$$

Therefore, as in section 11, we check that

$$
\begin{equation*}
\Theta(\tilde{h})=\Theta_{s}(\tilde{h}), \quad t(h)=t_{s}(h) \otimes \delta_{0}, \tag{245}
\end{equation*}
$$

where $\delta_{0}$ is the Dirac delta on $\mathrm{W}_{s}^{\perp}$.
Let $\mathfrak{z}_{s}$ denote the centralizer of $\mathfrak{h}$ in $\mathfrak{g}_{s}$. Then $\mathfrak{z}_{s}=\mathfrak{h} \oplus \mathfrak{g}_{s}^{\prime \prime}$, where $\mathfrak{g}_{s}^{\prime \prime}$ is the Lie algebra of the group $\mathrm{G}_{s}^{\prime \prime}$ of isometries of the restriction of the form $(\cdot, \cdot)$ to the $2\left(l-l^{\prime}\right)$-dimensional real vector space $\left(\mathrm{V}_{\overline{0}}^{0,0}\right)^{\perp}$. Then $\mathfrak{h}^{\prime \prime}$ is a Cartan subalgebra of $\mathfrak{g}_{s}^{\prime \prime}$. The following lemma is a variation of Lemma 27 in the present situation.

Lemma 39. Suppose $l>l^{\prime}$ and let $\mu$ be the Harish-Chandra parameter of a genuine irreducible representation of $\widetilde{\mathrm{O}}_{2 l+1}$. In terms of the decomposition (169)

$$
\begin{align*}
& \xi_{-s \mu}(\widehat{c}(x)) \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) \pi_{\mathfrak{z}_{s} / \mathfrak{h}(\mathfrak{g})}(x) \\
&=\left(\xi_{-s \mu}\left(\widehat{c}\left(x^{\prime}\right)\right) \operatorname{ch}^{2 l^{\prime}-2 l-1}\left(x^{\prime}\right)\right)\left(\xi_{-s \mu}\left(\widehat{c}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{2 l^{\prime}-2 l-1}\left(x^{\prime \prime}\right) \pi_{\mathfrak{g}_{s}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right)\right), \tag{246}
\end{align*}
$$

where $x=x^{\prime}+x^{\prime \prime} \in \mathfrak{h}(\mathfrak{g})$, with $x^{\prime} \in \mathfrak{h}$ and $x^{\prime \prime} \in \mathfrak{h}^{\prime \prime}$. Moreover,

$$
\begin{align*}
& \int_{\mathfrak{h}^{\prime \prime}} \xi_{-s \mu}\left(\widehat{c}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{2 l^{\prime}-2 l-1}\left(x^{\prime \prime}\right) \pi_{\mathfrak{g}_{s}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right) d x^{\prime \prime} \\
&=C \sum_{s^{\prime \prime} \in W\left(\mathfrak{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(s^{\prime \prime}\right) \mathbb{I}_{\{0\}}\left(-\left.(s \mu)\right|_{\mathfrak{h}^{\prime \prime}}+s^{\prime \prime} \rho^{\prime \prime}\right), \tag{247}
\end{align*}
$$

where $C$ is a constant, $\rho^{\prime \prime}$ is one half times the sum of the positive roots for $\left(\mathfrak{g}_{\mathbb{C}}^{\prime \prime}, \mathfrak{h}_{\mathbb{C}}^{\prime \prime}\right)$ and $\mathbb{I}_{\{0\}}$ is the indicator function of zero.

Proof. Formula (246) is obvious, because $\pi_{\mathfrak{z}_{s} / \mathfrak{h}(\mathfrak{g})}\left(x^{\prime}+x^{\prime \prime}\right)=\pi_{\mathfrak{g}_{s}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right)$. We shall verify (247). By (C.2) applied to $\mathfrak{g}^{\prime \prime} \supseteq \mathfrak{h}^{\prime \prime}$,

$$
\pi_{\mathfrak{s}_{s}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(x^{\prime \prime}\right)=C_{1}^{\prime \prime} \Delta^{\prime \prime}\left(\widehat{c}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{2\left(l-l^{\prime}\right)-1}\left(x^{\prime \prime}\right) \quad\left(x^{\prime \prime} \in \mathfrak{h}^{\prime \prime}\right),
$$

where $\Delta^{\prime \prime}$ is the Weyl denominator for $\mathrm{G}^{\prime \prime}$, see (172). Hence, the integral (247) is a constant multiple of

$$
\int_{\mathfrak{h}^{\prime \prime}} \xi_{-s \mu}\left(\widehat{c}\left(x^{\prime \prime}\right)\right) \Delta^{\prime \prime}\left(\widehat{c}\left(x^{\prime \prime}\right)\right) \operatorname{ch}^{-2}\left(x^{\prime \prime}\right) d x^{\prime \prime}=2^{\operatorname{dim} \mathfrak{h}^{\prime \prime}} \int_{\widehat{c}\left(\mathfrak{h}^{\prime \prime}\right)} \xi_{-s \mu}(h) \Delta^{\prime \prime}(h) d h,
$$

where $\widehat{c}\left(\mathfrak{h}^{\prime \prime}\right) \subseteq \widehat{-\mathrm{H}^{\prime \prime 0}}$. Noticing that $\operatorname{vol}\left(-\mathrm{H}^{\prime \prime 0}\right)=\operatorname{vol}\left(\mathrm{H}^{\prime \prime 0}\right)$, we obtain the right-hand side of (247) as in the proof of Lemma 27.

Proof of Theorem 9. As in section 12, we check that for $\phi \in \mathcal{S}(\mathrm{W})$,

$$
\begin{equation*}
\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) d g=\int_{\mathrm{G}_{s}^{0}} \check{\Theta}_{\Pi}(\tilde{g}) \operatorname{det}(1-g) T_{s}(\tilde{g})\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right) d g, \tag{248}
\end{equation*}
$$

which verifies the first equality in (101). On the other hand, similar computations as those done in section 12 and (245) imply that the right-hand side of (248) is a constant multiple of

$$
\begin{align*}
& \frac{1}{\left|W\left(\mathrm{G}^{0}, \mathfrak{h}(\mathfrak{g})\right)\right|} \int_{\mathfrak{h}(\mathfrak{g})}\left(\Theta_{\Pi}\left(\widehat{c}(x)^{-1}\right) \Delta\left(\widehat{c}(x)^{-1}\right)\right)\left(\frac{\Delta(\widehat{c}(x))}{\pi_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(x)} \Theta_{s}(\tilde{c}(x))\right) \\
& \times \pi_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(x) \int_{\mathrm{W}_{s}} \chi_{x}(w)\left(\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}\right)(w) d w \operatorname{ch}^{-2}(x) d x \tag{249}
\end{align*}
$$

where $c: \mathfrak{g}_{s} \rightarrow \mathrm{G}_{s}^{0}$ maps onto a dense subset of $\mathrm{G}_{s}^{0}$ and is such that $c\left(\mathfrak{h}_{s}\right)$ is a dense subset of $\mathrm{H}\left(\mathfrak{g}_{s}\right)^{0}$. Notice that $\mathrm{H}(\mathfrak{g})^{0}=\left\{\operatorname{diag}(h, 1) ; h \in \mathrm{H}\left(\mathfrak{g}_{s}\right)^{0}\right\}$. Lemma 23 shows that there is a constant $C_{1}$ such that

$$
\begin{aligned}
&\left.\pi_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(x) \int_{\mathrm{W}_{s}} \chi_{x}(w) \phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}(w) d w \\
&=\left.C_{1} \int_{\tau^{\prime}(\mathfrak{h} \overline{\mathrm{T}}} \sum^{r e g}\right) \\
& t W\left(\mathrm{Z}_{s}, \mathfrak{h}(\mathfrak{g})\right) \in W\left(\mathrm{G}_{s}, \mathfrak{h}(\mathfrak{g})\right) / W\left(\mathrm{Z}_{s}, \mathfrak{h}(\mathfrak{g})\right) \\
& \operatorname{sgn}_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}_{s} / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} \cdot x\right) e^{i B(x, t . y)} F_{\left.\phi^{\mathrm{G}}\right|_{\mathrm{W}_{s}}}(y) d y
\end{aligned}
$$

where $\mathfrak{z}_{s} \subseteq \mathfrak{g}_{s}$ is the centralizer of $\mathfrak{h}=\mathfrak{h}^{\prime}$. By (238), for a suitable constant $C_{1}$,

$$
\frac{\Delta(\widehat{c}(x))}{\pi_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(x)} \Theta_{s}(\tilde{c}(x)) \operatorname{ch}^{-2}(x)=C_{1} \operatorname{ch}^{2 l^{\prime}-2 l-1}(x)
$$

Hence (249) is equal to a constant multiple of

$$
\begin{aligned}
& \sum_{u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau^{\prime}\left(\mathfrak{h}_{\mathfrak{1}}^{r e g}\right)} \xi_{-\mu}\left(\widehat{c}\left(u^{-1} x\right)\right) \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) \\
& \quad \times \sum_{t W\left(Z_{s}, \mathfrak{h}(\mathfrak{g})\right) \in W\left(\mathrm{G}_{s}, \mathfrak{h}(\mathfrak{g})\right) / W\left(Z_{s}, \mathfrak{h}(\mathfrak{g})\right)} \operatorname{sgn}_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z} s / \mathfrak{h}(\mathfrak{g})}\left(t^{-1} x\right) e^{i B(x, t y)} F_{\phi^{\mathrm{G}} \mid \mathbf{W}_{s}}(y) d y d x .
\end{aligned}
$$

Interchanging the sums, changing the variable of integration $x$ to $t x$ and using that $\operatorname{ch}(t x)=\operatorname{ch}(x)$ and $B(t x, t y)=B(x, y)$, we see that (249) is a constant multiple of

$$
\begin{aligned}
& \sum_{t W\left(\mathrm{Z}_{s}, \mathfrak{h}(\mathfrak{g})\right) \in W\left(\mathrm{G}_{s}, \mathfrak{h}(\mathfrak{g})\right) / W\left(\mathrm{Z}_{s}, \mathfrak{h}(\mathfrak{g})\right)} \\
& \quad \sum_{u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(u) \operatorname{sgn}_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(t) \\
& \quad \times \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{\mathrm{T}}}^{r e g}\right)} \xi_{-\mu}\left(\widehat{c}\left(u^{-1} t x\right)\right) \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) \pi_{\mathfrak{z} s / \mathfrak{h}(\mathfrak{g})}(x) e^{i B(x, y)} F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y) d y d x .
\end{aligned}
$$

Now, replace $u \in W(G, \mathfrak{h}(\mathfrak{g}))$ with $t u$, where $t \in W\left(\mathrm{G}_{s}, \mathfrak{h}(\mathfrak{g})\right) \subseteq W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$. Since $\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(t u) \operatorname{sgn}_{\mathfrak{g}_{s} / \mathfrak{h}(\mathfrak{g})}(t)=\operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(u)$, we conclude that (249) is a constant multiple of

$$
\begin{align*}
& \sum_{u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau^{\prime}\left(\mathfrak{h}_{\mathrm{T}} \text { reg }\right)} \xi_{-\mu}\left(\widehat{c}\left(u^{-1} x\right)\right) \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) \\
& \times\left.\pi_{\mathfrak{z}_{s} / \mathfrak{h}(\mathfrak{g})}(x) e^{i B(x, y)} F_{\phi^{\mathrm{G}}}\right|_{\mathbf{W}_{s}}(y) d y d x . \tag{250}
\end{align*}
$$

Lemma 39, together with the identification (43) of $\mathfrak{h}$ and $\mathfrak{h}^{\prime}$, implies that this last expression is a constant multiple of

$$
\begin{align*}
& \sum_{u \in W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(u)\left(\sum_{u^{\prime \prime} \in W\left(\mathbf{G}^{\prime \prime}, \mathfrak{h}^{\prime \prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime \prime} / \mathfrak{h}^{\prime \prime}}\left(u^{\prime \prime}\right) \mathbb{I}_{\{0\}}\left(-\left.(u \mu)\right|_{\mathfrak{h}^{\prime \prime}}+u^{\prime \prime} \rho^{\prime \prime}\right)\right) \\
& \times \int_{\mathfrak{h}^{\prime}} \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{\mathrm{T}}}{ }^{\text {reg }}\right)} \xi_{-u \mu}(\widehat{c}(x)) \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) e^{i B(x, y)} F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y) d y d x \\
& =\left.\sum_{\substack{\left.u \in W(G, \mathfrak{h}(\mathfrak{g})) \\
(u \mu)\right|_{\mathfrak{h}^{\prime \prime}=\rho^{\prime \prime}}}} \operatorname{sgn}_{\mathfrak{g} / \mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}^{\prime}} \xi_{-u \mu}(\widehat{c}(x)) \operatorname{ch}^{2 l^{\prime}-2 l-1}(x) \int_{\tau^{\prime}\left(\mathfrak{h} \overline{\mathrm{T}}^{r e g}\right)} e^{i B(x, y)} F_{\phi^{\mathrm{G}}}\right|_{\mathbf{W}_{s}}(y) d y d x . \tag{251}
\end{align*}
$$

From Corollary 28 we see that (251) is zero unless there is $u \in W(G, \mathfrak{h}(\mathfrak{g}))$ satisfying (178). As in Theorem 5, this imposes the condition (a) on the highest weight $\lambda$ of $\Pi$. Moreover, $\left.u\right|_{\mathfrak{h}^{\prime \prime}}=1$ can be idenfied with its restriction to $\mathfrak{h}^{\prime}$. The sum is hence over $u \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$. For $u \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right) \subseteq W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ and $1 \leq j \leq l$, recall the parameters $a_{u, j}$ and $b_{u, j}$ introduced in (78). Notice that $Q_{b_{u, j}, a_{u, j}}=0$ for $1 \leq j \leq l$ because $a_{u, j}+b_{u, j}=2\left(l-l^{\prime}\right)+1 \geq 2$ for $l>l^{\prime}$.

Hence, as in Theorem 5 (with the parameters $a_{u, j}$ and $b_{u, j}$ reversed because we are evaluating $\xi_{-u \mu}$ at $\widehat{c}(x)$ and not at $\widehat{c}_{-}(x)$ ), formula (251) becomes a constant multiple of

$$
\left.\sum_{u \in W\left(\mathfrak{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(u) \int_{\tau^{\prime}\left(\mathfrak{h}_{\mathrm{T}}{ }^{\text {reg }}\right)}\left(\prod_{j=1}^{l^{\prime}} P_{b_{u, j}, a_{u, j}}\left(2 \pi y_{j}\right)\right) e^{-2 \pi \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} F_{\phi^{\mathrm{G}}}\right|_{\mathrm{W}_{s}}(y) d y
$$

We can replace $u \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$ with $u_{0} u \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$, where $u_{0}=-1$. By the $W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)$ skew invariance of $F_{\left.\phi^{\mathrm{G}}\right|_{\mathrm{w}_{s}}}(y)$ the above integral is equal to

$$
\begin{gathered}
\sum_{u \in W\left(\mathrm{G}^{\prime}, \mathfrak{h}^{\prime}\right)} \operatorname{sgn}_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}(u) \int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{\mathrm{T}}}{ }^{r e g}\right)}\left(\prod_{j=1}^{l^{\prime}} P_{a_{u, j}, b_{u, j}}\left(2 \pi y_{j}\right)\right) e^{-2 \pi \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y) d y \\
=\int_{\tau^{\prime}\left(\mathfrak{h}_{\overline{\mathrm{T}}}{ }^{\text {reg }}\right)}\left(\prod_{j=1} P_{a_{j}, b, j}\left(2 \pi y_{j}\right)\right) e^{-2 \pi \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} F_{\phi^{\mathrm{G}} \mid \mathrm{W}_{s}}(y) d y
\end{gathered}
$$

Finally, by (D.4) and (41), the last integral is a nonzero constant multiple of

$$
\left.\int_{\tau^{\prime}\left(\mathfrak{h}_{\mathrm{T}}{ }^{\text {reg }}\right)}\left(\prod_{j=1}^{l^{\prime}} P_{a_{j}, b, j, 2}\left(2 \pi y_{j}\right)\right) e^{-2 \pi \sum_{j=1}^{l^{\prime}}\left|y_{j}\right|} F_{\phi^{\mathrm{G}}}\right|_{\mathrm{w}_{s}}(y) d y
$$

The final formula (101), as well as the non-vanishing of the distribution $\int_{\mathrm{G}^{0}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g$ when the highest weight $\lambda$ satisfies condition (a), now follows using the steps leading to Theorem 5.

## 14. Proof of Corollary 12

We need to distinguish three cases:
(a) $p=0$, i.e. $l \leq q=l^{\prime}$,
(b) $p \geq 1$ and $p<l=p+q$,
(c) $p \geq 1$ and $p<l<p+q$.

In case (a), the dual pair is $\left(\mathrm{U}_{l}, \mathrm{U}_{l^{\prime}}\right)$, which has both compact members. The condition on $\lambda_{q+1}$ requires that $q+1 \leq l$, which is not possible here. On the other hand, as proved in [MPP23a, Corollary 3] using the explicit intertwining operators for this case, $\Pi$ occurs in Howe's correspondence for ( $\mathrm{U}_{l}, \mathrm{U}_{l^{\prime}}$ ) if and only if $\lambda_{1} \geq \cdots \geq \lambda_{l} \geq \frac{l^{\prime}}{2}=\frac{q}{2}$. This proves, in particular, that $\lambda_{l}<\frac{q}{2}$ cannot occur in Howe's correspondence.

In the cases (b) and (c), we prove that if

$$
\begin{equation*}
\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\widetilde{g}) T(\widetilde{g}) d g \neq 0 \tag{252}
\end{equation*}
$$

then $\lambda_{q+1} \leq \frac{q-p}{2}$ and $\lambda_{l-p} \geq \frac{q-p}{2}$.
Consider first case (b). Then $a_{j}+b_{j}=-2 \delta+2=1$ for all $1 \leq j \leq l$. So $Q_{a_{j}, b_{j}}=0$ for all $1 \leq j \leq l$, and hence, in the notation of (71),

$$
\begin{equation*}
\left.\prod_{j=1}^{l}\left(p_{j}\left(y_{j}\right)\right)+q_{j}\left(-\partial\left(J_{j}\right)\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi}(y)=\left(\prod_{j=1}^{l} P_{a_{j}, b_{j}}\left(\beta y_{j}\right)\right) e^{-\sum_{j=1}^{l}\left|y_{j}\right|} F_{\phi}(y) \tag{253}
\end{equation*}
$$

Moreover, by Lemma D.1, for every $1 \leq j \leq l$, at most one between $P_{a_{j}, b_{j}, 2}$ and $P_{a_{j}, b_{j},-2}$ can be nonzero. By [MPP20, Lemma 3.5] and because $l>p=l-q>0$,

$$
\begin{align*}
\mathfrak{h} \cap \tau(\mathrm{W}) & =W(\mathrm{G}, \mathfrak{h})\left\{y=\sum_{j=1}^{l} y_{j} J_{j}: y_{1}, \ldots, y_{l-q} \geq 0 \geq y_{p+1}, \ldots, y_{l}\right\}  \tag{254}\\
& =\left\{y=\sum_{j=1}^{l} y_{j} J_{j}: p \text { coordinates } y_{j} \text { are } \geq 0 \text { and } q \text { coordinates } y_{j} \text { are } \leq 0\right\} .
\end{align*}
$$

If (252) holds, then $P_{a_{j}, b_{j}, 2} \neq 0$ for $p$ coordinates $y_{j}$ and $P_{a_{j}, b_{j},-2} \neq 0$ for $q$ coordinates $y_{j}$. The first condition is equivalent to $a_{j} \geq 1$ for $p$ values of $j$, which in turn implies that $b_{j}<0$ for $p$ values of $j$. The second condition is equivalent to $b_{j} \geq 1$ for $q(=l-p)$ values of $j$. Since the $b_{j}$ 's are strictly decreasing, we conclude that if (252) holds, then

$$
b_{1}>\cdots>b_{l-p} \geq 1>0>b_{q+1}>\cdots>b_{l} .
$$

Hence $a_{q+1} \geq 0$. By (68) with $\mu=\lambda+\rho$ and (37),

$$
\begin{aligned}
& 1 \leq b_{l-p}=\lambda_{l-p}+\frac{l+1}{2}-(l-p)+\frac{1}{2} \quad \text { if and only if } \quad \lambda_{l-p} \geq \frac{q-p}{2} \\
& 1 \geq a_{q+1}=-\lambda_{q+1}-\frac{l+1}{2}+q+1+\frac{1}{2} \quad \text { if and only if } \quad \lambda_{q+1} \leq \frac{q-p}{2}
\end{aligned}
$$

This proves the claim in the case (b).
Let us now come to case (c). Then $Q_{a_{j}, b_{j}} \neq 0$ for all $1 \leq j \leq l$ because $a_{j}+b_{j}=$ $-2 \delta+2<1$. By (254),

$$
\mathfrak{h} \cap \tau(\mathrm{W})=\bigcup_{s \in W(\mathrm{G}, \mathfrak{h})} Y_{s}
$$

where

$$
\begin{equation*}
Y_{s}=\left\{y=\sum_{j=1}^{l} y_{j} J_{j}: y_{s(1)}, \ldots, y_{s(l-q)} \geq 0 \geq y_{s(p+1)}, \ldots, y_{s(l)}\right\} \tag{255}
\end{equation*}
$$

Hence the integral in (71) is a sum of the integrals over the $Y_{s}$ 's. We consider each of them separately. Let then $s \in W(\mathrm{G}, \mathfrak{h})$ be fixed. For $\gamma \subseteq\{1,2, \ldots, l\}$, let $|\gamma|$ denote its cardinality and set $\gamma^{c}=\{1,2, \ldots, l\} \backslash \gamma$. Hence the integral over $Y_{s}$ is equal to

$$
\begin{equation*}
\sum_{\gamma \subseteq\{1,2, \ldots, l\}} \int_{Y_{s}}\left(\prod_{j \in \gamma^{c}} p_{j}\left(y_{j}\right)\right)\left(\prod_{j \in \gamma} q_{j}\left(-\partial\left(J_{j}\right)\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi}(y) d y \tag{256}
\end{equation*}
$$

where empty products are equal to 1 . As in case (b), by Lemma D.1, for every $1 \leq j \leq l$, at most one between $P_{a_{j}, b_{j}, 2}$ and $P_{a_{j}, b_{j},-2}$ can be nonzero. By (255), if the integral (256) is nonzero then

$$
\begin{array}{ll}
j \in\{s(1), \ldots, s(l-q)\} \cap \gamma^{c} & \text { corresponds to a term } P_{a_{j}, b_{j}, 2} \neq 0, \text { i.e. } a_{j} \geq 1 \text { and } b_{j}<0, \\
j \in\{s(p+1), \ldots, s(l)\} \cap \gamma^{c} & \text { corresponds to a term } P_{a_{j}, b_{j},-2} \neq 0, \text { i.e. } b_{j} \geq 1 .
\end{array}
$$

For $\Gamma \in\left\{\gamma, \gamma^{c}\right\}$, define

$$
Y_{s, \Gamma}=\left\{y_{\Gamma}=\sum_{j \in \Gamma} y_{j} J_{j}:\left\{\begin{array}{l}
y_{j} \geq 0 \text { for all } j \in\{s(1), \ldots, s(l-q)\} \cap \Gamma, \\
y_{j} \leq 0 \text { for all } j \in\{s(p+1), \ldots, s(l)\} \cap \Gamma
\end{array}\right\} .\right.
$$

Then $Y_{s}=Y_{s, \gamma} \times Y_{s, \gamma^{c}}$ and (256) becomes

$$
\begin{align*}
& \sum_{\gamma \subseteq\{1,2, \ldots, l\}}(2 \pi)^{l-|\gamma|} \int_{Y_{s, \gamma^{c}}}\left(\prod_{j \in\{s(1), \ldots, s(l-q)\} \cap \gamma^{c}} P_{a_{j}, b_{j}, 2}\left(\beta y_{j}\right) \mathbb{I}_{\mathbb{R}^{+}}\left(y_{j}\right)\right) \\
& \times\left(\prod_{j \in\{s(p+1), \ldots, s(l)\} \cap \gamma^{c}} P_{a_{j}, b_{j},-2}\left(\beta y_{j}\right) \mathbb{I}_{\mathbb{R}^{-}}\left(y_{j}\right)\right) e^{-\sum_{j \in \gamma^{c}}\left|y_{j}\right|} \\
& \times \int_{Y_{s, \gamma}} e^{-\sum_{j \in \gamma}\left|y_{j}\right|}\left(\prod_{j \in \gamma} q_{j}\left(-\partial\left(J_{j}\right)\right) \delta_{0}\left(y_{j}\right)\right) F_{\phi}(y) d y_{\gamma} d y_{\gamma^{c}} \\
&=\sum_{\gamma \subseteq\{1,2, \ldots, l\}}(2 \pi)^{l-|\gamma|} \int_{Y_{s, \gamma}}\left(\prod_{j \in\{s(1), \ldots, s(l-q)\} \cap \gamma^{c}} P_{a_{j}, b_{j}, 2}\left(\beta y_{j}\right)\right) \\
& \times\left(\prod_{j \in\{s(p+1), \ldots, s(l)\} \cap \gamma^{c}} P_{a_{j}, b_{j},-2}\left(\beta y_{j}\right)\right) e^{-\sum_{j \in \gamma^{c}}\left|y_{j}\right|} \\
& \times\left(\left.\prod_{j \in \gamma} q_{j}\left(-\partial\left(J_{j}\right)\right) F_{\phi}(y)\right|_{y_{j}=0, j \in \gamma}\right) d y_{\gamma^{c}}, \tag{257}
\end{align*}
$$

where empty products are equal to 1 .
Suppose there is $j_{\gamma} \in\{s(1), \ldots, s(l-q)\} \cap \gamma$. Every $y=\sum_{j=1}^{l} y_{j} J_{j}$ with $y_{j} \geq 0$ for $j \in\{s(1), \ldots, s(l-q)\} \cap \gamma^{c}, y_{j} \leq 0$ for $j \in\{s(p+1), \ldots, s(l)\} \cap \gamma^{c}$ and $y_{j}=0$ for $j \in \gamma$ belongs to

$$
\left\{y=\sum_{j=1}^{l} y_{j} J_{j}:\left\{\begin{array}{l}
y_{j} \geq 0 \text { for all } j \in\{s(1), \ldots, s(l-q)\} \backslash\left\{j_{\gamma}\right\}, \\
y_{j} \leq 0 \text { for all } j \in\{s(p+1), \ldots, s(l)\}, \\
y_{j_{\gamma}}=0
\end{array}\right\} \subseteq \partial(\mathfrak{h} \cap \tau(\mathrm{W})),\right.
$$

where $\partial(\mathfrak{h} \cap \tau(\mathrm{W}))$ denotes the boundary of $\mathfrak{h} \cap \tau(\mathrm{W})$. For all $1 \leq j \leq l$,

$$
\operatorname{deg} Q_{a_{j}, b_{j}}=a_{j}+b_{j}=-2 \delta+2=l-p-q+1 \leq p+q-l-1,
$$

because $l<p+q$. Hence the term $\left.\left(\prod_{j \in \gamma} q_{j}\left(-\partial\left(J_{j}\right)\right) F_{\phi}(y)\right)\right|_{y_{j}=0, j \in \gamma}$ is zero on $\partial(\mathfrak{h} \cap \tau(\mathrm{W}))$ by [MPP20, Theorem 3.6]. Choosing $j=j_{\gamma}$, we see that the integral corresponding to $\gamma$ in (257) vanishes. Similarly, the integral corresponding to $\gamma$ vanishes if there is $j_{\gamma} \in\{s(p+1), \ldots, s(l)\} \cap \gamma$. The sum in (257) therefore reduces to a sum over the $\gamma$ having no intersection with $\{s(1), \ldots, s(l-q)\} \cup\{s(p+1), \ldots, s(l)\}$. For these $\gamma$ 's,

$$
\begin{aligned}
& \{s(1), \ldots, s(l-q)\} \cap \gamma^{c}=\{s(1), \ldots, s(l-q)\} \\
& \{s(p+1), \ldots, s(l)\} \cap \gamma^{c}=\{s(p+1), \ldots, s(l)\} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& b_{s(j)}<0 \quad \text { for } 1 \leq j \leq l-q \\
& b_{s(j)} \geq 1 \quad \text { for } p+1 \leq j \leq l
\end{aligned}
$$

This means that, if $j_{0}$ is chosen such that

$$
b_{1}>\cdots>b_{j_{0}} \geq 1>0>b_{j_{0}+1}>\cdots>b_{l}
$$

then

$$
\{s(p+1), \ldots, s(l)\} \subseteq\left\{1, \ldots, j_{0}\right\} \quad \text { and } \quad\{s(1), \ldots, s(l-q)\} \subseteq\left\{j_{0}+1, \ldots, l\right\}
$$

In particular, there are at least $l-p$ elements $b_{j} \geq 1$. So $b_{l-p} \geq 1$. Similarly, there are at least $l-q$ elements $b_{j}<0$. So $b_{q+1}<0$, i.e. $a_{q+1} \geq 1$. As in the case (b), we conclude that if the integral over $Y_{s}$ corresponing to this $\gamma$ is not zero, then $\lambda_{l-p} \geq \frac{q-p}{2}$ and $\lambda_{q+1} \leq \frac{q-p}{2}$. This applies to all $\gamma$ and all $s$. Hence, if (252) holds, then $\lambda_{l-p} \geq \frac{q-p}{2}$ and $\lambda_{q+1} \leq \frac{q-p}{2}$.

## 15. A sketch of a computation of the wave front of $\Pi^{\prime}$

Corollary 40. For any representation $\Pi \otimes \Pi^{\prime}$ which occurs in the restriction of the Weil representation to the dual pair $\left(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}^{\prime}\right)$,

$$
W F\left(\Pi^{\prime}\right)=\tau^{\prime}\left(\tau^{-1}(0)\right)
$$

Here $W F\left(\Pi^{\prime}\right)$ stands for the wave front of the character $\Theta_{\Pi^{\prime}}$ at the identity and $0=$ $W F(\Pi)$ since $\Pi$ is finite dimensional.

The complete proof is rather lengthy but unlike the one provided in [Prz91, Theorem $6.11]$, it is independent of [ $\operatorname{Vog} 78]$. We sketch the main steps below. The details are going to appear in [MPP23b].

The variety $\tau^{-1}(0) \subseteq \mathrm{W}$ is the closure of a single $\mathrm{GG}^{\prime}$-orbit $\mathcal{O}$; see e.g. [Prz91, Lemma 2.16]. There is a positive $\mathrm{GG}^{\prime}$-invariant measure $\mu_{\mathcal{O}}$ on this orbit which defines a homogeneous distribution. We denote its degree by $\operatorname{deg} \mu_{\mathcal{O}}$.

Recall that if V is a $n$-dimensional real vector space, $t>0$ and $M_{t} v=t v$ for $v \in \mathrm{~V}$, then the pullback of $u \in \mathcal{S}^{\prime}(V)$ by $M_{t}$ is $M_{t}^{*} u \in \mathcal{S}^{\prime}(\mathrm{V})$, defined by

$$
\left(M_{t}^{*} u\right)(\phi)=t^{-n} u\left(\phi \circ M_{t^{-1}}\right) \quad(\phi \in \mathcal{S}(\mathrm{V}))
$$

In particular, for $\mathrm{V}=\mathrm{W}$

$$
M_{t}^{*} \mu_{\mathcal{O}}=t^{\operatorname{deg} \mu_{\mathcal{O}}} \mu_{\mathcal{O}}
$$

Define $\tau_{*}^{\prime}: \mathcal{S}^{\prime}(\mathrm{W}) \rightarrow \mathcal{S}^{\prime}\left(\mathfrak{g}^{\prime}\right)$ by $\tau_{*}^{\prime}(u)(\psi)=u\left(\psi \circ \tau^{\prime}\right)$. Then, for $t>0$,

$$
\begin{equation*}
t^{2} \operatorname{dim} \mathfrak{g}^{\prime} M_{t^{2}}^{*} \circ \tau_{*}^{\prime}=t^{\operatorname{dim} \mathrm{W}} \tau_{*}^{\prime} \circ M_{t}^{*} \tag{258}
\end{equation*}
$$

A rather lengthy but straightforward computation based on Theorems 4, 5 and 7 , shows that

$$
\begin{equation*}
t^{\operatorname{deg} \mu_{\mathcal{O}}} M_{t^{-1}}^{*} f_{\Pi \otimes \Pi^{\prime}} \underset{t \rightarrow 0}{\rightarrow} C \mu_{\mathcal{O}} \tag{259}
\end{equation*}
$$

as tempered distributions on W , where $C$ is a non-zero constant.
Let $\mathcal{F}$ indicate a Fourier transform on $\mathcal{S}^{\prime}\left(\mathfrak{g}^{\prime}\right)$. Then, for $t>0$,

$$
\begin{equation*}
M_{t}^{*} \circ \mathcal{F}=t^{-\operatorname{dim} \mathfrak{g}^{\prime}} \mathcal{F} \circ M_{t^{-1}} \tag{260}
\end{equation*}
$$

Hence, in the topology of $\mathcal{S}^{\prime}\left(\mathfrak{g}^{\prime}\right)$,

$$
\begin{equation*}
t^{2 \operatorname{deg} \mu_{\mathcal{O}^{\prime}}} M_{t^{2}}^{*} \mathcal{F} \tau_{*}^{\prime}\left(f_{\Pi \otimes \Pi^{\prime}}\right) \underset{t \rightarrow 0+}{\rightarrow} C \mathcal{F} \mu_{\mathcal{O}^{\prime}} \tag{261}
\end{equation*}
$$

where $C \neq 0$ and $\mathcal{O}^{\prime}=\tau^{\prime}(\mathcal{O})$.
There is an easy to verify inclusion $W F\left(\Pi^{\prime}\right) \subseteq \overline{\mathcal{O}^{\prime}},[\operatorname{Prz91,~(6.14)]~and~a~formula~for~the~}$ character $\Theta_{\Pi^{\prime}}$ in terms of $\mathcal{F}\left(\tau_{*}^{\prime}\left(f_{\Pi \otimes \Pi^{\prime}}\right)\right)$,

$$
\begin{equation*}
\frac{1}{\sigma} \cdot \tilde{c}_{-}^{*} \Theta_{\Pi^{\prime}}=\widehat{\tau_{*}^{\prime}\left(f_{\Pi \otimes \Pi^{\prime}}\right)} \tag{262}
\end{equation*}
$$

where $\sigma$ is a smooth function, [Prz91, Theorem 6.7]. By combining this with the following elementary lemma, one completes the argument.

Lemma 41. Suppose $f, u \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and $u$ is homogeneous of degree $d \in \mathbb{C}$. Suppose

$$
\begin{equation*}
t^{d} M_{t^{-1}}^{*} f(\psi) \underset{t \rightarrow 0+}{\rightarrow} u(\psi) \quad\left(\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)\right) \tag{263}
\end{equation*}
$$

Then

$$
\begin{equation*}
W F_{0}(\hat{f}) \supseteq \operatorname{supp} u \tag{264}
\end{equation*}
$$

where the subscript 0 indicates the wave front set at zero and

$$
f(x)=\int_{\mathbb{R}^{n}} \hat{f}(y) e^{2 \pi i x \cdot y} d y
$$

## Appendix A. Products of positive roots

Keep the notation introduced in section 3. Recall, in particular, that $\left.\sum_{j=1}^{l^{\prime \prime}} y_{j} J_{j} \in \mathfrak{h}_{\frac{2}{1}}^{2} \right\rvert\, \mathrm{V}_{\overline{0}}$ and $\sum_{j=1}^{l^{\prime \prime}} y_{j} J_{j}^{\prime} \in \mathfrak{h}_{\overline{1}}^{2} \mid v_{\overline{1}}$ are identified via (43). Here $l^{\prime \prime}=\min \left\{l, l^{\prime}\right\}$.

Suppose $l \leq l^{\prime}$. We can choose the system of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ so that their product is given by the formula

$$
\begin{align*}
& \pi_{\mathfrak{g} / \mathfrak{h}}\left(\sum_{j=1}^{l} y_{j} J_{j}\right)  \tag{A.1}\\
& \quad= \begin{cases}\prod_{1 \leq j<k \leq l} i\left(-y_{j}+y_{k}\right) & \text { if } \quad \mathbb{D}=\mathbb{C}, \\
\prod_{1 \leq j<k \leq l}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l}\left(-2 i y_{j}\right) & \text { if } \\
\prod_{1 \leq j<k \leq l}\left(-y_{j}^{2}+y_{k}^{2}\right) & \text { if } \quad \mathbb{D}=\mathbb{R} \text { and } \mathfrak{g}=\mathfrak{s o}_{2 l}, \\
\prod_{1 \leq j<k \leq l}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l}\left(-i y_{j}\right) & \text { if } \\
\mathbb{D}=\mathbb{R} \text { and } \mathfrak{g}=\mathfrak{s o}_{2 l+1} .\end{cases}
\end{align*}
$$

Let $\mathfrak{z}^{\prime} \subseteq \mathfrak{g}^{\prime}$ be the centralizer of $\mathfrak{h}$. We may choose the order of roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}^{\prime} / \mathfrak{z}_{\mathbb{C}}^{\prime}$ so that the product of all of them is equal to

$$
\begin{align*}
& \pi_{\mathfrak{g}^{\prime} / \mathfrak{z}^{\prime}}\left(\sum_{j=1}^{l} y_{j} J_{j}^{\prime}\right)  \tag{A.2}\\
& =\left\{\begin{array}{lll}
\prod_{1 \leq j<k \leq l} i\left(-y_{j}+y_{k}\right) \cdot \prod_{j=1}^{l}\left(-i y_{j}\right)^{d^{\prime}-d} & \text { if } & \mathbb{D}=\mathbb{C}, \\
\prod_{1 \leq j<k \leq l}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l}\left(-y_{j}^{2}\right)^{d^{\prime}-d} & \text { if } & \mathbb{D}=\mathbb{H}, \\
\prod_{1 \leq j<k \leq l}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l}\left(-2 i y_{j}\right) \cdot \prod_{j=1}^{l}\left(-i y_{j}\right)^{d^{\prime}-d} & \text { if } & \mathbb{D}=\mathbb{R} \text { and } \mathfrak{g}=\mathfrak{s o}_{2 l}, \\
\prod_{1 \leq j<k \leq l}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l}\left(-2 i y_{j}\right) \cdot \prod_{j=1}^{l}\left(-i y_{j}\right)^{d^{\prime}-d+1} & \text { if } & \mathbb{D}=\mathbb{R} \text { and } \mathfrak{g}=\mathfrak{s o}_{2 l+1} .
\end{array}\right.
\end{align*}
$$

Suppose $l>l^{\prime}$. We can choose the system of the positive roots of $\mathfrak{h}^{\prime}$ in $\mathfrak{g}_{\mathbb{C}}^{\prime}$ so that their product is given by the formula

$$
\pi_{\mathfrak{g}^{\prime} / \mathfrak{h}^{\prime}}\left(\sum_{j=1}^{l^{\prime}} y_{j} J_{j}^{\prime}\right)= \begin{cases}\prod_{1 \leq j<k \leq l^{\prime}} i\left(-y_{j}+y_{k}\right) & \text { if } \quad \mathbb{D}=\mathbb{C}  \tag{A.3}\\ \prod_{1 \leq j<k \leq l^{\prime}}\left(-y_{j}^{2}+y_{k}^{2}\right) & \text { if } \mathbb{D}=\mathbb{H} \\ \prod_{1 \leq j<k \leq l^{\prime}}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l^{\prime}}\left(-2 i y_{j}\right) & \text { if } \quad \mathbb{D}=\mathbb{R}\end{cases}
$$

Moreover, let $\mathfrak{z} \subseteq \mathfrak{g}$ be the centralizer of $\mathfrak{h}$. We may choose the positive roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}} / \mathfrak{z} \mathbb{C}$ so that their product is equal to

$$
\begin{align*}
& \pi_{\mathfrak{g} / \mathfrak{s}}\left(\sum_{j=1}^{l^{\prime}} y_{j} J_{j}\right)  \tag{A.4}\\
& \quad= \begin{cases}\prod_{1 \leq j<k \leq l^{\prime}} i\left(-y_{j}+y_{k}\right) \cdot \prod_{j=1}^{l^{\prime}}\left(-i y_{j}\right)^{d-d^{\prime}} & \text { if } \quad \mathbb{D}=\mathbb{C}, \\
\prod_{1 \leq j<k \leq l^{\prime}}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l^{\prime}} 2 i y_{j} \cdot \prod_{j=1}^{l^{\prime}}\left(-y_{j}^{2}\right)^{d-d^{\prime}} & \text { if } \quad \mathbb{D}=\mathbb{H}, \\
\prod_{1 \leq j<k \leq l^{\prime}}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l^{\prime}}\left(-i y_{j}\right)^{d-d^{\prime}} & \text { if } \quad \mathbb{D}=\mathbb{R} \text { and } \mathfrak{g}=\mathfrak{s o}_{2 l}, \\
\prod_{1 \leq j<k \leq l}\left(-y_{j}^{2}+y_{k}^{2}\right) \cdot \prod_{j=1}^{l^{\prime}}\left(-i y_{j}\right) \cdot \prod_{j=1}^{l^{\prime}}\left(-i y_{j}\right)^{d-d^{\prime}} & \text { if } \quad \mathbb{D}=\mathbb{R} \text { and } \mathfrak{g}=\mathfrak{s o}_{2 l+1} .\end{cases}
\end{align*}
$$

## Appendix B. The Jacobian of the Cayley transform

Here we determine the Jacobian of the Cayley transform $c_{-}: \mathfrak{g} \rightarrow$ G. A straightforward computation shows that for a fixed $x \in \mathfrak{g}$,

$$
c_{-}(x+y) c_{-}(x)^{-1}-1=(1-x-y)^{-1} 2 y(1+x)^{-1} \quad(y \in \mathfrak{g}) .
$$

Hence the derivative (tangent map) is given by

$$
\begin{equation*}
c_{-}^{\prime}(x) y=(1-x)^{-1} 2 y(1-x)^{-1} \quad(y \in \mathfrak{g}) . \tag{B.1}
\end{equation*}
$$

Recall that G is the isometry group of a hermitian form $(\cdot, \cdot)$ on V. Hence we have the adjoint

$$
\operatorname{End}_{\mathbb{D}}(\mathrm{V}) \ni g \rightarrow g^{*} \in \operatorname{End}_{\mathbb{D}}(\mathrm{V})
$$

defined by

$$
(g u, v)=\left(u, g^{*} v\right) \quad(u, v \in \mathrm{~V}) .
$$

Let us view the Lie algebra $\mathfrak{g}$ as a real vector space and consider the map

$$
\gamma: \mathrm{GL}_{\mathbb{D}}(\mathrm{V}) \rightarrow \mathrm{GL}(\mathfrak{g}), \quad \gamma(g)(y)=g y g^{*}
$$

Then det o $\gamma: \operatorname{GL}_{\mathbb{D}}(\mathrm{V}) \rightarrow \mathbb{R}^{\times}$is a group homomorphism. Hence there is a number $s \in \mathbb{R}$ such that

$$
\operatorname{det}(\gamma(g))=\left(\operatorname{det}(g)_{\mathrm{V}_{\mathbb{R}}}\right)^{s} \quad\left(g \in \mathrm{GL}_{\mathbb{D}}(\mathrm{V})\right),
$$

where the subscript $\mathbb{R}$ indicates that we are viewing $V$ as a vector space over $\mathbb{R}$. On the other hand, for a fixed number $a \in \mathbb{R}^{\times}$,

$$
\operatorname{det}\left(\gamma\left(a I_{\mathrm{V}}\right)\right)=a^{2 \operatorname{dim} \mathfrak{g}} \quad \text { and } \quad \operatorname{det}\left(a I_{\mathrm{V}}\right)_{\mathrm{V}_{\mathbb{R}}}=a^{\operatorname{dim} \mathrm{V}_{\mathbb{R}}}
$$

Hence,

$$
\operatorname{det}(\gamma(g))=\left(\operatorname{det}(g)_{\mathrm{V}_{\mathbb{R}}}\right)^{\frac{2 \operatorname{dim} \mathrm{~m}}{\operatorname{dim} V_{\mathbb{R}}}} \quad\left(g \in \mathrm{GL}_{\mathbb{D}}(\mathrm{V})\right)
$$

If $x \in \mathfrak{g}$, then $1 \pm x \in \mathrm{GL}_{\mathbb{D}}(\mathrm{V})$ and

$$
(1 \pm x)^{*}=1 \mp x \quad \text { and } \quad\left((1 \pm x)^{-1}\right)^{*}=(1 \mp x)^{-1}
$$

Hence

$$
c_{-}^{\prime}(x) y=2(1-x)^{-1} y(1+x)^{-1} c_{-}(x)=2\left(\gamma\left((1-x)^{-1}\right) y\right) c_{-}(x) \quad(y \in \mathfrak{g}) .
$$

Notice that $\left|\operatorname{det}\left(c_{-}(x)\right)\right|=1$ because $c(\mathfrak{g}) \subseteq G$. Therefore

$$
\begin{equation*}
\left|\operatorname{det}\left(c_{-}^{\prime}(x)\right)\right|=2^{\operatorname{dim} \mathfrak{g}} \operatorname{det}(1-x)_{V_{\mathbb{R}}}^{-\frac{2 \operatorname{dim} \mathfrak{g}}{\operatorname{dim} V_{\mathbb{R}}}}=2^{\operatorname{dim} \mathfrak{g}} \operatorname{ch}(x)^{-2 r} \quad(x \in \mathfrak{g}), \tag{B.2}
\end{equation*}
$$

where ch and $r$ are as in (138) and (65), respectively.

## Appendix C. The Weyl denominator lifted by the Cayley transform

Consider the orthogonal matrix group

$$
\mathrm{G}=\mathrm{O}_{2 l+1}=\left\{g \in \mathrm{GL}_{2 l+1}(\mathbb{R}) ; g g^{t}=I\right\}
$$

and keep the notation introduced in sections 9 and 12. In particular, recall the Cayley transform $c: \mathfrak{h}_{s} \rightarrow \mathrm{H}_{s}^{0}$ defined in (236). In this appendix, we and verify the following two formulas:

$$
\begin{align*}
\Delta\left(\widehat{c}_{-}(x)\right) & =C_{1} \pi_{\mathfrak{g} / \mathfrak{h}}(x) \operatorname{ch}^{-2 l+1}(x) & (x \in \mathfrak{h}),  \tag{C.1}\\
\Delta(\widehat{c}(x)) & =C_{2} \pi_{\mathfrak{g}_{s} / \mathfrak{h}_{s}}(x) \operatorname{ch}^{-2 l+1}(x) & \left(x \in \mathfrak{h}_{s}=\mathfrak{h}\right), \tag{C.2}
\end{align*}
$$

where $C_{1}=(-1)^{l} 2^{l^{2}}$ and $C_{2}=i^{l} 2^{l^{2}}$. Notice that (C.1) is a special case of (139) for $\mathrm{G}=\mathrm{O}_{2 l+1}$. The proofs of formula (139) for the other cases are similar to the proof of (C.1).

We identify

$$
a+i b=\left(\begin{array}{cc}
a & b \\
-b & a
\end{array}\right) \quad(a, b \in \mathbb{R})
$$

Then

$$
\mathrm{SO}_{2}(\mathbb{R})=\{u \in \mathbb{C} ;|u|=1\}
$$

Fix the diagonal Cartan subgroup

$$
\mathrm{H}=\left\{\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, \pm 1\right) ; u_{j} \in \mathrm{SO}_{2}(\mathbb{R}), 1 \leq j \leq l\right\}
$$

Then the connected identity component of H is

$$
\mathrm{H}^{0}=\left\{\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1\right) ; u_{j} \in \mathrm{SO}_{2}(\mathbb{R}) ; 1 \leq j \leq l\right\}
$$

The group $\widehat{\mathrm{H}^{0}}$ may be realized as

$$
\widehat{\mathrm{H}^{0}}=\left\{\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1 ; \xi\right) ; u_{j}, \xi \in \mathrm{SO}_{2}(\mathbb{R}), 1 \leq j \leq l ; \xi^{2}=u_{1} u_{2} \cdots u_{l}\right\}
$$

and the covering map (127) is

$$
\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1 ; \xi\right) \longrightarrow \operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1\right)
$$

In these terms, the usual choice of the positive roots $e_{j} \pm e_{k}$, with $1 \leq j<k \leq l$, and $e_{j}$, with $1 \leq j \leq l$, gives

$$
\begin{aligned}
\xi_{-e_{j}+e_{k}}\left(\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1 ; \xi\right)\right) & =u_{j}^{-1} u_{k} \\
\xi_{-e_{j}-e_{k}}\left(\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1 ; \xi\right)\right) & =u_{j}^{-1} u_{k}^{-1} \\
\xi_{-e_{j}}\left(\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1 ; \xi\right)\right) & =u_{j}^{-1} \\
\xi_{\rho}\left(\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1 ; \xi\right)\right) & =u_{1}^{l-1} u_{2}^{l-2} \cdots u_{l-1} \xi
\end{aligned}
$$

We now prove (C.1). If $x=\operatorname{diag}\left(i x_{1}, i x_{2}, \ldots, i x_{l}, 0\right) \in \mathfrak{h}$, then $c_{-}(x)=\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1\right)$ has coordinates

$$
u_{j}=\frac{1+z_{j}}{1-z_{j}}, \quad z_{j}=i x_{j}, \quad 1 \leq j \leq l
$$

with $\left|u_{j}\right|=1$ and $u_{j} \neq-1$ for all $j$. We shall use the branch of the square root

$$
\sqrt{r e^{i \theta}}=\sqrt{r} e^{i \frac{\theta}{2}} \quad(r>0,-\pi<\theta<\pi)
$$

applied to the coordinates $u_{j}$ of the elements in $c_{-}(\mathfrak{h})$. It is easy to check that, for this choice,

$$
\begin{align*}
& \sqrt{u_{j}}=\sqrt{\frac{1+z_{j}}{1-z_{j}}}=\frac{\sqrt{1+z_{j}}}{\sqrt{1-z_{j}}}, \quad 1+z_{j}=\sqrt{1+z_{j}} \sqrt{1+z_{j}}, \quad \sqrt{1+x_{j}^{2}}=\sqrt{1+z_{j}} \sqrt{1-z_{j}} \\
&\left(z_{j}=i x_{j}, x_{j} \in \mathbb{R}\right) . \tag{C.3}
\end{align*}
$$

We choose the section $\sigma: c_{-}(\mathfrak{h}) \rightarrow \widehat{\mathrm{H}^{0}}$ of the covering map for which $\xi=\sqrt{u_{1}} \sqrt{u_{2}} \cdots \sqrt{u_{l}}$, i.e.

$$
\sigma\left(\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1\right)\right)=\operatorname{diag}\left(u_{1}, u_{2}, \ldots, u_{l}, 1 ; \sqrt{u_{1}} \sqrt{u_{2}} \cdots \sqrt{u_{l}}\right)
$$

Recall that $\widehat{c}_{-}(x)=\left(\sigma\left(c_{-}(x)\right), 1\right)$, see (132). We shall use the polynomial identity

$$
\begin{equation*}
\prod_{1 \leq j<k \leq l} a_{j} b_{k}=\left(\prod_{j=1}^{l} a_{j}^{l-j}\right)\left(\prod_{k=1}^{l} b_{k}^{k-1}\right) \tag{C.4}
\end{equation*}
$$

when either $b_{j}=1$ or $b_{j}=a_{j}$ for all $1 \leq j \leq l$. By (128) and (C.4),

$$
\begin{aligned}
\Delta\left(\widehat{c}_{-}(x)\right) & =\xi\left(\prod_{j=1}^{l-1} u_{j}^{l-j}\right) \prod_{1 \leq j<k \leq l}\left(1-u_{j}^{-1} u_{k}^{-1}\right)\left(1-u_{j}^{-1} u_{k}\right) \prod_{j=1}^{l}\left(1-u_{j}^{-1}\right) \\
& =\xi \prod_{1 \leq j<k \leq l}\left(u_{j}-u_{k}^{-1}\right)\left(1-u_{j}^{-1} u_{k}\right) \prod_{j=1}^{l}\left(1-u_{j}^{-1}\right) .
\end{aligned}
$$

By (C.3),

$$
\begin{aligned}
u_{j}-u_{k}^{-1} & =\frac{1+z_{j}}{1-z_{j}}-\frac{1-z_{k}}{1+z_{k}}=\frac{2\left(z_{j}+z_{k}\right)}{\left(1-z_{j}\right)\left(1+z_{k}\right)}, \\
1-u_{j}^{-1} u_{k} & =1-\frac{1-z_{j}}{1+z_{j}} \frac{1+z_{k}}{1-z_{k}}=\frac{2\left(z_{j}-z_{k}\right)}{\left(1+z_{j}\right)\left(1-z_{k}\right)}, \\
1-u_{j}^{-1} & =1-\frac{1-z_{j}}{1+z_{j}}=\frac{2 z_{j}}{1+z_{j}}=\frac{2 z_{j}}{\sqrt{1+z_{j}} \sqrt{1+z_{j}}} .
\end{aligned}
$$

Since $\xi=\prod_{j=1}^{l} \sqrt{u_{j}}$, we obtain by (C.3), (C.4) and (A.1),

$$
\begin{aligned}
\Delta\left(\widehat{c}_{-}(x)\right) & =2^{l^{2}} \prod_{1 \leq j<k \leq l} \frac{1}{\left(1-z_{j}^{2}\right)\left(1-z_{k}^{2}\right)} \prod_{j=1}^{l} \frac{1}{\sqrt{1-z_{j}} \sqrt{1+z_{j}}} \prod_{1 \leq j<k \leq l}\left(z_{k}+z_{j}\right)\left(z_{j}-z_{k}\right) \prod_{j=1}^{l} z_{j} \\
& =(-1)^{l} 2^{l^{2}}\left(\prod_{j=1}^{l} \frac{1}{\left(1-z_{j}^{2}\right)^{l-1}} \prod_{j=1}^{l} \frac{1}{1+x_{j}^{2}}\right) \pi_{\mathfrak{g} / \mathfrak{h}}(x) \quad(x \in \mathfrak{h}),
\end{aligned}
$$

which gives (C.1).
Let us now consider (C.2). If $x=\operatorname{diag}\left(i x_{1}, i x_{2}, \ldots, i x_{l}, 0\right) \in \mathfrak{h}$, then $c(x)=\operatorname{diag}\left(v_{1}, v_{2}, \ldots, v_{l}, 1\right)$ has coordinates

$$
v_{j}=\frac{z_{j}+1}{z_{j}-1}=-u_{j}, \quad z_{j}=i x_{j}, \quad 1 \leq j \leq l
$$

with $\left|v_{j}\right|=1$ and $v_{j} \neq 1$ for all $j$. We shall use the branch of the square root

$$
\sqrt{r e^{i \theta}}=\sqrt{r} e^{i \frac{\theta}{2}} \quad(r>0,0<\theta<2 \pi)
$$

applied to the coordinates $v_{j}$ of the elements in $c(\mathfrak{h})$. Here $\sqrt{r}$ is the usual square root of positive numbers, so that $\sqrt{ }$ has a one-sided continuous extension to $0 \leq \theta<2 \pi$. It is easy to check that for this choice of square roots

$$
\begin{align*}
& \sqrt{v_{j}=} \sqrt{\frac{z_{j}+1}{z_{j}-1}}=-\operatorname{sign}\left(x_{j}\right) \frac{\sqrt{z_{j}+1}}{\sqrt{z_{j}-1}}, \quad z_{j}+1=\sqrt{z_{j}+1} \sqrt{z_{j}+1} \\
& \quad i \operatorname{sign}\left(x_{j}\right) \sqrt{1+x_{j}^{2}}=\sqrt{z_{j}+1} \sqrt{z_{j}-1} \quad\left(z_{j}=i x_{j}, x_{j} \in \mathbb{R} \backslash\{0\}\right), \tag{C.5}
\end{align*}
$$

where $\operatorname{sign}(x)=x /|x|$. We choose the section $\sigma: c(\mathfrak{h}) \rightarrow \widehat{\mathrm{H}}^{0}$ of the covering map for which $\xi=\sqrt{v_{1}} \sqrt{v_{2}} \cdots \sqrt{v_{l}}$, and recall that $\widehat{c}(x)=(\sigma(c(x)), 1)$, see (132).

As before,

$$
\Delta(\widehat{c}(x))=\xi \prod_{1 \leq j<k \leq l}\left(v_{j}-v_{k}^{-1}\right)\left(1-v_{j}^{-1} v_{k}\right) \prod_{j=1}^{l}\left(1-v_{j}^{-1}\right),
$$

where, by (C.5),

$$
\begin{aligned}
v_{j}-v_{k}^{-1} & =\frac{z_{j}+1}{z_{j}-1}-\frac{z_{k}-1}{z_{k}+1}=\frac{2\left(z_{j}+z_{k}\right)}{\left(z_{j}-1\right)\left(z_{k}-1\right)} \\
1-v_{j}^{-1} v_{k} & =1-\frac{z_{j}-1}{z_{j}+1} \frac{z_{k}+1}{z_{k}-1}=\frac{2\left(z_{j}-z_{k}\right)}{\left(z_{j}+1\right)\left(z_{k}-1\right)} \\
1-v_{j}^{-1} & =1-\frac{z_{j}-1}{z_{j}+1}=\frac{2}{z_{j}+1}=\frac{2}{\sqrt{z_{j}+1} \sqrt{z_{j}+1}} .
\end{aligned}
$$

Since $\xi=\prod_{j=1}^{l} \sqrt{v_{j}}$, we obtain by (C.3), (C.4) and (A.1),

$$
\begin{aligned}
\Delta(\widehat{c}(x)) & =2^{l^{2}} \prod_{j=1}^{l} \sqrt{\frac{z_{j}+1}{z_{j}-1}} \prod_{1 \leq j<k \leq l} \frac{1}{\left(z_{j}^{2}-1\right)\left(z_{k}^{2}-1\right)} \prod_{j=1}^{l} \frac{1}{\sqrt{z_{j}+1} \sqrt{z_{j}+1}} \prod_{1 \leq j<k \leq l}\left(z_{k}+z_{j}\right)\left(z_{j}-z_{k}\right) \\
& =2^{l^{2}}(-1)^{l} \prod_{j=1}^{l} \operatorname{sign}\left(x_{j}\right) \frac{\sqrt{z_{j}+1}}{\sqrt{z_{j}-1}} \frac{1}{\sqrt{z_{j}+1} \sqrt{z_{j}+1}} \prod_{j=1}^{l} \frac{1}{\left(1-z_{j}^{2}\right)^{l-1}} \pi_{\mathfrak{g}_{s} / \mathfrak{h}_{s}}(x) \\
& =2^{l^{2}}(-1)^{l} \prod_{j=1}^{l} \operatorname{sign}\left(x_{j}\right) \frac{1}{\sqrt{z_{j}-1} \sqrt{z_{j}+1}} \prod_{j=1}^{l} \frac{1}{\left(1-z_{j}^{2}\right)^{l-1}} \pi_{\mathfrak{g}_{s} / \mathfrak{h}_{s}}(x) \\
& =2^{l^{2}} i^{l}\left(\prod_{j=1}^{l} \frac{1}{\sqrt{1+x_{j}^{2}}} \frac{1}{\left(1+x_{j}^{2}\right)^{l-1}}\right) \pi_{\mathfrak{g}_{s} / h_{s}}(x) \quad(x \in \mathfrak{h} \backslash\{0\}),
\end{aligned}
$$

which extends by continuity to $\mathfrak{h}$ and gives (C.2).

## Appendix D. The special functions $P_{a, b}$ and $Q_{a, b}$

For two integers $a$ and $b$ define the following functions in the real variable $\xi$,

$$
\begin{align*}
P_{a, b, 2}(\xi) & = \begin{cases}\sum_{k=0}^{b-1} \frac{a(a+1) \cdots(a+k-1)}{k!(b-1-k)!} 2^{-a-k} \xi^{b-1-k} & \text { if } b \geq 1 \\
0 & \text { if } b \leq 0,\end{cases}  \tag{D.1}\\
P_{a, b,-2}(\xi) & = \begin{cases}(-1)^{a+b-1} \sum_{k=0}^{a-1} \frac{b(b+1) \cdots(b+k-1)}{k!(a-1-k)!}(-2)^{-b-k} \xi^{a-1-k} & \text { if } a \geq 1 \\
0 & \text { if } a \leq 0\end{cases} \tag{D.2}
\end{align*}
$$

where $a(a+1) \cdots(a+k-1)=1$ if $k=0$. Notice that

$$
\begin{equation*}
P_{a, b,-2}(\xi)=P_{b, a, 2}(-\xi) \quad(\xi \in \mathbb{R}, a, b \in \mathbb{Z}) \tag{D.3}
\end{equation*}
$$

Set

$$
\begin{align*}
P_{a, b}(\xi) & =2 \pi\left(P_{a, b, 2}(\xi) \mathbb{I}_{\mathbb{R}^{+}}(\xi)+P_{a, b,-2}(\xi) \mathbb{I}_{\mathbb{R}^{-}}(\xi)\right)  \tag{D.4}\\
& =2 \pi\left(P_{a, b, 2}(\xi) \mathbb{I}_{\mathbb{R}^{+}}(\xi)+P_{b, a, 2}(-\xi) \mathbb{I}_{\mathbb{R}^{+}}(-\xi)\right),
\end{align*}
$$

where $\mathbb{I}_{S}$ denotes the indicator function of the set $S$. Also, let

$$
Q_{a, b}(i y)=2 \pi \begin{cases}0 & \text { if } a+b \geq 1  \tag{D.5}\\ \sum_{k=b}^{-a} \frac{a(a+1) \cdots(a+k-1)}{} 2^{-a-k}(1-i y)^{k-b} & \text { if }-a>b-1 \geq 0 \\ \sum_{k=a}^{-b} \frac{b(b+1) \cdots(b+k-1)}{k!} 2^{-b-k}(1+i y)^{k-a} & \text { if }-b>a-1 \geq 0 \\ (1+i y)^{-a}(1-i y)^{-b} & \text { if } a \leq 0 \text { and } b \leq 0\end{cases}
$$

Observe also that

$$
\begin{equation*}
P_{b, a}(\xi)=P_{a, b}(-\xi) \quad \text { and } \quad Q_{b, a}(i y)=Q_{a, b}(-i y) \tag{D.6}
\end{equation*}
$$

The following elementary fact will be crucial at several points.
Lemma D.1. Suppose that $a+b \leq 1$. Then at most one between $P_{a, b, 2}$ and $P_{a, b,-2}$ can be non-zero. Hence $P_{a, b}$ is either 0 or a the restriction of a polynomial to a half line.

Remark 13. Let $\Gamma$ denote the gamma function. If $k$ is a nonnegative integer, then

$$
a(a+1) \cdots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)}
$$

which is often shortened by the Pochhammer symbol $(a)_{k}$. Another useful formula is

$$
a(a+1) \cdots(a+k-1)=(-1)^{k}(-a)(-a-1) \cdots(-a-k+1)=(-1)^{k} \frac{\Gamma(-a+1)}{\Gamma(-a+1-k)}
$$

In this notation, for an integer $b \geq 1$ and $h=0,1, \ldots, b-1$,

$$
(b-1-h)!=\frac{(b-1)!}{(-b+1)_{h}} \quad \text { and } \quad \Gamma(-a-b+2+h)=\Gamma(-a-b+2)(-a-b+2)_{h} .
$$

Hence

$$
\begin{aligned}
P_{a, b, 2}(\xi) & =\sum_{k=0}^{b-1}(-1)^{k} \frac{\Gamma(-a+1)}{\Gamma(-a+1-k)} \frac{1}{k!(b-1-k)!} 2^{-a-k} \xi^{b-1-k} \\
& =\Gamma(-a+1) \sum_{h=0}^{b-1}(-1)^{b-1-h} \frac{1}{\Gamma(-a-b+2+h)} \frac{1}{(b-1-h)!h!} 2^{-a-b+1+h} \xi^{h} \\
& =(-1)^{b-1} 2^{-a-b+1} \frac{\Gamma(-a+1)}{\Gamma(-a-b+2)(b-1)!} \sum_{h=0}^{b-1} \frac{(-b+1)_{h}}{(-a-b+2)_{h} h!}(2 \xi)^{h} \\
& =(-1)^{b-1} 2^{-a-b+1} \frac{\Gamma(-a+1)}{\Gamma(-a-b+2)(b-1)!}{ }_{1} F_{1}(-b+1 ;-a-b+2 ; 2 \xi) \\
& =(-1)^{b-1} 2^{-a-b+1} L_{b-1}^{-a-b+1}(2 \xi),
\end{aligned}
$$

where ${ }_{1} F_{1}$ is the confluent hypergeometric function and $L_{n}^{\alpha}(x)$ is a Laguerre polynomial. See [Erd53, 6.9(36), §10.12].

Proposition D.2. For any $a, b \in \mathbb{Z}$, the formula

$$
\begin{equation*}
\int_{\mathbb{R}}(1+i y)^{-a}(1-i y)^{-b} \phi(y) d y \quad(\phi \in \mathcal{S}(\mathbb{R})) \tag{D.7}
\end{equation*}
$$

defines a tempered distribution on $\mathbb{R}$. The restriction of the Fourier transform of this distribution to $\mathbb{R} \backslash\{0\}$ is a function given by

$$
\begin{equation*}
\int_{\mathbb{R}}(1+i y)^{-a}(1-i y)^{-b} e^{-i y \xi} d y=P_{a, b}(\xi) e^{-|\xi|} \tag{D.8}
\end{equation*}
$$

The right-hand side of (D.8) is an absolutely integrable function on the real line and thus defines a tempered distribution on $\mathbb{R}$. Furthermore,

$$
\begin{equation*}
(1+i y)^{-a}(1-i y)^{-b}=\frac{1}{2 \pi} \int_{\mathbb{R}} P_{a, b}(\xi) e^{-|\xi|} e^{i y \xi} d y+\frac{1}{2 \pi} Q_{a, b}(i y) \tag{D.9}
\end{equation*}
$$

and hence,

$$
\begin{equation*}
\int_{\mathbb{R}}(1+i y)^{-a}(1-i y)^{-b} e^{-i y \xi} d y=P_{a, b}(\xi) e^{-|\xi|}+Q_{a, b}\left(-\frac{d}{d \xi}\right) \delta_{0}(\xi) \tag{D.10}
\end{equation*}
$$

Proof. Since, $|1 \pm i y|=\sqrt{1+y^{2}}$, (D.7) is clear. The integral (D.8) is equal to

$$
\begin{align*}
& \frac{1}{i} \int_{i \mathbb{R}}(1+z)^{-a}(1-z)^{-b} e^{-z \xi} d z \\
& \quad=2 \pi\left(-\mathbb{I}_{\mathbb{R}^{+}}(\xi) \operatorname{res}_{z=1}(1+z)^{-a}(1-z)^{-b} e^{-z \xi}+\mathbb{I}_{\mathbb{R}^{-}}(\xi) \operatorname{res}_{z=-1}(1+z)^{-a}(1-z)^{-b} e^{-z \xi}\right) \tag{D.11}
\end{align*}
$$

The computation of the two residues is straightforward and (D.8) follows.
Since

$$
\int_{0}^{\infty} e^{-\xi} e^{i \xi y} d \xi=(1-i y)^{-1}
$$

we have

$$
\begin{equation*}
\int_{0}^{\infty} \xi^{m} e^{-\xi} e^{i \xi y} d \xi=\left(\frac{d}{d(i y)}\right)^{m}(1-i y)^{-1}=m!(1-i y)^{-m-1} \quad(m=0,1,2, \ldots) \tag{D.12}
\end{equation*}
$$

Thus, if $b \geq 1$, then

$$
\begin{aligned}
\int_{0}^{\infty} & P_{a, b, 2}(\xi) e^{-\xi} e^{i \xi y} d \xi \\
& =\sum_{k=0}^{b-1} \frac{a(a+1) \cdots(a+k-1)}{k!} 2^{-a-k}(1-i y)^{-b+k} \\
& =(1-i y)^{-b} 2^{-a} \sum_{k=0}^{b-1} \frac{(-a)(-a-1) \cdots(-a-k+1)}{k!}\left(-\frac{1}{2}(1-i y)\right)^{k} .
\end{aligned}
$$

Also, if $a \leq 0$, then

$$
\begin{aligned}
2^{a}(1+i y)^{-a} & =\left(1-\frac{1}{2}(1-i y)\right)^{-a}=\sum_{k=0}^{-a}\binom{-a}{k}\left(-\frac{1}{2}(1-i y)\right)^{k} \\
& =\sum_{k=0}^{-a} \frac{(-a)(-a-1) \cdots(-a-k+1)}{k!}\left(-\frac{1}{2}(1-i y)\right)^{k}
\end{aligned}
$$

Hence,

$$
\begin{align*}
& \int_{0}^{\infty} P_{a, b, 2}(\xi) e^{-\xi} e^{i \xi y} d \xi-(1+i y)^{-a}(1-i y)^{-b}  \tag{D.13}\\
& =(1-i y)^{-b} 2^{-a}\left(\sum_{k=0}^{b-1} \frac{(-a)(-a-1) \cdots(-a-k+1)}{k!}\left(-\frac{1}{2}(1-i y)\right)^{k}\right. \\
& \left.\quad-\sum_{k=0}^{-a} \frac{(-a)(-a-1) \cdots(-a-k+1)}{k!}\left(-\frac{1}{2}(1-i y)\right)^{k}\right) .
\end{align*}
$$

Recall that $P_{a, b,-2}=0$ if $a \leq 0$. Hence, (D.8) shows that (D.13) is the inverse Fourier transform of a distribution supported at $\{0\}$, hence a polynomial.

Suppose $-a<b-1$. Then (D.13) is equal to

$$
2^{-a} \sum_{k=-a+1}^{b-1} \frac{(-a)(-a-1) \cdots(-a-k+1)}{k!}\left(-\frac{1}{2}\right)^{k}(1-i y)^{k-b},
$$

which is zero because $(-a)(-a-1) \cdots(-a-k+1)=0$ for $k \geq-a+1$. If $-a=b-1$, then (D.13) is obviously zero.

Suppose $-a>b-1$. Then (D.13) is equal to

$$
\begin{equation*}
-2^{-a} \sum_{k=b}^{-a} \frac{(-a)(-a-1) \cdots(-a-k+1)}{k!}\left(-\frac{1}{2}\right)^{k}(1-i y)^{k-b} . \tag{D.14}
\end{equation*}
$$

As in (D.12) we have

$$
\int_{-\infty}^{0} \xi^{m} e^{\xi} e^{i \xi y} d \xi=\left(\frac{d}{d(i y)}\right)^{m}(1+i y)^{-1}=(-1)^{m} m!(1+i y)^{-m-1} \quad(m=0,1,2, \ldots)
$$

Suppose $a \geq 1$. Then

$$
\begin{aligned}
\int_{-\infty}^{0} & P_{a, b,-2}(\xi) e^{\xi} e^{i \xi y} d \xi \\
& =(-1)^{a+b-1} \sum_{k=0}^{a-1} \frac{b(b+1) \cdots(b+k-1)}{k!}(-2)^{-b-k}(-1)^{a-1+k}(1+i y)^{-a+k} \\
& =(1+i y)^{-a} 2^{-b} \sum_{k=0}^{a-1} \frac{(-b)(-b-1) \cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+i y)\right)^{k}
\end{aligned}
$$

Also, if $b \leq 0$, then

$$
2^{b}(1-i y)^{-b}=\sum_{k=0}^{-b} \frac{(-b)(-b-1) \cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+i y)\right)^{k} .
$$

Hence,

$$
\begin{align*}
\int_{-\infty}^{0} & P_{a, b,-2}(\xi) e^{\xi} e^{i \xi y} d \xi-(1+i y)^{-a}(1-i y)^{-b}  \tag{D.15}\\
=(1+i y)^{-a} 2^{-b} & \left(\sum_{k=0}^{a-1} \frac{(-b)(-b-1) \cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+i y)\right)^{k}\right. \\
& \left.-\sum_{k=0}^{-b} \frac{(-b)(-b-1) \cdots(-b-k+1)}{k!}\left(-\frac{1}{2}(1+i y)\right)^{k}\right)
\end{align*}
$$

As before, we show that (D.15) is zero if $-b \leq a-1$. If $-b>a-1$, then (D.15) is equal to

$$
-2^{-b} \sum_{k=a}^{-b} \frac{(-b)(-b-1) \cdots(-b-k+1)}{k!}\left(-\frac{1}{2}\right)^{k}(1+i y)^{k-a}
$$

If $a \geq 1$ and $b \geq 1$, then our computations show that

$$
\begin{equation*}
\int_{0}^{\infty} P_{a, b, 2}(\xi) e^{-\xi} e^{i \xi y} d \xi+\int_{-\infty}^{0} P_{a, b,-2}(\xi) e^{\xi} e^{i \xi y} d \xi-(1+i y)^{-a}(1-i y)^{-b} \tag{D.16}
\end{equation*}
$$

is a polynomial which tends to zero if $y$ goes to infinity. Thus (D.16) is equal zero. This completes the proof of (D.9). The statement (D.10) is a direct consequence of (D.9).

The test functions which occur in Proposition D. 2 need not be in the Schwartz space. In fact the test functions we shall use in our applications are not necessarily smooth. Therefore we shall need a more precise version of the formula (D.10). This requires a definition and two well-known lemmas.

Following Harish-Chandra denote by $\mathcal{S}\left(\mathbb{R}^{\times}\right)$the space of the smooth complex valued functions defined on $\mathbb{R}^{\times}$whose all derivatives are rapidly decreasing at infinity and have limits at zero from both sides. For $\psi \in \mathcal{S}\left(\mathbb{R}^{\times}\right)$let

$$
\psi(0+)=\lim _{x \rightarrow 0+} \psi(\xi), \quad \psi(0-)=\lim _{x \rightarrow 0-} \psi(\xi), \quad\langle\psi\rangle_{0}=\psi(0+)-\psi(0-)
$$

In particular the condition $\langle\psi\rangle_{0}=0$ means that $\psi$ extends to a continuous function on $\mathbb{R}$.

Lemma D.3. Let $c=0,1,2, \ldots$ and let $\psi \in \mathcal{S}\left(\mathbb{R}^{\times}\right)$. Suppose

$$
\begin{equation*}
\langle\psi\rangle_{0}=\cdots=\left\langle\psi^{(c-1)}\right\rangle_{0}=0 . \tag{D.17}
\end{equation*}
$$

(The condition (D.17) is empty if $c=0$.) Then

$$
\begin{equation*}
\left|\int_{\mathbb{R}^{x}} e^{-i y \xi} \psi(\xi) d \xi\right| \leq \min \left\{1,|y|^{-c-1}\right\}\left(\left|\left\langle\psi^{(c)}\right\rangle_{0}\right|+\left\|\psi^{(c+1)}\right\|_{1}+\|\psi\|_{1}\right) \tag{D.18}
\end{equation*}
$$

Proof. Integration by parts shows that for $z \in \mathbb{C}^{\times}$

$$
\begin{aligned}
& \int_{\mathbb{R}^{+}} e^{-z \xi} \psi(\xi) d \xi=z^{-1} \psi(0+)+\cdots+z^{-c-1} \psi^{(c)}(0+)+z^{-c-1} \int_{\mathbb{R}^{+}} e^{-z \xi} \psi^{(c+1)}(\xi) d \xi \\
& \int_{\mathbb{R}^{-}} e^{-z \xi} \psi(\xi) d \xi=-z^{-1} \psi(0-)-\cdots-z^{-c-1} \psi^{(c)}(0-)+z^{-c-1} \int_{\mathbb{R}^{-}} e^{-z \xi} \psi^{(c+1)}(\xi) d \xi
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \int_{\mathbb{R}^{\times}} e^{-z \xi} \psi(\xi) d \xi \\
= & z^{-1}\langle\psi\rangle_{0}+\cdots+z^{-c}\left\langle\psi^{(c-1)}\right\rangle_{0}+z^{-c-1}\left\langle\psi^{(c)}\right\rangle_{0}+z^{-c-1} \int_{\mathbb{R}^{\times}} e^{-z \xi} \psi^{(c+1)}(\xi) d \xi
\end{aligned}
$$

and (D.18) follows.
Lemma D.4. Under the assumptions of Lemma D.3, with $1 \leq c$,

$$
\int_{\mathbb{R}^{x}} \int_{\mathbb{R}^{x}}(i y)^{k} e^{-i y \xi} \psi(\xi) d \xi d y=2 \pi \psi^{(k)}(0) \quad(0 \leq k \leq c-1)
$$

where each consecutive integral is absolutely convergent.
Proof. Since

$$
\int_{\mathbb{R}}|y|^{c-1} \min \left\{1,|y|^{-c-1}\right\} d y<\infty
$$

the absolute convergence follows from Lemma D.3. Since the Fourier transform of $\psi$ is absolutely integrable and since $\psi$ is continuous at zero, Fourier inversion formula [Hör83, (7.1.4)] shows that

$$
\begin{equation*}
\int_{\mathbb{R}} \int_{\mathbb{R}^{x}} e^{-i y \xi} \psi(\xi) d \xi d y=2 \pi \psi(0) \tag{D.19}
\end{equation*}
$$

Also, for $0<k$,

$$
\begin{aligned}
& \int_{\mathbb{R}^{x}}(i y)^{k} e^{-i y \xi} \psi(\xi) d \xi=\int_{\mathbb{R}^{x}}\left(-\partial_{\xi}\right)\left((i y)^{k-1} e^{-i y \xi}\right) \psi(\xi) d \xi \\
= & \int_{\mathbb{R}^{+}}\left(-\partial_{\xi}\right)\left((i y)^{k-1} e^{-i y \xi}\right) \psi(\xi) d \xi+\int_{\mathbb{R}^{-}}\left(-\partial_{\xi}\right)\left((i y)^{k-1} e^{-i y \xi}\right) \psi(\xi) d \xi \\
= & (i y)^{k-1} \psi(0+)+\int_{\mathbb{R}^{+}}(i y)^{k-1} e^{-i y \xi} \psi^{\prime}(\xi) d \xi \\
- & (i y)^{k-1} \psi(0-)+\int_{\mathbb{R}^{-}}(i y)^{k-1} e^{-i y \xi} \psi^{\prime}(\xi) d \xi \\
= & (i y)^{k-1}\langle\psi\rangle_{0}+\int_{\mathbb{R}^{x}}(i y)^{k-1} e^{-i y \xi} \psi^{\prime}(\xi) d \xi
\end{aligned}
$$

Hence, by induction on $k$ and by our assumption

$$
\begin{aligned}
\int_{\mathbb{R}^{x}}(i y)^{k} e^{-i y \xi} \psi(\xi) d \xi & =(i y)^{k-1}\langle\psi\rangle_{0}+(i y)^{k-2}\left\langle\psi^{\prime}\right\rangle_{0}+\cdots+\left\langle\psi^{(k-1)}\right\rangle_{0} \\
& +\int_{\mathbb{R}^{x}} e^{-i y \xi} \psi^{(k)}(\xi) d \xi \\
& =\int_{\mathbb{R}^{x}} e^{-i y \xi} \psi^{(k)}(\xi) d \xi
\end{aligned}
$$

Therefore our lemma follows from (D.19).
The following proposition is an immediate consequence of Lemmas D.3, D.4, and the formula (D.9).

Proposition D.5. Fix two integers $a, b \in \mathbb{Z}$ and a function $\psi \in \mathcal{S}\left(\mathbb{R}^{\times}\right)$. Let $c=-a-b$. If $c \geq 0$ assume that

$$
\begin{equation*}
\langle\psi\rangle_{0}=\cdots=\left\langle\psi^{(c)}\right\rangle_{0}=0 \tag{D.20}
\end{equation*}
$$

Then

$$
\begin{align*}
& \int_{\mathbb{R}} \int_{\mathbb{R}^{x}}(1+i y)^{-a}(1-i y)^{-b} e^{-i y \xi} \psi(\xi) d \xi d y  \tag{D.21}\\
= & \int_{\mathbb{R}^{\times}} P_{a, b}(\xi) e^{-|\xi|} \psi(\xi) d \xi+\left.Q_{a, b}\left(\partial_{\xi}\right) \psi(\xi)\right|_{\xi=0} \\
= & \int_{\mathbb{R}}\left(P_{a, b}(\xi) e^{-|\xi|}+Q_{a, b}\left(-\partial_{\xi}\right) \delta_{0}(\xi)\right) \psi(\xi) d \xi
\end{align*}
$$

where $\delta_{0}$ denotes the Dirac delta at 0 .
(Recall that $Q_{a, b}=0$ if $c<0$ and $Q_{a, b}$ is a polynomial of degree if $c$, if $c \geq 0$.)
Let $\mathcal{S}\left(\mathbb{R}^{+}\right)$be the space of the smooth complex valued functions whose all derivatives are rapidly decreasing at infinity and have limits at zero. Then $\mathcal{S}\left(\mathbb{R}^{+}\right)$may be viewed as the subspace of the functions in $\mathcal{S}\left(\mathbb{R}^{\times}\right)$which are zero on $\mathbb{R}^{-}$. Similarly we define $\mathcal{S}\left(\mathbb{R}^{-}\right)$. The following propositions are direct consequences of Proposition D.5. We sketch independent proofs below.

Proposition D.6. There is a seminorm $p$ on the space $\mathcal{S}\left(\mathbb{R}^{+}\right)$such that

$$
\begin{equation*}
\left|\int_{\mathbb{R}^{+}} e^{-z \xi} \psi(\xi) d \xi\right| \leq \min \left\{1,|z|^{-1}\right\} p(\psi) \quad\left(\psi \in \mathcal{S}\left(\mathbb{R}^{+}\right), \text {Re } z \geq 0\right) \tag{D.22}
\end{equation*}
$$

and similarly for $\mathcal{S}\left(\mathbb{R}^{-}\right)$.
Fix integers $a, b \in \mathbb{Z}$ with $a+b \geq 1$. Then for any function $\psi \in \mathcal{S}\left(\mathbb{R}^{+}\right)$,

$$
\begin{equation*}
\int_{\mathbb{R}}(1+i y)^{-a}(1-i y)^{-b} \int_{\mathbb{R}^{+}} e^{-i y \xi} \psi(\xi) d \xi d y=2 \pi \int_{\mathbb{R}^{+}} P_{a, b, 2}(\xi) e^{-\xi} \psi(\xi) d \xi \tag{D.23}
\end{equation*}
$$

and any function $\psi \in \mathcal{S}\left(\mathbb{R}^{-}\right)$,

$$
\begin{equation*}
\int_{\mathbb{R}}(1+i y)^{-a}(1-i y)^{-b} \int_{\mathbb{R}^{-}} e^{-i y \xi} \psi(\xi) d \xi d y=2 \pi \int_{\mathbb{R}^{-}} P_{a, b,-2}(\xi) e^{\xi} \psi(\xi) d \xi \tag{D.24}
\end{equation*}
$$

where each consecutive integral is absolutely convergent.

Proof. Clearly

$$
\left|\int_{\mathbb{R}^{+}} e^{-z \xi} \psi(\xi) d \xi\right| \leq \int_{\mathbb{R}^{+}} e^{-\operatorname{Re} z \xi}|\psi(\xi)| d \xi \leq\|\psi\|_{1}
$$

Integration by parts shows that for $z \neq 0$,

$$
\int_{\mathbb{R}^{+}} e^{-z \xi} \psi(\xi) d \xi=z^{-1} \psi(0)+z^{-1} \int_{\mathbb{R}^{+}} e^{-z \xi} \psi^{\prime}(\xi) d \xi
$$

Hence (D.22) follows with $p(\psi)=|\psi(0)|+\|\psi\|_{1}+\left\|\psi^{\prime}\right\|_{1}$.
Let $a, b \in \mathbb{Z}$ be such that $a+b \geq 1$. Then the function

$$
(1+z)^{-a}(1-z)^{-b} \int_{\mathbb{R}^{+}} e^{-z \xi} \psi(\xi) d \xi
$$

is continuous on $\operatorname{Re} z \geq 0$ and meromorphic on $\operatorname{Re} z>0$ and (D.22) shows that it is dominated by $|z|^{-2}$. Therefore Cauchy's Theorem implies that the left-hand side of (D.23) is equal to

$$
-2 \pi \operatorname{res}_{z=1}\left((1+z)^{-a}(1-z)^{-b} \int_{\mathbb{R}^{+}} e^{-z \xi} \psi(\xi) d \xi\right)
$$

The computation of this residue is straightforward. This verifies (D.23). The proof of (D.24) is entirely analogous.

## Appendix E. The covering $\widetilde{\mathrm{G}} \rightarrow \mathrm{G}$

In this appendix we recall some results about the splitting of the restrictions $\widetilde{L} \rightarrow L$ of the metaplectic covering

$$
\begin{equation*}
1 \rightarrow\{ \pm 1\} \rightarrow \widetilde{\mathrm{Sp}}(\mathrm{~W}) \rightarrow \mathrm{Sp}(\mathrm{~W}) \rightarrow 1 \tag{E.1}
\end{equation*}
$$

to a subgroup $L$ of the compact member $G$ of a dual pair ( $G, G^{\prime}$ ) as in (2). This is well known, but we could not find a reference sketching the proofs of the results we are using in this paper. We are therefore providing a short and complete argument.

If K is a maximal compact subgroup of $\mathrm{Sp}(\mathrm{W})$, then $\widetilde{\mathrm{K}}$ is a maximal compact subgroup of $\widetilde{\mathrm{Sp}}(\mathrm{W})$. The group $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is connected, noncompact, semisimple and with finite center $\widetilde{Z}$. (Since $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a double cover of $\mathrm{Sp}(\mathrm{W})$, only the connectedness needs to be commented. It follows from the fact that the covering (E.1) does not split; see e.g. [AP14, Proposition 4.20] or the original proof [Wei64, p. 199]). The maximal compact subgroup $\widetilde{\mathrm{K}}$ is therefore connected; see e.g. [Hel78, Chapter VI, Theorem 1.1]. Hence the covering

$$
\begin{equation*}
\widetilde{\mathrm{K}} \rightarrow \mathrm{~K} \tag{E.2}
\end{equation*}
$$

does not split.
As is well known, K is isomorphic to a compact unitary group. In fact, if $\mathrm{W}=\mathbb{R}^{2 n}$ and

$$
J_{2 n}=\left(\begin{array}{cc}
0 & I_{n}  \tag{E.3}\\
-I_{n} & 0
\end{array}\right)
$$

then

$$
\mathrm{Sp}_{2 n}(\mathbb{R})^{J_{2 n}}=\left\{\left(\begin{array}{cc}
a & -b  \tag{E.4}\\
b & a
\end{array}\right) ; a, b \in \mathrm{GL}_{n}(\mathbb{R}), a b^{t}=b a^{t}, a a^{t}+b b^{t}=I_{n}\right\}
$$

is a maximal compact subgroup of $\mathrm{Sp}_{2 n}(\mathbb{R})$ and

$$
\mathrm{Sp}_{2 n}(\mathbb{R})^{J_{2 n}} \ni\left(\begin{array}{cc}
a & -b  \tag{E.5}\\
b & a
\end{array}\right) \rightarrow a+i b \in \mathrm{U}_{n}
$$

is a Lie group isomorphism. Any two maximal compact subgroups of $\mathrm{Sp}(\mathrm{W})$ are conjugate by an inner automorphism. Let $\mathrm{K} \rightarrow \mathrm{Sp}_{2 n}(\mathbb{R})^{J_{2 n}}$ be the corresponding isomorphism. Composition with (E.5) fixes then an isomorphism $\phi: \mathrm{K} \rightarrow \mathrm{U}_{n}$. Set

$$
\begin{equation*}
\widetilde{\mathrm{K}}^{\phi}=\left\{(u, \zeta) \in \mathrm{K} \times \mathbb{C}^{\times} ; \operatorname{det}(\phi(u))=\zeta^{2}\right\} \tag{E.6}
\end{equation*}
$$

Recall the bijection between equivalence classes of $n$-fold path-connected coverings and the conjugacy classes of index- $n$ subgroups of the fundamental group (see e.g. [Hat01, Theorem 1.38]). Then, up to an isomorphism of coverings, $\mathrm{U}_{n}$ has only one connected double cover. Hence (E.2) is isomorphic to

$$
\begin{equation*}
\widetilde{\mathrm{K}}^{\phi} \ni(u, \zeta) \rightarrow u \in \mathrm{~K} . \tag{E.7}
\end{equation*}
$$

Let $\mathrm{L} \subseteq \mathrm{K}$ be any subgroup and

$$
\begin{equation*}
\widetilde{\mathrm{L}} \rightarrow \mathrm{~L} \tag{E.8}
\end{equation*}
$$

the restriction of the covering (E.2) to L. Let $\widetilde{\mathrm{L}}^{\phi}$ be the preimage of L in $\widetilde{\mathrm{K}}^{\phi}$. Then (E.8) splits if and only if

$$
\begin{equation*}
\widetilde{\mathrm{L}}^{\phi} \rightarrow \mathrm{L} \tag{E.9}
\end{equation*}
$$

splits, i.e. there is a group homomorphism $\mathrm{L} \ni g \rightarrow \zeta(g) \in \mathrm{U}_{1} \subset \mathbb{C}^{\times}$such that $\zeta(g)^{2}=$ $\operatorname{det}(\phi(g))$ for all $g \in \mathrm{~L}$. For instance, if L is a connected subgroup of K such that

$$
\begin{equation*}
\mathrm{L} \subseteq\{u \in \mathrm{~K} ; \operatorname{det}(\phi(u))=1\} \tag{E.10}
\end{equation*}
$$

then (E.8) splits.
To fix $\phi$, let $(\mathrm{V},(\cdot, \cdot))$ and $\left(\mathrm{V}^{\prime},(\cdot, \cdot)^{\prime}\right)$ be the defining spaces of G and $\mathrm{G}^{\prime}$, respectively, with $\operatorname{dim}_{\mathbb{D}} \mathrm{V}=d$ and $\operatorname{dim}_{\mathbb{D}} \mathrm{V}^{\prime}=d^{\prime}$. Realize W as $\mathrm{V} \otimes_{\mathbb{D}} \mathrm{V}^{\prime}$, considered as a real symplectic space, with symplectic form $\langle\cdot, \cdot\rangle=\operatorname{tr}_{\mathbb{D} / \mathbb{R}}\left((\cdot, \cdot) \otimes(\cdot, \cdot)^{\prime}\right)$, where $\operatorname{tr}_{\mathbb{D} / \mathbb{R}}$ denotes the reduced trace; see [How79, §5] and [Wei73, p. 169]. Then the group G is viewed as a subgroup of $\mathrm{Sp}(\mathrm{W})$ via the identification $\mathrm{G} \ni g \rightarrow g \otimes 1 \in \mathrm{Sp}(\mathrm{W}) .{ }^{1}$ Similarly, $\mathrm{G}^{\prime}$ is viewed as a subgroup of $\mathrm{Sp}(\mathrm{W})$ via the identification $\mathrm{G}^{\prime} \ni g^{\prime} \rightarrow 1 \otimes g^{\prime} \in \mathrm{Sp}(\mathrm{W})$. Recall that $n$-by- $n$ matrices over $\mathbb{C}$ can be identified with $2 n$-by- $2 n$ matrices over $\mathbb{R}$ under the isomorphism

$$
\alpha: M \rightarrow\left(\begin{array}{cc}
\operatorname{Re} M & -\operatorname{Im} M \\
\operatorname{Im} M & \operatorname{Re} M
\end{array}\right)
$$

Moreover, $n$-by- $n$-matrices over $\mathbb{H}$ can be identified with $2 n$-by- $2 n$ matrices over $\mathbb{C}$ under the isomorphism

$$
\beta: M \rightarrow\left(\begin{array}{cc}
z_{1}(M) & -\overline{z_{2}(M)} \\
z_{2}(M) & \overline{z_{1}(M)}
\end{array}\right) .
$$

Here, for $v \in \mathbb{H}$, we write $v=z_{1}(v)+j z_{2}(v)$ with $z_{1}(v), z_{2}(v) \in \mathbb{C}$, and we similarly define $z_{1}(M)$ and $z_{2}(M)$ if $M$ is a matrix over $\mathbb{H}$.

Since G is compact, there is a compatible positive complex structure $J$ on W such that the maximal compact subgroup $\mathrm{K}=\mathrm{Sp}(\mathrm{W})^{J}$ of $\mathrm{Sp}(\mathrm{W})$ contains $G$. Moreover, since $G$

[^0]commutes with $J$, there is $J^{\prime} \in \mathrm{G}^{\prime}$ such that $J=1 \otimes J^{\prime}$. Set $I_{p, q}=\left(\begin{array}{cc}I_{p} & 0 \\ 0 & -I_{q}\end{array}\right)$. Then, the explicit expressions of $J^{\prime}$ with respect to to the standard basis of $\mathrm{V} \simeq \mathbb{D}^{d}$ and of $J$ with respect to the standard basis of $\mathrm{W} \simeq \mathbb{R}^{2 n}$ are given as follows:

| $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ | $J^{\prime}$ | $n$ | $J$ |
| :---: | :---: | :---: | :---: |
| $\left(\mathrm{O}_{d}, \mathrm{Sp}_{2 m}(\mathbb{R})\right)$ | $J_{2 m}$ | $m d$ | $J_{2 m d}$ |
| $\left(\mathrm{U}_{d}, \mathrm{U}_{p, q}\right)$ | $-i I_{p, q}$ | $d(p+q)$ | $\left(\begin{array}{cc}0 & I_{d p, d q} \\ -I_{d p, d q} & 0\end{array}\right)$ |
| $\left(\mathrm{Sp}_{d}, \mathrm{O}_{2 m}^{*}\right)$ | $-j I_{m}$ | $2 m d$ | $\left(\begin{array}{cc}J_{2 p m} & 0 \\ 0 & J_{2 p m}\end{array}\right)$ |

Notice that in the $\left(\mathrm{U}_{d}, \mathrm{U}_{p, q}\right)$-case we have $S J S^{-1}=J_{2 d(p+q)}$ for $S=\left(\begin{array}{cc}I_{d(p+q)} & 0 \\ 0 & I_{d p, d q}\end{array}\right)$; in the $\left(\mathrm{Sp}_{d}, \mathrm{O}_{2 m}^{*}\right)$-case, $T J T^{-1}=J_{4 p m}$ for $T=\left(\begin{array}{cccc}I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I\end{array}\right)$. Hence, in all cases we can embed G in (E.4) from the identification $g \rightarrow g \otimes 1 \in \operatorname{Sp}(\mathrm{~W})^{J}$ followed by the isomorphism of $\mathrm{Sp}(\mathrm{W})^{J}$ and $\mathrm{Sp}_{2 n}(\mathbb{R})^{J_{2 n}}$ corresponding to the conjugations by $S$ or $T$, and then apply (E.5). We obtain:

$$
\operatorname{det}(\phi(g))= \begin{cases}\operatorname{det}(g)_{\mathrm{V}}^{m} & \text { if }\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{d}, \mathrm{Sp}_{2 m}(\mathbb{R})\right)  \tag{E.11}\\ \operatorname{det}(g)_{\mathrm{V}}^{p-q} & \text { if }\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{d}, \mathrm{U}_{p, q}\right) \\ 1 & \text { if }\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{Sp}_{d}, \mathrm{O}_{2 m}^{*}\right)\end{cases}
$$

where $\operatorname{det}(g)_{\mathrm{V}}$ denotes the determinant of $g$ as an element of $\mathrm{G} \subseteq \mathrm{GL}_{\mathbb{D}}(\mathrm{V})$. (The determinant of an $n$-by- $n$ matrix over $\mathbb{H}$ can be reduced to a determinant of a $2 n$-by- $2 n$ matrix over $\mathbb{C}$ via the isomorphism $\beta$. For elements of $\operatorname{Sp}(d)$, this notion of determinant coincides with other possible notions of quaternionic determinants; see [Asl96] for additional information.)
Proposition E.1. The covering $\widetilde{\mathrm{G}} \rightarrow \mathrm{G}$ splits if and only if $\operatorname{det}(\phi(g))$ is a square. This happens for all pairs $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)$ different from $\left(\mathrm{O}_{d}, \mathrm{Sp}_{2 m}(\mathbb{R})\right)$ with $m$ odd and $\left(\mathrm{U}_{d}, \mathrm{U}_{p, q}\right)$ with $p+q$ odd. In these two non-splitting cases, the covering $\widetilde{\mathrm{G}} \rightarrow \mathrm{G}$ is isomorphic to the $\operatorname{det}^{1 / 2}$-covering

$$
\begin{equation*}
\sqrt{\mathrm{G}} \ni(g, \zeta) \rightarrow g \in \mathrm{G} \tag{E.12}
\end{equation*}
$$

where

$$
\begin{equation*}
\sqrt{\mathrm{G}}=\left\{(g, \zeta) \in \mathrm{G} \times \mathbb{C}^{\times} ; \zeta^{2}=\operatorname{det}(g)_{\mathrm{v}}\right\} \tag{E.13}
\end{equation*}
$$

Proof. By (E.11) there is a group homomorphism $\mathrm{G} \ni g \rightarrow \zeta(g) \in \mathrm{U}_{1} \subseteq \mathbb{C}^{\times}$so that $\zeta(g)^{2}=\operatorname{det}(\phi(g))$ for all pairs (G, $\left.\mathrm{G}^{\prime}\right)$ except at most the two cases listed in the statement of the Proposition.

Suppose that $\mathrm{G}^{\prime}=\mathrm{Sp}_{2 m}(\mathbb{R})$, and let $\zeta: \mathrm{O}_{d} \rightarrow \mathrm{U}_{1}$ be a continuous group homomorphism so that $\zeta(g)^{2}=\operatorname{det}(g)_{\mathrm{V}}^{m}=( \pm 1)^{m}$. Then $\zeta\left(\mathrm{O}_{d}\right) \subseteq\{ \pm 1, \pm i\}$ and it is a subgroup with
at most two elements. So $\zeta\left(\mathrm{O}_{d}\right) \subseteq\{ \pm 1\}$. On the other hand, if $g \in \mathrm{O}_{d} \backslash \mathrm{SO}_{d}$, then $\operatorname{det}(g)_{\mathrm{G}}=-1$. Thus $\zeta(g)^{2} \neq \operatorname{det}(g)_{\mathrm{V}}^{m}$ if $m$ is odd.

Suppose now that $\mathrm{G}^{\prime}=\mathrm{U}_{p, q}$, and let $\zeta: \mathrm{U}_{d} \rightarrow \mathrm{U}_{1}$ be a continuous group homomorphism so that $\zeta(g)^{2}=\operatorname{det}(g)_{\mathrm{V}}^{p-q}$. Restriction to $\mathrm{U}_{1} \equiv\left\{\operatorname{diag}(h, 1 \ldots, 1) ; h \in \mathrm{U}_{1}\right\} \subseteq \mathrm{U}_{d}$ yields a continuous group homomorphism $h \in \mathrm{U}_{1} \rightarrow \zeta(h) \in \mathrm{U}_{1}$. Thus, there is $k \in \mathbb{Z}$ so that $\zeta(h)=h^{k}$ for all $h \in \mathrm{U}_{1}$. So $h^{2 k}=\zeta(h)^{2}=\operatorname{det}(\operatorname{diag}(h, 1, \ldots, 1))^{p-q}$ implies that $p+q$ must be even.

For the last statement, consider for $k \in \mathbb{Z}$ the covering $\mathrm{M}_{k}=\left\{(g, \zeta) \in \mathrm{G} \times \mathbb{C}^{\times} ; \zeta^{2}=\right.$ $\left.\operatorname{det}(g)_{\mathrm{V}}^{2 k+1}\right\}$ of G. Then $(g, \zeta) \rightarrow\left(g, \zeta^{\frac{1}{2 k+1}}\right)$ is a covering isomorphism between $\mathrm{M}_{k}$ and $\mathrm{M}_{0}$.
Remark 14. Keep the notation of (E.6) and let $\alpha: \widetilde{\mathrm{K}}^{\phi} \rightarrow \widetilde{\mathrm{K}}$ be the isomorphism lifting $\phi^{-1}: \mathrm{U}_{n} \rightarrow \mathrm{~K}$. Then, by [Fol89, Proposition 4.39] or [Prz89, (1.4.17)], the map

$$
(u, \zeta) \rightarrow \zeta^{-1} \omega(\alpha(u, \zeta))
$$

is independent of $\zeta$.

## Appendix F. On the nonoccurrence of the determinant character of $\mathrm{O}_{d}$ in Howe's correspondence

Consider the reductive dual pair $\left(\mathrm{O}_{d}, \mathrm{Sp}_{2 n}(\mathbb{R})\right)$ where $d>2 n$. Let $\mathrm{M}_{d, n}(\mathbb{R})$ denote the space of $d \times n$ matrices with real coefficients and consider the Schrödinger model for the Weil representation $\omega$, with space of smooth vectors $\mathcal{S}=\mathcal{S}\left(\mathrm{M}_{d, n}(\mathbb{R})\right)$. Moreover, let $\chi_{+}$ be the character of $\widetilde{\mathrm{O}}_{d}$ defined in (79). As recalled on page 5 , the representation $\omega \otimes \chi_{+}^{-1}$ descends to a representation $\omega_{0}$ of $\mathrm{O}_{d}$ given by

$$
\begin{equation*}
\omega_{0}(g) f(x)=f\left(g^{-1} x\right) \quad\left(g \in \mathrm{O}_{d}, f \in \mathcal{S}, x \in \mathrm{M}_{d, n}(\mathbb{R})\right) \tag{F.1}
\end{equation*}
$$

In this appendix, we prove that, under the assumption that $d>2 n$, the determinant character det does not occur in $\omega_{0}$. This property is a consequence of [Prz89, (C.43) Corollary] (which considers the more general case of the pseudo-orthogonal groups $\mathrm{O}_{p, q}$, where $p+q=d>2 n)$. However, the proof in [Prz89] uses part of the classification of the K-types of representations occurring in Howe's correspondence, determined by [KV78]. The proof below, which follows the $p$-adic case in [Ral84, p. 399], is classification-free.

Proposition F.1. If $d>2 n$, then det does not occur in $\omega_{0}$. In other words: if $d>2 n$, then there is no character $\sigma$ of $\widetilde{\mathrm{O}}_{d}$ occurring in Howe's correspondence such that $\sigma \otimes \chi_{+}^{-1}$ descends to the determinant character det of $\mathrm{O}_{d}$.

Proof. We argue by contradiction. Suppose that det occurs in $\omega_{0}$ and let $f_{0} \in \mathcal{S}$ be a non-zero function satisfying

$$
f_{0}\left(g^{-1} x\right)=\operatorname{det}(g) f_{0}(x) \quad\left(g \in \mathrm{O}_{d}, x \in \mathrm{M}_{d, n}(\mathbb{R})\right)
$$

Fix $s \in \mathrm{O}_{d}$ satisfying $s^{2}=1$ and $\operatorname{det}(s)=-1$, and consider the two element subgroup $\{1, s\}$ of $\mathrm{O}_{d}$. By (F.1), the restriction of $\omega_{0}$ to $\{1, s\}$ yields the decomposition

$$
\mathcal{S}=\mathcal{S}_{\mathrm{tr}} \oplus \mathcal{S}_{\mathrm{det}}
$$

into $\{1, s\}$-isotypic components. Since $\mathrm{O}_{d}=\mathrm{SO}_{d} \cup s \mathrm{SO}_{d}$, the space $\mathcal{S}_{\text {det }}$ contains $f_{0}$.

Define $\mathrm{Z}=\left\{x \in \mathrm{M}_{d, n}(\mathbb{R}): x\right.$ has maximal rank $\left.n\right\}$. Since Z is $\mathrm{O}_{d}$-invariant, $\omega_{0}$ restricts to a representation of $\mathrm{O}_{d}$ on $\left.\mathcal{S}\right|_{\mathrm{Z}}=\left\{\left.f\right|_{\mathrm{Z}} ; f \in \mathcal{S}\right\}$. Restricting to Z also yields the decomposition

$$
\left.\mathcal{S}\right|_{\mathrm{Z}}=\left.\left.\mathcal{S}_{\mathrm{tr}}\right|_{\mathrm{Z}} \oplus \mathcal{S}_{\mathrm{det}}\right|_{\mathrm{Z}}
$$

The restriction map $\left.f \rightarrow f\right|_{\mathrm{Z}}$ is injective by the density of Z in $\mathrm{M}_{d, n}(\mathbb{R})$. Hence $\left.f_{0}\right|_{\mathrm{Z}} \neq 0$. Therefore $\mathcal{S}_{\text {det }} \mid \mathrm{Z} \neq\{0\}$ because it contains $\left.f_{0}\right|_{\mathrm{z}}$.

Decompose Z as a union of $\mathrm{O}_{d}$-orbits $\mathcal{O}$. Then there is an $\mathrm{O}_{d \text {-orbit }} \mathcal{O}$ such that $\left.f_{0}\right|_{\mathcal{O}} \neq 0$. Set $\varphi=\left.f_{0}\right|_{\mathcal{O}}$. Then

$$
\begin{equation*}
\varphi\left(g^{-1} x\right)=\operatorname{det}(g) \varphi(x) \quad\left(g \in \mathrm{O}_{d}, x \in \mathcal{O}\right) \tag{F.2}
\end{equation*}
$$

The centralizer of any element in $\mathcal{O}$ is isomorphic to $\mathrm{O}_{d-n}$. Hence $\mathcal{O}=\mathrm{O}_{d} / \mathrm{O}_{d-n}$ and $\varphi \in \operatorname{Ind}_{\mathrm{O}_{d-n}}^{\mathrm{O}_{d}}(1)$. By (F.2), det occurs in $\operatorname{Ind}_{\mathrm{O}_{d-n}}^{\mathrm{O}_{d}}(1)$. Frobenius' reciprocity then implies that the character $\left.\operatorname{det}\right|_{\mathrm{O}_{d-n}}$ contains 1, i.e. $\left.\operatorname{det}\right|_{\mathrm{O}_{d-n}}=1$. This is clearly impossibile, and we have reached a contradiction. Thus det cannot occur in $\omega_{0}$.

## Appendix G. Tensor product decomposition of $T$ over complementary invariant symplectic subspaces of W

We keep the notation introduced in section 1 . Let

$$
\begin{equation*}
\chi_{+}(\widetilde{g})=\frac{\Theta(\widetilde{g})}{|\Theta(\widetilde{g})|} \quad(g \in \mathrm{Sp}(\mathrm{~W})) \tag{G.1}
\end{equation*}
$$

(Recall that $\chi_{+}$is not a character on $\widetilde{\mathrm{Sp}}(\mathrm{W})$, since $\widetilde{\mathrm{Sp}}(\mathrm{W})$ does not have any nontrivial character. However, $\chi_{+}$becomes a character when restricted to specific subgroups of $\widetilde{\mathrm{Sp}}(\mathrm{W})$, such as $\widetilde{\mathrm{O}}_{d}$; see (79).) By definition, see (14),

$$
\begin{equation*}
\chi_{+}^{-1}(\widetilde{g}) T(\widetilde{g})=|\Theta(\widetilde{g})| \chi_{c(g)} \mu_{(g-1) \mathrm{W}} \quad(g \in \mathrm{Sp}(\mathrm{~W})) \tag{G.2}
\end{equation*}
$$

descends to a distribution on $\mathrm{Sp}(\mathrm{W})$.
Let $\mathrm{W}=\mathrm{W}_{1} \oplus \mathrm{~W}_{2}$ be an orthogonal decomposition of W , and endow each subspace $\mathrm{W}_{j}$ (where $j=1,2$ ) of the symplectic form $\langle\cdot, \cdot\rangle_{j}=\left.\langle\cdot, \cdot\rangle\right|_{W_{j} \times \mathrm{W}_{j}}$. Suppose that $g \in \mathrm{Sp}(\mathrm{W})$ preserves $\mathrm{W}_{1}$ and $\mathrm{W}_{2}$. Let $g_{1}$ and $g_{2}$ respectively denote the restrictions $\left.g\right|_{\mathrm{W}_{1}}$ and $\left.g\right|_{\mathrm{W}_{2}}$ or $g$ to theses subspaces. Suppose we have chosen a complete polarization $\mathrm{W}=\mathrm{X} \oplus \mathrm{Y}$ of W such that $X=X_{1} \oplus X_{2}$ and $Y=Y_{1} \oplus Y_{2}$, where $W_{1}=X_{1} \oplus Y_{1}$ and $W_{2}=X_{2} \oplus Y_{2}$ are complete polarizations. Similarly, suppose that the compatible positive complex structures $J, J_{1}$, $J_{2}$ on $\mathrm{W}, \mathrm{W}_{1}, \mathrm{~W}_{2}$, respectively, satisfy $J=J_{1} \times J_{2}$. Then $J(\mathrm{X})=\mathrm{Y}$ if and only if $J\left(\mathrm{X}_{1}\right)=\mathrm{Y}_{1}$ and $J\left(\mathrm{X}_{2}\right)=\mathrm{Y}_{2}$, which we assume.

Write $T_{\mathrm{W}}, T_{\mathrm{W}_{1}}$ and $T_{\mathrm{W}_{2}}$ for the distributions corresponding to $\widetilde{\mathrm{Sp}}(\mathrm{W}), \widetilde{\mathrm{Sp}}\left(\mathrm{W}_{1}\right), \widetilde{\mathrm{Sp}}\left(\mathrm{W}_{2}\right)$, respectively. Similar notation will apply to other symbols occurring in the computations below. For the tensor product of tempered distributions, we refer to [Trè67, Corollary of Theorem 51.6, especially (51.7)].

Lemma G.1. In the above notations,

$$
\left|\Theta_{\mathrm{W}}(\widetilde{g})\right| \chi_{c(g)} \mu_{(g-1) \mathrm{W}}=\left|\Theta_{\mathrm{W}_{1}}\left(\widetilde{g_{1}}\right)\right| \chi_{c\left(g_{1}\right)} \mu_{\left(g_{1}-1\right) \mathrm{W}} \otimes\left|\Theta_{\mathrm{W}_{2}}\left(\widetilde{g_{2}}\right)\right| \chi_{c\left(g_{1}\right)} \mu_{\left(g_{2}-1\right) \mathrm{W}_{2}} .
$$

Consequently, independently of the choice of the preimages of $g, g_{1}$ and $g_{2}$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$, $\widetilde{\mathrm{Sp}}\left(\mathrm{W}_{1}\right), \widetilde{\mathrm{Sp}}\left(\mathrm{W}_{2}\right)$, respectively,

$$
\chi_{+}^{-1}(\widetilde{g}) T_{\mathrm{W}}(\widetilde{g})=\chi_{+}^{-1}\left(\widetilde{g_{1}}\right) T_{\mathrm{W}_{1}}\left(\widetilde{g_{1}}\right) \otimes \chi_{+}^{-1}\left(\widetilde{g_{2}}\right) T_{\mathrm{W}_{2}}\left(\widetilde{g_{2}}\right) .
$$

Proof. Since $\mathrm{W}=\mathrm{W}_{1} \oplus \mathrm{~W}_{2}$ and $g_{1}=\left.g\right|_{\mathrm{W}_{1}}, g_{2}=\left.g\right|_{\mathrm{W}_{2}}$, we have $(g-1) \mathrm{W}=\left(g_{1}-1\right) \mathrm{W}_{1} \oplus$ $\left(g_{2}-1\right) \mathrm{W}_{2}$. Recall from [AP14, Definitions 4.16, 4.18 and 4.23] that

$$
\Theta(\widetilde{g})^{2}=\Theta^{2}(g) \quad(g \in \operatorname{Sp}(\mathrm{~W}))
$$

Thus $\left|\Theta_{\mathrm{V}}(\widetilde{g})\right|^{2}=\left|\Theta_{\mathrm{V}}^{2}(g)\right|$ for $\mathrm{V} \in\left\{\mathrm{W}, \mathrm{W}_{1}, \mathrm{~W}_{2}\right\}$. It follows that $\left|\Theta_{\mathrm{W}}(\widetilde{g})\right|=\left|\Theta_{\mathrm{W}_{1}}\left(\widetilde{g_{1}}\right)\right|\left|\Theta_{\mathrm{W}_{2}}\left(\widetilde{g_{2}}\right)\right|$, and this independently of the choice of the preimages of $g, g_{1}$ and $g_{2}$ in $\widetilde{\mathrm{Sp}}(\mathrm{W}), \widetilde{\mathrm{Sp}}\left(\mathrm{W}_{1}\right)$, $\widetilde{\mathrm{Sp}}\left(\mathrm{W}_{2}\right)$, respectively. Since the decomposition $\mathrm{W}=\mathrm{W}_{1} \oplus \mathrm{~W}_{2}$ is orthogonal,
$\langle c(g) w, w\rangle=\left\langle c\left(g_{1}\right) w_{1}, w_{1}\right\rangle_{1}+\left\langle c\left(g_{2}\right) w_{2}, w_{2}\right\rangle_{2} \quad\left(w_{j}=\left(g_{j}-1\right) \mathrm{W}_{j}, j=1,2, w=w_{1}+w_{2}\right)$,
where $c$ denotes the Cayley transform. Therefore $\chi_{c(g)}=\chi_{c\left(g_{1}\right)} \otimes \chi_{c\left(g_{2}\right)}$ on $\mathrm{W}=\mathrm{W}_{1} \oplus \mathrm{~W}_{2}$. Finally, the normalization of measures on subspaces of W fixed at the beginning of section 1 is such that $\mu_{(g-1) \mathrm{W}}=\mu_{\left(g_{1}-1\right) \mathrm{W}_{1}} \otimes \mu_{\left(g_{2}-1\right) \mathrm{W}_{2}}$.

## Appendix H. Highest weights of irreducible genuine representations of $\widetilde{G}$

In this appendix we collect some root/weight for the irreducible genuine representations of $\widetilde{G}$, where $G$ is a compact member of a reductive dual pair ( $G, G^{\prime}$ ). Let $\mathfrak{h}$ be a fixed Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G . We denote by $\Delta^{+}$a choice of positive roots for $\left(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}\right)$ and by $\rho$ the one-half of their sum. The genuine irreducible representations of $\widetilde{\mathrm{G}}$ are parametrized by their highest weights $\lambda=\sum_{j=1}^{l} \lambda_{j} e_{j}$ listed below.
$\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{U}_{l}, \mathrm{U}_{p, q}\right), l \geq 1, q \geq p \geq 0, p+q \geq 1$ :
If $l=1$, then $\mathfrak{h}_{\mathbb{C}}=\mathfrak{g}_{\mathbb{C}}$. If $l \geq 2$, then:

$$
\begin{aligned}
& \Delta^{+}=\left\{e_{j}-e_{k} ; 1 \leq j<k \leq l\right\}\left(\text { type } A_{l-1}\right), \quad \rho=\sum_{j=1}^{l}\left(\frac{l+1}{2}-j\right) e_{j}, \\
& \lambda_{j}=\frac{q-p}{2}+\nu_{j}, \quad \nu_{j} \in \mathbb{Z}, \quad \nu_{1} \geq \nu_{2} \geq \cdots \geq \nu_{l} . \\
\left(\mathrm{G}, \mathrm{G}^{\prime}\right)= & \left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right), l \geq 0, l^{\prime} \geq 1:
\end{aligned}
$$

If $l=0$, then $\mathfrak{g}=0$. If $l \geq 1$, then:
$\Delta^{+}=\left\{e_{j} \pm e_{k} ; 1 \leq j<k \leq l\right\} \cup\left\{e_{j} ; 1 \leq j \leq l\right\}\left(\right.$ type $\left.B_{l}\right), \quad \rho=\sum_{j=1}^{l}\left(l+\frac{1}{2}-j\right) e_{j}$,
$\lambda_{j} \in \mathbb{Z}, \quad \lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{l} \geq 0$.
There are two irreducible genuine representations of highest weight $\lambda$.
$\underline{\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{Sp}_{l}, \mathrm{O}_{2 l^{\prime}}^{*}\right), l \geq 1, l^{\prime} \geq 2:}$
$\Delta^{+}=\left\{e_{j} \pm e_{k} ; 1 \leq j<k \leq l\right\} \cup\left\{2 e_{j} ; 1 \leq j \leq l\right\}\left(\right.$ type $\left.C_{l}\right), \quad \rho=\sum_{j=1}^{l}(l+1-j) e_{j}$,
$\lambda_{j} \in \mathbb{Z}, \quad \lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{l} \geq 0$.
$\underline{\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right), l \geq 1, l^{\prime} \geq 1}:$

If $l=1$, then $\mathfrak{h}_{\mathbb{C}}=\mathfrak{g}_{\mathbb{C}}$. If $l \geq 2$, then:
$\Delta^{+}=\left\{e_{j} \pm e_{k} ; 1 \leq j<k \leq l\right\}\left(\right.$ type $\left.D_{l}\right), \quad \rho=\sum_{j=1}^{l}(l-j) e_{j}$,
$\lambda_{j} \in \mathbb{Z}, \quad \lambda_{1} \geq \lambda_{2} \geq \cdots \geq\left|\lambda_{l}\right|$.
If $\lambda_{l}=0$, there are two irreducible genuine representations of highest weight $\lambda$.

## Appendix I. Integration on the quotient space $S / S_{\overline{1}}^{\mathfrak{h}}$

We retain the notation of sections 3 and 4 . The purpose of this appendix is to prove the following lemma.

Lemma I.1. Suppose first that $\mathrm{G} \neq \mathrm{O}_{2 l+1}$ with $l<l^{\prime}$. Then are positive constants $C_{1}$ and $C_{2}$ such that for all $\phi \in C_{c}(\mathrm{~W})$ and $w \in \mathfrak{h}_{\overline{1}}{ }^{\text {reg }}$

$$
\begin{array}{ll}
\int_{\mathrm{S} / \mathrm{S}^{\mathrm{h}_{\overline{\mathrm{T}}}}} \phi(s \cdot w) d\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right)=C_{1} \int_{\mathrm{G}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime}} \phi\left(\left(g, g^{\prime}\right) \cdot w\right) d g d\left(g^{\prime} \mathrm{Z}^{\prime}\right) & \\
{\text { if } l \leq l^{\prime}}^{\int_{\mathrm{S} / \mathrm{S}^{\mathrm{b}_{\bar{T}}}} \phi(s . w) d\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right)=C_{2} \int_{\mathrm{G} / \mathrm{Z}} \int_{\mathrm{G}^{\prime}} \phi\left(\left(g, g^{\prime}\right) \cdot w\right) d(g \mathrm{Z}) d g^{\prime}} \quad \text { if } l>l^{\prime} . \tag{I.2}
\end{array}
$$

Now, let $\mathrm{G} \neq \mathrm{O}_{2 l+1}$ with $l<l^{\prime}$ and let $w_{0} \in \mathfrak{s}_{1}\left(\mathrm{~V}^{0}\right)$ be a nonzero element. Then there is a positive constant $C_{3}$ such that for all $\phi \in C_{c}(\mathrm{~W})$ and $w \in \mathfrak{h}_{\overline{1}}{ }^{\text {reg }}$

$$
\begin{equation*}
\int_{\mathrm{S}_{/ \mathrm{S}^{\mathrm{b}_{\mathrm{T}}+w_{0}}}} \phi(s \cdot w) d\left(s \mathrm{~S}^{\mathfrak{h}_{\bar{T}}+w_{0}}\right)=C_{3} \int_{\mathrm{G}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime n}} \phi\left(\left(g, g^{\prime}\right) \cdot w\right) d g d\left(g^{\prime} \mathrm{Z}^{\prime n}\right), \tag{I.3}
\end{equation*}
$$

where $\mathrm{Z}^{\prime n}$ is the centralizer in $\mathrm{Z}^{\prime}$ of $n=\tau^{\prime}\left(w_{0}\right)$.
Before proving Lemma I.1, let us consider the special case of the dual pair ( $\mathrm{G}, \mathrm{G}^{\prime}$ ) $=$ $\left(\mathrm{O}_{1}, \mathrm{Sp}_{2 n}(\mathbb{R})\right)$, which is not included in this lemma but will be needed in its proof. In the notation of section $3, \mathrm{~V}=\mathrm{V}_{\overline{0}} \oplus \mathrm{~V}_{\overline{1}}$, where $\operatorname{dim} \mathrm{V}_{\overline{0}}=1$ and $\operatorname{dim} \mathrm{V}_{\overline{1}}=2 n$. We have the identifications

$$
\mathrm{S}=\mathrm{G} \times \mathrm{G}^{\prime}=\mathrm{O}\left(\mathrm{~V}_{\overline{0}}\right) \times \operatorname{Sp}\left(\mathrm{V}_{\overline{1}}\right), \quad \mathrm{W}=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}, \mathrm{~V}_{\overline{0}}\right) .
$$

Let $0 \neq w_{0} \in \mathrm{~W}$. We shall describe $\operatorname{Stab}_{\mathrm{G}^{\prime}}\left(w_{0}\right)$, the stabilizer of $w_{0}$ in $\mathrm{G}^{\prime}=\operatorname{Sp}\left(\mathrm{V}_{\overline{1}}\right)$, as well as $\left(\mathrm{O}\left(\mathrm{V}_{\overline{0}}\right) \times \mathrm{Sp}\left(\mathrm{V}_{\overline{1}}\right)\right)^{w_{0}^{2}}$ and $\left(\mathrm{O}\left(\mathrm{V}_{\overline{0}}\right) \times \mathrm{Sp}\left(\mathrm{V}_{\overline{1}}\right)\right)^{w_{0}}$.

Since $\operatorname{dim} \operatorname{Ker} w_{0}=\operatorname{dim} \mathrm{W}-1$, we see that $\operatorname{dim}\left(\operatorname{Ker} w_{0}\right)^{\perp}=1$. Let $\mathrm{X}=\left(\operatorname{Ker} w_{0}\right)^{\perp}$. Since $\operatorname{dim} \mathrm{X}=1$, this is an isotropic subspace of W . Furthermore $\operatorname{Ker} w_{0}=\mathrm{X}^{\perp}$. Let $\mathrm{Y} \subseteq \mathrm{W}$ be a subspace of dimension 1 such that $\mathrm{W}=\operatorname{Ker} w_{0} \oplus \mathrm{Y}$. Set $\mathrm{U}=(\mathrm{X}+\mathrm{Y})^{\perp}$. Then the restriction of the symplectic form of W to U is non-degenerate and

$$
\begin{equation*}
\mathrm{V}_{\overline{\mathrm{I}}}=\mathrm{X} \oplus \mathrm{U} \oplus \mathrm{Y} \tag{I.4}
\end{equation*}
$$

Let $P_{\mathrm{Y}} \subseteq \mathrm{G}^{\prime}$ be the parabolic subgroup preserving Y . Then we have an isomorphism

$$
P_{\mathrm{Y}}=\mathrm{GL}_{1}(\mathrm{Y}) \times \mathrm{Sp}(\mathrm{U}) \times \mathrm{N}
$$

where N is the uniponent radical, isomorphic to a Heisenberg group. We see from (I.4) that

$$
\begin{equation*}
\operatorname{Stab}_{\mathrm{G}^{\prime}}\left(w_{0}\right)=\{1\} \times \operatorname{Sp}(\mathrm{U}) \times \mathrm{N} . \tag{I.5}
\end{equation*}
$$

If $w_{1}, w_{2} \in \mathfrak{s}_{\overline{1}}(\mathrm{~V})$ are non-nonzero and such that $w_{1}^{2}=w_{2}^{2}$, then $w_{2}= \pm w_{1}$. Equivalently, let $\tau^{\prime}: \mathrm{W} \rightarrow \mathfrak{g}^{\prime}=\mathfrak{s p}(\mathrm{W})$ denote the unnormalized moment map. Then $\tau^{\prime}\left(w_{1}\right)=\tau^{\prime}\left(w_{2}\right)$ implies $w_{2}= \pm w_{1}$, because $\mathrm{O}_{1}$ acts transitively on the fibers of $\tau^{\prime}$. Equivalently, if one thinks of W as $\mathrm{M}_{1,2 n}(\mathbb{R})$ and setting $w^{*}=J w^{t}$ for $J=\left(\begin{array}{cc}0 & 1_{n} \\ -1_{n} & 0\end{array}\right)$, one has that $w_{1}^{*} w_{1}=w_{2}^{*} w_{2}$ implies $w_{2}= \pm w_{1}$.

Now, one readily checks that $g^{\prime} \in \operatorname{Sp}\left(\mathrm{V}_{\overline{1}}\right)^{w_{0}^{2}}$ if and only if $g^{\prime} \tau^{\prime}\left(w_{0}\right) g^{\prime-1}=\tau^{\prime}\left(w_{0}\right)$. Since, for $g^{\prime} \in \operatorname{Sp}\left(\mathrm{V}_{\overline{1}}\right)$,

$$
g^{\prime} \tau^{\prime}\left(w_{0}\right) g^{\prime-1}=g^{\prime} w_{0}^{*} w_{0} g^{\prime-1}=\left(w_{0} g^{\prime-1}\right)^{*}\left(w_{0} g^{\prime-1}\right)=\tau^{\prime}\left(w_{0} g^{\prime-1}\right),
$$

this is equivalent to $\tau^{\prime}\left(w_{0} g^{\prime-1}\right)=\tau^{\prime}\left(w_{0}\right)$, i.e. $w_{0} g^{\prime-1}= \pm w_{0}$. In turn, this means that $\pm g^{\prime} \in \operatorname{Stab}_{\mathrm{G}^{\prime}}\left(w_{0}\right)$. Thus

$$
\begin{equation*}
\operatorname{Sp}\left(\mathrm{V}_{\overline{1}}\right)^{w_{0}^{2}}=\{ \pm 1\} \times \operatorname{Sp}(\mathrm{U}) \times \mathrm{N} \tag{I.6}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}\right) \times \mathrm{Sp}\left(\mathrm{~V}_{\overline{1}}\right)\right)^{w_{0}^{2}}=\{ \pm 1\} \times(\{ \pm 1\} \times \mathrm{Sp}(\mathrm{U}) \times \mathrm{N}) \tag{I.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}\right) \times \operatorname{Sp}\left(\mathrm{V}_{\overline{1}}\right)\right)^{w_{0}}=\{(\varepsilon ; \varepsilon, m, n) ; \varepsilon= \pm 1, m \in \mathrm{Sp}(\mathrm{U}), n \in \mathrm{~N}\} \tag{I.8}
\end{equation*}
$$

Notice that they do not depend on the choice of $0 \neq w_{0} \in \mathrm{~W}$. Moreover,

$$
\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}\right) \times \mathrm{Sp}\left(\mathrm{~V}_{\overline{1}}\right)\right)^{w_{0}^{2}} /\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}\right) \times \mathrm{Sp}\left(\mathrm{~V}_{\overline{1}}\right)\right)^{w_{0}}=(\{ \pm 1\} \times\{ \pm 1\}) /\{ \pm(1,1)\}
$$

is a group isomorphic to $\mathrm{O}_{1}$.
Proof of Lemma I.1. We now prove (I.1), excluding for the moment the pair (G, $\left.\mathrm{G}^{\prime}\right)=$ $\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l}(\mathbb{R})\right)$.

If $l \leq l^{\prime}$, then $\mathfrak{h}^{\prime}=\mathfrak{h} \oplus \mathfrak{h}^{\prime \prime}$. Write $\mathfrak{z}^{\prime}=\mathfrak{h} \oplus \mathfrak{z}^{\prime \prime}$ and, for the corresponding groups, $Z^{\prime}=H \times Z^{\prime \prime}$. Then $S^{b^{\frac{2}{1}}}=\mathrm{H} \times \mathrm{Z}^{\prime}$.

Let $\Delta: \mathrm{H} \rightarrow \mathrm{G} \times \mathrm{G}^{\prime}$ be defined by $\Delta(h)=\left(h,\left(h, 1_{l^{\prime}-l}\right)\right)$, where $1_{r}$ denotes the identity matrix of size $r$. Then $\mathrm{S}^{\mathrm{h}^{\mathrm{T}}}=\Delta(\mathrm{H})\left(\left\{1_{l}\right\} \times\left(\left\{1_{l}\right\} \times \mathrm{Z}^{\prime \prime}\right)\right)$. Set

$$
\mathrm{L}=\mathrm{S}^{\mathrm{h}^{\frac{2}{1}}} / \mathrm{S}^{\mathfrak{h}_{\overline{1}}}=\left(\mathrm{H} \times \mathrm{H} \times \mathrm{Z}^{\prime \prime}\right) / \mathrm{S}^{\mathfrak{h}} \overline{\mathrm{T}}=\left(\mathrm{H} \times \mathrm{H} \times\left\{1_{l^{\prime}-l}\right\}\right) / \Delta(\mathrm{H}),
$$

Then $L$ is a compact abelian group because so is $H$. It acts on $S / S_{\overline{1}}^{\mathfrak{h}}$ by

$$
\left(g, g^{\prime}\right) \mathrm{S}^{\mathfrak{h}_{\bar{T}}} \cdot\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \Delta(\mathrm{H})=\left(g h_{1}, g^{\prime}\left(h_{2}, 1_{l^{\prime}-l}\right)\right) \mathrm{S}^{\mathfrak{h}_{\mathrm{T}}}
$$

The action is proper and free. Hence the quotient space $\left(\mathrm{S} / \mathrm{S}_{\overline{1}}^{\mathfrak{h}}\right) / \mathrm{L}$, i.e. the space of orbits for this action, has a unique structure of smooth manifold such that the canonical projection $\mathrm{S} / \mathrm{S}^{\mathfrak{h}_{\bar{T}}} \rightarrow\left(\mathrm{~S} / \mathrm{S}^{\mathfrak{h}^{\top}}\right) / \mathrm{L}$ is a principal fiber bundle with structure group L. Since we have fixed a Haar measure on H , we also have Haar measures on $\mathrm{H} \times \mathrm{H} \times\left\{1_{l^{\prime}-l}\right\}$ and $\Delta(\mathrm{H})$. This fixes a quotient measure on on $\mathrm{L}=\left(\mathrm{H} \times \mathrm{H} \times\left\{1_{l^{\prime}-l}\right\}\right) / \Delta(\mathrm{H})$. Recall the notation $d\left(s S^{\mathfrak{h}} \overline{\overline{1}}\right)$ for the quotient measure of $\mathrm{S} / \mathrm{S}^{\mathfrak{h}_{\overline{1}}}$. Then there is a unique measure $d s^{\bullet}$
on $\left(\mathrm{S} / \mathrm{S}^{\mathfrak{h}_{\overline{1}}}\right) / \mathrm{L}$ such that for all $\Phi \in C_{c}\left(\mathrm{~S} / \mathrm{S}^{\mathfrak{h}_{\overline{1}}}\right)$

$$
\begin{aligned}
& \int_{\mathrm{S} / \mathrm{S}^{\mathfrak{h}} \overline{\mathrm{T}}} \Phi\left(s \mathrm{~S}^{\mathfrak{h} \overline{\mathrm{T}}}\right) d\left(s \mathrm{~S}^{\left.\mathfrak{h}_{\overline{\mathrm{T}}}\right)}\right. \\
& =\int_{\left(\mathrm{S} / \mathrm{S}^{\mathrm{h}} \overline{\mathrm{~T}}\right) / \mathrm{L}}\left(\int_{\left(\mathrm{H} \times \mathrm{H} \times\left\{1_{l^{\prime}-l}\right\}\right) / \Delta(\mathrm{H})} \Phi\left(\left(g, g^{\prime}\right)\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \mathrm{S}^{\mathfrak{h}_{\bar{T}}}\right) d\left(\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \Delta(\mathrm{H})\right)\right) d\left(g, g^{\prime}\right)^{\bullet} \\
& =\frac{1}{\operatorname{vol}(\Delta(\mathrm{H}))} \int_{\left(\mathrm{S} / \mathrm{S}^{\mathrm{h}} \overline{\mathrm{~T}}\right) / \mathrm{L}}\left(\int_{\mathrm{H} \times \mathrm{H}} \Phi\left(\left(g, g^{\prime}\right)\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \mathrm{S}^{\mathfrak{h}_{\bar{T}}}\right) d\left(h_{1}, h_{2}\right)\right) d\left(g, g^{\prime}\right)^{\bullet} ;
\end{aligned}
$$

see e.g. [DK00, §3.13, p. 183]. As a set,

$$
\begin{align*}
\left(\mathrm{S} / \mathrm{S}^{\mathfrak{h}_{\bar{T}}}\right) / \mathrm{L} & =\left(\left(\mathrm{G} \times \mathrm{G}^{\prime}\right) / \mathrm{S}^{\mathfrak{h}_{\bar{T}}}\right) /\left(\left(\mathrm{H} \times \mathrm{H} \times \mathrm{Z}^{\prime \prime}\right) / \mathrm{S}^{\mathfrak{h}_{\bar{T}}}\right) \\
& =\left(\mathrm{G} \times \mathrm{G}^{\prime}\right) /\left(\mathrm{H} \times \mathrm{H} \times \mathrm{Z}^{\prime \prime}\right) \\
& =\left(\mathrm{G} \times \mathrm{G}^{\prime}\right) /\left(\mathrm{H} \times \mathrm{Z}^{\prime}\right)=\mathrm{G} / \mathrm{H} \times \mathrm{G}^{\prime} / \mathrm{Z}^{\prime} \tag{I.9}
\end{align*}
$$

where the second equality holds under the identification $\left(g, g^{\prime}\right) \mathrm{S}^{\mathfrak{h}_{\overline{1}}} L=\left(g, g^{\prime}\right)\left(\mathrm{H} \times \mathrm{H} \times \mathrm{Z}^{\prime \prime}\right)$. Since the measure $d\left(s S^{\mathfrak{h}_{\overline{1}}}\right)$ on $\mathrm{S} / \mathrm{S}^{\mathfrak{h}_{\bar{T}}}$ is invariant with respect to the action of S by lefttranslation and this action commutes with the right-action of L on $\mathrm{S} / \mathrm{S}^{\mathrm{h}^{\mathrm{T}}}$, the measure $d s^{\bullet}$ is left S-invariant. By the above identification, $\left(\mathrm{G} \times \mathrm{G}^{\prime}\right) /\left(\mathrm{H} \times \mathrm{Z}^{\prime}\right)$ is endowed with an S-invariant measure, which must be a positive multiple of the quotient measure of those of $\mathrm{G} \times \mathrm{G}^{\prime}$ and $\mathrm{H} \times \mathrm{Z}^{\prime}$. Thus $d s^{\bullet}$ is a positive multiple of the product measure of the quotient measures of $\mathrm{G} / \mathrm{H}$ and $\mathrm{G}^{\prime} / \mathrm{Z}^{\prime}$. In conclusion, there is a positive constant $C$ such that for every $\Phi \in C_{c}\left(\mathrm{~S} / \mathrm{S}^{\mathfrak{h}_{\mathrm{T}}}\right)$

$$
\begin{aligned}
& \int_{\mathrm{S} / \mathrm{S}^{\mathrm{h}_{\mathrm{T}}}} \Phi\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right) d\left(s \mathrm{~S}^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right) \\
& =C \int_{\mathrm{G} / \mathrm{H} \times \mathrm{G}^{\prime} / \mathrm{Z}^{\prime}}\left(\int_{\mathrm{H} \times \mathrm{H}} \Phi\left(\left(g, g^{\prime}\right)\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \mathrm{S}^{\mathfrak{h} \overline{\mathrm{T}}}\right) d\left(h_{1}, h_{2}\right)\right) d(g \mathrm{H}) d\left(g^{\prime} \mathrm{Z}^{\prime}\right) .
\end{aligned}
$$

Suppose that $\Phi(s)=\phi(s . w)$, where $\phi \in C_{c}(\mathrm{~W})$ and $w \in \mathfrak{h}_{\overline{1}}^{\text {reg }}$. Hence $\phi\left(s \mathrm{~S}^{\mathfrak{h}_{\bar{T}}} . w\right)=\phi(s . w)$. Observe that

$$
\left(g, g^{\prime}\right)\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \cdot w=g h_{1} w\left(h_{2}^{-1}, 1_{l^{\prime}-l}\right) g^{\prime-1}=g h_{1} h_{2}^{-1} w g^{\prime-1}=\left(g h_{1} h_{2}^{-1}, g^{\prime}\right) \cdot w .
$$

Hence

$$
\begin{aligned}
\int_{\mathrm{H} \times \mathrm{H}} \phi\left(\left(g, g^{\prime}\right)\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \cdot w\right) d\left(h_{1}, h_{2}\right) & =\int_{\mathrm{H}} \int_{\mathrm{H}} \phi\left(\left(g h_{1}, g^{\prime}\right) \cdot w\right) d h_{1} d h_{2} \\
& =\operatorname{vol}(\mathrm{H}) \int_{\mathrm{H}} \phi\left(\left(g h_{1}, g^{\prime}\right) \cdot w\right) d h_{1}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\mathrm{G} / \mathrm{H} \times \mathrm{G}^{\prime} / \mathrm{Z}^{\prime}} & \int_{\mathrm{H} \times \mathrm{H}} \phi\left(\left(g, g^{\prime}\right)\left(h_{1}, h_{2}, 1_{l^{\prime}-l}\right) \cdot w\right) d\left(h_{1}, h_{2}\right) d(g \mathrm{H}) d\left(g^{\prime} \mathrm{Z}^{\prime}\right) \\
& =\operatorname{vol}(\mathrm{H}) \int_{\mathrm{G} / \mathrm{H}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime}}\left(\int_{\mathrm{H}} \phi\left(\left(g h_{1}, g^{\prime}\right) \cdot w\right) d h_{1}\right) d(g \mathrm{H}) d\left(g^{\prime} \mathrm{Z}^{\prime}\right) \\
& =\operatorname{vol}(\mathrm{H}) \int_{\mathrm{G}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime}} \phi\left(\left(g, g^{\prime}\right) \cdot w\right) d g d\left(g^{\prime} \mathrm{Z}^{\prime}\right)
\end{aligned}
$$

In conclusion, there is a positive constant $C$ such that for all $\phi \in C_{c}(\mathrm{~W})$ and $w \in \mathfrak{h}_{\overline{1}}{ }^{\text {reg }}$

$$
\begin{equation*}
\int_{\mathrm{S}^{\prime} \mathrm{S}^{\mathrm{b}_{\bar{T}}}} \phi(s \cdot w) d\left(s \mathrm{~S}^{\mathrm{h}_{\overline{\mathrm{T}}}}\right)=C \int_{\mathrm{G}} \int_{\mathrm{G}^{\prime} / \mathrm{Z}^{\prime}} \phi\left(\left(g, g^{\prime}\right) \cdot w\right) d g d\left(g^{\prime} \mathrm{Z}^{\prime}\right) . \tag{I.10}
\end{equation*}
$$

Let us now consider the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}(\mathbb{R})\right)$ with $1 \leq l \leq l^{\prime}$. We keep the notation introduced on page 22. In particular, $\mathrm{V}^{0}=\mathrm{V}_{0}^{0} \oplus \mathrm{~V}_{1}^{0}$ where $\operatorname{dim} \mathrm{V}_{0}^{0}=1$ and $\operatorname{dim} \mathrm{V}_{\overline{1}}^{0}=2\left(l^{\prime}-l\right)$. Each $h \in \mathrm{H}^{0}$ fixes $\bigvee_{\overline{0}}^{0}$ and hence every $h \in \mathrm{H}$ is of the form $h=\left(h_{\bullet}, \varepsilon\right)$ where $h_{\bullet} \in \mathrm{O}\left(\mathrm{V}_{\overline{0}}^{1} \oplus \cdots \mathrm{~V}_{\overline{0}}^{l}\right) \simeq \mathrm{O}_{2 l}$ and $\varepsilon \in \mathrm{O}\left(\mathrm{V}_{\overline{0}}^{0}\right)$. The elements $h_{\bullet}$ form a Cartan subgroup H. of $\mathrm{O}\left(\mathrm{V}_{\overline{0}}^{1} \oplus \cdots \mathrm{~V}_{\overline{0}}^{l}\right)$. At the group level, the decomposition $\mathfrak{h}^{\prime}=\mathfrak{h} \oplus \mathfrak{h}^{\prime \prime}$ arising from the identification (43) corresponds to a decomposition $\mathrm{H}^{\prime}=\mathrm{H}_{\bullet} \times \mathrm{H}^{\prime \prime}$ of the Cartan subgroup $\mathrm{H}^{\prime}$ of $\mathrm{G}^{\prime}$.

If $l=l^{\prime}$, then $\mathfrak{h}^{\prime \prime}=0$ and the equality $\mathfrak{z}^{\prime}=\mathfrak{h}^{\prime}=\mathfrak{h}$ corresponds, at the group level, to $\mathrm{Z}^{\prime}=\mathrm{H}^{\prime}=\mathrm{H}_{\bullet}$. Hence $\mathrm{S}^{\mathfrak{h}_{1}^{2}}=\mathrm{H} \times \mathrm{Z}^{\prime}=\mathrm{H} \times \mathrm{H}_{\bullet} \cong \mathrm{H}_{\bullet} \times \mathrm{H}_{\bullet} \times \mathrm{O}\left(\mathrm{V}_{0}^{0}\right)$ and $\mathrm{S}^{\mathfrak{h}_{\mathrm{T}}}=$ $\left\{\left(h_{\bullet}, \varepsilon, h_{\bullet}\right) ; h_{\bullet} \in \mathrm{H}_{\bullet}\right\} \cong \Delta\left(\mathrm{H}_{\bullet}\right) \times \mathrm{O}\left(\mathrm{V}_{\overline{0}}^{0}\right)$, where $\Delta\left(\mathrm{H}_{\bullet}\right)=\left\{(h, h) ; h \in \mathrm{H}_{\bullet}\right\}$. Thus $\mathrm{L}=$ $\mathrm{S}^{\mathrm{h}_{\frac{2}{1}}^{2}} / \mathrm{S}^{\mathfrak{h}_{\overline{1}}} \cong\left(\mathrm{H}_{\bullet} \cong \mathrm{H}_{\bullet}\right) / \Delta\left(\mathrm{H}_{\bullet}\right)$ is a compact abelian group and, as a set,

$$
\left(\mathrm{S} / \mathrm{S}^{\mathfrak{h}_{\overline{\mathrm{T}}}}\right) / \mathrm{L}=\left(\left(\mathrm{G} \times \mathrm{G}^{\prime}\right) / \mathrm{S}^{\mathfrak{h}_{\overline{1}}}\right) /\left(\left(\mathrm{H} \times \mathrm{Z}^{\prime}\right) / \mathrm{S}^{\mathfrak{h}} \overline{\mathrm{T}}\right)=\mathrm{G} / \mathrm{H} \times \mathrm{G}^{\prime} / \mathrm{Z}^{\prime}
$$

as in (I.9). Hence (I.1) follows as in the general case $l \leq l^{\prime}$.
Let us now consider the dual pair $\left(\mathrm{G}, \mathrm{G}^{\prime}\right)=\left(\mathrm{O}_{2 l+1}, \mathrm{Sp}_{2 l^{\prime}}\right)$ with $1 \leq l<l^{\prime}$. Let $0 \neq w_{0} \in$ $\mathfrak{s}_{1}\left(\mathrm{~V}^{0}\right)=\operatorname{Hom}\left(\mathrm{V}_{\overline{1}}^{0}, \mathrm{~V}_{\overline{0}}^{0}\right)$. We shall describe $\mathrm{S}^{\left(\mathfrak{h}_{\overline{1}}+w_{0}\right)^{2}}$ and its subgroup $\mathrm{S}^{\mathfrak{h}_{1}+w_{0}}$.

Since $\mathfrak{h}_{\overline{1}}$ preserves the decomposition (35), we see that $\left(\mathfrak{h}_{\overline{1}}+w_{0}\right)^{2}=\mathfrak{h}_{\frac{1}{1}}^{2}+w_{0}^{2}$ and hence

$$
\begin{align*}
\mathrm{S}^{\left(\mathfrak{h}_{1}^{1}+w_{0}\right)^{2}}=\mathrm{S}^{\mathfrak{h}_{\frac{1}{1}}^{2}+w_{0}^{2}}=\left(\mathrm{S}^{\mathfrak{h}^{\frac{2}{1}}}\right)^{w_{0}^{2}} & =\mathrm{H} \bullet \times \mathrm{O}\left(\mathrm{~V}_{\frac{0}{0}}^{0}\right) \times \mathrm{H}_{\bullet} \times \mathrm{Sp}\left(\mathrm{~V}_{\overline{1}}^{0}\right)^{n} \\
& \simeq \mathrm{H} \bullet \times \mathrm{H}_{\bullet} \times\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}^{0}\right) \times \mathrm{Sp}\left(\mathrm{~V}_{\frac{0}{1}}^{0}\right)\right)^{w_{0}^{2}} \tag{I.11}
\end{align*}
$$

where $\mathrm{O}\left(\mathrm{V}_{\frac{0}{0}}^{0}\right)=\{ \pm 1\}$ and $\mathrm{Sp}\left(\mathrm{V}_{\frac{1}{1}}\right)^{n}$ is the centralizer of $n=\tau^{\prime}\left(w_{0}\right)$ in the symplectic group $\mathrm{Sp}\left(\mathrm{V}_{\frac{1}{\mathrm{I}}}^{0}\right)$. Notice that we can also write

$$
\begin{equation*}
\mathrm{S}^{\left(\mathfrak{h}_{\overline{1}}+w_{0}\right)^{2}}=\mathrm{H} \times \mathrm{Z}^{\prime n} \tag{I.12}
\end{equation*}
$$

where $\mathrm{Z}^{\prime n}$ is the centralizer of $n$ in $\mathrm{Z}^{\prime}$. In the identification (I.11),

$$
\begin{align*}
\mathrm{S}^{\mathrm{h}_{\mathrm{T}}+w_{0}} & =\left\{(h, h, s) ; h \in \mathrm{H}_{\bullet}, s \in\left(\mathrm{O}\left(\mathrm{~V}_{\frac{0}{0}}^{0}\right) \times \mathrm{Sp}\left(\mathrm{~V}_{\frac{1}{1}}^{0}\right)\right)^{w_{0}}\right\} \\
& =\Delta(\mathrm{H} \bullet) \times\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}^{0}\right) \times \operatorname{Sp}\left(\mathrm{V}_{\overline{1}}^{0}\right)\right)^{w_{0}} \tag{I.13}
\end{align*}
$$

The groups $\left(\mathrm{O}\left(\mathrm{V}_{\frac{0}{0}}^{0}\right) \times \operatorname{Sp}\left(\mathrm{V}_{\frac{0}{1}}^{0}\right)\right)^{w_{0}^{2}}$ and $\left(\mathrm{O}\left(\mathrm{V}_{\frac{0}{0}}^{0}\right) \times \operatorname{Sp}\left(\mathrm{V}_{\frac{1}{1}}^{0}\right)\right)^{w_{0}}$ are computed as in (I.7) and (I.8), respectively, with V replaced by $\mathrm{V}^{0}$. Then

$$
\begin{aligned}
\mathrm{L}=\mathrm{S}^{\left(\mathfrak{h}_{\overline{1}}+w_{0}\right)^{2}} / \mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_{0}} & \simeq\left(\mathrm{H}_{\bullet} \times \mathrm{H}_{\bullet}\right) / \Delta\left(\mathrm{H}_{\bullet}\right) \times\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}^{0}\right) \times \mathrm{Sp}\left(\mathrm{~V}_{\overline{1}}^{0}\right)\right)^{w_{0}^{2}} /\left(\mathrm{O}\left(\mathrm{~V}_{\overline{0}}^{0}\right) \times \mathrm{Sp}\left(\mathrm{~V}_{\overline{1}}^{0}\right)\right)^{w_{0}} \\
& \cong\left(\mathrm{H}_{\bullet} \times \mathrm{H}_{\bullet}\right) / \Delta\left(\mathrm{H}_{\bullet}\right) \times\{ \pm 1\},
\end{aligned}
$$

which is a compact abelian group. By (I.12), we therefore obtain that, as a set,

$$
\left(\mathrm{S} / \mathrm{S}^{\mathrm{h}_{\overline{1}}+w_{0}}\right) / \mathrm{L}=\left(\mathrm{G} \times \mathrm{G}^{\prime}\right) /\left(\mathrm{H} \times \mathrm{Z}^{\prime n}\right)=\mathrm{G} / \mathrm{H} \times \mathrm{G}^{\prime} / \mathrm{Z}^{\prime n},
$$

and (I.3) follows as in the general case $l \leq l^{\prime}$.
The proof of (I.2) is similar to that of (I.1) and left to reader.

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[^0]:    ${ }^{1}$ Following the notation at the beginning of Section 3 , one should identify $g$ and $\left(g^{-1}\right)^{t} \otimes 1$.

