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## 1. Introduction

A goal of these lectures is to explain the ingredients of Harish-Chandra's Plancherel formula, explain how they fit together, study particular cases and go through all the details for the group of the real unimodular matrices of size two.

All the necessary information in its original nearly perfect form is contained in Harish-Chandra's articles [HC14a], [HC14b], [HC14d], [HC14c], [HC18]. The example  $\mathrm{SL}_2(\mathbb{R})$  is explained in classical books such as [Lan75]. For all of that, a good understanding of the Fourier Transform and Distribution theory on an Euclidean space is needed. Here Hörmander's "The Analysis of Linear Partial Differential Operators I" is one of the best references, [Hör83].

## 2. The Fourier Transform on the Schwartz space $\mathcal{S}(\mathbb{R})$

Here we follow [Hör83, section 7.1]. Recall that the Schwartz space  $\mathcal{S}(\mathbb{R})$  consists of all infinitely many times differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that for any two integers  $n, k \geq 0$

$$\sup_{x \in \mathbb{R}} |x^n \partial_x^k f(x)| < \infty.$$

In particular  $\mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R})$  and we have the well defined

**Theorem 1.** *The Fourier transform*

$$\hat{f}(y) = \mathcal{F}f(y) = \int_{\mathbb{R}} e^{-2\pi ixy} f(x) dx \quad (y \in \mathbb{R}, f \in \mathcal{S}(\mathbb{R})) \quad (1)$$

maps the Schwartz space  $\mathcal{S}(\mathbb{R})$  into itself, is invertible, and the inverse is given by

$$f(x) = \int_{\mathbb{R}} e^{2\pi ixy} \mathcal{F}f(y) dy \quad (x \in \mathbb{R}, f \in \mathcal{S}(\mathbb{R})). \quad (2)$$

Since

$$\frac{d}{dy} \mathcal{F}f(y) = \int_{\mathbb{R}} e^{-2\pi ixy} (-2\pi ix) f(x) dx \quad (3)$$

and

$$\int_{\mathbb{R}} e^{-2\pi ixy} f'(x) dx = 2\pi iy \mathcal{F}f(y), \quad (4)$$

the inclusion  $\mathcal{F}\mathcal{S}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$  is easy to check. For the rest we need a few lemmas.

**Lemma 2.** *Let  $f \in \mathcal{S}(\mathbb{R})$  be such that  $f(0) = 0$ . Set  $g(x) = \int_0^1 f'(tx) dt$ . Show that  $f(x) = xg(x)$  and  $g \in \mathcal{S}(\mathbb{R})$ .*

*Proof.* The equality  $f(x) = xg(x)$  is immediate from the Fundamental Theorem of Calculus, via a change of variables  $y = tx$ .

Fix two non-negative integers  $n$  and  $k$ . Suppose  $|x| \leq 1$ . Then

$$|x^n g^{(k)}(x)| = \left| x^n \int_0^1 t^k f^{(k+1)}(tx) dt \right| \leq \int_0^1 |f^{(k+1)}(tx)| dt \leq \max_{y \in \mathbb{R}} |f^{(k+1)}(y)| < \infty.$$

Notice that

$$\begin{aligned} x^n g^{(k)}(x) &= x^n \left( \frac{d}{dx} \right)^k (x^{-1} f(x)) \\ &= x^n \sum_{p=0}^k \frac{k!}{p!(k-p)!} (-1)(-2)\dots(-p) x^{-p-1} f^{(k-p)}(x) \end{aligned}$$

and that

$$\max_{|x| \geq 1} |x^{n-p-1} f^{(k-p)}(x)| < \infty.$$

Hence

$$\max_{x \in \mathbb{R}} |x^n g^{(k)}(x)| < \infty.$$

□

**Corollary 3.** Fix  $y \in \mathbb{R}$ . Let  $\phi \in \mathcal{S}(\mathbb{R})$  be such that  $\phi(y) = 0$ . Then that there is  $\psi \in \mathcal{S}(\mathbb{R})$  such that

$$\phi(x) = (x - y)\psi(x).$$

**Lemma 4.** Let  $T : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  be a linear map with the property that if  $\phi(y) = 0$  for some  $y \in \mathbb{R}$  then  $T\phi(y) = 0$  for the same  $y$ . Then there is a function  $c(x)$  such that

$$T\phi(x) = c(x)\phi(x) \quad (\phi \in \mathcal{S}(\mathbb{R})).$$

(In other words,  $T$  is the multiplication by the function  $c$ .)

*Proof.* Let  $\phi_1(x) = e^{-x^2}$ . As we know this function belongs to  $\mathcal{S}(\mathbb{R})$ . Fix  $x \in \mathbb{R}$ . Then for any  $\phi_2 \in \mathcal{S}(\mathbb{R})$

$$(\phi_2(x)\phi_1 - \phi_1(x)\phi_2)(x) = \phi_2(x)\phi_1(x) - \phi_1(x)\phi_2(x) = 0.$$

Hence, by the assumption on  $T$ ,

$$0 = T(\phi_2(x)\phi_1 - \phi_1(x)\phi_2)(x). \quad (5)$$

Since  $T$  is linear

$$T(\phi_2(x)\phi_1) = \phi_2(x)T(\phi_1) \quad \text{and} \quad T(\phi_1(x)\phi_2) = \phi_1(x)T(\phi_2).$$

Thus evaluation at  $x$  and using (5) we see that

$$0 = \phi_2(x)T(\phi_1)(x) - \phi_1(x)T(\phi_2)(x).$$

Therefore

$$T(\phi_2)(x) = \frac{T(\phi_1)(x)}{\phi_1(x)} \phi_2(x).$$

Thus the claim holds with

$$c(x) = \frac{T(\phi_1)(x)}{\phi_1(x)}.$$

□

**Lemma 5.** *Suppose  $T : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  is a linear map which commutes with the multiplication by  $x$ :*

$$T(xf(x)) = xT(f)(x) \quad (f \in \mathcal{S}(\mathbb{R})).$$

*Then there is a function  $c(x)$  such that*

$$T\phi(x) = c(x)\phi(x) \quad (\phi \in \mathcal{S}(\mathbb{R})).$$

*Proof.* It'll suffice to show that  $T$  satisfies the assumptions of Lemma 4. Fix  $y \in \mathbb{R}$ . Let  $\phi \in \mathcal{S}(\mathbb{R})$  be such that  $\phi(y) = 0$ . Then, by Corollary 3 there is  $\psi \in \mathcal{S}(\mathbb{R})$  such that

$$\phi(x) = (x - y)\psi(x).$$

Hence

$$T(\phi)(x) = T((x - y)\psi(x)) = (x - y)T(\phi)(x),$$

which is zero if  $x = y$ . □

**Proposition 6.** *Suppose  $T : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  is a linear map which commutes with the multiplication by  $x$ :*

$$T(xf(x)) = xT(f)(x) \quad (f \in \mathcal{S}(\mathbb{R}))$$

*and with the derivative*

$$T(f') = T(f)' \quad (f \in \mathcal{S}(\mathbb{R})).$$

*Then there is a constant  $c$  such that*

$$T\phi(x) = c\phi(x) \quad (\phi \in \mathcal{S}(\mathbb{R})).$$

*Proof.* We know from Lemma 5. that  $T$  coincides with the multiplication by a function  $c(x)$ . Since  $T$  commutes with the derivative we see that for any  $f \in \mathcal{S}(\mathbb{R})$

$$c(x)f'(x) = (c(x)f(x))'.$$

Since the multiplication by  $c(x)$  preserves the Schwartz space,  $c(x)$  is differentiable and

$$(c(x)f(x))' = c'(x)f(x) + c(x)f'(x).$$

Hence  $c'(x) = 0$ . therefore  $c(x)$  is a constant. □

**Lemma 7.** *Let  $Rf(x) = f(-x)$ . The map  $T = R\mathcal{F}^2 : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  commutes with the multiplication by  $x$  and with the derivative.*

*Proof.* Since

$$R(xf(x)) = -xf(-x) = -xR(f)(x) \quad \text{and} \quad R(f') = -R(f),$$

it'll suffice to check that

$$\mathcal{F}^2(xf(x)) = -x\mathcal{F}^2(f)(x) \quad \text{and} \quad \mathcal{F}^2(f') = -(\mathcal{F}^2(f))',$$

which follows from the relations (3) and (4). □

**Lemma 8.** *Let  $f(x) = e^{-\pi x^2}$ . Then  $\mathcal{F}f = f$ . (Fourier transform of the normalized Gaussian is the same Gaussian.)*

*Proof.* Since

$$\frac{d}{dx} f(x) = -2\pi x f(x)$$

the formulas (3) and (4) show that

$$\frac{d}{dx} (\mathcal{F}f(x) \cdot f(x)) = 0.$$

Hence

$$\mathcal{F}f(x) = \text{const} f(x).$$

Evaluating at  $x = 0$  gives

$$\int_{\mathbb{R}} f(y) dy = \text{const}.$$

By squaring the integral and using polar coordinates we show that  $\text{const} = 1$ . □

Now we are ready to prove the inversion formula in Theorem 1. We see from that the map  $R\mathcal{F}^2 : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  is the identity,

$$R\mathcal{F}^2 f = f \quad (f \in \mathcal{S}(\mathbb{R})).$$

We know from Lemma 7 and Proposition 6 that the map  $R\mathcal{F}^2$  is a constant multiple of the identity:  $cI = R\mathcal{F}^2$ . Now Lemma 8 shows that with  $f(x) = e^{-\pi x^2}$

$$cf = R\mathcal{F}^2 f = Rf = f.$$

Thus  $c = 1$ . Hence

$$R\mathcal{F} = \mathcal{F}^{-1}$$

and the formula2 follows. This completes the proof of Theorem 1.

### 3. Magnetic Resonance Imaging

Suppose a source at  $s \in \mathbb{R}$  is emitting a signal with frequency  $ks \in \mathbb{R}$  and amplitude  $A(s)$ . The collective signal received is

$$B(x) = \int_{\mathbb{R}} A(s) e^{2\pi i x k s} ds.$$

By Fourier inversion,

$$A(s) = k \int_{\mathbb{R}} B(x) e^{-2\pi i x k s} dx.$$

Hence we can recover  $A(s)$  from  $B(x)$ . In particular, if we the function  $A(s)$  is linear (in some large interval contained in  $[0, \infty)$ ) and if we know  $A(s)$  then we know  $s$ , i.e. the location of the source. For the related physics see

[youtube.com/watch?v=pGcZvSG805Y](https://www.youtube.com/watch?v=pGcZvSG805Y) [youtube.com/watch?v=djAxjtN\\_7VE](https://www.youtube.com/watch?v=djAxjtN_7VE).

#### 4. The Fourier Transform on the space $C^\infty(U_1)$

Here  $U_1 = \{u \in \mathbb{C}; |u| = 1\}$  is the group of the unitary matrices of size 1. We shall use the following identification of groups

$$\mathbb{R}/2\pi\mathbb{Z} \ni \theta + 2\pi\mathbb{Z} \rightarrow e^{i\theta} \in U_1.$$

Here we follow [SS03, section 2.2]. Recall that a function  $f : \mathbb{Z} \rightarrow \mathbb{C}$  is called rapidly decreasing if

$$\sup_{n \in \mathbb{Z}} |(1 + |n|)^k|f(n)| < \infty \quad (n \in \mathbb{Z}).$$

**Theorem 9.** *The Fourier transform*

$$\hat{f}(n) = \mathcal{F}f(n) = \int_0^1 e^{-2\pi i x n} f(x) dx \quad (n \in \mathbb{Z}, f \in C^\infty(U_1)) \quad (6)$$

maps the space  $C^\infty(U_1)$  into the space of the rapidly decreasing functions on  $\mathbb{Z}$ , is invertible, and the inverse given by

$$f(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i x n} \mathcal{F}f(n) dx \quad (x \in \mathbb{R}, f \in C^\infty(U_1)). \quad (7)$$

(Notice that the integral (6) converges as long as the function  $f$  is absolutely integrable. Hence we have the Fourier  $\mathcal{F}f$  transform for any  $f \in L^1(U_1)$ .) Since

$$\int_0^1 e^{-2\pi i x n} f'(x) dx = 2\pi i n \mathcal{F}f(n), \quad (8)$$

the rapid decrease of  $\mathcal{F}f$  is easy to check. For the inversion formula we need the following Lemma.

**Lemma 10.** *If  $f : U_1 \rightarrow \mathbb{C}$  is continuous and  $\mathcal{F}f = 0$ , then  $f = 0$ .*

*Proof.* Since

$$\int_0^1 e^{-2\pi i x n} f(x + y) dx = e^{2\pi i y n} \int_0^1 e^{-2\pi i x n} f(x) dx$$

it'll suffice to show that  $f(0) = 0$ .

Suppose the claim is false. We may assume that  $f$  is real valued and that  $f(0) > 0$ . We shall arrive at a contradiction. Choose  $0 < \delta \leq \frac{\pi}{2}$  so that

$$f(x) > \frac{f(0)}{2} \quad (|x| < \delta).$$

Let  $\epsilon > 0$  be so small that the function

$$p(x) = \epsilon + \cos x$$

satisfies

$$|p(x)| < 1 - \frac{\epsilon}{2} \quad (\delta \leq |x| \leq \pi).$$

Choose  $0 < \eta < \delta$  so that

$$|p(x)| \geq 1 + \frac{\epsilon}{2} \quad (|x| \leq \eta).$$

Then for  $k = 0, 1, 2, \dots$ ,

$$\begin{aligned} \left| \int_{\delta \leq |x| \leq \pi} f(x)p(x)^k dx \right| &\leq 2\pi \|f\|_\infty \left(1 - \frac{\epsilon}{2}\right)^k, \\ \int_{\eta \leq |x| < \delta} f(x)p(x)^k dx &\geq 0, \\ \int_{|x| < \eta} f(x)p(x)^k dx &\geq 2\eta \frac{f(0)}{2} \left(1 + \frac{\epsilon}{2}\right)^k. \end{aligned}$$

Therefore

$$\lim_{k \rightarrow \infty} \int_{|x| \leq \pi} f(x)p(x)^k dx = +\infty.$$

However  $p(x)^k$  is a trigonometric polynomial (linear combination of powers of exponentials). Hence the assumption  $\mathcal{F}f = 0$  implies

$$\int_{|x| \leq \pi} f(x)p(x)^k dx = 0.$$

□

## 5. Some Functional Analysis on a Hilbert space

Here we follow [Lan85, sections 1.2 and 1.3]. Let  $V$  be a Hilbert space. A continuous linear map  $A : V \rightarrow V$  is called compact if it any bounded sequence  $v_n \in V$  to a sequence  $Av_n$  that has a convergent subsequence.

**Theorem 11.** *Let  $A$  be a compact hermitian operator on the Hilbert space  $V$ . Then the family of eigenspaces  $V_\lambda$ , where  $\lambda$  ranges over all eigenvalues (including 0), is an orthogonal decomposition of  $E$ :*

$$V = \bigoplus_{\lambda} V_\lambda.$$

For  $\lambda \neq 0$  the eigenspace  $V_\lambda$  is finite dimensional.

Let  $S$  be a set of operators (continuous linear maps) on  $V$ . We say that  $V$  is  $S$ -irreducible if  $V$  has no closed  $S$ -invariant subspace other than  $\{0\}$  and  $V$  itself. We say that  $V$  is completely reducible for  $S$  if  $V$  is the orthogonal direct sum of  $S$ -irreducible subspaces. Two subspaces  $V_1, V_2 \subseteq E$  are called  $S$ -isomorphic if there is an isometry from  $V_1$  onto  $V_2$  which intertwines the action of  $S$ . The number of elements in such an isomorphism class is called the multiplicity of that isomorphism class in  $V$ .

A subalgebra  $\mathcal{A}$  of operators on  $V$  is said to be  $*$ -closed if whenever  $A \in \mathcal{A}$ , then  $A^* \in \mathcal{A}$ . As explained in [Lan85, section 1.2], the following Theorem is a consequence of Theorem 11.

**Theorem 12.** *Let  $\mathcal{A}$  be a  $*$ -closed subalgebra of compact operators on a Hilbert space  $V$ . Then  $V$  is completely reducible for  $\mathcal{A}$ , and each irreducible subspace occurs with finite multiplicity.*

The following theorem (and the corollary below) is known as Schur's Lemma. For a proof see [Lan85, Appendix 1, Theorem 4]

**Theorem 13.** *Let  $S$  be a set of operators acting irreducibly on the Hilbert space  $\mathbf{V}$ . Let  $A$  be a hermitian operator such that  $AB = BA$  for all  $B \in S$ . Then  $A = cI$  for some real number  $c$ .*

**Corollary 14.** *Let  $S$  be a set of operators acting irreducibly on the Hilbert space  $\mathbf{V}$ . Let  $A$  be an operator such that  $AB = BA$  and  $A^*B = BA^*$  for all  $B \in S$ . Then  $A = cI$  for some  $c \in \mathbb{C}$ .*

*Proof.* We write  $A = B + iC$ , where

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2i}(A - A^*).$$

Then  $B$  and  $C$  commute with  $S$ . Hence there are real numbers  $b$  and  $c$  such that  $B = bI$  and  $C = cI$ . Hence  $A = (b + ic)I$ .  $\square$

We shall also use a few facts concerning integral kernel operators. The proofs may be found in [Lan85, section 1.3]

**Theorem 15.** *Let  $(X, M, dx)$  and  $(Y, N, dy)$  be measured spaces, and assume that  $L^2(X)$ ,  $L^2(Y)$  have countable orthogonal bases. Let  $q \in L^2(X \times Y)$ . Then the formula*

$$Qf(x) = \int_Y q(x, y)f(y)dy$$

*defines a bounded, compact operator from  $L^2(Y)$  into  $L^2(X)$  with  $\|Q\| \leq \|q\|_2$ . If  $(X, M, dx) = (Y, N, dy)$  then  $Q$  is of trace class. If  $X$  is a topological space,  $dx$  is a Borel measure and in addition  $q$  is continuous, then*

$$\text{tr } Q = \int_X q(x, x) dx.$$

## 6. Representations of locally compact groups

Let  $G$  be a locally compact group and let  $\mathbf{V}$  be a topological vector space. We shall always assume that  $\mathbf{V} \neq \{0\}$ . Let  $\text{GL}(\mathbf{V})$  denote the group of the invertible continuous endomorphisms of  $\mathbf{V}$ . A representation of  $G$  on  $\mathbf{V}$  is a pair  $(\pi, \mathbf{V})$ , where  $\pi : G \rightarrow \text{GL}(\mathbf{V})$  is a group homomorphism such that the map

$$G \times \mathbf{V} \ni (g, v) \rightarrow \pi(g)v \in \mathbf{V}$$

is continuous. A subspace  $W \subseteq \mathbf{V}$  is called invariant if  $\rho(G)W \subseteq W$ . The representation  $(\rho, \mathbf{V})$  is called irreducible if  $\mathbf{V}$  does not contain any closed invariant subspaces other than  $\{0\}$  and  $\mathbf{V}$ .

If  $(\pi_1, \mathbf{V}_1)$ ,  $(\pi_2, \mathbf{V}_2)$  are representations of  $G$ , then a continuous linear map  $T\mathbf{V}_1 \rightarrow \mathbf{V}_2$  such that

$$T\pi_1(g) = \pi_2(g)T \quad (g \in G)$$



is called an intertwining operator, or a  $G$ -homomorphism. The vector space of all the intertwining maps will be denoted  $\text{Hom}_G(\mathbf{V}_1, \mathbf{V}_2)$  or more precisely,  $\text{Hom}_G((\pi_1, \mathbf{V}_1), (\pi_2, \mathbf{V}_2))$ . The representations  $(\pi_1, \mathbf{V}_1), (\pi_2, \mathbf{V}_2)$  are equivalent if there exists a bijective  $T \in \text{Hom}_G(\mathbf{V}_1, \mathbf{V}_2)$ .

A basic example of a representation of  $G$  is the right regular representation  $(R, C(G))$ . Here  $C(G)$  is the space of the continuous, complex valued, functions on  $G$  (with the topology of uniform convergence on compact sets) and

$$R(g)f(x) = f(xg) \quad (g, x \in G, f \in C(G)).$$

Similarly, we have the left regular representation  $(L, C(G))$ ,

$$L(g)f(x) = f(g^{-1}x) \quad (g, x \in G, f \in C(G)).$$

Given an irreducible representation  $(\pi, \mathbf{V})$  and an element  $\lambda \in \mathbf{V}'$  (a continuous linear functional on  $\mathbf{V}$ ) the formula

$$Tv(x) = \lambda(\pi(x)v) \quad (x \in G)$$

defines an injective map  $T \in \text{Hom}_G((\pi, \mathbf{V}), (R, C(G)))$ . Thus every abstract representation of  $G$  may be viewed as a subrepresentation of the right regular representation on the continuous functions on  $G$ . The function

$$G \ni x \rightarrow \lambda(\pi(x)v) \in \mathbb{C}$$

is called a matrix coefficient of the representation  $(\pi, \mathbf{V})$ .

If  $\mathbf{V}$  is a Hilbert space, then a representation  $(\pi, \mathbf{V})$  is called unitary if every operator  $\pi(g), g \in G$ , is unitary. Two unitary representations  $(\pi_1, \mathbf{V}_1), (\pi_2, \mathbf{V}_2)$  are called unitarily equivalent if there is a bijective and isometric  $G$ -homomorphism  $T : \mathbf{V}_1 \rightarrow \mathbf{V}_2$ .

Here is the classical version of Schur's Lemma, which is an immediate consequence of Corollary 14.

**Theorem 16.** *Let  $(\pi, \mathbf{V})$  be an irreducible unitary representation of  $G$ . Then*

$$\text{Hom}_G(\mathbf{V}, \mathbf{V}) = \mathbb{C}I.$$

**Corollary 17.** *For two irreducible unitary representations  $(\rho, \mathbf{V})$  and  $(\rho', \mathbf{V}')$  of  $G$ ,*

$$\dim \text{Hom}_G(\mathbf{V}, \mathbf{V}') = \begin{cases} 1 & \text{if } (\rho, \mathbf{V}) \simeq (\rho', \mathbf{V}') \\ 0 & \text{if } (\rho, \mathbf{V}) \not\simeq (\rho', \mathbf{V}') \end{cases}$$

(Here  $\simeq$  stands for unitary equivalence.)

**Proposition 18.** *If two unitary representations  $(\pi, \mathbf{V}), (\pi', \mathbf{V}')$  are equivalent then they are unitarily equivalent.*

*Proof.* Let  $(\cdot, \cdot)$  be the invariant scalar product on  $\mathbf{V}$  and let  $(\cdot, \cdot)'$  be the invariant scalar product on  $\mathbf{V}'$ . Pick an isomorphism  $T : \mathbf{V} \rightarrow \mathbf{V}'$ . Define  $T^* : \mathbf{V}' \rightarrow \mathbf{V}$  by

$$(Tu, v)' = (u, T^*v) \quad (u \in \mathbf{V}, v \in \mathbf{V}).$$

Then  $T^*\mathbf{V}' \rightarrow \mathbf{V}$  is also a morphism. Hence,  $T^*T : \mathbf{V} \rightarrow \mathbf{V}$  commutes with the action of  $G$ . Hence, there is  $\lambda \in \mathbb{C}$  such that  $T^*T = \lambda I$ . Thus, for any  $u, v \in \mathbf{V}$ ,

$$(Tu, Tv)' = (u, T^*Tv) = \bar{\lambda}(u, v).$$

In particular, by taking  $u = v \neq 0$  we see that  $\lambda > 0$ . Hence,

$$\frac{1}{\sqrt{\lambda}}T : \mathbb{V} \rightarrow \mathbb{V}'$$

is an isometry and a morphism. □

Here is one more version of Schur's Lemma. For a proof see [Wal88a, 1.2.2].

**Theorem 19.** *Let  $(\pi, \mathbb{V})$  be an irreducible unitary representation of  $G$ . Let  $\mathbb{V}_0 \subseteq \mathbb{V}$  be a dense  $G$ -invariant subspace and let  $A : \mathbb{V}_0 \rightarrow \mathbb{V}$  be a linear  $G$ -intertwining map.*

*Suppose  $\mathbb{V}_1 \subseteq \mathbb{V}$  is a dense subspace and let  $B : \mathbb{V}_1 \rightarrow \mathbb{V}$  a linear map such that*

$$(Av_0, v_1) = (v_0, Bv_1) \quad (v_0 \in \mathbb{V}_0, v_1 \in \mathbb{V}_1).$$

*Then  $A$  is a scalar multiple of the identity on  $\mathbb{V}$ , restricted to  $\mathbb{V}_0$ .*

## 7. Haar measures and extension of a representation of $G$ to a representation of $L^1(G)$

A proof of the following theorem may be found in [HR63, (15.5)-(15.11)]

**Theorem 20.** *There is a positive Borel measure  $dx$  on  $G$  such that*

$$\int_G f(gx) dx = \int_G f(x) dx \quad (g \in G, f \in C_c(G)).$$

*This measure is unique up to a constant multiple. Furthermore there is a group homomorphism  $\Delta : G \rightarrow \mathbb{R}^+$  such that*

$$\int_G f(xg) dx = \Delta(g) \int_G f(x) dx \quad (g \in G, f \in C_c(G)).$$

The measure  $dx$  is called the left invariant Haar measure on  $G$ . The group  $G$  is called unimodular if  $\Delta = 1$ . This is certainly the case if  $G$  is compact. The Lebesgue measures we used in sections 2 and 4 are Haar measures and the groups are unimodular. An example of a non-unimodular group is

$$P = \left\{ p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

Here  $|a|^{-2} da db$  is the left invariant Haar measure and  $\Delta(p) = a^2$ .

The convolution  $\phi * \psi$  of two functions  $\phi, \psi \in L^1(G)$  is defined by

$$\phi * \psi(y) = \int_G \phi(x)\psi(x^{-1}y) dx.$$

The Banach space  $L^1(G)$  with the product defined by the convolution is a Banach algebra. This algebras may have no identity, but they always has an approximate identity.

**Theorem 21.** *There is a sequence  $\phi_n \in C_c(G)$  such that*

- (1)  $\phi_n \geq 0 \quad (n = 0, 1, 2, \dots),$
- (2)  $\int_G \phi_n(x) dx = 1 \quad (n = 0, 1, 2, \dots),$
- (3) *Given a neighborhood  $U$  of the identity in  $G$ ,  
the support of  $\phi_n$  is contained in  $U$  for all  $n$  sufficiently large.*

(See [Lan85, section 1.1] for a proof.) Given a representation  $(\pi, \mathbf{V})$  the formula

$$\pi(\phi) = \int_G \phi(x)\pi(x) \quad (\phi \in L^1(G))$$

defines a representation of the Banach algebra  $L^1(G)$  on  $\mathbf{V}$ , i.e.

$$\pi(\phi * \psi) = \pi(\phi)\pi(\psi) \quad (\phi, \psi \in L^1(G)).$$

By restriction,  $(\pi, \mathbf{V})$  is also a representation of the convolution algebra  $C_c(G)$ . As straightforward consequence of Theorem 21 we see that a subspace  $\mathbf{W} \subseteq \mathbf{V}$  is  $G$ -invariant if and only if it is  $C_c(G)$ -invariant. Also,  $(\pi, \mathbf{V})$  is  $G$ -irreducible if and only if it is  $C_c(G)$ -irreducible.

## 8. Representations of a compact group

Let  $G$  be a compact group with the Haar measure of total mass equal to 1. If  $(\pi, \mathbf{V})$  is a representation of  $G$  on a Hilbert space  $\mathbf{V}$ , then, by averaging the norm on  $\mathbf{V}$  over  $G$ , we obtain another norm with respect to which the representation is unitary.

Hence, while considering representations of  $G$  on Hilbert spaces, we may assume that they are unitary.

**Theorem 22.** *Any irreducible unitary representation  $(\pi, \mathbf{V})$  of  $G$  is finite dimensional.*

*Proof.* Let  $v \in \mathbf{V}$  be a unit vector and let  $P$  be the orthogonal projection on the one-dimensional space  $\mathbb{C}v$ . Let  $Q$  be the continuous linear map defined by

$$Q = \int_G \pi(x)^{-1}P\pi(x) dx.$$

Then  $Q = Q^*$  commutes with the action of  $G$ . Hence Schur's Lemma, Theorem 15, implies that there is a constant  $c \in \mathbb{R}$  such that  $Q = cI$ . Since  $\pi$  is unitary,

$$(Qv, v) = \int_G (P\pi(x)v, \pi(x)v) dx = \int_G (\pi(x)v, v)(v, \pi(x)v) dx = \int_G |(\pi(x)v, v)|^2 dx > 0.$$

Hence  $c > 0$ .

Let  $v_1, v_2, \dots$  be an orthonormal basis of  $\mathbf{V}$ . Then for each  $x \in G$ ,  $\pi(x)v_1, \pi(x)v_2, \dots$  is also an orthonormal basis of  $\mathbf{V}$ . Hence

$$\sum_{n=1}^{\infty} (P\pi(x)v_n, \pi(x)v_n) = \sum_{n=1}^{\infty} ((\pi(x)v_n, v)v, \pi(x)v_n) = \sum_{n=1}^{\infty} |(\pi(x)v_n, v)|^2 = \|v\|^2 = 1.$$

But  $(P\pi(x)v_n, \pi(x)v_n) = (\pi(x)^{-1}P\pi(x)v_n, v_n)$ . Hence, after integration over  $G$  we see that

$$\sum_{n=1}^{\infty} (Qv_n, v_n) = 1.$$

Thus  $c$  times the number of the elements in the basis is equal to 1. Hence,  $\dim \mathbf{V} < \infty$ .  $\square$

By combining Theorem 22 with the Spectral Theorem from Linear Algebra we deduce the following corollary.

**Corollary 23.** *An irreducible unitary representation of a compact abelian group is one dimensional.*

If  $\dim \mathbf{V} < \infty$ , we let  $\mathbf{V}^c$  denote the vector space dual to  $\mathbf{V}$ . The contragredient representation  $(\pi^c, \mathbf{V}^c)$  is defined by

$$\pi^c(g)v^c(v) = v^c(\pi(g^{-1})v) \quad (v \in \mathbf{V}, v^c \in \mathbf{V}^c, g \in G).$$

Given two finite dimensional representations  $(\pi, \mathbf{V})$  and  $(\pi', \mathbf{V}')$  define their direct sum  $(\pi \oplus \pi', \mathbf{V} \oplus \mathbf{V}')$  by

$$(\pi \oplus \pi')(g)(v, v') = (\pi(g)(v), \pi'(g)(v')) \quad (g \in G, v \in \mathbf{V}, v' \in \mathbf{V}')$$

and the tensor product  $(\pi \otimes \pi', \mathbf{V} \otimes \mathbf{V}')$  by

$$(\pi \otimes \pi')(g)[v \otimes v'] = [\pi(g)(v)] \otimes [\pi'(g)(v')] \quad (g \in G, v \in \mathbf{V}, v' \in \mathbf{V}').$$

By definition the character  $\Theta_{\mathbf{V}} = \Theta_{\pi}$  of a finite dimensional representation  $(\pi, \mathbf{V})$  is the following complex valued function on the group:

$$\Theta_{\mathbf{V}}(g) = \text{tr}(\pi(g)) \quad (g \in G).$$

This function is invariant under conjugation

$$\Theta(hgh^{-1}) = \Theta(g) \quad (h, g \in G).$$

Also, we have

$$\Theta_{\mathbf{V} \oplus \mathbf{V}'} = \Theta_{\mathbf{V}} + \Theta_{\mathbf{V}'}, \quad \Theta_{\mathbf{V} \otimes \mathbf{V}'} = \Theta_{\mathbf{V}}\Theta_{\mathbf{V}'}, \quad \text{and} \quad \Theta_{\pi^c} = \overline{\Theta_{\pi}}. \quad (9)$$

Denote by  $L^2(G)^G \subseteq L^2(G)$  the subspace of the functions invariant by the conjugation by all the elements of  $G$ . Our characters live in  $L^2(G)^G$ .

**Lemma 24.** *Suppose  $(\pi, \mathbf{V})$  is a non-trivial irreducible unitary representation of  $G$ . Then*

$$\int_G \pi(x)v \, dx = 0 \quad (v \in \mathbf{V}).$$

*Proof.* Notice that the integral defines a  $G$ -invariant vector  $u \in \mathbf{V}$ . Since  $\pi$  is irreducible, either  $u = 0$  or  $\mathbf{V} = \mathbb{C}u$ . Since  $\pi$  is non-trivial, the second option is impossible.  $\square$

**Corollary 25.** *Suppose  $(\pi, \mathbf{V})$  is a non-trivial irreducible unitary representation of  $G$ . Then*

$$\int_G \pi(x) \, dx = 0.$$

**Corollary 26.** *Suppose  $(\pi, \mathbf{V})$  is a finite dimensional representation of  $G$ . Then*

$$\int_G \Theta_\pi(x) dx = \dim \mathbf{V}^G,$$

where  $\mathbf{V}^G \subseteq \mathbf{V}$  is the space of the  $G$ -invariant vectors.

**Proposition 27.** *The characters of irreducible representations form an orthonormal set in  $L^2(G)^G$ .*

*Proof.* Consider two such representations  $(\pi, \mathbf{V})$  and  $(\pi', \mathbf{V}')$ . The group  $G$  acts on this vector space  $\text{Hom}_G(\mathbf{V}, \mathbf{V}')$  by

$$gT(v) = \pi'(g)T\pi(g^{-1})v \quad (g \in G, T \in \text{Hom}_G(\mathbf{V}, \mathbf{V}'), v \in \mathbf{V}).$$

This way  $\text{Hom}_G(\mathbf{V}, \mathbf{V}')$  becomes a representation of  $G$ . It is easy to check that as such it is isomorphic to  $(\pi^c \otimes \pi', \mathbf{V}^c \otimes \mathbf{V}')$ . Hence, by (9),

$$\Theta_{\text{Hom}(\mathbf{V}, \mathbf{V}')} (g) = \overline{\Theta_\pi(g)} \Theta_{\pi'}(g) \quad (g \in G).$$

Therefore,

$$\begin{aligned} (\Theta'_{\pi'}, \Theta_\pi) &= \int_G \Theta_{\text{Hom}(\mathbf{V}, \mathbf{V}')} (g) dg \\ &= \text{dimension of the space of the } G\text{-invariants in } \text{Hom}_G(\mathbf{V}, \mathbf{V}'). \end{aligned}$$

Thus the formula follows from Corollary 17.  $\square$

**Proposition 28.** *Any finite dimensional representation of  $G$  decomposes into the direct sum of irreducible representations.*

*Proof.* This follows from the fact that the orthogonal complement of a  $G$ -invariant subspace is  $G$ -invariant.  $\square$

Let  $(\pi, \mathbf{V})$  be a finite dimensional representation of  $G$  and let

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \dots \oplus \mathbf{V}_n$$

be the decomposition into irreducibles. The number of the  $\mathbf{V}_j$  which are isomorphic to  $\mathbf{V}_1$  is called the multiplicity of  $\mathbf{V}_1$  in  $\mathbf{V}$ , denoted  $m_1$ . We may collect the isomorphic representations in the above formula and (after changing the indices appropriately) get the following decomposition

$$\mathbf{V} = \mathbf{V}_1^{\oplus m_1} \oplus \mathbf{V}_2^{\oplus m_2} \oplus \dots \oplus \mathbf{V}_k^{\oplus m_k},$$

where the  $\mathbf{V}_j$  are irreducible and mutually non-isomorphic and

$$\mathbf{V}_j^{\oplus m_j} = \mathbf{V}_j \oplus \mathbf{V}_j \oplus \dots \oplus \mathbf{V}_j \quad (m_j \text{ summands}).$$

This is the primary decomposition of  $\mathbf{V}$ , which is also denoted by

$$\mathbf{V} = m_1 \mathbf{V}_1 \oplus m_2 \mathbf{V}_2 \oplus \dots \oplus m_k \mathbf{V}_k = m_1 \cdot \mathbf{V}_1 \oplus m_2 \cdot \mathbf{V}_2 \oplus \dots \oplus m_k \cdot \mathbf{V}_k,$$

for brevity.

**Proposition 29.** *Let*

$$\mathbf{V} = m_1 \mathbf{V}_1 \oplus m_2 \mathbf{V}_2 \oplus \dots \oplus m_k \mathbf{V}_k,$$

*be the primary decomposition of  $\mathbf{V}$ . Then*

$$\Theta_{\mathbf{V}} = m_1 \Theta_{\mathbf{V}_1} + m_2 \Theta_{\mathbf{V}_2} + \dots + m_k \Theta_{\mathbf{V}_k}$$

*and*

$$m_j = (\Theta_{\mathbf{V}}, \Theta_{\mathbf{V}_j}).$$

*In particular  $\mathbf{V}$  is irreducible if and only if  $(\Theta_{\mathbf{V}}, \Theta_{\mathbf{V}}) = 1$ .*

*Proof.* This follows from (9) and from Lemma 27. □

Given a finite dimensional representation  $(\pi, \mathbf{V})$  define a map

$$M_{\pi} : \text{End}(\mathbf{V}) \rightarrow L^2(\mathbf{G}), \quad M_{\pi}(T)(g) = \text{tr}(\pi(g)T) = \text{tr}(T\pi(g)).$$

In particular, if  $I \in \text{End}(\mathbf{V})$  is the identity, then

$$M_{\pi}(I) = \Theta_{\pi}.$$

Also, it is easy to see that the subspace  $M_{\pi}(\text{End}(\mathbf{V})) \subseteq L^2(\mathbf{G})$  depends only on the equivalence class of  $\pi$ .

**Theorem 30.** *Suppose  $(\pi, \mathbf{V})$  and  $(\pi', \mathbf{V}')$  are two irreducible unitary representations of  $\mathbf{G}$ . Then*

- a) *if  $(\pi, \mathbf{V})$  is not equivalent to  $(\pi', \mathbf{V}')$ , then  $M_{\pi}(\text{End}(\mathbf{V})) \perp M_{\pi'}(\text{End}(\mathbf{V}'))$ ,*
- b) *for any  $S, T \in \text{End}(\mathbf{V})$ ,  $(M_{\pi}(S), M_{\pi}(T)) = \frac{1}{\dim \mathbf{V}} \text{tr}(ST^*)$ .*

*Proof.* Define a representation  $(\Pi, \text{End}(\mathbf{V}))$  of the group  $\mathbf{G} \times \mathbf{G}$  on the vector space  $\text{End}(\mathbf{V})$  by

$$\Pi(g_1, g_2)T = \pi(g_2)T\pi(g_1^{-1}) \quad (g_1, g_2 \in \mathbf{G}, T \in \text{End}(\mathbf{V})).$$

Notice that  $(\Pi, \text{End}(\mathbf{V}))$  is isomorphic to outer tensor product  $(\pi^c \otimes \pi, \mathbf{V}^c \otimes \mathbf{V})$ . Hence,

$$\begin{aligned} & \int_{\mathbf{G} \times \mathbf{G}} |\Theta_{\Pi}(x_1, x_2)|^2 dx_1 dx_2 = \int_{\mathbf{G} \times \mathbf{G}} |\overline{\Theta_{\pi}(x_1)} \Theta_{\pi}(x_2)|^2 dx_1 dx_2 \\ & = \int_{\mathbf{G}} |\Theta_{\pi}(x_1)|^2 dx_1 \int_{\mathbf{G}} |\Theta_{\pi}(x_2)|^2 dx_2 = 1. \end{aligned}$$

Hence, by Proposition 29,  $(\Pi, \text{End}(\mathbf{V}))$  is irreducible.

We view  $\text{End}(\mathbf{V})$  as a Hilbert space with the following scalar product

$$(S_1, S_2) = \text{tr}(S_1 S_2^*) \quad (S_1, S_2 \in \text{End}(\mathbf{V})),$$

and similarly for  $\mathbf{V}'$ . In particular, we have the adjoint map

$$M_{\pi'}^* : L^2(\mathbf{G}) \rightarrow \text{End}(\mathbf{V}).$$

Notice also that

$$M_{\pi'}^* M_{\pi} : \text{End}(\mathbf{V}) \rightarrow \text{End}(\mathbf{V}')$$

is a  $G \times G$ -intertwining map. If the two representations  $(\pi, \mathbf{V})$  and  $(\pi', \mathbf{V}')$  are not isomorphic then The  $G \times G$ -modules  $\text{End}(\mathbf{V})$  and  $\text{End}(\mathbf{V}')$  are not isomorphic. Hence Corollary 17 shows that  $M_{\pi'}^* M_{\pi} = 0$ . Thus for  $S \in \text{End}(\mathbf{V})$  and  $T \in \text{End}(\mathbf{V}')$ ,

$$(M_{\pi}(S), M_{\pi'}(T)) = (S, M_{\pi'}^* M_{\pi}(T)) = 0$$

and a) follows.

Similarly, there is  $\lambda \in \mathbb{C}$  such that  $M_{\pi}^* M_{\pi} = \lambda I$ . Furthermore,

$$\lambda \cdot \dim \mathbf{V} = (I, M_{\pi}^* M_{\pi}(I)) = (M_{\pi}(I), M_{\pi}(I)) = (\Theta_{\pi}, \Theta_{\pi}) = 1$$

and b) follows.  $\square$

**Theorem 31.** *There are the following direct sum orthogonal decompositions of Hilbert spaces*

- a)  $L^2(G) = \sum_{\pi \in \hat{G}} M_{\pi}(\text{End}(\mathbf{V}))$ ,
- b)  $L^2(G)^G = \sum_{\pi \in \hat{G}} \mathbb{C}\Theta_{\pi}$ .

*Proof.* Part b) follows from part a), because, by Theorem 16,  $\text{End}(\mathbf{V})^G = \mathbb{C}I$ . Indeed,

$$L^2(G)^G = \sum_{\pi \in \hat{G}} (M_{\pi}(\text{End}(\mathbf{V})))^G = \sum_{\pi \in \hat{G}} M_{\pi}(\text{End}(\mathbf{V})^G) = \sum_{\pi \in \hat{G}} \mathbb{C}\Theta_{\pi}.$$

where only the middle equation

$$(M_{\pi}(\text{End}(\mathbf{V})))^G = M_{\pi}(\text{End}(\mathbf{V})^G)$$

requires an explanation. We need to show that if

$$\text{tr}(T\pi(ghg^{-1})) = \text{tr}(T\pi(h))$$

for all  $h \in G$ , then  $\pi(g^{-1})T\pi(g) = T$ . Since  $\text{tr}(T\pi(ghg^{-1})) = \text{tr}(\pi(g^{-1})T\pi(g)\pi(h))$ , we'll be done as soon as we show that if  $\text{tr}(S\pi(h)) = 0$  for all  $h \in G$ , then  $S = 0$ . But, since the map  $M_{\pi}$  is injective (because  $\pi$  is irreducible), this is indeed the case.

Part a) requires some work. We follow the argument in [Kna86, Theorem 1.12]. Let  $U = \sum_{\pi \in \hat{G}} M_{\pi}(\text{End}(\mathbf{V}))$ . This is a subspace of  $L^2(G)$  closed under the left and right translations and under the  $*$  operation,  $\phi \rightarrow \phi^*$ ,  $\phi^*(x) = \overline{\phi(x^{-1})}$ . Hence so is the orthogonal complement  $U^{\perp} \subseteq L^2(G)$ .

Suppose  $U^{\perp} \neq \{0\}$ . We shall arrive at a contradiction. Let  $\phi \in U^{\perp}$  be non-zero. Let  $\phi_n \in C_c(G)$  be an approximate identity, as in Theorem 21. Then a straightforward argument shows that

$$\lim_{n \rightarrow \infty} \|\phi_n * \phi - \phi\|_2 = 0.$$

Hence there is  $n$  such that  $\phi_n * \phi \neq 0$ . But this function is continuous. Thus we may assume that  $\phi$  is continuous. Furthermore, applying the translations and the  $*$  operation we may assume that  $\phi = \phi^*$  and  $\phi(1) > 0$ . Replacing  $\phi$  by the integral

$$\int_G L(x)R(x)\phi dx$$

we may assume that  $\phi$  is invariant under conjugation. Let

$$T\psi(x) = \int_G \phi(x^{-1}y)\psi(y) dy.$$

Since the integral kernel  $\phi(x^{-1}y)$  is continuous on  $G \times G$ ,  $T$  is a compact operator on  $L^2(G)$ . Furthermore,  $T = T^* \neq 0$ . Hence, by Theorem 11,  $T$  has a non-zero finite dimensional eigenspace  $V_\lambda \subseteq L^2(G)$ . Since  $T$  commutes with  $L(G)$ ,  $V_\lambda$  is closed under  $L(G)$ . Hence Theorem 28 shows that there is a  $G$ -irreducible subspace  $W_\lambda \subseteq V_\lambda$ . Let  $f_j$  be an orthonormal basis of  $W_\lambda$ . Set

$$h_{i,j}(x) = (L(x)f_i, f_j) = \int_G f_i(x^{-1}y)\overline{f_j(y)} dy.$$

This is a matrix coefficient of an irreducible representation of  $G$ , thus it belongs to  $U$ . Therefore,

$$0 = (\phi, h_{i,i}) = \int_G \int_G \phi(x^{-1}y)\overline{f_i(y)} dy dx = \lambda \int_G f_i(x)\overline{f_i(y)} dx,$$

which is a contradiction.  $\square$

**Theorem 32.** *Any unitary representation  $(\rho, W)$  of  $G$  is completely reducible, i.e. the Hilbert space  $W$  is the orthogonal sum of irreducible finite dimensional representations of  $G$ .*

*Proof.* We follow [Kna86, Theorem 1.12]. Suppose  $(\rho, W)$  is not completely reducible. By Zorn's Lemma we may choose a maximal orthogonal set of finite dimensional irreducible invariant subspaces. Let  $U$  denote the closure of their sum. Suppose  $0 \neq v \in U^\perp$ . We'll arrive at a contradiction.

Let  $\phi_n \in C_c(G)$  be an approximate identity, as in Theorem 21. Then, as we have seen before, there is  $n$  such that  $\rho(\phi_n)v \neq 0$ . We fix this  $n$ .

Theorem 31 a) implies that there is a finite set  $F \subseteq \hat{G}$  and  $\phi \in \bigoplus_{\pi \in F} M_\pi(\text{End}(V))$ , such that

$$\| \phi_n - \phi \|_2 \leq \frac{1}{2 \| v \|} \| \rho(\phi_n)v \|.$$

Since the total mass of  $G$  is 1, we have

$$\| \phi_n - \phi \|_1 \leq \| \phi_n - \phi \|_2.$$

Hence

$$\| \rho(\phi_n)v - \rho(\phi)v \| \leq \| \phi_n - \phi \|_1 \| v \| \leq \frac{1}{2} \| \rho(\phi_n)v \|.$$

Therefore

$$\| \rho(\phi)v \| \geq \| \rho(\phi_n)v \| - \| \rho(\phi_n)v - \rho(\phi)v \| \geq \frac{1}{2} \| \rho(\phi_n)v \| > 0.$$

This is a contradiction because  $\rho(\phi)v$  lies in a finite dimensional invariant subspace of  $W$ .  $\square$



For a function  $\phi \in L^1(\mathbf{G})$  define the Fourier transform

$$\mathcal{F}\phi(\pi) = \int_{\mathbf{G}} \phi(x)\pi(x) dx = \int_{\mathbf{G}} \phi(x)\pi(x) dx = \pi(\phi). \quad (10)$$

Notice that in order to be consistent with the theory of Fourier series, section 4, we should replace  $\pi$  by  $\pi^c$  in this definition. However we shall follow the tradition and not do that.

Thus for a representation  $(\pi, \mathbf{V})$ ,  $\mathcal{F}\phi(\pi) \in \text{End}(\mathbf{V})$ . Notice that

$$\pi(g)\mathcal{F}\phi(\pi) = \mathcal{F}(L(g)\phi) \quad (g \in \mathbf{G}). \quad (11)$$

Set  $d(\pi) = \dim \mathbf{V} = \Theta_\pi(1)$ . This is the degree of the representation  $(\pi, \mathbf{V})$ .

**Theorem 33.** (*Fourier inversion for  $\mathbf{G}$* ) For any  $\phi \in \bigoplus_{\pi \in \hat{\mathbf{G}}} M_\pi(\text{End}(\mathbf{V}))$  (algebraic sum),

$$\phi(g) = \sum_{\pi \in \hat{\mathbf{G}}} d(\pi) \cdot \text{tr}(\mathcal{F}\phi(\pi)\pi(g^{-1})) \quad (g \in \mathbf{G}), \quad (12)$$

or equivalently

$$\phi(1) = \sum_{\pi \in \hat{\mathbf{G}}} d(\pi) \cdot \Theta_\pi(\phi) \quad (g \in \mathbf{G}), \quad (13)$$

Notice that in the case of the classical Fourier series, section 4, the above formula (12) refers only to trigonometric polynomials,  $\phi$ .

*Proof.* Clearly (13) is a particular case of (12). Also, (12) follows from (13) and (11).

Let us write  $(\pi, \mathbf{V}_\pi)$  for  $(\pi, \mathbf{V})$  in order to indicate the dependence of the vector space on  $\pi$ . By definition, there is a finite set  $F \subseteq \hat{\mathbf{G}}$  and operators  $T_\pi \in \text{End}(\mathbf{V}_\pi)$  such that

$$\phi(g) = \sum_{\rho \in F} \text{tr}(T_\rho \rho(g)) \quad (g \in \mathbf{G}).$$

Hence

$$\mathcal{F}\phi(\pi) = \sum_{\rho \in F} \int_{\mathbf{G}} \text{tr}(T_\rho \rho(g))\pi(g) dg.$$

Therefore, by Theorem 30

$$\begin{aligned} \text{tr} \mathcal{F}\phi(\pi) &= \sum_{\rho \in F} \int_{\mathbf{G}} \text{tr}(T_\rho \rho(g)) \text{tr} \pi(g) dg = \int_{\mathbf{G}} \text{tr}(T_\pi \pi(g)) \text{tr} \pi(g) dg \\ &= (T_\pi \pi(g), \pi(g)) = \frac{1}{d_\pi} \text{tr}(T_\pi \pi(g)\pi(g)^*) = \frac{1}{d_\pi} \text{tr}(T_\pi). \end{aligned}$$

Hence

$$\sum_{\pi \in \hat{\mathbf{G}}} d(\pi) \cdot \Theta_\pi(\phi) = \sum_{\pi \in \hat{\mathbf{G}}} d(\pi) \cdot \text{tr} \mathcal{F}\phi(\pi) = \sum_{\pi \in \hat{\mathbf{G}}} \text{tr}(T_\pi) = \phi(1).$$

□

**Theorem 34.** (*Parseval formula*) For any  $\phi \in \bigoplus_{\pi \in \hat{\mathbf{G}}} M_\pi(\text{End}(\mathbf{V}))$ ,

$$\|\phi\|_2^2 = \sum_{\pi \in \hat{\mathbf{G}}} d_\pi \|\mathcal{F}\phi(\pi)\|^2.$$

*Proof.* That the left hand side is equal to

$$\phi * \phi^*(1) = \sum_{\pi \in \hat{G}} d_{\pi^c} \operatorname{tr}(\mathcal{F}\phi(\pi)\mathcal{F}\phi(\pi)^*),$$

which coincides with the right hand side.  $\square$

## 9. Square integrable representations of a locally compact group

In this section  $G$  is a locally compact unimodular group. We follow [Wal88a, section 1.3].

**Lemma 35.** *Let  $(\tau, \mathbf{V})$  be an irreducible unitary representation of  $G$ . Suppose that for some non-zero vectors  $u_0, v_0 \in \mathbf{V}$*

$$\int_G |(\tau(x)u_0, v_0)|^2 dx < \infty. \quad (14)$$

*Then for arbitrary  $u, v \in \mathbf{V}$ ,*

$$\int_G |(\tau(x)u, v)|^2 dx < \infty. \quad (15)$$

*Moreover, the map  $T : \mathbf{V} \rightarrow L^2(G)$  defined by*

$$Tu(x) = (\tau(x)u, v_0) \quad (u \in \mathbf{V}, x \in G)$$

*is  $G$ -intertwining and has the property that there is  $t > 0$  such that*

$$(Tu, Tv) = t(u, v) \quad (u, v \in \mathbf{V}). \quad (16)$$

*Proof.* Let  $W_0$  be the linear span of all the vectors  $\pi(g)u_0$ ,  $g \in G$ . This is a  $G$ -invariant subspace of  $\mathbf{V}$ . Since  $\mathbf{V}$  is irreducible,  $W_0$  is dense in  $\mathbf{V}$ . Let

$$W = \{u \in \mathbf{V}; \int_G |(\tau(x)u, v_0)|^2 dx < \infty\}.$$

Then  $W$  is also a  $G$ -invariant subspace of  $\mathbf{V}$  and  $W$  contains  $W_0$ . Hence  $W$  is dense in  $\mathbf{V}$ .

Clearly

$$T\tau(g)u = R(g)Tu \quad (g \in G).$$

Define an inner product on  $W$  by

$$\langle w_1, w_2 \rangle = (w_1, w_2) + (Tw_1, Tw_2) \quad (w_1, w_2 \in W).$$

One checks that  $W$  with this inner product is complete. (Every Cauchy sequence in  $W$  has a limit in  $W$ .) Thus  $(\tau, W)$ , with this new inner product, is a unitary representation of  $G$ .

The inclusion

$$I_W : W \ni w \rightarrow w \in \mathbf{V}$$

is a bounded linear map which intertwines the actions of  $G$ . The adjoint

$$I_W^* : \mathbf{V} \rightarrow W$$

is also a bounded linear map which intertwines the actions of  $G$ . Now we apply Theorem 19 with  $A = I_W^*$ ,  $V_0 = V$ ,  $B = I_W$  and  $V_1 = W$ . The conclusion is that  $I_W^*$  is a scalar multiple of the identity. Hence  $W = V$ . This means that (14) holds for the fixed  $v_0 \in V_0$  and arbitrary  $u \in V$  in place of the  $u_0$ . Now we fix arbitrary  $u \in V$  and, since the group is unimodular, we may apply the same argument to see that (14) holds with the  $u_0$  replaced by  $u$  and  $v_0$  replaced by an arbitrary  $v \in V$ . This verifies (15).

Now  $V$  is equipped with two inner products  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$  preserved by  $G$ . Define a map  $S : V \rightarrow V$  by

$$\langle u, v \rangle = (Su, v).$$

Then  $S$  commutes with the action of  $G$ . Hence  $S = sI$  for some  $s \in \mathbb{C}$ . Explicitly,

$$(u, v) + (Tu, Tv) = s(u, v).$$

Since  $T \neq 0$ , by taking  $u = v$ , we see that  $s = 1 + t$  with  $t > 0$ . This implies (16).  $\square$

An irreducible unitary representation of  $G$  is called square integrable if one, or equivalently all, its matrix coefficients are square integrable.

**Remark 1.** *The assumption that the group  $G$  is unimodular is essential. Lemma 35 fails for example for  $G = "ax + b"$  group. Recall, [Fol95, page 188 and 242], that  $G = \mathbb{R}^+ \times \mathbb{R}$  with the multiplication*

$$(a, b)(a', b') = (aa', b + ab')$$

and the left invariant Haar measure

$$a^{-2} da db.$$

The following formula defines an irreducible unitary representation of  $G$  on  $L^2(0, \infty)$ ,

$$\pi(a, b)u(s) = a^{\frac{1}{2}} e^{2\pi i b s} u(as) \quad (a \in \mathbb{R}^+, b \in \mathbb{R}, s \in (0, \infty), u \in L^2(0, \infty)),$$

see [Fol95, page 242]. Hence for two functions  $u, v \in L^2(0, \infty)$  the corresponding matrix coefficient has the  $L^2$  norm squared equal to

$$\begin{aligned} \int |(\pi(a, b)u, v)|^2 a^{-2} da db &= \int \left| \int a^{\frac{1}{2}} e^{2\pi i b s} u(as) \bar{v}(s) ds \right|^2 a^{-2} da db \\ &= \int \left( \left| \int e^{2\pi i b s} u(as) \bar{v}(s) ds \right|^2 db \right) a^{-1} da = \int \left( \int |u(as) \bar{v}(s)|^2 ds \right) a^{-1} da \\ &= \int \int |u(as) \bar{v}(s)|^2 a^{-1} da ds = \int |u(a)|^2 a^{-1} da \int |v(s)|^2 ds. \end{aligned}$$

This expression is finite if and only if

$$\int |u(a)|^2 a^{-1} da < \infty,$$

which is the case if  $u \in C_c(0, \infty)$  but is not the case if  $u$  is the indicator function of the interval  $(0, 1)$ .

**Proposition 36.** *Let  $(\sigma, \mathbf{U})$  and  $(\tau, \mathbf{V})$  be irreducible unitary square integrable representations of  $G$ . Then*

$$\int_{\mathbf{G}} (\sigma(x)u_1, u_2)(v_2, \tau(x)v_1) dx = 0 \quad (u_1, u_2 \in \mathbf{U}, v_1, v_2 \in \mathbf{V}) \text{ if } \sigma \not\sim \tau \quad (17)$$

There is a positive number  $d(\tau)$  such that

$$\int_{\mathbf{G}} (\tau(x)u_1, u_2)(v_2, \tau(x)v_1) dx = \frac{1}{d(\tau)}(u_1, v_1)(v_2, u_2) \quad (u_1, u_2, v_1, v_2 \in \mathbf{V}). \quad (18)$$

*Proof.* Define

$$Su(x) = (\sigma(x)u, u_2) \quad (u \in \mathbf{U}, x \in \mathbf{G})$$

and

$$Tv(x) = (\tau(x)v, v_2) \quad (v \in \mathbf{V}, x \in \mathbf{G}).$$

Then (16) shows that there is a positive constant  $C$  such that for  $u \in \mathbf{U}$  and  $v \in \mathbf{V}$

$$|(Su, Tv)| \leq \|Su\|_2 \|Tv\|_2 \leq C \|u\| \|v\|.$$

Hence there is a bounded  $G$ -intertwining linear map  $A : \mathbf{V} \rightarrow \mathbf{U}$  such that

$$(Su, Tv) = (u, Av) \quad (u \in \mathbf{U}, v \in \mathbf{V}).$$

This map is zero if the representations are not equivalent. Since

$$\int_{\mathbf{G}} (\sigma(x)u_1, u_2)(v_2, \tau(x)v_1) dx = (Su_1, Tv_1) = (u_1, Av_1),$$

(17) follows. Suppose  $\sigma = \tau$ . In this case let us write  $T_{v_2}$  for  $T$  and  $T_{u_2}$  for  $S$ . Then

$$(T_{u_2}u, T_{v_2}v) = (Su, Tv) = (u, Av) \quad (u, v \in \mathbf{V}),$$

where  $A$  is a constant because the representation is irreducible. Thus  $A = aI$  for some  $a_{u_2, v_2} \in \mathbb{C}$ . Explicitly

$$\int_{\mathbf{G}} (\tau(x)u, u_2) \overline{(\tau(x)v, v_2)} dx = a_{u_2, v_2}(u, v) \quad (u, u_2, v, v_2 \in \mathbf{V}).$$

Since  $G$  is unimodular,

$$\begin{aligned} \int_{\mathbf{G}} (\tau(x)u, u_2) \overline{(\tau(x)v, v_2)} dx &= \int_{\mathbf{G}} (\tau(x^{-1})u, u_2) \overline{(\tau(x^{-1})v, v_2)} dx \\ &= \int_{\mathbf{G}} (u, \tau(x)u_2) \overline{(v, \tau(x)v_2)} dx = \int_{\mathbf{G}} (\tau(x)v_2, v) \overline{(\tau(x)u_2, u)} dx. \end{aligned}$$

Previous argument implies that there is a constant  $b(u, v)$  such that

$$\int_{\mathbf{G}} (\tau(x)v_2, v) \overline{(\tau(x)u_2, u)} dx = b(v, u)(v_2, u_2) \quad (u, u_2, v, v_2 \in \mathbf{V}).$$

Hence

$$a_{u_2, v_2}(u, v) = b(v, u)(v_2, u_2) \quad (u, u_2, v, v_2 \in \mathbf{V}).$$

Now we take  $u = v$  to be a unit vector. Then

$$a_{u_2, v_2}(v, v) = b(v, v)(v_2, u_2) = b(v_2, u_2) \quad (u, u_2, v, v_2 \in \mathbf{V}).$$

Thus there is a constant  $b$  such that

$$a_{u_2, v_2} = b(v_2, u_2) \quad (u_2, v_2 \in \mathbf{V}).$$

By taking  $u_2 = v_2$  we see that  $b > 0$ . Thus

$$\int_G (\tau(x)u_1, u_2)(v_2, \tau(x)v_1) dx = \overline{a_{u_2, v_2}}(u_1, v_1) = b(u_1, v_1)(v_2, u_2).$$

□

The constant  $d(\tau)$  is called the formal degree of  $\tau$ . It depends on the choice of the Haar measure on  $G$ . If  $G$  is compact with the Haar measure of total mass 1 then, as we have seen in Theorem 30,  $d(\tau) = \dim \mathbf{V}$ . Hence the name.

## 10. $G = \mathrm{SL}_2(\mathbb{R})$

### 10.1. The maximal compact subgroup $K = \mathrm{SO}_2(\mathbb{R})$ .

We let

$$k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and define

$$K = \mathrm{SO}_2(\mathbb{R}) = \{k(\theta); \theta \in \mathbb{R}\}.$$

This is a maximal compact subgroup of  $G$ , which is unique up to conjugation. The group  $K$  is commutative. Define the characters

$$\chi_n(k(\theta)) = e^{in\theta} \quad (\theta \in \mathbb{R}, n \in \mathbb{Z}).$$

For two integers  $n$  and  $m$  define

$$S_{n,m} = \{f \in C_c(G); f(k(\theta_1)gk(\theta_2)) = e^{-in\theta_1} f(g)e^{-im\theta_2}\}.$$

**Lemma 37.** *The algebraic sum  $\bigoplus_{n,m} S_{n,m}$  is  $L^1$ -dense in  $C_c(G)$ . In fact, given  $\epsilon > 0$  and  $f \in C_c(G)$ , there exists a function  $g \in \bigoplus_{n,m} S_{n,m}$  such that the support of  $g$  is contained in  $K(\mathrm{supp} f)K$ , and such that  $\|f - g\|_\infty < \epsilon$ .*

*Proof.* This follows from the properties of the Fourier series. See [Lan85, page 20]. □

**Lemma 38.** *The following formulas hold.*

$$\begin{aligned} S_{n,m} * S_{l,q} &= \{0\} \text{ if } m \neq l, \\ S_{n,m}^* &= S_{m,n}, \\ S_{n,m} * S_{m,q} &\subseteq S_{n,q}. \end{aligned}$$

*Proof.* This is straightforward [Lan85, page 22]. □

**Lemma 39.** *Let  $S = \{x \in G; x = x^t\}$ . Then the map*

$$K \times S \ni (k, s) \rightarrow ks \in G$$

*is a bijective diffeomorphism.*

*Proof.* This is well known from Linear Algebra. See straightforward [Lan85, page 22].  $\square$

**Lemma 40.** *The algebra  $S_{0,0}$  is commutative.*

*Proof.* The argument is due to I. M. Gelfand. Consider an element  $x \in G$ . Then there is  $k \in K$  and  $s \in S$  such that  $x = ks$ . Hence the transpose

$$x^t = k^t s^t = k^{-1} s = k^{-1} x k^{-1}. \quad (19)$$

Also, the group  $G$  is unimodular. Hence the Haar measure is invariant under the change of variables  $x \rightarrow x^t$ . Let  $f, g \in S_{0,0}$ . Set  $f^t(x) = f(x^t)$ . It is easy to check that

$$(f * g)^t = g^t * f^t.$$

Notice that  $f^t = f$ . Indeed,

$$f^t(x) = f(x^t) = f(k^{-1} x k^{-1}) = f(x),$$

because elements of  $S_{0,0}$  are  $K$ -bi-invariant. Hence

$$g^t * f^t = g * f.$$

By Lemma 38,  $f * g \in S_{0,0}$ . Hence  $(f * g)^t = f * g$ . The conclusion follows.  $\square$

**Lemma 41.** *For any  $n \in \mathbb{Z}$ , the algebra  $S_{n,n}$  is commutative.*

*Proof.* The argument is due to S. Lang, see [Lan85]. Let

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly  $\gamma = \gamma^{-1}$ . Let  $s = s^t \in G$ . Then there is  $k \in K$  such that  $ksk^{-1} = d$  is diagonal. Therefore

$$\gamma s \gamma = \gamma k^{-1} d k \gamma = \gamma k^{-1} \gamma \gamma d \gamma k \gamma = \gamma k^{-1} \gamma d \gamma k \gamma.$$

Thus there is  $k_s \in K$  such that

$$\gamma s \gamma = k_s^{-1} s k_s.$$

Notice that

$$\gamma k \gamma = k^{-1} \quad (k \in K).$$

Recall that  $f^t(x) = f(x^t)$  and let  $f^\gamma(x) = f(\gamma x \gamma)$ . Then, for  $f \in S_{n,n}$

$$f^\gamma(x) = f^t(x)$$

Indeed, write  $x = ks$ . Then

$$\begin{aligned} f^\gamma(ks) &= f(\gamma ks \gamma) = f(k^{-1} \gamma s \gamma) = \chi_n(k) f(\gamma s \gamma) \\ &= \chi_n(k) f(k_s^{-1} s k_s) = \chi_n(k) \chi_n(k_s) f(s) \chi_n(k_s)^{-1} \\ &= \chi_n(k) f(s) = f(s) \chi_n(k) = f(sk^{-1}) = f((ks)^t) \\ &= f^t(ks). \end{aligned}$$

Therefore for any  $f, g \in S_{n,n}$ ,

$$f^\gamma * g^\gamma = f^t * g^t = (g * f)^t = (g * f)^\gamma = g^\gamma * f^\gamma$$

and we are done.  $\square$

For a representation  $(\pi, \mathbf{V})$  of  $G$  on a Banach space  $\mathbf{V}$  define

$$\mathbf{V}_n = \{v \in \mathbf{V}; \pi(k(\theta))v = e^{in\theta}v, \theta \in \mathbb{R}\}.$$

This is the  $\chi_n$  isotypic component of  $\mathbf{V}$ .

**Lemma 42.** *Let  $(\pi, \mathbf{V})$  be a representation of  $G$  on a Banach space  $\mathbf{V}$ . Then*

$$\pi(S_{n,m})\mathbf{V} \subseteq \mathbf{V}_n$$

and for  $q \neq m$ ,

$$\pi(S_{n,m})\mathbf{V}_q = 0.$$

*Proof.* See [Lan85, page 23].  $\square$

**Lemma 43.** *Let  $(\pi, \mathbf{V})$  be an irreducible representation of  $G$  on a Banach space  $\mathbf{V}$ . Then for any  $q$  such that  $\mathbf{V}_q \neq 0$ , the space  $\mathbf{V}_q$  is  $S_{q,q}$ -irreducible. Also,  $\pi(S_{q,q})\mathbf{V}_q \neq 0$ .*

*Proof.* This follows from the fact that the  $*$  algebra

$$\bigoplus_{n,m} S_{n,m}$$

is  $L^1$ -dense in  $C_c(G)$  see [Lan85, page 24].  $\square$

**Theorem 44.** *Let  $(\pi, \mathbf{V})$  be an irreducible representation of  $G$  on a Banach space  $\mathbf{V}$ . Fix an integer  $n$ . If  $\dim \mathbf{V}_n < \infty$  then  $\dim \mathbf{V}_n = 1$  or  $0$ .*

*Proof.* We know from Lemma 43 that the commutative algebra  $\pi(S_{n,n})$  acts irreducibly on  $\mathbf{V}_n$ . Hence finite dimensional Linear Algebra (Jordan Canonical Form of a complex matrix) implies the claim.  $\square$

Warning: there exists a Banach space  $\mathbf{V}$  of infinite dimension and a linear operator  $T \in \text{End}(\mathbf{V})$  acting irreducibly on  $\mathbf{V}$ , see [Enf87].

**Definition 45.** *A representation  $(\pi, \mathbf{V})$  of  $G$  on a Banach space  $\mathbf{V}$  is admissible if  $\dim \mathbf{V}_n < \infty$  for every integer  $n$ .*

**Theorem 46.** *Let  $(\pi, \mathbf{V})$  be an irreducible unitary representation of  $G$  on a Hilbert space  $\mathbf{V}$ . Fix an integer  $n$ . Then  $\dim \mathbf{V}_n = 1$  or  $0$ .*

*Proof.* In this case  $\pi(S_{n,n})$  is a commutative  $*$  algebra. Hence the claim follows from Theorem 13.  $\square$

**Theorem 47.** *Let  $(\pi, \mathbf{V})$  be an irreducible representation of  $G$  on a Banach space  $\mathbf{V}$ . Then the space of finite sums*

$$\bigoplus_{n \in \mathbb{Z}} \mathbf{V}_n \subseteq \mathbf{V}$$

is dense.

If  $(\pi, \mathbf{V})$  be an irreducible unitary representation of  $G$  on a Hilbert space  $\mathbf{V}$ , then

$$\mathbf{V} = \sum_{n \in \mathbb{Z}} \mathbf{V}_n$$

is a Hilbert space direct sum orthogonal decomposition.

*Proof.* This follows from the fact that the  $*$  algebra

$$\bigoplus_{n,m} S_{n,m}$$

is dense in  $C_c(G)$  see [Lan85, page 25]. □

**10.2. Induced representations.** Recall the following subgroups of  $G$ :

$$\mathbf{K} = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}; \theta \in \mathbb{R} \right\}, \quad \mathbf{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 0 \right\}, \quad \mathbf{N} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{R} \right\}.$$

The following map, is a bijective diffeomorphism

$$\mathbf{A} \times \mathbf{N} \times \mathbf{K} \ni (a, n, k) \rightarrow ank \in G$$

giving the Iwasawa decomposition  $G = \mathbf{A}\mathbf{N}\mathbf{K}$ . Let us fix Haar measures on these groups

$$d \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \frac{1}{2\pi} d\theta, \quad d \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{da}{a}, \quad d \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = dn.$$

Then, in terms of the Iwasawa decomposition, the formula

$$dx = da \, dn \, dk \quad (x = ank, a \in \mathbf{A}, n \in \mathbf{N}, k \in \mathbf{K})$$

defines a Haar measure on  $G$ . Furthermore see

$$\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^2, \quad \rho \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a.$$

Let  $\mathbf{P} = \mathbf{A}\mathbf{N}$ . This is a subgroup of  $G$  with the modular function

$$\Delta \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) = a^2.$$

**Theorem 48.** Let  $(\sigma, \mathbf{V})$  be a representation of  $\mathbf{P}$  on a Hilbert space  $\mathbf{V}$ . Denote by  $\mathbf{V}(\sigma)$  the space of functions  $f : G \rightarrow \mathbf{V}$  such that

$$f(px) = \Delta(p)^{\frac{1}{2}} \sigma(p) f(x) \quad (x \in G)$$

whose restriction to  $\mathbf{K}$  is square integrable:

$$\int_{\mathbf{K}} \|f(k)\|^2 dk < \infty.$$

Set

$$\pi(g)f(x) = f(xg) \quad (g, x \in G)$$

Then  $(\pi, \mathbf{V}(\sigma))$  is a representation of  $G$ . If  $\sigma$  is unitary then so is  $\pi$ .



*Proof.* See [Lan85, Theorem 2, page 44].  $\square$

Our main example is going to be the case when  $V = \mathbb{C}$  and for some fixed complex number  $s$ ,

$$\sigma(an) = \rho(a)^s \quad (a \in A, n \in N).$$

Then we shall write  $V(s) = V(\sigma)$  and  $\pi_s = \pi$ . Then the transformation property of functions in  $V(s)$  looks as follows

$$f( anx ) = \rho(a)^{1+s} f(x) \quad (a \in A, n \in N, x \in G). \quad (20)$$

The resulting representation  $(\pi_s, V(s))$  is called the principal series representation.

**Lemma 49.** *Let*

$$\rho_s(ank) = \rho(a)^{1+s} \quad (a \in A, n \in N, k \in K).$$

*Then  $\rho_s \in V(s)$  is  $K$ -invariant and*

$$(\pi_s(x)\rho_s, \rho_s) = \int_K \rho_s(kx) dk = \int_K \rho_0(kx)^{1+s} dk.$$

*Proof.* See [Lan85, page 47].  $\square$

The function

$$\phi_s = \int_K \rho_0(kx)^{1+s} dk \quad (21)$$

is known as a spherical function.

**Lemma 50.** *For a function  $\psi \in C_c(G)$ ,  $\pi_s(\psi)$  is an integral kernel operator with the integral kernel*

$$q_\psi(k, k') = \int_A \int_N \psi(k'^{-1}ank) \rho(a)^{1+s} da dn.$$

*Furthermore the operator  $\pi_s(\psi)$  is of trace class and*

$$\text{tr } \pi_s(\psi) = \int_K q_\psi(k, k) dk = \int_A \int_N \int_K \psi(kank^{-1}) dk \rho(a)^{1+s} da dn.$$

*Proof.* The formula for the integral kernel is obtained via a straightforward computation in [Lan85, page 48]. It is a continuous function. Hence the operator is of trace class.  $\square$

By computing a few Jacobians, as in [Lan85, page 68], we obtain the following lemma.

**Lemma 51.** *For  $\phi \in C_c(G)$*

$$\int_N \phi(ana^{-1}n^{-1}) dn = \frac{1}{|\alpha(a) - 1|} \int_N \phi(n) dn, \quad (22)$$

*and therefore*

$$\int_{A \setminus G} \phi(x^{-1}ax) dx = \frac{\rho(a)}{|D(a)|} \int_N \int_K \phi(kank^{-1}) dk dn, \quad (23)$$

*where*

$$D(a) = \rho(a) - \rho(a)^{-1} \quad (a \in A). \quad (24)$$

**Theorem 52.** *Let  $\Theta_{\pi_s}$  be a function on  $G$  defined as follows.*

$$\Theta_{\pi_s}(x) = \begin{cases} \frac{\rho(a)^s + \rho(a)^{-s}}{|D(a)|} & \text{if } x \text{ is conjugate to } a \in A, \\ 0 & \text{if } x \text{ is not conjugate to any element of } A. \end{cases} \quad (25)$$

Then

$$\mathrm{tr} \pi_s(\psi) = \int_G \Theta_{\pi_s}(x) \psi(x) dx \quad (\psi \in C_c(G)). \quad (26)$$

*Proof.* By combining Lemmas 50 and 51 we see that

$$\mathrm{tr} \pi_s(\psi) = \int_A |D(a)| \int_{A \setminus G} \psi(x^{-1}ax) dx \rho(a)^s da. \quad (27)$$

Changing the variable  $a$  to  $a^{-1}$  gives

$$\mathrm{tr} \pi_s(\psi) = \int_A |D(a^{-1})| \int_{A \setminus G} \psi(x^{-1}a^{-1}x) dx \rho(a)^{-s} da.$$

However  $|D(a^{-1})| = |D(a)|$  and

$$\int_{A \setminus G} \psi(x^{-1}a^{-1}x) dx = \int_{A \setminus G} \psi(x^{-1}ax) dx.$$

Therefore

$$\mathrm{tr} \pi_s(\psi) = \int_A |D(a)| \int_{A \setminus G} \psi(x^{-1}ax) dx \frac{\rho(a)^s + \rho(a)^{-s}}{2} da.$$

On the other hand,

$$\begin{aligned} \int_G \Theta_{\pi_s}(x) \psi(x) dx &= \frac{1}{2} \int_A |D(a)|^2 \int_{A \setminus G} \Theta_{\pi_s}(x^{-1}ax) \psi(x^{-1}ax) dx da \\ &= \frac{1}{2} \int_A |D(a)| \int_{A \setminus G} \psi(x^{-1}ax) dx \Theta_{\pi_s}(a) |D(a)| da \end{aligned}$$

and (28) follows.  $\square$

Notice that the trace is given via integration against a  $G$ -invariant locally integrable function. Let  ${}^0M = \{1, -1\} \subseteq G$ . This is the centralizer  $A$  in  $K$ . If  $\delta$  is a character of the group  ${}^0M$ , we define  $V(\delta, s) \subseteq V(s)$  to be the subspace of functions  $\phi$  such that

$$\phi(mx) = \delta(m) \psi(x) \quad (m \in {}^0M, x \in G).$$

Denote by  $(\pi_{\delta, s}, V(\delta, s))$  the resulting representation of  $G$ . Clearly  $(\pi_s, V(s))$  is the direct sum of the two subrepresentations  $(\pi_{\delta, s}, V(\delta, s))$ .

**Theorem 53.** *Let  $\Theta_{\pi_{\delta, s}}$  be a function on  $G$  defined as follows.*

$$\Theta_{\pi_s}(mx) = \begin{cases} \delta(m) \frac{\rho(a)^s + \rho(a)^{-s}}{D(a)} & \text{if } x \text{ is conjugate to } ma, \text{ where } m \in {}^0M \text{ and } a \in A, \\ 0 & \text{if } x \text{ is not conjugate to any element of } \pm A. \end{cases} \quad (28)$$

Then

$$\mathrm{tr} \pi_{\delta,s}(\psi) = \int_{\mathbf{G}} \Theta_{\pi_{\delta,s}}(x) \psi(x) dx \quad (\psi \in C_c(\mathbf{G})). \quad (29)$$

**10.3. Finite dimensional representations.** Define

$$D(k(\theta)) = e^{i\theta} - e^{-i\theta} \quad (k(\theta) \in \mathbf{K}). \quad (30)$$

**Lemma 54.** Fix an integer  $n \geq 1$ . For non-negative integers  $p, q$  with  $p + q = n - 1$  let  $f_{p,q}$  be a function of

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that  $f_{p,q}(x) = c^p d^q$ . Then  $f_{p,q} \in \mathbf{V}(\delta, -n)$ , where  $\delta(-1) = (-1)^{p+q}$ . The functions  $f_{p,q}$  span a  $\mathbf{G}$ -invariant finite dimensional subspace

$$\mathbf{U}(\delta, -n) = \bigoplus_{p+q=n-1} \mathbb{C} f_{p,q} \subseteq \mathbf{V}(\delta, -n)$$

of dimension  $n$ . Denote the resulting representation by  $(\sigma_{\delta,-n}, \mathbf{U}(\delta, -n))$ . Then

$$\mathrm{tr} \sigma_{\delta,-n}(ma) = \delta(m) \frac{\rho(a)^n - \rho(a)^{-n}}{D(a)} \quad (m \in {}^0M, a \in \mathbf{A}). \quad (31)$$

Also,

$$\mathrm{tr} \sigma_{\delta,-n}(k(\theta)) = \frac{e^{in\theta} - e^{-in\theta}}{D(k(\theta))} \quad (k(\theta) \in \mathbf{K}). \quad (32)$$

*Proof.* Everything till (31) is straightforward. See [Lan85, page 151]. The formula (32) follows from (31). Indeed, let  $\mathbf{G}_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$ . Then  $\mathbf{G} \subseteq \mathbf{G}_{\mathbb{C}}$  is a subgroup and the representation  $(\sigma_{\delta,-n}, \mathbf{U}(\delta, -n))$  extends to a representation of  $\mathbf{G}_{\mathbb{C}}$  so that  $\sigma_{\delta,-n} : \mathbf{G}_{\mathbb{C}} \rightarrow \mathrm{GL}(\mathbf{U}(\delta, -n))$  is a polynomial map. Notice that

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Thus  $k(\theta)$  is conjugate to

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

within  $\mathbf{G}_{\mathbb{C}}$ . Hence Lemma 54 shows that

$$\mathrm{tr} \sigma_{\delta,-n}(k(\theta)) = \frac{e^{in\theta} - e^{-in\theta}}{D(k(\theta))} \quad (k(\theta) \in \mathbf{K}). \quad (33)$$

□

**10.4. Smooth vectors and analytic vectors.** Let  $(\pi, \mathbf{V})$  be a representation of  $G$  on a Banach space  $\mathbf{V}$ . Denote by  $\mathbf{V}^\infty \subseteq \mathbf{V}$  the subspace of all vectors  $v$  such that the map

$$G \ni x \rightarrow \pi(x)v \in \mathbf{V} \quad (34)$$

is smooth (infinitely many times differentiable). This subspace is dense because, as is easy to check,  $\pi(C_c^\infty(G))\mathbf{V} \subseteq \mathbf{V}^\infty$ . For  $X \in \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ , the Lie algebra of  $G$ , define

$$d\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v|_{t=0} \quad (v \in \mathbf{V}^\infty). \quad (35)$$

One checks without difficulties that the map

$$d\pi : \mathfrak{g} \rightarrow \text{End}(\mathbf{V}^\infty) \quad (36)$$

is a Lie algebra homomorphism. By linearity we extend it to the complexification  $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$$d\pi : \mathfrak{g}_\mathbb{C} \rightarrow \text{End}(\mathbf{V}^\infty) \quad (37)$$

and obtain a representation  $(d\pi, \mathbf{V}^\infty)$  of the Lie algebra  $\mathfrak{g}_\mathbb{C}$ . One drawback of this representation is that the closure of a  $\mathfrak{g}$ -invariant subspace  $U \subseteq \mathbf{V}^\infty$  in  $\mathbf{V}$  does not need to be  $G$ -invariant. Indeed, consider the right regular representation  $(R, L^2(G))$  of  $G$ . Let  $U \subseteq G$ ,  $U \neq G$ , be a non-empty open set. Then the space  $U = C_c^\infty(U)$  is closed under the action of  $dR(\mathfrak{g})$ , but the closure  $L^2(U)$  is not  $R(G)$  invariant. For this reason one is lead to study the space  $\mathbf{V}^{an} \subseteq \mathbf{V}$  of all vectors  $v$  such that the map

$$G \ni x \rightarrow \pi(x)v \in \mathbf{V} \quad (38)$$

is analytic.

**Theorem 55.** *Let  $X \subseteq \mathbf{V}^{an}$  be a  $\mathfrak{g}$ -invariant vector subspace. Then the closure of  $X$  in  $\mathbf{V}$  is  $G$ -invariant*

*Proof.* We follow [Var89, Theorem 2, page 108]. It is enough to prove that for any  $v \in U$  and for any  $x \in G$ ,  $\pi(x)v$  belongs to the closure of  $U$ . Suppose not. Then by Haan-Banach Theorem there is  $\lambda$  in the dual of  $\mathbf{V}$  such that  $\lambda$  is equal to zero on the closure of  $U$ , but for some  $x_0 \in G$ ,  $\lambda(\pi(x_0)v) \neq 0$ .

By assumption the function

$$G \ni x \rightarrow \lambda(\pi(x)v) \in \mathbb{C}$$

is analytic. By the choice of  $\lambda$ , its Taylor series at  $x = 1$  is zero. Indeed, for any  $X_1, X_2, \dots, X_n \in \mathfrak{g}$

$$d\pi(X_1)d\pi(X_2)\dots d\pi(X_n)v \in U$$

Hence

$$\lambda(d\pi(X_1)d\pi(X_2)\dots d\pi(X_n)v) = 0$$

Also,  $\lambda(v) = 0$ . Thus the function is zero because  $G$  is connected. This is a contradiction.  $\square$

**10.5. The derivative of the right regular representation.** Every element  $g \in \mathrm{GL}^+(\mathbb{R})$  has a unique decomposition as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (39)$$

where  $u > 0$ ,  $y > 0$ ,  $x \in \mathbb{R}$  and  $\theta \in [-\pi, \pi)$ . We extend any function defined on  $\mathrm{SL}_2(\mathbb{R})$  to a function on  $\mathrm{GL}^+(\mathbb{R})$  by making it independent of the variable  $u$ . Also, we extend the right regular representation of  $\mathrm{SL}_2(\mathbb{R})$  to act on such defined functions on  $\mathrm{GL}^+(\mathbb{R})$ . Then a straightforward computation, see [Lan85, 113-116] verifies the following formulas,

$$\begin{aligned} dR \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= y \cos 2\theta \partial_x + y \sin 2\theta \partial_y + \sin^2 \theta \partial_\theta, \\ dR \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \partial_\theta, \\ dR \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= 2y \cos 2\theta \partial_x + 2y \sin 2\theta \partial_y - \cos 2\theta \partial_\theta, \\ dR \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= y \cos 2\theta \partial_x + y \sin 2\theta \partial_y - \cos^2 \theta \partial_\theta, \\ dR \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= -2y \sin 2\theta \partial_x + 2y \cos 2\theta \partial_y + \sin 2\theta \partial_\theta. \end{aligned} \quad (40)$$

**10.6. The universal enveloping algebra.** Here  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$  and  $G = \mathrm{SL}_2(\mathbb{R})$ . The universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  is the complex tensor algebra of  $\mathfrak{g}$  divided by the ideal generated by element  $AB - BA - [A, B]$ ,  $A, B \in \mathfrak{g}$ . The Poincare-Birkhoff-Witt Theorem says that if  $A, B, C$  form a basis of the vector space  $\mathfrak{g}_{\mathbb{C}}$ , then the elements

$$A^a B^b C^c \quad (0 \leq a, b, c \in \mathbb{Z})$$

form a basis of the vector space  $\mathcal{U}(\mathfrak{g})$ . (Here  $A^0 = B^0 = C^0 = 1$ .) Furthermore, for any  $A, B, C \in \mathfrak{g}_{\mathbb{C}}$  satisfying the commutation relations

$$[A, B] = 2B, \quad [A, C] = -2C, \quad [B, C] = A$$

The element, called the Casimir element,

$$\mathcal{C} = A^2 + 2(BC + CB)$$

generates the the center of  $\mathcal{U}(\mathfrak{g})$  and does not depend on the choice of the  $A, B, C$ . Thus the center of  $\mathcal{U}(\mathfrak{g})$  is equal to  $\mathbb{C}[\mathcal{C}]$ . Also, any representation of the Lie algebra  $\mathfrak{g}$  extends to a representation of the algebra  $\mathcal{U}(\mathfrak{g})$ . In terms of the coordinates used in section 10.5,

$$dR(\mathcal{C}) = 4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x\partial_\theta \quad (41)$$

Proofs may be found in [Lan85, pages 191-198].

**10.7. K-multiplicity 1 representations.** Here  $G = \mathrm{SL}_2(\mathbb{R})$  and we consider only representations  $(\pi, \mathbf{V})$  of  $G$  on Banach spaces  $\mathbf{V}$  such that for each integer  $n$  the isotypic component  $\mathbf{V}_n \subseteq \mathbf{V}$  has dimension at most 1.

**Lemma 56.** *For any  $n \in \mathbb{Z}$ , the space  $S_{n,n}^\infty = S_{n,n} \cap C^\infty(G)$  is dense in  $S_{n,n}$*

*Proof.* See [Lan85, page 101]. □

**Theorem 57.** *Let  $(\pi, \mathbf{V})$  be a representation  $G$  on a Banach space  $\mathbf{V}$  such that for each integer  $n$  the isotypic component  $\mathbf{V}_n \subseteq \mathbf{V}$  has dimension at most 1. Then*

$$\mathbf{V}_K = \bigoplus_{n \in \mathbb{Z}} \mathbf{V}_n \subseteq \mathbf{V}^{an}.$$

*In particular  $\mathbf{V}_K$  is a  $(\mathfrak{g}, K)$  module.*

*Proof.* It'll suffice to show that for a fixed  $n$  and a non-zero vector  $v \in \mathbf{V}_n$  the function

$$f_v(g) = \pi(g)v \quad (g \in G)$$

is analytic. Notice that

$$f_v(k(\theta)g) = e^{in\theta} f_v(g).$$

Hence (41) shows that

$$dR(\mathcal{C})f_v(g) = (4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x in)f_v(g).$$

Since, by assumption,  $\dim \mathbf{V}_n = 1$ , we see from Lemmas 56, 42 and 43 that

$$\mathbf{V}_n = \pi(S_{n,n}^\infty)\mathbf{V}_n.$$

In particular,  $\mathbf{V}_n \subseteq \mathbf{V}^\infty$ . On the other hand  $d\pi(\mathcal{C})$  commutes with  $\pi(K)$ , hence preserves  $\mathbf{V}_n$ . Since, by assumption,  $\dim \mathbf{V}_n = 1$ ,  $d\pi(\mathcal{C})$  acts on  $\mathbf{V}_n$  via multiplication by a scalar, call it  $c_n$ . Thus

$$d\pi(\mathcal{C})v = c_nv.$$

Since

$$R(h)f_v(g) = f_{\pi(h)v}(g),$$

this implies

$$dR(\mathcal{C})f_v = c_nf_v.$$

Since the characteristic variety of the system of differential equations

$$(4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x in)f_v = c_nf_v, \quad \partial_\theta f_v = in$$

is zero the function  $f_v$  is analytic. □

**Theorem 58.** *Let  $(\pi, \mathbf{V})$  be a representation  $G$  on a Banach space  $\mathbf{V}$  such that for each integer  $n$  the isotypic component  $\mathbf{V}_n \subseteq \mathbf{V}$  has dimension at most 1. Then the map*

$$\mathbf{V} \supseteq \mathbf{U} \rightarrow \mathbf{U}_K \subseteq \mathbf{V}_K$$

*is a bijection between closed  $G$ -invariant subspaces of  $\mathbf{V}$  and  $(\mathfrak{g}, K)$ -submodules of  $\mathbf{V}_K$ . The inverse is given by*

$$\mathbf{V}_K \supseteq \mathbf{X} \rightarrow Cl(\mathbf{X}) \subseteq \mathbf{V}.$$

where  $Cl(\mathbf{X})$  denotes the closure of  $\mathbf{X}$  in  $\mathbf{V}$ . In particular  $(\pi, \mathbf{V})$  is irreducible if and only if the  $(\mathfrak{g}, \mathbf{K})$ -module  $\mathbf{V}_{\mathbf{K}}$  is irreducible.

*Proof.* This is clear from Theorems 55 and 57.  $\square$

**Lemma 59.** *For any irreducible  $(\mathfrak{g}, \mathbf{K})$ -module  $\mathbf{X}$  and any integer  $n$  such that  $\mathbf{X}_n \neq 0$ , any non-zero vector  $v \in \mathbf{X}_n$  is cyclic for the action of  $\mathcal{U}(\mathfrak{g})$  on  $\mathbf{X}$ .*

*Proof.* Since  $\mathcal{U}(\mathfrak{g})v$  is a submodule of  $\mathbf{X}$ , the claim is obvious.  $\square$

Let  $\mathbf{X} = \bigoplus_n \mathbf{X}_n$  be a  $(\mathfrak{g}, \mathbf{K})$ -module with  $\dim \mathbf{X}_n \leq 1$ . A hermitian form  $(\cdot, \cdot)$  on  $\mathbf{X}$  is called  $\mathfrak{g}$ -invariant if

$$(Xu, v) = -(u, Xv) \quad (u, v \in \mathbf{X}, X \in \mathfrak{g}). \quad (42)$$

A  $(\mathfrak{g}, \mathbf{K})$  module is called unitarizable if it admits an invariant positive definite hermitian form.

**Theorem 60.** *Let  $\mathbf{X} = \bigoplus_n \mathbf{X}_n$  be an irreducible  $(\mathfrak{g}, \mathbf{K})$ -module with  $\dim \mathbf{X}_n \leq 1$ . Then any two positive definite  $\mathfrak{g}$ -invariant hermitian products on  $\mathbf{X}$  are positive multiples of each other (assuming they exist).*

*Proof.* Denote the two  $\mathfrak{g}$ -invariant positive hermitian products on  $\mathbf{X}$  by  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$ . Then the spaces  $\mathbf{V}_n$  are mutually orthogonal with respect to  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$ . Let  $P_n : \mathbf{V} \rightarrow \mathbf{V}_n$  denote the orthogonal projection. Fix  $m \in \mathbb{Z}$  such that  $\mathbf{V}_m \neq 0$  and a vector  $0 \neq v_m \in \mathbf{V}_m$ . Lemma 59 implies that for any  $n$  there is  $a \in (\mathfrak{g})$  such that  $P_n a v_m \neq 0$ . Then

$$(a v_m, a v_m) = (v_m, a^* a v_m) = (v_m, P_m a^* a v_m) = c \langle v_m, P_m a^* a v_m \rangle = c \langle a v_m, a v_m \rangle,$$

where

$$c = \frac{(v_m, v_m)}{\langle v_m, v_m \rangle}.$$

$\square$

**Theorem 61.** *Two irreducible unitary representations of  $G$ , of  $\mathbf{K}$ -multiplicity at most 1, are unitarily isomorphic if and only if their  $(\mathfrak{g}, \mathbf{K})$  modules are isomorphic.*

*Proof.* Let  $(\pi, \mathbf{V}), (\sigma, \mathbf{U})$  be the two representations. If they are isomorphic, then clearly so are the  $(\mathfrak{g}, \mathbf{K})$ -modules. Conversely, suppose

$$L : \mathbf{V}_{\mathbf{K}} \rightarrow \mathbf{U}_{\mathbf{K}}$$

is a  $\mathfrak{g}$ -intertwining map. Then  $(\cdot, \cdot)_{\mathbf{V}}$  and the pull-back of  $(\cdot, \cdot)_{\mathbf{U}}$  via  $L$  are positive definite  $\mathfrak{g}$ -invariant hermitian products on  $\mathbf{V}_{\mathbf{K}}$ . Theorem 60 shows that there is a constant  $c > 0$  such that

$$(v, v')_{\mathbf{V}} = c(Lv, Lv')_{\mathbf{U}} \quad (v, v' \in \mathbf{V}).$$

Hence

$$T = \sqrt{c}L : \mathbf{V}_{\mathbf{K}} \rightarrow \mathbf{U}_{\mathbf{K}}$$

is a  $\mathfrak{g}$ -intertwining isometry. We need to check that  $T$  is also  $G$ -intertwining.

Since  $K$ -finite vectors are analytic, we have

$$\pi(\exp(X))v = \sum_{n=0}^{\infty} \frac{1}{n!} (d\pi(X))^n v \quad (v \in \mathbf{V}_K).$$

For  $X \in \mathfrak{g}$  in some small neighborhood of zero. Hence, in this neighborhood,

$$T\pi(\exp(X))v = \sum_{n=0}^{\infty} \frac{1}{n!} (d\sigma(X))^n Tv = \sigma(\exp(X))Tv.$$

Thus there is an open neighborhood  $U$  of  $1 \in G$  such that

$$T\pi(g)v = \sigma(g)Tv \quad (g \in U).$$

Since  $G$  is connected  $U$  generates  $G$ , so the proof is complete.  $\square$

### 10.8. The character of a $(\mathfrak{g}, K)$ -module.

**Theorem 62.** *Let  $(\pi, \mathbf{V})$  be a representation of  $G$  on a Hilbert space  $\mathbf{V}$ , with  $\dim \mathbf{V}_n \leq 1$  for all  $n \in \mathbb{Z}$ . Then for any  $\phi \in C_c^\infty(G)$ , the operator  $\pi(\phi)$  is of trace class and the map*

$$C_c^\infty(G) \ni \phi \rightarrow \text{tr } \pi(\phi) \in \mathbb{C}$$

*is a distribution on  $G$ .*

*Proof.* Recall that

$$\int_G \phi(g)\pi(g) dg = \int_G \phi(g)\pi(gk^{-1}k) dg = \int_G R(k)\phi(g)\pi(g) dg \pi(k).$$

Hence by taking derivatives at  $k = 1$ ,

$$0 = \int_G dR(J)\phi(g)\pi(g) dg + \int_G \phi(g)\pi(g) dg d\pi(J).$$

Let  $v_n \in \mathbf{V}_n$  be a unite vector. Then

$$0 = \int_G dR(J)\phi(g)\pi(g) dg v_n + \int_G \phi(g)\pi(g) dg in v_n.$$

By iterating we see that for  $m = 0, 1, 2, \dots$

$$(-in)^m \pi(\phi)v_n = \pi(dR(J)^m \phi).$$

Thus

$$|(\pi(\phi)v_n, v_n)| \leq (1 + |n|)^{-m} \| \pi(dR(J)^m \phi) \| \quad (n \in \mathbb{Z}).$$

Since for  $m \geq 2$ ,

$$\sum_{n \in \mathbb{Z}} (1 + |n|)^{-m} < \infty,$$

the claim follows.  $\square$



**Theorem 63.** *Let  $(\pi, \mathbf{V})$ ,  $(\sigma, \mathbf{U})$  be representations of  $G$  on Hilbert spaces  $\mathbf{V}$  and  $\mathbf{U}$ , with  $\dim \mathbf{V}_n \leq 1$  and  $\dim \mathbf{U}_n \leq 1$  for all  $n \in \mathbb{Z}$ . Assume that the  $(\mathfrak{g}, \mathbf{K})$ -modules  $\mathbf{V}_{\mathbf{K}}$  and  $\mathbf{U}_{\mathbf{K}}$  are isomorphic. Then,*

$$\mathrm{tr} \pi(\phi) = \mathrm{tr} \sigma(\phi) \quad (\phi \in C_c^\infty(G)).$$

*Proof.* Given a linear map  $A : \mathbf{U} \rightarrow \mathbf{V}$ , recall the conjugate map  $T^* : \mathbf{V} \rightarrow \mathbf{U}$ ,

$$(Au, v)_{\mathbf{V}} = (u, A^*v)_{\mathbf{U}}.$$

Let

$$T : \mathbf{V}_{\mathbf{K}} \rightarrow \mathbf{U}_{\mathbf{K}}$$

be a  $(\mathfrak{g}, \mathbf{K})$ -intertwining isomorphism. Pick unit vectors  $v_n \in \mathbf{V}_n$ . Notice that  $Tv_n$ ,  $n \in \mathbb{Z}$ , form a basis of  $\mathbf{U}$  and  $(T^*)^{-1}v_n$  form a dual basis. Then

$$\mathrm{tr} \pi(\phi) = \sum_{n \in \mathbb{Z}} (\pi(\phi)v_n, v_n)_{\mathbf{V}} \quad \text{and} \quad \mathrm{tr} \sigma(\phi) = \sum_{n \in \mathbb{Z}} (\sigma(\phi)Tv_n, (T^{-1})^*v_n)_{\mathbf{U}}.$$

By definition

$$(\sigma(\phi)Tv_n, (T^{-1})^*v_n)_{\mathbf{U}} = (T^{-1}\sigma(\phi)Tv_n, v_n)_{\mathbf{U}}$$

Since  $v_n$  is an analytic vector,

$$T^{-1}\sigma(\exp(X))Tv_n = \pi(\exp(X))v_n$$

in a neighborhood of  $0 \in \mathfrak{g}$ . Hence

$$(\pi(g)v_n, v_n)_{\mathbf{V}} = (\sigma(g)Tv_n, (T^{-1})^*v_n)_{\mathbf{U}}$$

in a neighborhood of  $1 \in G$ . Since both functions are analytic, they are equal everywhere. Thus

$$(\pi(\phi)v_n, v_n)_{\mathbf{V}} = (\sigma(\phi)Tv_n, (T^{-1})^*v_n)_{\mathbf{U}}.$$

□

In general one says that two representations  $(\pi, \mathbf{V})$  and  $(\sigma, \mathbf{U})$  are infinitesimally equivalent if the  $(\mathfrak{g}, \mathbf{K})$ -modules  $\mathbf{V}_{\mathbf{K}}$  and  $\mathbf{U}_{\mathbf{K}}$  are isomorphic.

Given a representation of  $G$  (with  $\mathbf{K}$ -multiplicities at most 1) we define the character of  $\pi$  as

$$\Theta_\pi(\phi) = \mathrm{tr} \pi(\phi) \quad (\phi \in C_c^\infty(G)).$$

This is a distribution on  $G$  which, as shown in Theorem 63, does not depend on the infinitesimal equivalence class of  $(\pi, \mathbf{V})$ . Therefore we shall also write  $\Theta_\pi = \Theta_{\mathbf{V}_{\mathbf{K}}}$ .

### 10.9. The unitary dual.

**Lemma 64.** *Let*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

*Then*

$$[E^+, E^-] = -4iJ, \quad [J, E^+] = 2iE^+, \quad [J, E^-] = -2iE^-.$$

*Proof.* This is straightforward. See [Lan85, page 102].

□

**Corollary 65.** For any  $(\mathfrak{g}, \mathbb{K})$  module  $X$ , with the  $\mathbb{K}$ -isotypic components  $X_n$ ,

$$JX_n \subseteq X_n, \quad E^+X_n \subseteq X_{n+2}, \quad E^-X_n \subseteq X_{n-2}.$$

*Proof.* This is clear from Corollary 64 □

Recall the principal series representation  $(\pi_s, \mathbb{V}(s))$ .

**Lemma 66.** In terms of (39), for  $n \in \mathbb{Z}$  define

$$v_n \left( \left( \begin{array}{cc} u & 0 \\ 0 & u \end{array} \right) \left( \begin{array}{cc} y & x \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \right) = y^{1+s} e^{in\theta}.$$

Then  $v_n \in \mathbb{V}(s)$  and

$$\begin{aligned} d\pi_s(J)v_n &= inv_n, \\ d\pi_s(E^-)v_n &= (s+1-n)v_{n-2}, \\ d\pi_s(E^+)v_n &= (s+1+n)v_{n+2}. \end{aligned}$$

*Proof.* We see from (40) that

$$\begin{aligned} dR(J)v_n &= inv_n, \\ dR(E^-)v_n &= (s+1-n)v_{n-2}, \\ dR(E^+)v_n &= (s+1+n)v_{n+2}. \end{aligned}$$

But the right regular action coincides with  $\pi_s$ , hence the formulas follow. □

By combining Lemma 66 with Theorem 57 we deduce the following Corollary.

**Corollary 67.** The  $(\mathfrak{g}, \mathbb{K})$  module of the principal series  $(\pi_s, \mathbb{V}(s))$  is equal to

$$\mathbb{V}(s)_{\mathbb{K}} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}v_n.$$

Lemma 66 implies the following Proposition, see [Lan85, pages 119-121].

**Proposition 68.** If  $s$  is not an integer then

$$\mathbb{V}(s)_{\mathbb{K}}^+ = \bigoplus_{n \in 2\mathbb{Z}} \mathbb{C}v_n \quad \text{and} \quad \mathbb{V}(s)_{\mathbb{K}}^- = \bigoplus_{n \in 2\mathbb{Z}+1} \mathbb{C}v_n$$

are irreducible submodules of  $\mathbb{V}(s)_{\mathbb{K}}$  and  $\mathbb{V}(s)_{\mathbb{K}}$  is the direct sum of them.

If  $s = 0$ , then  $\mathbb{V}(0)_{\mathbb{K}}$  is the direct sum of three irreducible submodules

$$\mathbb{V}(0)_{\mathbb{K}} = \bigoplus_{n \in 2\mathbb{Z}} \mathbb{C}v_n \oplus \bigoplus_{1 \leq n \in 2\mathbb{Z}+1} \mathbb{C}v_n \oplus \bigoplus_{-1 \geq n \in 2\mathbb{Z}+1} \mathbb{C}v_n.$$

If  $m \geq 2$  is an integer and  $s = m - 1$ , then  $\mathbb{V}(m-1)_{\mathbb{K}}$  contains three irreducible submodules

$$X^m = \bigoplus_{m \leq n, n-m \in 2\mathbb{Z}} \mathbb{C}v_n, \quad X^{-m} = \bigoplus_{-m \geq n, n-m \in 2\mathbb{Z}} \mathbb{C}v_n, \quad \bigoplus_{n-m \in 2\mathbb{Z}+1} \mathbb{C}v_n.$$

The quotient module,  $\mathbf{V}(m-1)_{\mathbf{K}}$  divided by the three submodules is irreducible, finite dimensional of dimension  $m-1$ . It has a basis represented by the elements

$$v_{-m+2}, \quad v_{-m+4}, \quad \dots, \quad v_{m-2}.$$

If  $m \geq 2$  is an integer and  $s = -m+1$ , then  $\mathbf{V}(m-1)_{\mathbf{K}}$  contains the finite dimensional submodule

$$\mathbb{C}v_{-m+2} \oplus \mathbb{C}v_{-m+4} \oplus \dots \oplus \mathbb{C}v_{m-2}.$$

The quotient module,  $\mathbf{V}(m-1)_{\mathbf{K}}$  divided by this module is isomorphic to the direct sum of modules  $\mathbf{X}^m$  and  $\mathbf{X}^{-m}$  plus the sum of all  $\mathbf{K}$ -types of parity opposite to  $m$ .

Thus we have the highest weight modules, lowest weight modules, finite dimensional modules and modules with unbounded  $\mathbf{K}$ -types on both side.

The commutation relations Lemma 64 and the formulas Lemma 66 with some work imply the following theorem, due to Bargmann, [Bar47]. See [Lan85, page 123].

**Theorem 69.** *Here is a complete list of the irreducible  $(\mathfrak{g}, \mathbf{K})$  modules, up to equivalence.*

- (1) *Lowest weight module  $\mathbf{X}^m$  with lowest weight  $m \geq 1$  and the highest weight module  $\mathbf{X}^m$  with highest weight  $m \leq -1$*
- (2) *Principal series  $\mathbf{V}(i\tau)_{\mathbf{K}}^+$  and  $\mathbf{V}(i\tau)_{\mathbf{K}}^-$ ,  $\tau \in \mathbb{R} \setminus \{0\}$ ;*
- (3) *Principal series  $\mathbf{V}(0)_{\mathbf{K}}^+$ ;*
- (4) *Complementary series  $\mathbf{V}(s)_{\mathbf{K}}^+$ ,  $-1 < s < 1$ ;*
- (5) *Trivial module.*

A similar result was obtained by Dan Barbasch, [Bar89] for the complex classical groups.

The closures of these modules in the corresponding principal series representation are representations of  $\mathbf{G}$  on Hilbert spaces. They are unitary representations except the highest and lowest weight representations. In these cases the inner product inherited from the principal series is not  $\mathfrak{g}$  invariant. Therefore there is a problem of constructing the unitary representations of  $\mathbf{G}$  whose  $(\mathfrak{g}, \mathbf{K})$ -modules are the highest and lowest weight modules. This is explained in the theorem below. See [Lan85, page 181].

**Theorem 70.** *Define*

$$\tilde{\alpha} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(a+d-ic+ib), \quad \tilde{\beta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(c+b-ia+id).$$

Let  $m \geq 2$ . Then the closure  $\mathbf{V}^{(m)}$  in  $L^2(\mathbf{G})$  of the space

$$\bigoplus_{r=0}^{\infty} \mathbb{C}\tilde{\alpha}^{-m-r}\tilde{\beta}^r$$

is invariant under the left action

$$L(g)\phi(x) = \phi(g^{-1}x).$$

The resulting representation  $(L, \mathbf{V}^{(m)})$  of  $\mathbf{G}$  is irreducible and unitary. Its  $(\mathfrak{g}, \mathbf{K})$ -module  $\mathbf{V}_{\mathbf{K}}^{(m)}$  is isomorphis to the lowest weight module  $\mathbf{X}^m$  with the lowest weight  $m$ . In order to realize the lowest weight representations we take the complex conjugate of  $\mathbf{V}^{(m)}$ .

We shall skip the construction of the unitary representations whose  $(\mathfrak{g}, \mathbb{K})$ -modules are  $\mathbf{X}^1$  and  $\mathbf{X}^{-1}$ . They are not square integrable. One may find them in [Kna86, page 36].

**10.10. The character of the sum of discrete series.** For an integer  $m \geq 2$  let  $\Theta_{\mathbf{X}^{(m)}}$  denote the the character of the  $(\mathfrak{g}, \mathbb{K})$ -module  $\mathbf{X}^{(m)}$ , and  $\Theta_{\mathbf{X}^{(-m)}}$  denote the the character of the  $(\mathfrak{g}, \mathbb{K})$ -module  $\mathbf{X}^{(-m)}$ . Let

$$h_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad t \in \mathbb{R}.$$

**Proposition 71.** *The character of the sum of the two discrete series  $(\mathfrak{g}, \mathbb{K})$ -modules  $\mathbf{X}^{(m)}$  and  $\mathbf{X}^{(-m)}$  is represented by the  $G$ -conjugation invariant function  $\Theta_{\mathbf{X}^{(m)} \oplus \mathbf{X}^{(-m)}}$  given by*

$$\begin{aligned} \Theta_{\mathbf{X}^{(m)} \oplus \mathbf{X}^{(-m)}}(k(\theta)) &= -\frac{e^{i(m-1)\theta} - e^{-i(m-1)\theta}}{e^{i\theta} - e^{-i\theta}} \quad (\theta \in \mathbb{R}), \\ \Theta_{\mathbf{X}^{(m)} \oplus \mathbf{X}^{(-m)}}(zh_t) &= z^m \frac{e^{-|t|m-1}}{|e^t - e^{-t}|} \quad (z = \pm 1, t \in \mathbb{R}). \end{aligned}$$

*Proof.* From Proposition 68 we know the structure of the principal series  $\mathbf{V}(m-1)_{\mathbb{K}}$ , from Theorem 52, the character of it, from Lemma 54 the character of the finite dimensional representation which is in the principal series. Now Theorem 63 justifies the formula

character of principal series

= character of the sum of discrete series + character of finite dimensional module .

This completes the proof □

For convenience define The following  $G$ -invariant functions on  $G$

$$\begin{aligned} \Theta_{\mathbf{X}^{(m)}}(k(\theta)) &= -\frac{e^{i(m-1)\theta}}{e^{i\theta} - e^{-i\theta}} \quad (\theta \in \mathbb{R}), \\ \Theta_{\mathbf{X}^{(m)}}(zh_t) &= z^m \frac{e^{-|t|m-1}}{|e^t - e^{-t}|} \quad (z = \pm 1, t \in \mathbb{R}). \end{aligned}$$

and

$$\begin{aligned} \Theta_{\mathbf{X}^{(-m)}}(k(\theta)) &= \frac{e^{i(m-1)\theta}}{e^{i\theta} - e^{-i\theta}} \quad (\theta \in \mathbb{R}), \\ \Theta_{\mathbf{X}^{(-m)}}(zh_t) &= z^m \frac{e^{-|t|m-1}}{|e^t - e^{-t}|} \quad (z = \pm 1, t \in \mathbb{R}). \end{aligned}$$

These are the characters of the individual discrete series representations, but we don't need to know it to explain Harish-Chandra's Plancharel formula in the  $\mathrm{SL}_2(\mathbb{R})$  case.

10.11. **Harish-Chandra's Plancherel formula.** It is not difficult to check that  $K$  and  $H = A \cup (-A)$  are the only Cartan subgroups of  $G$  up to conjugacy. Let  $A' = \{a \in A; a \neq 1\}$  and let  $K' = \{t \in K; t \neq \pm 1\}$ . We begin by recalling Harish-Chandra's orbital integrals of a function  $\phi \in C_c^\infty(G)$ , using the notation of [Lan85],

$$\begin{aligned} H_\phi^A(zh_t) &= |D(h_t)| \int_{A \backslash G} \phi(x^{-1}zh_tx) dx \quad (z = \pm 1, t \in \mathbb{R}, t \neq 0), \\ H_\phi^K(k) &= D(k) \int_{K \backslash G} \phi(x^{-1}kx) dx \quad (k \in K'), \end{aligned} \quad (43)$$

The Haar measure on  $G$  may be expressed in terms of these integrals by

$$\int_G \phi(x) dx = \int_K H_\phi^K(k) \overline{D(k)} dk + \frac{1}{2} \int_{A'} H_\phi^A(a) |D(a)| da + \frac{1}{2} \int_{A'} H_\phi^A(-a) |D(a)| da. \quad (44)$$

**Theorem 72.** *The function  $H_\phi^A$  extends to a smooth function on  $H$ . The function  $H_\phi^K$  is smooth on  $H'$  and its derivatives have one sided limits on the boundary. In these terms,*

$$\partial_\theta H_\phi^K(k(\theta))|_{\theta=0} = -i\phi(1). \quad (45)$$

*Proof.* This is a problem concerning integrals on a 3 dimensional manifold,  $G = \mathrm{SL}_2(\mathbb{R})$ . Notice that we may replace  $\phi$  by a  $K$  conjugation invariant function  $\int_K \phi(kxk^{-1}) dk$ . This leads to analysis on the two dimensional manifold  $G/K$ . The computations are done in [Lan85, page 164 -167].  $\square$

Recall the character  $\chi_n(k(\theta)) = e^{in\theta}$ .

**Theorem 73.** *For any integer  $m \geq 2$  and  $\phi \in C_c^\infty(G)$*

$$\begin{aligned} \int_G \Theta_{\chi^{(m)}}(x) \phi(x) dx &= \int_K H_\phi^K(k) \chi_{m-1}(k) dk \\ &+ \frac{1}{2} \int_{\mathbb{R}} H_\phi^A(h_t) e^{-|t|^{m-1}} dt + \frac{1}{2} \int_{\mathbb{R}} (-1)^m H_\phi^A(-h_t) e^{-|t|^{m-1}} dt \end{aligned}$$

and

$$\begin{aligned} \int_G \Theta_{\chi^{(-m)}}(x) \phi(x) dx &= - \int_K F_\phi^K(t) \chi_{-m+1}(k) dk \\ &+ \frac{1}{2} \int_{\mathbb{R}} H_\phi^A(h_t) e^{-|t|^{m-1}} dt + \frac{1}{2} \int_{\mathbb{R}} (-1)^m H_\phi^A(-h_t) e^{-|t|^{m-1}} dt. \end{aligned}$$

*Proof.* This follows from (44) and the formulas for  $\Theta_{\chi^{(\pm m)}}$  in previous section.  $\square$

Theorem 73 gives formulas for the Fourier coefficients of  $H_\phi^K$ :

$$\hat{H}_\phi^K(n) = \int_K H_\phi^K(k) \chi_n(k) dk \quad (n \neq 0).$$

Indeed, for  $n \geq 1$ ,

$$\hat{H}_\phi^K(n) = \Theta_{\chi^{(n+1)}}(\phi) - \frac{1}{2} \int_{\mathbb{R}} H_\phi^A(h_t) e^{-|t|^n} dt - \frac{1}{2} \int_{\mathbb{R}} (-1)^m H_\phi^A(-h_t) e^{-|t|^n} dt$$

and

$$\hat{H}_\phi^K(-n) = \Theta_{\mathbf{X}(-n-1)}(\phi) - \frac{1}{2} \int_{\mathbb{R}} H_\phi^\Lambda(h_t) e^{-|t||n|} dt - \frac{1}{2} \int_{\mathbb{R}} (-1)^m H_\phi^\Lambda(-h_t) e^{-|t||n|} dt$$

In particular,

$$\begin{aligned} \hat{H}_\phi^K(n) - \hat{H}_\phi^K(-n) &= \Theta_{\mathbf{X}(n+1)+\mathbf{X}(-n-1)}(\phi) \\ &\quad - \int_{\mathbb{R}} H_\phi^\Lambda(h_t) e^{-|t|n} dt - \int_{\mathbb{R}} (-1)^m H_\phi^\Lambda(-h_t) e^{-|t|n} dt. \end{aligned} \quad (46)$$

On the other hand, for  $k \neq \pm 1$

$$H_\phi^K(k) = \sum_{n \in \mathbb{Z}} \hat{H}_\phi^K(n) \chi_{-n}(k) = \sum_{n \in \mathbb{Z}} \hat{H}_\phi^K(n) \chi_n(k^{-1}) = \sum_{n \in \mathbb{Z}} \hat{H}_\phi^K(-n) \chi_{-n}(k^{-1})$$

and therefore

$$H_\phi^K(k) - H_\phi^K(k^{-1}) = \sum_{0 \neq n \in \mathbb{Z}} (H_\phi^K(n) - H_\phi^K(-n)) \chi_{-n}(k).$$

Thus,

$$H_\phi^K(k(\theta)) - H_\phi^K(k(-\theta)) = \sum_{n=1}^{\infty} (H_\phi^K(n) - H_\phi^K(-n)) (-i) \sin(n\theta). \quad (47)$$

Continuing this way (and correcting the constants, if necessary) one obtains the following lemma, see [Lan85, page 174],

**Lemma 74.**

$$\begin{aligned} \frac{\pi}{i} (H_\phi^K(k(\theta)) - H_\phi^K(k(-\theta))) &= - \sum_{n=1}^{\infty} \Theta_{\mathbf{X}(n+1)+\mathbf{X}(-n-1)}(\phi) \sin(n\theta) \\ &\quad + \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{1}{2} (H_\phi^\Lambda(h_t) + (-1)^{n+1} H_\phi^\Lambda(-h_t)) \sin(n\theta) e^{-|t|n} dt. \end{aligned} \quad (48)$$

Let

$$\phi^+(x) = \frac{\phi(x) + \phi(-x)}{2} \quad \text{and} \quad \phi^-(x) = \frac{\phi(x) - \phi(-x)}{2}.$$

Then a straightforward argument shows that

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{1}{2} (H_\phi^\Lambda(h_t) + (-1)^{n+1} H_\phi^\Lambda(-h_t)) \sin(n\theta) e^{-|t|n} dt \\ = \int_{\mathbb{R}} H_{\phi^+}^\Lambda(h_t) \frac{\sin(\theta) \cosh(t)}{\cosh(2t) - \cos(2\theta)} dt + \int_{\mathbb{R}} H_{\phi^+}^\Lambda(h_t) \frac{\sin(2\theta)}{\cosh(2t) - \cos(2\theta)} dt \end{aligned}$$

For  $\psi \in C_c^\infty(\mathbb{G})$  we have the Fourier transform

$$\hat{H}_\psi^\Lambda(\lambda) = \int_{\mathbb{R}} H_\psi^\Lambda(h_t) e^{it\lambda} dt = \Theta_{\pi_i \lambda}(\psi), \quad (49)$$

where the second equality follows from (27). One computes the Fourier transforms of the functions

$$\frac{\sin(\theta) \cosh(t)}{\cosh(2t) - \cos(2\theta)} \quad \text{and} \quad \frac{\sin(2\theta)}{\cosh(2t) - \cos(2\theta)}$$

to deduce from Lemma (74) and (49) the following theorem. See [Lan85, page 173]

**Theorem 75.**

$$\begin{aligned} \frac{\pi}{i} (H_\phi^K(k(\theta)) - H_\phi^K(k(-\theta))) &= - \sum_{n=1}^{\infty} \Theta_{\mathbf{X}^{(n+1)} + \mathbf{X}^{(-n-1)}}(\phi) \sin(n\theta) \\ &+ \frac{1}{2} \int_0^\infty \Theta_{\pi_+, i\lambda}(\phi) \frac{\cosh((\frac{\pi}{2} - \theta)\lambda)}{\cosh(\frac{\pi\lambda}{2})} d\lambda + \frac{1}{2} \int_0^\infty \Theta_{\pi_-, i\lambda}(\phi) \frac{\sinh((\frac{\pi}{2} - \theta)\lambda)}{\sinh(\frac{\pi\lambda}{2})} d\lambda \end{aligned}$$

Now we take the derivative with respect to  $\theta$  of both sides, go to limit with  $\theta \rightarrow 0$  and apply (45) to deduce the Harish-Chandra's Plancherel formula.

**Theorem 76.** *For any  $\phi \in C_c^\infty(\mathbf{G})$ ,*

$$\begin{aligned} 2\pi\phi(1) &= \sum_{n=1}^{\infty} n \Theta_{\mathbf{X}^{(n+1)} + \mathbf{X}^{(-n-1)}}(\phi) \\ &+ \frac{1}{2} \int_0^\infty \Theta_{\pi_+, i\lambda}(\phi) \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda + \frac{1}{2} \int_0^\infty \Theta_{\pi_-, i\lambda}(\phi) \lambda \coth\left(\frac{\pi\lambda}{2}\right) d\lambda \quad (50) \end{aligned}$$

This Theorem was published by Harish-Chandra in 1952, [HC52]. Notice that knowing the composition series of the principal series was crucial for this proof.

## 11. Harish-Chandra's Plancherel formula for a real reductive group

Here we follow [Wal88b, page 248].

**Theorem 77.** *Let  $\mathbf{G}$  be a real reductive group. Then for  $f \in C_c^\infty(\mathbf{G})$*

$$f(1) = \sum_{(\mathbf{P}, \mathbf{A}) \succ (\mathbf{P}_0, \mathbf{A}_0)} C_{\mathbf{A}} \sum_{\omega \in \mathcal{E}_2({}^0\mathbf{M}_{\mathbf{P}})} d(\omega) \int_{\mathfrak{a}^*} \Theta_{\mathbf{P}, \omega, i\nu}(f) \mu(\omega, i\nu) d\nu, \quad (51)$$

*the summation is over standard parabolic subgroups  $\mathbf{P} = {}^0\mathbf{M}_{\mathbf{P}}\mathbf{A}\mathbf{N}$  containing a fixed minimal Parabolic subgroup  $\mathbf{P}_0 = {}^0\mathbf{M}_{\mathbf{P}_0}\mathbf{A}_0\mathbf{N}_0$  including  $\mathbf{P} = \mathbf{G}$ ,  $\mathcal{E}_2({}^0\mathbf{M}_{\mathbf{P}})$  is the set of the equivalence classes of discrete series representations of  ${}^0\mathbf{M}_{\mathbf{P}}$ ,  $d(\omega)$  is the formal degree of  $\omega$ ,  $\Theta_{\mathbf{P}, \omega, i\nu}$  is the character of the induced representation with parameters  $\omega$  and  $i\nu$ , which is irreducible, and  $\mu(\omega, i\nu)$  is described in [Har76, section 36].*

This theorem was proven by Harish-Chandra in 1976, [Har76]. One basic idea was to replace  $C_c^\infty(\mathbf{G})$  by a larger space  $\mathcal{C}(\mathbf{G})$  where functions decay at infinity sort of like matrix coefficients of discrete series and realize that the character of discrete series applies to them. In fact Harish-Chandra showed, [Har66, Lemma 81, page 93] that if  $\mathcal{C}_\omega(\mathbf{G})$  is

a space spanned by the matrix coefficients of a discrete series representation  $\omega$  then for  $\phi \in \mathcal{C}_{\omega'}(\mathbb{G})$

$$d(\omega)\Theta_{\omega}(\phi) = \begin{cases} \phi(1) & \text{if } \omega' \simeq \omega^*, \\ 0 & \text{otherwise} \end{cases}. \quad (52)$$

## 12. A smooth compactly supported positive definite function whose integral is negative

We refer to [Lan85, pages 62-65] for the definition and basic properties of positive definite functions on a group. By Bochner's theorem, the integral of a positive definite  $L^1$  function on the real line is positive. As we'll see below, this is not true for any non-compact semisimple Lie group.

Let  $\mathbb{G}$  be a real non-compact semisimple Lie group. Fix an Iwasawa decomposition  $\mathbb{G} = \mathbb{K}\mathbb{A}\mathbb{N}$  and let  $W = W(\mathbb{G}, \mathbb{A})$  denote the Weyl group acting on  $\mathbb{A}$  and on the Lie algebra  $\mathfrak{a}$ . This action extends to an action on the complexification  $\mathfrak{a}_{\mathbb{C}}$  by  $\mathbb{C}$ -linearity and dualizes to an action on  $\mathfrak{a}_{\mathbb{C}}^*$ . For  $g \in \mathbb{G}$  define  $H(g) \in \mathfrak{a}$  by  $g \in \mathbb{K} \cdot \exp(H(g)) \cdot \mathbb{N}$ . Let  $\mathfrak{n}$  denote the Lie algebra of  $\mathbb{N}$  and define  $\rho \in \mathfrak{a}^*$  by

$$\rho(H) = \frac{1}{2} \operatorname{tr} \operatorname{ad}(H)|_{\mathfrak{n}} \quad (H \in \mathfrak{a}).$$

In these terms the Haar measure on  $\mathbb{G}$  may be expressed as follows

$$\int_{\mathbb{G}} f(g) dg = \int_{\mathbb{N}} \int_{\mathbb{A}} \int_{\mathbb{K}} a^{2\rho} f(kan) dk da dn.$$

Recall the spherical functions, [GV88, page 104],

$$\phi_{\lambda}(g) = \int_{\mathbb{K}} e^{(\lambda - \rho)(H(gk))} dk \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*).$$

The spherical transform of a function  $f : \mathbb{G} \rightarrow \mathbb{C}$  is defined as

$$\mathcal{H}f(\lambda) = \frac{1}{|W|} \int_{\mathbb{G}} f(g) \phi_{\lambda}(g) dg \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*)$$

whenever the integral is absolutely convergent, [GV88, page 106]. More explicitly, if the function  $f$  is  $\mathbb{K}$ -bi invariant then, [GV88, page 107],

$$\mathcal{H}f(\lambda) = \int_{\mathbb{A}} a^{\lambda} \left[ a^{\rho} \int_{\mathbb{N}} f(an) dn \right] da \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*)$$

and the result is  $W$ -invariant. Moreover, if  $\mathbb{A}$  is a Cartan subgroup of  $\mathbb{G}$ , then the expression in the square brackets, usually called the Abel transform of  $f$ , denoted  $\mathcal{A}f$ , is the Harish-Chandra orbital integral of  $f$  evaluated at  $a$ , [Wal88a, page 249].

Let  $C_c(\mathbb{G}/\mathbb{K})$  and  $\mathcal{C}(\mathbb{G}/\mathbb{K})$  denote the commutative convolution algebras of the  $\mathbb{K}$ -bi invariant and continuous compactly supported and Harish-Chandra Schwartz functions on  $\mathbb{G}$ , respectively. Recall that the convolution of two functions  $f_1$  and  $f_2$  is defined by

$$f_1 * f_2(g) = \int_{\mathbb{G}} f_1(gh^{-1}) f_2(h) dh.$$



Notice by the way that the convolution of  $f$  with  $f^*(g) = \overline{f(g^{-1})}$  is equal to the diagonal matrix coefficient of the left regular representation:

$$f * f^*(g) = \int_{\mathbf{G}} f(gh^{-1}) \overline{f(h^{-1})} dh = (f, L(g)f)_{L^2(\mathbf{G})}.$$

Let  $\mathcal{PW}(\mathfrak{a}_{\mathbb{C}}^*)$  be the Paley-Wiener space of all entire functions  $F : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$  for which there is a constant  $R > 0$  such that

$$|F(\lambda)| \leq C_N (1 + |\lambda|)^{-N} e^{R|\operatorname{Re}(\lambda)|} \quad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*, N = 0, 1, 2, \dots).$$

This is a commutative multiplicative algebra and so is the Schwartz space  $\mathcal{S}(i\mathfrak{a}^*)$ .

The spherical transform is a (bijective) algebra isomorphism

$$\mathcal{H} : C_c(\mathbf{G}/\mathbf{K}) \rightarrow \mathcal{PW}(\mathfrak{a}_{\mathbb{C}}^*)^{\mathbf{W}}$$

which extends to a (bijective) algebra isomorphism

$$\mathcal{H} : \mathcal{C}(\mathbf{G}/\mathbf{K}) \rightarrow \mathcal{S}(i\mathfrak{a}^*)^{\mathbf{W}}.$$

The inverse is given by

$$f(g) = \frac{1}{|\mathbf{W}|} \int_{i\mathfrak{a}^*} \mathcal{H}f(\lambda) \phi_{-\lambda}(g) |c(\lambda)|^{-2} d\lambda,$$

where  $c$  is the Harish-Chandra  $c$ -function and  $d\lambda$  is a Lebesgue measure on the vector space  $i\mathfrak{a}^*$ , see [GV88, Proposition 3.1.4, Proposition 3.3.2 and Theorem 6.4.1]. Recall also that  $\phi_{-\lambda}$  is positive definite for  $\lambda \in i\mathfrak{a}^*$ . Hence  $f \neq 0$  is positive definite if  $\mathcal{H}f$  is non-negative on  $i\mathfrak{a}^*$ .

Furthermore,

$$\mathcal{H}(f^*)(\lambda) = \overline{\mathcal{H}f(-\bar{\lambda})} = \overline{\mathcal{H}f(\bar{\lambda})}$$

for the  $\lambda$  in the domain of  $\mathcal{H}f$ . This is because  $\phi_{\lambda}^*(g) = \phi_{-\bar{\lambda}}(g)$  and  $-\lambda = w_0\lambda$ , where  $w_0$  is the reflection with respect to the longest root.

Recall Harish-Chandra homomorphism, [GV88, Theorem 2.6.7],

$$\gamma : \mathcal{U}(\mathfrak{g})^{\mathbf{K}} \rightarrow \mathcal{U}(\mathfrak{a})^{\mathbf{W}} = \mathbb{C}[\mathfrak{a}_{\mathbb{C}}^*]^{\mathbf{W}}.$$

Then for  $q \in \mathcal{U}(\mathfrak{g})^{\mathbf{K}}$ , acting as a differential operator via the right regular representation,

$$\mathcal{H}(qf)(\lambda) = \gamma(q)(\lambda) \mathcal{H}f(\lambda),$$

see [GV88, Proposition 3.2.1] and [Hel84, Theorem 5.18, page 306].

**Proposition 78.** *Let  $q \in \mathcal{U}(\mathfrak{g})^{\mathbf{K}}$  be such that  $\gamma(q)(\rho) < 0$  and  $\gamma(q)(\lambda) \geq 0$  for all  $\lambda \in i\mathfrak{a}^*$  and  $\gamma(q)(\lambda) > 0$  for some  $\lambda \in i\mathfrak{a}^*$ . Then for any  $f \in C_c(\mathbf{G}/\mathbf{K})$  with a non-zero integral over  $\mathbf{G}$ , the function  $q(f * f^*) \in C_c^\infty(\mathbf{G}/\mathbf{K})$  is positive definite and*

$$\int_{\mathbf{G}} q(f * f^*)(g) dg < 0.$$

*Proof.* Since  $q(f * f^*) = (qf) * f^*$ , this function is in  $C_c^\infty(G//K)$ . Furthermore,

$$\begin{aligned} \int_G q(f * f^*)(g) dg &= \mathcal{H}(q(f * f^*))(\rho) = \gamma(q)(\rho) \mathcal{H}(f * f^*)(\rho) \\ &= \gamma(q)(\rho) \mathcal{H}(f)(\rho) \mathcal{H}(f^*)(\rho) \\ &= \gamma(q)(\rho) |\mathcal{H}(f)(\rho)|^2 \\ &= \gamma(q)(\rho) \left| \int_G f(g) dg \right|^2 < 0. \end{aligned}$$

Since,

$$\mathcal{H}(q(f * f^*))(\lambda) = \gamma(q)(\lambda) |\mathcal{H}f(\lambda)|^2,$$

the inversion formula for the spherical transform implies that the function  $q(f * f^*)$  is positive definite.  $\square$

Let  $B$  be a  $G$ -invariant non-degenerate symmetric bilinear form on the real vector space  $\mathfrak{g}$ , [GV88, page 94], and let  $C_B \in \mathcal{U}(\mathfrak{g})^G$  and  $C_{B,\mathfrak{a}} \in \mathcal{U}(\mathfrak{a})^W$  be the corresponding Casimir elements. Then

$$\gamma(C_B) = C_{B,\mathfrak{a}} - B(\rho, \rho),$$

where we dualize  $B$  from  $\mathfrak{a}$  to  $\mathfrak{a}^*$ , see [GV88, Lemma 2.6.10]. Assuming  $B$  is positive definite on  $\mathfrak{a}^*$ , we see that the polynomial  $C_{B,\mathfrak{a}}$  has negative values on  $i\mathfrak{a}^* \setminus \{0\}$  and is positive on  $\rho$ . Hence,

$$q = -C_B - B(\rho, \rho)$$

satisfies the conditions of Proposition 78.

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