

Boundedness of the Cauchy Harish-Chandra Integral

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Abstract. This is the first of our three articles showing that the Cauchy Harish-Chandra integral maps invariant eigendistributions to invariant eigendistributions with the correct infinitesimal character. Here we prove that all the derivatives of the Cauchy Harish-Chandra integral are bounded.

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Introduction

Let (\tilde{G}, \tilde{G}') be a real reductive dual pair in a metaplectic group $\tilde{\mathrm{Sp}}$. Let Θ be the character of an oscillator representation of $\tilde{\mathrm{Sp}}$. Assume \tilde{G}' is compact and let (π, π') be two irreducible representations of \tilde{G} and \tilde{G}' in Howe's correspondence. In these terms the First Fundamental Theorem of Classical Invariant Theory can be written as the following equality of distributions

$$\int_{\tilde{G}'} \Theta(gg') \overline{\Theta_{\pi'}(g')} dg' = \Theta_{\pi}(g)$$

where Θ_{π} and $\Theta_{\pi'}$ stand for the characters of π and π' . For a smooth compactly supported function ϕ on \tilde{G} , the formula

$$\phi'(g') = \int_{\tilde{G}} \phi(g) \Theta(gg') dg \tag{0.1}$$

defines a smooth compactly supported function on \tilde{G}' and

$$\int_{\tilde{G}'} \overline{\Theta_{\pi'}(g')} \phi'(g') dg' = \int_{\tilde{G}} \Theta_{\pi}(g) \phi(g) dg.$$

The Cauchy Harish-Chandra integral (*Chc*) extends formula (0.1) to all dual pairs with rank of \tilde{G}' less or equal to rank of \tilde{G} . One of the goals of this project is to

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prove that this analogue of formula (0.1) provides a smooth compactly supported function on \widetilde{G}' . A key step in this direction is to consider a similar question on the Lie algebra.

More precisely, let f be a smooth compactly supported function on \mathfrak{g} , the Lie algebra of G . Let H be a Cartan subgroup and \mathfrak{h} be its Lie algebra. The orbital integral of f on \mathfrak{h} is

$$\phi(x) = \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{G/H} f(g.x) d\mu$$

where $x \in \mathfrak{h}^{\text{reg}}$, $\pi_{\mathfrak{g}/\mathfrak{h}}$ is the product of positive roots, $g.x$ is the adjoint action of g on x and $d\mu$ is the canonical measure on the quotient space G/H . The functions obtained this way are characterized (see [3]) by the following four properties

1. ϕ is smooth on $\mathfrak{h}^{\text{reg}}$,
2. all the derivatives of ϕ are locally bounded on \mathfrak{h} (boundedness property),
3. ϕ is not necessarily continuous but satisfies certain jump relations (2.3),
4. the support of ϕ is bounded.

In this paper, we prove that functions obtained via *Chc* satisfy the boundedness property. The forthcoming two papers deal with the remaining problems.

Let (G, G') be one of the following dual pairs of type *I*:

$$\begin{aligned} & (O_{2p+1,2q}, \text{Sp}_{2n}(\mathbb{R})), (\text{Sp}_{2n}(\mathbb{R}), O_{2p+1,2q}), (O_{2p,2q}, \text{Sp}_{2n}(\mathbb{R})), (\text{Sp}_{2n}(\mathbb{R}), O_{2p,2q}), \\ & (U_{p,q}, U_{p',q'}), (O_{2m}^*, \text{Sp}_{p,q}), (\text{Sp}_{p,q}, O_{2m}^*). \end{aligned} \quad (0.2)$$

In particular, G' is the isometry group of a form $(,)'$ on a vector space V' over a division algebra $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . Let $H' \subseteq G'$ be a compact Cartan subgroup, with the Lie algebra \mathfrak{h}' . Let

$$V' = V'_0 \oplus \sum_{j=1}^{n'} V'_j \quad (0.3)$$

be the decomposition into H' -irreducibles over \mathbb{D} . Here $V'_0 = 0$ unless $G' = O_{2p+1,2q}$, in which case H' acts trivially on V'_0 and $\dim(V'_0) = 1$. There is a complex structure J' on V' which belongs to \mathfrak{h}' . More precisely, the restriction of J' to V'_0 is zero and the restriction of J' to $\sum_{j=1}^{n'} V'_j$ is a complex structure. Let $J'_j = J'_j|_{V'_j}$ be the restriction of J' to V'_j . Then, any element $x' \in \mathfrak{h}'$ may be written uniquely as

$$x' = \sum_{j=1}^{n'} x'_j J'_j,$$

where $x'_j \in \mathbb{R}$. In particular n' is the rank of G' . Let n denote the rank of G . We shall always assume that $n' \leq n$. Define the integer p as in [11, (1.12)]. Let $\pi_{\mathfrak{g}'/\mathfrak{h}'}$ be the product of the positive roots of $\mathfrak{h}'_{\mathbb{C}}$ in $\mathfrak{g}'_{\mathbb{C}}$, with respect to some ordering

of the roots. Recall the Cauchy Harish-Chandra integral chc , [11] (see (0.7) and (0.8)).

Let $\mathfrak{h}^{\text{reg}}$ be the set of regular elements in \mathfrak{h}' . We denote by μ a Lebesgue measure on \mathfrak{g} .

Let P be a polynomial function on \mathfrak{h}' . If $\mathbb{D} \neq \mathbb{C}$, we make the following assumption:

$$\begin{aligned} &\text{The degree of } P \text{ in each variable } x'_j, 1 \leq j \leq n', \text{ is less than } \underline{p}, \text{ and} \\ &\text{if } \mathbb{D} \neq \mathbb{C} \text{ then the function } P \text{ is even in each variable } x'_j, 1 \leq j \leq n'. \end{aligned} \tag{0.4}$$

The evenness of P is not needed until the proof of Theorem 1 in section 10.

Theorem 1. *Let $n' \geq 1$ and let P be a polynomial function on \mathfrak{h}' satisfying (0.4). Then, for any constant coefficient differential operator \mathcal{D} on \mathfrak{h}' and for any $\psi \in S(\mathfrak{g})$,*

$$\sup_{x' \in \mathfrak{h}'^{\text{reg}}} \left| \mathcal{D} \left(P(x') \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x' + x) \psi(x) d\mu(x) \right) \right| < \infty.$$

Moreover, the above quantity defines a continuous seminorm on $S(\mathfrak{g})$.

Theorem 1 holds also for the pair $(O_{2p+1,2q+1}, Sp_{2n}(\mathbb{R}))$, with $n' \leq p+q+1$. Since the group $O_{2p+1,2q+1}$ does not have any compact Cartan subgroups the proof requires some additional notation. We explain the argument in section 11.

Since we do not expect the reader to be familiar with the notion of the Cauchy Harish-Chandra integral, we offer a brief explanation and motivation for Theorem 1.

Recall that, given a dual pair (G, G') , there is a real symplectic space W with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ such that the groups G, G' are mutual centralizers in the symplectic group $Sp(W)$. Let $\mathfrak{sp}(W)$ denote the Lie algebra of $Sp(W)$ and let

$$\chi_x(w) = \exp\left(i\frac{\pi}{2}\langle xw, w \rangle\right) \quad (x \in \mathfrak{sp}(W), w \in W).$$

Then, for any $\phi \in S(W)$, the function

$$\mathfrak{sp}(W) \ni x' \rightarrow \int_W \chi_{x'}(w) \phi(w) dw \in \mathbb{C}, \tag{0.5}$$

and all its derivatives, are bounded. In fact, the van der Corput Lemma, [16, page 332] (or the formula for the Fourier transform of a Gaussian, [6, Theorem 7.6.1]), implies a precise description of the behavior of the function (0.5) as x' tends to infinity. This leads to the well known estimates for the matrix coefficients of the oscillator representation, [7, section 8]. (See [10] for more details.)

Suppose the group G' is compact. Then, for any $\psi \in S(\mathfrak{g})$, the formula

$$\phi(w) = \int_{\mathfrak{g}} \chi_x(w) \psi(x) d\mu(x) \quad (w \in W) \tag{0.6}$$

defines a Schwartz function $\phi \in S(\mathbf{W})$, see [9, Section 6]. Thus, since $\chi_{x'+x}(w) = \chi_{x'}(w)\chi_x(w)$, the function

$$\mathfrak{sp}(\mathbf{W}) \ni x' \rightarrow \int_{\mathbf{W}} \int_{\mathfrak{g}} \chi_{x'+x}(w) \psi(x) d\mu(x) dw \in \mathbb{C}, \quad (0.7)$$

and all its derivatives, are bounded (and decay at infinity according to the van der Corput Lemma). Hence, as explained in [11], the Cauchy Harish-Chandra integral maps the character of an irreducible unitary representation of $\tilde{\mathbf{G}}'$ to the character of the corresponding (via Howe's correspondence) irreducible unitary representation of $\tilde{\mathbf{G}}$.

Suppose the group G' is not compact. Since the function ϕ , defined in (0.6), is G' -invariant, it is not integrable (unless $\psi = 0$). Thus the above arguments break down. In particular, the integral over \mathbf{W} in (0.7) does not converge. However, if we take the test function $\psi \in S(\mathfrak{sp}(\mathbf{W}))$ then for every $x' \in \mathfrak{sp}(\mathbf{W})$ the formula (0.6) makes sense and defines a temperate distribution on $\mathfrak{sp}(\mathbf{W})$. Suppose x' is a regular element in a fundamental Cartan subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}'$. Then, as shown in [11, Proposition 1.8], the wave front set of this distribution is disjoint with the conormal bundle of the embedding $\mathfrak{g} \ni x \rightarrow x' + x \in \mathfrak{sp}(\mathbf{W})$. Hence, we may restrict this last distribution to \mathfrak{g} and thus give a meaning to the formula (0.7). This is the Cauchy Harish-Chandra integral (corresponding to the Cartan subalgebra \mathfrak{h}')

$$\mathfrak{h}'^{\text{reg}} \ni x' \rightarrow \int_{\mathbf{W}} \int_{\mathfrak{g}} \chi_{x'+x}(w) \psi(x) d\mu(x) dw = \int_{\mathfrak{g}} chc(x' + x) \psi(x) d\mu(x) \in \mathbb{C}. \quad (0.8)$$

The function (0.8) is not bounded, but it becomes bounded if we multiply it by $\pi_{\mathfrak{g}'/\mathfrak{h}'}$, the product of all the positive roots. This is the essence of Theorem 1, which is stronger than Theorem 1.13 in [11]. Our proof of Theorem 1 is a refinement of the proof of Theorem 1.13 in [11] and is based on Stokes Theorem and a careful study of the jump relations of the Harish-Chandra orbital integrals. Also, in Appendix A we provide a short argument verifying the bounds and the jump relations for the pair $(U_{1,1}, U_{1,1})$ obtained previously in [1].

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1. Preliminaries

Let θ be a Cartan involution on \mathfrak{g} . Let $H \subseteq G$ be a compact Cartan subgroup with the Lie algebra \mathfrak{h} , stable under θ . Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition corresponding to θ . Denote by $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ the set of roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Thus

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}, \alpha},$$

where $\mathfrak{g}_{\mathbb{C}, \alpha}$ is the α -eigenspace for the adjoint action of $\mathfrak{h}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$. Since \mathfrak{h} is θ -stable, so is each $\mathfrak{g}_{\mathbb{C}, \alpha}$. Since $\dim(\mathfrak{g}_{\mathbb{C}, \alpha}) = 1$, we must have either $\mathfrak{g}_{\mathbb{C}, \alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$, or $\mathfrak{g}_{\mathbb{C}, \alpha} \subseteq \mathfrak{p}_{\mathbb{C}}$. The root α is called compact in the first case, and non-compact in the the second case. Let $\Delta^c \subseteq \Delta$ denote the set of all the compact roots, and let $\Delta^n \subseteq \Delta$ denote the set of all non-compact roots. Clearly, Δ is the disjoint union of Δ^c and Δ^n .

For $x \in \mathfrak{g}_{\mathbb{C}}$, let \bar{x} denote the conjugate of x with respect to the real form $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$. As in [13, (2.7)], for each root $\alpha \in \Delta$ we fix elements $H_{\alpha} \in \mathfrak{ih}$, $X_{\alpha} \in \mathfrak{g}_{\mathbb{C},\alpha}$ such that

$$\begin{aligned} [H_{\alpha}, X_{\alpha}] &= 2X_{\alpha}, & [H_{\alpha}, X_{-\alpha}] &= -2X_{-\alpha}, \\ [X_{\alpha}, X_{-\alpha}] &= H_{\alpha}, & \bar{H}_{\alpha} &= -H_{\alpha} = H_{-\alpha}, \\ \bar{X}_{\alpha} &= -X_{-\alpha} & \text{if } \alpha \in \Delta^c, \\ \bar{X}_{\alpha} &= X_{-\alpha} & \text{if } \alpha \in \Delta^n. \end{aligned}$$

Let

$$\begin{aligned} W(\mathbb{H}_{\mathbb{C}}) &= \text{Normalizer}_{\mathbb{G}_{\mathbb{C}}}(\mathfrak{h}_{\mathbb{C}})/\mathbb{H}_{\mathbb{C}}, & \text{and} \\ W(\mathbb{H}) &= \text{Normalizer}_{\mathbb{G}}(\mathfrak{h})/\mathbb{H}. \end{aligned}$$

Clearly, $W(\mathbb{H}_{\mathbb{C}})$ acts on $\mathfrak{h}_{\mathbb{C}}$, and on the dual $\mathfrak{h}_{\mathbb{C}}^*$. Also, as we will see later, one may assume that $W(\mathbb{H}_{\mathbb{C}})$ preserves \mathfrak{h} (and therefore also \mathfrak{h}^*). Clearly, $W(\mathbb{H})$ is a subgroup of $W(\mathbb{H}_{\mathbb{C}})$.

Recall that two roots α and β are called strongly orthogonal ($\alpha \perp \beta$) if

$$\alpha \neq \pm\beta \text{ and } \alpha \pm \beta \notin \Delta.$$

A set of roots is called strongly orthogonal if all the pairs of roots in it are strongly orthogonal.

Fix a positive root system

$$\Psi \subseteq \Delta. \tag{1.1}$$

Let $\Psi^c = \Psi \cap \Delta^c$ and let $\Psi^n = \Psi \cap \Delta^n$. The group $W(\mathbb{H})$ acts on Δ^n and therefore also on the subsets of Δ^n . Let Ψ_{st}^n denote the family of strongly orthogonal subsets of Ψ^n . Let Δ_{st}^n denote the family of all the subset of the form $\mathcal{S} \cup (-\mathcal{S})$, where $\mathcal{S} \in \Psi_{st}^n$. There is a bijection between Δ_{st}^n and Ψ_{st}^n given by

$$\Delta_{st}^n \ni \mathcal{S} \cup (-\mathcal{S}) \rightarrow (\mathcal{S} \cup (-\mathcal{S})) \cap \Psi \in \Psi_{st}^n. \tag{1.2}$$

The inverse of (1.2) is

$$\Psi_{st}^n \ni \mathcal{S} \rightarrow \mathcal{S} \cup (-\mathcal{S}) \in \Delta_{st}^n.$$

For $\mathcal{S} \in \Psi_{st}^n$ let

$$[\mathcal{S}] = (W(\mathbb{H})(\mathcal{S} \cup (-\mathcal{S}))) \cap \Psi. \tag{1.3}$$

Clearly, $[\mathcal{S}] \subseteq \Psi_{st}^n$ and Ψ_{st}^n is a disjoint union of the sets of the form (1.3). Thus, we have an equivalence relation on Ψ_{st}^n , where $[\mathcal{S}]$ is the equivalence class of \mathcal{S} .

Recall that for each $\alpha \in \Psi$, we have the Cayley transform

$$c(\alpha) = \exp(-i\frac{\pi}{4} \text{ad}(X_{\alpha} + X_{-\alpha})) \in \text{End}(\mathfrak{g}_{\mathbb{C}}).$$

For $\mathcal{S} \in \Psi_{st}^n$, let

$$c(\mathcal{S}) = \prod_{\alpha \in \mathcal{S}} c(\alpha) \tag{1.4}$$

and let

$$\mathfrak{h}(\mathcal{S}) = \mathfrak{g} \cap c(\mathcal{S})(\mathfrak{h}_{\mathbb{C}}).$$

This is a Cartan subalgebra of \mathfrak{g} , with the corresponding Cartan subgroup $H(\mathcal{S}) \subseteq G$, and the Weyl group

$$W(H(\mathcal{S})) = \text{Normalizer}_G(\mathfrak{h}(\mathcal{S}))/H(\mathcal{S}). \tag{1.5}$$

For any subset A of Ψ , we put

$$\mathfrak{h}^A = \{x \in \mathfrak{h} \mid \alpha(x) = 0 \text{ for all } \alpha \in A\}.$$

Proposition 1.1. *[13, (2.16)] Every Cartan subalgebra of \mathfrak{g} is conjugate to one of the form $\mathfrak{h}(\mathcal{S})$. Two Cartan subalgebras $\mathfrak{h}(\mathcal{S}), \mathfrak{h}(\mathcal{S}')$ are conjugate if and only if $[\mathcal{S}] = [\mathcal{S}']$. Thus, the conjugacy classes of the Cartan subalgebras in \mathfrak{g} are parametrized by the equivalence classes of the sets of the strongly orthogonal roots in Ψ^n .*

Let

$$\mathfrak{h}_{\mathcal{S}} = c(\mathcal{S})^{-1}(\mathfrak{h}(\mathcal{S})) \subseteq \mathfrak{h}_{\mathbb{C}}. \tag{1.6}$$

Then,

$$\mathfrak{h}_{\mathcal{S}} = \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\mathcal{S}} + \sum_{\alpha \in \mathcal{S}} \mathbb{R}H_{\alpha}. \tag{1.7}$$

Let

$$\begin{aligned} \Delta_{\mathcal{S},i\mathbb{R}} &= \{\alpha \in \Delta \mid \alpha(\mathfrak{h}_{\mathcal{S}}) \subseteq i\mathbb{R}\} = \mathcal{S}^{\perp} \cap \Delta, \\ \Delta_{\mathcal{S},\mathbb{R}} &= \{\alpha \in \Delta \mid \alpha(\mathfrak{h}_{\mathcal{S}}) \subseteq \mathbb{R}\}, \\ \Delta_{\mathcal{S},\mathbb{C}} &= \Delta \setminus (\Delta_{\mathcal{S},i\mathbb{R}} \cup \Delta_{\mathcal{S},\mathbb{R}}). \end{aligned}$$

By the composition with $c(\mathcal{S})$, the elements of $\Delta_{\mathcal{S},i\mathbb{R}}$ correspond to the imaginary roots of $\mathfrak{h}(\mathcal{S})$ in $\mathfrak{g}_{\mathbb{C}}$, the elements of $\Delta_{\mathcal{S},\mathbb{R}}$ to the real roots, and the elements of $\Delta_{\mathcal{S},\mathbb{C}}$ to the complex roots. Let

$$\begin{aligned} \Delta_{\mathcal{S},i\mathbb{R}}^c &= \{\alpha \in \Delta_{\mathcal{S},i\mathbb{R}} \mid c(\mathcal{S})\mathfrak{g}_{\mathbb{C},\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}\}, \\ \Delta_{\mathcal{S},i\mathbb{R}}^n &= \{\alpha \in \Delta_{\mathcal{S},i\mathbb{R}} \mid c(\mathcal{S})\mathfrak{g}_{\mathbb{C},\alpha} \subseteq \mathfrak{p}_{\mathbb{C}}\}. \end{aligned}$$

As above, these are the sets of the compact roots and the non-compact roots for $\mathfrak{h}(\mathcal{S})$. Define the following subsets of the set Ψ of positive roots:

$$\begin{aligned} \Psi_{\mathcal{S},i\mathbb{R}} &= \Psi \cap \Delta_{\mathcal{S},i\mathbb{R}}, \quad \Psi_{\mathcal{S},\mathbb{R}} = \Psi \cap \Delta_{\mathcal{S},\mathbb{R}}, \quad \Psi_{\mathcal{S},\mathbb{C}} = \Psi \cap \Delta_{\mathcal{S},\mathbb{C}}, \\ \Psi_{\mathcal{S},i\mathbb{R}}^c &= \Psi \cap \Delta_{\mathcal{S},i\mathbb{R}}^c, \quad \Psi_{\mathcal{S},i\mathbb{R}}^n = \Psi \cap \Delta_{\mathcal{S},i\mathbb{R}}^n. \end{aligned}$$

Let

$$\begin{aligned} \Psi_{\mathcal{S},i\mathbb{R},st}^n &= \{\alpha \in \Psi_{\mathcal{S},i\mathbb{R}}^n \mid \alpha \perp \mathcal{S}\}, \\ \Psi_{\mathcal{S},i\mathbb{R},nst}^n &= \{\alpha \in \Psi_{\mathcal{S},i\mathbb{R}}^n \mid \alpha \not\perp \mathcal{S}\}, \end{aligned}$$

where $\alpha \not\perp \mathcal{S}$ means that there exists $\beta \in \mathcal{S}$ such that α and β are not strongly orthogonal.

Clearly, $\Psi_{\mathcal{S},\mathbb{R}}^n$ is the disjoint union of $\Psi_{\mathcal{S},\mathbb{R},st}^n$ and $\Psi_{\mathcal{S},\mathbb{R},nst}^n$. Recall, [13, (2.61)], that

$$\Psi_{\mathcal{S},\mathbb{R}}^n = (\Psi^n \cap \mathcal{S}^\perp) \cup ((\Psi^c \cap \mathcal{S}^\perp) \setminus \mathcal{S}^\perp).$$

Lemma 1.2. *For every $\alpha \in \Psi_{\mathcal{S},\mathbb{R},nst}^n$ there is exactly one $\alpha' \in \mathcal{S}$ such that $\alpha' \not\perp \alpha$. Suppose $G \neq O_{2p+1,2q}$. If the set $\Psi_{\mathcal{S},\mathbb{R},nst}^n$ is not empty, then G is a real symplectic group. Moreover,*

$$\Psi_{\mathcal{S},\mathbb{R},nst}^n = \{\alpha \in \Psi^c \mid \alpha \perp \mathcal{S} \text{ and there is exactly one } \alpha' \in \mathcal{S} \text{ such that } \alpha + \alpha' \in \Psi^n\}.$$

Proof. The first part follows from [13, (2.61)]. Let $G \neq O_{2p+1,2q}$ and let $\alpha \in \Psi_{\mathcal{S},\mathbb{R},nst}^n$. Then α is orthogonal to α' and $\alpha \pm \alpha'$ are roots. Therefore, both α and α' are short roots, and $\alpha \pm \alpha'$ are long roots. This can happen if and only if $\mathfrak{g}_{\mathbb{C}}$ is a complex symplectic Lie algebra (see (0.2)). Thus $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{R})$ or $\mathfrak{sp}_{p,q}$. By inspection, we see that $\Psi_{\mathcal{S},\mathbb{R},nst}^n = \emptyset$ in the second case. Thus $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{R})$ and therefore the long roots $\alpha \pm \alpha'$ are not compact. Clearly, $\alpha + \alpha' \in \Psi$. Since $[\mathfrak{p}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}] \subseteq \mathfrak{k}_{\mathbb{C}}$, and since $\mathfrak{g}_{\mathbb{C},\alpha'} \subseteq \mathfrak{p}_{\mathbb{C}}$, we must have $\mathfrak{g}_{\mathbb{C},\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$. Thus $\alpha \in \Psi^c$. ■

Definition 1.3. Let $\mathcal{S} \in \Psi_{st}^n$ and $\alpha \in \Psi_{\mathcal{S},\mathbb{R}}^n$. Let

$$\mathcal{S} \vee \alpha = \begin{cases} (\mathcal{S} \setminus \alpha') \cup \{\alpha + \alpha'\} \cup \{\pm(\alpha - \alpha')\} \cap \Psi & \text{if } \alpha \in \Psi_{\mathcal{S},\mathbb{R},nst}^n, \\ \mathcal{S} \cup \alpha & \text{if } \alpha \in \Psi_{\mathcal{S},\mathbb{R},st}^n. \end{cases}$$

Here, and in the rest of this paper, $\mathcal{S} \setminus \alpha' = \mathcal{S} \setminus \{\alpha'\}$, $\mathcal{S} \cup \alpha = \mathcal{S} \cup \{\alpha\}$ and α^\perp is the orthogonal complement to α .

Lemma 1.4. [8, Proposition 6.72] *Let $\mathcal{S} \in \Psi_{st}^n$ and $\alpha \in \Psi_{\mathcal{S},\mathbb{R}}^n$. We have $\mathcal{S} \vee \alpha \in \Psi_{st}^n$ and*

$$\Psi_{\mathcal{S} \vee \alpha, \mathbb{R}}^n = \Psi_{\mathcal{S}, \mathbb{R}}^n \cap \alpha^\perp \cup \{\beta \in \Psi_{\mathcal{S}, \mathbb{R}}^c \mid \beta \perp \alpha \text{ and } \beta \not\perp \alpha\}.$$

Let $\mathcal{S} \in \Psi_{st}^n$ and let $\alpha \in \Psi_{\mathcal{S},\mathbb{R}}^n$. The Cayley transform mapping $\mathfrak{h}(\mathcal{S})_{\mathbb{C}}$ to $\mathfrak{h}(\mathcal{S} \vee \alpha)_{\mathbb{C}}$, as in [3] is equal to

$$\exp(-i\frac{\pi}{4}(c(\mathcal{S})X_\alpha + c(\mathcal{S})X_{-\alpha})) = c(\mathcal{S})c(\alpha)c(\mathcal{S})^{-1}. \tag{1.8}$$

Indeed, if $\alpha \perp \mathcal{S}$ then $c(\mathcal{S})X_{\pm\alpha} = X_{\pm\alpha}$ and (1.8) coincides with $c(\alpha)$. If $\alpha \not\perp \mathcal{S}$, then $\alpha \in \Psi^c$ and

$$\overline{c(\mathcal{S})X_\alpha} = \overline{c(\mathcal{S})} \overline{X_\alpha} = c(\mathcal{S})^{-1}(-X_\alpha) = -c(\mathcal{S})^3 X_{-\alpha} = -c(\mathcal{S})c(\mathcal{S})^2 X_{-\alpha} = c(\mathcal{S})X_{-\alpha},$$

where the last equation follows from the proof of Lemma 2.61 in [13].

Define

$$u(\mathcal{S}, \alpha) = c(\mathcal{S} \vee \alpha)^{-1}(c(\mathcal{S})c(\alpha)c(\mathcal{S})^{-1})c(\mathcal{S}) = c(\mathcal{S} \vee \alpha)^{-1}c(\mathcal{S})c(\alpha).$$

This is the isomorphism of $\mathfrak{h}_{\mathbb{C}}$ obtained via the composition of (1.8) with $c(\mathcal{S} \vee \alpha)^{-1}$ on the left and $c(\mathcal{S})$ on the right. Clearly, $u(\mathcal{S}, \alpha) = 1$ if $\alpha \in \Psi_{\mathcal{S}, i\mathbb{R}, st}^n$. Otherwise, $u(\mathcal{S}, \alpha)$ is not trivial.

In terms of (1.7) define a linear map $L = L_{\mathcal{S}} : \mathfrak{h}_{\mathcal{S}} \rightarrow \mathfrak{h}_{\mathcal{S}}$ by

$$L_{\mathcal{S}}(x) = \begin{cases} -x & \text{if } x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\mathcal{S}}, \\ x & \text{if } x \in \bigoplus_{\alpha \in \mathcal{S}} \mathbb{R}H_{\alpha}. \end{cases}$$

Then, for $\alpha \in \Delta$,

$$\begin{aligned} \alpha \in \Delta_{\mathcal{S}, i\mathbb{R}} & \text{ if and only if } & \alpha \circ L_{\mathcal{S}} = -\alpha, \\ \alpha \in \Delta_{\mathcal{S}, \mathbb{R}} & \text{ if and only if } & \alpha \circ L_{\mathcal{S}} = \alpha, \\ \alpha \in \Delta_{\mathcal{S}, \mathbb{C}} & \text{ if and only if } & \alpha \circ L_{\mathcal{S}} \neq -\alpha \text{ and } \alpha \circ L_{\mathcal{S}} \neq \alpha. \end{aligned}$$

We shall write $L\alpha$ instead of $\alpha \circ L$ for convenience.

Lemma 1.5. *Let $C = (-1)^{\frac{1}{2}|(L_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}}) \cap (-\Psi_{\mathcal{S}, \mathbb{C}})|}$. Then, for $x \in \mathfrak{h}_{\mathcal{S}}$,*

$$\prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{C}}} \alpha(x) = C \prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{C}}} |\alpha(x)|, \tag{1.9}$$

$$\prod_{\alpha \in \Psi} \alpha(x) = C \prod_{\alpha \in \Psi_{\mathcal{S}, i\mathbb{R}}} \frac{\alpha(x)}{|\alpha(x)|} \prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}} \frac{\alpha(x)}{|\alpha(x)|} \prod_{\alpha \in \Psi} |\alpha(x)|, \tag{1.10}$$

$$\prod_{\alpha \in \Psi} \overline{\alpha(x)} = (-1)^{|\Psi_{\mathcal{S}, \mathbb{R}}|} \prod_{\alpha \in \Psi} (-\alpha(x)). \tag{1.11}$$

Proof. The equality (1.9) is verified in the proof of Lemma 9 in [5]. Clearly (1.10) follows from (1.9). Notice that

$$(-\Psi)_{-\mathcal{S}, \mathbb{C}} = -\Psi_{\mathcal{S}, \mathbb{C}}.$$

Hence,

$$\begin{aligned} L_{\mathcal{S}}((-\Psi)_{-\mathcal{S}, \mathbb{C}}) \cap (-(-\Psi)_{-\mathcal{S}, \mathbb{C}}) &= (L_{\mathcal{S}}(-\Psi)_{-\mathcal{S}, \mathbb{C}}) \cap \Psi_{\mathcal{S}, \mathbb{C}} \\ &= (-L_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}}) \cap \Psi_{\mathcal{S}, \mathbb{C}} = -((L_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}}) \cap (-\Psi_{\mathcal{S}, \mathbb{C}})). \end{aligned}$$

Thus

$$|L_{\mathcal{S}}((-\Psi)_{-\mathcal{S}, \mathbb{C}}) \cap (-(-\Psi)_{-\mathcal{S}, \mathbb{C}})| = |(L_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}}) \cap (-\Psi_{\mathcal{S}, \mathbb{C}})|,$$

and therefore (1.10) implies (1.11). ■

Definition 1.6. For a subset $A \subseteq \Delta$ define the support of A

$$\underline{A} = \{j = 1, 2, 3, \dots, n \mid \text{there is } \alpha \in A \text{ such that } \alpha(J_j) \neq 0\}.$$

Lemma 1.7. For any subset $A \subseteq \Delta$, let

$$\mathcal{A}(A) = \prod_{\alpha \in A} \frac{\alpha}{|\alpha|}.$$

For $\beta \in \Psi_{\mathcal{S}, \mathbb{i}\mathbb{R}}^n$, let $\epsilon(\Psi, \mathcal{S}, \beta) = (-1)^{c_1+c_2+c_3}$, where

$$c_1 = \frac{1}{2}|L_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}} \cap (-\Psi_{\mathcal{S}, \mathbb{C}})|, \quad c_2 = \frac{1}{2}|L_{S\vee\beta}\Psi_{S\vee\beta, \mathbb{C}} \cap (-\Psi_{S\vee\beta, \mathbb{C}})| \quad \text{and}$$

$$c_3 = |\{\alpha \in \Psi_{\mathcal{S}, \mathbb{i}\mathbb{R}}^n \mid \alpha(H_\beta) < 0\}|.$$

Let $x \in \mathfrak{h}_{\mathcal{S}}$ be semiregular, and such that $\beta(x) = 0$. Let $t \in \mathbb{R}$. Assume that

$$\mathcal{A}(\Psi_{S\vee\beta, \mathbb{R}} \setminus \beta)(x + tH_\beta) = \mathcal{A}(\Psi_{S\vee\beta, \mathbb{R}} \setminus \beta)(x). \tag{1.12}$$

Then

$$\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}})(x) \frac{\beta(x + tH_\beta)}{|\beta(x + tH_\beta)|} \epsilon(\Psi, \mathcal{S}, \beta) = \mathcal{A}(\Psi_{S\vee\beta, \mathbb{R}})(x + tH_\beta). \tag{1.13}$$

The assumption (1.12) is satisfied if

$$\frac{\alpha(x + tH_\beta)}{|\alpha(x + tH_\beta)|} = \frac{\alpha(x)}{|\alpha(x)|} \quad (\alpha \in \Psi_{S\vee\beta, \mathbb{R}} \setminus \beta). \tag{1.14}$$

The condition (1.14) holds if $\mathbb{D} = \mathbb{C}$, $\mathbb{D} = \mathbb{H}$, or if $\mathbb{D} = \mathbb{R}$, $|\beta| = 2$ and $\underline{\mathcal{S}} \cap \underline{\beta} = \emptyset$. If $\mathbb{D} = \mathbb{R}$ and $|\beta| = 1$, then (1.14) holds if

$$|\beta(\tilde{x})| \leq |\gamma(\tilde{x})| \quad \text{for all } \gamma \in \Psi_{\mathcal{S}, \mathbb{R}}, \text{ with } |\underline{\gamma}| = 1, \tag{1.15}$$

where $\tilde{x} = x + tH_\beta$.

Also, $\epsilon(-\Psi, -\mathcal{S}, -\beta) = \epsilon(\Psi, \mathcal{S}, \beta)$, and replacing $(\Psi, \mathcal{S}, \beta)$ by $(-\Psi, -\mathcal{S}, -\beta)$ and t by $-t$ in (1.13), we obtain

$$\mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}})(x) \frac{\beta(x + tH_\beta)}{|\beta(x + tH_\beta)|} \epsilon(\Psi, \mathcal{S}, \beta) = -\mathcal{A}(-\Psi_{S\vee\beta, \mathbb{R}})(x + tH_\beta). \tag{1.16}$$

Proof. Let $\Psi(\beta) = \{\alpha \in \Delta \mid \alpha(H_\beta) > 0, \text{ or } \alpha(H_\beta) = 0 \text{ and } \alpha \in \Psi\}$. This is a positive root system adopted to β (as defined in [3, section 3.1]). In particular,

$$\Psi_{S\vee\beta, \mathbb{i}\mathbb{R}} = \Psi(\beta)_{\mathcal{S}, \mathbb{i}\mathbb{R}} \cap \beta^\perp.$$

Dividing the right hand side of (1.13) by the left hand side of (1.13), evaluating at x and using the formulas (1.9) and (1.10), we get the following expression:

$$\begin{aligned} (-1)^{c_3} \frac{|\beta|}{\beta} \frac{(-1)^{c_2} \prod_{\alpha \in \Psi_{S\vee\beta, \mathbb{R}}} \frac{\alpha}{|\alpha|}}{(-1)^{c_1} \prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}} \frac{\alpha}{|\alpha|}} &= (-1)^{c_3} \frac{|\beta|}{\beta} \frac{\prod_{\alpha \in \Psi_{S\vee\beta, \mathbb{R}} \cup \Psi_{S\vee\beta, \mathbb{C}}} \frac{\alpha}{|\alpha|}}{\prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}} \cup \Psi_{\mathcal{S}, \mathbb{C}}} \frac{\alpha}{|\alpha|}} \\ &= (-1)^{c_3} \frac{|\beta|}{\beta} \frac{\prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}} \frac{\alpha}{|\alpha|}}{\prod_{\alpha \in \Psi_{S\vee\beta, \mathbb{i}\mathbb{R}}} \frac{\alpha}{|\alpha|}} \\ &= \frac{|\beta|}{\beta} \frac{\prod_{\alpha \in \Psi(\beta)_{\mathcal{S}, \mathbb{i}\mathbb{R}}} \frac{\alpha}{|\alpha|}}{\prod_{\alpha \in \Psi(\beta)_{\mathcal{S}, \mathbb{i}\mathbb{R}} \cap \beta^\perp} \frac{\alpha}{|\alpha|}} = \prod_{\alpha \in \Delta_{\mathcal{S}, \beta}} \frac{\alpha}{|\alpha|}, \end{aligned}$$

where $\Delta_{\mathcal{S}, \beta} = \{\alpha \in \Delta \mid \alpha(H_\beta) > 0, \alpha \perp \mathcal{S} \text{ and } \alpha \neq \beta\}$. Let s_β denote the reflection with respect to β . The set $\Delta_{\mathcal{S}, \beta}$ is closed under the operation

$$\alpha \rightarrow -s_\beta(\alpha),$$

which has no fixed points in this set. Indeed,

$$-s_\beta(\alpha) = -\alpha + 2\frac{\alpha(H_\beta)}{\beta(H_\beta)}\beta = -\alpha + \alpha(H_\beta)\beta.$$

Hence, $\alpha \perp \mathcal{S}$ implies $-s_\beta(\alpha) \perp \mathcal{S}$. Also,

$$-s_\beta(\alpha)(H_\beta) = \alpha(H_\beta) > 0.$$

If $\alpha = -s_\beta(\alpha)$, then $\alpha = -\alpha + \alpha(H_\beta)\beta$, which implies that α is proportional to β , a contradiction.

Thus, there is a subset $\mathcal{Z} \subseteq \Delta_{\mathcal{S},\beta}$ such that

$$\Delta_{\mathcal{S},\beta} = \mathcal{Z} \cup (-s_\beta(\mathcal{Z})), \text{ and } \mathcal{Z} \cap (-s_\beta(\mathcal{Z})) = \emptyset.$$

Therefore,

$$\prod_{\alpha \in \Delta_{\mathcal{S},\beta}} \frac{\alpha}{|\alpha|} = \prod_{\alpha \in \mathcal{Z}} \left(\frac{\alpha}{|\alpha|} \frac{-s_\beta(\alpha)}{|-s_\beta(\alpha)|} \right).$$

Notice that for $\alpha \in \mathcal{Z}$,

$$\alpha(x)(-s_\beta(\alpha)(x)) = \alpha(x)(-\alpha(x)) = |\alpha(x)|^2,$$

because $\alpha(x) \in i\mathbb{R}$. Hence,

$$\prod_{\alpha \in \Delta_{\mathcal{S},\beta}} \frac{\alpha}{|\alpha|} = 1.$$

This verifies (1.13). The rest is easy. ■

Definition 1.8. Let $\beta \in \Psi_{\mathcal{S},i\mathbb{R}}^n$ and let $x \in \mathfrak{h}_{\mathcal{S}}$ be such that $\beta(x) = 0$, but $\alpha(x) \neq 0$ for all $\alpha \in \Psi_{\mathcal{S},i\mathbb{R}}^n \setminus \beta$. We say that x is semiregular with respect to β . Let

$$d(\beta) = \begin{cases} 1 & \text{if } s_\beta \notin \mathbf{W}(\mathfrak{H}_{\mathcal{S}}), \\ 2 & \text{otherwise.} \end{cases}$$

If the Lie algebras $\mathfrak{so}_{2p,2q}$ and $\mathfrak{sp}_{2n}(\mathbb{R})$, have the same rank ($n = p + q$), we may assume that they have a common fundamental Cartan subalgebra \mathfrak{h} , so that

$$\Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{so}_{2p,2q,\mathbb{C}}) \subseteq \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{sp}_{2n}(\mathbb{R})_{\mathbb{C}}).$$

Then it may happen that two strongly orthogonal roots $\alpha, \beta \in \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{so}_{2p,2q,\mathbb{C}})$ are not strongly orthogonal in $\Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{sp}_{2n}(\mathbb{R})_{\mathbb{C}})$.

Lemma 1.9. *Suppose $G \neq O_{2p+1,2q}$. Then*

$$d(\beta) = \begin{cases} 2 & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \beta \text{ is not strongly orthogonal to } \mathcal{S} \text{ in } \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{sp}_{2n}(\mathbb{R})_{\mathbb{C}}), \\ 1 & \text{otherwise.} \end{cases}$$

If $G = O_{2p+1,2q}$ then

$$d(\beta) = \begin{cases} 1 & \text{if } \beta \text{ is a long root and } \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If G is connected then we have the result by Theorem 4.19 of [2]. Otherwise, an explicit computation can be done. The result follows. \blacksquare

Let

$$W(\mathfrak{H}_{\mathcal{S}}) = c(\mathcal{S})^{-1}W(\mathfrak{H}(\mathcal{S}))c(\mathcal{S}).$$

This is a subgroup of $W(\mathfrak{H}_{\mathbb{C}})$, isomorphic to the Weyl group $W(\mathfrak{H}(\mathcal{S}))$ defined in (1.5). Let $W(\Delta_{\mathcal{S},\mathbb{R}}) \subseteq W(\mathfrak{H}_{\mathbb{C}})$ be the subgroup generated by the reflections with respect to the elements of $\Delta_{\mathcal{S},\mathbb{R}}$. According to [13, (2.36)], we have

$$W(\mathfrak{H}_{\mathcal{S}}) = \text{Stab}_{W(\mathfrak{H})}(\mathcal{S} \cup (-\mathcal{S}))W(\Delta_{\mathcal{S},\mathbb{R}}).$$

2. Orbital integrals

Let κ be a Killing form on \mathfrak{g} . Then

$$\tilde{\kappa}(x, y) = -\kappa(\theta x, y) \quad (x, y \in \mathfrak{g}),$$

is a positive definite symmetric form on \mathfrak{g} . We shall normalize the Lebesgue measure μ on \mathfrak{g} , so that the volume of the unit cube (with respect to $\tilde{\kappa}$) is 1.

For any unimodular Lie subgroup $E \subseteq G$, the measure μ induces the left invariant Haar measure on E and a left invariant measure on the quotient G/E . We shall denote these induced measures also by μ .

For $\psi \in \mathcal{S}(\mathfrak{g})$ and $x \in \mathfrak{h}_{\mathcal{S}} \setminus \mathfrak{h}^{\Psi}$, define

$$\begin{aligned} \psi_{\mathcal{S}}(x) &= \prod_{\alpha \in \Psi} \alpha(x) \int_{G/H(\mathcal{S})} \psi(g \cdot c(\mathcal{S})(x)) d\mu(gH(\mathcal{S})), \\ \psi^{\mathcal{S}}(x) &= \prod_{\alpha \in \Psi} |\alpha(x)| \int_{G/H(\mathcal{S})} \psi(g \cdot c(\mathcal{S})(x)) d\mu(gH(\mathcal{S})), \end{aligned}$$

where $g \cdot c(\mathcal{S}) = gc(\mathcal{S})g^{-1}$. Let

$$\mathcal{H}_{\mathcal{S}}\psi = \mathcal{A}(\Psi_{\mathcal{S},\mathbb{R}})\psi_{\mathcal{S}}. \tag{2.1}$$

This is the pull-back of the Harish-Chandra orbital integral of ψ , from $\mathfrak{h}(\mathcal{S})$ to $\mathfrak{h}_{\mathcal{S}}$ via $c(\mathcal{S})$. For $\beta \in \Psi_{\mathcal{S},\mathbb{R}}^n$, x a semiregular element of $\mathfrak{h}_{\mathcal{S}}$ with respect to β , \mathcal{U} a neighborhood of x in $\mathfrak{h}_{\mathcal{S}} \setminus \mathfrak{h}^{\beta}$ and a function $f : \mathcal{U} \rightarrow \mathbb{C}$, define

$$\langle f \rangle_{\beta}(x) = \langle f \rangle_{H_{\beta}}(x) = \lim_{t \rightarrow 0^+} f(x + itH_{\beta}) - \lim_{t \rightarrow 0^+} f(x - itH_{\beta}), \tag{2.2}$$

whenever these limits exist.

Theorem 2.1. (Harish-Chandra,[14],[3, Section 3.1]) Let $\beta \in \Psi_{\mathcal{S},\mathbb{R}}^n$ and $x \in \mathfrak{h}_{\mathcal{S}}$ semiregular with respect to β . Then, for any $w \in \text{Sym}(\mathfrak{h}_{\mathbb{C}})$, the symmetric algebra of $\mathfrak{h}_{\mathbb{C}}$,

$$\langle \partial(w)\mathcal{H}_{\mathcal{S}}\psi \rangle_{\beta}(x) = \epsilon(\Psi, \mathcal{S}, \beta)\text{id}(\beta)\partial(u(\mathcal{S}, \beta)(w))\mathcal{H}_{\mathcal{S} \vee \beta}\psi(x). \tag{2.3}$$

Moreover, the function $\mathcal{H}_{\mathcal{S}}\psi$ is smooth on the set $\mathfrak{h}_{\mathcal{S}} \setminus \mathfrak{h}^{\Psi_{\mathcal{S},\mathbb{R}}^n}$. The above relation is called the “jump relation”.

Here $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathcal{S}} \oplus i\mathfrak{h}_{\mathcal{S}}$ on the left hand side and $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathcal{S} \vee \beta} \oplus i\mathfrak{h}_{\mathcal{S} \vee \beta}$ on the right hand side. In particular, if $\beta \in \Psi_{\mathcal{S}, i\mathbb{R}, st}^n$, then for $w = iH_{\beta}$, $\partial(w) = \partial(iH_{\beta})$ on the left hand side and $\partial(w) = i\partial(H_{\beta})$ on the right hand side.

Proof. Let $\Psi(\beta) = \{\alpha \in \Delta \mid \alpha(H_{\beta}) > 0, \text{ or } \alpha(H_{\beta}) = 0 \text{ and } \alpha \in \Psi\}$. This is a positive root system adapted to β . Therefore,

$$\langle \partial(w)(\mathcal{A}(\Psi(\beta)_{\mathcal{S}, i\mathbb{R}})\psi^{\mathcal{S}}) \rangle_{\beta}(x) = id(\beta)\partial(u(\mathcal{S}, \beta)(w))(\mathcal{A}(\Psi(\beta)_{\mathcal{S}, i\mathbb{R}} \cap \beta^{\perp})\psi^{S \vee \beta})(x),$$

(see [3, (ii) of Section 3.1]). Notice that

$$\Psi(\beta)_{\mathcal{S}, i\mathbb{R}} \cap \beta^{\perp} = \Psi_{\mathcal{S} \vee \beta, i\mathbb{R}} \quad \text{and} \quad \Psi(\beta)_{\mathcal{S}, \mathbb{R}} = \Psi_{\mathcal{S}, \mathbb{R}}.$$

Furthermore, by (1.10),

$$\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}})\psi_{\mathcal{S}} = (-1)^{\frac{1}{2}|\eta_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}} \cap (-\Psi_{\mathcal{S}, \mathbb{C}})|} \mathcal{A}(\Psi_{\mathcal{S}, i\mathbb{R}})\psi^{\mathcal{S}}$$

and the same with the \mathcal{S} replaced by $\mathcal{S} \vee \beta$. Also, it is easy to check that

$$\mathcal{A}(\Psi_{\mathcal{S}, i\mathbb{R}}) = (-1)^{|\{\alpha \in \Psi_{\mathcal{S}, i\mathbb{R}} \mid \alpha(H_{\beta}) < 0\}|} \mathcal{A}(\Psi(\beta)_{\mathcal{S}, i\mathbb{R}}).$$

Hence, the left hand side of (2.3) is equal to

$$\begin{aligned} & (-1)^{\frac{1}{2}|\eta_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}} \cap (-\Psi_{\mathcal{S}, \mathbb{C}})|} \langle \partial(w)(\mathcal{A}(\Psi_{\mathcal{S}, i\mathbb{R}})\psi^{\mathcal{S}}) \rangle_{\beta}(x) \\ &= (-1)^{\frac{1}{2}|\eta_{\mathcal{S}}\Psi_{\mathcal{S}, \mathbb{C}} \cap (-\Psi_{\mathcal{S}, \mathbb{C}})| + |\{\alpha \in \Psi_{\mathcal{S}, i\mathbb{R}} \mid \alpha(H_{\beta}) < 0\}|} id(\beta)\partial(u(\mathcal{S}, \beta)(w))(\mathcal{A}(\Psi_{\mathcal{S} \vee \beta, i\mathbb{R}})\psi^{S \vee \beta})(x), \end{aligned}$$

which coincides with the right hand side. ■

Corollary 2.2. For $N \in \mathbb{N}$ (the set of non negative integers), set

$$e_N(u) = \sum_{m=0}^N \frac{1}{m!} u^m \quad (u \in \mathfrak{h}_{\mathcal{S}, \mathbb{C}}).$$

For $\psi \in S(\mathfrak{g})$, $x \in \mathfrak{h}_{\mathcal{S}} \setminus \mathfrak{h}^{\Psi_{\mathcal{S}, i\mathbb{R}}^n}$, $y \in \mathfrak{h}_{\mathcal{S}}$ and $N \in \mathbb{N}$, let

$$(\mathcal{H}_{\mathcal{S}}\psi)_N(x + iy) = e_N(iy)(\mathcal{H}_{\mathcal{S}}\psi)(x).$$

Let $\beta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$ and let $x \in \mathfrak{h}_{\mathcal{S}}$ be such that $\beta(x) = 0$, but $\alpha(x) \neq 0$ for all $\alpha \in \Psi_{\mathcal{S}, i\mathbb{R}} \setminus \beta$. Then

$$\langle (\mathcal{H}_{\mathcal{S}}\psi)_N \rangle_{\beta}(x + iy) = \epsilon(\Psi, \mathcal{S}, \beta) id(\beta)\partial(e_N(iy))\mathcal{H}_{\mathcal{S} \vee \beta}\psi(x) \quad (y = \sum_{i \notin \mathcal{S}} y_j J_j \in \mathfrak{h}_{\mathcal{S}}).$$

Proof. We need to check that

$$u(\mathcal{S}, \beta)y = y. \tag{2.4}$$

This is clear if $\beta \perp \mathcal{S}$. If $\beta \not\perp \mathcal{S}$, then either $G = \text{Sp}_{2n}(\mathbb{R})$ (see Lemma 1.2) or $G = \text{O}_{2p+1, 2q}$. In the first case $\underline{\beta} \subseteq \underline{\mathcal{S}}$, which easily implies (2.4). In the second

case we may assume that $G = O_{3,2}$, $\mathcal{S} = \{iJ_2^*\}$, $\beta = iJ_1^*$ and $y = y_1J_1$. Then $\mathcal{S} \vee \beta = \{iJ_1^* + iJ_2^*, iJ_1^* - iJ_2^*\}$ and

$$u(\mathcal{S}, \beta)y = c(iJ_1^* + iJ_2^*)^{-1}c(iJ_1^* - iJ_2^*)^{-1}c(iJ_2^*)c(iJ_1^*).$$

By a straightforward computation one checks that

$$u(\mathcal{S}, \beta)J_1 = J_1,$$

thus (2.4) follows. ■

Lemma 2.3. [5, Lemma 11] Define a character ϵ_0 of the group $W(H_S)$ by

$$\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}})(s \cdot x) \prod_{\alpha \in \Psi} \alpha(s \cdot x) = \epsilon_0(s)\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}})(x) \prod_{\alpha \in \Psi} \alpha(x) \quad (s \in W(H_S), x \in \mathfrak{h}_S \setminus \mathfrak{h}^\Psi).$$

Then, ϵ_0 is trivial on the subgroup $W(\Delta_{\mathcal{S}, \mathbb{R}})$, and for $\psi \in S(\mathfrak{g})$,

$$\mathcal{H}_S\psi(s \cdot x) = \epsilon_0(s)\mathcal{H}_S\psi(x) \quad (s \in W(H_S), x \in \mathfrak{h}_S).$$

In particular, the Harish-Chandra orbital integral, $\mathcal{H}_S\psi$, is $W(\Delta_{\mathcal{S}, \mathbb{R}})$ -invariant. Furthermore, if $s \in W(H_S)$ is the reflection with respect to an imaginary root, then $\epsilon_0(s) = -1$.

3. The Weyl Groups

Let $(V, (\cdot, \cdot))$ be the defining module for G . Thus G is the isometry group of the form (\cdot, \cdot) on the vector space V over \mathbb{D} . Let $H \subseteq G$ be a compact Cartan subgroup, with the Lie algebra \mathfrak{h} . Let

$$V = V_0 \oplus \sum_{j=1}^n V_j \tag{3.1}$$

be the decomposition of V into H -irreducibles over \mathbb{D} . Here $V_0 = 0$ unless $G = O_{2p+1, 2q}$, in which case $\dim(V_0) = 1$ and H acts trivially on V_0 . Recall the decomposition (0.3). We may and shall assume that

$$V_j = V'_j \quad (1 \leq j \leq n'). \tag{3.2}$$

There is a complex structure J on V , which belongs to \mathfrak{h} . (More precisely, the restriction of J to V_0 is zero and the restriction of J to $\sum_{j=1}^n V_j$ is a complex structure.) We may assume

$$J|_{V'_j} = J'|_{V'_j} \quad (1 \leq j \leq n'). \tag{3.3}$$

Furthermore, we have the following orthogonal direct sum of symplectic spaces

$$W = \text{Hom}(V, V') = \sum_{j=0}^n \sum_{j'=0}^{n'} \text{Hom}(V_j, V'_{j'}).$$

Let $J_j = J|_{V_j}$, $1 \leq j \leq n$.

The following lemma is well known and easy to check.

Lemma 3.1. For $1 \leq j \leq n$ and $1 \leq j' \leq n$, there are invertible elements $\eta_{j',j} \in \text{Hom}(\mathbf{V}_j, \mathbf{V}_{j'})^J$ such that

$$\eta_{j',j} \eta_{j,j'} = 1, \tag{3.4}$$

and, for $u, v \in \mathbf{V}_j$,

$$(\eta_{j',j} u, \eta_{j',j} v) = \begin{cases} (u, v) & \text{if } G = \text{Sp}_{2n}(\mathbb{R}) \text{ or } \text{O}_{2n}^*, \\ \frac{\text{sgn}(\cdot, \cdot)_{\mathbf{V}_{j'}}}{\text{sgn}(\cdot, \cdot)_{\mathbf{V}_j}}(u, v) & \text{if } G = \text{O}_{2p,2q}, \text{O}_{2p+1,2q}, \text{U}_{p,q} \text{ or } \text{Sp}_{p,q}. \end{cases}$$

Let Σ_n denote the permutation group on n letters. Define an embedding

$$\Sigma_n \rightarrow \text{GL}(\mathbf{V}), \quad \sigma : \mathbf{V}_j \ni v \rightarrow \eta_{\sigma(j),j} v \in \mathbf{V}_{\sigma(j)} \quad (\sigma \in \Sigma_n, 1 \leq j \leq n). \tag{3.5}$$

It is easy to see from (3.4) that by composing the injection (3.5) with the usual inclusion $\text{GL}(\mathbf{V}) \subseteq \text{GL}(\mathbf{W})$, obtained via the pre-multiplication by the inverse, we get the following embedding:

$$\begin{aligned} \Sigma_n &\rightarrow \text{GL}(\mathbf{W}) \\ \sigma &\mapsto (\text{Hom}(\mathbf{V}_j, \mathbf{V}_{j'}) \ni w \rightarrow w \eta_{j,\sigma(j)} \in \text{Hom}(\mathbf{V}_{\sigma(j)}, \mathbf{V}_{j'})), \quad (1 \leq j \leq n, 1 \leq j' \leq n'). \end{aligned} \tag{3.6}$$

Lemma 3.2. If $\mathbb{D} = \mathbb{R}$ or \mathbb{H} , then there are elements $\tilde{s}_j \in \text{GL}(\mathbf{V}_j)$, $1 \leq j \leq n$, such that

$$\tilde{s}_j J_j \tilde{s}_j^{-1} = -J_j,$$

$$\tilde{s}_j^2 = \begin{cases} 1 & \text{if } \mathbb{D} = \mathbb{R}, \\ -1 & \text{if } \mathbb{D} = \mathbb{H}. \end{cases}$$

and for $u, v \in \mathbf{V}_j$,

$$(\tilde{s}_j u, \tilde{s}_j v) = \begin{cases} (u, v) & \text{if } G = \text{O}_{2p,2q}, \text{O}_{2p+1,2q} \text{ or } \text{Sp}_{p,q}, \\ -(u, v) & \text{if } G = \text{Sp}_{2n}(\mathbb{R}) \text{ or } \text{O}_{2n}^*. \end{cases}$$

Proof. This is straightforward via a case by case analysis. ■

For $1 \leq j, j' \leq n$ define

$$s_{j',j} \in \text{GL}(\text{Hom}(\mathbf{V}_j, \mathbf{V}_{j'})), \quad s_{j',j}(w) = \begin{cases} w \tilde{s}_j^{-1} & \text{if } \mathbb{D} = \mathbb{R}, \\ J w \tilde{s}_j^{-1} & \text{if } \mathbb{D} = \mathbb{H}. \end{cases} \tag{3.7}$$

Let $\mathbb{Z}_2 = \{0, 1\}$ with the addition modulo 2. Define an embedding

$$\begin{aligned} \mathbb{Z}_2^n &\rightarrow \text{GL}(\mathbf{W}), \\ \epsilon : \text{Hom}(\mathbf{V}_j, \mathbf{V}_{j'}) \ni w &\rightarrow s_{j',j}^{\epsilon_j}(w) \in \text{Hom}(\mathbf{V}_j, \mathbf{V}_{j'}), \end{aligned} \tag{3.8}$$

where $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{Z}_2^n$. By (3.7), we have $s_{j',j}^2 = 1$. Thus (3.8) is a group homomorphism. Recall that $\mathfrak{h} \subseteq \mathfrak{gl}(\mathbf{W})$ via the following action on \mathbf{W} :

$$x(w) = -wx \quad (w \in \mathbf{W}, x \in \mathfrak{h}).$$

For $\epsilon \in \mathbb{Z}_2^n$ and for $1 \leq j \leq n$ let

$$\hat{\epsilon}_j = (-1)^{\epsilon(j)}. \tag{3.9}$$

Corollary 3.3. *The groups Σ_n and \mathbb{Z}_2^n act on \mathfrak{h} via the conjugation as follows*

$$\begin{aligned} \sigma \cdot \left(\sum_{j=1}^n x_j J_j \right) &= \sum_{j=1}^n x_j J_{\sigma(j)} \quad (\sigma \in \Sigma_n), \\ \epsilon \cdot \left(\sum_{j=1}^n x_j J_j \right) &= \sum_{j=1}^n \hat{\epsilon}_j x_j J_j \quad (\epsilon \in \mathbb{Z}_2^n). \end{aligned}$$

In particular, the formulas (3.6) and (3.8) define an embedding

$$W(\mathbb{H}_{\mathbb{C}}) \rightarrow \text{GL}(W),$$

where

$$W(\mathbb{H}_{\mathbb{C}}) = \begin{cases} \Sigma_n & \text{if } \mathbb{D} = \mathbb{C}, \\ \Sigma_n \times \mathbb{Z}_2^n & \text{otherwise.} \end{cases}$$

Recall the character $\det : W(\mathbb{H}_{\mathbb{C}}) \rightarrow \{\pm 1\}$ of the Weyl group $W(\mathbb{H}_{\mathbb{C}})$, defined by

$$\begin{aligned} \det(\sigma) &= \text{the sign of the permutation } \sigma \in \Sigma_n \\ \det(\epsilon) &= \prod_{j=1}^n (-1)^{\epsilon_j}, \quad \epsilon \in \mathbb{Z}_2^n \\ \det(\sigma\epsilon) &= \det(\sigma) \det(\epsilon). \end{aligned} \tag{3.10}$$

Definition 3.4. Suppose $n' = n$. Define a function

$$W(\mathbb{H}_{\mathbb{C}}) \ni s \rightarrow y_s \in \mathfrak{h}$$

as follows. If $\mathbb{D} = \mathbb{C}$ and $s = \sigma \in \Sigma_n$, let

$$y_s = \sum_{j=1}^n \text{sgn} \langle J, \cdot \rangle_{\sigma(\text{Hom}(V_j, V_j))} J_{\sigma(j)}.$$

If $\mathbb{D} \neq \mathbb{C}$ and $s = \sigma\epsilon$, with $\sigma \in \Sigma_n$ and $\epsilon \in \mathbb{Z}_2^n$, let

$$y_s = \sum_{j=1}^n \text{sgn} \langle J, \cdot \rangle_{s(\text{Hom}(V_j, V_j)^J)} J_{\sigma(j)}.$$

If $n' < n$, then we enlarge the defining module $(V', (,)')$, by adding a space with a non-degenerate form of the same type as $(,)'$, and define y_s as above.

Here, by definition, $\text{sgn} \langle J, \cdot \rangle_U = 1$ if the restriction of the symmetric form $\langle J, \cdot \rangle$ to U is positive definite, and $\text{sgn} \langle J, \cdot \rangle_U = -1$ if this restriction is negative definite.

The following three lemmas may be verified via a case by case analysis. We omit the details.

Lemma 3.5. *Suppose $G \neq O_{2p+1,2q}$. Let $\beta \in \Delta^n$ and let $s_\beta \in W(\mathbb{H}_\mathbb{C})$ be the corresponding reflection. Then*

$$\beta(y_{s_\beta s}) = \beta(y_s) \quad (s \in W(\mathbb{H}_\mathbb{C})).$$

If $\alpha \in \Delta^n$ and $\underline{\alpha} \cap \underline{\beta} = \emptyset$, then

$$\beta(y_{s_\alpha s}) = \beta(y_s) \quad (s \in W(\mathbb{H}_\mathbb{C})).$$

Let $\mathcal{S} \in \Psi_{st}^n$, $s \in W(\mathbb{H}_\mathbb{C})$ and let $y = y_s$. Define

$$\mathcal{S}(y) = \{\alpha \in \mathcal{S} \mid \alpha(y) \neq 0 \text{ and } \underline{\alpha} \cap \underline{\mathcal{S} \setminus \alpha} = \emptyset\}$$

Then

$$\mathcal{S}(y) = \{\alpha \in \mathcal{S} \mid \alpha(y) \neq 0\}$$

unless $G = O_{2p,2q}$. Moreover,

$$\underline{\mathcal{S}} \setminus \underline{\mathcal{S}(y)} = \underline{\mathcal{S}} \setminus \underline{\mathcal{S}(y)} \tag{3.11}$$

and

$$|\underline{\beta}| = 2 \quad (\beta \in \mathcal{S} \setminus \mathcal{S}(y)). \tag{3.12}$$

Also, if $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$, then

$$\mathcal{S}(y_{s_\beta s}) = \mathcal{S}(y_s).$$

For $y \in \mathfrak{h}$ and $\mathcal{S} \in \Psi_{st}^n$, let

$$pr_{\underline{\mathcal{S}}^\perp}(y) = \sum_{j \notin \underline{\mathcal{S}}} J_j^*(y) J_j,$$

where $J_1^*, J_2^*, \dots, J_n^*$ is the basis of \mathfrak{h}^* , dual to J_1, J_2, \dots, J_n . Also, for $s \in W(\mathbb{H}_\mathbb{C})$, let

$$y_{s,\mathcal{S}} = pr_{\underline{\mathcal{S}}^\perp}(y_s). \tag{3.13}$$

Lemma 3.6. *Suppose $G = O_{2p+1,2q}$. Let $s \in W(\mathbb{H}_\mathbb{C})$ and let $y = y_s$. For $\mathcal{S} \in \Psi_{st}^n$ define*

$$\begin{aligned} \mathcal{S}_1 &= \{\alpha, \beta \in \Psi \mid \alpha \in \Psi^n, \beta \in \Psi^c, \alpha \not\perp \beta, \alpha \vee \beta \subseteq \mathcal{S}\}, \\ \mathcal{S}_2(y) &= \{\beta \in \mathcal{S} \mid \beta(y) \neq 0, \underline{\beta} \cap \underline{(\mathcal{S} \setminus \beta)} = \emptyset\}, \\ \mathcal{S}(y) &= \mathcal{S}_1 \cup \mathcal{S}_2(y). \end{aligned}$$

Then

$$\mathcal{S}_2(y) = \{\beta \in \mathcal{S} \mid \beta(pr_{\underline{(\mathcal{S} \setminus \beta)}^\perp}(y)) \neq 0\},$$

and

$$\mathcal{S}(y) \supseteq \{\beta \in \Psi \mid \text{there is } \mathcal{S}' \in \Psi_{st}^n \text{ such that } \beta \in \Psi_{\mathcal{S}',i\mathbb{R}}^n, \mathcal{S} = \mathcal{S}' \vee \beta \text{ and } \beta(pr_{\underline{\mathcal{S}'^\perp}}(y)) \neq 0\}.$$

If $\gamma \in \Psi_{\mathcal{S}, \mathbb{R}}^n$ and $\gamma(\text{pr}_{\mathcal{S}^\perp}(y)) \neq 0$ then

$$(\mathcal{S} \vee \gamma)(y) = \mathcal{S}(y) \cup \{\gamma\}. \quad (3.14)$$

Let

$$\mathcal{S}_3(y) = \mathcal{S}(y) \cup \{\beta \in \mathcal{S} \mid \underline{\beta} \cap \underline{(\mathcal{S} \setminus \beta)} \neq \emptyset\}.$$

Then

$$\begin{aligned} \mathcal{S} \setminus \mathcal{S}_3(y) &= \{\beta \in \mathcal{S} \mid \beta(y) = 0, \underline{\beta} \cap \underline{(\mathcal{S} \setminus \beta)} = \emptyset\}, \\ \underline{\mathcal{S}} \setminus \underline{\mathcal{S}(y)} &= \underline{\mathcal{S}} \setminus \underline{\mathcal{S}_3(y)}, \\ |\underline{\beta}| &= 2 \text{ for all } \beta \in \mathcal{S} \setminus \mathcal{S}_3(y). \end{aligned} \quad (3.15)$$

Notice by the way that

$$\mathcal{S}_1 = \Psi_{\mathcal{S}, \mathbb{R}}(\text{short}) \setminus \mathcal{S}(\text{short}).$$

Lemma 3.7. Let $\beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n$, $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$, $s \in W(\mathbb{H}_{\mathbb{C}})$, $\beta(y_s) \neq 0$. Then

$$y_{s\beta s, \mathcal{S} \vee \beta} = y_{s, \mathcal{S} \vee \beta}, \quad (3.16)$$

Suppose $G = O_{2p+1, 2q}$. Then

$$\begin{aligned} \beta(y_{s\beta s}) &= \begin{cases} \beta(y_s) & \text{if } \beta \text{ is long,} \\ -\beta(y_s) & \text{if } \beta \text{ is short.} \end{cases} \\ \alpha(y_{s\beta s}) &= \alpha(y_s) \text{ if } \alpha \in \Psi, \underline{\alpha} \cap \underline{\beta} = \emptyset, \\ \mathcal{S}(y_{s\beta s}) &= \mathcal{S}(y_s). \end{aligned}$$

Let

$$\mathcal{S}'' = \begin{cases} \mathcal{S}(\text{long}) & \text{if } G = \text{Sp}_{2n}(\mathbb{R}), \\ \{\alpha \in \mathcal{S} \mid \underline{\alpha} \subseteq \underline{\mathcal{S}} \setminus \underline{\alpha}\} & \text{if } G = O_{2p, 2q}. \end{cases} \quad (3.17)$$

Let $\beta \in \mathcal{S} \setminus \mathcal{S}(y_s)$. Then, by (3.12), we may write the support of β as,

$$\underline{\beta} = \{a(\beta), b(\beta)\}, \text{ with } 1 \leq a(\beta) < b(\beta) \leq n.$$

Define the following function:

$$\mathcal{B}_{\mathcal{S}}: W(\mathbb{H}_{\mathbb{C}}) \rightarrow \{R \subset \mathcal{S} \cup (-\mathcal{S})\}, \quad (3.18)$$

$$\mathcal{B}_{\mathcal{S}}(s) = \begin{cases} \{\pm\beta \mid \beta \in (\mathcal{S} \setminus \mathcal{S}(y_s)) \cap (\mathcal{S} \setminus \mathcal{S}''), \sigma^{-1}(a(\beta)) \leq n' \text{ and } \sigma^{-1}(b(\beta)) \leq n'\} \\ \hspace{15em} \text{if } G = O_{2p, 2q}, \\ \{\pm\beta \mid \beta \in \mathcal{S} \setminus \mathcal{S}_3(y_s), \sigma^{-1}(a(\beta)) \leq n' \text{ and } \sigma^{-1}(b(\beta)) \leq n'\} \\ \hspace{15em} \text{if } G = O_{2p+1, 2q}, \\ \{\pm\beta \mid \beta \in \mathcal{S} \setminus \mathcal{S}(y_s), \sigma^{-1}(a(\beta)) \leq n' \text{ and } \sigma^{-1}(b(\beta)) \leq n'\} \text{ otherwise.} \end{cases}$$

where $s = \sigma \in \Sigma_n$ if $\mathbb{D} = \mathbb{C}$ and $s = \sigma\epsilon$, $\epsilon \in \mathbb{Z}_2^n$, if $\mathbb{D} \neq \mathbb{C}$.

For R in the image of $\mathcal{B}_{\mathcal{S}}$, let $W(R) \subseteq W(\mathbb{H}_{\mathbb{C}})$ be the group generated by all the reflections s_{β} , $\beta \in R$. If $R = \emptyset$, then $W(R) = 1$.

The fiber $\mathcal{B}_S^{-1}(R) \subseteq W(\mathbb{H}_\mathbb{C})$ is invariant under the left multiplication by elements of $W(R)$.

Indeed, let $\alpha \in R$ and let $s \in \mathcal{B}_S^{-1}(R)$. We need to check that

$$\mathcal{B}_S(s_\alpha s) = \mathcal{B}_S(s). \tag{3.19}$$

Let $\sigma_\alpha \in W(\mathbb{H}_\mathbb{C})$ denote the transposition of the $a(\alpha)$ and $b(\alpha)$. Then $s_\alpha = \sigma_\alpha$ if $\mathbb{D} = \mathbb{C}$, and $s_\alpha = \sigma_\alpha \epsilon_\alpha$, for some $\epsilon_\alpha \in \mathbb{Z}_2^n$, if $\mathbb{D} \neq \mathbb{C}$. Therefore,

$$(\sigma_\alpha \sigma)^{-1} a(\alpha) = \sigma^{-1} b(\alpha) \quad \text{and} \quad (\sigma_\alpha \sigma)^{-1} b(\alpha) = \sigma^{-1} a(\alpha). \tag{3.20}$$

Hence, $\alpha \in \mathcal{B}_S(s_\alpha s)$.

Let $\beta \in \mathcal{S} \setminus \mathcal{S}(y_s)$, if $G \neq O_{2p+1,2q}$, and $\beta \in \mathcal{S} \setminus \mathcal{S}_2(y_s)$, if $G = O_{2p+1,2q}$. Suppose $\beta \neq \alpha$. By (3.12) and (3.15), $|\beta| = |\alpha| = 2$. Therefore, since $\beta \perp \alpha$, we must have $\underline{\beta} \cap \underline{\alpha} = \emptyset$ or $\underline{\beta} = \underline{\alpha}$. In the first case

$$(\sigma_\alpha \sigma)^{-1} a(\beta) = \sigma^{-1} a(\beta) \quad \text{and} \quad (\sigma_\alpha \sigma)^{-1} b(\beta) = \sigma^{-1} b(\beta).$$

In the second case (3.20) holds. Thus, (3.19) follows.

Therefore, we may decompose the Weyl group $W(\mathbb{H}_\mathbb{C})$ into the disjoint union of the fibers of the map \mathcal{B}_S , and each fiber into the disjoint union of the corresponding left cosets:

$$W(\mathbb{H}_\mathbb{C}) = \bigcup_{R \in \mathcal{B}_S(W(\mathbb{H}_\mathbb{C}))} \bigcup_{\substack{W(R)s \in W(R) \setminus W(\mathbb{H}_\mathbb{C}) \\ \mathcal{B}_S(s) = R}} W(R)s. \tag{3.21}$$

If $(G, G') = (O_{2p,2q}, Sp_{2n'}(\mathbb{R}))$ or $(Sp_{2n}(\mathbb{R}), O_{2p+1,2q})$, define $W(s, \mathcal{S}) \subseteq \mathbb{Z}_2^n \subseteq W(\mathbb{H}_\mathbb{C})$ to be the subgroup generated by the reflections with respect to all the iJ_k^* , with $1 \leq k \leq n'$ and $\sigma(k) \in \underline{\mathcal{S}}''$. Thus for $\delta \in W(s, \mathcal{S})$, $\hat{\delta}_k \neq 1$ may happen if and only if $1 \leq k \leq n'$ and $\sigma(k) \in \underline{\mathcal{S}}''$. For the remaining dual pairs we let $W(s, \mathcal{S}) = \{1\}$. This leads to a refinement of the decomposition (3.21) into a disjoint union of subsets:

$$W(\mathbb{H}_\mathbb{C}) = \bigcup_{R \in \mathcal{B}_S(W(\mathbb{H}_\mathbb{C}))} \bigcup_{\substack{W(R)sW(s, \mathcal{S}) \in W(R) \setminus W(\mathbb{H}_\mathbb{C}) / W(s, \mathcal{S}) \\ \mathcal{B}_S(s) = R}} W(R)sW(s, \mathcal{S}). \tag{3.22}$$

4. Boundaries and orientations

Fix a strongly orthogonal set $\mathcal{S} \in \Psi_{st}^n$.

Lemma 4.1. *Suppose $G \neq O_{2p+1,2q}$. Let $y = \sum_{j=1}^n y_j J_j$, with $y_j = \pm 1$. Then*

$$\frac{i\alpha(y)}{2} \in \{\pm 1, 0\} \quad (\alpha \in \Psi^n), \tag{4.1}$$

$$pr_{\alpha^\perp}(y) = y + \frac{i\alpha(y)}{2} iH_\alpha \quad (\alpha \in \Psi^n, \alpha(y) \neq 0). \tag{4.2}$$

$$\text{If } \beta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n \text{ and } \beta(pr_{\underline{\mathcal{S}}^\perp}(y)) \neq 0, \text{ then } \beta \in \Psi^n \text{ and } \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset. \tag{4.3}$$

In particular, $\beta \perp \mathcal{S}$.

Proof. Let $\alpha \in \Psi^n$. Then (up to a sign) $-\alpha \in \{\pm 2J_a^*, \pm J_a^* \pm J_b^*\}$ for some $a \neq b$. This, obviously, implies (4.1). The relation (4.2) is clear if $-\alpha = 2J_a^*$. Suppose $-\alpha = J_a^* + J_b^*$, with $a \neq b$. Then

$$\frac{i\alpha(y)}{2}iH_\alpha = -\frac{y_a + y_b}{2}(J_a + J_b) = -y_a J_a - y_b J_b,$$

because the condition $\alpha(y) = 0$ implies $y_a = y_b$. Hence the right hand side of (4.2) is equal to

$$y - y_a J_a - y_b J_b = pr_{\underline{\alpha}^\perp}(y).$$

Similarly, we check the case $-\alpha = J_a^* - J_b^*$.

It remains to verify (4.3). Suppose $\beta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$ and $\underline{\beta} \cap \underline{\mathcal{S}} \neq \emptyset$. If $\underline{\beta} \cap \underline{\mathcal{S}}$ consisted of a single point then β could not be orthogonal to \mathcal{S} . Hence we may assume that $\beta = i(J_a^* \pm J_b^*)$ for some $a \neq b$, and $i(J_a^* \mp J_b^*) \in \mathcal{S}$. Hence, $pr_{\underline{\mathcal{S}}^\perp}(y)$ does not contain the terms $y_a J_a, y_b J_b$. Therefore $\beta(pr_{\underline{\mathcal{S}}^\perp}(y)) = 0$, this is a contradiction. We verified the following implication

$$\beta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n \text{ and } \beta(pr_{\underline{\mathcal{S}}^\perp}(y)) \neq 0 \Rightarrow \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset.$$

By [8, Proposition 6.72],

$$(\beta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n \text{ and } \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset) \Rightarrow \beta \in \Psi^n.$$

Thus (4.3) follows. ■

Similarly we have the following lemma.

Lemma 4.2. *Suppose $G = O_{2p+1, 2q}$. Let $y \in \mathfrak{h}$ such that $y = \sum_{j=1}^n y_j J_j$, with $y_j = \pm 1$. Let $\beta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$.*

1. *If β is short, then $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$, and therefore $\beta(pr_{\underline{\mathcal{S}}^\perp}(y)) \neq 0$.*
2. *If β is long and $\beta(pr_{\underline{\mathcal{S}}^\perp}(y)) \neq 0$, then $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$. In particular, β is strongly orthogonal to \mathcal{S} .*
3. *If $\beta(y) \neq 0$, then $pr_{\underline{\mathcal{S}}^\perp}(y) = y + \frac{i\beta(y)}{2}iH_\beta$.*

Lemma 4.3. *Suppose $G \neq O_{2p+1, 2q}$. Let $y \in \mathfrak{h}$ such that $y = \sum_{j=1}^n y_j J_j$, with $y_j = \pm 1$. Define (as in Lemma 3.5)*

$$\mathcal{S}(y) = \{\alpha \in \mathcal{S} \mid \alpha(y) \neq 0 \text{ and } \underline{\alpha} \cap \underline{\mathcal{S}} \setminus \underline{\alpha} = \emptyset\}.$$

Suppose $\beta \in \Psi^n$ is such that $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$ and $\beta(y) \neq 0$. Then

$$(\mathcal{S} \cup \beta)(y) = \mathcal{S}(y) \cup \beta. \tag{4.4}$$

Define

$$m_{\mathcal{S}(y)}(x) = \min \left\{ 1, \frac{|\alpha(x)|}{2} \mid \alpha \in \mathcal{S}(y) \right\} \quad (x \in \mathfrak{h}_{\mathcal{S}}).$$

Then, for β as in (4.4), $x \in \mathfrak{h}_{\mathcal{S}}$, with $\beta(x) = 0$, and $t \in \mathbb{R}$,

$$m_{(\mathcal{S} \cup \beta)(y)}(x + t m_{\mathcal{S}(y)}(x) \frac{i\beta(y)}{2} H_\beta) = |t| m_{\mathcal{S}(y)}(x). \tag{4.5}$$

Proof. The equality (4.4) is obvious. The left hand side of (4.5) is equal to the minimum of

$$1; \frac{|\alpha(x)|}{2}, \alpha \in \mathcal{S}(y); \frac{1}{2}|\beta(x + t m_{\mathcal{S}(y)}(x) \frac{i\beta(y)}{2} H_\beta)| = |t| m_{\mathcal{S}(y)}(x),$$

which, by (4.1), coincides with the right hand side. ■

Lemma 4.4. *Suppose $G = O_{2p+1,2q}$. For $y = \sum_{j=1}^n y_j J_j$, with $y_j = \pm 1$, and for $\mathcal{S} \in \Psi_{st}^n$ define $\mathcal{S}(y)$ as in Lemma 3.6, and let*

$$m_{\mathcal{S}(y)}(x) = \min \left\{ 1, \frac{|\alpha(x)|}{2}, |\beta(x)| \mid \alpha \in \mathcal{S}(y)(\text{long}), \beta \in \mathcal{S}(y)(\text{short}) \right\} \quad (x \in \mathfrak{h}_{\mathcal{S}}).$$

Let $\beta \in \Psi_{\mathcal{S},i\mathbb{R}}^n$ be such that $\beta(\text{pr}_{\underline{\mathcal{S}}^\perp}(y)) \neq 0$. Let $x \in \mathfrak{h}_{\mathcal{S}}$ be semiregular with $\beta(x) = 0$. Then, for $t \in \mathbb{R}$,

$$m_{(\mathcal{S} \vee \beta)(y)}(x + t m_{\mathcal{S}(y)}(x) \frac{i\beta(y)}{2} H_\beta) = |t| m_{\mathcal{S}(y)}(x).$$

Proof. Suppose β is long. Then

$$\begin{aligned} \frac{1}{2} \left| \beta(x + t m_{\mathcal{S}(y)}(x) \frac{i\beta(y)}{2} H_\beta) \right| &= \frac{1}{2} |t| m_{\mathcal{S}(y)}(x) \left| \frac{i\beta(y)}{2} \beta(H_\beta) \right| \\ &= |t| m_{\mathcal{S}(y)}(x) \left| \frac{i\beta(y)}{2} \right| = |t| m_{\mathcal{S}(y)}(x), \end{aligned}$$

because $\frac{i\beta(y)}{2} = \pm 1$.

Suppose β is short. Then

$$\begin{aligned} \left| \beta(x + t m_{\mathcal{S}(y)}(x) \frac{i\beta(y)}{2} H_\beta) \right| &= |t| m_{\mathcal{S}(y)}(x) \left| \frac{i\beta(y)}{2} \beta(H_\beta) \right| \\ &= |t| m_{\mathcal{S}(y)}(x) |\beta(y)| = |t| m_{\mathcal{S}(y)}(x), \end{aligned}$$

because $|\beta(y)| = 1$.

By definition, $\beta \perp \mathcal{S}$. Hence, $\beta \perp \mathcal{S}_1$ (see Lemma 3.6). Thus $\beta \perp \mathcal{S}(y)$. In other words, $\alpha(H_\beta) = 0$ for $\alpha \in \mathcal{S}(y)$. Since, as we have shown in (3.14) that $(\mathcal{S} \vee \beta)(y) = \mathcal{S}(y) \cup \beta$, the equality follows. ■

Lemma 4.5. *Let $I = [-1, 1] \subseteq \mathbb{R}$. For $s \in W(H_{\mathbb{C}})$ define the following function (chain):*

$$\mathcal{C}_{s,\mathcal{S}} : I \times \mathfrak{h}_{\mathcal{S}} \ni (t, x) \rightarrow x + it m_{\mathcal{S}(y_s)}(x) y_{s,\mathcal{S}} \in \mathfrak{h}_{\mathcal{S}} + iI y_{s,\mathcal{S}}.$$

Then, for $\beta \in \Psi_{\mathcal{S},i\mathbb{R}}^n$ such that $\beta(y_{s,\mathcal{S}}) \neq 0$ and for $x \in \mathfrak{h}_{\mathcal{S}}$ with $\beta(x) = 0$,

$$\mathcal{C}_{s,\mathcal{S}}(t, x) = \mathcal{C}_{s,\mathcal{S} \vee \beta}(1, x + t m_{\mathcal{S}(y_s)}(x) \frac{i\beta(y_s)}{2} H_\beta) \quad (t \in I).$$

Proof. We compute:

$$\begin{aligned}
\mathcal{C}_{s,\mathcal{S}}(t, x) &= x + it m_{\mathcal{S}(y_s)}(x) y_{s,\mathcal{S}} \\
&= (x + t m_{\mathcal{S}(y_s)}(x) \frac{i\beta(y_s)}{2} H_\beta) + it m_{\mathcal{S}(y_s)}(x) (y_{s,\mathcal{S}} + \frac{i\beta(y_s)}{2} iH_\beta) \\
&= (x + t m_{\mathcal{S}(y_s)}(x) \frac{i\beta(y_s)}{2} H_\beta) + it m_{\mathcal{S}(y_s)}(x) y_{s,\mathcal{S}\vee\beta} \\
&= \mathcal{C}_{s,\mathcal{S}\vee\beta}(1, x + t m_{\mathcal{S}(y_s)}(x) \frac{i\beta(y_s)}{2} H_\beta),
\end{aligned}$$

and this completes the proof. \blacksquare

Let $[J_1, J_2, \dots, J_n]$ be the orientation of the real vector space \mathfrak{h} . Define a linear map $c_{\mathcal{S}} : \mathfrak{h} \rightarrow \mathfrak{h}_{\mathcal{S}}$ by

$$\begin{aligned}
c_{\mathcal{S}}(x) &= x \text{ for } x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\mathcal{S}}, \\
c_{\mathcal{S}}(iH_\alpha) &= H_\alpha \text{ for } \alpha \in \mathcal{S}.
\end{aligned}$$

Let

$$[c_{\mathcal{S}}J_1, c_{\mathcal{S}}J_2, \dots, c_{\mathcal{S}}J_n] \quad (4.6)$$

be the orientation of $\mathfrak{h}_{\mathcal{S}}$. For $\beta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$, choose an orientation

$$[J_1^\beta, J_2^\beta, \dots, J_{n-1}^\beta]$$

of $\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\beta$ so that

$$[iH_\beta, J_1^\beta, J_2^\beta, \dots, J_{n-1}^\beta] = [c_{\mathcal{S}}J_1, c_{\mathcal{S}}J_2, \dots, c_{\mathcal{S}}J_n]. \quad (4.7)$$

For $s \in W(\mathbb{H}_{\mathbb{C}})$, let $[iy_{s,\mathcal{S}}, c_{\mathcal{S}}J_1, c_{\mathcal{S}}J_2, \dots, c_{\mathcal{S}}J_n]$ be the orientation of $\mathfrak{h}_{\mathcal{S}} \oplus i\mathbb{R}y_{s,\mathcal{S}}$, and let

$$[iy_{s,\mathcal{S}}, J_1^\beta, J_2^\beta, \dots, J_{n-1}^\beta]$$

be the orientation of $\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\beta + i\mathbb{R}y_{s,\mathcal{S}}$. Let us orient $\mathcal{C}_{s,\mathcal{S}}(1 \times \mathfrak{h}_{\mathcal{S}})$ via the identification

$$\mathfrak{h}_{\mathcal{S}} \ni x \rightarrow \mathcal{C}_{s,\mathcal{S}}(1, x) \in \mathcal{C}_{s,\mathcal{S}}(1 \times \mathfrak{h}_{\mathcal{S}}).$$

Further, let

$$\begin{aligned}
\mathfrak{h}_{\mathcal{S}, -i\beta > 0} &= \{x \in \mathfrak{h}_{\mathcal{S}} \mid -i\beta(x) > 0\}, \\
\mathfrak{h}_{\mathcal{S}, -i\beta < 0} &= \{x \in \mathfrak{h}_{\mathcal{S}} \mid -i\beta(x) < 0\}, \\
\mathfrak{h}_{\mathcal{S}, \beta = 0} &= \{x \in \mathfrak{h}_{\mathcal{S}} \mid \beta(x) = 0\}.
\end{aligned}$$

Lemma 4.6. *With the above notation, we have*

$$\partial(\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, -i\beta > 0})) = \mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta = 0}) - \mathcal{C}_{s,\mathcal{S}}(0 \times \mathfrak{h}_{\mathcal{S}, -i\beta > 0}) + \mathcal{C}_{s,\mathcal{S}}(1 \times \mathfrak{h}_{\mathcal{S}, -i\beta > 0}) \quad (4.8)$$

and

$$\begin{aligned}
&\partial(\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, -i\beta < 0})) \\
&= -\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta = 0}) - \mathcal{C}_{s,\mathcal{S}}(0 \times \mathfrak{h}_{\mathcal{S}, -i\beta < 0}) + \mathcal{C}_{s,\mathcal{S}}(1 \times \mathfrak{h}_{\mathcal{S}, -i\beta < 0}),
\end{aligned} \quad (4.9)$$

where the manifolds on the left hand side are equipped with the induced orientation (see [15, p.325]).

Proof. We shall check the sign by the first term on the right hand side of (4.8). By definition,

$$\begin{aligned} \mathcal{C}_{s,S}(I \times \mathfrak{h}_{S,-i\beta>0}) &= \{ity_{s,S} + x \mid t \in I, x \in \mathfrak{h}_{S,-i\beta>0}\} \\ &= \{ity_{s,S} + t_\beta iH_\beta + x \mid t \in I, t_\beta > 0, x \in \mathfrak{h}_S \cap \mathfrak{h}^\beta\} \\ &= \{-it_\beta H_\beta + ity_{s,S} + \sum_{k=1}^{n-1} t_k J_k^\beta \mid t \in I, t_\beta < 0, t_k \in \mathbb{R}\}. \end{aligned}$$

Since, by (4.7),

$$\begin{aligned} [iy_{s,S}, c_S J_1, c_S J_2, \dots, c_S J_n] &= [iy_{s,S}, iH_\beta, J_1^\beta, J_2^\beta, \dots, J_{n-1}^\beta] \\ &= [-iH_\beta, iy_{s,S}, J_1^\beta, J_2^\beta, \dots, J_{n-1}^\beta], \end{aligned}$$

we see that the term $\mathcal{C}_{s,S}(I \times \mathfrak{h}_{S,\beta=0})$ enters with the plus sign. Similarly, we check the sign by the first term on the right hand side of (4.9). The remaining four signs in (4.8) and (4.9) are obvious. ■

Let $\mathfrak{F}\Psi_{\mathcal{S},\mathbb{R}}^n$ the set of all functions $\gamma : \Psi_{\mathcal{S},\mathbb{R}}^n \rightarrow \{\pm 1\}$. An analogous argument verifies the following lemma.

Lemma 4.7. *Let $\gamma \in \mathfrak{F}\Psi_{\mathcal{S},\mathbb{R}}^n$, $\epsilon > 0$, and $\alpha \in \Psi_{\mathcal{S},\mathbb{R}}^n$. Define*

$$\begin{aligned} \mathfrak{h}_{S,\gamma,\epsilon} &= \{x \in \mathfrak{h}_S \mid -i\beta(x)\gamma(\beta) > \epsilon, \beta \in \Psi_{\mathcal{S},\mathbb{R}}^n\}, \\ \mathfrak{h}_{S,\gamma,-i\gamma(\alpha)\alpha=\epsilon} &= \{x \in \mathfrak{h}_S \mid -i\beta(x)\gamma(\beta) > \epsilon, \beta \in \Psi_{\mathcal{S},\mathbb{R}}^n \setminus \alpha, -i\alpha(x)\gamma(\alpha) = \epsilon\}. \end{aligned}$$

Then,

$$\partial(\mathcal{C}_{s,S}(I \times \mathfrak{h}_{S,\gamma,\epsilon})) = \sum_{\alpha \in \Psi_{\mathcal{S},\mathbb{R}}^n} \gamma(\alpha) \mathcal{C}_{s,S}(I \times \mathfrak{h}_{S,\gamma,-i\gamma(\alpha)\alpha=\epsilon}) - \mathcal{C}_{s,S}(0 \times \mathfrak{h}_{S,\gamma,\epsilon}) + \mathcal{C}_{s,S}(1 \times \mathfrak{h}_{S,\gamma,\epsilon}).$$

5. The *chc* as a rational function, for pairs (G, G') with $G' \neq O_{2p+1,2q}$

In this section, suppose $G' \neq O_{2p+1,2q}$. We shall view the symplectic space W as a vector space over \mathbb{C} , by

$$iw = J'(w) \quad (w \in W).$$

Then

$$\begin{aligned} \mathfrak{gl}_{\mathbb{C}}(W) &= \text{End}(W)^{J'}, \\ \mathfrak{g}_{\mathbb{C}} &= \mathfrak{g} + J'\mathfrak{g} \subseteq \mathfrak{gl}_{\mathbb{C}}(W), \\ \mathfrak{h}'_{\mathbb{C}} &= \mathfrak{h}' + J'\mathfrak{h} \subseteq \mathfrak{gl}_{\mathbb{C}}(W), \\ \text{GL}_{\mathbb{C}}(W) &= \text{GL}(W)^{J'}. \end{aligned}$$

Let $2p_+$ be the maximal dimension of a subspace of W (over \mathbb{R}) on which the symmetric form $\langle J', \cdot \rangle$ is positive definite. Let $\det : \mathfrak{gl}_{\mathbb{C}}(W) \rightarrow \mathbb{C}$ be the determinant. Recall the following proposition (see [11, (10.10)]).

Proposition 5.1. *The distribution chc_W extends to a rational function of $z' \in \mathfrak{h}'_{\mathbb{C}}$ and $z \in \mathfrak{g}_{\mathbb{C}}$,*

$$chc_W(z' + z) = \frac{(-1)^{p_+} \sqrt{2}^{\dim_{\mathbb{R}} W}}{\det(z' + z)}.$$

Let

$$\mathfrak{z} = \mathfrak{g}^{b'}, Z = G^{b'}, \text{ and } W(\mathbb{H}_{\mathbb{C}}, Z_{\mathbb{C}}) = \text{Normalizer}_{Z_{\mathbb{C}}}(\mathbb{H}_{\mathbb{C}})/\mathbb{H}_{\mathbb{C}} \subseteq W(\mathbb{H}_{\mathbb{C}}).$$

Lemma 5.2. *The Weyl group $W(\mathbb{H}_{\mathbb{C}}, Z_{\mathbb{C}})$ acts trivially on the space*

$$W^{b'} = \sum_{j=1}^{n'} \text{Hom}(V_j, V_j)^J \subseteq W.$$

Proof. In terms of section 3,

$$W(\mathbb{H}_{\mathbb{C}}, Z_{\mathbb{C}}) = \begin{cases} \Sigma_{n-n'} & \text{if } \mathbb{D} = \mathbb{C}, \\ \Sigma_{n-n'} \times \mathbb{Z}_2^{n-n'} & \text{if } \mathbb{D} \neq \mathbb{C}, \end{cases}$$

where the group $\Sigma_{n-n'}$ acts trivially on $\{1, 2, \dots, n'\}$ and the elements of $\mathbb{Z}_2^{n-n'}$ are viewed as elements of \mathbb{Z}_2^n by the embedding

$$\mathbb{Z}_2^{n-n'} \ni (\epsilon_1, \epsilon_2, \dots, \epsilon_{n-n'}) \rightarrow (0, 0, \dots, 0, \epsilon_1, \epsilon_2, \dots, \epsilon_{n-n'}) \in \mathbb{Z}_2^n.$$

Clearly, the lemma follows. ■

Recall the positive root system Ψ , (1.1). Let $\Phi = -\Psi$. Let $\Phi(\mathfrak{h}_{\mathbb{C}}, \mathfrak{z}_{\mathbb{C}}) = \{\alpha \in \Phi \mid \mathfrak{g}_{\mathbb{C}, \alpha} \subseteq \mathfrak{z}_{\mathbb{C}}\}$. This is a positive root system of the roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{z}_{\mathbb{C}}$. Set

$$\pi_{\mathfrak{g}/\mathfrak{h}} = \prod_{\alpha \in \Phi} \alpha, \quad \pi_{\mathfrak{z}/\mathfrak{h}} = \prod_{\alpha \in \Phi(\mathfrak{h}, \mathfrak{z})} \alpha. \tag{5.1}$$

Proposition 5.3. *There is a constant u , with $u^4 = 1$, such that, for any polynomial P satisfying (0.4), $z' \in \mathfrak{h}'_{\mathbb{C}}$ and $z \in \mathfrak{h}_{\mathbb{C}}$, we have*

$$\begin{aligned} & P(z') \pi_{\mathfrak{g}'/\mathfrak{h}'}(z') chc_W(z' + z) \pi_{\mathfrak{g}/\mathfrak{h}}(z) \\ &= \frac{u \sqrt{2}^{\dim_{\mathbb{R}} W}}{|W(\mathbb{H}_{\mathbb{C}}, Z_{\mathbb{C}})|} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \text{sgn}(s) \frac{P(s^{-1} \cdot z) \pi_{\mathfrak{z}/\mathfrak{h}}(s^{-1} \cdot z)}{\det(z' + z)_{sW^{b'}}}, \end{aligned}$$

where the “sgn” is defined by

$$\pi_{\mathfrak{g}/\mathfrak{h}}(s \cdot z) = \text{sgn}(s) \pi_{\mathfrak{g}/\mathfrak{h}}(z).$$

This proposition is proved case by case in Appendix B (see Propositions B.1, B.2, B.3 and B.4).

6. The chc for pairs $(G, G') = (Sp_{2n}(\mathbb{R}), O_{2p+1,2q})$

In this case

$$\begin{aligned} & chc_W(x' + x) \\ = & chc_{\text{Hom}(V, V'_0)}(x' + x) \cdot chc_{\text{Hom}(V, V'_0{}^\perp)}(x' + x) \quad (x' \in \mathfrak{h}^{\text{reg}}, x \in \mathfrak{g}), \end{aligned} \tag{6.1}$$

where the product of distributions is well defined (see [11, (1.8)]). The distribution

$$chc_{\text{Hom}(V, V'_0)}(x' + x) = chc_{\text{Hom}(V, V'_0)}(x)$$

is described (for example) in [11, Proposition 9.3]. From now on we assume that $V'_0{}^\perp \neq 0$ and concentrate on the second factor in (6.1).

We shall view the symplectic space $\text{Hom}(V, V'_0{}^\perp)$ as a vector space over \mathbb{C} , by

$$iw = J'(w) \quad (w \in \text{Hom}(V, V'_0{}^\perp)).$$

Let $2p_+$ be the maximal dimension of a subspace of W (over \mathbb{R}) on which the symmetric form $\langle J', \cdot \rangle$ is positive definite. (Since the restriction of J' to $\text{Hom}(V, V'_0)$ is zero, the value $2p_+$ does not change if we replace the W by $\text{Hom}(V, V'_0{}^\perp)$ in the above definition.) Let $\det : \mathfrak{gl}_{\mathbb{C}}(\text{Hom}(V, V'_0{}^\perp)) \rightarrow \mathbb{C}$ be the determinant. Recall the following proposition (see [11, (10.10)]).

Proposition 6.1. *The distribution $chc_{\text{Hom}(V, V'_0{}^\perp)}$ extends to a rational function of $z' \in \mathfrak{h}'_{\mathbb{C}}$ and $z \in \mathfrak{g}(V'_0{}^\perp)_{\mathbb{C}}$,*

$$chc_{\text{Hom}(V, V'_0{}^\perp)}(z' + z) = \frac{(-1)^{p_+} \sqrt{2}^{\dim_{\mathbb{R}} \text{Hom}(V, V'_0{}^\perp)}}{\det(z' + z)_{\text{Hom}(V, V'_0{}^\perp)}}.$$

Let

$$\begin{aligned} \mathfrak{z} &= \mathfrak{g}^{\mathfrak{h}'}, \quad Z = G^{\mathfrak{h}'}, \quad \text{and} \\ W(H_{\mathbb{C}}, Z_{\mathbb{C}}) &= \text{Normalizer}_{Z_{\mathbb{C}}}(H_{\mathbb{C}})/H_{\mathbb{C}} \subseteq W(H_{\mathbb{C}}), \end{aligned}$$

as in the previous section.

Lemma 6.2. *The Weyl group $W(H_{\mathbb{C}}, Z_{\mathbb{C}})$ acts trivially on the space*

$$W^{\mathfrak{h}'} = \sum_{j=1}^{n'} \text{Hom}(V_j, V_j)^J \subseteq W.$$

This is verified by the same argument as Lemma 5.2.

Recall the positive root system Ψ , (1.1). Let $\Phi = -\Psi$. Let $\Phi(\mathfrak{h}_{\mathbb{C}}, \mathfrak{z}_{\mathbb{C}}) = \{\alpha \in \Phi \mid \mathfrak{g}_{\mathbb{C}, \alpha} \subseteq \mathfrak{z}_{\mathbb{C}}\}$. Set

$$\pi_{\mathfrak{g}/\mathfrak{h}} = \prod_{\alpha \in \Phi} \alpha, \quad \pi_{\mathfrak{z}/\mathfrak{h}} = \prod_{\alpha \in \Phi(\mathfrak{h}, \mathfrak{z})} \alpha.$$

Proposition 6.3. *There is a constant u , with $u^4 = 1$, such that, for any polynomial P satisfying (0.4), $z' \in \mathfrak{h}'_{\mathbb{C}}$ and $z \in \mathfrak{h}_{\mathbb{C}}$, we have*

$$\begin{aligned}
 & P(z')\pi_{\mathfrak{g}'/\mathfrak{h}'}(z')chc_{\text{Hom}(V,V'_0{}^\perp)}(z'+z)\pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(z) \\
 &= \frac{u\sqrt{2}^{\dim_{\mathbb{R}} \text{Hom}(V,V'_0{}^\perp)}}{2^{n'}|W(\mathbb{H}_{\mathbb{C}},\mathbb{Z}_{\mathbb{C}})|} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \text{sgn}(\text{short})(s) \frac{P(s^{-1} \cdot z)\pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(s^{-1} \cdot z)}{\det(z'+z)_{sW^{b'}}},
 \end{aligned}$$

where $\pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})$ is the product of all the positive short roots, and the “ $\text{sgn}(\text{short})$ ” is defined by

$$\pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(s \cdot z) = \text{sgn}(\text{short})(s) \pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(z).$$

This proposition is proved in Appendix B, Proposition B.5.

7. Convex cones and chc as the boundary value of a holomorphic function

Fix $\mathcal{S} \in \Psi_{st}^n$. Let

$$\begin{aligned}
 \mathcal{S}^{\vee\perp} &= \{\xi \in \mathfrak{h}^* \mid \xi(H_\alpha) = 0 \text{ for all } \alpha \in \mathcal{S}\}, \\
 \underline{\mathcal{S}}^{\vee\perp} &= \{\xi \in \mathfrak{h}^* \mid \xi(J_j) = 0 \text{ for all } j \in \underline{\alpha}, \alpha \in \mathcal{S}\}.
 \end{aligned}$$

For a set $C \subseteq \mathfrak{h}^*$, let

$$C^o = \{y \in \mathfrak{h} \mid \xi(y) > 0 \text{ for all } \xi \in C \setminus 0\}.$$

Recall the moment map

$$\tau_{\mathfrak{h}} : W \rightarrow \mathfrak{h}^*, \quad \tau_{\mathfrak{h}}(w)(x) = \langle x(w), w \rangle \quad (x \in \mathfrak{h}, w \in W).$$

Lemma 7.1. *Let $s \in W(\mathbb{H}_{\mathbb{C}})$ and let*

$$\Gamma_{s,\mathcal{S}} = (\mathcal{S}^{\vee\perp} \cap \tau_{\mathfrak{h}}(sW^{b'}))^o. \tag{7.1}$$

Then

$$\Gamma_{s,\mathcal{S}} = (\underline{\mathcal{S}}^{\vee\perp} \cap \tau_{\mathfrak{h}}(sW^{b'}))^o, \tag{7.2}$$

$$\Gamma_{s,\mathcal{S}} = \{y \in \mathfrak{h} \mid \langle y, \cdot \rangle_{sW^{b'} \cap \sum_{j \notin \underline{\mathcal{S}}} \text{Hom}(V_j, V')} > 0\}. \tag{7.3}$$

Moreover,

$$y_{s,\mathcal{S}} \in \Gamma_{s,\mathcal{S}}, \tag{7.4}$$

where $y_{s,\mathcal{S}}$ was defined in (3.13).

If $\mathbb{D} = \mathbb{C}$ and $s = \sigma \in \Sigma_n$, then

$$\Gamma_{s,\mathcal{S}} = \sum_{j=1, \sigma(j) \notin \underline{\mathcal{S}}}^{n'} (0, \infty) J_{\sigma(j)}^*(y_\sigma) J_{\sigma(j)} + \sum_{j=1, \sigma(j) \in \underline{\mathcal{S}}}^{n'} \mathbb{R} J_{\sigma(j)} + \sum_{j=n'+1}^n \mathbb{R} J_{\sigma(j)}. \tag{7.5}$$

If $\mathbb{D} \neq \mathbb{C}$ and $s = \sigma\epsilon$, with $\sigma \in \Sigma_n$ and $\epsilon \in \mathbb{Z}_2^n$, then

$$\Gamma_{s,\mathcal{S}} = \sum_{j=1, \sigma(j) \notin \underline{\mathcal{S}}}^{n'} (0, \infty) J_{\sigma(j)}^*(y_s) J_{\sigma(j)} + \sum_{j=1, \sigma(j) \in \underline{\mathcal{S}}}^{n'} \mathbb{R} J_{\sigma(j)} + \sum_{j=n'+1}^n \mathbb{R} J_{\sigma(j)}. \tag{7.6}$$

Proof. Clearly (7.2) and (7.3) are equivalent. Moreover, (7.5), (7.6) and (7.3) imply (7.4). Also, (7.5) and (7.6) imply (7.2). Hence it will suffice to show that (7.1) implies (7.5) and (7.6).

Suppose $\mathbb{D} = \mathbb{C}$. By Definition 3.4,

$$\tau_{\mathfrak{h}}(\sigma \operatorname{Hom}(\mathbf{V}_j, \mathbf{V}_j)) = [0, \infty) J_{\sigma(j)}^*(y_\sigma) J_{\sigma(j)}^*,$$

and, by Lemmas 5.2 and 6.2,

$$\tau_{\mathfrak{h}}(\sigma \mathbf{W}^{b'}) = \sum_{j=1}^{n'} [0, \infty) J_{\sigma(j)}^*(y_\sigma) J_{\sigma(j)}^*.$$

Therefore

$$\mathcal{S}^{\vee\perp} \cap \tau_{\mathfrak{h}}(\sigma \mathbf{W}^{b'}) = \sum_{j=1, \sigma(j) \notin \underline{\mathcal{S}}}^{n'} [0, \infty) J_{\sigma(j)}^*(y_\sigma) J_{\sigma(j)}^*$$

and (7.5) follows.

Suppose $\mathbb{D} \neq \mathbb{C}$. Let $1 \leq j \leq n'$. Then,

$$s(\operatorname{Hom}(\mathbf{V}_j, \mathbf{V}_j)^0) = \operatorname{Hom}(\mathbf{V}_{\sigma(j)}, \mathbf{V}_j)^{\epsilon_j},$$

so that

$$\tau_{\mathfrak{h}}(s \mathbf{W}^{b'}) = \sum_{j=1}^{n'} [0, \infty) J_{\sigma(j)}^*(y_s) J_{\sigma(j)}^*.$$

Hence,

$$\mathcal{S}^{\vee\perp} \cap \tau_{\mathfrak{h}}(s \mathbf{W}^{b'}) = \sum_{j=1, \sigma(j) \notin \underline{\mathcal{S}}}^{n'} [0, \infty) J_{\sigma(j)}^*(y_s) J_{\sigma(j)}^*$$

and (7.6) follows. ■

Lemma 7.2. *Suppose $G' \neq O_{2p+1, 2q}$. Let $x \in \mathfrak{h}_{\mathcal{S}}$ and let $x' \in \mathfrak{h}'$. Then*

$$\ker(x' + x) = \ker(x' + c(\mathcal{S})x) \subseteq \mathbf{W} \tag{7.7}$$

and for x regular,

$$ch_{C_{\mathbf{W}}}(x' + c(\mathcal{S})x) = ch_{C_{\mathbf{W}}}(x' + x). \tag{7.8}$$

If $G' = O_{2p+1, 2q}$, then (7.7) and (7.8) hold with \mathbf{W} replaced by $\operatorname{Hom}(\mathbf{V}, \mathbf{V}_0^\perp)$.

Proof. Suppose $G' \neq O_{2p+1, 2q}$. Notice that for a root α , the Cayley transform $c(\alpha) = \operatorname{Ad}(\tilde{c}(\alpha))$, where

$$\tilde{c}(\alpha) = \exp\left(-i\frac{\pi}{4}(X_\alpha + X_{-\alpha})\right) \in G_{\mathbb{C}} \subseteq \operatorname{GL}(\mathbf{W}).$$

Hence,

$$c(\mathcal{S}) = \operatorname{Ad}(\tilde{c}(\mathcal{S})), \quad \text{where } \tilde{c}(\mathcal{S}) = \prod_{\alpha \in \mathcal{S}} \tilde{c}(\alpha).$$

Therefore,

$$\ker(x' + c(\mathcal{S})x) = \ker(\tilde{c}(\mathcal{S})(x' + x)\tilde{c}(\mathcal{S})^{-1}) = \tilde{c}(\mathcal{S}) \ker(x' + x) = \ker(x' + x),$$

and (7.7) follows.

The left hand side of (7.8) is equal to

$$(-1)^{p+} \sqrt{2}^{\dim_{\mathbb{R}}(W)} \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\det(x' + c(\mathcal{S})x)_W} = (-1)^{p+} \sqrt{2}^{\dim_{\mathbb{R}}(W)} \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\det(x' + x)_W},$$

which coincides with the right hand side.

The case $G' = O_{2p+1,2q}$ is analogous. ■

As in section 2, the positive definite symmetric form $\tilde{\kappa}$ determines a Lebesgue measure μ on $\mathfrak{h}(\mathcal{S})$. We transport this measure to $\mathfrak{h}_{\mathcal{S}}$ via the isomorphism $c(\mathcal{S})$ (see (1.6)).

Let

$$m_{\mathcal{S}} = \frac{|Stab_{W(\mathbb{H})}(\mathcal{S} \cup (-\mathcal{S}))|}{|W(\mathbb{H})||W(\mathbb{H}_{\mathcal{S}})|}, \tag{7.9}$$

so that for any $\psi \in S(\mathfrak{g})$, we have the following version of the Weyl integration formula,

$$\int_{\mathfrak{g}} \psi(x) d\mu(x) = \sum_{\mathcal{S} \in \Psi_{st}^n} m_{\mathcal{S}} \int_{\mathfrak{h}_{\mathcal{S}}} |\pi_{\mathfrak{g}/\mathfrak{h}}(x)| \psi^{\mathcal{S}}(x) d\mu(x). \tag{7.10}$$

For $s \in W(\mathbb{H}_{\mathbb{C}})$, define

$$m_{\mathcal{S}}(s) = \begin{cases} \frac{m_{\mathcal{S}} u \sqrt{2}^{\dim_{\mathbb{R}} W}}{|W(\mathbb{H}_{\mathbb{C}}, \mathbb{Z}_{\mathbb{C}})|} \text{sgn}(s) & \text{if } G' \neq O_{2p+1,2q}, \\ \frac{m_{\mathcal{S}} u \sqrt{2}^{\dim_{\mathbb{R}} W}}{2^{n'-n} |W(\mathbb{H}_{\mathbb{C}}, \mathbb{Z}_{\mathbb{C}})|} \text{sgn}(short)(s) & \text{if } G' = O_{2p+1,2q} \text{ and } n' = p + q. \end{cases} \tag{7.11}$$

The constant u is as in Proposition 5.3 or 6.3. Recall that n' is the rank of G' .

We denote by $\Psi(short)$ ($\Psi_{\mathcal{S},\mathbb{R}}(short)$) the set of all the short roots of Ψ ($\Psi_{\mathcal{S},\mathbb{R}}$). Let

$$\tilde{\Psi} = \begin{cases} \Psi(short) & \text{if } G' = O_{2p+1,2q}, \\ \Psi & \text{otherwise.} \end{cases}$$

(and similarly for $\Psi_{\mathcal{S},\mathbb{R}}(short)$) and let

$$\tilde{\pi}_{\mathfrak{g}/\mathfrak{h}} = \prod_{\alpha \in \tilde{\Psi}} \alpha.$$

Put for $s \in W(\mathbb{H}_{\mathbb{C}})$, $\widetilde{\text{sgn}}(s) \in \{\pm 1\}$ such that

$$\widetilde{\text{sgn}}(s) \tilde{\pi}_{\mathfrak{g}/\mathfrak{h}} \circ s = \tilde{\pi}_{\mathfrak{g}/\mathfrak{h}}.$$

Theorem 7.3. *Let P be a polynomial on \mathfrak{h}' satisfying (0.4), and let $\psi \in S(\mathfrak{g})$ and $x' \in \mathfrak{h}'^{\text{reg}}$. If $G' = O_{2p+1,2q}$ with $p + q \geq 1$, we can assume that the symplectic space $\langle \cdot, \cdot \rangle$ and the positive root system Ψ are such that*

$$\langle J, \cdot \rangle_{\text{Hom}(\mathbb{V}, \mathbb{V}'_0)} > 0 \text{ and } i\beta(J) < 0 \text{ for all } \beta \in \Psi. \tag{7.12}$$

Then, for any dual pairs (G, G') (see (0.2)),

$$\begin{aligned}
 P(x')\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x' + x)\psi(x) d\mu(x) &= \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} m_{\mathcal{S}}(s) \\
 \lim_{y \in \Gamma_{s, \mathcal{S}}, y \rightarrow 0} \int_{\mathfrak{h}_{\mathcal{S}}} \frac{P(s^{-1} \cdot (x + iy))\tilde{\pi}_{\mathfrak{h}/\mathfrak{h}}(s^{-1} \cdot (x + iy))}{\det(x' + x + iy)_{sW\mathfrak{h}'}} \mathcal{A}(-\tilde{\Psi}_{\mathcal{S}, \mathbb{R}})(x)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x), & \tag{7.13}
 \end{aligned}$$

where $A(-\tilde{\Psi}_{\mathcal{S}, \mathbb{R}}) = (-1)^{|\tilde{\Psi}_{\mathcal{S}, \mathbb{R}}|} A(\tilde{\Psi}_{\mathcal{S}, \mathbb{R}}) = \prod_{\alpha \in \tilde{\Psi}_{\mathcal{S}, \mathbb{R}}} \frac{-\alpha}{|\alpha|}$.

Proof. First, we verify (7.13). Suppose ψ is compactly supported in a completely invariant open subset of \mathfrak{g} , which is disjoint with the singular support of the distribution $chc(x' + \cdot)$. Then there is $\epsilon > 0$ such that for any $\mathcal{S} \in \Psi_{st}^n$,

$$|\det(x' + x)_{sW\mathfrak{h}'}| > \epsilon \quad (x \in \text{supp } \psi_{\mathcal{S}}, s \in W(\mathbb{H}_{\mathbb{C}})).$$

Hence Proposition 5.3 and (7.10) imply the formula (7.13) with $y = 0$, and hence with $y \rightarrow 0$.

Let $\tilde{x} \in \mathfrak{g}$ be a semisimple element in the singular support of the distribution $chc(x' + \cdot)$. Since this distribution is conjugation invariant, we may assume that there is $\tilde{\mathcal{S}} \in \Psi_{st}^n$ such that $\mathfrak{h}(\tilde{\mathcal{S}})$ is a fundamental Cartan subalgebra of $\mathfrak{g}^{\tilde{x}}$. Let

$$\Psi_{\tilde{\mathcal{S}}, \mathbb{R}}^n(\mathfrak{g}^{\tilde{x}}) = \{\alpha \in \Psi_{\tilde{\mathcal{S}}, \mathbb{R}}^n \mid c(\tilde{\mathcal{S}})\mathfrak{g}_{\mathbb{C}, \alpha} \subseteq \mathfrak{g}_{\mathbb{C}}^{\tilde{x}}\} = \{\alpha \in \Psi_{\tilde{\mathcal{S}}, \mathbb{R}}^n \mid \alpha \circ c(\tilde{\mathcal{S}})^{-1}(\tilde{x}) = 0\}.$$

This is a positive root system of non-compact imaginary roots of $\mathfrak{h}(\tilde{\mathcal{S}})$ in $\mathfrak{g}_{\mathbb{C}}^{\tilde{x}}$. Schmid’s description of the Cartan subalgebras (see Section 2 of [13]) shows that any Cartan subalgebra of $\mathfrak{g}^{\tilde{x}}$ is G -conjugate to one of the form $\mathfrak{h}(\mathcal{S})$, with

$$\mathcal{S} = \cdots ((\tilde{\mathcal{S}} \vee \alpha_1) \vee \alpha_2) \cdots \vee \alpha_k \in \Psi_{st}^n \tag{7.14}$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subseteq \Psi_{\tilde{\mathcal{S}}, \mathbb{R}}^n(\mathfrak{g}^{\tilde{x}})$ is a strongly orthogonal set. We shall denote by $\Psi_{st}^n(\mathfrak{g}^{\tilde{x}})$ the set of all the strongly orthogonal sets which occur in (7.14). In particular, we have

$$\underline{\mathcal{S}} \supseteq \tilde{\mathcal{S}} \quad (\mathcal{S} \in \Psi_{st}^n(\mathfrak{g}^{\tilde{x}})). \tag{7.15}$$

Since the element $x' + \tilde{x} \in \mathfrak{sp}(W)$ is semisimple, the restriction of the symplectic form $\langle \cdot, \cdot \rangle$ to $\ker(x' + \tilde{x}) \subseteq W$ is non-degenerate. Furthermore, as checked in [11], the restriction of the pair $(G^{\tilde{x}}, G')$ to $\ker(x' + \tilde{x})$ is isomorphic to the direct sum of the pairs $(U_{p,q}, U_1)$, with the appropriate p and q . Let $\mathcal{U} \subseteq \mathfrak{g}^{\tilde{x}}$ be a slice through \tilde{x} , see [17, I, page 26], such that

$$\ker(x' + x) \subseteq \ker(x' + \tilde{x}) \quad (x \in \mathcal{U}).$$

From now on, we assume that

$$\text{supp } \psi \subseteq G \cdot \mathcal{U}. \tag{7.16}$$

Let $\mathfrak{z}(\mathfrak{g}^{\tilde{x}})$ denote the center of $\mathfrak{g}^{\tilde{x}}$. Then $\mathfrak{z}(\mathfrak{g}^{\tilde{x}})|_{\ker(x' + \tilde{x})} \subseteq \mathfrak{h}|_{\ker(x' + \tilde{x})}$. Define

$$\Gamma_{\tilde{x}} = \{y \in \mathfrak{z}(\mathfrak{g}^{\tilde{x}}) \mid \langle y, \cdot \rangle_{\ker(x' + \tilde{x})} > 0 \text{ and } y|_{\ker(x' + \tilde{x})^\perp} = 0\}. \tag{7.17}$$

where $\ker(x' + \tilde{x})^\perp \subseteq \mathbf{W}$ is the orthogonal complement of $\ker(x' + \tilde{x})$, with respect to the symplectic form. Then, by [11, (9.3)], the left hand side of the equation (7.13) coincides with

$$\lim_{y \in \Gamma_{\tilde{x}}, y \rightarrow 0} P(x') \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathbf{G}/\mathbf{G}^{\tilde{x}}} \int_{\mathcal{U}} chc(x' + x + iy) |\det(\text{ad}(x))_{\mathfrak{g}/\mathfrak{g}^{\tilde{x}}}| \psi(g \cdot x) d\mu(x) d\mu(g\mathbf{G}^{\tilde{x}}). \quad (7.18)$$

The formula (1.11) shows that for $x \in \mathfrak{h}_{\mathcal{S}}$,

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(x)|^2 = \pi_{\mathfrak{g}/\mathfrak{h}}(x) \overline{\pi_{\mathfrak{g}/\mathfrak{h}}(x)} = \pi_{\mathfrak{g}/\mathfrak{h}}(x) (-1)^{|\Psi_{\mathcal{S}, \mathbb{R}}|} \prod_{\alpha \in \Psi} \alpha(x).$$

Hence, by combining Proposition 5.3, (7.15) and [4, Theorem 1.5], we see that (7.18) coincides with

$$\lim_{y \in \Gamma_{\tilde{x}}, y \rightarrow 0} \sum_{\mathcal{S} \in \Psi_{st}^n(\mathfrak{g}^{\tilde{x}})} \sum_{s \in W(\mathbf{H}_{\mathbb{C}})} m_{\mathcal{S}}(s) \int_{\mathfrak{h}_{\mathcal{S}}} \frac{P(s^{-1} \cdot (x + iy)) \pi_{\mathfrak{g}/\mathfrak{h}}(s^{-1} \cdot (x + iy))}{\det(x' + x + iy)_{s\mathbf{W}\mathfrak{h}'}} (-1)^{|\Psi_{\mathcal{S}, \mathbb{R}}|} \psi_{\mathcal{S}}(x) d\mu(x). \quad (7.19)$$

In order to shed some more light at the situation recall that we have the following decompositions

$$\begin{aligned} \mathbf{W} &= \ker(x' + \tilde{x}) \oplus \ker(x' + \tilde{x})^\perp, \\ \mathfrak{g}^{\tilde{x}} &= \mathfrak{g}^{\tilde{x}}|_{\ker(x'+\tilde{x})} \oplus \mathfrak{g}^{\tilde{x}}|_{\ker(x'+\tilde{x})^\perp}, \\ \mathfrak{g}^{\tilde{x}} &= \mathfrak{g}_1^{\tilde{x}} \oplus \mathfrak{g}_2^{\tilde{x}} \oplus \cdots \oplus \mathfrak{g}_m^{\tilde{x}}, \end{aligned}$$

where each $\mathfrak{g}_i^{\tilde{x}}$ is isomorphic to some \mathfrak{u}_{p_i, q_i} , $1 \leq i \leq m$. Let $\mathfrak{g}_{m+1}^{\tilde{x}} = \mathfrak{g}^{\tilde{x}}|_{\ker(x'+\tilde{x})^\perp}$ and let

$$\mathfrak{h}_i = \mathfrak{h}(\tilde{\mathcal{S}}) \cap \mathfrak{g}_i^{\tilde{x}} \quad (1 \leq i \leq m+1).$$

Then

$$\mathfrak{h}(\tilde{\mathcal{S}}) = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_m \oplus \mathfrak{h}_{m+1},$$

where

$$\mathfrak{h}_i \subseteq \mathfrak{h} \quad (1 \leq i \leq m)$$

is an elliptic Cartan subalgebra of $\mathfrak{g}_i^{\tilde{x}}$. Moreover,

$$\Psi_{st}^n(\mathfrak{g}^{\tilde{x}}) = \bigcup_{i=1}^{m+1} \Psi_{st}^n(\mathfrak{g}_i^{\tilde{x}})$$

is a disjoint union, where for $1 \leq i \leq m$, the $\Psi_{st}^n(\mathfrak{g}_i^{\tilde{x}})$ is the set of all the strongly orthogonal sets of the positive non-compact (imaginary) roots of \mathfrak{h}_i in $\mathfrak{g}_i^{\tilde{x}}$ and $\Psi_{st}^n(\mathfrak{g}_{m+1}^{\tilde{x}})$ contains $\tilde{\mathcal{S}}$. In particular for any such root α the support $\underline{\alpha}$ has two elements and the function $\det(x' + x)^{-1}$ is locally integrable over the subspace

$$\left(\sum_{j \in \underline{\alpha}} \mathbb{C}J_j \right) \cap \mathfrak{h}_{\tilde{\mathcal{S}} \vee \alpha}.$$

We may repeat this argument adding more roots. Hence, the condition (7.16) on the support of ψ implies that for each $\mathcal{S} \in \Psi_{st}^n(\mathfrak{g}^{\tilde{x}})$,

$$\det(x' + x)^{-1} \psi_{\mathcal{S}}(x) \text{ is locally integrable over } \left(\sum_{j \in \underline{\alpha}} \mathbb{C} J_j \right) \cap \mathfrak{h}_{\mathcal{S}}.$$

Fix s and \mathcal{S} as in (7.19). Let $\Gamma_{s, \mathcal{S}, \tilde{x}} =$

$$\{y \in \mathfrak{h} \mid \langle y, \cdot \rangle_{s\mathcal{W}^{b'} \cap \ker(x'+\tilde{x})} > 0, y|_{\ker(x'+\tilde{x})^\perp} = 0, \text{ and } J_j^*(y) = 0 \text{ for all } j \in \underline{\mathcal{S}}\}.$$

Then every element $y \in \Gamma_{s, \mathcal{S}}$ may be written uniquely as $y = y_1 + y_2$, with $y_1 \in \Gamma_{s, \mathcal{S}, \tilde{x}}$ and $y_2|_{\ker(x'+\tilde{x})} = 0$. Hence, as a generalized function of $x \in \mathfrak{h}_{\mathcal{S}}$ applied to $\psi_{\mathcal{S}}$,

$$\begin{aligned} & \lim_{y \in \Gamma_{\tilde{x}}, y \rightarrow 0} \det(x' + x + iy)^{-1}_{s\mathcal{W}^{b'}} \\ &= \lim_{y \in \Gamma_{\tilde{x}}, y \rightarrow 0} \det(x' + x + iy)^{-1}_{s\mathcal{W}^{b'} \cap \ker(x'+\tilde{x})} \det(x' + x)^{-1}_{s\mathcal{W}^{b'} \cap \ker(x'+\tilde{x})^\perp} \\ &= \lim_{y \in \Gamma_{s, \mathcal{S}, \tilde{x}}, y \rightarrow 0} \det(x' + x + iy)^{-1}_{s\mathcal{W}^{b'} \cap \ker(x'+\tilde{x})} \det(x' + x)^{-1}_{s\mathcal{W}^{b'} \cap \ker(x'+\tilde{x})^\perp} \\ &= \lim_{y \in \Gamma_{s, \mathcal{S}}, y \rightarrow 0} \det(x' + x + iy_1)^{-1}_{s\mathcal{W}^{b'} \cap \ker(x'+\tilde{x})} \det(x' + x + iy_2)^{-1}_{s\mathcal{W}^{b'} \cap \ker(x'+\tilde{x})^\perp} \\ &= \lim_{y \in \Gamma_{s, \mathcal{S}}, y \rightarrow 0} \det(x' + x + iy)^{-1}_{s\mathcal{W}^{b'}}. \end{aligned}$$

Thus we may write the limit after the summation in (7.19), and replace $\Gamma_{\tilde{x}}$ by $\Gamma_{s, \mathcal{S}}$. Hence, by localization (see [17, I, page 18]) the equation (7.13) holds for an arbitrary function $\psi \in S(\mathfrak{g})$. This completes the proof of (7.13).

For (7.13), we sketch the idea, without going into the localization details as above. Let $\mathcal{S} \in \Psi_{st}^n$ and $x \in \mathfrak{h}_{\mathcal{S}}$. Then, by Lemma 7.2,

$$\begin{aligned} chc(x' + c(\mathcal{S})x) &= chc_{\text{Hom}(V, V'_0)}(c(\mathcal{S})x) chc_{\text{Hom}(V, V'_0^\perp)}(x' + c(\mathcal{S})x) \\ &= chc_{\text{Hom}(V, V'_0)}(c(\mathcal{S})x) chc_{\text{Hom}(V, V'_0^\perp)}(x' + x). \end{aligned}$$

Let $\Psi = \{i(J_j^* \pm J_k^*) \mid 1 \leq j < k \leq n\} \cup \{2iJ_j^* \mid 1 \leq j \leq n\}$. Then

$$\begin{aligned} & chc_{\text{Hom}(V, V'_0)}(x' + c(\mathcal{S})x) \\ &= \prod_{j \notin \underline{\mathcal{S}}} \frac{2i}{x_j} \cdot \prod_{j \in \underline{\mathcal{S}}} \frac{2}{|x_j|} = \prod_{j \notin \underline{\mathcal{S}}} \frac{2i}{x_j} \cdot \prod_{j \in \underline{\mathcal{S}}(long)} \frac{2}{|x_j|} \cdot \prod_{j \in \underline{\mathcal{S}}(short)} \frac{2}{|x_j|} \\ &= \prod_{j \notin \underline{\mathcal{S}}} \frac{2i}{x_j} \cdot \prod_{j \in \underline{\mathcal{S}}(long)} \frac{2}{x_j} \cdot \prod_{j \in \underline{\mathcal{S}}(long)} \frac{x_j}{|x_j|} \cdot \prod_{j \in \underline{\mathcal{S}}(short)} \frac{2}{x_j} \cdot (-1)^{|\mathcal{S}(short)|} \\ &= 2^n i^{n-|\underline{\mathcal{S}}|} (-1)^{|\mathcal{S}(short)|} \prod_{j=1}^n \frac{1}{x_j} \cdot \prod_{j \in \underline{\mathcal{S}}(long)} \frac{x_j}{|x_j|} \\ &= 4^n i^{-|\underline{\mathcal{S}}|+|\mathcal{S}(long)|} (-1)^{|\mathcal{S}(short)|} \prod_{\alpha \in \Psi(long)} \frac{1}{-\alpha(x)} \cdot \prod_{\alpha \in \mathcal{S}(long)} \frac{-\alpha(x)}{|\alpha(x)|} \\ &= 4^n \prod_{\alpha \in \Psi(long)} \frac{1}{-\alpha(x)} \cdot \prod_{\alpha \in \mathcal{S}(long)} \frac{-\alpha(x)}{|\alpha(x)|}. \end{aligned}$$

Hence,

$$\begin{aligned} & chc_{\text{Hom}(\mathbb{V}, \mathbb{V}'_0)}(c(\mathcal{S})x) \cdot \prod_{\alpha \in \Psi(\text{long})} (-\alpha(x)) \cdot \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}})(x) \quad (7.20) \\ &= 4^n \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}}(\text{short}))(x). \end{aligned}$$

Notice that in the formula (7.20) the symplectic form $\langle \cdot, \cdot \rangle$ and the positive root system Ψ are linked together by the relation (7.12). Therefore, by Proposition 6.3,

$$\begin{aligned} & P(x') \prod_{\alpha' \in \Psi'} \alpha'(x') \cdot chc(x' + c(\mathcal{S})x) \cdot \prod_{\alpha \in \Psi} (-\alpha(x)) \cdot \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}})(x) \\ &= chc_{\text{Hom}(\mathbb{V}, \mathbb{V}'_0)}(x' + c(\mathcal{S})x) \prod_{\alpha \in \Psi(\text{long})} (-\alpha(x)) \cdot \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}}(\text{short}))(x) \\ &\quad P(x') \prod_{\alpha' \in \Psi'} \alpha'(x') \cdot chc_{\text{Hom}(\mathbb{V}, \mathbb{V}'_0^\perp)}(x' + c(\mathcal{S})x) \prod_{\alpha \in \Psi(\text{short})} (-\alpha(x)) \\ &= 4^n \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}})(x) P(x') \prod_{\alpha' \in \Psi'} \alpha'(x') \cdot chc_{\text{Hom}(\mathbb{V}, \mathbb{V}'_0^\perp)}(x' + c(\mathcal{S})x) \pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(x) \\ &= 4^n \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}}(\text{short}))(x) \frac{u\sqrt{2}^{\dim_{\mathbb{R}} \text{Hom}(\mathbb{V}, \mathbb{V}'_0^\perp)}}{2^{n'} |W(\mathbb{H}_{\mathbb{C}}, \mathbb{Z}_{\mathbb{C}})|} \\ &\quad \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \text{sgn}(\text{short})(s) \frac{P(s^{-1} \cdot x) \pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(s^{-1} \cdot x)}{\det(x' + x)_{sW\mathfrak{h}'}} \\ &= \frac{u\sqrt{2}^{\dim_{\mathbb{R}} W}}{2^{n'-n} |W(\mathbb{H}_{\mathbb{C}}, \mathbb{Z}_{\mathbb{C}})|} \\ &\quad \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \text{sgn}(\text{short})(s) \frac{P(s^{-1} \cdot x) \pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(s^{-1} \cdot x)}{\det(x' + x)_{sW\mathfrak{h}'}} \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}}(\text{short}))(x). \end{aligned}$$

This, combined with the Weyl integration formula (7.10), implies (7.13). ■

Theorem 7.4. *Under the assumptions of Theorem 7.3, let $z = x + iy$, and let*

$$\tilde{F}_{s, x', \mathcal{S}}(z) = m_{\mathcal{S}}(s) \frac{\tilde{\pi}_{\mathfrak{g}/\mathfrak{h}}(s^{-1} \cdot z)}{\det(x' + z)_{sW\mathfrak{h}'}} \mathcal{A}(-\tilde{\Psi}_{\mathcal{S}, \mathbb{R}}(x)).$$

Then, for any $1 \leq j \leq n'$,

$$\begin{aligned} & \partial(J'_j) \left(P(x') \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x' + x) \psi(x) d\mu(x) \right) \\ &= \sum_{S \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \lim_{y \in \Gamma_{s, S}, y \rightarrow 0} \int_{\mathfrak{h}_S} \tilde{F}_{s, x', \mathcal{S}}(z) \partial(s \cdot J_j)(P(s^{-1} \cdot z) \mathcal{H}_S \psi(x)) d\mu(x). \end{aligned}$$

Remark. Notice that the function $\mathcal{H}_S \psi$ does not have to be smooth on \mathfrak{h}_S . However, it is smooth on the complement of the zero set of all the non-compact

imaginary roots. The derivative

$$\partial(s \cdot J_j)(P(s^{-1} \cdot z)\mathcal{H}_S\psi(x))$$

stands for the usual derivative on that set. In other words the integral is over $\mathfrak{h}_S \setminus \mathfrak{h}^{\Psi_{S, \mathbb{R}}^n}$ rather than \mathfrak{h}_S .

We shall prove this theorem via a case by case analysis in Appendix C.

8. Differential forms

Recall the symmetric form $\tilde{\kappa}$, which determines the measure μ . Define the following n -form on the n -dimensional complex vector space $\mathfrak{h}_{\mathbb{C}}$:

$$\nu = \prod_{j=1}^n \tilde{\kappa}(J_j, J_j)^{1/2} dJ_1^* dJ_2^* \cdots dJ_n^*.$$

For $\mathcal{S} \in \Psi_{st}^n$ and for $s \in W(\mathbb{H}_{\mathbb{C}})$, let

$$n_{\mathcal{S}}(s) = m_{\mathcal{S}}(s)|^{\mathcal{S}|}. \tag{8.1}$$

Let $N \in \mathbb{N}$. Define

$$\nu_{s, x', \mathcal{S}, N}(z) = n_{\mathcal{S}}(s) \frac{P(s^{-1} \cdot z) \tilde{\pi}_{\mathfrak{g}/\mathfrak{h}}(s^{-1} \cdot z)}{\det(x' + z)_{sW\mathfrak{h}'}} \mathcal{A}(-\tilde{\Psi}_{\mathcal{S}, \mathbb{R}})(x) (\mathcal{H}_S\psi)_N(z) \nu(z) \tag{8.2}$$

where $x \in \mathfrak{h}_S \setminus \mathfrak{h}^{\Psi_{S, \mathbb{R}} \cup \Psi_{S, \mathbb{R}}^n}$, $y \in \mathfrak{h}_S$, and $z = x + iy$.

Remark. Though the differential form (8.2) lives in the complexification of a Cartan subalgebra of the Lie algebra of the group G , we are going to use it in the context of a dual pair (G, G') , hence the distinction of the two cases $G' = O_{2p+1, 2q}$ and $G' \neq O_{2p+1, 2q}$.

For a function f defined on a subset of $\mathfrak{h}_{\mathbb{C}}$, and for a root $\beta \in \Psi$, let

$$\langle f\nu \rangle_{\beta} = \langle f \rangle_{\beta} \nu,$$

where $\langle f \rangle_{\beta}$ is the jump of f defined in (2.2).

Lemma 8.1. *Let $\beta \in \Psi_{S, \mathbb{R}}^n$, with $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$. Suppose $G \neq O_{2p+1, 2q}$. Then*

$$\frac{m_{\mathcal{S}}}{m_{\mathcal{S} \vee \beta}} = 2.$$

Suppose $G = O_{2p+1, 2q}$. Then

$$\frac{m_{\mathcal{S}}}{m_{\mathcal{S} \vee \beta}} = \begin{cases} 1 & \text{if } \mathcal{S} = \emptyset \text{ and } \beta \text{ is short,} \\ 2l + 1 & \text{if } \mathcal{S} \text{ consists of long roots only, } |\mathcal{S}_1| = 2l, \text{ and } \beta \text{ is short,} \\ 2 & \text{otherwise.} \end{cases} \tag{8.3}$$

Proof. If $G \neq O_{2p+1,2q}$, This can be easily proved using [13]. Otherwise, an explicit computation can be done. ■

Definition 8.2. Let $\beta \in \Psi_{\mathcal{S},i\mathbb{R}}^n$, $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$. Put

$$d(\mathcal{S}, \beta) = d(\beta) \frac{m_{\mathcal{S}}}{m_{S\vee\beta}}$$

Lemma 8.3. With the notations of the previous definition, suppose $G \neq O_{2p+1,2q}$. Then

$$d(\mathcal{S}, \beta) = 2.$$

Suppose $G = O_{2p+1,2q}$. Then

$$d(\mathcal{S}, \beta) = \begin{cases} (2l + 1)2 & \text{if } \mathcal{S} \text{ consists of long roots only, } |\mathcal{S}_1| = 2l, \text{ and } \beta \text{ is short,} \\ 4 & \text{if } \mathcal{S} \neq \emptyset \text{ and is not as in the previous case, and } \beta \text{ is short,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If G is a connected group then the results of [13] give a complete description of the Weyl groups. Otherwise these groups can be described explicitly. ■

Lemma 8.4. Fix $\mathcal{S} \in \Psi_{st}^n$, $s \in W(\mathbb{H}_{\mathbb{C}})$ and $\beta \in \Psi_{\mathcal{S},i\mathbb{R}}^n$, with $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$. Let $x \in \mathfrak{h}_{\mathcal{S}}$ be semiregular with $\beta(x) = 0$. For $y \in \mathfrak{h}_{\mathcal{S}} \setminus \mathfrak{h}^{\beta}$ set $z = x + iy$. Let $\tilde{x} = x + \frac{\beta(iy)}{2}H_{\beta}$ and let $\tilde{y} = y - \frac{\beta(y)}{2}H_{\beta}$, so that $z = \tilde{x} + i\tilde{y}$. If $\mathbb{D} = \mathbb{R}$ and $|\underline{\beta}| = 1$, assume that the element $\tilde{x} \in \mathfrak{h}_{S\vee\beta}$ satisfies the condition (1.15). Define

$$\begin{aligned} \tau_{s,x',S\vee\beta,\beta,N}(z) &= n_{S\vee\beta}(s) \frac{P(s^{-1} \cdot z) \tilde{\pi}_{\mathfrak{h}/\mathfrak{h}}(s^{-1} \cdot z)}{\det(x' + z)_{sW^{\mathfrak{h}'}}} \mathcal{A}(-\tilde{\Psi}_{S\vee\beta,\mathbb{R}})(\tilde{x}) \\ &\quad \partial(e_N(\frac{\beta(iy)}{2}H_{\beta} + i\tilde{y}))(\mathcal{H}_{S\vee\beta}\psi)(x) \nu(z). \end{aligned} \tag{8.4}$$

Then

$$\begin{aligned} &\langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta}(z) \tag{8.5} \\ &= -d(\mathcal{S}, \beta) \begin{cases} -\tau_{s,x',S\vee\beta,\beta,N}(z) & \text{if } \beta \text{ is a long root and } G' = O_{2p+1,2q}, \\ \frac{|\beta(\tilde{x})|}{\beta(\tilde{x})} \tau_{s,x',S\vee\beta,\beta,N}(z) & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Let $G' \neq O_{2p+1,2q}$. By Corollary 2.2, (8.1) and (8.2),

$$\begin{aligned} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta}(z) &= d(\mathcal{S}, \beta) n_{S\vee\beta}(s) \frac{P(s^{-1} \cdot z) \pi_{\mathfrak{h}/\mathfrak{h}}(s^{-1} \cdot z)}{\det(x' + z)_{sW^{\mathfrak{h}'}}} \mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x) \\ &\quad \epsilon(\Psi, \mathcal{S}, \beta) \partial(e_N(iy)) \mathcal{H}_{S\vee\beta}\psi(x) \nu(z). \end{aligned}$$

Since, by (1.16),

$$\begin{aligned} &\mathcal{A}(-\Psi_{S\vee\beta,\mathbb{R}})(\tilde{x}) \\ &= -\frac{\beta(\tilde{x})}{|\beta(\tilde{x})|} \mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x) \epsilon(\Psi, \mathcal{S}, \beta), \end{aligned}$$

the equality (8.5) follows.

Let $G' = O_{2p+1,2q}$. By Corollary 2.2, (8.1) and (8.2),

$$\langle \nu_{s,x',S,N} \rangle_\beta(z) = d(\mathcal{S}, \beta) n_{S \vee \beta}(s) \frac{P(s^{-1} \cdot z) \pi_{\mathfrak{h}/\mathfrak{h},short}(s^{-1} \cdot z)}{\det(x' + z)_{sW\mathfrak{h}'}} \mathcal{A}(-\Psi_{S,\mathbb{R}}(short))(x) \epsilon(\Psi, \mathcal{S}, \beta) \partial(e_N(iy)) \mathcal{H}_{S \vee \beta} \psi(x) \nu(z).$$

Since, by (1.16),

$$\begin{aligned} & \mathcal{A}(-\Psi_{S \vee \beta, \mathbb{R}}(short))(\tilde{x}) \\ = & \begin{cases} \mathcal{A}(-\Psi_{S, \mathbb{R}}(short))(x) \epsilon(\Psi, \mathcal{S}, \beta) & \text{if } \beta \text{ is a long root,} \\ -\frac{\beta(\tilde{x})}{|\beta(\tilde{x})|} \mathcal{A}(-\Psi_{S, \mathbb{R}}(short))(x) \epsilon(\Psi, \mathcal{S}, \beta) & \text{if } \beta \text{ is a short root.} \end{cases} \end{aligned}$$

The equality (8.5) follows. ■

Corollary 8.5. *With the notation of Lemma 8.4, let s_β be the reflection with respect to β . Then,*

$$\langle \nu_{s,x',S,N} \rangle_\beta(s_\beta \cdot z) = \widetilde{\text{sgn}}(s_\beta) \langle \nu_{s_\beta s, x', S, N} \rangle_\beta(z).$$

Proof. Let $G' \neq O_{2p+1,2q}$. The left hand side is equal to

$$\begin{aligned} & n_S(s) \frac{P((s_\beta s)^{-1} \cdot z) \pi_{\mathfrak{h}/\mathfrak{h}}((s_\beta s)^{-1} \cdot z)}{\det(x' + z)_{s_\beta s W \mathfrak{h}'}} \mathcal{A}(-\Psi_{S, \mathbb{R}})(s_\beta^{-1} \cdot x) \\ & \quad \epsilon(\Psi, \mathcal{S}, \beta) \text{id}(\beta) \partial(e_N(is_\beta^{-1} \cdot y)) \mathcal{H}_{S \vee \beta} \psi(s_\beta^{-1} \cdot x) \nu(s_\beta^{-1} \cdot z) \\ = & n_S(s_\beta s) \text{sgn}(s_\beta) \frac{P((s_\beta s)^{-1} \cdot z) \pi_{\mathfrak{h}/\mathfrak{h}}((s_\beta s)^{-1} \cdot z)}{\det(x' + z)_{s_\beta s W \mathfrak{h}'}} \mathcal{A}(-\Psi_{S, \mathbb{R}})(x) \\ & \quad \epsilon(\Psi, \mathcal{S}, \beta) \text{id}(\beta) \partial(e_N(iy)) \mathcal{H}_{S \vee \beta} \psi(x) \nu(z) \det(s_\beta), \end{aligned}$$

by Lemma 2.3. Since, by (3.10), $\text{sgn}(s_\beta) \det(s_\beta) = 1$, this coincides with the right hand side. The case $G' = O_{2p+1,2q}$ is entirely analogous. The difference is that

$$\text{sgn}(s_\beta) \det(s_\beta) = -\text{sgn}(s_\beta) = \begin{cases} 1 & \text{if } \beta \text{ is short,} \\ -1 & \text{if } \beta \text{ is long.} \end{cases} \quad \blacksquare$$

9. An application of Stokes Theorem

For a root α , let

$$\tilde{\alpha} = \begin{cases} \alpha & \text{if } \alpha \text{ is short and } G = O_{2p+1,2q}, \\ \frac{\alpha}{2} & \text{otherwise.} \end{cases} \tag{9.1}$$

For $\mathcal{S} \in \Psi_{st}^n$, $s \in W(H_{\mathbb{C}})$ and $\beta \in \mathcal{S}(y_s)$ define

$$T_{\mathcal{S}, \beta}(s) = \left\{ x \in \mathfrak{h}_{\mathcal{S}} \mid |\tilde{\beta}(x)| \leq \min\{1, |\tilde{\alpha}(x)|\}; \alpha \in \mathcal{S}(y_s) \setminus \beta \right\} \subseteq \mathfrak{h}_{\mathcal{S}}. \tag{9.2}$$

Also, let

$$T_{\mathcal{S}}(s) = \bigcup_{\beta \in \mathcal{S}(y_s)} T_{\mathcal{S},\beta}(s).$$

If there is $\mathcal{S}_\beta \in \Psi_{st}^n$, such that

$$\mathcal{S} = \mathcal{S}_\beta \vee \beta, \quad \beta \in \Psi_{\mathcal{S}_\beta, i\mathbb{R}}^n, \quad \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset, \tag{9.3}$$

then

$$T_{\mathcal{S},\beta}(s) = \{x + t m_{\mathcal{S}_\beta(y_s)}(x) \left| \frac{i\beta(y_s)}{2} \right| H_\beta \mid x \in \mathfrak{h}_{\mathcal{S}_\beta, \beta=0}, \quad -1 \leq t \leq 1\}.$$

The condition (9.3) holds, with $\mathcal{S}_\beta = \mathcal{S} \setminus \beta$ unless $G = O_{2p+1, 2q}$ and $\beta \notin \mathcal{S}$. If, in addition $\beta \in \Psi^c(short)$ and there is $\alpha \in \Psi^n(short)$ such that $\alpha \vee \beta \subseteq \mathcal{S}$, then (9.3) is satisfied with $\mathcal{S}_\beta = (\mathcal{S} \setminus (\alpha \vee \beta)) \vee \alpha$. In the remaining cases (9.3) does not hold.

Furthermore, for $G' \neq O_{2p+1, 2q}$, $x' \in \mathfrak{h}^{reg}$, $\mathcal{S} \in \Psi_{st}^n$, $s \in W(\mathbb{H}_\mathbb{C})$, $\beta \in \Psi_{\mathcal{S}, \mathbb{R}}$, $x \in \mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\beta$, $y \in \mathfrak{h}_\mathcal{S} \setminus \mathfrak{h}^\beta$, $z = x + iy$ and a non-negative integer N , set

$$\begin{aligned} & \tau_{s, x', \mathcal{S}, \beta, N}(z) \tag{9.4} \\ &= n_{\mathcal{S}}(s) \frac{P(s^{-1} \cdot z) \pi_{\mathfrak{g}/\mathfrak{h}}(s^{-1} \cdot z)}{\det(x' + z)_{sW\mathfrak{h}'}} \mathcal{A}(-\Psi_{\mathcal{S}, \mathbb{R}})(x_\beta) \partial(e_N(y)) (\mathcal{H}_\mathcal{S} \psi)(x) \nu(z), \end{aligned}$$

where $x_\beta = x + \frac{\beta(y)}{2} H_\beta$. Under the condition (9.3), the definition (9.4) coincides with (8.4).

Theorem 9.1. *Let $n' \geq 1$, $N \in \mathbb{N}$ and let P be a polynomial function on \mathfrak{h}' satisfying (0.4). Then, for $x' \in \mathfrak{h}^{reg}$ and $\psi \in S(\mathfrak{g})$,*

$$\begin{aligned} & P(x') \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x' + x) \psi(x) d\mu(x) \\ &= \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{C_{s, \emptyset}(1 \times \mathfrak{h})} \nu_{s, x', \emptyset, N} + \sum_{\emptyset \neq \mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{C_{s, \mathcal{S}}(1 \times (\mathfrak{h}_\mathcal{S} \setminus T_{\mathcal{S}}(s)))} \nu_{s, x', \mathcal{S}, N} \\ &+ \sum_{\emptyset \neq \mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \sum_{\beta \in \mathcal{S}(y_s)} \int_{C_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} (\nu_{s, x', \mathcal{S}, N} - \tau_{s, x', \mathcal{S}, \beta, N}) \\ &- \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{C_{s, \mathcal{S}}(I \times \mathfrak{h}_\mathcal{S})} d\nu_{s, x', \mathcal{S}, N} + E, \tag{9.5} \end{aligned}$$

where

$$E = \begin{cases} \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\substack{\beta \in \Psi_{\mathcal{S}, \mathbb{R}}(short), \\ i\beta(y_s) < 0}} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{C_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} 2 \frac{\beta}{|\beta|} \tau_{s, x', \mathcal{S}, \beta, N} & \text{if } G = O_{2p+1, 2q}, \\ \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\substack{\beta \in \mathcal{S}(long), \\ i\beta(y_s) > 0}} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{C_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} 2 \tau_{s, x', \mathcal{S}, \beta, N} & \text{if } G' = O_{2p+1, 2q}, \\ 0 & \text{otherwise.} \end{cases}$$

The relation (7.12) implies that $E = 0$, if $G' = O_{2p+1}$. Also, if G' is compact, then

$$\mathcal{S}(y_s) = \mathcal{S} \quad (\mathcal{S} \in \Psi_{st}^n, s \in W(\mathbb{H}_\mathbb{C})). \tag{9.6}$$

As we shall see in the next section, (9.6) implies that (9.5) is a smooth function of x' . If $G = O_{2p+1}$, then $E = 0$ because there is no non-empty \mathcal{S} . In fact, if G is compact then $\Psi_{st}^n = \emptyset$ and therefore, as we shall see in the next section, (9.5) is a smooth function of x' .

If $G = O_{2p+1,2q}$, then (9.17) below, implies that for any $s \in W(\mathbb{H}_\mathbb{C})$, $\mathcal{S} \in \Psi_{st}^n$, and $\beta \in \mathcal{S}_1$ (see Lemma 3.6 for the definition of \mathcal{S}_1),

$$\int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \frac{\beta}{|\beta|} \tau_{s,x',\mathcal{S},\beta,N} = - \int_{\mathcal{C}_{s_{\beta s},\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s_{\beta s}))} \frac{\beta}{|\beta|} \tau_{s_{\beta s},x',\mathcal{S},\beta,N}.$$

Also, $\beta(y_{s_{\beta s}}) = -\beta(y_s)$. Therefore,

$$\begin{aligned} E &= - \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\beta \in \mathcal{S}_1} \sum_{s \in W(\mathbb{H}_\mathbb{C}), i\beta(y_s) > 0} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} 2 \frac{\beta}{|\beta|} \tau_{s,x',\mathcal{S},\beta,N} \\ &= \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\beta \in \mathcal{S}_1} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \frac{-i\beta(y_s)}{|\beta(y_s)|} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \frac{\beta}{|\beta|} \tau_{s,x',\mathcal{S},\beta,N} \end{aligned}$$

Proof. Notice that, in terms of the form ν and the orientation (4.6), the Weyl integration formula (7.10), may be rewritten as follows:

$$\begin{aligned} \int_{\mathfrak{g}} \psi(x) d\mu(x) &= \sum_{\mathcal{S} \in \Psi_{st}^n} m_{\mathcal{S}} i^{|\mathcal{S}|} \int_{\mathfrak{h}_{\mathcal{S}}} |\pi_{\mathfrak{g}/\mathfrak{h}}| \psi^{\mathcal{S}} \nu \\ &= \sum_{\mathcal{S} \in \Psi_{st}^n} m_{\mathcal{S}} i^{|\mathcal{S}|} (-1)^{|\Psi_{\mathcal{S},\mathbb{R}}|} \int_{\mathfrak{h}_{\mathcal{S}}} \pi_{\mathfrak{g}/\mathfrak{h}} \psi^{\mathcal{S}} \nu \\ &= \sum_{\mathcal{S} \in \Psi_{st}^n} m_{\mathcal{S}} i^{|\mathcal{S}|} (-1)^{|\Psi_{\mathcal{S},\mathbb{R}}|} \int_{\mathfrak{h}_{\mathcal{S}}} \pi_{\mathfrak{g}/\mathfrak{h}} \mathcal{A}(\Psi_{\mathcal{S},\mathbb{R}}) (\mathcal{A}(\Psi_{\mathcal{S},\mathbb{R}}) \psi^{\mathcal{S}}) \nu \\ &= \sum_{\mathcal{S} \in \Psi_{st}^n} m_{\mathcal{S}} i^{|\mathcal{S}|} \int_{\mathfrak{h}_{\mathcal{S}}} \pi_{\mathfrak{g}/\mathfrak{h}} \mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}}) \mathcal{H}_{\mathcal{S}} \psi \nu, \end{aligned} \tag{9.7}$$

where the second equality follows from (1.11). By combining (9.7), Theorem 7.3, (8.1) and Lemma 4.7 with Stokes Theorem, [12, Theorem 16.1.6], we see that the left hand side of the equation (9.5) coincides with

$$\begin{aligned} &\sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \sum_{\gamma \in \mathfrak{F}\Psi_{\mathcal{S},\mathbb{R}}^n} \lim_{\epsilon \rightarrow 0} \left(\int_{\mathcal{C}_{s,\mathcal{S}}(1 \times \mathfrak{h}_{\mathcal{S},\gamma,\epsilon})} \nu_{s,x',\mathcal{S},N} \right. \\ &\quad \left. + \sum_{\beta \in \Psi_{\mathcal{S},\mathbb{R}}^n} \gamma(\beta) \int_{\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S},\gamma,-i\beta\gamma(\beta)=\epsilon})} \nu_{s,x',\mathcal{S},N} - \int_{\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S},\gamma,\epsilon})} d\nu_{s,x',\mathcal{S},N} \right). \end{aligned} \tag{9.8}$$

We see from Lemma 4.6 that

$$\lim_{\epsilon \rightarrow 0} \sum_{\gamma \in \mathfrak{F}\Psi_{\mathcal{S},\mathbb{R}}^n} \gamma(\beta) \int_{\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S},\gamma,-i\beta\gamma(\beta)=\epsilon})} \nu_{s,x',\mathcal{S},N} = \int_{\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S},\beta=0})} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta}. \tag{9.9}$$

Since ν is a form of top degree on the complex vector space $\mathfrak{h}_{\mathbb{C}}$, the restriction of ν to any proper complex subspace is zero. In particular, if $\beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n$ and $\beta(y_{\mathcal{S}}) = 0$, then the restriction of ν to $\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})$ is zero. Therefore, Definition 3.4, (4.3) and Lemma 4.2 imply that the integral on the right hand side of (9.9) is zero unless $\beta(y_{\mathcal{S}}) \neq 0$ and $\underline{\beta} \cap \underline{\mathcal{S}} = \emptyset$. Hence, (9.8) may be rewritten as

$$\sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \left(\int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}})} \nu_{s, x', \mathcal{S}, N} + \sum_{\substack{\beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \beta(y_{\mathcal{S}}) \neq 0 \\ \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset}} \int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})} \langle \nu_{s, x', \mathcal{S}, N} \rangle_{\beta} - \int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}})} d\nu_{s, x', \mathcal{S}, N} \right).$$

Corollary 8.5 implies that,

$$\int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})} \langle \nu_{s, x', \mathcal{S}, N} \rangle_{\beta} = - \widetilde{\text{sgn}}(s_{\beta}) \int_{s_{\beta} \mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})} \langle \nu_{s_{\beta} s, x', \mathcal{S}, N} \rangle_{\beta}.$$

Hence,

$$\begin{aligned} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})} \langle \nu_{s, x', \mathcal{S}, N} \rangle_{\beta} \\ = \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \frac{1}{2} \int_{(\mathcal{C}_{s, \mathcal{S}} - \widetilde{\text{sgn}}(s_{\beta}) s_{\beta} \mathcal{C}_{s_{\beta} s, \mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})} \langle \nu_{s, x', \mathcal{S}, N} \rangle_{\beta}. \end{aligned} \tag{9.10}$$

Let $x \in \mathfrak{h}_{\mathcal{S}, \beta=0}$ and let $t \in [-1, 1]$. Then, by Lemma 4.5,

$$\mathcal{C}_{s, \mathcal{S}}(t, x) = \mathcal{C}_{s, S \vee \beta}(1, x + tm_{\mathcal{S}(y_{\mathcal{S}})}(x) \frac{i\beta(y_{\mathcal{S}})}{2} H_{\beta}). \tag{9.11}$$

For $G = O_{2p+1, 2q}$, we have

$$\text{sgn}(short)(s_{\beta}) = \begin{cases} 1 & \text{if } \beta \text{ is long,} \\ -1 & \text{otherwise.} \end{cases}$$

By Lemmas 3.5, 3.7 and (3.16),

$$\begin{aligned} & s_{\beta} \mathcal{C}_{s_{\beta} s, \mathcal{S}}(t, x) \tag{9.12} \\ = & \begin{cases} \mathcal{C}_{s, S \vee \beta}(1, x - \text{sgn}(short)(s_{\beta}) tm_{\mathcal{S}(y_{\mathcal{S}})}(x) \frac{i\beta(y_{\mathcal{S}})}{2} H_{\beta}) & \text{if } G = O_{2p+1, 2q}, \\ \mathcal{C}_{s, S \vee \beta}(1, x - tm_{\mathcal{S}(y_{\mathcal{S}})}(x) \frac{i\beta(y_{\mathcal{S}})}{2} H_{\beta}) & \text{otherwise.} \end{cases} \end{aligned}$$

Let

$$T_{S \vee \beta, \beta}^{\pm}(s) = \left\{ x \pm tm_{\mathcal{S}(y_{\mathcal{S}})}(x) \left| \frac{i\beta(y_{\mathcal{S}})}{2} \right| H_{\beta} \mid 0 \leq t \leq 1, x \in \mathfrak{h}_{\mathcal{S}, \beta=0} \right\}.$$

We view $T_{S \vee \beta, \beta}^{\pm}(s)$ as a function of t and x , with the t varying from 0 to 1. We put

$$\epsilon_{\beta, s} = \frac{i\beta(y_{\mathcal{S}})}{|\beta(y_{\mathcal{S}})|}.$$

Then,

$$\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S},\beta=0}) = \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(s))$$

If $G = O_{2p+1,2q}$, then

$$s_{\beta}\mathcal{C}_{s_{\beta}s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S},\beta=0}) = \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-\text{sgn}(\text{short})(s_{\beta})\epsilon_{\beta,s}}(s))$$

If $G \neq O_{2p+1,2q}$, then

$$s_{\beta}\mathcal{C}_{s_{\beta}s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S},\beta=0}) = \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-\epsilon_{\beta,s}}(s))$$

Let $G' \neq O_{2p+1,2q}$. Then, by (9.11) and (9.12),

$$\begin{aligned} & (\mathcal{C}_{s,\mathcal{S}} + s_{\beta}\mathcal{C}_{s_{\beta}s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0}) \\ &= \begin{cases} \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{+}(s)) + \text{sgn}(\text{short})(s_{\beta})\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-}(s)) & \text{if } G = O_{2p+1,2q}, \\ \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{+}(s)) + \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-}(s)) & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, for $G = O_{2p+1,2q}$,

$$\begin{aligned} & (\mathcal{C}_{s,\mathcal{S}} + s_{\beta}\mathcal{C}_{s_{\beta}s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0}) \\ &= \begin{cases} \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{+}(s)) + \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-}(s)) & \text{if } \beta \text{ is long,} \\ 2\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(s)) & \text{if } \beta \text{ is short.} \end{cases} \end{aligned}$$

For $G \neq O_{2p+1,2q}$ and $G' \neq O_{2p+1,2q}$,

$$(\mathcal{C}_{s,\mathcal{S}} + s_{\beta}\mathcal{C}_{s_{\beta}s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0}) = \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{+}(s)) + \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-}(s)),$$

and for $G' = O_{2p+1,2q}$,

$$\begin{aligned} & (\mathcal{C}_{s,\mathcal{S}} - \text{sgn}(\text{short})(s_{\beta})s_{\beta}\mathcal{C}_{s_{\beta}s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0}) \\ &= \mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(s)) - \text{sgn}(\text{short})(s_{\beta})\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-\epsilon_{\beta,s}}(s)). \end{aligned}$$

Furthermore,

$$\frac{\beta(x \pm tm_{\mathcal{S}(y_s)}(x)H_{\beta})}{|\beta(x \pm tm_{\mathcal{S}(y_s)}(x)H_{\beta})|} = \text{sgn}(\pm t).$$

Notice that if $\mathbb{D} = \mathbb{R}$ and $|\underline{\beta}| = 1$ then the elements of the set $T_{\mathcal{S}\vee\beta,\beta}(s)$ satisfy the condition (1.15). Hence, by Lemma 8.4, if $G = O_{2p+1,2q}$ and β is long, then

$$\begin{aligned} & \int_{(\mathcal{C}_{s,\mathcal{S}} + s_{\beta}\mathcal{C}_{s_{\beta}s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0})} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta} \tag{9.13} \\ &= \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{+}(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta} + \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-}(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta} \\ &= \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{+}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta} - \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta} \\ &= \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_{\beta} \\ &= -d(\mathcal{S}, \beta) \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \tau_{s,x',\mathcal{S}\vee\beta,\beta,N}; \end{aligned}$$

if $G = O_{2p+1,2q}$ and β is short, then

$$\begin{aligned} \int_{(\mathcal{C}_{s,\mathcal{S}}+s_\beta\mathcal{C}_{s_\beta s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0})} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta &= 2 \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta \tag{9.14} \\ &= 2\epsilon_{\beta,s} \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta \\ &= -2d(\mathcal{S}, \beta)\epsilon_{\beta,s} \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(-s))} \tau_{s,x',\mathcal{S}\vee\beta,\beta,N}. \end{aligned}$$

if $G \neq O_{2p+1,2q}$ and $G' \neq O_{2p+1,2q}$, then

$$\begin{aligned} &\int_{(\mathcal{C}_{s,\mathcal{S}}+s_\beta\mathcal{C}_{s_\beta s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0})} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta \\ &= \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^+(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta + \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^-(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta \\ &= \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^+(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta - \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^-(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta \\ &= \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta; \\ &= -d(\mathcal{S}, \beta) \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \tau_{s,x',\mathcal{S}\vee\beta,\beta,N}; \tag{9.15} \end{aligned}$$

if $G' = O_{2p+1,2q}$, then

$$\begin{aligned} &\int_{(\mathcal{C}_{s,\mathcal{S}}-\text{sgn}(\text{short})(s_\beta)s_\beta\mathcal{C}_{s_\beta s,\mathcal{S}})(I \times \mathfrak{h}_{\mathcal{S},\beta=0})} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta \tag{9.16} \\ &= \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta - \text{sgn}(\text{short})(s_\beta) \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-\epsilon_{\beta,s}}(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta, \\ &= \epsilon_{\beta,s} \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{\epsilon_{\beta,s}}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta \\ &\quad + \epsilon_{\beta,s} \text{sgn}(\text{short})(s_\beta) \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}^{-\epsilon_{\beta,s}}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta, \\ &= \begin{cases} \epsilon_{\beta,s} \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta & \text{if } \beta \text{ is long,} \\ \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \frac{\beta}{|\beta|} \langle \nu_{s,x',\mathcal{S},N} \rangle_\beta & \text{if } \beta \text{ is short,} \end{cases} \\ &= d(\mathcal{S}, \beta) \begin{cases} \epsilon_{\beta,s} \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \tau_{s,x',\mathcal{S}\vee\beta,\beta,N} & \text{if } \beta \text{ is long,} \\ - \int_{\mathcal{C}_{s,\mathcal{S}\vee\beta}(1 \times T_{\mathcal{S}\vee\beta,\beta}(s))} \tau_{s,x',\mathcal{S}\vee\beta,\beta,N} & \text{if } \beta \text{ is short.} \end{cases} \end{aligned}$$

Suppose, $G \neq O_{2p+1,2q}$ and $G' \neq O_{2p+1,2q}$. Then, by (9.10), (9.15), Lemmas 8.3

and 3.5,

$$\begin{aligned}
 & \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\substack{s \in W(\mathbb{H}_{\mathbb{C}}), \beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n \\ \beta(y_s) \neq 0, \underline{\beta} \cap \underline{\mathcal{S}} \neq \emptyset}} \int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})} \langle \mathcal{V}_{s, x', \mathcal{S}, N} \rangle_{\beta} \\
 &= \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\substack{s \in W(\mathbb{H}_{\mathbb{C}}), \beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n \\ \beta(y_s) \neq 0, \underline{\beta} \cap \underline{\mathcal{S}} \neq \emptyset}} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} -\mathcal{T}_{s, x', S \vee \beta, \beta, N} \\
 &= \sum_{\emptyset \neq \mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \sum_{\beta \in \mathcal{S}(y_s)} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{\mathcal{S}, \beta}(s))} -\mathcal{T}_{s, x', S \vee \beta, \beta, N},
 \end{aligned}$$

and (9.5) follows. Suppose, $G' = O_{2p+1, 2q}$. Then, by (9.16), Lemmas 8.3 and 3.5,

$$\begin{aligned}
 & \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\substack{s \in W(\mathbb{H}_{\mathbb{C}}), \beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n \\ \beta(y_s) \neq 0, \underline{\beta} \cap \underline{\mathcal{S}} \neq \emptyset}} \int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}, \beta=0})} \langle \mathcal{V}_{s, x', \mathcal{S}, N} \rangle_{\beta} \\
 &= \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\substack{s \in W(\mathbb{H}_{\mathbb{C}}), \beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n(\text{short}) \\ \beta(y_s) \neq 0, \underline{\beta} \cap \underline{\mathcal{S}} \neq \emptyset}} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} -\mathcal{T}_{s, x', S \vee \beta, \beta, N} \\
 &+ \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\substack{s \in W(\mathbb{H}_{\mathbb{C}}), \beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n(\text{long}) \\ \beta(y_s) \neq 0, \underline{\beta} \cap \underline{\mathcal{S}} \neq \emptyset}} \epsilon_{\beta, s} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \mathcal{T}_{s, x', S \vee \beta, \beta, N} \\
 &= \sum_{\emptyset \neq \mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \sum_{\beta \in \mathcal{S}(y_s)} \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} -\mathcal{T}_{s, x', \mathcal{S}, \beta, N}, \\
 &+ \sum_{\emptyset \neq \mathcal{S} \in \Psi_{st}^n} \sum_{\substack{s \in W(\mathbb{H}_{\mathbb{C}}), \beta \in \mathcal{S}(y_s)(\text{long}) \\ i\beta(y_s) > 0}} \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} 2\mathcal{T}_{s, x', \mathcal{S}, \beta, N},
 \end{aligned}$$

and (9.5) follows. Suppose, $G = O_{2p+1, 2q}$. Let \mathcal{S} and β be as in (9.14) (In particular, β is short). Then,

$$\begin{aligned}
 & \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^+(s))} \mathcal{T}_{s, x', S \vee \beta, \beta, N} \tag{9.17} \\
 &= \int_{s_{\beta} \mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^+(s))} \mathcal{T}_{s, x', S \vee \beta, \beta, N} \circ S_{\beta} \\
 &= - \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^-(s))} \mathcal{T}_{s_{\beta} s, x', S \vee \beta, \beta, N} \\
 &= - \int_{\mathcal{C}_{s_{\beta} s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^-(s_{\beta} s))} \mathcal{T}_{s_{\beta} s, x', S \vee \beta, \beta, N},
 \end{aligned}$$

because, by Lemma 8.4 and Corollary 8.5,

$$\mathcal{T}_{s, x', S \vee \beta, \beta, N} \circ S_{\beta} = -\mathcal{T}_{s_{\beta} s, x', S \vee \beta, \beta, N},$$

and, by definition,

$$s_\beta \mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^+(s)) = \mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^-(s)) = \mathcal{C}_{s_\beta s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^-(s_\beta s)).$$

Hence,

$$\sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^+(s))} \tau_{s, x', S \vee \beta, \beta, N} = - \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^-(s))} \tau_{s, x', S \vee \beta, \beta, N},$$

so that

$$\begin{aligned} & \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^+(s))} \tau_{s, x', S \vee \beta, \beta, N} \\ &= \frac{1}{2} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \tau_{s, x', S \vee \beta, \beta, N}. \end{aligned} \tag{9.18}$$

Hence,

$$\begin{aligned} & \sum_{s \in W(\mathbb{H}_\mathbb{C}), i\beta(y_s) > 0} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^+(s))} \tau_{s, x', S \vee \beta, \beta, N} \\ & \quad - \sum_{s \in W(\mathbb{H}_\mathbb{C}), i\beta(y_s) < 0} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^-(s))} \tau_{s, x', S \vee \beta, \beta, N} \\ &= \frac{1}{2} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \tau_{s, x', S \vee \beta, \beta, N} \\ & - \sum_{s \in W(\mathbb{H}_\mathbb{C}), i\beta(y_s) < 0} \left(\int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^+(s))} \tau_{s, x', S \vee \beta, \beta, N} + \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}^-(s))} \tau_{s, x', S \vee \beta, \beta, N} \right) \\ &= \frac{1}{2} \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \tau_{s, x', S \vee \beta, \beta, N} \\ & \quad - \sum_{s \in W(\mathbb{H}_\mathbb{C}), i\beta(y_s) < 0} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \frac{\beta}{|\beta|} \tau_{s, x', S \vee \beta, \beta, N} \end{aligned}$$

Then, by (9.13), (9.14), (9.18) and Lemma 8.3,

$$\begin{aligned} & \sum_{S \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C}), \beta \in \Psi_{S, i\mathbb{R}}^n, \beta(y_s) \neq 0, \underline{\beta} \cap \underline{S} \neq \emptyset} \int_{\mathcal{C}_{s, S}(I \times \mathfrak{h}_{S, \beta=0})} \langle \nu_{s, x', S, N} \rangle_\beta \\ &= \sum_{S \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C}), \beta \in \Psi_{S, i\mathbb{R}}^n(\text{long}), \beta(y_s) \neq 0, \underline{\beta} \cap \underline{S} \neq \emptyset} -\frac{d(S, \beta)}{2} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \tau_{s, x', S \vee \beta, \beta, N} \\ & \quad + \sum_{S \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C}), \beta \in \Psi_{S, i\mathbb{R}}^n(\text{short}), \beta(y_s) \neq 0, \underline{\beta} \cap \underline{S} \neq \emptyset} -\frac{d(S, \beta)}{2} \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \tau_{s, x', S \vee \beta, \beta, N} \\ & \quad + \sum_{S \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_\mathbb{C}), \beta \in \Psi_{S, i\mathbb{R}}^n(\text{short}), i\beta(y_s) < 0, \underline{\beta} \cap \underline{S} \neq \emptyset} d(S, \beta) \int_{\mathcal{C}_{s, S \vee \beta}(1 \times T_{S \vee \beta, \beta}(s))} \frac{\beta}{|\beta|} \tau_{s, x', S \vee \beta, \beta, N} \end{aligned}$$

$$\begin{aligned}
 &= - \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \sum_{\beta \in \mathcal{S}(y_s)(long)} \int_{\mathcal{C}_{s, \mathcal{S} \vee \beta}(1 \times T_{\mathcal{S} \vee \beta}(s))} \tau_{s, x', \mathcal{S} \vee \beta, \beta, N} \\
 &- \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\mathcal{S}(short) = \{\beta\}} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} (|\mathcal{S}_1| + 1) \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} \tau_{s, x', \mathcal{S}, \beta, N} \tag{9.19}
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\mathcal{S}(short) = \{\beta\}} \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), i\beta(y_s) < 0} 2(|\mathcal{S}_1| + 1) \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} \frac{\beta}{|\beta|} \tau_{s, x', \mathcal{S}, \beta, N} \\
 &- \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\mathcal{S}_1 \neq \emptyset} \sum_{\mathcal{S}(short)} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \sum_{\beta \in \mathcal{S}_1 \cap \Psi^c(short)} 2 \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} \tau_{s, x', \mathcal{S}, \beta, N} \tag{9.20}
 \end{aligned}$$

$$+ \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\mathcal{S}_1 \neq \emptyset} \sum_{\mathcal{S}(short) = \emptyset} \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \sum_{\beta \in \mathcal{S}_1 \cap \Psi^c(short), i\beta(y_s) < 0} 4 \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} \frac{\beta}{|\beta|} \tau_{s, x', \mathcal{S}, \beta, N}.$$

If \mathcal{S} and β are such that $\mathcal{S} \cap \Psi^n(short) = \{\beta\} \neq \emptyset$, (9.19), then there are elements $s_i \in W(\Delta_{\mathcal{S}, \mathbb{R}})$, stabilizing $\Psi_{\mathcal{S}, \mathbb{R}}$ such that

$$\mathcal{S}_1 = \{s_1\beta, s_2\beta, \dots, s_{|\mathcal{S}_1|}\beta\}. \tag{9.21}$$

If \mathcal{S} and β are such that $\beta \in \mathcal{S}_1 \cap \Psi^c(short)$, (9.20), then there is an elements $s_1 \in W(\Delta_{\mathcal{S}, \mathbb{R}})$, stabilizing $\Psi_{\mathcal{S}, \mathbb{R}}$ such that

$$s_1\beta \in \Psi^n(short) \text{ and } s_1\beta \vee \beta \subseteq \mathcal{S}. \tag{9.22}$$

Let us assume, for the purpose of the argument, that Ψ is "standard", as in Theorem B.4. We may assume that in any case, (9.21) or (9.22), $s_i \in \Sigma_n$ is a transposition. Then, by a straightforward computation, for any $s \in W(\mathbb{H}_{\mathbb{C}})$,

$$s_i\mathcal{S}(y_s) = \mathcal{S}(y_s) = \mathcal{S}(y_{s_i s}).$$

Hence,

$$\begin{aligned}
 s_i T_{\mathcal{S}, \beta}(s) &= \{s_i \cdot x \mid x \in \mathfrak{h}_{\mathcal{S}} \mid |\tilde{\beta}(x)| \leq \min\{1, |\tilde{\alpha}(x)| \mid \alpha \in \mathcal{S}(y_s) \setminus \beta\}\} \\
 &= \{x \in \mathfrak{h}_{\mathcal{S}} \mid |\tilde{\beta}(s_i^{-1} \cdot x)| \leq \min\{1, |\tilde{\alpha}(s_i^{-1} \cdot x)| \mid \alpha \in \mathcal{S}(y_s) \setminus \beta\}\} \\
 &= \{x \in \mathfrak{h}_{\mathcal{S}} \mid |\widetilde{s_i\beta}(x)| \leq \min\{1, |\widetilde{s_i\alpha}(x)| \mid \alpha \in \mathcal{S}(y_s) \setminus \beta\}\} \\
 &= \{x \in \mathfrak{h}_{\mathcal{S}} \mid |\widetilde{s_i\beta}(x)| \leq \min\{1, |\tilde{\alpha}(x)| \mid s_i^{-1}\alpha \in \mathcal{S}(y_s) \setminus \beta\}\} \\
 &= \{x \in \mathfrak{h}_{\mathcal{S}} \mid |\widetilde{s_i\beta}(x)| \leq \min\{1, |\tilde{\alpha}(x)| \mid \alpha \in s_i\mathcal{S}(y_s) \setminus s_i\beta\}\} \\
 &= T_{\mathcal{S}, s_i\beta}(s_i s).
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 \tau_{s, x', \mathcal{S}, \beta, N}(s_i^{-1} \cdot z) &= n_{\mathcal{S}}(s) \frac{P((s_1 s)^{-1} \cdot z) \pi_{\mathfrak{z}/\mathfrak{h}}((s_1 s)^{-1} \cdot z)}{\det(x' + z)_{s_1 s W^{\mathfrak{h}'}}} \mathcal{A}(-\Psi_{-\mathcal{S}, \mathbb{R}})(s_1^{-1} \cdot x_{\beta}) \\
 &\quad \partial(e_N(is_i^{-1} \cdot y)) \mathcal{H}_{\mathcal{S}} \psi(s_i^{-1} \cdot x) \nu(s_i^{-1} \cdot z) \\
 &= n_{\mathcal{S}}(s_i s) \operatorname{sgn}(s_i) \frac{P((s_1 s)^{-1} \cdot z) \pi_{\mathfrak{z}/\mathfrak{h}}((s_1 s)^{-1} \cdot z)}{\det(x' + z)_{s_1 s W^{\mathfrak{h}'}}} \mathcal{A}(-\Psi_{-\mathcal{S}, \mathbb{R}})(x_{\beta}) \\
 &\quad \partial(e_N(iy)) \mathcal{H}_{\mathcal{S}} \psi(x) \nu(z) \det(s_i) \\
 &= \tau_{s_i s, x', \mathcal{S}, s_i \beta, N}(z),
 \end{aligned}$$

because

$$s_i^{-1} \cdot x + \frac{\beta(s_i^{-1} \cdot y)}{2} H_\beta = s_i^{-1} \cdot \left(x + \frac{s_i \beta(y)}{2} H_{s_i \beta}\right) = s_i^{-1} \cdot x_{s_i \beta},$$

and $\text{sgn}(s_i) \det(s_i) = 1$ for $G = O_{2p+1,2q}$. Hence,

$$\begin{aligned} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \tau_{s,x',\mathcal{S},\beta,N} &= \int_{s_i \mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \tau_{s_i s, x', \mathcal{S}, s_i \beta, N} \\ &= \int_{\mathcal{C}_{s_i s, \mathcal{S}}(1 \times T_{\mathcal{S}, s_i \beta}(s_i s))} \tau_{s_i s, x', \mathcal{S}, s_i \beta, N}, \end{aligned}$$

so that

$$\sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \tau_{s,x',\mathcal{S},\beta,N} = \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},s_i \beta}(s))} \tau_{s,x',\mathcal{S},s_i \beta,N}.$$

Therefore, if β is as in (9.19), then

$$(|\mathcal{S}_1| + 1) \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \tau_{s,x',\mathcal{S},\beta,N} = \sum_{s \in W(\mathbb{H}_\mathbb{C})} \sum_{s_i} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},s_i \beta}(s))} \tau_{s,x',\mathcal{S},s_i \beta,N},$$

and if β is as in (9.20), then

$$\begin{aligned} 2 \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \tau_{s,x',\mathcal{S},\beta,N} &= \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} \tau_{s,x',\mathcal{S},\beta,N} \\ &\quad + \sum_{s \in W(\mathbb{H}_\mathbb{C})} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},s_1 \beta}(s))} \tau_{s,x',\mathcal{S},s_1 \beta,N}, \end{aligned}$$

and (9.5) follows. ■

Suppose $G = O_{2p+1,2q}$ or $G' = O_{2p+1,2q}$. Let $\mathcal{S} \in \Psi_{st}^n$ and let $\beta \in \Psi_{\mathcal{S},\mathbb{R}}(\text{short})$ if $G = O_{2p+1,2q}$, or $\beta \in \mathcal{S}(\text{long})$ if $G' = O_{2p+1,2q}$. Recall that $\epsilon_{\beta,s} = \frac{i\beta(y_s)}{|\beta(y_s)|}$. Define

$$W(\mathbb{H}_\mathbb{C})_\beta = \begin{cases} \{s \in W(\mathbb{H}_\mathbb{C}) \mid \epsilon_{\beta,s} = 1\} & \text{if } G' = O_{2p+1,2q}, \\ \{s \in W(\mathbb{H}_\mathbb{C}) \mid \epsilon_{\beta,s} = -1\} & \text{if } G = O_{2p+1,2q}. \end{cases}$$

It is easy to see that

$$\underline{\beta} \cap \mathcal{B}_\mathcal{S}(s) = \emptyset \quad (s \in W(\mathbb{H}_\mathbb{C})).$$

Hence, $W(\mathbb{H}_\mathbb{C})_\beta$ is the union of the fibers of the map

$$\mathcal{B}_\mathcal{S} : W(\mathbb{H}_\mathbb{C}) \rightarrow \{P \subset \mathcal{S} \cup (-\mathcal{S})\}.$$

Therefore, as in (3.21), we have a decomposition into the disjoint union of sets :

$$W(\mathbb{H}_\mathbb{C})_\beta = \bigcup_{R \in \mathcal{B}_\mathcal{S}(W(\mathbb{H}_\mathbb{C})_\beta)} \bigcup_{\substack{W(R)s \in W(\mathbb{H}_\mathbb{C}) \\ \mathcal{B}_\mathcal{S}(s) = R}} W(R)s. \tag{9.23}$$

By combining Theorem 9.1 with the decomposition (3.22) of the Weyl group $W(\mathbb{H}_\mathbb{C})$ we obtain the following Corollary:

Corollary 9.2. *Let $n' \geq 1$, $N \in \mathbb{N}$ and let P be a polynomial function on \mathfrak{h}' satisfying (0.4). Then for $x' \in \mathfrak{h}'^{\text{reg}}$ and $\psi \in S(\mathfrak{g})$,*

$$\begin{aligned} & P(x')\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} \text{chc}(x' + x)\psi(x) d\mu(x) \\ &= \sum_{s \in W(\mathbb{H}_{\mathbb{C}})} \int_{\mathcal{C}_{s,\emptyset}(1 \times \mathfrak{h})} \nu_{s,x',\emptyset,N} \\ &+ \sum_{\emptyset \neq \mathcal{S} \in \Psi_{st}^n} \sum_{R \in \mathcal{B}_{\mathcal{S}}(W(\mathbb{H}_{\mathbb{C}}))} \sum_{W(R)sW(s,\mathcal{S}) \in W(R) \setminus W(\mathbb{H}_{\mathbb{C}})/W(s,\mathcal{S})} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times (\mathfrak{h}_{\mathcal{S}} \setminus T_{\mathcal{S}}(s)))} \tilde{\nu}_{s,x',\mathcal{S},N} \\ &+ \sum_{\emptyset \neq \mathcal{S} \in \Psi_{st}^n} \sum_{R \in \mathcal{B}_{\mathcal{S}}(W(\mathbb{H}_{\mathbb{C}}))} \sum_{\substack{W(R)sW(s,\mathcal{S}) \in W(R) \setminus W(\mathbb{H}_{\mathbb{C}})/W(s,\mathcal{S}) \\ \beta \in \mathcal{S}(y_s)}} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times (T_{\mathcal{S},\beta}(s)))} (\tilde{\nu}_{s,x',\mathcal{S},N} - \tilde{\tau}_{s,x',\mathcal{S},\beta,N}) \\ &- \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{R \in \mathcal{B}_{\mathcal{S}}(W(\mathbb{H}_{\mathbb{C}}))} \sum_{W(R)sW(s,\mathcal{S}) \in W(R) \setminus W(\mathbb{H}_{\mathbb{C}})/W(s,\mathcal{S})} \int_{\mathcal{C}_{s,\mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}})} d\tilde{\nu}_{s,x',\mathcal{S},N} \\ &+ E, \end{aligned}$$

where

$$\tilde{\nu}_{s,x',\mathcal{S},N} = \sum_{\eta \in W(R), \delta \in W(s,\mathcal{S})} \nu_{\eta s \delta, x', \mathcal{S}, N}, \quad \tilde{\tau}_{s,x',\mathcal{S},\beta,N} = \sum_{\eta \in W(R), \delta \in W(s,\mathcal{S})} \tau_{\eta s \delta, x', \mathcal{S}, \beta, N},$$

and E is as in Theorem 9.1. Moreover,

$$E = \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\beta \in \Psi_{\mathcal{S},\mathbb{R}}(\text{short})} \sum_{R \in \mathcal{B}_{\mathcal{S}}(W(\mathbb{H}_{\mathbb{C}})_{\beta})} \sum_{W(R)s \in W(R) \setminus W(\mathbb{H}_{\mathbb{C}}), \mathcal{B}_{\mathcal{S}}(s) = R} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} 2 \frac{\beta}{|\beta|} \tilde{\tau}_{s,x',\mathcal{S},\beta,N} \text{ if } G = O_{2p+1,2q},$$

$$E = \sum_{\mathcal{S} \in \Psi_{st}^n} \sum_{\beta \in \mathcal{S}(\text{long})} \sum_{R \in \mathcal{B}_{\mathcal{S}}(W(\mathbb{H}_{\mathbb{C}})_{\beta})} \sum_{W(R)sW(s,\mathcal{S}) \in W(R) \setminus W(\mathbb{H}_{\mathbb{C}})/W(s,\mathcal{S}), \mathcal{B}_{\mathcal{S}}(s) = R} \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} 2 \tilde{\tau}_{s,x',\mathcal{S},\beta,N} \text{ if } G' = O_{2p+1,2q}.$$

Proof. This follows from Theorem 9.1, (3.18) and (9.23). One only needs to notice that

$$T_{\mathcal{S},\beta}(\eta s \delta) = T_{\mathcal{S},\beta}(s) \quad (\beta \in \mathcal{S}(y_s), s \in W(\mathbb{H}_{\mathbb{C}}), \eta \in W(\mathcal{B}_{\mathcal{S}}(s)), \delta \in W(s, \mathcal{S})).$$

■

10. Proof of Theorem 1

We retain the notation of Corollary 9.2. Clearly, Theorem 1 follows from Theorem 7.4, Corollary 9.2 and Proposition 10.1 below.

Proposition 10.1. *Fix an element $s \in W(\mathbb{H}_{\mathbb{C}})$. Then*

$$\sup_{x' \in \mathfrak{h}^{\text{reg}}} \left| \int_{\mathcal{C}_{s, \emptyset}(1 \times \mathfrak{h})} \nu_{s, x', \emptyset, N} \right| < \infty \quad (N \in \mathbb{N}). \tag{10.1}$$

Suppose $\emptyset \neq \mathcal{S} \in \Psi_{st}^n$. Then, for all $N \in \mathbb{N}$ large enough,

$$\sup_{x' \in \mathfrak{h}^{\text{reg}}} \left| \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times (\mathfrak{h}_{\mathcal{S}} \setminus T_{\mathcal{S}}(s)))} \tilde{\nu}_{s, x', \mathcal{S}, N} \right| < \infty, \tag{10.2}$$

and

$$\sup_{x' \in \mathfrak{h}^{\text{reg}}} \left| \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times (T_{\mathcal{S}, \beta}(s)))} (\tilde{\nu}_{s, x', \mathcal{S}, N} - \tilde{\tau}_{s, x', \mathcal{S}, \beta, N}) \right| < \infty \quad (\beta \in \mathcal{S}(y_s)). \tag{10.3}$$

For any $\mathcal{S} \in \Psi_{st}^n$ and for all $N \in \mathbb{N}$ large enough,

$$\sup_{x' \in \mathfrak{h}^{\text{reg}}} \left| \int_{\mathcal{C}_{s, \mathcal{S}}(I \times \mathfrak{h}_{\mathcal{S}})} d\tilde{\nu}_{s, x', \mathcal{S}, N} \right| < \infty. \tag{10.4}$$

If $(G, G') = (\text{Sp}_{2n}(\mathbb{R}), \text{O}_{2p+1, 2q})$, $\beta \in \mathcal{S}(\text{long})$ and $i\beta(y_s) > 0$, then for N large enough,

$$\sup_{x' \in \mathfrak{h}^{\text{reg}}} \left| \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} \tilde{\tau}_{s, x', \mathcal{S}, \beta, N} \right| < \infty. \tag{10.5}$$

If $(G, G') = (\text{O}_{2p+1, 2q}, \text{Sp}_{2n'}(\mathbb{R}))$, $\beta \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{short})$ and $i\beta(y_s) < 0$, then for N large enough,

$$\sup_{x' \in \mathfrak{h}^{\text{reg}}} \left| \int_{\mathcal{C}_{s, \mathcal{S}}(1 \times T_{\mathcal{S}, \beta}(s))} 2 \frac{\beta}{|\beta|} \tilde{\tau}_{s, x', \mathcal{S}, \beta, N} \right| < \infty. \tag{10.6}$$

Moreover, the above quantities define continuous seminorms on $S(\mathfrak{g})$.

Lemma 10.2. *For a root α let $\tilde{\alpha}$ be as in (9.1). Let $\emptyset \neq \mathcal{S} \in \Psi_{st}^n$, let $\beta \in \mathcal{S}(y_s)$ and let $s \in W(\mathbb{H}_{\mathbb{C}})$. Then*

$$\min\{1, |\tilde{\beta}(x)|\} = m_{\mathcal{S}(y_s)}(x) \quad (x \in T_{\mathcal{S}, \beta}(s)), \tag{10.7}$$

$$\mathfrak{h}_{\mathcal{S}} \setminus T_{\mathcal{S}}(s) = \{x \in \mathfrak{h}_{\mathcal{S}} \mid |\tilde{\alpha}(x)| > 1 \text{ for all } \alpha \in \mathcal{S}(y_s)\}, \tag{10.8}$$

$$m_{\mathcal{S}(y_s)}(x) = 1 \quad (x \in \mathfrak{h}_{\mathcal{S}} \setminus T_{\mathcal{S}}(s)), \tag{10.9}$$

$$|r - x_a| \geq 1 \quad (x \in \mathfrak{h}_{\mathcal{S}} \setminus T_{\mathcal{S}}(s), a \in \underline{\mathcal{S}(y_s)}, r \in \mathbb{R}), \tag{10.10}$$

$$|\Im x_a| \geq |\tilde{\beta}(x)| \quad (x \in \mathfrak{h}_{\mathcal{S}}, a \in \underline{\beta}). \tag{10.11}$$

(Here $\Im z = \Im(z)$ stands for the imaginary part of the complex number z .)

Before proving this lemma, we have to prove the following statement.

Lemma 10.3. *Let $x_1, x_2, \dots, x_n \geq 0$. Suppose*

$$x_a > \min\{1, x_b \mid b \neq a\} \quad (a = 1, 2, 3, \dots, n). \tag{10.12}$$

Then

$$x_a > 1 \text{ for all } a = 1, 2, 3, \dots, n. \tag{10.13}$$

Proof. Indeed, if (10.13) were not true, then there would be an a such that $x_a < 1$. By (10.12), we must have

$$x_a > \min\{x_b \mid b \neq a\}.$$

Let us denote the right hand side of the above expression by x_c . But, again by (10.12),

$$x_c > \min\{x_b \mid b \neq c\},$$

which is impossible. Thus the lemma follows. ■

Proof. [of Lemma 10.2] The equality (10.7) follows from (9.2) and the definition of the function $m_{\mathcal{S}(y_s)}$ in Lemmas 4.3 and 4.4. We see from (9.2) that

$$\mathfrak{h}_{\mathcal{S}} \setminus T_{\mathcal{S},\beta}(s) = \left\{ x \in \mathfrak{h}_{\mathcal{S}} \mid |\tilde{\beta}(x)| > \min\{1, |\tilde{\alpha}(x)|; \alpha \in \mathcal{S}(y_s) \setminus \beta\} \right\}.$$

Hence, Lemma 10.3 implies (10.8). Clearly (10.9) follows from (10.8).

Let $a \in \underline{\mathcal{S}(y_s)}$. Then there is $\beta \in \mathcal{S}(y_s)$ such that $a \in \underline{\beta}$. Therefore, we may assume that $\beta = 2iJ_a^*$, $\beta = iJ_a^*$ or $\beta = i(J_a^* \pm J_b^*)$, for some $a \neq b$. In the first two cases,

$$|r - x_a| = |r + i\tilde{\beta}(x)| \geq |\tilde{\beta}(x)|,$$

and both (10.10) and (10.11) follow. In the second case, suppose $\beta = i(J_a^* - J_b^*)$, for some $a \neq b$. Then $iH_{\beta} = J_a - J_b$, so there are $u, v \in \mathbb{R}$ such that

$$x = u(J_a + J_b) - iv(J_a - J_b) + \tilde{x},$$

where $\tilde{x}_c = 0$ for all $c \notin \{a, b\}$. In particular,

$$x_a = u - iv \quad \text{and} \quad |v| = \frac{|\beta(x)|}{2}.$$

Thus

$$|r - x_a| = |u + r - iv| \geq |v| = \frac{|\beta(x)|}{2},$$

which verifies (10.10) and (10.11). ■

Lemma 10.4. *Let f be a smooth function, defined in a neighborhood of a point x_0 in a finite dimensional real vector space. Then, for each non-negative integer N there is a constant $\text{const}(f, N)$ such that for any vector y , any vector x and any real number b , with $|b| \leq 1$ and $x_0 + bx$ in the domain of f ,*

$$|\partial(e_N(iby))f(x_0 + bx) - \partial(e_N(bx + iby))f(x_0)| \leq \text{const}(f, N) |b|^{N+1}.$$

Proof. By Taylor's formula

$$|\partial(e_N(iby))f(x_0 + bx) - \partial(e_N(bx))\partial(e_N(iby))f(x_0)| \leq \text{const}_1(f, N) |b|^{N+1}.$$

Thus the left hand side of the inequality in question is not greater than

$$\text{const}_1(f, N) |b|^{N+1} + |\partial(e_N(bx))\partial(e_N(iby))f(x_0) - \partial(e_N(bx + iby))f(x_0)|.$$

But

$$\begin{aligned}
 & \left| \partial(e_N(bx))\partial(e_N(iby))f(x_0) - \partial(e_N(bx + iby))f(x_0) \right| \\
 = & \left| \partial \left(\sum_{0 \leq m, n \leq N} \frac{(bx)^m (iby)^n}{m!n!} - \sum_{0 \leq m, n; m+n \leq N} \frac{(bx)^m (iby)^n}{m!n!} \right) f(x_0) \right| \\
 = & \left| \partial \left(\sum_{0 \leq m, n \leq N < m+n} \frac{(bx)^m (iby)^n}{m!n!} \right) f(x_0) \right| \\
 \leq & \sum_{0 \leq m, n \leq N < m+n} |b|^{m+n} \left| \partial \left(\frac{x^m y^n}{m!n!} \right) f(x_0) \right| \leq \text{const}_2(f, N) |b|^{N+1}.
 \end{aligned}$$

Hence, the inequality follows, with $\text{const} = \text{const}_1 + \text{const}_2$. ■

The following lemma is known and may be deduced from the proof of Theorem 3.1.15 in [6].

Lemma 10.5. *Let f be a smooth function defined on an open set $\mathcal{U} \subseteq \mathbb{R}^d$. For a non-negative integer N , let f_N be the extension of f to $\mathcal{U} + i\mathbb{R}^d$ as defined in Corollary 2.2. Then for any holomorphic function F defined in a neighborhood of \mathcal{U} we have*

$$d(F f_N dz)(u + itv) = t^N F(u + itv) \left(-\frac{\partial(iv)^{N+1}}{N!} \right) f(u) dt du,$$

where $x = u$, $y = tv$, $t \in \mathbb{R}$, $z = x + iy$ and $dz = dx + idy$. (Here the evaluation at a point stands for the pull-back of the differential form via the indicated map.) Hence, for a positive number m ,

$$d(F f_N dz)(u + itmv) = (tm)^N F(u + itmv) \left(-\frac{\partial(iv)^{N+1}}{N!} \right) f(u) m dt du.$$

Proof. [of Proposition 10.1] Notice that, with $(y_s)_{\sigma(j)} = J_{\sigma(j)}^*(y_s)$,

$$|\det(x' + \mathcal{C}_{s, \emptyset}(1, x))_{sW^{b'}}| = \prod_{1 \leq j \leq n'} |x'_j \pm (x_{\sigma(j)} + i(y_s)_{\sigma(j)})| \geq \prod_{1 \leq j \leq n'} |i(y_s)_{\sigma(j)}| = 1.$$

Hence,

$$\frac{1}{\det(x' + \mathcal{C}_{s, \emptyset}(1, x))_{sW^{b'}}$$

is bounded as, a function of $x \in \mathfrak{h}$. Thus part (10.1) follows. Also, since the function $m_\emptyset = 1$, it is easy to see from Lemma 10.5 that (10.4) holds in the case $\mathcal{S} = \emptyset$. Thus, from now on, we assume that $\mathcal{S} \neq \emptyset$. Let

$$\mathcal{S}_4(y_s) = \begin{cases} \mathcal{S} \setminus \mathcal{S}_3(y_s) & \text{if } G = O_{2p+1, 2q}, \\ (\mathcal{S} \setminus \mathcal{S}(y_s)) \cap (\mathcal{S} \setminus \mathcal{S}'') & \text{if } G = O_{2p, 2q} \text{ or } G' = O_{2p+1, 2q}, \\ \mathcal{S} \setminus \mathcal{S}(y_s) & \text{otherwise,} \end{cases}$$

where $\mathcal{S}_3(y_s)$ is as in Lemma 3.6. Then, by (3.11), (3.15) and (3.18), we have the following decomposition of $\underline{\mathcal{S}}$ into a disjoint union of sets:

$$\underline{\mathcal{S}} = \begin{cases} \underline{\mathcal{S}(y_s)} \cup \left(\underline{\mathcal{S}_4(y_s)} \setminus \underline{\mathcal{B}_S(s)} \right) \cup \underline{\mathcal{B}_S(s)} & \text{if } G \neq O_{2p,2q}, \\ \underline{\mathcal{S}''} \cup \underline{\mathcal{S}(y_s)} \cup \left(\underline{\mathcal{S}_4(y_s)} \setminus \underline{\mathcal{B}_S(s)} \right) \cup \underline{\mathcal{B}_S(s)} & \text{if } G = O_{2p,2q}. \end{cases}$$

Let $z = \mathcal{C}_{s,\mathcal{S}}(t, x)$. If $\mathbb{D} = \mathbb{C}$ and $s = \sigma \in \Sigma_n$, define (according to the above decomposition)

$$\begin{aligned} A(s, x', z) &= \prod_{1 \leq j \leq n', \sigma(j) \notin \underline{\mathcal{S}}} (x'_j - x_{\sigma(j)} - itm_{\mathcal{S}(y_s)}(x)(y_s)_{\sigma(j)}) \\ B(s, x', z) &= \prod_{1 \leq j \leq n', \sigma(j) \in \underline{\mathcal{S}(y_s)}} (x'_j - x_{\sigma(j)}) \\ C(s, x', z) &= \prod_{1 \leq j \leq n', \sigma(j) \in \underline{\mathcal{S}_4(y_s)} \setminus \underline{\mathcal{B}_S(s)}} (x'_j - x_{\sigma(j)}) \\ D(s, x', z) &= \prod_{1 \leq j \leq n', \sigma(j) \in \underline{\mathcal{B}_S(s)}} (x'_j - x_{\sigma(j)}). \end{aligned}$$

If $\mathbb{D} \neq \mathbb{C}$ and $s = \sigma\epsilon$, with $\sigma \in \Sigma_n$ and $\epsilon \in \mathbb{Z}_2^n$, define

$$\begin{aligned} A(s, x', z) &= \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \notin \underline{\mathcal{S}}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)} - itm_{\mathcal{S}(y_s)}(x)\hat{\epsilon}_j(y_s)_{\sigma(j)}) \\ B(s, x', z) &= \begin{cases} \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \in \underline{\mathcal{S}(y_s)}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)}), & \text{if } G \neq O_{2p,2q}, \\ \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \in \underline{\mathcal{S}''} \cup \underline{\mathcal{S}(y_s)}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)}), & \text{if } G = O_{2p,2q} \end{cases} \\ C(s, x', z) &= \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \in \underline{\mathcal{S}_4(y_s)} \setminus \underline{\mathcal{B}_S(s)}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)}) \\ D(s, x', z) &= \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \in \underline{\mathcal{B}_S(s)}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)}), \end{aligned}$$

and let

$$\begin{aligned} B'(s, x', z) &= \begin{cases} \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \in \underline{\mathcal{S}(y_s)} \setminus \underline{\mathcal{S}''}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)}) & \text{if } (G, G') = (\mathrm{Sp}_{2n}(\mathbb{R}), O_{2p+1,2q}), \\ \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \in \underline{\mathcal{S}(y_s)}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)}), & \text{if } (G, G') = (O_{2p,2q}, \mathrm{Sp}_{2n'}(\mathbb{R})), \end{cases} \\ B''(s, x', z) &= \prod_{\substack{1 \leq j \leq n', \\ \sigma(j) \in \underline{\mathcal{S}''}}} (x'_j - \hat{\epsilon}_j x_{\sigma(j)}), \text{ if } (G, G') = (O_{2p,2q}, \mathrm{Sp}_{2n'}(\mathbb{R})) \text{ or } (\mathrm{Sp}_{2n}(\mathbb{R}), O_{2p+1,2q}). \end{aligned}$$

For the remaining pairs set

$$B'(s, x', z) = B(s, x', z) \text{ and } B''(s, x', z) = 1.$$

Then $B(s, x', z) = B'(s, x', z)B''(s, x', z)$ and

$$\det(x' + \mathcal{C}_{s, \mathcal{S}}(t, x))_{sWb'} = i^{n'} A(s, x', z)B(s, x', z)C(s, x', z)D(s, x', z).$$

Let

$$E(s, x', z) = \sum_{\eta \in W(\mathcal{B}_{\mathcal{S}}(s))} \frac{\det(\eta)}{D(\eta s, x', z)}.$$

By (3.12) and (3.15), each element $\beta \in \mathcal{S}_4(y_s)$ may be written uniquely (up to a sign) as

$$\beta = i(J_{a(\beta)}^* + \epsilon(\beta)J_{b(\beta)}^*),$$

where $a(\beta) < b(\beta)$ and $\epsilon(\beta) = \pm 1$. If $\mathbb{D} = \mathbb{C}$, then $\epsilon(\beta) = -1$.

Since $\mathcal{B}_{\mathcal{S}}(s) \subseteq \mathcal{S}_4(y_s)$,

$$D(s, x', z) = \prod_{\beta \in \mathcal{B}_{\mathcal{S}}(s) \cap \Psi} (x'_{\sigma^{-1}(a(\beta))} - x_{a(\beta)})(x'_{\sigma^{-1}(b(\beta))} - x_{b(\beta)})$$

if $\mathbb{D} = \mathbb{C}$, and

$$D(s, x', z) = \prod_{\beta \in \mathcal{B}_{\mathcal{S}}(s) \cap \Psi} (x'_{\sigma^{-1}(a(\beta))} - \hat{\epsilon}_{\sigma^{-1}(a(\beta))}x_{a(\beta)})(x'_{\sigma^{-1}(b(\beta))} - \hat{\epsilon}_{\sigma^{-1}(b(\beta))}x_{b(\beta)})$$

if $\mathbb{D} \neq \mathbb{C}$. Moreover, in both cases $x_{b(\beta)} = -\epsilon(\beta)\overline{x_{a(\beta)}}$.

The reflection s_{β} acts as follows

$$\begin{aligned} s_{\beta}(J_{a(\beta)} - \epsilon(\beta)J_{b(\beta)}) &= J_{a(\beta)} - \epsilon(\beta)J_{b(\beta)}, \\ s_{\beta}(J_{a(\beta)} + \epsilon(\beta)J_{b(\beta)}) &= -J_{a(\beta)} - \epsilon(\beta)J_{b(\beta)}, \\ s_{\beta}J_c &= J_c \text{ for } c \notin \{a(\beta), b(\beta)\}. \end{aligned}$$

Hence,

$$\begin{aligned} & s_{\beta}(iu(J_{a(\beta)} + \epsilon(\beta)J_{b(\beta)}) + v(J_{a(\beta)} - \epsilon(\beta)J_{b(\beta)})) \\ &= -iu(J_{a(\beta)} + \epsilon(\beta)J_{b(\beta)}) + v(J_{a(\beta)} - \epsilon(\beta)J_{b(\beta)}). \end{aligned}$$

Thus

$$(s_{\beta} \cdot x)_{a(\beta)} = \overline{x_{a(\beta)}}$$

so that, for $\mathbb{D} \neq \mathbb{C}$,

$$\begin{aligned} & \frac{1}{(x'_{\sigma^{-1}(a(\beta))} - (s_{\beta} \cdot x)_{a(\beta)})(x'_{\sigma^{-1}(b(\beta))} - (s_{\beta} \cdot x)_{b(\beta)})} \\ &= \frac{1}{(x'_{\sigma^{-1}(a(\beta))} - \overline{x_{a(\beta)}})(x'_{\sigma^{-1}(b(\beta))} - x_{a(\beta)})} \\ &= \frac{1}{(x'_{\sigma^{-1}(b(\beta))} - x_{a(\beta)})(x'_{\sigma^{-1}(a(\beta))} - \overline{x_{a(\beta)}})}. \end{aligned}$$

Furthermore, $\det(s_\beta) = -1$. Thus, there is a constant $\epsilon_1(\Psi) = \pm 1$, depending on the choice of the positive root system Ψ , such that

$$E(s, x', z) = \epsilon_1(\Psi) \prod_{\beta \in \mathcal{B}_S(s) \cap \Psi} \left(\frac{1}{(x'_{\sigma^{-1}(a(\beta))} - x_{a(\beta)})(x'_{\sigma^{-1}(b(\beta))} - \bar{x}_{a(\beta)})} \right. \\ \left. - \frac{1}{(x'_{\sigma^{-1}(b(\beta))} - x_{a(\beta)})(x'_{\sigma^{-1}(a(\beta))} - \bar{x}_{a(\beta)})} \right),$$

if $\mathbb{D} = \mathbb{C}$. Similarly,

$$E(s, x', z) = \epsilon_1(\Psi) \left(\prod_{\beta \in \mathcal{B}_S(s) \cap \Psi} \hat{\epsilon}_{\sigma^{-1}(a(\beta))}(-\hat{\epsilon}_{\sigma^{-1}(b(\beta))})\epsilon(\beta) \right) \prod_{\beta \in \mathcal{B}_S(s) \cap \Psi} \\ \left(\frac{1}{(\hat{\epsilon}_{\sigma^{-1}(a(\beta))}x'_{\sigma^{-1}(a(\beta))} - x_{a(\beta)})(-\hat{\epsilon}_{\sigma^{-1}(b(\beta))}\epsilon(\beta)x'_{\sigma^{-1}(b(\beta))} - \bar{x}_{a(\beta)})} \right. \\ \left. - \frac{1}{(-\hat{\epsilon}_{\sigma^{-1}(b(\beta))}\epsilon(\beta)x'_{\sigma^{-1}(b(\beta))} - x_{a(\beta)})(\hat{\epsilon}_{\sigma^{-1}(a(\beta))}x'_{\sigma^{-1}(a(\beta))} - \bar{x}_{a(\beta)})} \right).$$

if $\mathbb{D} \neq \mathbb{C}$. Let

$$E''(s, x', z) = \sum_{\delta \in W(s, \mathcal{S})} \frac{1}{B''(s\delta, x', z)}$$

and let

$$Q(z) = P(s^{-1} \cdot z)\tilde{\pi}_{\mathfrak{h}}(s^{-1} \cdot z)\mathcal{A}(-\tilde{\Psi}_{\mathcal{S}, \mathbb{R}})(x).$$

Since the terms

$$\frac{P(s^{-1} \cdot z)\tilde{\pi}_{\mathfrak{h}}(s^{-1} \cdot z)}{A(s, x', z)B(s, x', z)C(s, x', z)}$$

are invariant under the substitutions $s \rightarrow \eta s$, $\eta \in W(\mathcal{B}_S(s))$ and the terms

$$\frac{n(s)P(s^{-1} \cdot z)\tilde{\pi}_{\mathfrak{h}}(s^{-1} \cdot z)}{A(s, x', z)B'(s, x', z)C(s, x', z)}$$

are invariant under the substitutions $s \rightarrow s\delta$, $\delta \in W(s, \mathcal{S})$ (here is the only place where we use the evenness of P), we have

$$\tilde{\nu}_{s, x', \mathcal{S}, N}(z) \\ = \frac{n_{\mathcal{S}}(s)i^{-n'}}{A(s, x', z)B(s, x', z)C(s, x', z)} E(s, x', z)E''(s, x', z)Q(z)(\mathcal{H}_S\psi)_N(z)\nu(z).$$

Similarly, we have

$$\tilde{\tau}_{s, x', \mathcal{S}, \beta, N}(z) = \frac{n_{\mathcal{S}}(s)i^{-n'}}{A(s, x', z)B(s, x', z)C(s, x', z)} E(s, x', z)E''(s, x', z)Q(z) \\ \partial(e_N(\frac{\beta(y_s)}{2}H_\beta + im_{\mathcal{S}(y_s)}(x)ty_{s, s}))(\mathcal{H}_S\psi)(x - \frac{\beta(y_s)}{2}H_\beta)\nu(z), \quad (10.14)$$

and hence

$$\begin{aligned} & \tilde{\nu}_{s,x',\mathcal{S},N}(z) - \tilde{\tau}_{s,x',\mathcal{S},\beta,N}(z) \\ &= \frac{n_{\mathcal{S}}(s)i^{-n'}}{A(s,x',z)B'(s,x',z)C(s,x',z)} E(s,x',z)E''(s,x',z)Q(z) \\ & \quad \left((\mathcal{H}_{\mathcal{S}}\psi)_N(z) - \partial(e_N(\frac{\beta(y_s)}{2}H_{\beta} + im_{\mathcal{S}(y_s)}(x)ty_{s,\mathcal{S}}))(\mathcal{H}_{\mathcal{S}}\psi)(x - \frac{\beta(y_s)}{2}H_{\beta}) \right) \nu(z). \end{aligned}$$

The terms

$$\frac{1}{A(s,x',z)}, \quad \frac{1}{B(s,x',z)}$$

are bounded on $C_{s,\mathcal{S}}(1 \times (\mathfrak{h}_{\mathcal{S}} \setminus T_{\mathcal{S}}(s)))$, by (10.9) and (10.10). The term

$$\frac{1}{C(s,x',z)} \tag{10.15}$$

is absolutely integrable (with respect to the variables which occur in it), and the integral does not depend on x' .

Since,

$$\begin{aligned} E''(s,x',z) &= \prod_{1 \leq j \leq n', \sigma(j) \in \underline{\mathcal{S}}''} \left(\frac{1}{x'_j + x_{\sigma(j)}} + \frac{1}{x'_j - x_{\sigma(j)}} \right) \tag{10.16} \\ &= \prod_{1 \leq j \leq n', \sigma(j) \in \underline{\mathcal{S}}''} \frac{2x'_j}{x'^2_j + |x_{\sigma(j)}|^2}, \end{aligned}$$

the integral of $E''(s,x',z)$ with respect to the variables $x_{\sigma(j)}$, $\sigma(j) \in \underline{\mathcal{S}}''$, is finite and does not depend on x' .

The part of the integral corresponding to

$$E(x',z)Q(z)(\mathcal{H}_{\mathcal{S}}\psi)_N(z)$$

is bounded, for the N large enough, by Proposition A.1 if $\mathbb{D} \neq \mathbb{R}$ and Proposition A.5 if $\mathbb{D} = \mathbb{R}$. Hence, (10.2) follows.

Similarly, for $\beta \in \mathcal{S}(y_s)$,

$$\begin{aligned} & \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times (T_{\mathcal{S},\beta}(s)))} (\tilde{\nu}_{s,x',\mathcal{S},N} - \tilde{\tau}_{s,x',\mathcal{S},\beta,N}) \\ &= \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times (T_{\mathcal{S},\beta}(s)))} \frac{n_{\mathcal{S}}(s)i^{-n'}}{A(s,x',z)} \frac{1}{B(s,x',z)} \frac{1}{C_{\mathcal{S}}(s,x',z)} E(s,x',z)E''(s,x',z) \\ & \quad \left((\mathcal{H}_{\mathcal{S}}\psi)_N(z) - \partial(e_N(\frac{\beta(y_s)}{2}H_{\beta} + im_{\mathcal{S}(y_s)}(x)ty_{s,\mathcal{S}}))(\mathcal{H}_{\mathcal{S}}\psi)(x - \frac{\beta(y_s)}{2}H_{\beta}) \right) \nu. \end{aligned} \tag{10.17}$$

Notice that in the formula (10.17),

$$(\mathcal{H}_{\mathcal{S}}\psi)_N(z) = \partial(e_N(im_{\mathcal{S}(y_s)}(x)y_{s,\mathcal{S}}))(\mathcal{H}_{\mathcal{S}}\psi)(x).$$

Thus Lemma 10.4 implies

$$\begin{aligned} & |(\mathcal{H}_S\psi)_N(z) - \partial(e_N(\frac{\beta(y_s)}{2}H_\beta + im_{S(y_s)}(x)y_{s,S}))(\mathcal{H}_S\psi)(x - \frac{\beta(y_s)}{2}H_\beta)| \\ & \leq \text{const} \left| \frac{\beta(x)}{2} \right|^{N+1}, \end{aligned} \tag{10.18}$$

where the constant *const* depends on N and ψ . Thus, by (10.7) and (10.11), the term

$$\frac{1}{A(s, x', z)} \frac{1}{B(s, x', z)},$$

when multiplied by (10.18), is bounded for N large enough. By combining this with (10.15) and Proposition A.1, we see that (10.3) follows.

We shall apply Lemma 10.5 to the following situation.

- \mathbb{R}^d is identified with the span of J_a , $a \notin \underline{S}$ by $\mathbb{R}^d \ni x \rightarrow \sum_{a \notin \underline{S}} x_a J_a \in \mathfrak{h}$ ($d = n - |\underline{S}|$)
- \mathcal{U} is the complement of the zero set of the non-compact imaginary roots of \mathfrak{h}_S in \mathcal{U} ;
- F is the restriction of the function

$$\frac{n_S(s)i^{-n'}}{A(s, x', z)B(s, x', z)C(s, x', z)} E(s, x', z)E''(s, x', z)Q(z)$$

to \mathcal{U} ;

- f is the restriction of the function $\mathcal{H}_S\psi(x)$ to \mathcal{U} ;
- $m = m_{S(y_s)}(x)$, $v = y_{s,S}$.

Notice that $m_{S(y_s)}(x)$ does not depend on the variables x_a , $a \notin \underline{S}$. Furthermore,

$$\begin{aligned} & \left| \int_{\mathcal{C}_{s,S}(I \times \mathfrak{h}_S)} d\tilde{\nu}_{s,x',S,N} \right| \\ & = |n_S(s)| \left| \int_{\mathfrak{h}_S} \int_I (tm_{S(y_s)}(x))^N \frac{1}{A(s, x', \mathcal{C}_{s,S}(t, x))} \frac{1}{B(s, x', \mathcal{C}_{s,S}(t, x))} \right. \\ & \quad \frac{1}{C(s, x', \mathcal{C}_{s,S}(t, x))} E(s, x', \mathcal{C}_{s,S}(t, x))E''(s, x', \mathcal{C}_{s,S}(t, x)) \\ & \quad \left. Q(\mathcal{C}_{s,S}(t, x)) \left(-\frac{\partial(iy_{s,S})^{N+1}}{N!} \right) \mathcal{H}_S\psi(x)m_{S(y_s)}(x) dt d\mu(x) \right|. \end{aligned}$$

By (10.11), the term

$$(tm_{S(y_s)}(x))^N \frac{1}{A(s, x', \mathcal{C}_{s,S}(t, x))} \frac{1}{B(s, x', \mathcal{C}_{s,S}(t, x))}$$

is bounded for N large enough. The term

$$\frac{1}{C(s, x', \mathcal{C}_{s, \mathcal{S}}(t, x))}$$

is absolutely integrable, and the integral does not depend on x' . The term

$$E(x', \mathcal{C}_{s, \mathcal{S}}(t, x))$$

leads to a bounded integral via Proposition A.1 if $\mathbb{D} \neq \mathbb{R}$ and Proposition A.5 if $\mathbb{D} = \mathbb{R}$. Hence, (10.4) follows.

Consider the integral (10.5). In the formula (10.14) let $x \in T_{\mathcal{S}, \beta}(s)$. Then $\nu(z) = \nu(x)$. Moreover, by (10.7),

$$\left| \frac{1}{A(s, x', z)} \right| \leq \prod_{1 \leq j \leq n', \sigma(j) \notin \underline{\mathcal{S}}} \left| \frac{1}{-\hat{\epsilon}_j x'_j + x_{\sigma(j)} + i \min\{1, |\tilde{\beta}(x)|\}} \right|. \tag{10.19}$$

The set $\mathcal{S}(y_s) \setminus \mathcal{S}''$ consists of roots of the form $\eta = iJ_{a(\eta)}^* + iJ_{b(\eta)}^*$, $a(\eta) < b(\eta)$, which vanish on y_s . In these terms

$$\left| \frac{1}{B'(s, x', z)} \right| = \prod_{\substack{\eta \in \mathcal{S}(y_s) \setminus \mathcal{S}'' \\ \sigma^{-1}\eta \subseteq \{1, 2, \dots, n'\}}} \prod_{\substack{\sigma(j) \in \underline{\eta}, 1 \leq j \leq n'}} \left| \frac{1}{x'_j - \hat{\epsilon}_j x_{\sigma(j)}} \right| \prod_{\substack{\eta \in \mathcal{S}(y_s) \setminus \mathcal{S}'' \\ \sigma^{-1}\eta \subseteq \{1, 2, \dots, n'\}}} \left| \frac{1}{-\hat{\epsilon}_{\sigma^{-1}(a(\eta))} x'_{\sigma^{-1}(a(\eta))} + x_{a(\eta)}} \frac{1}{\hat{\epsilon}_{\sigma^{-1}(b(\eta))} x'_{\sigma^{-1}(b(\eta))} + \overline{x_{a(\eta)}}} \right|, \tag{10.20}$$

where $\Im(x_{a(\eta)}) \geq \min\{1, |\tilde{\beta}(x)|\}$.

Since

$$\int_0^1 |\log(y)|^m dy < \infty \quad (m = 0, 1, 2, \dots),$$

we see that

$$\int_0^1 \left| \int_{-1}^1 \frac{1}{|u + iy|} du \right|^k \left| \int_y^1 \int_{-1}^1 \frac{1}{|u^2 + v^2|} du dv \right|^l dy < \infty \tag{10.21}$$

($k, l = 0, 1, 2, \dots$).

By combining (10.16), (10.19)-(10.21), we see that the integral of

$$\left| \frac{1}{A(s, x', z)} \frac{1}{B'(s, x', z)} E''(s, x', z) \right|$$

with respect to the variables which occur in this product, over $T_{\mathcal{S}, \beta}(s)$, is finite and bounded independently of x' . This combined with Proposition A.1, as in the argument concerning previous cases, verifies (10.5).

Consider the integral (10.6). Let us rewrite the formula (9.4) as

$$\frac{\beta(z)}{|\beta(z)|} \tau_{s, x', \mathcal{S}, \beta, N}(z) = \frac{F_{s, \mathcal{S}, \beta, N}(z)}{\det(x' + z)_{sW^b}} \nu(z),$$

with $z = \mathcal{C}_{s,\mathcal{S}}(1, x)$ and $x \in T_{\mathcal{S},\beta}(s)$. Notice that

$$F_{s,\mathcal{S},\beta,N}(s_\beta \cdot z) = F_{s,\mathcal{S},\beta,N}(z) \text{ and } T_{\mathcal{S},\beta}(ss_\beta) = T_{\mathcal{S},\beta}(s).$$

Hence,

$$\begin{aligned} & \int_{\mathcal{C}_{s,\mathcal{S}}(1 \times T_{\mathcal{S},\beta}(s))} 2 \frac{\beta}{|\beta|} \tau_{s,x',\mathcal{S},\beta,N} \\ &= \int_{T_{\mathcal{S},\beta}(s)} \left(\frac{1}{\det(x' + z)_{sW^{b'}}} + \frac{1}{\det(x' + s_\beta \cdot z)_{sW^{b'}}} \right) F_{s,\mathcal{S},\beta,N}(z) \nu(z). \end{aligned}$$

Moreover,

$$\begin{aligned} & \left(\frac{1}{\det(x' + z)_{sW^{b'}}} + \frac{1}{\det(x' + s_\beta \cdot z)_{sW^{b'}}} \right) \\ &= \begin{cases} \frac{2}{\det(x' + z)_{sW^{b'}}} & \text{if } \underline{\beta} \cap \{1, 2, \dots, n'\} = \emptyset, \\ \frac{2x'_b}{x_b'^2 + |x_{\sigma(b)}|^2} \prod_{1 \leq j \leq n', j \neq b} \frac{1}{i(x'_j - \hat{\epsilon}_j z_{\sigma(j)})} & \text{if } 1 \leq b \leq n', \underline{\beta} = \{\sigma(b)\}. \end{cases} \end{aligned}$$

In the second case, $\sigma(b) \in \mathcal{S}(y_s)$ and the function

$$\frac{2x'_b}{x_b'^2 + |x_{\sigma(b)}|^2}$$

is absolutely integrable with respect to the variable $x_{\sigma(b)}$, and the integral can be dominated independently of x' . Furthermore, (10.19) holds and

$$|\mathfrak{S}(x_{\sigma(j)})| \geq \min\{1, |\beta(x)|\}, \quad (1 \leq j \leq n', \sigma(j) \in \mathcal{S}(y_s), x \in T_{\mathcal{S},\beta}(s)).$$

Hence, as before, (10.21) implies (10.6). ■

11. The pair $(\mathbf{O}_{2p+1,2q+1}, \mathbf{Sp}_{2n}(\mathbb{R}))$, with $n' \leq p + q + 1$.

In this case the defining module V for G has the following orthogonal direct sum decomposition

$$V = V_1 \oplus V'', \tag{11.1}$$

where $G(V_1)$ is isomorphic to $O_{1,1}$ and $G(V'')$ is isomorphic to $O_{2p,2q}$. Let \mathfrak{h}'' be an elliptic Cartan subalgebra of $\mathfrak{g}(V'')$. Let $\mathfrak{h} = \mathfrak{g}(V_1) \oplus \mathfrak{h}''$. This is a fundamental Cartan subalgebra of \mathfrak{g} . We retain the notation and the identifications (3.1) - (3.3).

Every Cartan subalgebra of \mathfrak{g} is conjugate to one which preserves the decomposition (11.1), and two such are G -conjugate if and only if they are $G(V'')$ -conjugate.

Let $\Psi(V'')$ be a system of positive roots of \mathfrak{h}'' in $\mathfrak{g}(V'')_{\mathbb{C}}$, $\Psi^n(V'') \subseteq \Psi(V'')$ be the subset of the non-compact roots and $\Psi_{st}^n(V'')$ be the set of the strongly orthogonal subsets of $\Psi^n(V'')$. Let Ψ be a system of positive roots of \mathfrak{h} in $\mathfrak{g}_{\mathbb{C}}$ such that, if we extend each root in $\Psi(V'')$ by zero to $\mathfrak{g}(V_1)$, then $\Psi(V'') \subseteq \Psi$. In order to simplify the notation we shall write Ψ_{st}^n for $\Psi_{st}^n(V'')$.

There is a complex structure J_1 on V_1 and for any $\mathcal{S} \in \Psi_{st}^n$ an element $\tilde{c}(\mathcal{S}) \in GL(V_{\mathbb{C}})$ such that

$$\begin{aligned} \tilde{c}(\mathcal{S})V_{1,\mathbb{C}} &= V_{1,\mathbb{C}}, \quad \tilde{c}(\mathcal{S})V''_{\mathbb{C}} = V''_{\mathbb{C}}, \\ \mathfrak{g}(V_1) &= \mathbb{R}\tilde{c}(\mathcal{S})iJ_1\tilde{c}(\mathcal{S})^{-1}, \\ \text{Ad}(\tilde{c}(\mathcal{S})|_{\mathfrak{g}(V''_{\mathbb{C}})}) &\text{ is the Cayley transform for } \mathcal{S} \text{ and } \mathfrak{g}(V''), \end{aligned}$$

Let $c(\mathcal{S}) = \text{Ad}(\tilde{c}(\mathcal{S})) \in GL(\mathfrak{g}_{\mathbb{C}})$ and let $\mathfrak{h}(\mathcal{S}) = \mathfrak{g}(V_1) \oplus \mathfrak{h}''(\mathcal{S})$. Define

$$\mathfrak{h}_{\mathcal{S}} = c(\mathcal{S})^{-1}(\mathfrak{h}(\mathcal{S})_{\mathbb{C}}) \cap \mathfrak{g}.$$

Then

$$\mathfrak{h}_{\mathcal{S}} = \mathbb{R}iJ_1 \oplus \mathfrak{h}''_{\mathcal{S}}$$

and

$$chc(x' + c(\mathcal{S})z) = chc(x' + z) \quad (x' \in \mathfrak{h}^{\text{reg}}, z \in \mathfrak{h}_{\mathcal{S}}).$$

Define the Harish-Chandra integral $\mathcal{H}_{\mathcal{S}}\psi$ as in (2.1). Then, Proposition 5.3 carries over without any change. One sees from (7.9) that

$$\begin{aligned} m_{\mathcal{S}} &= \frac{1}{2(|\mathcal{S}''| + 1)} m_{\mathcal{S}}^{\mathfrak{g}(V'')}, \text{ and} \\ m_{\mathcal{S}}^{\mathfrak{g}(V'')} &= \frac{1}{2^{(|\mathcal{S}''|/2)2^{|\mathcal{S}'|}} \cdot 1 \cdot 3 \cdot 5 \cdots (|\mathcal{S}''| - 1)} \frac{1}{2^p p! 2^q q!}, \end{aligned} \tag{11.2}$$

where, as in (3.17), $\mathcal{S}'' = \{\alpha \in \mathcal{S} \mid \underline{\alpha} \subseteq \mathcal{S} \setminus \alpha\}$ and $\mathcal{S}' = \mathcal{S} \setminus \mathcal{S}''$. In particular, Lemma 8.1 holds. Theorems 7.3 and 7.4 carry over with the $\Gamma_{s,\mathcal{S}}$ defined as in (7.6). The rest of the proof requires minor modification which essentially amount to a reduction to the subalgebra $\mathfrak{g}(V'')$.

Appendix A.

Let $\mathbb{R}^{2n,\text{reg}} = \{x''' = (x'_1, x''_1, x'_2, x''_2, \dots, x'_n, x''_n) \mid x'_k \neq x''_k, k = 1, 2, \dots, n\}$, and let $\mathbb{C}^+ = \{z = x + iy \in \mathbb{C} \mid y = \Im(z) > 0\}$. Let $S(\mathbb{C}^n)$ the Schwartz space on \mathbb{C}^n considered as a real vector space.

Proposition A.1. *For any constant coefficient differential operator D with respect to the variable $x''' \in \mathbb{R}^{2n,\text{reg}}$, the formula*

$$\begin{aligned} \sup_{x''' \in \mathbb{R}^{2n,\text{reg}}} |D \int_{(\mathbb{C}^+)^n} \prod_{k=1}^n \Im \left(\frac{1}{(z_k - x'_k)(\bar{z}_k - x''_k)} \right) \phi(z_1, z_2, \dots, z_n) \\ dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n| \end{aligned}$$

defines a continuous seminorm on the subspace of $S(\mathbb{C}^n)$ of the functions ϕ invariant under the transformations $z_k \rightarrow \bar{z}_k, k = 1, 2, \dots, n$.

Remark. Though we shall need the above result only with $D = 1$, it seems natural to include the general case for completeness.

For $\epsilon > 0$ define

$$D_\epsilon^+ = \{z \in \mathbb{C} \mid \Im(z) > 0, |z| < \epsilon\}.$$

For $v \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C} \setminus \{\pm v\}$ define

$$f_v(z) = \frac{1}{2i} \left(\frac{1}{(z-v)(\bar{z}+v)} - \frac{1}{(z+v)(\bar{z}-v)} \right).$$

For any two non-negative integers α, β let

$$\phi_{\alpha,\beta}(z) = z^\alpha \bar{z}^\beta + z^\beta \bar{z}^\alpha \quad (z \in \mathbb{C}).$$

Lemma A.2. *If $\alpha + \beta \in 2\mathbb{Z} + 1$, then*

$$\int_{D_1^+} f_v(z) \phi_{\alpha,\beta}(z) \, dx \, dy = 0. \tag{A.1}$$

Suppose $\alpha + \beta \in 2\mathbb{Z}$. Then, for $|v| < 1$,

$$\int_{D_1^+} f_v(z) \phi_{\alpha,\beta}(z) \, dx \, dy = (-1)^{\alpha+1} \pi^2 \operatorname{sgn}(v) v^{\alpha+\beta} + \sum_{k=0}^\infty c_k v^{2k+1}, \tag{A.2}$$

where

$$c_k = \frac{4(-1)^k}{2k+1-\alpha-\beta} \sum_{b=0}^k (-1)^b \left(\frac{1}{2b+1+\alpha-\beta} + \frac{1}{2b+1-\alpha+\beta} \right)$$

is a bounded sequence.

Proof. The set D_1^+ is invariant under the map $z \rightarrow -\bar{z}$. Since

$$f_v(-\bar{z}) = f_v(z) \quad \text{and} \quad \phi_{\alpha,\beta}(-\bar{z}) = (-1)^{\alpha+\beta} \phi_{\alpha,\beta}(z), \tag{A.3}$$

the equation (A.1) follows.

From now on we assume that $\alpha + \beta \in 2\mathbb{Z}$. Notice that for $r > 0$,

$$f_v(rz) = r^{-2} f_{\frac{v}{r}}(z) \quad \text{and} \quad f_{-v}(z) = -f_v(z),$$

so that

$$f_v(|v|z) = \operatorname{sgn}(v) |v|^{-2} f_1(z).$$

Hence,

$$\begin{aligned} \int_{D_{|v|}^+} f_v(z) \phi_{\alpha,\beta}(z) \, dx \, dy &= \int_{D_1^+} f_v(|v|z) \phi_{\alpha,\beta}(|v|z) |v|^2 \, dx \, dy \\ &= \operatorname{sgn}(v) v^{\alpha+\beta} \int_{D_1^+} f_1(z) \phi_{\alpha,\beta}(z) \, dx \, dy. \end{aligned} \tag{A.4}$$

Let $\epsilon > \delta > 0$ and let $a, b \in \mathbb{Z}$. Integration in polar coordinates shows that

$$\Im \int_{D_\epsilon^+ \setminus D_\delta^+} z^a \bar{z}^b \, dx \, dy = 0, \quad \text{if } a - b \in 2\mathbb{Z}, \tag{A.5}$$

and

$$\Im \int_{D_\epsilon^+ \setminus D_\delta^+} z^a \bar{z}^b dx dy = \frac{\epsilon^{a+b+2} - \delta^{a+b+2}}{a+b+2} \frac{2}{a-b}, \quad \text{if } a-b \in 2\mathbb{Z} + 1. \quad (\text{A.6})$$

Furthermore, by considering the geometric series for $\frac{1}{z-v}$ and for $\frac{1}{\bar{z}+v}$ one checks that

$$\begin{aligned} & f_v(z)\phi_{\alpha,\beta}(z) \tag{A.7} \\ = & \begin{cases} \sum_{a,b \geq 0} v^{-a-b-2} (-1)^{b+1} \Im(z^{a+\alpha} \bar{z}^{b+\beta} + z^{a+\beta} \bar{z}^{b+\alpha}) & \text{if } |z| < |v| \\ \sum_{a,b \geq 0} v^{a+b} (-1)^b \Im(z^{\alpha-a-1} \bar{z}^{\beta-b-1} + z^{\beta-a-1} \bar{z}^{\alpha-b-1}) & \text{if } |z| > |v|. \end{cases} \end{aligned}$$

Hence, by applying (A.5) and (A.6) with $\delta = 0$ and $\epsilon = \rho$, we see that

$$\begin{aligned} & \int_{D_1^+} f_1(z)\phi_{\alpha,\beta}(z) dx dy \\ = & \lim_{0 < \rho < 1, \rho \rightarrow 1} \int_{D_\rho^+} \sum_{a,b \geq 0} (-1)^{b+1} \Im(z^{a+\alpha} \bar{z}^{b+\beta} + z^{a+\beta} \bar{z}^{b+\alpha}) dx dy \\ = & -2 \lim_{0 < \rho < 1, \rho \rightarrow 1} \left(\sum_{b=0}^\infty \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b \rho^{s+2b+\alpha+\beta+2}}{s+2b+\alpha+\beta+2} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right) \right. \\ & \left. + \sum_{b=0}^\infty \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^{b+s} \rho^{s+2b+\alpha+\beta+2}}{s+2b+\alpha+\beta+2} \left(\frac{1}{-s+\alpha-\beta} + \frac{1}{-s-\alpha+\beta} \right) \right) \\ = & -4 \sum_{b=0}^\infty \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b+\alpha+\beta+2} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right). \tag{A.8} \end{aligned}$$

Similarly, (A.7) implies that for $|v| < 1$,

$$\begin{aligned} & \int_{D_1^+ \setminus D_{|v|}^+} f_v(z)\phi_{\alpha,\beta}(z) dx dy \\ = & \lim_{|v| < \rho < 1, \rho \rightarrow |v|} \int_{D_\rho^+ \setminus D_{|v|}^+} f_v(z)\phi_{\alpha,\beta}(z) dx dy \\ = & \sum_{a,b \geq 0; a+b \in 2\mathbb{Z}+1} v^{a+b} \frac{2(-1)^b}{a+b-\alpha-\beta} \left(\frac{1}{a-b+\alpha-\beta} + \frac{1}{a-b-\alpha+\beta} \right) \\ & - \frac{2 \lim_{|v| < \rho < 1, \rho \rightarrow |v|}}{\sum_{a,b \geq 0; a-b \in 2\mathbb{Z}+1} v^{a+b} \rho^{\alpha+\beta-a-b} (-1)^b} \left(\frac{1}{a-b+\alpha-\beta} + \frac{1}{a-b-\alpha+\beta} \right) \\ = & \sum_{0 < n \in 2\mathbb{Z}+1} v^n \frac{2}{n-\alpha-\beta} \sum_{b=0}^n (-1)^b \left(\frac{1}{n-2b+\alpha-\beta} + \frac{1}{n-2b-\alpha+\beta} \right) \\ & - 4 \operatorname{sgn}(v) v^{\alpha+\beta} \sum_{b=0}^\infty \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b-\alpha-\beta} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right). \tag{A.9} \end{aligned}$$

Notice that (for n odd)

$$\begin{aligned} &(-1)^{n-b} \left(\frac{1}{n-2(n-b)+\alpha-\beta} + \frac{1}{n-2(n-b)-\alpha+\beta} \right) \\ &= (-1)^b \left(\frac{1}{n-2b+\alpha-\beta} + \frac{1}{n-2b-\alpha+\beta} \right) \end{aligned}$$

Hence, with $n = 2k + 1$,

$$\begin{aligned} &\sum_{b=0}^n (-1)^b \left(\frac{1}{n-2b+\alpha-\beta} + \frac{1}{n-2b-\alpha+\beta} \right) \\ &= 2 \sum_{b=0}^k (-1)^{k-b} \left(\frac{1}{2b+1+\alpha-\beta} + \frac{1}{2b+1-\alpha+\beta} \right). \quad (\text{A.10}) \end{aligned}$$

Furthermore,

$$\begin{aligned} &- \sum_{b=0}^{\infty} \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b+\alpha+\beta+2} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right) \\ &= - \sum_{b=\alpha+\beta+1}^{\infty} \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^{b-\alpha-\beta-1}}{s+2b-\alpha-\beta} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right) \\ &= \sum_{b=\alpha+\beta+1}^{\infty} \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b-\alpha-\beta} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right) \end{aligned}$$

Therefore

$$\begin{aligned} &- \sum_{b=0}^{\infty} \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b+\alpha+\beta+2} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right) \\ &- \sum_{b=0}^{\infty} \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b-\alpha-\beta} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right) \\ &= - \sum_{b=0}^{\alpha+\beta} \sum_{1 \leq s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b-\alpha-\beta} \left(\frac{1}{s+\alpha-\beta} + \frac{1}{s-\alpha+\beta} \right) \\ &= - \sum_{b=0}^{\alpha+\beta} \sum_{s \in 2\mathbb{Z}+1} \frac{(-1)^b}{s+2b-\alpha-\beta} \frac{1}{s+\alpha-\beta} = -(-1)^\alpha \frac{\pi^2}{4}, \quad (\text{A.11}) \end{aligned}$$

because, by Plancherel’s Theorem for the Fourier series,

$$\sum_{s \in 2\mathbb{Z}+1} \frac{1}{s} \frac{1}{s+2a} = \begin{cases} \frac{\pi^2}{4} & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, (A.4), (A.8), (A.9), (A.10) and (A.11) imply the lemma. ■

Proof. [of Proposition A.1] For $N \in \mathbb{N}$ and $z \in \mathbb{C}$, Taylor’s formula ([6, (1.1.7)']) reads:

$$\begin{aligned} \phi(z) = & \sum_{\substack{a,b \geq 0 \\ a+b < N}} \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(0) x^a y^b + \sum_{\substack{a,b \geq 0 \\ a+b = N}} \left(N \int_0^1 (1-t)^{N-1} \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(tz) dt \right) x^a y^b. \end{aligned} \tag{A.12}$$

Let $D_1 = \{z \in \mathbb{C} \mid |z| < 1\}$. For $A \subseteq \mathbb{C}$ let $1_A(z)$ be the indicator function of A (equal to 1 for $z \in A$ and 0 for $z \in \mathbb{C} \setminus A$). Put $z = x + iy$ and define

$$\begin{aligned} T_0^{(N-1)} \phi(z) &= 1_{\mathbb{C} \setminus D_1}(z) \phi(z) \\ T_1^{(N-1)} \phi(z) &= 1_{D_1}(z) \sum_{a,b \geq 0; a+b < N} \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(0) x^a y^b \\ T_2^{(N-1)} \phi(z) &= 1_{D_1}(z) \sum_{a,b \geq 0; a+b = N} \left(N \int_0^1 (1-t)^{N-1} \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(tz) dt \right) x^a y^b \end{aligned}$$

so that

$$\phi = T_0^{(N-1)} \phi + T_1^{(N-1)} \phi + T_2^{(N-1)} \phi. \tag{A.13}$$

Since

$$\frac{1}{(z-v)(\bar{z}+v)} = \frac{1}{2x} \left(\frac{1}{z-v} + \frac{1}{\bar{z}+v} \right),$$

we have

$$\begin{aligned} & \int_{|x| > \frac{1}{\sqrt{2}}} \frac{1}{(z-v)(\bar{z}+v)} \phi(z) dx dy \\ &= \int_{|x+v| > \frac{1}{\sqrt{2}}} \frac{1}{2(x+v)} \frac{1}{z} \partial_v^N \phi(z+v) dx dy + \int_{|x-v| > \frac{1}{\sqrt{2}}} \frac{1}{2(x-v)} \frac{1}{\bar{z}} \partial_v^N \phi(z-v) dx dy. \end{aligned}$$

Thus, since $\frac{1}{\sqrt{2}} - \frac{1}{2} > 0$,

$$\sup_{0 < |v| \leq \frac{1}{2}} \left| \partial_v^N \int_{|x| > \frac{1}{\sqrt{2}}} \frac{1}{(z-v)(\bar{z}+v)} \phi(z) dx dy \right| < \infty. \tag{A.14}$$

Also,

$$\sup_{v \in \mathbb{R}^\times} \left| \partial_v^N \int_{y > \frac{1}{\sqrt{2}}} \frac{1}{(z-v)(\bar{z}+v)} \phi(z) dx dy \right| < \infty. \tag{A.15}$$

On the other hand,

$$\frac{1}{(z-v)(\bar{z}+v)} = \left(\frac{1}{z-v} - \frac{1}{\bar{z}+v} \right) \frac{1}{\bar{z}-z+2v}.$$

Therefore,

$$\begin{aligned} & \partial_v^N \int_{\mathbb{C}^+} \frac{1}{(z-v)(\bar{z}+v)} \phi(z) dx dy \\ &= \int_{\mathbb{C}^+} \frac{1}{z} \partial_v^N \left(\frac{1}{\bar{z}-z+2v} \phi(z+v) \right) dx dy \\ &+ \int_{\mathbb{C}^+} \frac{1}{\bar{z}} \partial_v^N \left(\frac{1}{\bar{z}-z+2v} \phi(z-v) \right) dx dy, \end{aligned}$$

so that

$$\sup_{|v| > \frac{1}{2}} \left| \partial_v^N \int_{\mathbb{C}^+} \frac{1}{(z-v)(\bar{z}+v)} \phi(z) dx dy \right| < \infty. \tag{A.16}$$

Since, $|x^a y^b| \leq |z|^{a+b}$ and since $N \int_0^1 (1-t)^{N-1} dt = 1$, we have

$$|T_2^{(N-1)} \phi(z)| \leq |z|^N \sum_{a,b \geq 0; a+b=N} \sup_{w \in D_1} \left| \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(w) \right|. \tag{A.17}$$

By combining (A.14), (A.15), (A.16), (A.17) with Lemma A.2 and the fact that, by the assumption that $\phi(\bar{z}) = \phi(z)$, the Taylor expansion (A.12) may be expressed in terms of the $\phi_{\alpha,\beta}$, we see that for each $N = 0, 1, 2, \dots$, and $\alpha = 0, 1, 2$, the formula

$$q_\alpha^{(N)}(\phi) = \sup_{v \in \mathbb{R}^\times} \left| \partial_v^N \int_{\mathbb{C}^+} f_v(z) T_\alpha^N \phi(z) dx dy \right|$$

defines a continuous seminorm on the space of the Schwartz functions on the real vector \mathbb{C} , invariant under the transformation $z \rightarrow \bar{z}$. In particular, by (A.13),

$$\sup_{v \in \mathbb{R}^\times} \left| \partial_v^N \int_{\mathbb{C}^+} f_v(z) \phi(z) dx dy \right| \leq q_0^{(N)}(\phi) + q_1^{(N)}(\phi) + q_2^{(N)}(\phi) < \infty.$$

For a Schwartz function ϕ on the real vector space \mathbb{C}^n and for $\alpha \in \{0, 1, 2\}^n$ let

$$T_\alpha^{(N)} \phi(z_1, z_2, \dots, z_n) = T_{\alpha_1}^{(N)} \otimes T_{\alpha_2}^{(N)} \otimes \dots \otimes T_{\alpha_n}^{(N)} \phi(z_1, z_2, \dots, z_n).$$

Then

$$\phi = \sum_{\alpha \in \{0,1,2\}^n} T_\alpha^{(N)} \phi.$$

Hence, for non-negative integers N_1, N_2, \dots, N_n and ϕ as in Proposition A.1,

$$\begin{aligned} & \left| \partial_{v_1}^{N_1} \partial_{v_2}^{N_2} \dots \partial_{v_n}^{N_n} \int_{(\mathbb{C}^+)^n} \prod_{k=1}^n f_{v_k}(z_k) \phi(z_1, z_2, \dots, z_n) dx_1 dy_1 dx_2 dy_2 \dots dx_n dy_n \right| \\ & \leq \sum_{\alpha \in \{0,1,2\}^n} q_{\alpha_1}^{(N_1)} \otimes q_{\alpha_2}^{(N_2)} \otimes \dots \otimes q_{\alpha_n}^{(N_n)}(\phi) < \infty \end{aligned} \tag{A.18}$$

Let $x'_k = u_k + v_k$ and let $x''_k = u_k - v_k$. Then

$$\begin{aligned} & \int_{(\mathbb{C}^+)^n} \prod_{k=1}^n \mathfrak{S} \left(\frac{1}{(z_k - x'_k)(\bar{z}_k - x''_k)} \right) \phi(z_1, z_2, \dots, z_n) dx_1 dy_1 dx_2 dy_2 \dots dx_n dy_n \\ & = \int_{(\mathbb{C}^+)^n} \left(\prod_{k=1}^n f_{v_k}(z_k) \right) \phi(z_1 + u_1, z_2 + u_2, \dots, z_n + u_n) dx_1 dy_1 dx_2 dy_2 \dots dx_n dy_n. \end{aligned} \tag{A.19}$$

Clearly, (A.18) and (A.19) imply Proposition A.1. ■

For a complex valued function F defined in a neighborhood of zero in \mathbb{R} define the jump at zero by

$$\langle F \rangle = \lim_{v \rightarrow 0^+} F(v) - \lim_{v \rightarrow 0^-} F(v),$$

whenever the limits exist.

Proposition A.3. *Let $\phi \in \mathcal{S}(\mathbb{C})$ be such that $\phi(z) = \phi(\bar{z})$. Then, for any non-negative integer k ,*

$$\langle \partial_v^k \int_{\mathbb{C}^+} f_v(z) \phi(z) dx dy \rangle = -\pi^2 \partial_{iy}^k \phi(0).$$

Proof. It is clear that

$$\langle \partial_v^k \int_{\mathbb{C}^+ \setminus D_1^+} f_v(z) \phi(z) dx dy \rangle = 0.$$

Let $N > k$. Then, (A.17) implies that

$$\langle \partial_v^k \int_{D_1^+} f_v(z) T_2^{(N-1)} \phi(z) dx dy \rangle = 0.$$

Hence

$$\langle \partial_v^k \int_{\mathbb{C}^+} f_v(z) \phi(z) dx dy \rangle = \langle \partial_v^k \int_{D_1^+} f_v(z) T_1^{(N-1)} \phi(z) dx dy \rangle.$$

Therefore it will suffice to show that

$$\langle \partial_v^k \int_{\mathbb{C}^+} f_v(z) \phi_{\alpha,\beta}(z) dx dy \rangle = -\pi^2 \partial_{iy}^k \phi_{\alpha,\beta}(0). \tag{A.20}$$

By (A.1), the left hand side of (A.20) is zero if $\alpha + \beta \in 2\mathbb{Z} + 1$ (and so is the right hand side). Suppose $\alpha + \beta \in 2\mathbb{Z}$. Then, (A.2) shows that

$$\begin{aligned} \langle \partial_v^k \int_{\mathbb{C}^+} f_v(z) \phi_{\alpha,\beta}(z) dx dy \rangle &= (-1)^{\alpha+1} \pi^2 \langle \partial_v^k (\text{sgn}(v) v^{\alpha+\beta}) \rangle \\ &= (-1)^{\alpha+1} \pi^2 \begin{cases} 2k! & \text{if } k = \alpha + \beta, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Also,

$$\begin{aligned} \partial_{iy}^k \phi_{\alpha,\beta}(0) &= \partial_{iy}^k ((iy)^\alpha (-iy)^\beta + (iy)^\beta (-iy)^\alpha) |_{y=0} \\ &= \begin{cases} (-1)^\alpha 2k! & \text{if } k = \alpha + \beta \in 2\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, the equality (A.20) holds and we are done. ■

Corollary A.4. *Let $\phi \in S(\mathbb{C})$ be such that $\phi(z) = \phi(\bar{z})$. Let*

$$\tilde{\phi}(v) = \int_{\mathbb{C}} f_v(z) \frac{y}{|y|} \phi(z) dx dy \quad (v \in \mathbb{R} \setminus 0).$$

Then

$$\langle \tilde{\phi} \rangle(0) = -2\pi^2 \phi(0).$$

Proof. Since, $f_v(\bar{z}) = -f_v(z)$,

$$\tilde{\phi}(v) = 2 \int_{\mathbb{C}^+} f_v(z) \phi(z) dx dy.$$

Thus, the Corollary follows from Proposition A.3. ■

We shall also need a fact similar to Proposition A.1, but with fewer assumptions.

Let V be a finite dimensional vector space over \mathbb{R} and let $A \subseteq V^*$ be a finite set such that no two elements of A are proportional. Let $V^A = \{x \in V \mid \text{there exists } \alpha \in A \text{ such that } \alpha(x) = 0\}$. We shall say that a function $\phi \in C^\infty(V \setminus V^A)$ is a Harish-Chandra Schwartz function with respect to A if and only if for every constant coefficient differential operator D on V and for every polynomial function P on V ,

$$\sup_{x \in V \setminus V^A} |P(x)D\phi(x)| < \infty, \tag{A.21}$$

and for every connected component $C \subseteq V \setminus V^A$ the restriction of $D\phi$ to C extends to a continuous function on \bar{C} , the closure of C in V . (Notice that this extension is a rapidly decreasing function on \bar{C} .) We shall denote by $\mathcal{HCS}(V \setminus V^A)$ the space of all the Harish-Chandra Schwartz functions with respect to A and equip this space with the topology induced by the seminorms (A.21). Our definition is motivated by a theorem of Harish-Chandra concerning his orbital integrals, see Theorem 23 on page 23 and the proof of Proposition 10 in the Appendix of part I of [17]. Let $\mathbb{C}^{++} = \{z \in \mathbb{C} \mid \text{Re}(z) > 0, \Im(z) > 0\}$.

Proposition A.5. *By the following formula, setting $d\mu = dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n$,*

$$\sup_{x'' \in \mathbb{R}^{2n, \text{reg}}} \left| \int_{(\mathbb{C}^{++})^n} \prod_{k=1}^n \Im \left(\frac{1}{(z_k - x'_k)(\bar{z}_k - x''_k)} \right) \phi(z_1, z_2, \dots, z_n) d\mu \right|$$

we define a continuous seminorm on the subspace of $\mathcal{HCS}(\mathbb{C}^n \setminus \bigcup_{k=1}^n \{z \in \mathbb{C}^n \mid \text{Re}(z_k) = 0\})$ of the functions ϕ such that for any $1 \leq k \leq n$

$$\phi(z) = \begin{cases} \phi(z') & \text{if } z' = (z_1, \dots, \bar{z}_k, \dots, z_n), \\ -\phi(z') & \text{if } z' = (z_1, \dots, -\bar{z}_k, \dots, z_n). \end{cases}$$

Lemma A.6. *Let $\phi \in L^\infty(\mathbb{C}^+) \cap L^1(\mathbb{C}^+)$. Then, for $t \in \mathbb{R}$,*

$$\int_{\mathbb{C}^+} \left| \frac{1}{z+t} \phi(z) \right| dx dy \leq \left(\int_{|z| \leq 1} \frac{1}{|z|} dx dy + 1 \right) (\|\phi\|_\infty + \|\phi\|_1).$$

Proof. The left hand side is equal to

$$\begin{aligned} \int_{|z|\leq 1} \left| \frac{1}{z} \phi(z-t) \right| dx dy + \int_{|z|\geq 1} \left| \frac{1}{z} \phi(z-t) \right| dx dy \\ \leq \int_{|z|\leq 1} \frac{1}{|z|} dx dy \|\phi\|_\infty + \int_{|z|\geq 1} |\phi(z-t)| dx dy, \end{aligned}$$

which is dominated by the right hand side. ■

Lemma A.7. Let $\phi \in L^\infty(\mathbb{C}^+) \cap L^1(\mathbb{C}^+)$. Then, for $u, v \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{C}^+, |x-u|>\frac{1}{\sqrt{2}} \text{ or } y>\frac{1}{\sqrt{2}}} \left| \frac{1}{(z-u-v)(\bar{z}-u+v)} \phi(z) \right| dx dy \\ \leq \sqrt{2} \left(\int_{|z|\leq 1} \frac{1}{|z|} dx dy + 1 \right) (\|\phi\|_\infty + \|\phi\|_1). \end{aligned}$$

Proof. Notice that

$$\begin{aligned} \frac{1}{(z-u-v)(\bar{z}-u+v)} &= \frac{1}{2(x-u)} \left(\frac{1}{z-u-v} + \frac{1}{\bar{z}-u+v} \right) \\ &= \frac{1}{2(v-iy)} \left(\frac{1}{z-u-v} - \frac{1}{\bar{z}-u+v} \right). \end{aligned}$$

Hence, the left hand side is less than or equal to

$$\frac{\sqrt{2}}{2} \left(\int_{\mathbb{C}^+} \left| \frac{1}{z-u-v} \phi(z) \right| dx dy + \int_{\mathbb{C}^+} \left| \frac{1}{\bar{z}-u+v} \phi(z) \right| dx dy \right).$$

Therefore the inequality follows from Lemma A.6. ■

Lemma A.8. For any $u, v \in \mathbb{R}$,

$$\int_{|z-u|\leq 1, x>0, y>0} \left| \frac{1}{(z-u-v)(\bar{z}-u+v)} \right| |z-u|^2 dx dy \leq 2 \int_{|z|\leq \sqrt{2}} \frac{1}{|z|} dx dy.$$

Proof. The left hand side is less than or equal to

$$\int_{|z|\leq 1} \left| \frac{1}{(z-v)(\bar{z}+v)} \right| |z|^2 dx dy. \tag{A.22}$$

We may assume that $v \geq 0$. Then,

$$\frac{|z|}{|z-v|} \frac{|z|}{|z-v|} \leq \begin{cases} \frac{1}{|z-v|} & \text{if } x > 0, \\ \frac{1}{|z+v|} & \text{if } x < 0. \end{cases}$$

Hence, (A.22) is less than or equal to

$$\begin{aligned} \int_{|z|<1, x>0} \frac{1}{|z-v|} dx dy + \int_{|z|<1, x<0} \frac{1}{|z+v|} dx dy \\ \leq \int_{|z-v|<\sqrt{2}} \frac{1}{|z-v|} dx dy + \int_{|z+v|<\sqrt{2}} \frac{1}{|z+v|} dx dy = 2 \int_{|z|\leq \sqrt{2}} \frac{1}{|z|} dx dy. \blacksquare \end{aligned}$$

Lemma A.9. *Let $D_1(u) = \{z \in \mathbb{C}; |z - u| \leq 1\}$. For $u \in \mathbb{R}$, $N = 0, 1, 2, \dots$, and ϕ as in Proposition A.5 define*

$$T_{2,u}^{(N-1)}\phi(z) = 1_{D_1}(z) \sum_{a,b \geq 0; a+b=N} \left(N \int_0^1 (1-t)^{N-1} \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(u+tz) dt \right) (x-u)^a y^b.$$

Then

$$|T_{2,u}^{(N-1)}\phi(z)| \leq 1_{D_1}(z) |z-u|^N \sum_{a,b \geq 0; a+b=N} \sup_{w \in \mathbb{C}^+} \left| \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(w) \right|.$$

Proof. This follows from the fact that

$$|(x-u)^a y^b| \leq |z|^{a+b}$$

and

$$N \int_0^1 (1-t)^{N-1} dt = 1.$$

■

By combining Lemmas A.8 and A.9, we deduce the following corollary.

Corollary A.10. *For ϕ as in Proposition A.5 and any $u, v \in \mathbb{R}$,*

$$\begin{aligned} \int_{\mathbb{C}^{++}} \left| \frac{1}{(z-u-v)(\bar{z}-u+v)} T_{2,u}^{(2-1)}\phi(z) \right| dx dy \\ \leq 2 \int_{|z| \leq \sqrt{2}} \frac{1}{|z|} dx dy \sum_{a,b \geq 0; a+b=2} \sup_{w \in \mathbb{C}^+} \left| \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(w) \right|. \end{aligned}$$

Lemma A.11. *For any $a, b \geq 0$,*

$$\sup_{u \in \mathbb{R}, v > 0} \left| \int_{\mathbb{C}^{++} \cap D_1(u)} f_v(z-u)(x-u)^a y^b dx dy \right| < \infty.$$

Proof. Notice that

$$f_v(z) = -\frac{2yv}{|z-v|^2 |z+v|^2}.$$

Hence, for $v > 0$,

$$|f_v(z)| = -f_v(z).$$

Therefore,

$$\begin{aligned} \left| \int_{\mathbb{C}^{++} \cap D_1(u)} f_v(z-u)(x-u)^a y^b dx dy \right| &\leq \int_{\mathbb{C}^{++} \cap D_1(u)} -f_v(z-u) |x-u|^a y^b dx dy \\ &\leq \int_{\mathbb{C}^{++} \cap D_1(u)} -f_v(z-u) dx dy \\ &= \int_{|z| \leq 1, x > u, y > 0} -f_v(z) dx dy \\ &\leq \int_{|z| \leq 1, y > 0} -f_v(z) dx dy. \end{aligned}$$

Thus, the lemma follows from Lemma A.2 and (A.15). ■

Proof. (of Proposition A.5) The quantity we are trying to estimate coincides with the integral (A.19), which, by (A.3), is equal to

$$\begin{aligned}
 & \int_{(\mathbb{C}^+)^n} \left(\prod_{k=1}^n f_{v_k}(z_k - u_k) \right) \phi(z_1, z_2, \dots, z_n) dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n \quad (\text{A.23}) \\
 &= \int_{(\mathbb{C}^+)^{n-1}} \prod_{k=2}^n f_{v_k}(z_k - u_k) \\
 & \quad \int_{\mathbb{C}^{++}} (f_{v_1}(z_1 - u_1) - f_{v_1}(-\bar{z}_1 - u_1)) \phi(z_1, z_2, \dots, z_n) dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n \\
 &= \int_{(\mathbb{C}^+)^{n-1}} \prod_{k=2}^n f_{v_k}(z_k - u_k) \\
 & \quad \int_{\mathbb{C}^{++}} (f_{v_1}(z_1 - u_1) - f_{v_1}(z_1 + u_1)) \phi(z_1, z_2, \dots, z_n) dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n \\
 & \quad \dots \\
 &= \int_{(\mathbb{C}^{++})^n} \prod_{k=1}^n (f_{v_k}(z_k - u_k) - f_{v_k}(z_k + u_k)) \cdot \phi(z_1, z_2, \dots, z_n) dx_1 dy_1 \cdots dx_n dy_n \\
 &= \sum_{\delta \in \{\pm 1\}^n} \int_{(\mathbb{C}^{++})^n} \prod_{k=1}^n \delta_k f_{v_k}(z_k - \delta_k u_k) \cdot \phi(z_1, z_2, \dots, z_n) dx_1 dy_1 \cdots dx_n dy_n \\
 &= \sum_{\delta \in \{\pm 1\}^n} \int_{\mathbb{C}^{++-\delta_1 u_1}} \cdots \int_{\mathbb{C}^{++-\delta_n u_n}} \prod_{k=1}^n \delta_k f_{v_k}(z_k) \cdot \phi(z_1 + \delta_1 u_1, \dots, z_n + \delta_n u_n) dx_1 \cdots dy_n.
 \end{aligned}$$

Taylor’s formula (A.12) implies that, for $n = 1$, (A.23) may be rewritten as

$$\begin{aligned}
 & \sum_{\delta \in \{\pm 1\}} \int_{(\mathbb{C}^{++-\delta u}) \setminus D_1(0)} \delta f_v(z) \phi(z + \delta u) dx dy \quad (\text{A.24}) \\
 &+ \sum_{\delta \in \{\pm 1\}} \int_{(\mathbb{C}^{++-\delta u}) \cap D_1(0)} \delta f_v(z) \sum_{a,b \geq 0; a+b \leq 1} \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(\delta u) x^a y^b \\
 &+ \sum_{\delta \in \{\pm 1\}} \int_{(\mathbb{C}^{++-\delta u}) \cap D_1(0)} \delta f_v(z) \sum_{a,b \geq 0; a+b=2} \left(2 \int_0^1 (1-t) \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(\delta u + tz) dt \right) x^a y^b.
 \end{aligned}$$

Let

$$\begin{aligned}
 T_{0,\delta,u,v} \phi(z) &= 1_{(\mathbb{C}^{++-\delta u}) \setminus D_1(0)}(z) \delta f_v(z) \phi(z + \delta u), \\
 T_{1,\delta,u,v} \phi(z) &= 1_{(\mathbb{C}^{++-\delta u}) \cap D_1(0)}(z) \delta f_v(z) \sum_{a,b \geq 0; a+b \leq 1} \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(\delta u) x^a y^b \\
 T_{2,\delta,u,v} \phi(z) &= 1_{(\mathbb{C}^{++-\delta u}) \cap D_1(0)}(z) \delta f_v(z) \left(2 \int_0^1 (1-t) \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(\delta u + tz) dt \right) x^a y^b.
 \end{aligned}$$

We see from (A.24) that (A.23) is equal to

$$\sum_{\delta \in \{\pm 1\}^n} \sum_{\alpha \in \{0,1,2\}^n} \int_{(\mathbb{C}^{++})^n} T_{\alpha_1, \delta_1, u_1, v_1} \otimes \cdots \otimes T_{\alpha_n, \delta_n, u_n, v_n} \phi(z_1, \dots, z_n) dx_1 \cdots dy_n. \tag{A.25}$$

Lemma A.7, Corollary A.10 and Lemma A.11 imply that for $n = 1$ there is a finite constant C independent of u, v, δ and ϕ such that

$$\begin{aligned} \left| \int_{\mathbb{C}^+} T_{0, \delta, u, v} \phi(z) dx dy \right| &\leq C(\|\phi\|_\infty + \|\phi\|_1), \\ \left| \int_{\mathbb{C}^+} T_{1, \delta, u, v} \phi(z) dx dy \right| &\leq C \sum_{a, b \geq 0; a+b \leq 1} \sup_{w \in \mathbb{C}^+} \left| \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(w) \right|, \\ \left| \int_{\mathbb{C}^+} T_{2, \delta, u, v} \phi(z) dx dy \right| &\leq C \sum_{a, b \geq 0; a+b=2} \sup_{w \in \mathbb{C}^+} \left| \frac{\partial(1)^a \partial(i)^b}{a!b!} \phi(w) \right|, \end{aligned} \tag{A.26}$$

The proposition follows from (A.25) and (A.26). ■

Appendix B.

In this appendix, we choose a positive root system Ψ' such that

$$\pi_{\mathfrak{g}'/\mathfrak{h}'} = \prod_{\alpha' \in \Psi'} \alpha'$$

and a positive root system Ψ such that

$$\pi_{\mathfrak{g}/\mathfrak{h}} = \prod_{\alpha \in \Psi} (-\alpha)$$

and compute the rational function

$$\frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z') \pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_W} \quad (z' \in \mathfrak{h}'_{\mathbb{C}}, z \in \mathfrak{h}_{\mathbb{C}}) \tag{B.1}$$

for all dual pairs (G, G') with $G' \neq O_{2p+1, 2q}$. In the case $G' = O_{2p+1, 2q}$ we compute the quantity (B.1) with the W replaced by $\text{Hom}(V, V_0^\perp)$, and $\pi_{\mathfrak{g}/\mathfrak{h}}$ by $\pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})$ - the product of the positive short roots.

The constant u of Propositions 5.3 and 6.3 is equal to

$$u = (-1)^{p+} u'', \tag{B.2}$$

where u'' is defined below (see (B.7), (B.11), (B.15), (B.19) and (B.22)).

From now on, P is a polynomial function on $\mathfrak{h}'_{\mathbb{C}}$ satisfying the condition (0.4).

The pairs $(G, G') = (U_{p,q}, U_{p',q'})$ with $p' + q' = n' \leq n = p + q$.

Theorem B.1. *Suppose*

$$\begin{aligned} \Psi &= \{iJ_j^* - iJ_k^* \mid 1 \leq j < k \leq n\} \text{ and} \\ \Psi' &= \{iJ_j^* - iJ_k^* \mid 1 \leq j < k \leq n'\}. \end{aligned} \tag{B.3}$$

Let

$$u' = (-1)^{\frac{n'(n'-1)}{2} + \frac{n(n-1)}{2}} i^{\frac{n'(n'-1)}{2} + \frac{n(n-1)}{2} - n'n}.$$

Then

$$\begin{aligned} &P(z') \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z') \pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_W} \\ &= \frac{u'}{|\Sigma_{n-n'}|} \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \frac{P(z_{\sigma(1)}, \dots, z_{\sigma(n')}) \prod_{n' < j < k \leq n} (z_{\sigma(j)} - z_{\sigma(k)})}{\prod_{j=1}^{n'} (z'_j - z_{\sigma(j)})}. \end{aligned} \tag{B.4}$$

Moreover,

$$\pi_{\mathfrak{g}/\mathfrak{h}}(z) = \prod_{n' < j < k \leq n} (-1)(iz_j - iz_k), \tag{B.5}$$

$$\det(z' + z)_{W^{\mathfrak{h}'}} = \prod_{j=1}^{n'} i(z'_j - z_j), \tag{B.6}$$

and

$$u'' = (-1)^{n'-n'n}. \tag{B.7}$$

Proof. In this case

$$\begin{aligned} \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z') \pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_W} &= \frac{\prod_{1 \leq j < k \leq n'} (iz'_j - iz'_k) \cdot \prod_{1 \leq j < k \leq n} (-1)(iz_j - iz_k)}{\prod_{j=1}^{n'} \prod_{k=1}^n i(z'_j - z_k)} \\ &= \frac{i^{\frac{n'(n'-1)}{2}} (-1)^{\frac{n(n-1)}{2}} i^{\frac{n(n-1)}{2}} \prod_{1 \leq j < k \leq n'} (z'_j - z'_k) \cdot \prod_{1 \leq j < k \leq n} (z_j - z_k)}{i^{n'n} \prod_{j=1}^{n'} \prod_{k=1}^n (z'_j - z_k)}. \end{aligned}$$

Therefore, by partial fractions, the left hand side of (B.4) is equal to

$$\sum_{\iota} \frac{(-1)^{\frac{n'(n'-1)}{2}} u'}{\prod_{j=1}^{n'} (z'_j - z_{\iota(j)})} \frac{P(z_{\iota(1)}, \dots, z_{\iota(n')}) \prod_{1 \leq j < k \leq n'} (z_{\iota(j)} - z_{\iota(k)}) \prod_{1 \leq j < k \leq n} (z_j - z_k)}{\prod_{j=1}^{n'} \prod_{k=1, k \neq \iota(j)}^n (z_{\iota(j)} - z_k)},$$

where the summation is over all injections $\iota : \{1, 2, \dots, n'\} \xrightarrow{\sim} \{1, 2, \dots, n\}$. Let us choose an element $\sigma \in \Sigma_n$ so that $\sigma(j) = \iota(j)$ for all $1 \leq j \leq n'$. Then

$$\prod_{j=1}^{n'} \prod_{k=1, k \neq \iota(j)}^n (z_{\iota(j)} - z_k) = \prod_{j=1}^{n'} \prod_{k=1, k \neq j}^n (z_{\sigma(j)} - z_{\sigma(k)}).$$

Therefore,

$$\frac{\prod_{1 \leq j < k \leq n} (z_j - z_k)}{\prod_{j=1}^{n'} \prod_{k=1, k \neq \iota(j)}^n (z_{\iota(j)} - z_k)} = \frac{\prod_{1 \leq j < k \leq n} (z_{\sigma(j)} - z_{\sigma(k)})}{\prod_{j=1}^{n'} \prod_{k=1, k \neq j}^n (z_{\sigma(j)} - z_{\sigma(k)})} \text{sgn}(\sigma),$$

where the ‘sgn’ is the character of the group Σ_n defined by

$$\prod_{1 \leq j < k \leq n} (z_{\sigma(j)} - z_{\sigma(k)}) = \text{sgn}(\sigma) \prod_{1 \leq j < k \leq n} (z_j - z_k).$$

Let us write $u_k = z_{\sigma(k)}$ for simplicity. Then,

$$\begin{aligned} & \frac{\prod_{1 \leq j < k \leq n'} (z_{\iota(j)} - z_{\iota(k)}) \prod_{1 \leq j < k \leq n} (z_{\sigma(j)} - z_{\sigma(k)})}{\prod_{j=1}^{n'} \prod_{k=1, k \neq j}^n (z_{\sigma(j)} - z_{\sigma(k)})} \\ &= \frac{\prod_{1 \leq j < k \leq n'} (u_j - u_k) \cdot \prod_{1 \leq j < k \leq n'} (u_j - u_k) \cdot \prod_{\substack{1 \leq j \leq n' \\ n' < k \leq n}} (u_j - u_k) \cdot \prod_{n' < j < k \leq n} (u_j - u_k)}{\prod_{1 \leq j < k \leq n'} (u_j - u_k) \cdot \prod_{1 \leq j < k \leq n'} (u_k - u_j) \cdot \prod_{\substack{1 \leq j \leq n' \\ n' < k \leq n}} (u_j - u_k)} \\ &= (-1)^{\frac{n'(n'-1)}{2}} \prod_{n' < j < k \leq n} (z_{\sigma(j)} - z_{\sigma(k)}). \end{aligned} \tag{B.8}$$

Hence,

$$\begin{aligned} & \frac{1}{\prod_{j=1}^{n'} (z'_j - z_{\iota(j)})} \frac{P(z_{\iota(1)}, \dots, z_{\iota(n')}) \prod_{1 \leq j < k \leq n'} (z_{\iota(j)} - z_{\iota(k)}) \prod_{1 \leq j < k \leq n} (z_j - z_k)}{\prod_{\substack{1 \leq j \leq n' \\ 1 \leq k \leq n \\ k \neq \iota(j)}} (z_{\iota(j)} - z_k)} \\ &= \frac{(-1)^{\frac{n'(n'-1)}{2}}}{\prod_{j=1}^{n'} (z'_j - z_{\sigma(j)})} \cdot P(z_{\sigma(1)}, \dots, z_{\sigma(n')}) \prod_{n' < j < k \leq n} (z_{\sigma(j)} - z_{\sigma(k)}) \cdot \text{sgn}(\sigma) \end{aligned} \tag{B.9}$$

Let $\eta \in \Sigma_n$ be such that $\eta(j) = j$ for all $1 \leq j \leq n'$. Then

$$z_{\sigma(j)} = z_{\sigma\eta(j)} \quad (1 \leq j \leq n')$$

and

$$\prod_{n' < j < k \leq n} (z_{\sigma(j)} - z_{\sigma(k)}) = \prod_{n' < j < k \leq n} (z_{\sigma\eta(j)} - z_{\sigma\eta(k)}) \text{sgn}(\eta).$$

Thus the quantity (B.9) is invariant under the substitution $\sigma \rightarrow \sigma\eta$. Hence (B.4) follows.

Furthermore, (B.5) follows from equations (5.1), (B.6) and Lemma 5.2, and (B.7) from (B.2) (B.4) and Proposition 5.1. ■

The pairs $(G, G') = (\text{Sp}_{2n}(\mathbb{R}), \text{O}_{2p,2q})$ and $(\text{Sp}_{r,s}, \text{O}_{2n'}^*)$ with $p+q = n' \leq n = r+s$.

Theorem B.2. *Suppose*

$$\begin{aligned} \Psi &= \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n\} \cup \{2iJ_j^* \mid 1 \leq j \leq n\} \text{ and} \\ \Psi' &= \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n'\}. \end{aligned}$$

Let

$$u' = (-1)^{\frac{n(n-1)}{2} - n'n} \mathbf{i}^{-n}.$$

Then

$$\begin{aligned}
 P(z') \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z')\pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_{\mathbb{W}}} &= \frac{u'}{|\Sigma_{n-n'} \times \mathbb{Z}_2^{n-n'}|} \\
 &\sum_{\sigma \epsilon \in \Sigma_n \times \mathbb{Z}_2^n} \operatorname{sgn}(\sigma \epsilon) \cdot \frac{P(z_{\sigma(1)}, \dots, z_{\sigma(n')}) \prod_{n' < j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{n' < j \leq n} 2\hat{\epsilon}(j)z_{\sigma(j)}}{\prod_{1 \leq j \leq n'} (z'_j - \hat{\epsilon}_j z_{\sigma(j)})}.
 \end{aligned}
 \tag{B.10}$$

Moreover,

$$\begin{aligned}
 \pi_{\mathfrak{g}/\mathfrak{h}}(z) &= \prod_{n' < j < k \leq n} ((iz_j)^2 - (iz_k)^2) \cdot \prod_{n' < j \leq n} (-2iz_j), \\
 \det(z' + z)_{\mathbb{W}^{\mathfrak{h}'}} &= \prod_{j=1}^{n'} i(z'_j - z_j),
 \end{aligned}$$

and

$$u'' = (-1)^{\frac{n'(n'-1)}{2} + n'}.
 \tag{B.11}$$

Proof. In this case

$$\begin{aligned}
 &\frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z')\pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_{\mathbb{W}}} \\
 &= \frac{\prod_{1 \leq j < k \leq n'} ((iz'_j)^2 - (iz'_k)^2) \cdot \prod_{1 \leq j < k \leq n} ((iz_j)^2 - (iz_k)^2) \cdot \prod_{j=1}^n (-2iz_j)}{\prod_{j=1}^{n'} \prod_{k=1}^n i(z'_j - z_k) i(z'_j + z_k)} \\
 &= \frac{(-1)^{\frac{n'(n'-1)}{2} + \frac{n(n-1)}{2}} i^{-n} \prod_{1 \leq j < k \leq n'} (z_j'^2 - z_k'^2) \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n (2z_j)}{(-1)^{n'n} \prod_{j=1}^{n'} \prod_{k=1}^n (z'_j - z_k)(z'_j + z_k)}.
 \end{aligned}$$

Therefore, by partial fractions, the left hand side of (B.10) is equal to

$$\sum \frac{(-1)^{\frac{n'(n'-1)}{2}} u'}{\prod_{j=1}^{n'} (z'_j - \delta(j)z_{\iota(j)})} \frac{P(z_{\iota(1)}, \dots, z_{\iota(n')}) \prod_{1 \leq j < k \leq n'} (z_{\iota(j)}^2 - z_{\iota(k)}^2) \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n 2z_j}{\prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta = \pm 1, (k, \delta) \neq (\iota(j), \delta(j))}} (\delta(j)z_{\iota(j)} - \delta z_k)},$$

where the summation is over all injections

$$\{1, 2, \dots, n'\} \ni j \mapsto (\iota(j), \delta(j)) \in \{1, 2, \dots, n\} \times \{\pm 1\}.$$

Let $\sigma \epsilon = (\sigma, \epsilon) \in \Sigma_n \times \mathbb{Z}_2^n$ be such that $\sigma(j) = \iota(j)$ and $\hat{\epsilon}_j = \delta(j)$ for all $1 \leq j \leq n'$.

Then

$$\prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta = \pm 1, (k, \delta) \neq (\iota(j), \delta(j))}} (\delta(j)z_{\iota(j)} - \delta z_k) = \prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta = \pm 1, (k, \delta \hat{\epsilon}_k) \neq (j, \hat{\epsilon}_j)}} (\hat{\epsilon}_j z_{\sigma(j)} - \delta \hat{\epsilon}_k z_{\sigma(k)}).$$

Hence,

$$\begin{aligned} & \frac{\prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n 2z_j}{\prod_{j=1}^{n'} \prod_{k=1}^n \prod_{\delta=\pm 1, (k,\delta) \neq (\iota(j), \delta(j))} (\delta(j)z_{\iota(j)} - \delta z_k)} \\ &= \frac{\prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{j=1}^n 2\hat{\epsilon}_j z_{\sigma(j)}}{\prod_{j=1}^{n'} \prod_{k=1}^n \prod_{\delta=\pm 1, (k,\delta\hat{\epsilon}_k) \neq (j,\hat{\epsilon}_j)} (\hat{\epsilon}_j z_{\sigma(j)} - \delta\hat{\epsilon}_k z_{\sigma(k)})} \text{sgn}(\sigma\epsilon), \end{aligned} \tag{B.12}$$

where the ‘sgn’ is the character of the group $\Sigma_n \times \mathbb{Z}_2^n$ defined by

$$\prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{j=1}^n 2\hat{\epsilon}_j z_{\sigma(j)} = \text{sgn}(\sigma\epsilon) \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n 2z_j.$$

Let $u_j = \hat{\epsilon}_j z_{\sigma(j)}$. Then,

$$\frac{\prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{1 \leq j \leq n} 2\hat{\epsilon}_j z_{\sigma(j)}}{\prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta=\pm 1, (k,\delta\hat{\epsilon}_k) \neq (j,\hat{\epsilon}_j)}} (\hat{\epsilon}_j z_{\sigma(j)} - \delta\hat{\epsilon}_k z_{\sigma(k)})} = \frac{\prod_{1 \leq j < k \leq n} (u_j^2 - u_k^2) \cdot \prod_{n' < j \leq n} 2u_j}{\prod_{1 \leq j \leq n'} \prod_{\substack{1 \leq k \leq n \\ k \neq j}} (u_j^2 - u_k^2)}$$

Hence, by (B.8),

$$\begin{aligned} & \frac{\prod_{1 \leq j < k \leq n'} (u_j^2 - u_k^2) \cdot \prod_{1 \leq j < k \leq n} (u_j^2 - u_k^2) \cdot \prod_{n' < j \leq n} 2u_j}{\prod_{1 \leq j \leq n'} \prod_{\substack{1 \leq k \leq n \\ k \neq j}} (u_j^2 - u_k^2)} \\ &= (-1)^{\frac{n'(n'-1)}{2}} \prod_{n' \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{n' < j \leq n} 2\hat{\epsilon}_j z_{\sigma(j)}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{\prod_{1 \leq j \leq n'} (z'_j - \delta(j)z_{\iota(j)})} \tag{B.13} \\ & \frac{P(z_{\iota(1)}, \dots, z_{\iota(n')}) \prod_{1 \leq j < k \leq n'} (z_{\iota(j)}^2 - z_{\iota(k)}^2) \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{1 \leq j \leq n} 2z_j}{\prod_{1 \leq j \leq n'} \prod_{1 \leq k \leq n, \delta=\pm 1, (k,\delta) \neq (\iota(j), \delta(j))} (\delta(j)z_{\iota(j)} - \delta z_k)} \\ &= \frac{(-1)^{\frac{n'(n'-1)}{2}}}{\prod_{1 \leq j \leq n'} (z'_j - \hat{\epsilon}_j z_{\sigma(j)})} P(z_{\sigma(1)}, \dots, z_{\sigma(n')}) \\ & \quad \prod_{n' \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{n' < j \leq n} 2\hat{\epsilon}_j z_{\sigma(j)} \cdot \text{sgn}(\sigma\epsilon). \end{aligned}$$

Let $\eta\rho = (\eta, \rho) \in \Sigma_n \times \mathbb{Z}_2^n$ be such that $\eta(j) = j$ and $\rho(j) = 0$ for all $1 \leq j \leq n'$. Then

$$\begin{aligned} & \prod_{n' < j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{n' < j \leq n} 2\hat{\epsilon}_j z_{\sigma(j)} \\ &= \prod_{n' < j < k \leq n} (z_{\sigma\eta(j)}^2 - z_{\sigma\eta(k)}^2) \cdot \prod_{n' < j \leq n} 2\hat{\epsilon}_j \rho(j) z_{\sigma\eta(j)} \cdot \text{sgn}(\eta\rho). \end{aligned}$$

Thus the right hand side of (B.13) is invariant under the substitution $\sigma\epsilon \rightarrow \sigma\eta\epsilon\rho$. Hence (B.10) follows. The rest is straightforward. \blacksquare

The pairs $(G, G') = (O_{2p,2q}, Sp_{2n'}(\mathbb{R}))$ **and** $(O_{2n}^*, Sp_{r,s})$, **with** $r + s = n' \leq n = p + q$

Theorem B.3. *Suppose*

$$\Psi = \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n\}$$

and

$$\Psi' = \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n'\} \cup \{2iJ_j^* \mid 1 \leq j \leq n'\}.$$

Let

$$u' = (-1)^{\frac{n(n-1)}{2} - n'n} i^{n'}.$$

Then

$$\begin{aligned} & P(z') \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z') \pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_W} \tag{B.14} \\ &= \frac{u'}{|\Sigma_{n-n'} \times \mathbb{Z}_2^{n-n'}|} \sum_{\sigma \in \Sigma_n \times \mathbb{Z}_2^n} \text{sgn}(\sigma\epsilon) \cdot \frac{P(z_{\sigma(1)}, \dots, z_{\sigma(n')}) \prod_{n' \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2)}{\prod_{j=1}^{n'} (z'_j - \hat{\epsilon}_j z_{\sigma(j)})}. \end{aligned}$$

Moreover,

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}}(z) &= \prod_{n' < j < k \leq n} ((iz_j)^2 - (iz_k)^2), \\ \det(z' + z)_{W_{\mathfrak{h}'}} &= \prod_{j=1}^{n'} i(z'_j - z_j), \end{aligned}$$

and

$$u'' = (-1)^{\frac{n'(n'-1)}{2}}. \tag{B.15}$$

Proof. In this case

$$\begin{aligned} & \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z') \pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_W} = \frac{\prod_{1 \leq j < k \leq n'} ((iz'_j)^2 - (iz'_k)^2) \cdot \prod_{j=1}^{n'} 2iz'_j \cdot \prod_{1 \leq j < k \leq n} ((iz_j)^2 - (iz_k)^2)}{\prod_{j=1}^{n'} \prod_{k=1}^n i(z'_j - z_k) i(z'_j + z_k)} \\ &= \frac{(-1)^{\frac{n'(n'-1)}{2} + \frac{n(n-1)}{2}} i^{n'}}{(-1)^{n'n}} \frac{\prod_{1 \leq j < k \leq n'} (z_j'^2 - z_k'^2) \cdot \prod_{j=1}^{n'} 2z'_j \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2)}{\prod_{j=1}^{n'} \prod_{k=1}^n (z'_j - z_k)(z'_j + z_k)}. \tag{B.16} \end{aligned}$$

Therefore, by partial fractions, the left hand side of (B.14) is equal to

$$\begin{aligned} & \sum_{1 \leq j \leq n'} \frac{(-1)^{\frac{n'(n'-1)}{2}} u'}{\prod_{1 \leq j \leq n'} (z'_j - \delta(j) z_{\iota(j)})} \\ & \frac{P(z_{\iota(1)}, \dots, z_{\iota(n')}) \prod_{1 \leq j < k \leq n'} (z_{\iota(j)}^2 - z_{\iota(k)}^2) \cdot \prod_{1 \leq j \leq n'} 2\delta(j) z_{\iota(j)} \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2)}{\prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta = \pm 1, (k, \delta) \neq (\iota(j), \delta(j))}} (\delta(j) z_{\iota(j)} - \delta z_k)}, \end{aligned}$$

where the summation is over all injections

$$\{1, 2, \dots, n'\} \ni j \mapsto (\iota(j), \delta(j)) \in \{1, 2, \dots, n\} \times \{\pm 1\}.$$

Let $\sigma\epsilon = (\sigma, \epsilon) \in \Sigma_n \times \mathbb{Z}_2^n$ be such that $\sigma(j) = \iota(j)$ and $\hat{\epsilon}_j = \delta(j)$ for all $1 \leq j \leq n'$. Then, as in (B.12), we check that

$$\begin{aligned} & \frac{\prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2)}{\prod_{1 \leq j \leq n' 1 \leq k \leq n} \prod_{\substack{\delta = \pm 1 \\ (k, \delta) \neq (\iota(j), \delta(j))}} (\delta(j)z_{\iota(j)} - \delta z_k)} \\ &= \frac{\prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2)}{\prod_{1 \leq j \leq n' 1 \leq k \leq n} \prod_{\substack{\delta = \pm 1 \\ (k, \delta \hat{\epsilon}_k) \neq (j, \hat{\epsilon}_j)}} (\hat{\epsilon}_j z_{\sigma(j)} - \delta \hat{\epsilon}_k z_{\sigma(k)})} \cdot \text{sgn}(\sigma\epsilon), \end{aligned}$$

where the ‘sgn’ is the character of the group $\Sigma_n \times \mathbb{Z}_2^n$ defined by

$$\prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) = \text{sgn}(\sigma\epsilon) \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2).$$

In particular sgn is trivial on the subgroup \mathbb{Z}_2^n . Then, by (B.8),

$$\begin{aligned} & \frac{\prod_{1 \leq j < k \leq n'} (z_{\iota(j)}^2 - z_{\iota(k)}^2) \cdot \prod_{j=1}^{n'} 2\delta(j)z_{\iota(j)} \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2)}{\prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta = \pm 1, (k, \delta) \neq (\iota(j), \delta(j))}} (\delta(j)z_{\iota(j)} - \delta z_k)} \\ &= (-1)^{\frac{n'(n'-1)}{2}} \prod_{n' < j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \text{sgn}(\sigma\epsilon), \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1 \cdot \prod_{1 \leq j < k \leq n'} (z_{\iota(j)}^2 - z_{\iota(k)}^2) \cdot \prod_{1 \leq j \leq n} 2\delta(j)z_{\iota(j)} \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2)}{\prod_{1 \leq j \leq n'} (z'_j - \delta(j)z_{\iota(j)}) \prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta = \pm 1, (k, \delta) \neq (\iota(j), \delta(j))}} (\delta(j)z_{\iota(j)} - \delta z_k)} \\ &= \frac{(-1)^{\frac{n'(n'-1)}{2}}}{\prod_{j=1}^{n'} (z'_j - \hat{\epsilon}_j z_{\sigma(j)})} \prod_{n' < j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \text{sgn}(\sigma\epsilon). \quad (\text{B.17}) \end{aligned}$$

Let $\eta\rho = (\eta, \rho) \in \Sigma_n \times \mathbb{Z}_2^n$ be such that $\eta(j) = j$ and $\rho(j) = 0$ for all $1 \leq j \leq n'$. Then

$$\prod_{n' \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) = \prod_{n' \leq j < k \leq n} (z_{\sigma\eta(j)}^2 - z_{\sigma\eta(k)}^2) \cdot \text{sgn}(\eta\rho).$$

Thus the quantity (B.17) is invariant under the substitution $\sigma\epsilon \rightarrow \sigma\eta\epsilon\rho$. Hence (B.14) follows. The rest is straightforward. ■

The pairs $(G, G') = (O_{2p+1, 2q}, Sp_{2n'}(\mathbb{R}))$ with $n' \leq n = p + q$.

Theorem B.4. *Suppose*

$$\Psi = \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n\} \cup \{iJ_j^* \mid 1 \leq j \leq n\} \text{ and}$$

$$\Psi' = \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n'\} \cup \{2iJ_j^* \mid 1 \leq j \leq n'\}.$$

Let

$$u' = (-1)^{\frac{n(n-1)}{2} - n'n} i^{-n}.$$

Then

$$P(z') \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z') \pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_{\mathbb{W}}} = \frac{u'}{|\Sigma_{n-n'} \times \mathbb{Z}_2^{n-n'}|}$$

$$\sum_{\sigma \in \Sigma_n \times \mathbb{Z}_2^n} \text{sgn}(\sigma \epsilon) \cdot \frac{P(z_{\sigma(1)}, \dots, z_{\sigma(n')}) \prod_{n' < j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{n' < k \leq n} \hat{\epsilon}_k z_{\sigma(k)}}{\prod_{j=1}^{n'} (z'_j - \hat{\epsilon}_j z_{\sigma(j)})}. \tag{B.18}$$

Moreover,

$$\pi_{\mathfrak{g}/\mathfrak{h}}(z) = \prod_{n' < j < k \leq n} ((iz_j)^2 - (iz_k)^2) \cdot \prod_{n' < j \leq n} (-iz_j),$$

$$\det(z' + z)_{\mathbb{W}^{\mathfrak{h}'}} = \prod_{j=1}^{n'} i(z'_j - z_j),$$

and

$$u'' = (-1)^{\frac{n'(n'+1)}{2}}. \tag{B.19}$$

Proof. In this case

$$\frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z') \pi_{\mathfrak{g}/\mathfrak{h}}(z)}{\det(z' + z)_{\mathbb{W}}}$$

$$= \frac{\prod_{1 \leq j < k \leq n'} ((iz'_j)^2 - (iz'_k)^2) \cdot \prod_{j=1}^{n'} 2iz'_j \cdot \prod_{1 \leq j < k \leq n} ((iz_j)^2 - (iz_k)^2) \cdot \prod_{j=1}^n (-iz_j)}{\prod_{j=1}^{n'} \prod_{k=1}^n i(z'_j - z_k) i(z'_j + z_k) \cdot \prod_{j=1}^{n'} iz'_j}$$

$$= \frac{2^{n'} (-1)^{\frac{n'(n'-1)}{2} + \frac{n(n-1)}{2}} i^{-n} \prod_{1 \leq j < k \leq n'} (z_j'^2 - z_k'^2) \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n z_j}{(-1)^{n'n} \prod_{j=1}^{n'} \prod_{k=1}^n (z'_j - z_k)(z'_j + z_k)}.$$

Therefore, by partial fractions, the left hand side of (B.18) is equal to

$$\sum \frac{2^{n'} (-1)^{\frac{n'(n'-1)}{2}} u'}{\prod_{j=1}^{n'} (z'_j - \delta(j) z_{\iota(j)})}$$

$$\frac{P(z_{\iota(1)}, \dots, z_{\iota(n')}) \prod_{1 \leq j < k \leq n'} (z_{\iota(j)}^2 - z_{\iota(k)}^2) \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n z_j}{\prod_{\substack{1 \leq j \leq n', 1 \leq k \leq n \\ \delta = \pm 1, (k, \delta) \neq (\iota(j), \delta(j))}} (\delta(j) z_{\iota(j)} - \delta z_k)},$$

where the summation is over all injections

$$\{1, 2, \dots, n'\} \ni j \mapsto (\iota(j), \delta(j)) \in \{1, 2, \dots, n\} \times \{\pm 1\}.$$

Let $\sigma\epsilon = (\sigma, \epsilon) \in \Sigma_n \times \mathbb{Z}_2^n$ be such that $\sigma(j) = \iota(j)$ and $\hat{\epsilon}_j = \delta(j)$ for all $1 \leq j \leq n'$. Then

$$\begin{aligned} & \frac{\prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n z_j}{\prod_{j=1}^{n'} \prod_{k=1}^n \prod_{\delta=\pm 1, (k,\delta) \neq (\iota(j), \delta(j))} (\delta(j)z_{\iota(j)} - \delta z_k)} \\ &= \frac{\prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{j=1}^n \hat{\epsilon}_j z_{\sigma(j)}}{\prod_{j=1}^{n'} \prod_{k=1}^n \prod_{\delta=\pm 1, (k,\delta \hat{\epsilon}_k) \neq (j, \hat{\epsilon}_j)} (\hat{\epsilon}_j z_{\sigma(j)} - \delta \hat{\epsilon}_k z_{\sigma(k)})} \text{sgn}(\sigma\epsilon), \end{aligned}$$

where the ‘sgn’ is the character of the group $\Sigma_n \times \mathbb{Z}_2^n$ defined by

$$\prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{j=1}^n \hat{\epsilon}_j z_{\sigma(j)} = \text{sgn}(\sigma\epsilon) \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n z_j.$$

Hence, by (B.8),

$$\begin{aligned} & \frac{\prod_{1 \leq j < k \leq n'} (z_{\iota(j)}^2 - z_{\iota(k)}^2) \cdot \prod_{1 \leq j < k \leq n} (z_j^2 - z_k^2) \cdot \prod_{j=1}^n z_j}{\prod_{j=1}^{n'} \prod_{k=1}^n \prod_{\delta=\pm 1, (k,\delta) \neq (\iota(j), \delta(j))} (\delta(j)z_{\iota(j)} - \delta z_k)} \\ &= \frac{\prod_{1 \leq j < k \leq n'} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{j=1}^n \hat{\epsilon}_j z_{\sigma(j)}}{\prod_{j=1}^{n'} \prod_{k=1}^n \prod_{\delta=\pm 1, (k,\delta) \neq (\iota(j), \delta(j))} (\hat{\epsilon}_j z_{\sigma(j)} - \delta \hat{\epsilon}_k z_{\sigma(k)})} \text{sgn}(\sigma\epsilon) \\ &= \frac{\prod_{1 \leq j < k \leq n'} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{1 \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{j=1}^n \hat{\epsilon}_j z_{\sigma(j)}}{\prod_{j=1}^{n'} \prod_{k=1, k \neq j}^n (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{j=1}^{n'} 2\hat{\epsilon}_j z_{\sigma(j)}} \text{sgn}(\sigma\epsilon) \\ &= 2^{-n'} (-1)^{\frac{n'(n'-1)}{2}} \prod_{n' \leq j < k \leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2) \cdot \prod_{n' < j \leq n} \hat{\epsilon}_j z_{\sigma(j)} \text{sgn}(\sigma\epsilon). \quad (\text{B.20}) \end{aligned}$$

Let $\eta\rho = (\eta, \rho) \in \Sigma_n \times \mathbb{Z}_2^n$ be such that $\eta(j) = j$ and $\rho(j) = 0$ for all $1 \leq j \leq n'$. Then the quantity (B.20) is invariant under the substitution $\sigma\epsilon \rightarrow \sigma\eta\epsilon\rho$. Hence (B.18) follows. The rest is straightforward. ■

The pairs $(G, G') = (\text{Sp}_{2n}(\mathbb{R}), \text{O}_{2p+1,2q})$ **with** $p + q = n' \leq n$.

Theorem B.5. *Suppose*

$$\Psi = \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n\} \cup \{2iJ_j^* \mid 1 \leq j \leq n\}$$

and

$$\Psi' = \{iJ_j^* \pm iJ_k^* \mid 1 \leq j < k \leq n'\} \cup \{iJ_j^* \mid 1 \leq j \leq n'\}.$$

Let

$$u' = (-1)^{\frac{n(n-1)}{2} - n'n} i^{n'}.$$

Then

$$\begin{aligned}
 P(z') \frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z')\pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(z)}{\det(z'+z)_{\text{Hom}(\mathbb{V},\mathbb{V}'_0^\perp)}} &= \frac{u'}{2^{n'|\Sigma_{n-n'}\times\mathbb{Z}_2^{n-n'}|}} \times \\
 \sum_{\sigma\epsilon\in\Sigma_n\times\mathbb{Z}_2^n} \text{sgn}(\text{short})(\sigma\epsilon) \cdot &\frac{P(z_{\sigma(1)},\dots,z_{\sigma(n')}) \prod_{n'<j<k\leq n} (z_{\sigma(j)}^2 - z_{\sigma(k)}^2)}{\prod_{j=1}^{n'} (z'_j - \hat{\epsilon}_j z_{\sigma(j)}}.
 \end{aligned}
 \tag{B.21}$$

Moreover,

$$\begin{aligned}
 \pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(z) &= \prod_{n'<j<k\leq n} ((iz_j)^2 - (iz_k)^2), \\
 \det(z'+z)_{\mathbb{W}\mathfrak{b}'} &= \prod_{j=1}^{n'} i(z'_j - z_j),
 \end{aligned}$$

and

$$u'' = (-1)^{\frac{n'(n'+1)}{2}}.
 \tag{B.22}$$

Proof. In this case

$$\begin{aligned}
 &\frac{\pi_{\mathfrak{g}'/\mathfrak{h}'}(z')\pi_{\mathfrak{g}/\mathfrak{h}}(\text{short})(z)}{\det(z'+z)_{\text{Hom}(\mathbb{V},\mathbb{V}'_0^\perp)}} \\
 &= \frac{\prod_{1\leq j<k\leq n'} ((iz'_j)^2 - (iz'_k)^2) \cdot \prod_{j=1}^{n'} iz'_j \cdot \prod_{1\leq j<k\leq n} ((iz_j)^2 - (iz_k)^2)}{\prod_{j=1}^{n'} \prod_{k=1}^n i(z'_j - z_k)i(z'_j + z_k)}.
 \end{aligned}$$

This expression is equal to $2^{-n'}$ times (B.16). Hence, (B.21) follows from Theorem B.3. ■

Appendix C.

In this Appendix, we prove Theorem 7.4, via a case by case analysis. We shall denote by Ψ the standard positive root system, as in Appendix B. Always, \mathcal{S} will stand for a strongly orthogonal subset of Ψ^n and s for an element of the Weyl group $W(\mathbb{H}_{\mathbb{C}})$. Furthermore,

$$x' \in \mathfrak{h}'^{\text{reg}}, \quad x \in \mathfrak{h}_{\mathcal{S}}, \quad y \in \mathfrak{h}, \quad z = z(x) = x + iy.
 \tag{C.1}$$

Let

$$\begin{aligned}
 f_{s,\mathcal{S}}(z) &= m_{\mathcal{S}}(s)P(s^{-1} \cdot z)\tilde{\pi}_{\mathfrak{g}/\mathfrak{h}}(s^{-1} \cdot z), \\
 \det(x'+z)_{s\mathbb{W}\mathfrak{b}'} &= \prod_{j=1}^{n'} i(x'_j - \hat{\epsilon}_j z_{\sigma(j)}) \quad (\hat{\epsilon}_j = 1 \text{ if } \mathbb{D} = \mathbb{C}), \\
 F_{s,x',\mathcal{S}}(z) &= \frac{f_{s,\mathcal{S}}(z)}{\det(x'+z)_{s\mathbb{W}\mathfrak{b}'}} \mathcal{A}(-\tilde{\Psi}_{\mathcal{S},\mathbb{R}})(x), \\
 \tilde{F}_{s,x',\mathcal{S}}(z) &= F_{s,x',\mathcal{S}}(z)/P(s^{-1} \cdot z).
 \end{aligned}$$

We shall normalize the Killing form κ so that

$$\tilde{\kappa}(J_j, J_j) = 1 \quad (1 \leq j \leq n). \tag{C.2}$$

In the proof we shall use the following consequence of the Fundamental Theorem of Calculus:

$$\int_{\mathbb{R} \setminus 0} \partial(1)f(x) dx = \int_{\mathbb{R} \setminus 0} f'(x) dx = -\langle f \rangle_1(0), \tag{C.3}$$

where $f : \mathbb{R} \setminus 0 \rightarrow \mathbb{C}$ is a differentiable function of bounded support such that the limits

$$\langle f \rangle_1(0) = \lim_{x \rightarrow 0^+} f(x) - \lim_{x \rightarrow 0^+} f(-x)$$

exist. Since we may think of f as of a compactly supported distribution on \mathbb{R} , we shall write the formula (C.3) as

$$\int_{\mathbb{R}} \partial(1)f(x) dx = -\langle f \rangle_1(0), \tag{C.4}$$

Furthermore, by combining (C.4) with Leibniz rule, we see that for any smooth function $\phi : \mathbb{R} \rightarrow \mathbb{C}$

$$\begin{aligned} \int_{\mathbb{R}} (\partial(1)\phi(x))f(x) dx &= \int_{\mathbb{R}} \partial(1)(\phi(x)f(x)) dx - \int_{\mathbb{R}} \phi(x)(\partial(1)f(x)) dx, \\ \int_{\mathbb{R}} \partial(1)(\phi(x)f(x)) dx &= -\phi(0)\langle f \rangle_1(0). \end{aligned} \tag{C.5}$$

Here the singularity at 0 may be replaced by the singularity at any other point in \mathbb{R} . This will be frequently combined with the "jump relation" (2.3).

As in the proof of Theorem 7.3 we shall appeal to a partition of unity and thus consider two cases. Either ψ is compactly supported in a completely invariant open subset of \mathfrak{g} , which is disjoint with the singular support of the distribution $chc(x'+ \cdot)$, or ψ satisfies the condition (7.16). In the first (non-singular) case, we take $y = 0$ in (C.1). In the second (singular) case, $y \in \Gamma_{\tilde{x}}$ as in (7.17). In both cases $\frac{1}{\det(x'+z)_{sW^{b'}}$ is a smooth function of x in an open neighborhood of the support of ψ .

Let $1 \leq j \leq n'$. Then

$$\partial(J'_j) \frac{1}{\det(x'+z)_{sW^{b'}}} = \partial(-\hat{\epsilon}_j J_{\sigma(j)}) \frac{1}{\det(x'+z)_{sW^{b'}}}.$$

Hence,

$$\begin{aligned} &\partial(J'_j)(F_{s,x',S}(z)\mathcal{H}_S\psi(x)) \\ &= \left(\partial(-\hat{\epsilon}_j J_{\sigma(j)}) \frac{1}{\det(x'+z)_{sW^{b'}}} \right) f_{s,S}(z)\mathcal{A}(-\tilde{\Psi}_{S,\mathbb{R}})(x)\mathcal{H}_S\psi(x). \end{aligned}$$

Let ϕ be the restriction of $\frac{1}{\det(x'+z)_{sW^{b'}}$ to $x - x_{\sigma(j)}J_{\sigma(j)} + \mathbb{R}J_{\sigma(j)}$ and let f be the restriction of $f_{s,S}(z)\mathcal{A}(-\tilde{\Psi}_{S,\mathbb{R}})(x)\mathcal{H}_S\psi(x)$ to the same set. Since $f_{s,S}$ is constant

on $x - x_{\sigma(j)}J_{\sigma(j)} + \mathbb{R}J_{\sigma(j)}$ and $\mathcal{A}(-\tilde{\Psi}_{\mathcal{S},\mathbb{R}})$ is locally constant on this set, (C.5) implies that, in terms of Theorem 7.4,

$$\partial(J_j) \left(P(x')\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x' + x)\psi(x) d\mu(x) \right) = D + E,$$

where, in the non-singular case,

$$\begin{aligned} D &= \sum_{\mathcal{S},s} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',s}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x), \\ E &= \sum_{\mathcal{S},s} \int_{\mathfrak{h}_{\mathcal{S}}} \tilde{F}_{s,x',s}(z)\partial(\hat{\epsilon}_j J_{\sigma(j)})(P(s^{-1} \cdot x)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x), \end{aligned} \tag{C.6}$$

and in the singular case

$$\begin{aligned} D &= \lim_{y \in \Gamma_{\hat{x}}, y \rightarrow 0} \sum_{\mathcal{S},s} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',s}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x), \\ E &= \lim_{y \in \Gamma_{\hat{x}}, y \rightarrow 0} \sum_{\mathcal{S},s} \int_{\mathfrak{h}_{\mathcal{S}} \setminus \mathfrak{h}^{\Psi_{\mathcal{S},i\mathbb{R}}^n}} \tilde{F}_{s,x',s}(z)\partial(\hat{\epsilon}_j J_{\sigma(j)})(P(s^{-1} \cdot x)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S},s} \int_{\mathfrak{h}_{\mathcal{S}} \setminus \mathfrak{h}^{\Psi_{\mathcal{S},i\mathbb{R}}^n}} \lim_{y \in \Gamma_{s,S}, y \rightarrow 0} \tilde{F}_{s,x',s}(z)\partial(\hat{\epsilon}_j J_{\sigma(j)})(P(s^{-1} \cdot x)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x). \end{aligned} \tag{C.7}$$

The argument for the equality (C.7) is as in the proof of Theorem 7.3. In any case, let

$$\tilde{D} = \sum_{\mathcal{S},s} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',s}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x). \tag{C.8}$$

Since, by Corollary 3.3, $s \cdot J_j = \hat{\epsilon}_j J_{\sigma(j)}$, the proof of Theorem 7.4 will be complete as soon as we show that

$$\tilde{D} = 0. \tag{C.9}$$

The pairs $(G, G') = (U_{p,q}, U_{p',q'})$ **with** $p' + q' = n' \leq n = p + q$.

Here

$$\Psi^n = \{e_a - e_b \mid 1 \leq a \leq p < b \leq n\},$$

where $e_c = iJ_c^*$ for $c \in \{1, \dots, n\}$.

Lemma C.1. *Let $c \in \{1, 2, 3, \dots, n\} \setminus \underline{\mathcal{S}}$. Then*

$$\begin{aligned} &\int_{\mathfrak{h}_{\mathcal{S}}} \partial(J_c)(F_{s,x',s}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= - \sum_{\eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset, c \in \underline{\eta}} \frac{i}{\sqrt{2}} e_c(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',s}(z)\mathcal{H}_{\mathcal{S} \vee \eta}\psi(x) d\mu(x). \end{aligned}$$

Proof. Let $\eta = e_a - e_b$ so that $iH_\eta = J_a - J_b$.

We begin by computing the determinant of the map

$$\mathfrak{h}_S = \mathbb{R}iH_\eta \oplus \mathfrak{h}_S \cap \mathfrak{h}^\eta \ni itH_\eta + u \rightarrow tJ_a + u \in \mathfrak{h}_S.$$

Since

$$\begin{aligned} \mathfrak{h}_S &= \mathbb{R}(J_a + J_b) \oplus \mathbb{R}(J_a - J_b) \oplus \mathfrak{h}^{J_a^*, J_b^*}, \text{ and} \\ \mathfrak{h}_S &= \mathbb{R}J_a \oplus \mathbb{R}J_b \oplus \mathfrak{h}^{J_a^*, J_b^*}, \end{aligned}$$

it will suffice to compute the determinant of the map

$$t(J_a + J_b) + s(J_a - J_b) \rightarrow tJ_a + s(J_a - J_b),$$

which is easily seen to be $\frac{1}{2}$.

Notice that the map

$$\mathfrak{h}_S \cap \mathfrak{h}^{J_c^*} \ni x \rightarrow -\frac{\eta(x)}{\eta(J_c)}J_c + x \in \mathfrak{h}_S \cap \mathfrak{h}^\eta$$

is a well defined linear bijection. Thus, for $x \in \mathfrak{h}_S \cap \mathfrak{h}^{J_c^*}$ and for $t \in \mathbb{R}$,

$$\begin{aligned} \tilde{\kappa}(J_c, J_c)^{1/2} dt d\mu(x) &= d\mu(tJ_c + x) \\ &= d\mu\left(\left(t + \frac{\eta(x)}{\eta(J_c)}\right)J_c - \frac{\eta(x)}{\eta(J_c)}J_c + x\right) \\ &= \frac{1}{2}d\mu\left(\left(t + \frac{\eta(x)}{\eta(J_c)}\right)iH_\eta - \frac{\eta(x)}{\eta(J_c)}J_c + x\right) \\ &= \frac{1}{2}\tilde{\kappa}(iH_\eta, iH_\eta)^{1/2} dt d\mu\left(-\frac{\eta(x)}{\eta(J_c)}J_c + x\right). \end{aligned}$$

Since, $\tilde{\kappa}(iH_\eta, iH_\eta)^{1/2} = \sqrt{2}\tilde{\kappa}(J_c, J_c)^{\frac{1}{2}}$, we see that

$$d\mu(x) = \frac{1}{\sqrt{2}}d\mu\left(-\frac{\eta(x)}{\eta(J_c)}J_c + x\right) \quad x \in \mathfrak{h}_S \cap \mathfrak{h}^{J_c^*}. \tag{C.10}$$

Since $\Psi_{S, \mathbb{R}}^n = \{\eta \in \Psi^n \mid \underline{\eta} \cap \underline{S} = \emptyset\}$, Theorem 2.1, (C.2), (C.4), and (C.10) show that

$$\begin{aligned} &\int_{\mathfrak{h}_S} \partial(J_c)(F_{s, x', S}(z)\mathcal{H}_S\psi(x)) d\mu(x) \\ &= \int_{\mathfrak{h}_S \cap \mathfrak{h}^{J_c^*}} \int_{\mathbb{R}} \partial(J_c)(F_{s, x', S}(z(tJ_c + x))\mathcal{H}_S\psi(z(tJ_c + x))) dt d\mu(x) \\ &= - \sum_{\eta \in \Psi^n, \underline{\eta} \cap \underline{S} = \emptyset, c \in \underline{\eta}} \int_{\mathfrak{h}_S \cap \mathfrak{h}^{J_c^*}} F_{s, x', S}\left(z\left(-\frac{\eta(x)}{\eta(J_c)}J_c + x\right)\right) \langle \mathcal{H}_S\psi \rangle_{J_c}\left(-\frac{\eta(x)}{\eta(J_c)}J_c + x\right) d\mu(x) \\ &= - \sum_{\eta \in \Psi^n, \underline{\eta} \cap \underline{S} = \emptyset, c \in \underline{\eta}} -\frac{1}{\sqrt{2}} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s, x', S}(z(x)) \langle \mathcal{H}_S\psi \rangle_{J_c}(x) d\mu(x). \end{aligned}$$

Notice that,

$$\langle \mathcal{H}_S \psi \rangle_{J_c}(x) = J_c^*(iH_\eta) \langle \mathcal{H}_S \psi \rangle_{iH_\eta}(x).$$

Furthermore, by Theorem 2.1,

$$\langle \mathcal{H}_S \psi \rangle_{iH_\eta}(x) = i\epsilon(\Psi, \mathcal{S}, \eta) \mathcal{H}_{S \vee \eta} \psi(x).$$

But Lemma 1.7 implies $\epsilon(\Psi, \mathcal{S}, \eta) = 1$. Hence, the lemma follows. ■

Lemma C.2. *Let $\eta \in \mathcal{S}$ and let $c \in \underline{\eta}$. Then,*

$$\int_{\mathfrak{h}_S} \partial(J_c)(F_{s,x',\mathcal{S}}(z) \mathcal{H}_S \psi(x)) d\mu(x) = \frac{i}{\sqrt{2}} e_c(H_\eta) \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',\mathcal{S} \setminus \eta}(z) \mathcal{H}_S \psi(x) d\mu(x).$$

Proof. First we will show that

$$\int_{\mathfrak{h}_S} \partial(H_\eta)(F_{s,x',\mathcal{S}}(z) \mathcal{H}_S \psi(x)) d\mu(x) = \sqrt{2} \int_{\mathfrak{h}_S \cap \mathcal{V}(\eta)} F_{s,x',\mathcal{S} \setminus \eta}(z) \mathcal{H}_S \psi(x) d\mu(x). \tag{C.11}$$

Indeed, by (C.2), $\tilde{\kappa}(H_\eta, H_\eta)^{1/2} = \sqrt{2}$. Moreover, $\Psi_{\mathcal{S}, \mathbb{R}} = \mathcal{S}$. Therefore, the left hand side of (C.11) is equal to

$$\begin{aligned} & \sqrt{2} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} \int_{\mathbb{R}} \partial(H_\eta)(F_{s,x',\mathcal{S}}(z(tH_\eta + x)) \mathcal{H}_S \psi(tH_\eta + x)) dt d\mu(x) \\ &= -\sqrt{2} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} \frac{f_{s,\mathcal{S}}(z(x))}{\det(x' + z(x))_{sW^{\mathfrak{h}'}}} \mathcal{A}(-(\mathcal{S} \setminus \eta))(x) \langle \frac{-\eta}{|\eta|} \rangle_{H_\eta}(x) \mathcal{H}_S \psi(x) d\mu(x). \end{aligned}$$

Lemma 8.1 shows that

$$\frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{\mathfrak{h}'}}} = \frac{1}{2} F_{s,x',\mathcal{S} \setminus \eta}(z).$$

Furthermore,

$$\langle \frac{-\eta}{|\eta|} \rangle_{H_\eta}(x) = -2.$$

Hence, (C.11) follows.

If $\eta = e_a - e_b$, let $\eta^c = e_a + e_b$. Then $\eta^c \in \Psi^c \setminus \Psi_{\mathcal{S}, i\mathbb{R}}^n$. Hence, Theorem 2.1 implies

$$\int_{\mathfrak{h}_S} \partial(iH_{\eta^c})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_S \psi(x)) d\mu(x) = 0. \tag{C.12}$$

where $iH_{\eta^c} = J_a + J_b$. Moreover,

$$J_c = \frac{i}{2} J_c^*(iH_\eta)(H_\eta - iH_{\eta^c}). \tag{C.13}$$

Clearly, Lemma C.2 follows from (C.11), (C.12) and (C.13). ■

Consider the non-singular case. The quantity (C.8) may be written as

$$\tilde{D} = D_1 + D_2, \tag{C.14}$$

where

$$D_1 = \sum_{\mathcal{S}, s, \sigma(j) \notin \underline{\mathcal{S}}} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-J_{\sigma(j)})(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x)$$

and

$$D_2 = \sum_{\mathcal{S}, s, \sigma(j) \in \underline{\mathcal{S}}} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-J_{\sigma(j)})(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x).$$

We see from Lemma C.1 that

$$D_1 = \sum_{\mathcal{S}, s; \eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset, \sigma(j) \in \underline{\eta}} \frac{i}{\sqrt{2}} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}_{\eta}} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x),$$

and from Lemma C.2 that

$$D_2 = \sum_{\mathcal{S}, s; \eta \in \mathcal{S}, \sigma(j) \in \underline{\eta}} -\frac{i}{\sqrt{2}} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}_{\eta}} F_{s, x', \mathcal{S} \setminus \eta}(z) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x).$$

Clearly, $D_1 + D_2 = 0$.

In the singular case the \mathcal{S} varies over $\Psi_{st}^n(\mathfrak{g}^{\tilde{x}})$ and the s over the complex Weyl group of $\mathfrak{g}_{\mathbb{C}}^{\tilde{x}}$. Thus the same decomposition (C.14) holds. Indeed, it suffices to realize that

$$\mathfrak{g}^{\tilde{x}} = \mathfrak{u}_{p_1, q_1} \oplus \cdots \oplus \mathfrak{u}_{p_m, q_m} \oplus \mathfrak{gl}_{n_1}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{gl}_{n_k}(\mathbb{R}),$$

so that

$$\Psi_{st}^n(\mathfrak{g}^{\tilde{x}}) = \bigcup_{i=1}^m \Psi_{st}^n(\mathfrak{g}_i^{\tilde{x}}) \cup \{\tilde{\mathcal{S}}\}.$$

Thus (C.9) follows.

The pairs $(G, G') = (\mathrm{Sp}_{p, q}, \mathrm{O}_{2n'}^*)$ **with** $n' \leq n = p + q$.

Here

$$\Psi^n = \{e_a \pm e_b \mid 1 \leq a \leq p < b \leq n\}$$

and

$$\mathcal{S} = \{e_{a_1} + \delta_1 e_{b_1}, e_{a_2} + \delta_2 e_{b_2}, \dots, e_{a_m} + \delta_m e_{b_m}\},$$

where $a_j \neq a_k$ and $b_j \neq b_k$ for $k \neq j$, and

$$1 \leq a_j \leq p < b_j \leq n, \delta_j = \pm 1 \quad (1 \leq j \leq m).$$

Hence, the proof of (C.9) is almost identical to the proof in the $(\mathrm{U}_{p, q}, \mathrm{U}_{p', q'})$ case.

The pairs $(G, G') = (\mathrm{O}_{2n}^*, \mathrm{Sp}_{p, q})$ **with** $p + q = n' \leq n$.

Here

$$\Psi^n = \{e_a + e_b; 1 \leq a < b \leq n\}$$

and

$$\mathcal{S} = \{e_{a_1} + e_{b_1}, e_{a_2} + e_{b_2}, \dots, e_{a_m} + e_{b_m}\}.$$

Thus, as before, the proof of (C.9) is almost identical to the proof in the $(U_{p,q}, U_{p',q'})$ case.

The pairs $(G, G') = (\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{2p+1,2q})$ **with** $p + q = n' \leq n$.

Here

$$\Psi^n = \{e_a + e_b \mid 1 \leq a < b \leq n\} \cup \{2e_c \mid 1 \leq c \leq n\}$$

and

$$\mathcal{S} = \{e_{a_1} + e_{b_1}, e_{a_2} + e_{b_2}, \dots, e_{a_m} + e_{b_m}\} \cup \{2e_{c_1}, 2e_{c_2}, \dots, 2e_{c_k}\}.$$

Lemma C.3. *Let $c \in \{1, 2, 3, \dots, n\} \setminus \underline{\mathcal{S}}$. Set $\gamma = 2e_c$ so that $iH_\gamma = J_c$. Then*

$$\begin{aligned} & \int_{\mathfrak{h}_\mathcal{S}} \partial(iH_\gamma)(F_{s,x',\mathcal{S}}(z)\mathcal{H}_\mathcal{S}\psi(x)) d\mu(x) = \\ & -2i \int_{\mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\gamma} F_{s,x',\mathcal{S} \vee \gamma}(z)\mathcal{H}_{\mathcal{S} \vee \gamma}\psi(x) d\mu(x) - \frac{i}{\sqrt{2}} \sum_{\substack{\eta \in \Psi^n(\text{short}), \\ \eta \cap \underline{\mathcal{S}} = \emptyset, \gamma \subset \eta}} \int_{\mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\eta} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \vee \eta}\psi(x) d\mu(x). \end{aligned}$$

Proof. Since an analog of (C.10) holds in this case, the left hand side coincides with

$$\begin{aligned} & \int_{\mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\gamma} \int_{\mathbb{R}} \partial(iH_\gamma)(F_{s,x',\mathcal{S}}(z(tiH_\gamma + x))\mathcal{H}_\mathcal{S}\psi(tiH_\gamma + x)) dt d\mu(x) \\ & = - \int_{\mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\gamma} F_{s,x',\mathcal{S}}(z)\langle \mathcal{H}_\mathcal{S}\psi \rangle_{iH_\gamma}(x) d\mu(x) \\ & - \sum_{\substack{\eta \in \Psi^n(\text{short}), \\ \eta \cap \underline{\mathcal{S}} = \emptyset, \gamma \subset \eta}} \int_{\mathfrak{h}_\mathcal{S} \cap \mathcal{V}(\gamma)} F_{s,x',\mathcal{S}}(z(-\frac{\eta(x)}{\eta(iH_\gamma)}iH_\gamma + x))\langle \mathcal{H}_\mathcal{S}\psi \rangle_{iH_\gamma}(-\frac{\eta(x)}{\eta(iH_\gamma)}iH_\gamma + x) d\mu(x) \\ & = - \int_{\mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\gamma} F_{s,x',\mathcal{S}}(z)\langle \mathcal{H}_\mathcal{S}\psi \rangle_{iH_\gamma}(x) d\mu(x) \\ & \quad - \sum_{\substack{\eta \in \Psi^n(\text{short}), \\ \eta \cap \underline{\mathcal{S}} = \emptyset, \gamma \subset \eta}} \frac{1}{\sqrt{2}} \int_{\mathfrak{h}_\mathcal{S} \cap \mathcal{V}(\eta)} F_{s,x',\mathcal{S}}(z(x))\langle \mathcal{H}_\mathcal{S}\psi \rangle_{iH_\gamma}(x) d\mu(x). \quad (\text{C.15}) \end{aligned}$$

Let $x \in \mathfrak{h}_\mathcal{S}$ be semiregular with respect to γ . Then, Lemma 1.7 shows that

$$\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x)\epsilon(\Psi, \mathcal{S}, \gamma) = \mathcal{A}(-(\Psi_{\mathcal{S} \vee \gamma, \mathbb{R}} \setminus \gamma))(x).$$

But,

$$\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x) = \mathcal{A}(-\tilde{\Psi}_{\mathcal{S},\mathbb{R}})(x)\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}}(\text{long}))(x),$$

and

$$\mathcal{A}(-(\Psi_{\mathcal{S} \vee \gamma, \mathbb{R}} \setminus \gamma))(x) = \mathcal{A}(-t\Psi_{\mathcal{S} \vee \gamma, \mathbb{R}})(x)\mathcal{A}(-(\Psi_{\mathcal{S} \vee \gamma, \mathbb{R}}(\text{long}) \setminus \gamma))(x).$$

However,

$$\begin{aligned} \mathcal{A}(-(\Psi_{\mathcal{S} \vee \gamma, \mathbb{R}}(\text{long}) \setminus \gamma))(x) & = \mathcal{A}(-((\mathcal{S} \vee \gamma)(\text{long}) \setminus \gamma))(x) \\ & = \mathcal{A}(-\mathcal{S}(\text{long}))(x) = \mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}}(\text{long}))(x). \end{aligned}$$

Thus

$$\mathcal{A}(-\tilde{\Psi}_{\mathcal{S},\mathbb{R}})(x)\epsilon(\Psi, \mathcal{S}, \gamma) = \mathcal{A}(-\tilde{\Psi}_{\mathcal{S}\vee\gamma,\mathbb{R}})(x).$$

Hence, by Theorem 2.1 and Lemma 8.1,

$$\begin{aligned} F_{s,x',\mathcal{S}}(z)\langle \mathcal{H}_{\mathcal{S}}\psi \rangle_{iH_\gamma}(x) &= F_{s,x',\mathcal{S}}(z)i\epsilon(\Psi, \mathcal{S}, \gamma)\mathcal{H}_{\mathcal{S}\vee\gamma}\psi(x) \\ &= 2i F_{s,x',\mathcal{S}\vee\gamma}(z)\mathcal{H}_{\mathcal{S}\vee\gamma}\psi(x). \end{aligned} \tag{C.16}$$

Similarly,

$$\epsilon(\Psi, \mathcal{S}, \eta) = 1 \quad (\eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset),$$

and since iH_γ and iH_η are in the same connected component of $\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\gamma$,

$$\langle \mathcal{H}_{\mathcal{S}}\psi \rangle_{iH_\gamma}(x) = \langle \mathcal{H}_{\mathcal{S}}\psi \rangle_{iH_\eta}(x) = i\epsilon(\Psi, \mathcal{S}, \gamma)\mathcal{H}_{\mathcal{S}\vee\eta}\psi(x) = i\mathcal{H}_{\mathcal{S}\vee\eta}\psi(x). \tag{C.17}$$

Clearly, (C.15), (C.16) and (C.17), imply the lemma. ■

Lemma C.4. *Under the assumptions of Lemma C.3, suppose $\gamma(y) = 0$. Then*

$$\sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\gamma} F_{s,x',\mathcal{S}\vee\gamma}(z)\mathcal{H}_{\mathcal{S}\vee\gamma}\psi(x) d\mu(x) = 0.$$

Proof. As an element of the Weyl group $W(\mathbb{H}_{\mathbb{C}}) = \Sigma_n \ltimes \mathbb{Z}_2^n$, the reflection s_γ , with respect to γ , may be identified with an element $\tilde{\epsilon} \in \mathbb{Z}_2^n$, such that

$$\hat{\tilde{\epsilon}}_k = 1 \text{ for } k \neq c \text{ and } \hat{\tilde{\epsilon}}_c = -1.$$

Recall that $s = \sigma\epsilon$. Hence

$$s_\gamma s = \sigma(\sigma^{-1}\tilde{\epsilon}\sigma)\epsilon,$$

and therefore

$$((\sigma^{-1}\tilde{\epsilon}\sigma)\epsilon)_j = (\sigma^{-1}\tilde{\epsilon}\sigma)_j \hat{\epsilon}_j = \hat{\tilde{\epsilon}}_{\sigma(j)} \hat{\epsilon}_j = -\hat{\epsilon}_j,$$

if $\sigma(j) = c$. Thus it will suffice to show that

$$F_{s,x',\mathcal{S}\vee\gamma}(z) = F_{s_\gamma s,x',\mathcal{S}\vee\gamma}(z). \tag{C.18}$$

Notice that $m_{\mathcal{S}\vee\gamma}(s_\gamma s) = m_{\mathcal{S}\vee\gamma}(s)$. Furthermore,

$$\det(x' + z)_{s_\gamma s W^{b'}} = i(x'_{\sigma^{-1}(c)} + \hat{\epsilon}_{\sigma^{-1}(c)} z_c) \cdot \prod_{1 \leq j \leq n', \sigma(j) \neq c} i(x'_j - \hat{\epsilon}_j z_{\sigma(j)}).$$

But for $x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\gamma$,

$$z_c = x_c + iy_c = 0.$$

Hence,

$$\det(x' + z)_{s_\gamma s W^{b'}} = \det(x' + z)_{s W^{b'}}$$

and (C.18) follows. ■

Lemmas C.3 and C.4 imply that in the non-singular case the following formula holds:

$$\begin{aligned} \sum_{\mathcal{S}, s, \sigma(j) \notin \underline{\mathcal{S}}} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{e}_j J_{\sigma(j)})(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ = \sum_{\substack{\mathcal{S}, s; \eta \in \Psi^n(\text{short}), \\ \eta \cap \underline{\mathcal{S}} = \emptyset, \sigma(j) \in \eta}} \frac{i\hat{e}_j}{\sqrt{2}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x). \end{aligned} \tag{C.19}$$

In the singular case, if $\gamma(y) \neq 0$ then c belongs to $\bigcup_{i=1}^m \overline{\Psi_{st}^n(\mathfrak{g}_i^{\tilde{x}})}$. Hence the support condition (7.16) implies that the first integral on the right hand side of the equation of Lemma C.3 is zero. Thus (C.19) holds too.

Lemma C.5. *Let $c \in \underline{\mathcal{S}}$ and let $\gamma = 2e_c$. Suppose $\gamma \in \mathcal{S}$. Then*

$$\begin{aligned} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(H_{\gamma})(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ = \sum_{\eta \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \sqrt{2} \eta(H_{\gamma}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sW^{\mathfrak{b}'}}} \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x). \end{aligned}$$

Proof. The left hand side is equal to

$$\begin{aligned} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\gamma}} \int_{\mathbb{R}} \partial(H_{\gamma})(F_{s, x', \mathcal{S}}(z(tH_{\gamma} + x)) \mathcal{H}_{\mathcal{S}} \psi(tH_{\gamma} + x)) dt d\mu(x) \\ = \sum_{\eta \in \tilde{\Psi}_{\mathcal{S}, \mathbb{R}}, \eta(H_{\gamma}) \neq 0} - \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z(-\frac{\eta(x)}{\eta(H_{\gamma})} H_{\gamma} + x))}{\det(x' + z(-\frac{\eta(x)}{\eta(H_{\gamma})} H_{\gamma} + x))_{sW^{\mathfrak{b}'}}} \\ \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S}, \mathbb{R}} \setminus \eta))(-\frac{\eta(x)}{\eta(H_{\gamma})} H_{\gamma} + x) \\ \langle \frac{-\eta}{|\eta|} \rangle_{H_{\gamma}}(-\frac{\eta(x)}{\eta(H_{\gamma})} H_{\gamma} + x) \mathcal{H}_{\mathcal{S}} \psi(-\frac{\eta(x)}{\eta(H_{\gamma})} H_{\gamma} + x) d\mu(x) \\ = \sum_{\eta \in \tilde{\Psi}_{\mathcal{S}, \mathbb{R}}, \eta(H_{\gamma}) \neq 0} - \frac{1}{\sqrt{2}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z(x))}{\det(x' + z(x))_{sW^{\mathfrak{b}'}}} \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \\ \langle \frac{-\eta}{|\eta|} \rangle_{H_{\gamma}}(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x), \end{aligned}$$

which coincides with the right hand side because

$$\langle \frac{-\eta}{|\eta|} \rangle_{H_{\gamma}}(x) = -2\eta(H_{\gamma}).$$

■

Lemma C.6. *Let $\eta = e_a + e_b \in \Psi^n$, $\eta^c = e_a - e_b \in \Psi^c$ and let $\beta = 2e_b \in \Psi^n$. Suppose $\eta, \eta^c \in \Psi_{\mathcal{S}, \mathbb{R}}$ and $\eta(y) = \eta^c(y) = 0$. Then, the map*

$$\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c} \ni x \rightarrow s_{\beta} \cdot x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta} \tag{C.20}$$

is a linear bijection and

$$\begin{aligned} & \frac{f_{s,\mathcal{S}}(z(s_\beta \cdot x))}{\det(x' + z(s_\beta \cdot x))_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S},\mathbb{R}} \setminus \eta))(s_\beta \cdot x) \mathcal{H}_{\mathcal{S}}\psi(s_\beta \cdot x) \\ &= \frac{f_{s_\beta s,\mathcal{S}}(z(x))}{\det(x' + z(x))_{s_\beta s\mathbb{W}^{b'}}} \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x) \mathcal{H}_{\mathcal{S}}\psi(x). \end{aligned} \quad (\text{C.21})$$

Proof. Since

$$s_\beta \cdot J_b = -J_b \quad \text{and} \quad s_\beta \cdot J_{b'} = -J_{b'} \quad (b \neq b'),$$

(C.20) is clear. Also, since $\eta \circ s_\beta = \eta^c$,

$$\mathcal{A}(-(\tilde{\Psi}_{\mathcal{S},\mathbb{R}} \setminus \eta))(s_\beta \cdot x) = \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x).$$

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$$\mathcal{H}_{\mathcal{S}}\psi(s_\beta \cdot x) = \mathcal{H}_{\mathcal{S}}\psi(x).$$

Furthermore, by definition,

$$f_{s,\mathcal{S}}(z(s_\beta \cdot x)) = f_{s,\mathcal{S}}(z(x)) = f_{s_\beta s,\mathcal{S}}(z(x)),$$

and

$$\det(x' + z(s_\beta \cdot x))_{s\mathbb{W}^{b'}} = \det(x' + z(x))_{s_\beta s\mathbb{W}^{b'}}.$$

Thus (C.21) follows. ■

Lemma C.7. *Let $\gamma = 2e_c$, $\eta = e_a + e_b$, $\eta^c = e_a - e_b$, $\eta(y) = \eta^c(y) = 0$. Suppose $\gamma, \eta, \eta^c \in \Psi_{\mathcal{S},\mathbb{R}}$. Then,*

$$\begin{aligned} & \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \left(\eta(H_\gamma) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \right. \\ &+ \left. \eta^c(H_\gamma) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \right) \\ &= \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} 2\hat{\epsilon}_j \eta^c(H_\gamma) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\tilde{\Psi}_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x). \end{aligned}$$

Proof. Let us abbreviate $I_{s,\eta}$ and I_{s,η^c} for the integrals on the left hand side of the equation (C.7). Then, with $\beta = 2e_b$, $I_{s_\beta s,\eta} = I_{s,\eta^c}$, by Lemma C.6. As in the proof of Lemma C.4 we check that if $\sigma(j) = c$, then

$$(\sigma^{-1} s_\beta \sigma \epsilon)_j = \hat{\epsilon}_j \eta^c(H_\gamma).$$

Thus the left hand side is equal to

$$\begin{aligned}
 & \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \eta(H_\gamma) I_{s,\eta} + \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \eta^c(H_\gamma) I_{s,\eta^c} \\
 = & \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j I_{s,\eta} + \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \eta^c(H_\gamma) I_{s,\eta^c} \\
 = & \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} (\sigma^{-1} s_\beta \sigma \epsilon)_j I_{s_\beta s, \eta} + \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \eta^c(H_\gamma) I_{s,\eta^c} \\
 = & \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} 2 \hat{\epsilon}_j \eta^c(H_\gamma) I_{s,\eta^c},
 \end{aligned}$$

which coincides with the right hand side. ■

Lemma C.8. *Suppose $\eta = e_a - e_b \in \Psi_{\mathcal{S}, \mathbb{i}\mathbb{R}}^n$ and $\underline{\eta} \subseteq \underline{\mathcal{S}}$. Let $(a, b) = \{a + 1, a + 2, \dots, b - 2, b - 1\}$.*

Then

$$\epsilon(\Psi, \mathcal{S}, \eta) = (-1)^{|(a,b) \cap \mathcal{S}(\text{long})|}.$$

Proof. Let $\alpha \in \Psi_{\mathcal{S}, \mathbb{i}\mathbb{R}}^n$ be such that $\alpha(H_\eta) \neq 0$. The $\alpha = \eta$ and therefore $\alpha(H_\eta) = 2$. Thus

$$\{\alpha \in \Psi_{\mathcal{S}, \mathbb{i}\mathbb{R}}^n \mid \alpha(H_\eta) < 0\} = \emptyset.$$

Hence, Lemma 1.7 implies that

$$\epsilon(\Psi, \mathcal{S}, \eta) = (-1)^{c_1 + c_2}, \tag{C.22}$$

where

$$\begin{aligned}
 c_1 &= \frac{1}{2} |\Psi_{\mathcal{S}, \mathbb{C}} \cap (-L_{\mathcal{S}} \Psi_{\mathcal{S}, \mathbb{C}})|, \\
 c_2 &= \frac{1}{2} |\Psi_{\mathcal{S} \vee \eta, \mathbb{C}} \cap (-L_{\mathcal{S} \vee \eta} \Psi_{\mathcal{S}, \mathbb{C}})|.
 \end{aligned} \tag{C.23}$$

Notice that

$$L_{\mathcal{S}} = \prod_{\beta \in \Psi_{\mathcal{S}, \mathbb{i}\mathbb{R}}^n, \underline{\beta} \subseteq \underline{\mathcal{S}}} s_\beta \cdot \prod_{\beta \in \Psi(\text{long}), \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset} s_\beta. \tag{C.24}$$

Hence,

$$L_{\mathcal{S}} = \begin{cases} e_c & \text{if } c \in \underline{\mathcal{S}(\text{long})}, \\ -e_c & \text{if } c \notin \underline{\mathcal{S}}, \\ e_{c'} & \text{if there is } \beta \in \Psi_{\mathcal{S}, \mathbb{i}\mathbb{R}}^n \text{ such that } \{c, c'\} = \underline{\beta} \subseteq \underline{\mathcal{S}}. \end{cases} \tag{C.25}$$

For a root α , we shall write $\alpha > 0$ if $\alpha \in \Psi$ and $\alpha < 0$ if $\alpha \in -\Psi$. It is easy to see that

$$\{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, s_\eta \alpha < 0\} = \{e_a - e_c \mid a < c < b\} \cup \{e_c - e_b \mid a < c < b\}. \tag{C.26}$$

Since $L_{Sv\eta} = L_S s_\eta = s_\eta L_S$, we see from (C.24), (C.25) and (C.26) that

$$\begin{aligned}
& (\Psi_{S,C} \cap (-L_S \Psi_{S,C})) \setminus (\Psi_{Sv\eta,C} \cap (-L_{Sv\eta} \Psi_{Sv\eta,C})) \\
= & \{ \alpha \in \Psi_{S,C} \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \notin \Psi_{Sv\eta,C}, L_S \alpha < 0 \} \\
& \cup \{ \alpha \in \Psi_{S,C} \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \in \Psi_{Sv\eta,C}, L_S \alpha < 0, s_\eta \alpha < 0 \} \\
= & \{ \alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap \underline{\mathcal{S}(long)} \neq \emptyset, L_S \alpha < 0 \} \\
& \cup \{ \alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \setminus (\underline{\mathcal{S}(long)} \cup \underline{\eta}) \neq \emptyset, L_S \alpha < 0, s_\eta \alpha < 0 \} \\
= & \{ \alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap \underline{\mathcal{S}(long)} \neq \emptyset, s_\eta \alpha < 0 \} \\
& \cup \{ \alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \setminus (\underline{\mathcal{S}(long)} \cup \underline{\eta}) \neq \emptyset, L_S \alpha < 0, s_\eta \alpha < 0 \} \\
= & \{ e_a - e_c \mid a < c < b, c \in \underline{\mathcal{S}(long)} \} \cup \{ e_c - e_b \mid a < c < b, c \in \underline{\mathcal{S}(long)} \} \\
& \cup \{ e_a - e_c \mid a < c < b, c \in \underline{\mathcal{S}(short)}, c' < b \} \\
& \cup \{ e_c - e_b \mid a < c < b, c \in \underline{\mathcal{S}(short)}, a < c' \} \\
& \cup \{ e_c - e_b \mid a < c < b, c \notin \underline{\mathcal{S}} \}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |(\Psi_{S,C} \cap (-L_S \Psi_{S,C})) \setminus (\Psi_{Sv\eta,C} \cap (-L_{Sv\eta} \Psi_{Sv\eta,C}))| = 2|(a, b) \cap \underline{\mathcal{S}(long)}| + |(a, b) \setminus \underline{\mathcal{S}}| \\
& + |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' < b\}| + |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' > a\}|. \quad (C.27)
\end{aligned}$$

Similarly, $(\Psi_{Sv\eta,C} \cap (-L_{Sv\eta} \Psi_{Sv\eta,C})) \setminus (\Psi_{S,C} \cap (-L_S \Psi_{S,C}))$

$$\begin{aligned}
& = \{ \alpha \in \Psi_{S,C} \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \in \Psi_{Sv\eta,C}, L_S \alpha > 0, s_\eta \alpha < 0 \} \\
& = \{ e_a - e_c \mid a < c < b, c \notin \underline{\mathcal{S}} \} \\
& \cup \{ e_a - e_c \mid a < c < b, c \in \underline{\mathcal{S}(short)}, c' > b \} \\
& \cup \{ e_c - e_b \mid a < c < b, c \in \underline{\mathcal{S}(short)}, a > c' \}.
\end{aligned}$$

and therefore,

$$\begin{aligned}
& |(\Psi_{Sv\eta,C} \cap (-L_{Sv\eta} \Psi_{Sv\eta,C})) \setminus (\Psi_{S,C} \cap (-L_S \Psi_{S,C}))| \quad (C.28) \\
= & |(a, b) \setminus \underline{\mathcal{S}}| + |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' > b\}| + |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' < a\}|.
\end{aligned}$$

We see from (C.23), (C.27) and (C.28) that

$$\begin{aligned}
& 2c_1 - 2c_2 = 2|(a, b) \cap \underline{\mathcal{S}(long)}| \quad (C.29) \\
& + |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' < b\}| - |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' > b\}| \\
& + |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' > a\}| - |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' < a\}| \\
= & 2|(a, b) \cap \underline{\mathcal{S}(long)}| + 2|(a, b) \cap \underline{\mathcal{S}(short)}| \\
& - 2|\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' > b\}| - 2|\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' < a\}| \\
= & 2|(a, b) \cap \underline{\mathcal{S}(long)}| + 2|(a, b) \cap \underline{\mathcal{S}(short)}| \quad (C.30) \\
& - 2|\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid c' > b \text{ or } c' < a\}| \\
= & 2|(a, b) \cap \underline{\mathcal{S}(long)}| + |\{c \in (a, b) \cap \underline{\mathcal{S}(short)} \mid a < c' < b\}|.
\end{aligned}$$

Notice that the number of the $c \in \underline{\mathcal{S}(short)}$ such that $a < c' < b$ is even. Hence, (C.29) and (C.22) imply the lemma. \blacksquare

Lemma C.9. *Let $\eta = e_a + e_b \in \Psi^n$ and let $\eta^c = e_a - e_b \in \Psi^c$. Suppose $2e_a \in \mathcal{S}$ and $2e_b \in \mathcal{S}$, so that $\eta, \eta^c \in \tilde{\Psi}_{\mathcal{S}, \mathbb{R}}$. Then, for $x \in \mathfrak{h}_{\mathcal{S}}$ semiregular with respect to η^c ,*

$$\begin{aligned} \frac{2f_{s, \mathcal{S}}(z(x))}{\det(x' + z(x))_{sW\mathfrak{h}'}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}}(\text{short}) \setminus \eta^c))(x) \\ = \epsilon(\Psi, (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \eta^c) F_{s, x', (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta}(z(x)). \end{aligned}$$

(Since $\eta \vee \eta^c = \{2e_a, 2e_b\} \subseteq \mathcal{S}$, the difference $\mathcal{S} \setminus (\eta \vee \eta^c)$ makes sense.)

Proof. Notice that

$$2f_{s, \mathcal{S}} = f_{s, (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta}. \tag{C.31}$$

Furthermore,

$$\begin{aligned} \frac{\mathcal{A}(-(\tilde{\Psi}_{\mathcal{S}, \mathbb{R}} \setminus \eta^c))(x)}{\mathcal{A}(-\tilde{\Psi}_{(\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \mathbb{R}})(x)} &= \frac{\prod_{\alpha \in \tilde{\Psi}_{\mathcal{S}, \mathbb{R}}, \alpha \neq \eta^c} \frac{-\alpha(x)}{|\alpha(x)|}}{\prod_{\alpha \in \tilde{\Psi}_{(\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \mathbb{R}}} \frac{-\alpha(x)}{|\alpha(x)|}} \\ &= \frac{\prod_{\alpha \in \tilde{\Psi}_{\mathcal{S}, \mathbb{R}}, \alpha \neq \eta^c, \alpha \neq \eta} \frac{-\alpha(x)}{|\alpha(x)|}}{\prod_{\alpha \in \tilde{\Psi}_{\mathcal{S} \setminus (\eta \vee \eta^c), \mathbb{R}}} \frac{-\alpha(x)}{|\alpha(x)|}} = \frac{\prod_{\alpha \in \tilde{\Psi}_{\mathcal{S}(long), \mathbb{R}}, \alpha \neq \eta^c, \alpha \neq \eta} \frac{-\alpha(x)}{|\alpha(x)|}}{\prod_{\alpha \in \tilde{\Psi}_{\mathcal{S}(long) \setminus (\eta \vee \eta^c), \mathbb{R}}} \frac{-\alpha(x)}{|\alpha(x)|}} \\ &= \prod_{\substack{\alpha \in \tilde{\Psi}_{\mathcal{S}(long), \mathbb{R}}, \\ \alpha \neq \eta^c, \alpha \neq \eta, \alpha \cap \eta \neq \emptyset}} \frac{-\alpha(x)}{|\alpha(x)|} = \prod_{\substack{\alpha \in \tilde{\Psi}_{\mathcal{S}(long), \mathbb{R}}, \\ \alpha \neq \eta^c, \alpha \neq \eta, \alpha \cap \eta \neq \emptyset}} \frac{\alpha(x)}{|\alpha(x)|}. \end{aligned} \tag{C.32}$$

Notice that the roots α which occur in (C.32) are of the form

$$\alpha = \begin{cases} e_a \pm e_c \text{ or } e_b \pm e_c & \text{if } b < c, \\ e_c \pm e_a \text{ or } e_c \pm e_b & \text{if } c < a, \\ e_a \pm e_c \text{ or } e_c \pm e_b & \text{if } a < c < b, \end{cases}$$

where $c \in \mathcal{S}(long)$.

Let $y_k = e_k(x)$. Then $y_a = y_b, y_c \in \mathbb{R}$ and (C.32) is equal to

$$\begin{aligned} &\prod_{b < c, c \in \mathcal{S}(long)} \frac{y_a^2 - y_c^2}{|y_a^2 - y_c^2|} \frac{y_b^2 - y_c^2}{|y_b^2 - y_c^2|} \cdot \prod_{c < a, c \in \mathcal{S}(long)} \frac{y_c^2 - y_a^2}{|y_c^2 - y_a^2|} \frac{y_c^2 - y_b^2}{|y_c^2 - y_b^2|} \\ &\cdot \prod_{a < c < b, c \in \mathcal{S}(long)} \frac{y_a^2 - y_c^2}{|y_a^2 - y_c^2|} \frac{y_c^2 - y_b^2}{|y_c^2 - y_b^2|} \\ &= \prod_{b < c, c \in \mathcal{S}(long)} \left(\frac{y_a^2 - y_c^2}{|y_a^2 - y_c^2|} \right)^2 \cdot \prod_{c < a, c \in \mathcal{S}(long)} \left(\frac{y_c^2 - y_a^2}{|y_c^2 - y_a^2|} \right)^2 \\ &\cdot \prod_{a < c < b, c \in \mathcal{S}(long)} (-1) \\ &= (-1)^{|(a, b) \cap \mathcal{S}(long)|}. \end{aligned} \tag{C.33}$$

Since $(\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta = \mathcal{S}(long) \setminus \{a, b\}$, (C.33) coincides with $\epsilon(\Psi, (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \eta^c)$, by Lemma C.8. ■

In the non-singular case $y = 0$, so the condition $\eta(y) = \eta^c(y) = 0$ of Lemmas C.6 and C.7 is satisfied. This condition is also satisfied in the singular case, because $\eta, \eta^c \in \Psi_{\mathcal{S}, \mathbb{R}}$ implies

$$\underline{\eta} \cap \left(\bigcup_{i=1}^m \underline{\Psi}_{st}^n(\mathfrak{g}_i^{\tilde{x}}) \right) = \emptyset,$$

and hence $\eta(y) = \eta^c(y) = 0$. Thus, by combining Lemmas C.5 - C.9, we obtain the following formula

$$\begin{aligned} & \sum_{\mathcal{S}, s, \sigma(j) \in \underline{\mathcal{S}}(long)} \int_{\mathfrak{h}_S} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s, x', \mathcal{S}}(z) \mathcal{H}_S \psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S}, s, \sigma(j) \in \underline{\mathcal{S}}(long); \eta \in \tilde{\Psi}_{\mathcal{S}, \mathbb{R}} \cap \Psi^n} -\hat{\epsilon}_j \sqrt{2} \eta^c(J_{\sigma(j)}) \epsilon(\Psi, (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \eta^c) \\ & \quad \int_{\mathfrak{h}_S \cap \mathfrak{h}^{\eta^c}} F_{s, x', (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta}(z(x)) \mathcal{H}_S \psi(x) d\mu(x). \end{aligned} \tag{C.34}$$

Lemma C.10. *Let $\eta \in \mathcal{S}(short)$. Then*

$$\int_{\mathfrak{h}_S} \partial(H_\eta)(F_{s, x', \mathcal{S}}(z) \mathcal{H}_S \psi(x)) d\mu(x) = \sqrt{2} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S} \setminus \eta}(z(x)) \mathcal{H}_S \psi(x) d\mu(x).$$

Proof. Since $\tilde{\kappa}(H_\eta, H_\eta)^{1/2} = \sqrt{2}$, the left hand side is equal to

$$\begin{aligned} & \sqrt{2} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} \int_{\mathbb{R}} \partial(H_\eta)(F_{s, x', \mathcal{S}}(z(tH_\eta + x)) \mathcal{H}_S \psi(tH_\eta + x)) dt d\mu(x) \\ &= \sqrt{2} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} \int_{\mathbb{R}} \partial(H_\eta) \left(\frac{f_{s, \mathcal{S}}(z(tH_\eta + x))}{\det(x' + z(tH_\eta + x))_{sWb'}} \mathcal{A}(-\tilde{\Psi}_{\mathcal{S} \setminus \eta, \mathbb{R}})(tH_\eta + x) \right. \\ & \quad \left. \left(\frac{-\eta(tH_\eta)}{|\eta(tH_\eta)|} \right) \mathcal{H}_S \psi(tH_\eta + x) \right) dt d\mu(x) \\ &= 2\sqrt{2} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} \frac{f_{s, \mathcal{S}}(z(x))}{\det(x' + z(x))_{sWb'}} \mathcal{A}(-\tilde{\Psi}_{\mathcal{S} \setminus \eta, \mathbb{R}})(x) \mathcal{H}_S \psi(x) d\mu(x). \end{aligned}$$

Since,

$$\frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sWb'}} \mathcal{A}(-\tilde{\Psi}_{\mathcal{S} \setminus \eta, \mathbb{R}})(x) = \frac{1}{2} F_{s, x', \mathcal{S} \setminus \eta}(z),$$

the equation follows. ■

Lemma C.11. *Let $\eta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$, $\underline{\eta} \subseteq \underline{\mathcal{S}}$. Then*

$$\begin{aligned} & \int_{\mathfrak{h}_S} \partial(iH_\eta)(F_{s, x', \mathcal{S}}(z) \mathcal{H}_S \psi(x)) d\mu(x) \\ &= -2\sqrt{2}i \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z(x)) \mathcal{H}_{\mathcal{S} \setminus \eta} \psi(x) d\mu(x). \end{aligned}$$

Proof. By Theorem 2.1, Lemma 1.9, (C.5) and (2.3), the left hand side is equal to

$$\begin{aligned} & \sqrt{2} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \int_{\mathbb{R}} \partial(iH_{\eta})(F_{s,x',\mathcal{S}}(z(itH_{\eta} + x))\mathcal{H}_{\mathcal{S}}\psi(itH_{\eta} + x)) dt d\mu(x) \\ &= -\sqrt{2} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z(x)) \langle \mathcal{H}_{\mathcal{S}}\psi \rangle_{iH_{\eta}}(x) d\mu(x) \\ &= -\sqrt{2} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z(x)) i2\epsilon(\Psi, \mathcal{S}, \eta) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x), \end{aligned}$$

which coincides with the right hand side. ■

Since, for $\eta = e_a + e_b$, $\eta^c = e_a - e_b$ and $\sigma(j) \in \underline{\eta}$,

$$-\hat{\epsilon}_j J_{\sigma(j)} = \frac{\hat{\epsilon}_j}{2i} (H_{\eta} - \eta^c(J_{\sigma(j)})iH_{\eta^c}),$$

Lemmas C.10 and C.11 imply the following formula

$$\begin{aligned} & \sum_{\mathcal{S},s,\sigma(j) \in \underline{\mathcal{S}(short)}} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \tag{C.35} \\ &= \sum_{\mathcal{S},s;\eta \in \mathcal{S}(short), \sigma(j) \in \underline{\eta}} -\hat{\epsilon}_j \frac{i}{\sqrt{2}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S} \setminus \eta}(z)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S},s;\eta \in \mathcal{S}(short), \sigma(j) \in \underline{\eta}} \sqrt{2}\hat{\epsilon}_j \epsilon(\Psi, \mathcal{S}, \eta^c) \eta^c(J_{\sigma(j)}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \vee \eta^c} \psi(x) d\mu(x). \end{aligned}$$

Clearly, (C.19), (C.34) and (C.35) imply (C.9).

The pairs $(G, G') = (\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{2p,2q})$ **with** $p + q = n' \leq n$.

Here Ψ^n and \mathcal{S} are as in the previous section, and the following statements (which hold in both the non-singular and the singular case) may be verified the same way as the corresponding statements there.

Lemma C.12. *Let $c \in \{1, 2, 3, \dots, n\} \setminus \underline{\mathcal{S}}$. Put $\gamma = 2e_c$. Then*

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}}} \partial(iH_{\gamma})(F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= -i \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\gamma}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}\mathfrak{h}^{\gamma}}} \mathcal{A}(-(\Psi_{\mathcal{S} \vee \gamma, \mathbb{R}} \setminus \gamma))(x)\mathcal{H}_{\mathcal{S} \vee \gamma} \psi(x) d\mu(x) \\ &- \sum_{\eta \in \Psi^n(short), \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset, \underline{\gamma} \subset \underline{\eta}} \frac{i}{\sqrt{2}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z(x))\mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x). \end{aligned}$$

Lemma C.13. *Let $c \in \underline{\mathcal{S}}$ and put $\gamma = 2e_c$. Suppose $\gamma \in \mathcal{S}$. Then*

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}}} \partial(H_{\gamma})(F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= 2 \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}_{\gamma}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \gamma))(x)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\ &+ \sum_{\eta \in \Psi_{\mathcal{S},\mathbb{R}}(\text{short})} \sqrt{2}\eta(H_{\gamma}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}_{\eta}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x). \end{aligned}$$

Lemma C.14. *Let $\eta = e_a + e_b \in \Psi^n$, $\eta^c = e_a - e_b \in \Psi^c$ and let $\beta = 2e_b \in \Psi^n$. Suppose $\eta, \eta^c \in \Psi_{\mathcal{S},\mathbb{R}}$ and $\eta(y) = \eta^c(y) = 0$. Then the map*

$$\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c} \ni x \rightarrow s_{\beta}x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta} \tag{C.36}$$

is a linear bijection and

$$\begin{aligned} & \frac{f_{s,\mathcal{S}}(z(s_{\beta} \cdot x))}{\det(x' + z(s_{\beta} \cdot x))_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(s_{\beta} \cdot x)\mathcal{H}_{\mathcal{S}}\psi(s_{\beta} \cdot x) \\ &= \frac{f_{s_{\beta}s,\mathcal{S}}(z(x))}{\det(x' + z(x))_{s_{\beta}sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x)\mathcal{H}_{\mathcal{S}}\psi(x). \end{aligned} \tag{C.37}$$

Proof. This is similar to the Lemma C.6, except that

$$\mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(s_{\beta} \cdot x) = -\mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x)$$

and

$$f_{s,\mathcal{S}}(z(s_{\beta} \cdot x)) = f_{s,\mathcal{S}}(z(x)) = -f_{s_{\beta}s,\mathcal{S}}(z(x)).$$

Thus the negative signs cancel and the lemma follows. ■

Lemma C.15. *Let $\eta = e_a + e_b \in \Psi^n$, $\eta^c = e_a - e_b \in \Psi^c$. Suppose $2e_a, 2e_b \in \mathcal{S}$, so that $\eta, \eta^c \in \Psi_{\mathcal{S},\mathbb{R}}(\text{short})$. Then for $x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}$,*

$$2 \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x) = \epsilon(\Psi, (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \eta^c) F_{s,x',(\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta}(z(x)).$$

Lemma C.16. *With the notation of Lemmas C.13 and C.14 we have*

$$\begin{aligned} & \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \left(\eta(H_{\gamma}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}_{\eta}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \right. \\ &+ \left. \eta^c(H_{\gamma}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \right) \\ &= \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} 2\hat{\epsilon}_j \eta^c(H_{\gamma}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta^c))(x)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\ &= \sum_{s \in W(\mathbb{H}_{\mathbb{C}}), \sigma(j)=c} \hat{\epsilon}_j \eta^c(H_{\gamma}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} \epsilon(\Psi, (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \eta^c) \\ & F_{s,x',(\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta}(z(x))\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x). \end{aligned}$$

Lemma C.17. *The following formula holds:*

$$\begin{aligned} & \sum_{\mathcal{S},s,\sigma(j) \in \underline{\mathcal{S}}(short)} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ = & \sum_{\mathcal{S},s;\eta \in \mathcal{S}(short),\sigma(j) \in \underline{\eta}} -\hat{\epsilon}_j \frac{i}{\sqrt{2}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S} \setminus \eta}(z) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\ + & \sum_{\mathcal{S},s;\eta \in \mathcal{S}(short),\sigma(j) \in \underline{\eta}} \sqrt{2} \hat{\epsilon}_j \epsilon(\Psi, \mathcal{S}, \eta^c) \eta^c(J_{\sigma(j)}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \setminus \eta^c} \psi(x) d\mu(x). \end{aligned}$$

We see from Lemma C.12 that

$$\begin{aligned} & \sum_{\mathcal{S},s,\sigma(j) \notin \underline{\mathcal{S}}} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ = & \sum_{\mathcal{S},s;\sigma(j) \notin \underline{\mathcal{S}}} i \hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{e_{\sigma(j)}}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \\ & \mathcal{A}(-(\Psi_{\mathcal{S} \vee 2e_{\sigma(j)}, \mathbb{R}} \setminus 2e_{\sigma(j)}))(x) \mathcal{H}_{\mathcal{S} \vee 2e_{\sigma(j)}} \psi(x) d\mu(x) \\ & + \sum_{\substack{\mathcal{S},s;\eta \in \Psi^n(short), \\ \sigma(j) \in \underline{\eta}, \eta \cap \underline{\mathcal{S}} = \emptyset}} \frac{i}{\sqrt{2}} \hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \setminus \eta} \psi(x) d\mu(x). \quad (C.38) \end{aligned}$$

Furthermore, Lemma C.13 implies

$$\begin{aligned} & \sum_{\mathcal{S},s,\sigma(j) \in \underline{\mathcal{S}}(long)} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ = & \sum_{\mathcal{S},s;\sigma(j) \in \underline{\mathcal{S}}(long)} -i \hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{e_{\sigma(j)}}} \frac{f_{s,\mathcal{S} \setminus 2e_{\sigma(j)}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \\ & \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus 2e_{\sigma(j)}))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\ + & \sum_{\mathcal{S},s;\sigma(j) \in \underline{\mathcal{S}}(long)} -\sqrt{2} \hat{\epsilon}_j \eta^c(J_{\sigma(j)}) \epsilon(\Psi, (\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta, \eta^c) \\ & \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^c}} F_{s,x',(\mathcal{S} \setminus (\eta \vee \eta^c)) \vee \eta}(z(x)) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x). \quad (C.39) \end{aligned}$$

Lemma C.17, (C.38) and (C.39) imply (C.9).

The pairs $(G, G') = (O_{2p+1,2q}, Sp_{2n'}(\mathbb{R}))$ **with** $n' \leq n = p + q$. Here

$$\Psi^n = \{e_a \pm e_b \mid 1 \leq a \leq p < b \leq n\} \cup \{e_c \mid p + 1 \leq c \leq n\}$$

and the strongly orthogonal set \mathcal{S} is as in one of the following cases

$$\{e_{a_1} \pm e_{b_1}, \dots, e_{a_l} \pm e_{b_l}, e_{a_{l+1}} + \delta_{l+1} e_{b_{l+1}}, \dots, e_{a_m} + \delta_m e_{b_m}\}, \quad (C.40)$$

$$\{e_{a_1} \pm e_{b_1}, \dots, e_{a_l} \pm e_{b_l}, e_{a_{l+1}} + \delta_{l+1} e_{b_{l+1}}, \dots, e_{a_m} + \delta_m e_{b_m}\} \cup \{e_{b_{m+1}}\}, \quad (C.41)$$

where $a_i \leq p < b_i$, $p < b_{m+1}$, $\delta_i = \pm 1$, and l or m could be equal to zero.

The proofs of some of the following statements are similar to the proofs of the corresponding statements in the $(\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{2p+1,2q})$ case. In such situations we skip the details.

Lemma C.18. *Suppose $c \in \{1, 2, 3, \dots, n\} \setminus \underline{\mathcal{S}}$. Let $\gamma = e_c$, so that $iH_\gamma = 2J_c$. Then*

$$\int_{\mathfrak{h}_S} \partial(iH_\gamma)(F_{s,x',S}(z)\mathcal{H}_S\psi(x)) d\mu(x) = A + B,$$

where

$$A = \begin{cases} -2id(\mathcal{S}, \gamma) \int_{\mathfrak{h}_S \cap \mathcal{V}(\gamma)} \frac{f_{s,S\vee\gamma}(z)}{\det(x'+z)_{sW\mathfrak{h}'}} \mathcal{A}(-(\Psi_{S\vee\gamma} \setminus \gamma))(x)\mathcal{H}_{S\vee\gamma}\psi(x) d\mu(x), \\ \text{if } \gamma \in \Psi_{\mathcal{S},\mathbb{R}}^n, \\ 0 \text{ otherwise,} \end{cases}$$

$$B = \sum_{\substack{\eta \in \Psi^n, c \in \eta \\ \eta \cap \underline{\mathcal{S}} = \emptyset}} -2\sqrt{2}i\gamma(H_\eta) \int_{\mathfrak{h}_S \cap \mathcal{V}(\eta)} F_{s,x',S}(z)\mathcal{H}_{S\vee\eta}\psi(x) d\mu(x),$$

and $d(\mathcal{S}, \gamma)$ is as in Lemma 8.3.

Corollary C.19. *The following formula holds*

$$\begin{aligned} & \sum_{\mathcal{S},s;\sigma(j) \notin \underline{\mathcal{S}}} \int_{\mathfrak{h}_S} \partial(-\hat{e}_j J_{\sigma(j)})(F_{s,x',S}(z)\mathcal{H}_S\psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S},s;\gamma \in \Psi_{\mathcal{S},\mathbb{R}}(\text{short}), \underline{\gamma} = \{\sigma(j)\}} 2i\hat{e}_j \int_{\mathfrak{h}_S \cap \mathfrak{h}^\gamma} \frac{f_{s,S}(z)}{\det(x'+z)_{sW\mathfrak{h}'}} \\ & \quad \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \gamma))(x)\mathcal{H}_S\psi(x) d\mu(x) \\ & + \sum_{\mathcal{S},s;\eta \in \Psi^n(\text{long}), \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset} \sqrt{2}e_{\sigma(j)}(H_\eta)i\hat{e}_j \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',S}(z)\mathcal{H}_{S\vee\eta}\psi(x) d\mu(x). \end{aligned} \tag{C.42}$$

Proof. By Lemma C.18, the left hand side is equal to

$$\begin{aligned} & \sum_{\substack{\mathcal{S},s;\gamma \in \Psi_{\mathcal{S},\mathbb{R}}^n, \\ \underline{\gamma} \cap \underline{\mathcal{S}} = \emptyset, \underline{\gamma} = \{\sigma(j)\}}} i\hat{e}_j d(\mathcal{S}, \gamma) \int_{\mathfrak{h}_{S\vee\gamma} \cap \mathfrak{h}^\gamma} \frac{f_{s,S\vee\gamma}(z)}{\det(x'+z)_{sW\mathfrak{h}'}} \mathcal{A}(-(\Psi_{S\vee\gamma,\mathbb{R}} \setminus \gamma))(x)\mathcal{H}_{S\vee\gamma}\psi(x) d\mu(x) \\ & + \sum_{\substack{\mathcal{S},s;\eta \in \Psi^n(\text{long}), \\ \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset}} \sqrt{2}e_{\sigma(j)}(H_\eta)i\hat{e}_j \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',S}(z)\mathcal{H}_{S\vee\eta}\psi(x) d\mu(x), \end{aligned} \tag{C.43}$$

because $\mathfrak{h}_S \cap \mathfrak{h}^\eta = \mathfrak{h}_{S\vee\eta} \cap \mathfrak{h}^\eta$. Let \mathcal{S} be as in (C.40). Assume first that

$$\mathcal{S} = \mathcal{S}(\text{long}), \gamma \in \Psi^n(\text{short}), \underline{\gamma} \cap \underline{\mathcal{S}} = \emptyset.$$

Let $\beta \in \Psi_{S\vee\gamma,\mathbb{R}}(\text{short}) \setminus \gamma$. Then $\gamma - \beta$ is a real root. Let $w = w_\beta$ denote the corresponding reflection. Then,

$$\gamma \circ w = \beta, w^2 = 1, w(\mathfrak{h}_{S\vee\gamma}) = \mathfrak{h}_{S\vee\gamma}, w(\mathfrak{h}_{S\vee\gamma} \cap \mathfrak{h}^\beta) = \mathfrak{h}_{S\vee\gamma} \cap \mathfrak{h}^\gamma.$$

Let $(\gamma - \beta)_{pos} \in \Psi$ be either $\gamma - \beta$ or $\beta - \gamma$, depending which of the two roots is positive. Notice that for $x \in \mathfrak{h}_{S \vee \gamma}$,

$$\prod_{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}, \alpha \neq \gamma} \alpha(w \cdot x) = \prod_{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}, \underline{\alpha} \cap (\underline{\gamma} \cup \underline{\beta}) = \emptyset} \alpha(x) \prod_{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}, \underline{\alpha} \cap (\underline{\gamma} \cup \underline{\beta}) \neq \emptyset, \alpha \neq \gamma} \alpha(w \cdot x)$$

and that

$$\begin{aligned} & \prod_{\substack{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}, \\ \underline{\alpha} \cap (\underline{\gamma} \cup \underline{\beta}) \neq \emptyset, \alpha \neq \gamma}} \alpha(w \cdot x) = (\gamma - \beta)_{pos}(w \cdot x)(\gamma + \beta)(x)\beta(w \cdot x) \\ \prod_{\substack{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}(long), \\ \underline{\alpha} \cap \underline{\gamma} \neq \emptyset, \underline{\alpha} \cap \underline{\beta} = \emptyset}} \alpha(w \cdot x) & \prod_{\substack{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}(long), \\ \underline{\alpha} \cap \underline{\gamma} = \emptyset, \underline{\alpha} \cap \underline{\beta} \neq \emptyset}} \alpha(w \cdot x) = -(\gamma - \beta)_{pos}(x)(\gamma + \beta)(x)\gamma(x) \\ \prod_{\substack{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}(long), \\ \underline{\alpha} \cap \underline{\gamma} = \emptyset, \underline{\alpha} \cap \underline{\beta} \neq \emptyset}} \alpha(x) & \prod_{\substack{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}(long), \\ \underline{\alpha} \cap \underline{\gamma} \neq \emptyset, \underline{\alpha} \cap \underline{\beta} = \emptyset}} \alpha(x) = - \prod_{\substack{\alpha \in \Psi_{S \vee \gamma, \mathbb{R}}, \\ \underline{\alpha} \cap (\underline{\gamma} \cup \underline{\beta}) \neq \emptyset, \alpha \neq \beta}} \alpha(x). \end{aligned}$$

Thus,

$$\mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \gamma))(w \cdot x) = -\mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \beta))(x).$$

In the non-singular case $y = 0$. In the singular case $\beta(y) = 0$ for any $\beta \in \Psi_{S \vee \gamma, \mathbb{R}}(short)$. Hence,

$$\begin{aligned} & \int_{\mathfrak{h}_{S \cap \mathfrak{h} \gamma}} \frac{f_{s, S \vee \gamma}(z(x))}{\det(x' + z(x))_{sW^{b'}}} \mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{S \vee \gamma} \psi(x) d\mu(x) \tag{C.44} \\ &= \int_{\mathfrak{h}_{S \vee \gamma} \cap \mathfrak{h} \beta} \frac{f_{s, S \vee \gamma}(z(w \cdot x))}{\det(x' + z(w \cdot x))_{sW^{b'}}} \mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \gamma))(w \cdot x) \mathcal{H}_{S \vee \gamma} \psi(w \cdot x) d\mu(x) \\ &= \int_{\mathfrak{h}_{S \vee \gamma} \cap \mathfrak{h} \beta} \frac{-f_{ws, S \vee \gamma}(z(x))}{\det(x' + z(x))_{wsW^{b'}}} (-\mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \beta))(x)) \mathcal{H}_{S \vee \gamma} \psi(x) d\mu(x) \\ &= \int_{\mathfrak{h}_{S \vee \gamma} \cap \mathfrak{h} \beta} \frac{f_{ws, S \vee \gamma}(z(x))}{\det(x' + z(x))_{wsW^{b'}}} \mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \beta))(x) \mathcal{H}_{S \vee \gamma} \psi(x) d\mu(x). \end{aligned}$$

Furthermore, by Lemma 8.3,

$$d(\mathcal{S}, \gamma) = 2|\Psi_{S \vee \gamma, \mathbb{R}}(short)|.$$

Hence, (C.44) shows that

$$\begin{aligned} & \sum_{s \in W(\mathbb{H}_C), \underline{\gamma} = \{\sigma(j)\}} i\hat{\epsilon}_j d(\mathcal{S}, \gamma) l_{s, \mathcal{S}} \int_{\mathfrak{h}_{S \vee \gamma} \cap \mathfrak{h} \gamma} \frac{f_{s, S \vee \gamma}(z(x))}{\det(x' + z(x))_{sW^{b'}}} \\ & \quad \mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{S \vee \gamma} \psi(x) d\mu(x) \\ &= \sum_{s \in W(\mathbb{H}_C), \beta \in \Psi_{S \vee \gamma, \mathbb{R}}(short), \underline{\beta} = \{\sigma(j)\}} i\hat{\epsilon}_j 2l_{s, \mathcal{S}} \int_{\mathfrak{h}_{S \vee \gamma} \cap \mathfrak{h} \beta} \frac{f_{ws, S \vee \gamma}(z(x))}{\det(x' + z(x))_{wsW^{b'}}} \\ & \quad \mathcal{A}(-(\Psi_{S \vee \gamma, \mathbb{R}} \setminus \beta))(x) \mathcal{H}_{S \vee \gamma} \psi(x) d\mu(x). \tag{C.45} \end{aligned}$$

If $\mathcal{S} = \mathcal{S}(long) \vee \beta$, $\beta \in \Psi^n(short)$, $\gamma \in \Psi^c(short)$, $\underline{\gamma} \cap \underline{\mathcal{S}} = \emptyset$, then we also proceed as above. Since, $d(\mathcal{S}, \gamma) = 4$ in this case, (C.45) holds. Finally, we replace $\mathcal{S} \vee \gamma$ by \mathcal{S} and deduce that (C.43) coincides with the right hand side of the equality of Corollary C.19. ■

Lemma C.20. *Let $\eta \in \Psi_{S,\mathbb{R}}(long)$. Assume that for every $c \in \underline{\eta}$, $e_c \in \Psi_{S,\mathbb{R}}(short)$. Let $s = \sigma\epsilon \in W(H_{\mathbb{C}}) = \Sigma_n \times \mathbb{Z}_2^n$. Define $\tilde{\epsilon} \in \mathbb{Z}_2^n$ by*

$$\hat{\tilde{\epsilon}}|_{\underline{\eta}} = -1, \quad \hat{\tilde{\epsilon}}|_{\{1,2,3,\dots,n\}\setminus\underline{\eta}} = 1. \text{ (see(3.9))}$$

Then

$$\hat{\epsilon}_j = -((\sigma^{-1}\tilde{\epsilon}\sigma)\epsilon)_j \quad (\sigma(j) \in \underline{\eta}), \tag{C.46}$$

$$\tilde{\epsilon}(\mathfrak{h}_S \cap \mathfrak{h}^\eta) = \mathfrak{h}_S \cap \mathfrak{h}^\eta, \tag{C.47}$$

$$\int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',S}(z)\mathcal{H}_S\psi(x) d\mu(x) = \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{\tilde{\epsilon}s,x',S}(z)\mathcal{H}_S\psi(x) d\mu(x), \tag{C.48}$$

$$\hat{\epsilon}_j \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',S}(z)\mathcal{H}_S\psi(x) d\mu(x) \tag{C.49}$$

$$= -((\sigma^{-1}\tilde{\epsilon}\sigma)\epsilon)_j \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{\tilde{\epsilon}s,x',S}(z)\mathcal{H}_S\psi(x) d\mu(x) \quad (\sigma(j) \in \underline{\eta}). \tag{C.50}$$

Proof. Notice that in both the non-singular and the singular case we have $e_c(y) = 0$ for $c \in \underline{\eta}$.

Suppose $\sigma(j) \in \underline{\eta}$. Then

$$((\sigma^{-1}\tilde{\epsilon}\sigma)\epsilon)_j = (\sigma^{-1}\tilde{\epsilon}\sigma)_j \hat{\epsilon}_j = \hat{\tilde{\epsilon}}_{\sigma(j)} \hat{\epsilon}_j = -\hat{\epsilon}_j.$$

This verifies (C.46). Since $\tilde{\epsilon} \in W(H_S)$ and $\mathfrak{h}^\eta = \mathfrak{h}^{\eta \circ \tilde{\epsilon}}$, (C.47) follows. Also, by Harish-Chandra,

$$\begin{aligned} \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',S}(z(x))\mathcal{H}_S\psi(x) d\mu(x) &= \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',S}(z(\tilde{\epsilon} \cdot x))\mathcal{H}_S\psi(\tilde{\epsilon} \cdot x) d\mu(x) \\ &= \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} F_{s,x',S}(z(\tilde{\epsilon} \cdot x))\mathcal{H}_S\psi(x) d\mu(x). \end{aligned}$$

Since,

$$\mathcal{A}(-\Psi_{S,\mathbb{R}})(\tilde{\epsilon} \cdot x) = \mathcal{A}(-\Psi_{S,\mathbb{R}})(x),$$

we see that

$$F_{s,x',S}(z(\tilde{\epsilon} \cdot x)) = F_{\tilde{\epsilon}s,x',S}(z(x)).$$

Thus (C.48) follows. Clearly (C.46) and (C.48) imply (C.50). ■

Lemma C.21. *Let $\gamma \in \Psi_{S,\mathbb{R}}(short)$. Then*

$$\begin{aligned} &\int_{\mathfrak{h}_S} \partial(H_\gamma)(F_{s,x',S}(z)\mathcal{H}_S\psi(x)) d\mu(x) \\ &= 4 \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} \frac{f_{s,S}(z)}{\det(x' + z)_{sW\mathfrak{h}'}} \mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \gamma))(x)\mathcal{H}_S\psi(x) d\mu(x) \\ &+ \sum_{\eta \in \Psi_{S,\mathbb{R}}(long), \eta(H_\gamma) \neq 0} 2\sqrt{2}\gamma(H_\eta) \int_{\mathfrak{h}_S \cap \mathfrak{h}^\eta} \frac{f_{s,S}(z)}{\det(x' + z)_{sW\mathfrak{h}'}} \mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \eta))(x)\mathcal{H}_S\psi(x) d\mu(x). \end{aligned}$$

Proof. The left hand side is equal to

$$\begin{aligned} & \int_{\mathfrak{h}_S \cap \mathfrak{h}_\gamma} \int_{\mathbb{R}} \partial(H_\gamma)(F_{s,x',S}(z(tH_\gamma + x))) \mathcal{H}_S \psi(tH_\gamma + x) dt d\mu(x) 2 \\ &= -2 \int_{\mathfrak{h}_S \cap \mathfrak{h}_\gamma} \frac{f_{s,S}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \gamma))(x) \langle \frac{-\gamma}{|\gamma|} \rangle_{H_\gamma}(x) \mathcal{H}_S \psi(x) d\mu(x) \\ & \quad + \sum_{\eta \in \Psi_{S,\mathbb{R}}(long), \eta(H_\gamma) \neq 0} -2 \int_{\mathfrak{h}_S \cap \mathfrak{h}_\gamma} \frac{f_{s,S}(z(\frac{-\eta(x)}{\eta(H_\gamma)} H_\gamma + x))}{\det(x' + z(\frac{-\eta(x)}{\eta(H_\gamma)} H_\gamma + x))_{sW^{b'}}} \\ & \mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \eta))(\frac{-\eta(x)}{\eta(H_\gamma)} H_\gamma + x) \langle \frac{-\gamma}{|\gamma|} \rangle_{H_\gamma}(\frac{-\eta(x)}{\eta(H_\gamma)} H_\gamma + x) \mathcal{H}_S \psi(\frac{-\eta(x)}{\eta(H_\gamma)} H_\gamma + x) d\mu(x), \end{aligned}$$

which coincides with the right hand side, because of (C.10). ■

Lemma C.22. *Let $\gamma \in \Psi_{S,\mathbb{R}}(short)$, $\eta \in \Psi_{S,\mathbb{R}}(long) \cap \Psi^c$, $\eta(H_\gamma) \neq 0$. Then, there is $\tilde{s} \in W(H_S)$ such that*

$$\eta \circ \tilde{s} \in \Psi_{S,\mathbb{R}}(long) \cap \Psi^n, \eta \circ \tilde{s}(H_\gamma) \neq 0, \tag{C.51}$$

and for $x \in \mathfrak{h}_S$,

$$\begin{aligned} & \frac{f_{s,S}(z(\tilde{s} \cdot x))}{\det(x' + z(\tilde{s} \cdot x))_{sW^{b'}}} \mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \eta))(\tilde{s} \cdot x) \mathcal{H}_S \psi(\tilde{s} \cdot x) \\ &= \frac{f_{\tilde{s}s,S}(z(x))}{\det(x' + z(x))_{\tilde{s}sW^{b'}}} \mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \eta \circ \tilde{s}))(x) \mathcal{H}_S \psi(x). \end{aligned} \tag{C.52}$$

Proof. In case when S is as in (C.40),

$$\begin{aligned} & \gamma \in \{e_{a_1}, e_{b_1}, \dots, e_{a_l}, e_{b_l}\} \\ & \text{and } \eta = e_\mu \pm e_\nu, \text{ with } \mu, \nu \in \{e_{a_1}, e_{b_1}, \dots, e_{a_l}, e_{b_l}\} \text{ and } \mu < \nu. \end{aligned}$$

In case when S is as in (C.41),

$$\begin{aligned} & \gamma \in \{e_{a_1}, e_{b_1}, \dots, e_{a_l}, e_{b_l}, e_{b_{m+1}}\} \\ & \text{and } \eta = e_\mu \pm e_\nu, \text{ with } \mu, \nu \in \{e_{a_1}, e_{b_1}, \dots, e_{a_l}, e_{b_l}, e_{b_{m+1}}\} \text{ and } \mu < \nu. \end{aligned}$$

Our assumptions on γ and η place them in one of the following cases:

$$\gamma = e_{a_i} \quad \eta = e_{a_i} + e_{a_j} \tag{C.53}$$

$$\gamma = e_{a_j} \quad \eta = e_{a_i} + e_{a_j} \tag{C.54}$$

$$\gamma = e_{a_i} \quad \eta = e_{a_i} - e_{a_j} \tag{C.55}$$

$$\gamma = e_{a_j} \quad \eta = e_{a_i} - e_{a_j} \tag{C.56}$$

$$\gamma = e_{b_i} \quad \eta = e_{b_i} + e_{b_j} \tag{C.57}$$

$$\gamma = e_{b_j} \quad \eta = e_{b_i} + e_{b_j} \tag{C.58}$$

$$\gamma = e_{b_i} \quad \eta = e_{b_i} - e_{b_j} \tag{C.59}$$

$$\gamma = e_{b_j} \quad \eta = e_{b_i} - e_{b_j} \tag{C.60}$$

where $a_i < a_j$ and $b_i < b_j$.

Let

$$\xi = \begin{cases} e_{a_j} - e_{b_j} & \text{in cases (C.53), (C.55), (C.57), (C.59)} \\ e_{a_i} - e_{b_i} & \text{in cases (C.54), (C.56), (C.58), (C.60)} \end{cases}$$

and let

$$\tilde{s} = \begin{cases} -s_\xi & \text{in cases (C.56), (C.59)} \\ s_\xi & \text{otherwise.} \end{cases}$$

Then $\xi \in \Psi_{\mathcal{S}, \mathbb{R}}$ and $s_\xi \in W(H_{\mathcal{S}})$ is the a_j, b_j (or a_i, b_i) transposition, $\tilde{s} \in W(H_{\mathcal{S}})$ and (C.51) holds. Also, $\det(\tilde{s}) = -1$. Hence,

$$f_{s, \mathcal{S}}(z(\tilde{s} \cdot x)) = -f_{\tilde{s}s, \mathcal{S}}(z(x)). \tag{C.61}$$

Since in both the non-singular and the singular case $e_c(y) = 0$ for $c \in \underline{\eta}$,

$$\det(x' + z(\tilde{s} \cdot x))_{sW^{b'}} = \det(x' + z(x))_{\tilde{s}sW^{b'}}. \tag{C.62}$$

Furthermore,

$$\mathcal{H}_{\mathcal{S}}\psi(\tilde{s} \cdot x) = \mathcal{H}_{\mathcal{S}}\psi(x). \tag{C.63}$$

Notice that the only roots in $\Psi_{\mathcal{S}, \mathbb{R}} \setminus \xi$ which become negative under the action of s_ξ are

$$e_{a_k} - e_c, \quad e_c - e_{b_k}, \quad a_k < c < b_k,$$

where $k = i$ or j . Since, the number of these roots is even,

$$\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \xi) \circ s_\xi = \mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \xi).$$

Since $\xi \circ s_\xi = -\xi$, we have

$$\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}}) \circ s_\xi = -\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}}).$$

Since the cardinality of the set $\Psi_{\mathcal{S}, \mathbb{R}}$ is even, the last equality implies

$$\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}}) \circ \tilde{s}_\xi = -\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}}). \tag{C.64}$$

Hence,

$$\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta) \circ \tilde{s}_\xi = -\mathcal{A}(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta \circ \tilde{s}_\xi). \tag{C.65}$$

Clearly, (C.61) - (C.65) imply (C.52). ■

Lemma C.23. *Suppose $\eta = e_a + \delta e_b \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$, $\delta = \pm 1$, $\underline{\eta} \subseteq \underline{\mathcal{S}}$. Then*

$$\epsilon(\Psi, \mathcal{S}, \eta) = -\delta(-1)^{|(a,b) \cap \underline{\Psi}_{\mathcal{S}, \mathbb{R}}(\text{short})|}.$$

Proof. Let $\alpha \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$ be such that $\alpha(H_\eta) \neq 0$. Then $\alpha = \eta$, so $\{\alpha \in \Psi_{\mathcal{S}, i\mathbb{R}}^n; \alpha(H_\eta) < 0\} = \emptyset$. Therefore

$$\epsilon(\Psi, \mathcal{S}, \eta) = (-1)^{c_1 - c_2},$$

where c_1 and c_2 are as in Lemma 1.7. Notice that

$$L_S = \prod_{\beta \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \underline{\beta} \subseteq \underline{\mathcal{S}}} s_\beta \cdot \prod_{\beta \in \Psi(\text{short}), \underline{\beta} \cap \underline{\mathcal{S}} = \emptyset} s_\beta.$$

Hence,

$$L_S = \begin{cases} e_c & \text{if } c \in \underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})}, \\ -e_c & \text{if } c \notin \underline{\mathcal{S}}, \\ e_{c'} & \text{if } e_c - e_{c'} \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \text{ or } e_{c'} - e_c \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \\ -e_{c'} & \text{if } e_c + e_{c'} \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \end{cases}$$

and $L_{S \vee \eta} = L_S s_\eta = s_\eta L_S$.

For a root α we shall write $\alpha < 0$ if $\alpha \in -\Psi$ and $\alpha > 0$ if $\alpha \in \Psi$.

Suppose $\eta = e_a - e_b$. Then

$$\{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, s_\eta \alpha < 0\} = \{e_a - e_c \mid a < c < b\} \cup \{e_c - e_b \mid a < c < b\}.$$

Therefore, with the notation $\mathcal{S}' = \{\alpha \in \mathcal{S}(\text{long}) \mid \underline{\alpha} \cap \underline{\mathcal{S}} \setminus \alpha = \emptyset\}$, we have

$$\begin{aligned} & (\Psi_{\mathcal{S}, \mathbb{C}} \cap (-L_S \Psi_{\mathcal{S}, \mathbb{C}})) \setminus (\Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}})) \\ &= \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap \underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \neq \emptyset, L_S \alpha < 0\} \\ &\cup \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap (\underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \cup \underline{\eta}) \neq \emptyset, L_S \alpha < 0, s_\eta \alpha < 0\} \\ &= \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap \underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \neq \emptyset, s_\eta \alpha < 0\} \\ &\cup \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap (\underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \cup \underline{\eta}) \neq \emptyset, L_S \alpha < 0, s_\eta \alpha < 0\} \\ &= \{e_a - e_c \mid a < c < b, c \in \underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})}\} \cup \{e_c - e_b \mid a < c < b, c \in \underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})}\} \\ &\cup \{e_a - e_c \mid a < c < b, c \in \underline{(\mathcal{S} \vee \eta)'} , L_S(e_a - e_c) < 0\} \\ &\cup \{e_c - e_b \mid a < c < b, c \in \underline{(\mathcal{S} \vee \eta)'} , L_S(e_c - e_b) < 0\} \\ &\cup \{e_c - e_b \mid a < c < b, c \notin \underline{\mathcal{S}}\}. \end{aligned}$$

Hence,

$$\begin{aligned} & |(\Psi_{\mathcal{S}, \mathbb{C}} \cap (-L_S \Psi_{\mathcal{S}, \mathbb{C}})) \setminus \Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}})| = 2|(a, b) \cap \underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})}| + |(a, b) \setminus \underline{\mathcal{S}}| \\ &+ |\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b - L_S(e_c) < 0\}| + |\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'} \mid L_S(e_c) - e_a < 0\}| \end{aligned}$$

Similarly,

$$\begin{aligned} & (\Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}})) \setminus (\Psi_{\mathcal{S}, \mathbb{C}} \cap (-L_S \Psi_{\mathcal{S}, \mathbb{C}})) \\ &= \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \setminus (\underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \cup \underline{\eta}) \neq \emptyset, L_S \alpha > 0, s_\eta \alpha < 0\} \\ &= \{e_a - e_c \mid a < c < b, c \notin (\underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \cup \underline{\eta}), L_S(e_a - e_c) > 0\} \\ &\cup \{e_c - e_b \mid a < c < b, c \notin (\underline{\Psi_{\mathcal{S}, \mathbb{R}}(\text{short})} \cup \underline{\eta}), L_S(e_c - e_b) > 0\} \\ &= \{e_a - e_c \mid a < c < b, c \notin \underline{\mathcal{S}}\}. \\ &\cup \{e_a - e_c \mid a < c < b, c \in \underline{(\mathcal{S} \vee \eta)'} , e_b - L_S(e_c) > 0\} \\ &\cup \{e_c - e_b \mid a < c < b, c \in \underline{(\mathcal{S} \vee \eta)'} , L_S(e_c) - e_a > 0\}. \end{aligned}$$

Hence,

$$\begin{aligned} & |(\Psi_{S\vee\eta, \mathbb{C}} \cap (-L_{S\vee\eta} \Psi_{S\vee\eta, \mathbb{C}})) \setminus (\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}}))| \\ &= |(a, b) \setminus \underline{\mathcal{S}}| \\ &+ |\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid e_b - L_S(e_c) > 0\}| \\ &+ |\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid L_S(e_c) - e_a > 0\}| \end{aligned}$$

Therefore,

$$\begin{aligned} 2c_1 - 2c_2 &= |(\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}})) \setminus (\Psi_{S\vee\eta, \mathbb{C}} \cap (-L_{S\vee\eta} \Psi_{S\vee\eta, \mathbb{C}}))| \\ &- |(\Psi_{S\vee\eta, \mathbb{C}} \cap (-L_{S\vee\eta} \Psi_{S\vee\eta, \mathbb{C}})) \setminus (\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}}))| \\ &= 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| \\ &+ |\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid L_S(e_c) - e_b > 0\}| \\ &- |\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid e_b - L_S(e_c) > 0\}| \\ &+ |\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid e_a - L_S(e_c) > 0\}| \\ &- |\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid L_S(e_c) - e_a > 0\}| \\ &= 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 2|(a, b) \cap (\underline{\mathcal{S} \vee \eta})'| \\ &- 2|\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid e_b - L_S(e_c) > 0\}| \\ &- 2|\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid L_S(e_c) - e_a > 0\}| \\ &= 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 2|(a, b) \cap (\underline{\mathcal{S} \vee \eta})'| \\ &- 2|\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid e_b - L_S(e_c) > 0, \text{ or } L_S(e_c) - e_a > 0\}| \\ &= 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| \\ &+ 2|\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid e_b - L_S(e_c) < 0, \text{ and } L_S(e_c) - e_a < 0\}|. \end{aligned}$$

Notice that $e_b - L_S(e_c) < 0$ implies that $L_S(e_c) = e_{c'}$ for some $c' \neq c$. Thus $e_b - L_S(e_c) < 0$ and $L_S(e_c) - e_a < 0$ happens if and only if $a < c' < b$. Therefore, the number

$$|\{c \in (a, b) \cap (\underline{\mathcal{S} \vee \eta})' \mid e_b - L_S(e_c) < 0, \text{ and } L_S(e_c) - e_a < 0\}|$$

is even. Hence the lemma follows.

Suppose $\eta = e_a + e_b$. Then

$$\begin{aligned} & \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, s_\eta \alpha < 0\} \\ &= \{e_a, e_b\} \cup \{e_a + e_c \mid b < c\} \cup \{e_b + e_c \mid a < c\} \cup \{e_a - e_c \mid a < c\} \cup \{e_b - e_c \mid b < c\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & (\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}})) \setminus (\Psi_{S\vee\eta, \mathbb{C}} \cap (-L_{S\vee\eta} \Psi_{S\vee\eta, \mathbb{C}})) \\ &= \{\alpha \in \Psi_{S, \mathbb{C}} \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \notin \Psi_{S\vee\eta, \mathbb{C}}, L_S \alpha < 0\} \\ &\cup \{\alpha \in \Psi_{S, \mathbb{C}} \cap \Psi_{S\vee\eta, \mathbb{C}} \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, L_S \alpha < 0, s_\eta \alpha < 0\} \end{aligned}$$

$$\begin{aligned}
&= \{\alpha \in \Psi \mid \underline{\alpha} \subseteq \underline{\eta}, L_S \alpha < 0\} \\
&\cup \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap \underline{\Psi_{S, \mathbb{R}}(short)} \neq \emptyset, L_S \alpha < 0\} \\
&\cup \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \setminus (\underline{\Psi_{S, \mathbb{R}}(short)} \cup \underline{\eta}) \neq \emptyset, L_S \alpha < 0, s_\eta < 0\} \\
&= \{e_a, e_b\} \\
&\cup \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \cap \underline{\Psi_{S, \mathbb{R}}(short)} \neq \emptyset, s_\eta \alpha < 0\} \\
&\cup \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \setminus (\underline{\Psi_{S, \mathbb{R}}(short)} \cup \underline{\eta}) \neq \emptyset, L_S \alpha < 0, s_\eta < 0\} \\
&= \{e_a, e_b\} \\
&\cup \{e_a + e_c \mid b < c, c \in \underline{\Psi_{S, \mathbb{R}}(short)}\} \cup \{e_b + e_c; a < c, c \in \underline{\Psi_{S, \mathbb{R}}(short)}\} \\
&\cup \{e_a - e_c \mid a < c, c \in \underline{\Psi_{S, \mathbb{R}}(short)}\} \cup \{e_b - e_c; b < c, c \in \underline{\Psi_{S, \mathbb{R}}(short)}\} \\
&\cup \{e_a + e_c \mid b < c, c \notin \underline{\mathcal{S}}\} \cup \{e_b + e_c; a < c, c \notin \underline{\mathcal{S}}\} \\
&\cup \{e_a - e_c \mid b < c, c \notin \underline{\mathcal{S}}\} \cup \{e_b - e_c; b < c, c \notin \underline{\mathcal{S}}\} \\
&\cup \{e_a + e_c \mid b < c, c \in \underline{(\mathcal{S} \vee \eta)'} , e_b - L_S(e_c) > 0\} \\
&\cup \{e_b + e_c \mid a < c, c \in \underline{(\mathcal{S} \vee \eta)'} , e_a - L_S(e_c) > 0\} \\
&\cup \{e_a - e_c \mid a < c, c \in \underline{(\mathcal{S} \vee \eta)'} , e_b + L_S(e_c) > 0\} \\
&\cup \{e_b - e_c \mid b < c, c \in \underline{(\mathcal{S} \vee \eta)'} , e_a + L_S(e_c) > 0\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|(\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}})) \setminus (\Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}}))| \\
&= 2 + 4|(b, n] \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + |(a, b) \setminus \underline{\mathcal{S}}| + 4|(b, n] \setminus \underline{\mathcal{S}}| \\
&+ |\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b - L_S(e_c) > 0\}| \\
&+ |\{c \in ((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_a - L_S(e_c) > 0\}| \\
&+ |\{c \in ((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b + L_S(e_c) > 0\}| \\
&+ |\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_a + L_S(e_c) > 0\}|.
\end{aligned}$$

Also,

$$\begin{aligned}
&(\Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}})) \setminus (\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}})) \\
&= \{\alpha \in \Psi_{S, \mathbb{C}} \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \in \Psi_{S \vee \eta, \mathbb{C}}, L_S \alpha > 0, s_\eta \alpha < 0\} \\
&= \{\alpha \in \Psi \mid \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \underline{\alpha} \setminus (\underline{\Psi_{S, \mathbb{R}}(short)} \cup \underline{\eta}) \neq \emptyset, L_S \alpha > 0, s_\eta \alpha < 0\} \\
&= \{e_a + e_c \mid b < c, c \notin \underline{\Psi_{S, \mathbb{R}}(short)} \cup \underline{\eta}, -e_b + L_S(e_c) > 0\} \\
&\cup \{e_b + e_c \mid a < c, c \notin \underline{\Psi_{S, \mathbb{R}}(short)} \cup \underline{\eta}, -e_a + L_S(e_c) > 0\} \\
&\cup \{e_a - e_c \mid a < c, c \notin \underline{\Psi_{S, \mathbb{R}}(short)} \cup \underline{\eta}, -e_b - L_S(e_c) > 0\} \\
&\cup \{e_b - e_c \mid b < c, c \notin \underline{\Psi_{S, \mathbb{R}}(short)} \cup \underline{\eta}, -e_a - L_S(e_c) > 0\} \\
&= \{e_a - e_c \mid a < c < b, c \notin \underline{\mathcal{S}}\} \\
&\cup \{e_a + e_c \mid b < c, c \in \underline{(\mathcal{S} \vee \eta)'} , -e_b + L_S(e_c) > 0\} \\
&\cup \{e_b + e_c \mid a < c, c \in \underline{(\mathcal{S} \vee \eta)'} , -e_a + L_S(e_c) > 0\} \\
&\quad \cup \{e_a - e_c \mid a < c, c \in \underline{(\mathcal{S} \vee \eta)'} , -e_b - L_S(e_c) > 0\} \\
&\quad \cup \{e_b - e_c \mid b < c, c \in \underline{(\mathcal{S} \vee \eta)'} , -e_b - L_S(e_c) > 0\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |(\Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}})) \setminus (\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}}))| \\
= & |(a, b) \setminus \underline{\mathcal{S}}| \\
+ & |\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b - L_S(e_c) < 0\}| \\
+ & |\{c \in ((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_a - L_S(e_c) < 0\}| \\
+ & |\{c \in ((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b + L_S(e_c) < 0\}| \\
+ & |\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_a + L_S(e_c) < 0\}|.
\end{aligned}$$

Hence,

$$\begin{aligned}
2c_1 - 2c_2 &= |(\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}})) \setminus (\Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}}))| \\
&\quad - |(\Psi_{S \vee \eta, \mathbb{C}} \cap (-L_{S \vee \eta} \Psi_{S \vee \eta, \mathbb{C}})) \setminus (\Psi_{S, \mathbb{C}} \cap (-L_S \Psi_{S, \mathbb{C}}))| \\
&= 2 + 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 4|(b, n] \cap \underline{\Psi_{S, \mathbb{R}}(short)}| \\
&\quad + 4|(b, n] \setminus \underline{\mathcal{S}}| + 2|(b, n] \cap \underline{(\mathcal{S} \vee \eta)'}| \\
&\quad - 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid -e_b + L_S(e_c) > 0\}| \\
&\quad - 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid -e_a - L_S(e_c) > 0\}| \\
&\quad + 2|((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'}| \\
&\quad - 2|\{c \in ((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'} \mid -e_a + L_S(e_c) > 0\}| \\
&\quad - 2|\{c \in ((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'} \mid -e_b - L_S(e_c) > 0\}| \\
&= 2 + 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 4|(b, n] \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 4|(b, n] \setminus \underline{\mathcal{S}}| \\
&\quad + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b - L_S(e_c) > 0, e_a + L_S(e_c) > 0\}| \\
&\quad + 2|\{c \in ((a, b) \cup (b, n]) \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b + L_S(e_c) > 0, e_a - L_S(e_c) > 0\}| \\
&= 2 + 2|(a, b) \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 4|(b, n] \cap \underline{\Psi_{S, \mathbb{R}}(short)}| + 4|(b, n] \setminus \underline{\mathcal{S}}| \\
&\quad + 2|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b + L_S(e_c) > 0, e_a - L_S(e_c) > 0\}| \\
&\quad + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b - L_S(e_c) > 0, e_a + L_S(e_c) > 0\}| \\
&\quad + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'} \mid e_b + L_S(e_c) > 0, e_a - L_S(e_c) > 0\}|.
\end{aligned}$$

The following two sets are disjoint

$$(\mathcal{S} \vee \eta)'_+ = \{e_a + e_c \in (\mathcal{S} \vee \eta)'\}, \quad (\mathcal{S} \vee \eta)'_- = \{e_a - e_c \in (\mathcal{S} \vee \eta)'\},$$

and their union coincides with $(\mathcal{S} \vee \eta)'$. Thus,

$$\begin{aligned}
& |(\Psi_{\mathcal{S},\mathbb{C}} \cap (-L_{\mathcal{S}}\Psi_{\mathcal{S},\mathbb{C}})) \setminus (\Psi_{\mathcal{S}\vee\eta,\mathbb{C}} \cap (-L_{\mathcal{S}\vee\eta}\Psi_{\mathcal{S}\vee\eta,\mathbb{C}}))| \\
& - |(\Psi_{\mathcal{S}\vee\eta,\mathbb{C}} \cap (-L_{\mathcal{S}\vee\eta}\Psi_{\mathcal{S}\vee\eta,\mathbb{C}})) \setminus (\Psi_{\mathcal{S},\mathbb{C}} \cap (-L_{\mathcal{S}}\Psi_{\mathcal{S},\mathbb{C}}))| \\
& = 2 + 2|(a, b) \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \setminus \underline{\mathcal{S}}| \\
& + 2|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'}, e_b + L_{\mathcal{S}}(e_c) > 0, e_a - L_{\mathcal{S}}(e_c) > 0\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'}, e_b - L_{\mathcal{S}}(e_c) > 0, e_a + L_{\mathcal{S}}(e_c) > 0\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'}, e_b + L_{\mathcal{S}}(e_c) > 0, e_a - L_{\mathcal{S}}(e_c) > 0\}| \\
& = 2 + 2|(a, b) \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \setminus \underline{\mathcal{S}}| \\
& + 2|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, e_b - e_{c'} > 0, e_a + e_{c'} > 0\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, e_b + e_{c'} > 0, e_a - e_{c'} > 0\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, e_b - e_{c'} > 0, e_a + e_{c'} > 0\}| \\
& + 2|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, e_b + e_{c'} > 0, e_a - e_{c'} > 0\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, e_b - e_{c'} > 0, e_a + e_{c'} > 0\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, e_b + e_{c'} > 0, e_a - e_{c'} > 0\}| \\
& = 2 + 2|(a, b) \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \setminus \underline{\mathcal{S}}| \\
& + 2|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, b < c'\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, a < c'\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, b < c'\}| \\
& + 2|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, a < c'\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, b < c'\}| \\
& + 2|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, a < c'\}| \\
& = 2 + 2|(a, b) \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 4|(b, n] \setminus \underline{\mathcal{S}}| \\
& + 4|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, b < c'\}| \\
& + 4|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{+}}, b < c'\}| \\
& + 4|\{c \in (a, b) \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, b < c'\}| \\
& + 4|\{c \in (b, n] \cap \underline{(\mathcal{S} \vee \eta)'_{-}}, b < c'\}|.
\end{aligned}$$

Therefore,

$$c_1 - c_2 = 1 + |(a, b) \cap \underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}| + 2m,$$

for some integer m . Hence, the lemma follows. \blacksquare

Lemma C.24. *Let \mathcal{S} be as in (C.41), $b = b_{m+1}$, $\gamma = e_b$, $\eta = e_{a_1} + \delta e_b$, $\delta = \pm 1$, and $\xi = e_{b_1} - e_b$. Then $\mathcal{S} \circ s_{\xi} \in \Psi_{st}^n$,*

$$\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta} = \mathfrak{h}_{\mathcal{S} \circ s_{\xi}} \cap \mathfrak{h}^{\eta}, \quad (\text{C.66})$$

and for $x \in \mathfrak{h}_S \cap \mathfrak{h}^\eta$,

$$\mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \eta))(x) = \epsilon(\Psi, \mathcal{S} \circ s_\xi \setminus \eta, \eta) \mathcal{A}(-\Psi_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}})(x). \tag{C.67}$$

Proof. Recall that $\underline{\mathcal{S}}_1 = \{a_1, b_1, \dots, a_l, b_l\}$. For $c \in \underline{\mathcal{S}}_1$ let

$$\epsilon(c) = \begin{cases} 1 & \text{if } a_1 < c, \\ -1 & \text{if } c < a_1, \end{cases} \quad \epsilon'(c) = \begin{cases} 1 & \text{if } b < c, \\ -1 & \text{if } c < b. \end{cases}$$

For simplicity we shall write

$$a =_s b \text{ if and only if } \frac{a}{|a|} = \frac{b}{|b|} \quad (a, b \in \mathbb{R}^\times).$$

Let $\eta' = e_{a_1} - \delta e_b$. Then $\mathcal{A}(-(\Psi_{S,\mathbb{R}} \setminus \eta))(x)$

$$\begin{aligned} &=_s \prod_{\alpha \in \Psi_{S,\mathbb{R}}, \alpha \neq \eta} (-\alpha(x)) && \tag{C.68} \\ &= \prod_{\substack{\alpha \in \Psi_{S,\mathbb{R}}, \\ \underline{\alpha} \cap \underline{\eta} = \emptyset}} (-\alpha(x)) \cdot (-e_{a_1}(x))(-e_b(x))(-\eta'(x)) \prod_{\substack{\alpha \in \Psi_{S,\mathbb{R}}(long), \\ \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \neq \eta}} (-\alpha(x)) \\ &=_s -\delta \prod_{\substack{\alpha \in \Psi_{S,\mathbb{R}}, \\ \underline{\alpha} \cap \underline{\eta} = \emptyset}} (-\alpha(x)) \cdot (-\eta'(x)) \cdot \prod_{\substack{\alpha \in \Psi_{S,\mathbb{R}}(long), \\ \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \neq \eta}} (-\alpha(x)). \end{aligned}$$

Notice that

$$\begin{aligned} &\{\alpha \in \Psi_{S,\mathbb{R}}(long) \mid \underline{\alpha} \cap \underline{\eta} = \emptyset\} \\ &= \{e_\mu \pm e_\nu \mid \mu, \nu \in \underline{\mathcal{S}}_1 \setminus \{a_1, b_1\}, \mu < \nu\} \cup \{e_\mu \pm e_{b_1} \mid \mu \in \underline{\mathcal{S}}_1 \setminus \{a_1\}, \mu < b_1\} \\ &\quad \cup \{e_{b_1} \pm e_\nu \mid \nu \in \underline{\mathcal{S}}_1 \setminus \{b_1\}, b_1 < \nu\}, \end{aligned}$$

$$\{\alpha \in \Psi_{S,\mathbb{R}}(short) \mid \underline{\alpha} \cap \underline{\eta} = \emptyset\} = \{e_\mu \mid \mu \in \underline{\mathcal{S}}_1 \setminus \{a_1\}\},$$

$$\mathcal{S} \circ s_\xi \setminus \eta = \{e_{a_1} - \delta e_b = \eta', e_{a_2} \pm e_{b_2}, \dots, e_{a_l} \pm e_{b_l}, e_{a_{l+1}} + \delta_{l+1} e_{b_{l+1}}, \dots, e_{a_m} + \delta_m e_{b_m}, e_{b_1}\},$$

so that

$$\Psi_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}} = \{\alpha \in \Psi_{S,\mathbb{R}} \mid \underline{\alpha} \cap \underline{\eta} = \emptyset\} \cup \{\eta'\}.$$

Thus,

$$\prod_{\alpha \in \Psi_{S,\mathbb{R}}, \underline{\alpha} \cap \underline{\eta} = \emptyset} (-\alpha(x)) \cdot (-\eta'(x)) =_s \mathcal{A}(-\Psi_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}})(x). \tag{C.69}$$

Moreover, since $x_{a_1}^2 = x_b^2$,

$$\begin{aligned} \prod_{\substack{\alpha \in \Psi_{S,\mathbb{R}}(long), \\ \underline{\alpha} \cap \underline{\eta} \neq \emptyset, \alpha \neq \eta}} (-\alpha(x)) &= \prod_{c \in \underline{\mathcal{S}}_1 \setminus a_1} \epsilon(c)(x_{a_1}^2 - x_c^2) \epsilon'(c)(x_b^2 - x_c^2) && \tag{C.70} \\ &=_s \prod_{c \in \underline{\mathcal{S}}_1 \setminus a_1} \epsilon(c) \epsilon'(c) = (-1)^{|(a_1, b) \cap \underline{\mathcal{S}}_1|}. \end{aligned}$$

Since

$$\underline{\Psi}_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}}(\text{short}) = \{b_1, a_2, b_2, \dots, a_l, b_l\},$$

we see that

$$(a_1, b) \cap \underline{\mathcal{S}}_1 = (a_1, b) \cap \underline{\Psi}_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}}(\text{short}),$$

and therefore, by Lemma C.23,

$$-\delta(-1)^{|(a_1, b) \cap \underline{\mathcal{S}}_1|} = \epsilon(\Psi, \mathcal{S} \circ s_\xi \setminus \eta, \eta). \tag{C.71}$$

The equality (C.66) is obvious because $\mathfrak{h}_\mathcal{S} = \mathfrak{h}_{\mathcal{S} \circ s_\xi}$. The equality (C.67) follows from (C.68), (C.69), (C.70) and (C.71). ■

Lemma C.25. *Let \mathcal{S} be as in (C.40), $\gamma = e_{b_j}$, $\eta = e_{a_1} + \delta e_{b_j}$, $\delta = \pm 1$, $\xi = e_{b_1} - e_{b_j}$, $1 \leq j \leq l$. Then $\eta \in \mathcal{S} \circ s_\xi$,*

$$\mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\eta = \mathfrak{h}_{\mathcal{S} \circ s_\xi} \cap \mathfrak{h}^\eta, \tag{C.72}$$

and for $x \in \mathfrak{h}_\mathcal{S} \cap \mathfrak{h}^\eta$,

$$\mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) = \epsilon(\Psi, \mathcal{S} \circ s_\xi \setminus \eta, \eta) \mathcal{A}(-(\Psi_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}}))(x). \tag{C.73}$$

Proof. For $c \in \underline{\mathcal{S}}_1 \setminus \{a_1, b_j\}$ let

$$\epsilon(c) = \begin{cases} 1 & \text{if } a_1 < c, \\ -1 & \text{if } c < a_1, \end{cases} \quad \epsilon'(c) = \begin{cases} 1 & \text{if } b_j < c, \\ -1 & \text{if } c < b_j. \end{cases}$$

Let $\eta' = e + a - \delta e_b$. Then $\mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x)$

$$\begin{aligned} &=_s \prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}, \alpha \neq \eta} (-\alpha(x)) \tag{C.74} \\ &= \prod_{\substack{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}, \\ \underline{\alpha} \cap \eta = \emptyset}} (-\alpha(x)) \cdot (-e_{a_1}(x))(-e_{b_j}(x))(-\eta'(x)) \prod_{\substack{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{long}), \\ \underline{\alpha} \cap \eta \neq \emptyset, \alpha \neq \eta}} (-\alpha(x)) \\ &=_s -\delta \prod_{\substack{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}, \\ \underline{\alpha} \cap \eta = \emptyset}} (-\alpha(x)) \cdot (-\eta'(x)) \cdot \prod_{\substack{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{long}), \\ \underline{\alpha} \cap \eta \neq \emptyset, \alpha \neq \eta}} (-\alpha(x)). \end{aligned}$$

Notice that

$$\begin{aligned} &\{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}} \mid \underline{\alpha} \cap \eta = \emptyset\} \\ &= \{e_\mu \pm e_\nu; \{\mu, \nu \in \underline{\mathcal{S}}_1 \setminus \{a_1, b_j\}, \mu < \nu\} \cup \{e_\mu; \mu \in \underline{\mathcal{S}}_1 \setminus \{a_1, b_j\}\}\}, \\ &\mathcal{S} \circ s_\xi \setminus \eta \\ &= \{e_{a_1} \mp e_{b_j} = \eta', e_{a_2} \pm e_{b_2}, \dots, e_{a_j} \pm e_{b_1}, \dots, e_{a_l} \pm e_{b_l}, e_{a_{l+1}} + \delta_{l+1} e_{b_{l+1}}, \dots, e_{a_m} + \delta_m e_{b_m}\}, \end{aligned}$$

so that

$$\Psi_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}} = \{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}} \mid \underline{\alpha} \cap \eta = \emptyset\} \cup \{\eta'\}. \tag{C.75}$$

Thus,

$$\prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}, \alpha \cap \eta = \emptyset} (-\alpha(x)) \cdot (-\eta'(x)) =_s \mathcal{A}(-\Psi_{\mathcal{S} \circ s_\xi \setminus \eta, \mathbb{R}})(x). \tag{C.76}$$

Moreover, since $x_{a_1}^2 = x_{b_j}^2$, we have as in the previous lemma,

$$\begin{aligned} & -\delta \prod_{\alpha \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{long}), \alpha \cap \eta \neq \emptyset, \alpha \neq \eta} (-\alpha(x)) \\ &= -\delta \prod_{c \in \underline{\mathcal{S}}_1 \setminus \{a_1, a_j\}} \epsilon(c)(x_{a_1}^2 - x_c^2) \epsilon'(c)(x_{b_j}^2 - x_c^2) \\ &= -\delta \prod_{c \in \underline{\mathcal{S}}_1 \setminus \{a_1, b_j\}} \epsilon(c) \epsilon'(c) = -\delta (-1)^{|(a_1, b_j) \cap \underline{\mathcal{S}}_1|} = \epsilon(\Psi, \mathcal{S} \circ s_\xi \setminus \eta, \eta). \end{aligned} \tag{C.77}$$

The equality (C.72) is obvious because $\mathfrak{h}_{\mathcal{S}} = \mathfrak{h}_{\mathcal{S} \circ s_\xi}$. The equality (C.73) follows from (C.74) - (C.77). ■

Lemma C.26. *The following equality holds*

$$\begin{aligned} & \sum_{\mathcal{S}, s; \sigma(j) \in \underline{\Psi}_{\mathcal{S}, \mathbb{R}}(\text{short})} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{e}_j J_{\sigma(j)})(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S}, s; \sigma(j) \in \underline{\Psi}_{\mathcal{S}, \mathbb{R}}(\text{short})} -2i\hat{e}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{e_{\sigma(j)}}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus e_{\sigma(j)}))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s, \eta \in \Psi^n(\text{long}), \underline{\mathcal{S}} \cap \eta = \emptyset} -\frac{\sqrt{2}}{2} i\hat{e}_j e_{\sigma(j)}(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s; \eta \in \Psi_{\mathbb{R}}^n(\text{long}), \underline{\eta} \subseteq \underline{\mathcal{S}}} -\sqrt{2} i\hat{e}_j e_{\sigma(j)}(H_\eta) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s, \eta \in \Psi_{\mathbb{R}}^n(\text{long}), \underline{\eta} \subseteq \underline{\mathcal{S}}} -\frac{\sqrt{2}}{2} i\hat{e}_j e_{\sigma(j)}(H_\eta) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x). \end{aligned}$$

Proof. By Lemma C.21 the left hand side is equal to

$$\begin{aligned} & \sum_{\substack{\mathcal{S}, s; \sigma(j) \in \underline{\Psi}_{\mathcal{S}, \mathbb{R}}(\text{short}), \\ \gamma = e_{\sigma(j)}}} -2i\hat{e}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\gamma} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\ &+ \sum_{\substack{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{long}), \\ \gamma = e_{\sigma(j)}}} -\sqrt{2} i\hat{e}_j \gamma(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\mathcal{S}, s; \sigma(j) \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{short}), \\ \gamma = e_{\sigma(j)}}} -2i\hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\gamma}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
&+ \sum_{\substack{\mathcal{S}, s; \eta \in \mathcal{S}(\text{long}), \\ \eta \cap (\mathcal{S} \setminus \eta) = \emptyset, \gamma = e_{\sigma(j)}}} -\sqrt{2}i\hat{\epsilon}_j \gamma(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-\Psi_{\mathcal{S} \setminus \eta, \mathbb{R}})(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
&+ \sum_{\substack{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{long}), \\ \eta \subseteq \Psi_{\mathcal{S}, \mathbb{R}}(\text{short}), \gamma = e_{\sigma(j)}}} -\sqrt{2}i\hat{\epsilon}_j \gamma(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x).
\end{aligned}$$

By applying Lemma 8.3 to the second sum after the last equality we get

$$\begin{aligned}
&\sum_{\substack{\mathcal{S}, s; \sigma(j) \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{short}), \\ \gamma = e_{\sigma(j)}}} -2i\hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\gamma}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
&+ \sum_{\substack{\mathcal{S}, s; \eta \in \Psi^n(\text{long}), \\ \eta \cap (\mathcal{S}) = \emptyset}} -\frac{\sqrt{2}}{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) \\
&+ \sum_{\substack{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{long}), \\ \eta \subseteq \Psi_{\mathcal{S}, \mathbb{R}}(\text{short}), \gamma = e_{\sigma(j)}}} -\sqrt{2}i\hat{\epsilon}_j \gamma(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
&= \sum_{\substack{\mathcal{S}, s; \sigma(j) \in \Psi_{\mathcal{S}, \mathbb{R}}(\text{short}), \\ \gamma = e_{\sigma(j)}}} -2i\hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\gamma}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
&+ \sum_{\substack{\mathcal{S}, s; \eta \in \Psi^n(\text{long}), \\ \eta \cap (\mathcal{S}) = \emptyset}} -\frac{\sqrt{2}}{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) \\
&+ \sum_{\substack{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}^n(\text{long}), \\ \eta \subseteq \Psi_{\mathcal{S}, \mathbb{R}}(\text{short})}} -\sqrt{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
&+ \sum_{\substack{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}^c(\text{long}), \\ \eta \subseteq \Psi_{\mathcal{S}, \mathbb{R}}(\text{short})}} -\sqrt{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{s\mathbb{W}^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x).
\end{aligned}$$

By Lemma C.22 the above is equal to

$$\begin{aligned}
 & \sum_{\substack{\mathcal{S},s;\sigma(j)\in\Psi_{\mathcal{S},\mathbb{R}}(short), \\ \gamma=e_{\sigma(j)}}} -2i\hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\gamma} \frac{f_{s,\mathcal{S}}(z)}{\det(x'+z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\
 + & \sum_{\substack{\mathcal{S},s;\eta\in\Psi^n(long), \\ \underline{\eta}\cap(\underline{\mathcal{S}})=\emptyset}} -\frac{\sqrt{2}}{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}\vee\eta}\psi(x) d\mu(x) \\
 + & \sum_{\substack{\mathcal{S},s;\eta\in\Psi_{\mathcal{S},\mathbb{R}}^n(long)\setminus\mathcal{S}, \\ \underline{\eta}\subseteq\underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}}} -2\sqrt{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} \frac{f_{s,\mathcal{S}}(z)}{\det(x'+z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\
 + & \sum_{\substack{\mathcal{S},s;\eta\in\mathcal{S}(long), \\ \underline{\eta}\subseteq\underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}}} -\sqrt{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} \frac{f_{s,\mathcal{S}}(z)}{\det(x'+z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x).
 \end{aligned}$$

Now we apply Lemmas C.27, C.28, 8.3 and obtain

$$\begin{aligned}
 & \sum_{\substack{\mathcal{S},s;\sigma(j)\in\Psi_{\mathcal{S},\mathbb{R}}(short), \\ \gamma=e_{\sigma(j)}}} -2i\hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\gamma} \frac{f_{s,\mathcal{S}}(z)}{\det(x'+z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \gamma))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\
 + & \sum_{\substack{\mathcal{S},s;\eta\in\Psi^n(long), \\ \underline{\eta}\cap(\underline{\mathcal{S}})=\emptyset}} -\frac{\sqrt{2}}{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}\vee\eta}\psi(x) d\mu(x) \\
 + & \sum_{\substack{\mathcal{S},s;\eta\in\Psi_{\mathcal{S},\mathbb{R}}^n, \\ \underline{\eta}\subseteq\underline{\mathcal{S}}}} -2\sqrt{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_\eta) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}\vee\eta}\psi(x) d\mu(x) \\
 + & \sum_{\substack{\mathcal{S},s;\eta\in\mathcal{S}(long), \\ \underline{\eta}\subseteq\underline{\Psi_{\mathcal{S},\mathbb{R}}(short)}}} -\sqrt{2}i\hat{\epsilon}_j e_{\sigma(j)}(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} \frac{f_{s,\mathcal{S}}(z)}{\det(x'+z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x),
 \end{aligned}$$

which, by the previous argument, coincides with the right hand side. ■

Lemma C.27. *Let $\eta \in \mathcal{S}(long)$ with $\underline{\eta} \cap (\underline{\mathcal{S}} \setminus \eta) = \emptyset$. Then*

$$\begin{aligned}
 & \int_{\mathfrak{h}_{\mathcal{S}}} (H_\eta)(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\
 & = 2\sqrt{2} \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} \frac{f_{s,\mathcal{S}}(z)}{\det(x'+z)_{sW^{b'}}} \mathcal{A}(-\Psi_{\mathcal{S}\setminus\eta,\mathbb{R}})(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x).
 \end{aligned}$$

Proof. The left hand side is equal to

$$\begin{aligned}
 & \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} \int_{\mathbb{R}} \partial(H_\eta)(F_{s,x',\mathcal{S}}(z(tH_\eta+x)) \mathcal{H}_{\mathcal{S}}\psi(tH_\eta+x)) dt d\mu(x) \sqrt{2} \\
 & = 2 \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\eta} \frac{f_{s,\mathcal{S}}(z(x))}{\det(x'+z(x))_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S}\setminus\eta,\mathbb{R}}))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \sqrt{2},
 \end{aligned}$$

which coincides with the right hand side. ■

Lemma C.28. *Let $\eta \in \Psi_{\mathcal{S}, \mathbb{R}}^n$ with $\underline{\eta} \subseteq \underline{\mathcal{S}}$. Then*

$$\int_{\mathfrak{h}_{\mathcal{S}}} \partial(iH_{\eta})(F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) = -2\sqrt{2}i\epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \vee \eta}\psi(x) d\mu(x).$$

Proof. The left hand side is equal to

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \int_{\mathbb{R}} \partial(iH_{\eta})(F_{s,x',\mathcal{S}}(z(iH_{\eta} + x))\mathcal{H}_{\mathcal{S}}\psi(itH_{\eta} + x)) dt d\mu(x)\sqrt{2} \\ &= - \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z) \langle \mathcal{H}_{\mathcal{S}}\psi \rangle_{iH_{\eta}} d\mu(x)\sqrt{2} \\ &= - \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z) 2i\epsilon(\Psi, \mathcal{S}, \eta)\mathcal{H}_{\mathcal{S} \vee \eta}\psi(x) d\mu(x)\sqrt{2}, \end{aligned}$$

which coincides with the right hand side. ■

Corollary C.29. *The following equality holds*

$$\begin{aligned} & \sum_{\mathcal{S}, s; \eta \in \mathcal{S}(long), \sigma(j) \in \underline{\eta}, \underline{\eta} \cap (\underline{\mathcal{S}} \setminus \eta) = \emptyset} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S}, s; \eta \in \Psi^n(long), \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset} -i\hat{\epsilon}_j \frac{\sqrt{2}}{2} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \vee \eta}\psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \underline{\eta} \subseteq \underline{\mathcal{S}}} i\hat{\epsilon}_j \sqrt{2} e_{\sigma(j)}(H_{\eta}) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \vee \eta}\psi(x) d\mu(x). \end{aligned}$$

Proof. If $iH_{\eta} = J_a + \delta J_b$ and $iH_{\eta'} = J_a - \delta J_b$, where $\delta = \pm 1$, then for $c \in \{a, b\}$,

$$J_c = \frac{1}{2}(J_c^*(iH_{\eta})iH_{\eta} + J_c^*(iH_{\eta'})iH_{\eta'}).$$

Hence, if $\eta \in \mathcal{S}$, $\underline{\eta} \cap \underline{\mathcal{S}} \setminus \eta = \emptyset$ and $c = \sigma(j)$, then

$$\partial(-\hat{\epsilon}_j J_{\sigma(j)}) = -\frac{\hat{\epsilon}_j}{2} J_{\sigma(j)}^*(iH_{\eta})i\partial(H_{\eta}) - \frac{\hat{\epsilon}_j}{2} J_{\sigma(j)}^*(iH_{\eta'})\partial(iH_{\eta'}).$$

Thus Lemmas C.27 and C.28 show that the left hand side is equal to

$$\begin{aligned} & \sum_{\substack{\mathcal{S}, s, \eta \in \mathcal{S}(long), \\ \underline{\eta} \cap (\underline{\mathcal{S}} \setminus \eta) = \emptyset}} -i\hat{\epsilon}_j \sqrt{2} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z(x))_{s\mathbb{W}b'}} \mathcal{A}(-\Psi_{\mathcal{S} \setminus \eta, \mathbb{R}})(x)\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \underline{\eta} \subseteq \underline{\mathcal{S}}} i\hat{\epsilon}_j \sqrt{2} e_{\sigma(j)}(H_{\eta}) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \vee \eta}\psi(x) d\mu(x). \end{aligned}$$

Lemma 8.3 shows that the first sum coincides with the first sum on the right hand side of the equation of the corollary. ■

We see from Corollary C.19, Lemma C.27 and Corollary C.29 that (C.8) is equal to

$$\sum_{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S}, \mathbb{R}}^n, \underline{\eta} \subseteq \underline{\mathcal{S}}} -i\sqrt{2}e_{\sigma(j)}(H_\eta)\epsilon(\Psi, \mathcal{S}, \eta)\hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S} \vee \eta}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x).$$

But Lemma C.20 shows that the above sum is zero. Hence, (C.9) follows.

The pairs $(G, G') = (O_{2p, 2q}, Sp_{2n'}(\mathbb{R}))$ **with** $n' \leq n = p + q$. Here

$$\Psi^n = \{e_a \pm e_b \mid 1 \leq a \leq p < b \leq n\}$$

and the strongly orthogonal set \mathcal{S} looks as follows

$$\{e_{a_1} \pm e_{b_1}, \dots, e_{a_l} \pm e_{b_l}, e_{a_{l+1}} + \delta_{l+1}e_{b_{l+1}}, \dots, e_{a_m} + \delta_me_{b_m}\}, \quad (C.78)$$

where $a_i \leq p < b_i$, $\delta_i = \pm 1$, and l could be equal to zero.

Lemma C.30. *Let $c \in \{1, 2, 3, \dots, n\} \setminus \underline{\mathcal{S}}$. Then*

$$\begin{aligned} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(J_c)(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ = \sum_{\eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset, c \in \underline{\eta}} -\frac{i}{\sqrt{2}}e_c(H_\eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x). \end{aligned}$$

Proof. As in (C.10) we check that

$$d\mu(x) = \frac{1}{\sqrt{2}}d\mu\left(-\frac{\eta(x)}{\eta(J_c)}J_c + x\right) \quad (x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{J_c^*}).$$

Also, by Theorem 4.44 in [2],

$$\epsilon(\Psi, \mathcal{S}, \eta) = 1.$$

Hence, the left hand side of the equation we are trying to verify is equal to

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{J_c^*}} \int_{\mathbb{R}} \partial(J_c)(F_{s, x', \mathcal{S}}(z(tJ_c + x)) \mathcal{H}_{\mathcal{S}} \psi(tJ_c + x)) dt d\mu(x) \\ &= - \sum_{\eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset, c \in \underline{\eta}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{J_c^*}} F_{s, x', \mathcal{S}}(z(-\frac{\eta(x)}{\eta(J_c)}J_c + x)) \langle \mathcal{H}_{\mathcal{S}} \psi \rangle_{J_c}(-\frac{\eta(x)}{\eta(J_c)}J_c + x) d\mu(x) \\ &= - \sum_{\eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset, c \in \underline{\eta}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z(x)) \langle \mathcal{H}_{\mathcal{S}} \psi \rangle_{iH_\eta}(x) d\mu(x) \frac{1}{\sqrt{2}}J_c^*(iH_\eta) \\ &= - \sum_{\eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset, c \in \underline{\eta}} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z(x)) i\mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) \frac{1}{\sqrt{2}}J_c^*(iH_\eta). \quad \blacksquare \end{aligned}$$

Corollary C.31. *The following equality holds:*

$$\begin{aligned} & \sum_{\mathcal{S}, s; \sigma(j) \notin \underline{\mathcal{S}}} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S}, s; \eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset} \frac{i\hat{\epsilon}_j}{\sqrt{2}} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S} \cap \mathfrak{h}^{\eta}}} F_{s, x', \mathcal{S}}(z(x)) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x). \end{aligned}$$

Lemma C.32. *Let $\eta \in \Psi_{\mathcal{S}, i\mathbb{R}}^n$, $\underline{\eta} \subseteq \underline{\mathcal{S}}$, $s = \sigma\epsilon \in W(\mathbb{H}_{\mathbb{C}})$. Define $\tilde{\epsilon} \in \mathbb{Z}_2^n \subseteq W(\mathbb{H}_{\mathbb{C}})$ by*

$$\hat{\epsilon}|_{\underline{\eta}} = -1, \quad \hat{\epsilon}|_{\{1, 2, \dots, n\} \setminus \underline{\eta}} = 1. \quad (\text{see (3.9)})$$

Then, if $\sigma(j) \in \underline{\eta}$,

$$\hat{\epsilon}_j \int_{\mathfrak{h}_{\mathcal{S} \cap \mathfrak{h}^{\eta}}} F_{s, x', \mathcal{S}}(z(x)) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) = -((\sigma^{-1} \tilde{\epsilon} \sigma) \epsilon)_j \int_{\mathfrak{h}_{\mathcal{S} \cap \mathfrak{h}^{\eta}}} F_{\tilde{\epsilon} s, x', \mathcal{S}}(z(x)) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x).$$

This is verified the same way as Lemma C.20. Let $\mathcal{S}' = \{\eta \in \mathcal{S} \mid \underline{\eta} \cap \underline{\mathcal{S}} \setminus \eta = \emptyset\}$ and let $\mathcal{S}'' = \mathcal{S} \setminus \mathcal{S}'$, as in (3.17).

Lemma C.33. *Let $c \in \underline{\mathcal{S}}''$. Then*

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}}} \partial(J_c)(F_{s, x', \mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ &= \sum_{\eta \in \Psi_{\mathcal{S}, \mathbb{R}}, \eta(J_c) \neq 0} i\sqrt{2} \eta(-iJ_c) \int_{\mathfrak{h}_{\mathcal{S} \cap \mathfrak{h}^{\eta}}} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sW\mathfrak{b}'}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x). \end{aligned}$$

Proof. The left hand side is equal to

$$\begin{aligned} & i \int_{\mathfrak{h}_{\mathcal{S} \cap \mathfrak{h}^{J_c^*}}} \int_{\mathbb{R}} \partial(-iJ_c)(F_{s, x', \mathcal{S}}(z(t(-iJ_c) + x)) \mathcal{H}_{\mathcal{S}} \psi(t(-iJ_c) + x)) dt d\mu(x) \\ &= -i \sum_{\eta \in \Psi_{\mathcal{S}, \mathbb{R}}, \eta(J_c) \neq 0} \int_{\mathfrak{h}_{\mathcal{S} \cap \mathfrak{h}^{J_c^*}}} \frac{f_{s, \mathcal{S}}(z(-\frac{\eta(x)}{\eta(-iJ_c)}(-iJ_c) + x))}{\det(x' + z(-\frac{\eta(x)}{\eta(-iJ_c)}(-iJ_c) + x))_{sW\mathfrak{b}'}} \\ & \quad \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(-\frac{\eta(x)}{\eta(-iJ_c)}(-iJ_c) + x) \langle -\frac{\eta}{|\eta|} \rangle_{-iJ_c} (-\frac{\eta(x)}{\eta(-iJ_c)}(-iJ_c) + x) \\ & \quad \mathcal{H}_{\mathcal{S}} \psi(-\frac{\eta(x)}{\eta(-iJ_c)}(-iJ_c) + x) d\mu(x) \\ &= -i \sum_{\eta \in \Psi_{\mathcal{S}, \mathbb{R}}, \eta(J_c) \neq 0} \int_{\mathfrak{h}_{\mathcal{S} \cap \mathfrak{h}^{\eta}}} \frac{f_{s, \mathcal{S}}(z(x))}{\det(x' + z(x))_{sW\mathfrak{b}'}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \\ & \quad (-2)\eta(-iJ_c) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \frac{1}{\sqrt{2}}, \end{aligned}$$

which is equal to the right hand side. ■

As in the case of the pair $(O_{2p+1,2q}, \text{Sp}_{2n'}(\mathbb{R}))$, we verify the following two lemmas.

Lemma C.34. *Let $c \in \underline{\mathcal{S}}''$, $\eta \in \Psi_{\mathcal{S},\mathbb{R}} \cap \Psi^c$ and $c \in \underline{\eta}$. Then, there is an element $\tilde{s} \in W(\mathfrak{H}_{\mathcal{S}})$ such that*

$$\eta \circ \tilde{s} \in \Psi_{\mathcal{S},\mathbb{R}} \cap \Psi^n \quad \text{and} \quad c \in \underline{\eta \circ \tilde{s}}.$$

Moreover, for $x \in \mathfrak{h}_{\mathcal{S}}$,

$$\begin{aligned} \frac{f_{s,\mathcal{S}}(z(\tilde{s}.x))}{\det(x' + z(\tilde{s}.x))_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(\tilde{s}.x) \mathcal{H}_{\mathcal{S}}\psi(\tilde{s}.x) \\ = \frac{f_{\tilde{s}s,\mathcal{S}}(z(x))}{\det(x' + z(x))_{\tilde{s}sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta \circ \tilde{s}))(x) \mathcal{H}_{\mathcal{S}}\psi(x). \end{aligned}$$

Lemma C.35. *Let \mathcal{S} be as in (C.78). Let $1 < j \leq l$ and let $\eta = e_{a_1} \pm e_{b_j}$, $\zeta = e_{b_1} - e_{b_j}$. Then*

$$\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta} = \mathfrak{h}_{\mathcal{S} \circ s_{\zeta}} \cap \mathfrak{h}^{\eta}$$

and, for $x \in \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}$,

$$\mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) = \mathcal{A}(-\Psi_{\mathcal{S} \circ s_{\zeta} \setminus \eta, \mathbb{R}})(x) \epsilon(\Psi, \mathcal{S} \circ s_{\zeta} \setminus \eta, \eta).$$

Corollary C.36. *The following equality holds:*

$$\begin{aligned} & \sum_{\mathcal{S}, s; \sigma(j) \in \underline{\mathcal{S}}''} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{e}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S},\mathbb{R}}^n, \underline{\eta} \subseteq \underline{\mathcal{S}}} -\hat{e}_j i \sqrt{2} e_{\sigma(j)}(H_{\eta}) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S},\mathbb{R}}^n, \underline{\eta} \subseteq \underline{\mathcal{S}}} -i \frac{\sqrt{2}}{2} e_{\sigma(j)}(H_{\eta}) \epsilon(\Psi, \mathcal{S}, \eta) \hat{e}_j \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x). \end{aligned}$$

Proof. By Lemma C.33, the left hand side is equal to

$$\sum_{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S},\mathbb{R}}^n, \sigma(j) \in \underline{\mathcal{S}}''} -\hat{e}_j i \sqrt{2} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x).$$

Lemma C.34 shows that the above is equal to

$$\begin{aligned} & \sum_{\mathcal{S}, s; \eta \in \Psi_{\mathcal{S},\mathbb{R}}^n \setminus \mathcal{S}''} -2\hat{e}_j i \sqrt{2} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s; \eta \in \mathcal{S}''} -\hat{e}_j i \sqrt{2} e_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} \frac{f_{s,\mathcal{S}}(z)}{\det(x' + z)_{sW^{b'}}} \mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}} \setminus \eta))(x) \mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x). \end{aligned}$$

By Lemma C.36 this coincides with

$$\begin{aligned} & \sum_{\mathcal{S}, s; \eta \in \mathcal{S}''} -2\hat{\epsilon}_j i \sqrt{2} e_{\sigma(j)}(H_\eta) \epsilon(\Psi, \mathcal{S} \setminus \eta, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sW^{\mathfrak{h}'}}} \mathcal{A}(-\Psi_{\mathcal{S} \setminus \eta, \mathbb{R}})(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\ & + \sum_{\mathcal{S}, s; \eta \in \mathcal{S}''} -\hat{\epsilon}_j i \sqrt{2} e_{\sigma(j)}(H_\eta) \epsilon(\Psi, \mathcal{S} \setminus \eta, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \frac{f_{s, \mathcal{S}}(z)}{\det(x' + z)_{sW^{\mathfrak{h}'}}} \mathcal{A}(-\Psi_{\mathcal{S} \setminus \eta, \mathbb{R}})(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x). \end{aligned}$$

This is equal to the right hand side by Lemma 8.3. \blacksquare

Lemma C.37. *Suppose $\eta \in \Psi_{\mathcal{S}, \mathbb{R}}^n$ and $\underline{\eta} \subseteq \underline{\mathcal{S}}$. Then*

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}}} \partial(iH_\eta)(F_{s, x', \mathcal{S}}(z(x)) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\ & = \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} -2\sqrt{2} i \epsilon(\Psi, \mathcal{S}, \eta) F_{s, x', \mathcal{S}}(z(x)) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x). \end{aligned}$$

Proof. The left had side is equal to

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \int_{\mathbb{R}} \partial(iH_\eta)(F_{s, x', \mathcal{S}}(z(itH_\eta + x)) \mathcal{H}_{\mathcal{S}} \psi(itH_\eta + x)) dt d\mu(x) \sqrt{2} \\ & = - \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z(x)) \langle \mathcal{H}_{\mathcal{S}} \psi \rangle_{iH_\eta}(x) d\mu(x) \sqrt{2} \\ & = - \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} F_{s, x', \mathcal{S}}(z(x)) i2\epsilon(\Psi, \mathcal{S}, \eta) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x) \sqrt{2}, \end{aligned}$$

which is equal to the right hand side. \blacksquare

Lemma C.38. *Suppose $\eta \in \mathcal{S}'$. Then,*

$$\int_{\mathfrak{h}_{\mathcal{S}}} \partial(H_\eta)(F_{s, x', \mathcal{S}}(z(x)) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) = \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \sqrt{2} F_{s, x', \mathcal{S} \setminus \eta}(z(x)) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x).$$

Proof. Since $\tilde{\kappa}(H_\eta, H_\eta)^{1/2} = \sqrt{2}$, the left had side is equal to

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \int_{\mathbb{R}} \partial(H_\eta)(F_{s, x', \mathcal{S}}(z(tH_\eta + x)) \mathcal{H}_{\mathcal{S}} \psi(tH_\eta + x)) dt d\mu(x) \sqrt{2} \\ & = - \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \frac{f_{s, \mathcal{S}}(z(x))}{\det(x' + z(x))_{sW^{\mathfrak{h}'}}} \mathcal{A}(-(\Psi_{\mathcal{S}, \mathbb{R}} \setminus \eta))(x) \langle -\frac{\eta}{|\eta|} \rangle_{H_\eta}(x) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \sqrt{2} \\ & = - \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^\eta} \frac{1}{2} F_{s, x', \mathcal{S} \setminus \eta}(z(x)) (-2) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \sqrt{2}, \end{aligned}$$

which is equal to the right hand side. \blacksquare

For some $a < b$, let $\eta^+ = e_a + e_b$ and $\eta^- = e_a - e_b$.

Lemma C.39. *Suppose $\eta^+ \in \mathcal{S}$, $\eta^- \notin \mathcal{S}$, $\underline{\eta}^+ = \underline{\eta}^-$, $\sigma(j) \in \underline{\eta}^+$. Then*

$$-\hat{\epsilon}_j J_{\sigma(j)} = \frac{\hat{\epsilon}_j}{2i} (H_{\eta^+} + J_{\sigma(j)}^*(H_{\eta^-})iH_{\eta^-})$$

and

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z(x))\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= \frac{\hat{\epsilon}_j}{\sqrt{2}i} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^+}} F_{s,x',\mathcal{S} \setminus \eta^+}(z(x))\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x) \\ &+ \hat{\epsilon}_j \sqrt{2}i\eta^- (-iJ_{\sigma(j)})\epsilon(\Psi, \mathcal{S}, \eta^-) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^-}} F_{s,x',\mathcal{S}}(z(x))\mathcal{H}_{\mathcal{S} \setminus \eta^-}\psi(x) d\mu(x). \end{aligned}$$

Proof. This is clear from Lemmas C.37 and C.38. ■

Similarly, we have the following lemma.

Lemma C.40. *Suppose $\eta^- \in \mathcal{S}$, $\eta^+ \notin \mathcal{S}$, $\underline{\eta}^+ = \underline{\eta}^-$, $\sigma(j) \in \underline{\eta}^+$. Then*

$$-\hat{\epsilon}_j J_{\sigma(j)} = -\frac{\hat{\epsilon}_j}{2} (iH_{\eta^+} - J_{\sigma(j)}^*(H_{\eta^-})H_{\eta^-})$$

and

$$\begin{aligned} & \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z(x))\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= \hat{\epsilon}_j i\sqrt{2}\epsilon(\Psi, \mathcal{S}, \eta^+) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^+}} F_{s,x',\mathcal{S}}(z(x))\mathcal{H}_{\mathcal{S} \setminus \eta^+}\psi(x) d\mu(x) \\ &\quad - \frac{\hat{\epsilon}_j}{\sqrt{2}}i\eta^- (-iJ_{\sigma(j)}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^-}} F_{s,x',\mathcal{S} \setminus \eta^-}(z(x))\mathcal{H}_{\mathcal{S}}\psi(x) d\mu(x). \end{aligned}$$

Corollary C.41. *The following equality holds:*

$$\begin{aligned} & \sum_{\mathcal{S}, s; \sigma(j) \in \underline{\mathcal{S}}'} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S}}\psi(x)) d\mu(x) \\ &= \sum_{\mathcal{S}, s; \eta \in \Psi^n, \underline{\eta} \cap \underline{\mathcal{S}} = \emptyset} -\frac{\hat{\epsilon}_j}{\sqrt{2}}ie_{\sigma(j)}(H_{\eta}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \setminus \eta}\psi(x) d\mu(x) \\ &+ \sum_{\mathcal{S}, s; \eta \in \Psi^n, \underline{\eta} \subseteq \underline{\mathcal{S}}} \hat{\epsilon}_j i\sqrt{2}e_{\sigma(j)}(H_{\eta})\epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z)\mathcal{H}_{\mathcal{S} \setminus \eta}\psi(x) d\mu(x). \end{aligned}$$

Proof. The left hand side is equal to

$$\begin{aligned}
 & \sum_{\mathcal{S},s;\eta^+ \in \mathcal{S}, \eta^- \notin \mathcal{S}, \sigma(j) \in \underline{\eta}^+} \frac{\hat{\epsilon}_j}{\sqrt{2}i} \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^+}} F_{s,x',\mathcal{S} \setminus \eta^+}(z) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
 + & \sum_{\mathcal{S},s;\eta^+ \in \mathcal{S}, \eta^- \notin \mathcal{S}, \sigma(j) \in \underline{\eta}^+} i\hat{\epsilon}_j \sqrt{2} \eta^- (-iJ_{\sigma(j)}) \epsilon(\Psi, \mathcal{S}, \eta^-) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^-}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta^-} \psi(x) d\mu(x) \\
 + & \sum_{\mathcal{S},s;\eta^+ \notin \mathcal{S}, \eta^- \in \mathcal{S}, \sigma(j) \in \underline{\eta}^-} i\hat{\epsilon}_j \sqrt{2} \epsilon(\Psi, \mathcal{S}, \eta^+) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^+}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta^+} \psi(x) d\mu(x) \\
 + & \sum_{\mathcal{S},s;\eta^+ \notin \mathcal{S}, \eta^- \in \mathcal{S}, \sigma(j) \in \underline{\eta}^-} \frac{\hat{\epsilon}_j}{\sqrt{2}} i \eta^- (-iJ_{\sigma(j)}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta^-}} F_{s,x',\mathcal{S} \setminus \eta^-}(z) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
 = & \sum_{\mathcal{S},s;\eta \in \mathcal{S}', \sigma(j) \in \underline{\eta}} -\frac{\hat{\epsilon}_j}{\sqrt{2}} i \eta (-iJ_{\sigma(j)}) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S} \setminus \eta}(z) \mathcal{H}_{\mathcal{S}} \psi(x) d\mu(x) \\
 + & \sum_{\mathcal{S},s;\eta \in \Psi_{\mathcal{S},\mathbb{R}}^n, \sigma(j) \in \underline{\eta} \subseteq \mathcal{S}} \hat{\epsilon}_j i \sqrt{2} \eta (-iJ_{\sigma(j)}) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x),
 \end{aligned}$$

which is equal to the right hand side. ■

Notice that we may rewrite the quantity (C.6) as $\tilde{D} = D_1 + D_2 + D_3$, where

$$\begin{aligned}
 D_1 &= \sum_{\mathcal{S},s;\sigma(j) \notin \underline{\mathcal{S}}} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\
 D_2 &= \sum_{\mathcal{S},s;\sigma(j) \in \underline{\mathcal{S}}''} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x) \\
 D_3 &= \sum_{\mathcal{S},s;\sigma(j) \in \underline{\mathcal{S}}'} \int_{\mathfrak{h}_{\mathcal{S}}} \partial(-\hat{\epsilon}_j J_{\sigma(j)})(F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S}} \psi(x)) d\mu(x).
 \end{aligned}$$

Thus, we see from Corollaries C.31, C.36 and C.41 that

$$\begin{aligned}
 & D_1 + D_2 + D_3 \\
 = & \sum_{\mathcal{S},s;\eta \in \Psi_{\mathcal{S},\mathbb{R}}^n, \underline{\eta} \subseteq \mathcal{S}} -\hat{\epsilon}_j i \frac{\sqrt{2}}{2} e_{\sigma(j)}(H_{\eta}) \epsilon(\Psi, \mathcal{S}, \eta) \int_{\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\eta}} F_{s,x',\mathcal{S}}(z) \mathcal{H}_{\mathcal{S} \vee \eta} \psi(x) d\mu(x),
 \end{aligned}$$

which, by Lemma C.32, is zero. This verifies (C.9).

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