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CHARACTERS, DUAL PAIRS, AND UNITARY REPRESENTATIONS

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0. Introduction. The main inspiration for this work is an open problem of constructing irreducible unitary representations of classical groups, attached to nilpotent coadjoint orbits. At present, it is not clear what the word "attached" means. We would like to suggest an approach motivated by Howe's description of the oscillator representation ω in terms of the Weyl transform and the Cayley transform [H2] (see 1.7). This description implies immediately a "Cayley-Kirillov-Rossmann"-type character formula for each of the irreducible pieces ω_+, ω_- of ω, where the Fourier transform of character is supported on the closure of a single nilpotent coadjoint orbit [P1, (5.4), (6.7)]. Thus the representations ω_+ and ω_- are attached to this orbit in a classical, easily acceptable way. This phenomenon persists for a number of other irreducible unitary representations of classical groups [P1] (see 6.13), but we do not follow this (thorny) path in this work. Instead, we concentrate on the associated varieties and the wave front sets.

Let W be a symplectic space over $\mathbf{R}$ and let G, G' be an irreducible dual pair in the symplectic group $Sp = Sp(W)$, [H7]. Let $\mathfrak{g}$, $\mathfrak{g}'$ denote the Lie algebras of G, G' respectively. There are canonical moment maps (see (2.6))

$$(0.1)\qquad \tau_{\mathfrak{g}}\colon W \to \mathfrak{g}^*, \qquad \tau_{\mathfrak{g}'}\colon W \to \mathfrak{g}'^*,$$

which intertwine the actions of G, G' on W with the coadjoint actions on $\mathfrak{g}^*$, $\mathfrak{g}'^*$ respectively. Here is an interesting and easily verifiable [H9] property of these maps:

$$\tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(\text{a nilpotent coadjoint orbit in } \mathfrak{g}^*))$$

$$(0.2)\qquad = \text{union of nilpotent coadjoint orbits in } \mathfrak{g}'^*.$$

The maps (0.1) extend canonically to the complexifications

$$(0.3)\qquad \tau_{\mathfrak{g}}\colon W_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}^*, \qquad \tau_{\mathfrak{g}'}\colon W_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}'^*$$

and intertwine the appropriate actions of the complexified algebraic groups $G_{\mathbf{C}}$, $G'_{\mathbf{C}}$. The first fundamental theorem of the classical invariant theory asserts that $\tau_{\mathfrak{g}}$, $\tau_{\mathfrak{g}'}$ are quotient maps (under $G_{\mathbf{C}}$, $G'_{\mathbf{C}}$), in the sense of algebraic geometry [KP1, 2.2],

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[KP2, 1.2]. The orbit structure of these maps (0.3) is well known [KP1], [KP2]. In particular, we have the following result.

THEOREM 0.4 [DKP]. *Let $O \subseteq \mathfrak{g}_{\mathbf{C}}^* \cap \tau_{\mathfrak{g}}(W_{\mathbf{C}})$ be a nilpotent coadjoint orbit. Then there is a unique coadjoint orbit $O' \subseteq \mathfrak{g}_{\mathbf{C}}'^*$ such that*

$$\tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(\bar{O})) = \overline{O'}.$$

Here the "$^-$" stands for the closure.

Let us recall Howe's correspondence [H1]. Let ω be an oscillator representation of the metaplectic group $\widetilde{Sp}$ on a Hilbert space H_ω. For a reductive subgroup $E \subseteq Sp$, let $\tilde{E}$ be the preimage of E in $\widetilde{Sp}$, under the covering map $\widetilde{Sp} \to Sp$. Denote by $\mathscr{R}(E, \omega)$ the set of infinitesimal equivalence classes of continuous irreducible representations on a locally convex space, which can be realized as quotients of H_ω^∞ by $\omega^\infty(\tilde{E})$-invariant closed subspaces. Since the groups $\tilde{G}$, $\tilde{G}'$ commute with one another (in $\widetilde{Sp}$), one can express elements of $\mathscr{R}(G \cdot G', \omega)$ as $\Pi \otimes \Pi'$, where $\Pi \in \mathscr{R}(G, \omega)$ and $\Pi' \in \mathscr{R}(G', \omega)$. Howe's theorem follows.

HOWE'S DUALITY THEOREM 0.5 [H1]. *The set $\mathscr{R}(G \cdot G', \omega)$ is the graph of bijection between all of $\mathscr{R}(G, \omega)$ and all of $\mathscr{R}(G', \omega)$. Moreover, an element $\Pi \otimes \Pi' \in \mathscr{R}(G \cdot G', \omega)$ occurs as a quotient of ω^∞ in a unique way.*

The bijection $\Pi \to \Pi'$ defined by this theorem is called Howe's correspondence [MVW].

Judging from the title of [H1], one is led to believe that Howe's correspondence should be compatible with the maps (0.1) and (0.3). Our first result in this direction is Corollary 2.8, which says that

$$WF(\Pi) \subseteq \tau_{\mathfrak{g}}(W) \qquad (\Pi \in \mathscr{R}(G, \omega)). \tag{0.6}$$

Here $WF(\Pi)$ stands for the wave front set of the distribution character Θ_Π of Π at the identity [H5]. We include $0 \in \mathfrak{g}^*$ in the $WF(\Pi)$. Notice that the notion of the wave-front set depends on the choice of a character of the additive group $\mathbf{R}$, and so does the oscillator representation ω. We assume that both ω and the Fourier transform are associated to the same character χ (see 1.7 and (2.5)).

In 7.10 we show that

$$WF(\Pi') = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(WF(\Pi)))\ (= \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(0)) = \text{the closure of a single orbit}) \tag{0.7}$$

if the pair G, G' is in the stable range with G the smaller member ([8], [Li2]), $\Pi \in \mathscr{R}(G, \omega)$ is unitary, and $dim\ \Pi < \infty$. A part of this result was previously shown in [P1, (8.2)].

The equation (0.7) is not true is general. (Consider the pair $GL(1, \mathbf{R})$, $GL(2, \mathbf{R})$. Here $WF(\Pi) = \{0\}$ for any $\Pi \in \mathscr{R}(G, \omega)$. However, the corresponding representation $\Pi' \in \mathscr{R}(G', \omega)$ may be either finite-dimensional or infinite-dimensional. In the first case $WF(\Pi') = \{0\}$, and in the second one $WF(\Pi') \neq \{0\}$.)

A detailed comparison of the two sides of (0.7) in general seems to be a formidable, but not hopeless, task. Let I_Π denote the annihilator of the Harish-Chandra module of Π in the universal enveloping algebra of $\mathfrak{g}$. We define $I_{\Pi'}$ similarly. Let us consider the associated varieties $Ass(I_\Pi) \subseteq \mathfrak{g}_{\mathbf{C}}^*$ and $Ass(I_{\Pi'}) \subseteq \mathfrak{g}_{\mathbf{C}}'^*$ ([V2], [Ma]) instead of the wave front sets. By a theorem of Borho, Brylinski, and Joseph [V2], $Ass(I_\Pi)$ is the closure of a single nilpotent coadjoint orbit. Thus in view of Theorem 0.4 one may investigate the equation

$$(0.8) \qquad Ass(I_{\Pi'}) = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(Ass(I_\Pi)))$$

for a given representation $\Pi \otimes \Pi' \in \mathscr{R}(G \cdot G', \omega)$. Again, there are plenty of easy counterexamples. We prove the equation (0.8) under some very general assumptions. Curiously, unitarity seems to play a role in the argument. One should remark here that the variety on the left-hand side of (0.8) is always contained in the one on the right-hand side, [P1, (7.1)].

We recall some notation. The groups G, G' act on their defining modules V, V', which are finite-dimensional vector spaces over a division algebra $\mathbf{D}$, over $\mathbf{R}$ (see Sec. 8 and Sec. 9). Let $d' = dim\ V'$. We attach a number r (4.10) to the Lie algebra $\mathfrak{g}$ and a number ι (5.7) to the algebra $\mathbf{D}$. Let $\mu_\kappa \in S^*(\mathfrak{g}^*)$ be the Fourier transform of the lowest term in the asymptotic expansion of the distribution character Θ_Π of Π (see (2.5) and (5.10)). Denote by $max\,sup\,\mu_\kappa \subseteq \mathfrak{g}^*$ the union of orbits of maximal dimension in $supp\,\mu_\kappa$. In the discussion preceding 7.9, we define an open set $W_{\mathfrak{g}\mathfrak{g}} \subseteq W$ on which the map $\tau_{\mathfrak{g}}$ (0.1) is submersive. Let G_1 denote the Zariski-identity component of G. If $G \neq G_1$, then G is an orthogonal group over $\mathbf{D} = \mathbf{R}$ or $\mathbf{C}$. In this case the determinant character det of G may be viewed as a character of $\tilde{G}$, with the kernel equal to $\tilde{G}_1$. The following theorem is the intersection of the main results of this paper. We shall prove it in the last section, Section 10.

THEOREM 0.9. *Let Π be an irreducible unitary representation of $\tilde{G}$. Suppose the character Θ_Π has the rate of growth $\gamma \geqslant 0$ (see Sec. 4) such that*

$$\text{(a)} \qquad d' > \gamma(r-1) + r - \iota .$$

Further, assume that there is a vector u in the Harish-Chandra module of Π such that

(b)

$$\int_{\tilde{G}} (u, \Pi(g)u)(\omega(g)v, v)\, dg \geqslant 0 \quad v \in H_\omega^\infty, \text{ and this integral is nonzero for some } v.$$

(It follows from (a) *that the function under the integral* (b) *is absolutely integrable.) Then $\Pi \in \mathscr{R}(G, \omega)$ and the corresponding representation $\Pi' \in \mathscr{R}(G', \omega)$ (via 0.5) is unitary.*

Moreover, if

(c) $$max\,supp\ \mu_\kappa \cap \tau_{\mathfrak{g}}(W_{\mathfrak{g}\mathfrak{g}}) \neq 0,$$

then (0.8) *holds, except (possibly?) the case when* $G \neq G_1$ *and* $\Pi|_{\tilde{G}_1}$ *is irreducible. In this case* (0.8) *holds with* $Ass(I_{\Pi'})$ *replaced by* $Ass(I_{\Pi'}) \cup Ass((I_{(\Pi \otimes det)'})$.

The nonvanishing of the integral 0.9(b) for some $v \in H^\infty_\omega$ forces Π to be a genuine representation of $\tilde{G}$, in the sense that the restriction of Π to the kernel of the covering map $\tilde{G} \to G$ (a two element group) is a multiple of the unique nontrivial character of this kernel.

For such Π the positivity 0.9(b) holds for pairs of type II without any further assumptions (see 9.2). For pairs of type I it holds in the stable range, with G the smaller member [Li1] (8.1, 8.6), and in many cases well beyond the stable range (see 8.6, 8.7).

In the stable range, with G the smaller member, all three conditions (a), (b), and (c) are satisfied.

The representations Π, Π' (in the stable range) have been studied by Howe [H3], [H6] via his theory of rank. This theory has been completed and shown to be compatible with Howe's correspondence by Scaramuzzi [Sc] and Li [Li1], [Li2]. In all this work, Mackey's theory of unitary induction plays a crucial role. In our approach we do not use it at all.

A reader interested only in preservation of unitarity under Howe's correspondence is referred to Theorems 3.1, 8.9, 9.3, and Example 8.10. The proof of these results occupies a small part of this paper (Sections 3, 8, and 9). Most of the effort is spent on calculating the associated varieties and wave-front sets (Theorem 7.9 and Corollary 7.10). The main reason for success is Howe's duality theorem, which (via Theorem 3.1) enables us to express Π' in terms of the character of Π.

1. The oscillator representation, Weyl transform, and Gaussian functions. We begin by recalling some results about the oscillator representation, in a form suitable for our applications. Our main references are [H2], [H3], [H4].

Let W be a finite-dimensional vector space over the reals, with a nondegenerate symplectic form $\langle\ ,\ \rangle$. Fix a unitary character χ of the additive group $\mathbf{R}$, $\chi(x) = exp(2\pi i x)$, $x \in \mathbf{R}$, and a Lebesgue measure dw on W. For $\phi_1, \phi_2 \in S(W)$ define a product $\phi_1 \natural \phi_2 \in S(W)$ by the formula

$$\phi_1 \natural \phi_2(w') = \int_W \phi_1(w)\phi_2(w' - w)\chi\left(\frac{1}{2}\langle w, w'\rangle\right) dw. \tag{1.1}$$

Then $(S(W), \natural)$ is an associative algebra.

We embed $S(W)$ into the space $S^*(W)$ of tempered distributions on W by

$$f(\phi) = \int_W f(w)\phi(w)\,dw \qquad (f, \phi \in S(W)). \tag{1.2}$$

The symplectic group $Sp = Sp(W) = Sp(W, \langle\ ,\ \rangle)$ acts on $S(W)$ by algebra automorphisms as follows

$$\omega_{1,1}(g)\phi(w) = \phi(g^{-1}(w)) \qquad (w \in W, g \in Sp, \phi \in S(W)). \tag{1.3}$$

By dualizing (1.3) we obtain an action of Sp on $S^*(W)$

$$\omega_{1,1}(g)f(\phi) = f(\omega_{1,1}(g)^{-1}(\phi)), \tag{1.4}$$

where $g \in Sp$, $f \in S^*(W)$, $\phi \in S(W)$. The formula (1.2) implies that the action (1.4) is an extension of the action (1.3) from $S(W)$ to $S^*(W)$.

Let $\omega = \omega_\chi$ be the oscillator representation of the metaplectic group $\widetilde{Sp}$ attached to the character χ, [H2]. Let us choose a realization of ω on a Hilbert space H. Denote by H^∞ the space of smooth vectors in H and by $H^{\infty*}$ the linear topological dual of H^∞. The symbols $B(H)$, $H.S.(H)$, and $Hom(H^\infty, H^{\infty*})$ will stand respectively for the space of bounded operators on H, Hilbert Schmidt operators on H, and continuous linear maps from H^∞ to $H^{\infty*}$.

We recall a version of the classical Stone-von Neumann theorem [H4], combined with the Schwartz kernel theorem [Hö]:

THEOREM 1.5. *There is an algebra homomorphism*

$$\rho\colon S(W) \to B(H)$$

which extends to a surjective isometry

$$\rho\colon L^2(W) \to H.S.(H)$$

and even further to a linear bijection

$$\rho\colon S^*(W) \to Hom(H^\infty, H^{\infty*})$$

which has the intertwining property

$$\omega(\tilde{g})\rho(f)\omega(\tilde{g}^{-1}) = \rho(\omega_{1,1}(g)f),$$

where $f \in S^*(W)$ and $\tilde{g} \in \widetilde{Sp}$ is in the preimage of $g \in Sp$.

The map ρ is usually called the Weyl transform.

For $x \in End(W)$, such that $x - 1$ is invertible, define the Cayley transform

$$c(x) = (x + 1)(x - 1)^{-1}. \tag{1.6}$$

Let $sp = sp(W) = sp(W, \langle\ ,\ \rangle) \subset End(W)$ denote the Lie algebra of the symplectic group $Sp \subset End(W)$. Denote by sp^c the intersection of sp with the domain of the Cayley transform c (1.6). A simple argument shows that $c(sp^c) \subset Sp$, [H2].

THEOREM 1.7 (Howe [H2]). *One can choose a function Θ_ω on $\widetilde{Sp}$, representing the distribution character of ω via integration against the Haar measure on $\widetilde{Sp}$, so that for any $x \in sp^c$ and any $\tilde{c}(x) \in \widetilde{Sp}$ in the preimage of $c(x) \in Sp$, and $\phi \in S(W)$*

$$\text{(a)} \qquad \rho^{-1}\omega(\tilde{c}(x))(\phi) = \int_W \Theta_\omega(\tilde{c}(x))\chi\left(\frac{1}{4}\langle x(w), w\rangle\right)\phi(w)\,dw.$$

Moreover, there is a real analytic lifting $\tilde{c}\colon sp^c \to \widetilde{Sp}$ of $c\colon sp^c \to Sp$ such that

$$\text{(b)} \qquad \Theta_\omega(\tilde{c}(x)) = const\,|det(1 - x)|^{1/2} \qquad (x \in sp^c).$$

Thus, with the $\tilde{c}$ as in (b), *we may identify the distribution $\rho^{-1}\omega(\tilde{c}(x)) \in S^*(W)$, $x \in sp^c$, with the function*

$$\text{(c)} \qquad \rho^{-1}\omega(\tilde{c}(x))(w) = const\,|det(1 - x)|^{1/2}\chi\left(\frac{1}{4}\langle x(w), w\rangle\right),$$

of the variable $w \in W$.

Let J be a compatible positive complex structure on W; i.e., $J \in Sp(W)$, $\langle J(w), w\rangle > 0$ for a nonzero $w \in W$, and $J^2 = -I$, where I is the identity. Define an euclidian norm $|\ |$ on W by $|w|^2 = \langle J(w), w\rangle$, $w \in W$, and let

$$\gamma_{I/2}(w) = exp\left(-\frac{\pi}{2}|w|^2\right) \qquad (w \in W)$$

by a Gaussian function, as in [H2].

Denote by U the centralizer of J in Sp. Then U is a maximal compact subgroup of Sp, isomorphic to the unitary group $U(n)$, where $dim\ W = 2n$. The preimage $\tilde{U}$ of U in $\widetilde{Sp}$ is a maximal compact subgroup of $\widetilde{Sp}$, and the restriction of ω to $\tilde{U}$ decomposes into a direct sum of irreducible representations, each occuring with multiplicity (at most) one. These are the $\tilde{U}$-types of ω. The lowest $\tilde{U}$-type, in the sense of Vogan [V1], is one-dimensional. Pick a unit vector v in it and set

$$(1.8) \qquad \Omega(g) = |(\omega(g)v, v)| \qquad (g \in \widetilde{Sp}).$$

For reasons which shall soon become apparent, we shall refer to Ω as to the Gaussian function on the metaplectic group $\widetilde{Sp}$.

Choose a maximal isotropic subspace $Y \subset W$ and a basis $y_1, y_2, \ldots, y_n$ of Y. Let A be the vector part of a split Cartan subgroup of $\widetilde{Sp}$ which acts diagonally on Y, with respect to this basis. Define the characters α_i of A by

$$ay_i = \alpha_i(a)y_i \qquad (a \in A,\ 1 \leqslant i \leqslant n).$$

We assume that $U \cdot A \cdot U$ is a Cartan decomposition of Sp.

PROPOSITION 1.9. *The function Ω is real analytic, strictly positive, and $\tilde{U}$-bi-invariant. Moreover, the formulae*

(a) $$\Omega(g) = |\rho^{-1}\omega(g)\natural\gamma_{I/2}(0)| \qquad (g \in \widetilde{Sp}),$$

(b) $$\Omega(\tilde{c}(x)) = const\,|det(1-x)|^{1/2}\,|det(iI - Jx)|^{-1/2},$$

hold for $\tilde{c}(x) \in \widetilde{Sp}$ in the preimage of $c(x) \in Sp$, $x \in sp^c$, and

(c) $$\Omega(a) = const \prod_{i=1}^{n} (\alpha_i(a) + \alpha_i(a)^{-1})^{-1/2} \qquad (a \in A).$$

Proof. Formula (a) follows from [H2, (13.1b), (1.3.4)] and [H4, (1.7.18)]. Formula (b) follows from (a) and Theorem 1.7. Formula (c) is taken from [H3, page 91]. Since $\tilde{U}$ acts on v (1.8) via multiplication by a character, Ω is $\tilde{U}$-bi-invariant. Hence (c) shows that Ω is real analytic. ∎

Notice that, with the notation of 1.7, formula 1.9(b) may be rewritten as

(1.10) $$\Omega(\tilde{c}(x)) = const\,|\Theta_\omega(\tilde{c}(x))|\,|det(iI - Jx)|^{-1/2} \qquad (x \in sp^c).$$

Since for $x \in sp$ the map $Jx \in End(W)$ is symmetric with respect to the scalar product $\langle J, \ \rangle$ on W, one checks easily that for a fixed norm $|\ |$ on the real vector space sp there are consants C, C' such that

(1.11) $$C(1 + |x|^2) \leq |det(iI - Jx)|^2 \leqslant C'(1 + |x|^2)^{2n} \qquad (x \in sp).$$

Let $\mathfrak{a}$ be the Lie algebra of A. The elements $d\alpha_1, d\alpha_2, \ldots, d\alpha_n$ form a basis of $\mathfrak{a}^*$. Define a norm $|\ |_\Omega$ on the real vector space $\mathfrak{a}$

$$|x|_\Omega = \frac{1}{2}\sum_{i=1}^{n} |d\alpha_i(x)| \qquad (x \in \mathfrak{a}).$$

This norm is invariant under all permutations and sign changes of the $d\alpha_i$'s, i.e., under the Weyl group $W(\widetilde{Sp}, A)$. Thus 1.9(c) implies that there are constants C, C' such that

(1.12) $$C\,exp(-|x|_\Omega) \leqslant \Omega(exp(x)) \leqslant C'\,exp(-|x|_\Omega) \qquad (x \in \mathfrak{a}).$$

PROPOSITION 1.13. *There is a seminorm q on $S(W)$ such that*

(a) $$|\rho^{-1}\omega(g)\natural\phi(0)| \leqslant \Omega(g)q(\phi) \qquad (g \in \widetilde{Sp}, \phi \in S(W)).$$

For any seminorm q on $S(W)$ there is an integer $p \geqslant 0$ and a seminorm q' on $S(W)$

such that

$$\text{(b)} \qquad q(\rho^{-1}\omega(g)\natural\phi) \leqslant \Omega(g)^{-p}q'(\phi) \qquad (g \in \widetilde{Sp}, \phi \in S(W)).$$

Proof. This is an elaboration on Howe's estimates for the matrix coefficients of ω [H3, (8.5)]. Since the function

$$\tilde{U} \times S(W) \times \tilde{U} \ni (k_1, \phi, k_2) \to \rho^{-1}(\omega(k_1)\rho(\phi)\omega(k_2)) \in S(W)$$

is continuous (see [H7, (11.4)] and [Wa, 4.11]) and $\tilde{U}$ is compact, it will suffice to prove the proposition with $\widetilde{Sp}$ replaced by the closed positive Weyl chamber $A^+ = \{a \in A; \alpha_1(a) \geqslant \alpha_2(a) \geqslant \cdots \geqslant \alpha_n(a) \geqslant 1\}$.

Choose a maximal isotropic subspace $X \subset W$ complementary to Y. Consider the Schrödinger model of ω attached to the complete polarization $W = X \oplus Y$, so that the Hilbert space $H = L^2(X)$, [H3]. Then for each $\phi \in S(W)$, $\rho(\phi)$ is an integral kernel operator with kernel $K_\rho(\phi)$ in the Schwartz space $S(X \times X)$. The formula [H3, (1.8)] implies that

$$tr(\omega(a)\rho(\phi)) = \prod_{i=1}^{n} \alpha_i(a)^{-1/2} \int_X K_\rho(\phi)(a^{-1}(x), x)\, dx$$

for $a \in A$ and $\phi \in S(W)$. Here we identify A with the corresponding subgroup of Sp. Let q be a seminorm on $S(W)$ such that

$$\int_X sup_{x' \in X} |K_\rho(\phi)(x', x)|\, dx \leqslant q(\phi) \qquad (\phi \in S(W).$$

Then for $a \in A^+$ and $\phi \in S(W)$

$$|tr(\omega(a)\rho(\phi))| \leqslant \prod_{i=1}^{n} \alpha_i(a)^{-1/2} q(\phi) \leqslant const\, \Omega(a)q(\phi),$$

where the last inequality follows from 1.9(c). This verifies (a).

For part (b) we notice that, by the procedure of doubling [H7, Sec. 11], it will suffice to check that for any seminorm q on H^∞ there is an integer $p \geqslant 0$ and a seminorm q' on H^∞ such that for $a \in A$ and $v \in H^\infty$

$$\text{(b}') \qquad q(\omega(a)v) \leqslant \Omega(a)^{-p}q'(v).$$

In our Schrödinger model, $H^\infty = S(X)$ and

$$\omega(a)v(x) = \prod_{i=1}^{n} \alpha_i(a)^{-1/2} v(a^{-1}(x)) \qquad (a \in A, v \in S(X), x \in X).$$

Let us identify X with $\mathbf{R}^n$ by choosing a basis. Then for some multi-indices η, η', $q(v) = sup_{x \in X}|x^\eta \partial_x^{\eta'} v(x)|$, [Hö, 7.1.2]. Hence

$$q(\omega(a)v) = \prod_{i=1}^{n} \alpha_i(a)^{-1/2+\eta_i-\eta'_i} q(v) \qquad (a \in A^+, v \in S(X)),$$

and (b′) follows from 1.9(c). ∎

2. A Cayley-Bochner theorem for $\tilde{K}$-finite matrix coefficients. Let G, G' be a reductive dual pair in Sp. For any subgroup $E \subset Sp$ denote by $\tilde{E}$ the preimage of E in Sp. Fix a representation $\Pi \otimes \Pi' \in R(G \cdot G', \omega)$ see [H1]. Since a smooth version of $\Pi \otimes \Pi'$ may be realized on a subspace of $H^{\infty *}$ [P2, Proposition 1.2.19], Theorem 1.5 implies that there is a tempered distribution $f = f_{\Pi \otimes \Pi'} = f_\Pi \in S^*(W)$, such that the operator $\rho(f)$: $H^\infty \to H^{\infty *}$ intertwines the restriction of ω^∞ to $\tilde{G}\tilde{G}'$ with that realization of $\Pi \otimes \Pi'$. This distribution f is unique up to a nonzero scalar multiple (see 0.5).

Let K be a maximal compact subgroup of G. Since the Harish-Chandra module of Π is obtained as a quotient of the restriction to $(\mathfrak{g}, \tilde{K})$ of the Harish-Chandra module of the oscillator representation ω of $\widetilde{Sp}$, we see from Theorem 1.5 that for any $\tilde{K}$-finite matrix coefficient μ of Π there is a function $\phi \in S(W)$ such that

$$\mu(g) = f \natural (\rho^{-1}\omega(g) \natural \phi)(0) \qquad (g \in \tilde{G}). \tag{2.1}$$

Clearly, the function on the right-hand side of the formula (2.1) is well defined for all $g \in \widetilde{Sp}$. We shall denote it by μ_ϕ. Thus μ_ϕ is an extension of μ from $\tilde{G}$ to $\widetilde{Sp}$. Proposition 1.13(b) implies that there is an integer $p \geqslant 0$ and a seminorm q on $S(W)$ such that for any $\phi \in S(W)$

$$|\mu_\phi(g)| \leqslant q(\phi)\Omega(g)^{-p} \qquad (g \in \widetilde{Sp}). \tag{2.2}$$

Notice that, since we may assume that $|\Omega| \leqslant 1$, the function on the right-hand side of (2.2) increases with p.

Let us fix a measurable lifting $\tilde{c}$: $sp^c \to \widetilde{Sp}$ of the Cayley transform c: $sp^c \to Sp$. Then the formulas (1.10) and (1.11) show that for p and q as in (2.2)

$$|\Theta_\omega^p(\tilde{c}(x))\mu_\phi(\tilde{c}(x))| \leqslant const\ q(\phi)(1 + |x|^2)^{pn}, \tag{2.3}$$

where $x \in sp^c$ and $\phi \in S(W)$. It is important to notice that for p odd the function $\Theta_\omega^p(\tilde{c}(x))\mu_\phi(\tilde{c}(x))$, $x \in sp^c$, does not depend on the choice of the lifting $\tilde{c}$. It is also clear that $\tilde{c}(\mathfrak{g}^c) \subset \tilde{G}$ and $\tilde{c}(\mathfrak{g}'^c) \subset \tilde{G}'$, where $\mathfrak{g}^c$ stands for the domain of the Cayley transform c in $\mathfrak{g}$, and similarly for $\mathfrak{g}'$. Thus for p odd and where (2.3) holds, we have a well-defined tempered distribution $\tilde{c}^*(\Theta_\omega^p \mu) \in S^*(\mathfrak{g})$:

$$\tilde{c}^*(\Theta_\omega^p \mu)(\psi) = \int_{\mathfrak{g}} \Theta_\omega^p(\tilde{c}(x))\mu(\tilde{c}(x))\psi(x)\, dx, \tag{2.4}$$

where $\psi \in S(\mathfrak{g})$. The notation $\tilde{c}^*(\ )$ (2.4) is consistent with the usual terminology concerning pullbacks of distributions [Hö, 6.1.2].

For a function $\psi \in S(\mathfrak{g})$ define the Fourier transform $\mathscr{F}_{\mathfrak{g}}(\psi) \in S(\mathfrak{g}^*)$:

$$\mathscr{F}_{\mathfrak{g}}(\psi)(\xi) = \int_{\mathfrak{g}} \chi(\xi(x))\psi(x)\,dx \qquad (\xi \in \mathfrak{g}^*). \tag{2.5}$$

The adjoint of $\mathscr{F}_{\mathfrak{g}}^{-1}$ is a Fourier transform on tempered distributions $(\mathscr{F}_{\mathfrak{g}}^{-1})^*$: $S^*(\mathfrak{g}) \to S^*(\mathfrak{g}^*)$.

The formula 1.7(a) suggests the following definition of a moment map $\tau_{\mathfrak{g}}: W \to \mathfrak{g}^*$:

$$\tau_{\mathfrak{g}}(x) = \frac{1}{4}\langle x(w), w\rangle \qquad (w \in W, x \in \mathfrak{g}). \tag{2.6}$$

Notice that the subset $\tau_{\mathfrak{g}}(W) \subset \mathfrak{g}^*$ is closed. Indeed, the subset $\tau_{sp}(W) \subset sp^*$ is closed and conical because it is linearly isomorphic to the closed set of endomorphisms of W which are symmetric with respect to the scalar product $\langle J, \ \rangle$. Since $\tau_{\mathfrak{g}}(W)$ is the image of $\tau_{sp}(W)$ under the restriction map $sp^* \to \mathfrak{g}^*$, it is closed too. The title of this section refers to the following theorem.

THEOREM 2.7. *Suppose $p \geqslant 0$ is an integer such that* (2.3) *holds and such that $p + 1$ is divisible by* 4. *Then*

$$supp((\mathscr{F}_{\mathfrak{g}}^{-1})^*(\tilde{c}^*(\Theta_\omega^p \mu))) \subseteqq \tau_{\mathfrak{g}}(W).$$

Proof. Fix a function $\psi \in S(\mathfrak{g})$ with

$$\int_{\mathfrak{g}} \chi(\tau_{\mathfrak{g}}(w))\psi(x)\,dx = 0 \qquad (w \in W). \tag{1}$$

We want to show that

$$\int_{\mathfrak{g}} \Theta_\omega^p(\tilde{c}(x))\mu(\tilde{c}(x))\psi(x)\,dx = 0. \tag{$*$}$$

Since the restriction of the Killing form from sp to $\mathfrak{g}$ is nondegenerate, we have a direct sum decomposition

$$sp = \mathfrak{g} \oplus \mathfrak{g}^\perp. \tag{2}$$

Let δ_m ($m = 1, 2, 3, \ldots$) be a (Dirac) sequence of smooth compactly supported nonnegative functions on $\mathfrak{g}^\perp$ with the integral equal to one, and the support contained in the ball centered at the origin and radius $1/m$. Set

$$\psi_m(x + y) = \psi(x)\delta_m(y) \qquad (x \in \mathfrak{g}, y \in \mathfrak{g}^\perp; m = 1, 2, 3, \ldots). \tag{3}$$

Then $\psi_m \in S(sp)$; so by [P1, (6.1)] $\tau_{sp}^*\mathscr{F}_{sp}(\psi_m) = \mathscr{F}_{sp}(\psi_m) \circ \tau_{sp} \in S(W)$, and by (1), (2), and (3)

$$(4) \qquad \tau_{sp}^*\mathscr{F}_{sp}(\psi_m)(w) = \int_{sp} \chi(\tau_{sp}(w)(x))\psi_m(x)\,dx$$

$$= \int_{\mathfrak{g}}\int_{\mathfrak{g}^\perp} \chi(\tau_{sp}(w)(x))\chi(\tau_{sp}(w)(y))\psi(x)\delta_m(y)\,dy\,dx = 0.$$

By Theorem 1.7(b), $\tilde{c}^*\Theta_\omega^{p+1}(x) = \Theta_\omega^{p+1}(\tilde{c}(x)) = const\,det(1-x)^{(p+1)/2}$ is a polynomial function on sp because $(p+1)/2$ is an even integer. Thus $\tilde{c}^*\Theta_\omega^{p+1}\psi_m \in S(sp)$ and

$$supp\ \mathscr{F}_{sp}(\tilde{c}^*\Theta_\omega^{p+1}\cdot\psi_m) \subseteq supp\ \mathscr{F}_{sp}(\psi_m) \subseteq sp^*\backslash\tau_{sp}(W).$$

Hence

$$\int_{sp} \Theta_\omega^p(\tilde{c}(x))\mu_\phi(\tilde{c}(x))\psi_m(x)\,dx = f\natural\tau_{sp}^*\mathscr{F}_{sp}(\tilde{c}^*\Theta_\omega^{p+1}\cdot\psi_m)\natural\phi(0) = f\natural 0\natural\phi(0) = 0.$$

Therefore

$$\left|\int_{\mathfrak{g}} \Theta_\omega^p(\tilde{c}(x))\mu(\tilde{c}(x))\psi(x)\,dx\right| = \left|\int_{\mathfrak{g}} \Theta_\omega^p(\tilde{c}(x))\mu(\tilde{c}(x))\psi(x)\,dx\right.$$

$$\left. - \int_{\mathfrak{g}}\int_{\mathfrak{g}^\perp} \Theta_\omega^p(\tilde{c}(x+y))\mu_\phi(\tilde{c}(x+y))\psi_m(x+y)\,dx\,dy\right|$$

$$\leqslant \int_{\mathfrak{g}^\perp} \delta_m(y)\int_{\mathfrak{g}} |\psi(x)|\,|\Theta_\omega^p(\tilde{c}(x+y))\mu_\phi(\tilde{c}(x+y)) - \Theta_\omega^p(\tilde{c}(x))\mu_\phi(\tilde{c}(x))|\,dx\,dy.$$

In the last expression the integral over $\mathfrak{g}$ is a continuous function of $y \in \mathfrak{g}^\perp$, by (2.3). Hence when $m \to \infty$, the limit of the whole expression is zero, and (*) follows. ∎

COROLLARY 2.8. *Suppose* $\Pi \in \mathscr{R}(\tilde{G}, \omega)$, [H7]. *Let* $WF(\Pi) \subseteq \mathfrak{g}^*$ *denote the wave-front set of the distribution character* Θ_Π *of* Π *at the identity of* $\tilde{G}$, [H5]. *Then* $WF(\Pi) \subseteq \tau_{\mathfrak{g}}(W)$.

Proof. Suppose Π is realized on a Hilbert space H_Π. For a trace class operator T on H_Π we have a continuous function $tr_\Pi(T)(g) = tr(T\Pi(g))$, $g \in \tilde{G}$, which may be viewed as a distribution via integration against the Haar measure dg. From the well-known proof of the existence of the character Θ_Π, [W, 8.1.1], we know that there is an orthonormal basis $u_1, u_2, u_3, \ldots$ of $\tilde{K}$-finite vectors of H_Π, a summable operator T, in the sense [W, 8.A.1.4] that

$$(1)\qquad \sum_{k,l=1}^{\infty} |(Tu_k, u_l)| < \infty,$$

and a differential operator D on $\tilde{G}$ such that

$$\Theta_\Pi = D\, tr_\Pi(T).$$

Hence, by [Hö, (8.1.11)], it will suffice to show that for some open neighborhood $O \subset \tilde{G}$ of the identity of $\tilde{G}$

$$(*)\qquad WF(tr_\Pi(T)|_O) \subseteq O \times \tau_{\mathfrak{g}}(W).$$

Set $t_{lk} = (Tu_l, u_k)$, $\mu_{kl}(g) = (\Pi(g)u_k, u_l)$, $g \in \tilde{G}$; $k, l = 1, 2, 3, \ldots$. Then

$$tr_\Pi(T)(g) = \sum_{k,l=1}^{\infty} t_{lk}\mu_{kl}(g) \qquad (g \in \tilde{G}),$$

where the series is absolutely convergent, by (1). We also have some control of the rate of growth of the $\tilde{K}$-finite matrix coefficients μ_{kl}, independent of the indices k, l. Indeed, for a fixed norm $\|\ \|$ on $\tilde{G}$ [Wa, 2.A.2], there is $s \geqslant 0$ such that

$$|\mu_{kl}(g)| = |(\Pi(g)u_k, u_l)| \leqslant \|\Pi(g)\| \leqslant const\, \|g\|^s$$

for $g \in \tilde{G}$ and k, $l = 1, 2, 3, \ldots$. It follows from [W, 2.A.2.3] and (1.12) that $\|g\|^s \leqslant const \cdot \Omega(g)^{-p}$, $g \in \tilde{G}$, for some $p \geqslant 0$. Hence the series

$$\Omega^p\, tr_\Pi(T) = \sum_{k,l=1}^{\infty} t_{lk}\Omega^p\mu_{kl}$$

is absolutely and uniformly convergent. Thus, by (1.10) and (1.11), if $p \geqslant 0$ is large enough, the series

$$(2)\qquad \tilde{c}^*(\Theta_\omega^p \cdot tr_\Pi(T)) = \sum_{k,l=1}^{\infty} t_{lk}\tilde{c}^*(\Theta_\omega^p\mu_{kl})$$

converges in $S^*(\mathfrak{g})$. We may choose $p \geqslant 0$ such that the conditions of the Theorem 2.7 are satisfied. Then (2) implies

$$(3)\qquad supp(\mathscr{F}_{\mathfrak{g}}^{-1})^*(\tilde{c}^*(\Theta_\omega^p \cdot tr_\Pi(T))) \subseteq \tau_{\mathfrak{g}}(W).$$

The inclusion $(*)$ follows from (3) and [Hö, 8.1.7]. ∎

It seems that the above proof could be sharpened to show that the Theorem 2.7 holds with the μ replaced by the character Θ_Π.

The above argument shows also that, if the representation Π is unitary, then the wave-front set of its restriction to the unipotent radical N of a maximal parabolic subgroup P of $\tilde{G}$ is contained in the projection of $\tau_{\mathfrak{g}}(W)$ on the dual $\mathfrak{n}^*$ of the Lie algebra $\mathfrak{n}$ of N, i.e. in $\tau_{\mathfrak{n}}(W)$. Thus 2.8 extends the notion of N-rank of a representation of $\tilde{G}$ from the setting of P-orbits in $\mathfrak{n}^*$ to the setting of $\tilde{G}$-orbits in $\mathfrak{g}^*$.

3. From a distribution character on $\tilde{G}$ to an irreducible unitary representation of $\tilde{G}'$, via Howe's correspondence. Let G, G' be a reductive dual pair in Sp, as in Section 2. We shall always assume, as we may, that $K = G \cap U$ and $K' = G' \cap U$ are maximal compact subgroups of G and G' respectively. Denote by $\tilde{G}$, $\tilde{G}'$, $\tilde{K}$, $\tilde{K}'$ the preimages of G, G', K, K' in $\widetilde{Sp}$, as usual. These are real reductive groups in the sense of Wallach [W, 2.1.1].

Let Π be an irreducible unitary representation of $\tilde{G}$ and let Θ_Π be the distribution character of Π. By the celebrated Harish-Chandra regularity theorem ([W, 8.4.1], [Bou]), the distribution Θ_Π coincides with the Haar measure on $\tilde{G}$ multiplied by a locally integrable function, which is real analytic on the set $\tilde{G}^{rs}$ of regular semisimple elements of $\tilde{G}$ and is equal to zero elsewhere. We shall identify Θ_Π with this function.

THEOREM 3.1. *Suppose*

$$\text{(a)} \qquad \int_{\tilde{G}} |\Theta_\Pi(g)|\,\Omega(g)\,dg < \infty .$$

Then the formula

$$(*) \qquad (\omega(\overline{\Theta}_\Pi)v, v') = \int_{\tilde{G}} \overline{\Theta}_\Pi(g)(\omega(g)v, v')\,dg \qquad (v, v' \in H^\infty)$$

defines a $\tilde{G}\cdot\tilde{G}'$-invariant hermitian form on H^∞. Let $R \subseteq H^\infty$ denote the radical of this form. Suppose that

(b) *the form* $(*)$ *is positive semidefinite and nontrivial.*

Then the $\tilde{G}\cdot\tilde{G}'$-module H^∞/R, equipped with the form induced by $(*)$, completes to an irreducible unitary representation of $\tilde{G}\cdot\tilde{G}'$, infinitesimally equivalent to $\Pi \otimes \Pi'$ for some $\Pi' \in \hat{\tilde{G}}'$, the unitary dual of $\tilde{G}'$. Thus Π corresponds to Π' via Howe's correspondence.

Moreover, under the assumption (b), *the intertwining distribution $f = f_{\Pi\otimes\Pi'} \in S^*(W)$ is given by the integral*

$$(**) \qquad f = \int_{\tilde{G}} \overline{\Theta}_\Pi(g)\rho^{-1}\omega(g)\,dg,$$

which converges in the topology of $S^*(W)$.

It is not easy to see what condition 3.1(a) means in terms of the Langlands parameters of the representation Π. We shall provide a slightly stronger condition in Proposition 4.11 in terms of the rate of growth of the character Θ_Π.

Proof. It follows from formula (2.1) and Proposition 1.13(a) that the integral (*) is absolutely convergent. Hence the form (*) is well defined.

Since $\overline{\Theta}_\Pi(g) = \Theta_\Pi(g^{-1})$, $g \in \tilde{G}^{rs}$, this form is hermitian, and it is obviously $\tilde{G} \cdot \tilde{G}'$-invariant.

Suppose for the rest of this proof that assumption (b) holds. Let X_ω be the space of $\tilde{U}$-finite vectors in H and let $X_{\overline{\Pi}}$ be the space of $\tilde{K}$-finite vectors in $H^{\overline{\Pi}}$, where $\overline{\Pi}$ is the irreducible unitary representation of $\tilde{G}$ with the distribution character $\Theta_{\overline{\Pi}} = \overline{\Theta}_\Pi$. As a consequence of Theorem 1.5 and Proposition 1.13(a), the form (*) is continuous as a function of the two variables $v, v' \in H^\infty$. Therefore the restriction of this form to X_ω is nontrivial.

Since the function Ω is $\tilde{K}$-bi-invariant (see Proposition 1.9) the Fourier components of $\Theta_{\overline{\Pi}}$ [W, 8.1.2(1)], and therefore all $\tilde{K}$-finite matrix coefficients of $\overline{\Pi}$, are absolutely integrable against Ω, over $\tilde{G}$. Thus for $v' \in X_\omega$ and for $u' \in X_{\overline{\Pi}}$ we may define a map $\Phi_{u',v'}\colon X_\omega \to X_{\overline{\Pi}}^*$, the vector space dual to $X_{\overline{\Pi}}$, by

$$\Phi_{u',v'}(v)(u) = \int_{\tilde{G}} (\overline{\Pi}(g)u, u')(\omega(g)v, v')\, dg \qquad (v \in X_\omega,\, u \in X_{\overline{\Pi}}).$$

The space $X_{\overline{\Pi}}^*$ carries the usual contragredient action of $\mathfrak{g}$ and $\tilde{K}$ [W, 3.3.6]. It is clear that $\Phi_{u',v'}$ is a $(\mathfrak{g}, \tilde{K})$-intertwining map. Hence the image of $\Phi_{u',v'}$ is contained in the subspace $X_{\overline{\Pi}}^{\sim} \subseteq X_{\overline{\Pi}}^*$ of $\tilde{K}$-finite vectors. But the $(\mathfrak{g}, \tilde{K})$-modules $X_{\overline{\Pi}}^{\sim}$ and X_Π are isomorphic [V1, 8.5.3]. Hence $X_{\overline{\Pi}}^{\sim}$ is irreducible and consequently the map

$$\Phi_{u',v'}\colon X_\omega \to X_{\overline{\Pi}}^{\sim} \cong X_\Pi \tag{1}$$

is either zero or surjective.

Choose an orthonormal basis of H_Π consisting of $\tilde{K}$-finite vectors $u_1, u_2, u_3, \ldots$. Then the usual argument [W, 8.1.1] shows that

$$(\omega(\overline{\Theta}_\Pi)v, v') = \sum_{n=1}^{\infty} \int_{\tilde{G}} (\overline{\Pi}(g)u_i, u_i)(\omega(g)v, v')\, dg \qquad (v, v' \in H^\infty). \tag{2}$$

Hence the space $R \cap X_\omega$ contains the intersection of the kernels of the maps $\Phi_{u_i,v'}$; $i = 1, 2, 3, \ldots$; $v' \in H^\infty$. Therefore there is a $(\mathfrak{g}', \tilde{K}')$-module X' such that

(3) $X_\omega/(R \cap X_\omega)$ is isomorphic to $X_\Pi \otimes X'$ as a $(\mathfrak{g} + \mathfrak{g}', \tilde{K} \cdot \tilde{K}')$-module.

Theorem 2.1 in [H1] implies that X' is admissible and has a unique irreducible quotient. Since, by (b), X' is unitarizable, it is completely reducible and hence irreducible.

The space of $\tilde{K} \cdot \tilde{K}'$-finite vectors in H^∞/R coincides with $X_\omega/R \cap X_\omega$ and is dense. Hence it is clear that H^∞/R completes to an irreducible unitary representation $\Pi \otimes \Pi' \in \mathscr{R}(\tilde{G} \cdot \tilde{G}', \omega)$.

As a consequence of assumption (a) and Proposition 1.9, the integral (**) is absolutely convergent. Under assumption (b), the radical R coincides with the kernel of the map

$$\rho(f) = \int_{\tilde{G}} \overline{\Theta}_\Pi(g)\omega(g)\, dg \in Hom(H^\infty, H^{\infty *}). \tag{4}$$

Thus the representation $\Pi \otimes \Pi'$ is realized on the image of this map, so that f is indeed the intertwining distribution corresponding to $\Pi \otimes \Pi'$. ■

LEMMA 3.2. *Condition* 3.1(b) *may be replaced by the requirement that there be u in the Harish-Chandra module X_Π of Π such that*

$$\int_{\tilde{G}} (u, \Pi(g)u)(\omega(g)v, v)\, dg \geqslant 0 \qquad (v \in X_\omega) \tag{b'}$$

and there be a vector v for which this integral is nonzero.

Proof. Since the $(\mathfrak{g}, \tilde{K})$-module X_Π is irreducible, it is generated by any nonzero vector. Thus if (b′) holds for one $u \in X_\Pi$, then it holds for all nonzero $u \in X_\Pi$.

Indeed, let us fix a nonzero vector $u \in X_\Pi$. Since X_Π is irreducible, any vector in it can be written as a finite sum

$$\sum_{j=1}^{m} \Pi(X_j)\Pi(k_j)u$$

for some positive integer m, $X_j \in \mathscr{U}(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$), $k_j \in \tilde{K}$, $1 \leqslant j \leqslant m$. The left and right invariance of the Haar measure on $\tilde{G}$ implies the formula

$$\int_{\tilde{G}} \left(\sum_{j=1}^{m} \Pi(X_j)\Pi(k_j)u, \Pi(g) \sum_{j=1}^{m} \Pi(X_j)\Pi(k_j)u\right)(\omega(g)v, v)\, dg$$
$$= \int_{\tilde{G}} (u, \Pi(g)u)\left(\omega(g) \sum_{j=1}^{m} \omega(k_j^{-1})\omega(\check{X}_j)v, \sum_{j=1}^{m} \omega(k_j^{-1})\omega(\check{X}_j)v\right) dg$$

where $\mathscr{U}(\mathfrak{g}) \ni X \to \check{X} \in \mathscr{U}(\mathfrak{g})$ is the antiautomorphism of $\mathscr{U}(\mathfrak{g})$ whose restriction to $\mathfrak{g}$ coincides with the map $X \to -X$. Since the $v \in X_\omega$ in (b′) is arbitrary, we see in the inequality (b′) that nonnegativity holds for all $u \in X_\Pi$. Hence the remark follows from formula (2) in the proof of Theorem 3.1. ■

Remark 3.3. Suppose G, G' is an irreducible pair [H7]. Let G_1 denote the Zariski identity component of G. The case when $G \neq G_1$ requires some explanation. From the classification of dual pair [H1] we know that G is isomorphic to an orthogonal group; i.e., G can be identified with the isometry group of a symmetric non-degenerate form (,) on a finite-dimensional space V over $\mathbf{D} = \mathbf{R}$ or $\mathbf{C}$. If $dim_{\mathbf{D}}\, V$ is odd, then the group G, and hence also $\tilde{G}$, is of inner type [W, 7.4.1] so that any element of $\tilde{G}^{rs}$ is conjugate to an element of a Cartan subgroup of $\tilde{G}$, and there is nothing to worry about.

Consider the case when $dim_{\mathbf{D}}\, V$ is even. In this case all the Cartan subgroups of $\tilde{G}$ are contained in $\tilde{G}_1$; thus there is a problem of understanding the character Θ_Π on the complement $\tilde{G}\backslash\tilde{G}_1$ of $\tilde{G}_1$. If the restriction $\Pi|_{\tilde{G}_1}$ of Π to $\tilde{G}_1$ is reducible, then it is a direct sum of two nonequivalent representations of $\tilde{G}_1$, which are permuted by any element of $\Pi(\tilde{G}\backslash\tilde{G}_1)$. Hence, as a function, $\Theta_\Pi|_{\tilde{G}\backslash\tilde{G}_1} = 0$.

Notice that since $\tilde{G}/\tilde{G}_1 \simeq G/G_1$, the determinant character of G may be viewed as a character $det\colon \tilde{G} \to \mathbf{C}^\times$ of $\tilde{G}$. If $\Pi|_{\tilde{G}_1}$ is irreducible, then $\Pi \otimes det$ has the same property, is not equivalent to Π, and

$$\Theta_\Pi|_{\tilde{G}\backslash\tilde{G}_1} + \Theta_{\Pi\otimes det}|_{\tilde{G}\backslash\tilde{G}_1} = 0.$$

If in this case we replace G by G_1 in 3.1(a) and (*), then the $\tilde{G}\tilde{G}'$ module H^∞/R is infinitesimally equivalent to $\Pi|_{\tilde{G}_1} \otimes (\Pi' \oplus (\Pi \otimes det)')$ or to $\Pi|_{\tilde{G}_1} \otimes \Pi'$.

Remark 3.4. Theorem 3.1 provides a strong criterion for preservation of unitarity under Howe's correspondence. We shall see in Sections 8 and 9 that positivity in 3.2(b′) (and hence 3.1(b)) holds in the stable range (as shown previously in [Li (50)]) and in many cases beyond the stable range. At present, it is not clear how far we can go. However, there is a reason for optimism.

Notice that in 3.2(b′) we integrate a positive definite function. Hence, motivated by abelian harmonic anaysis we expect such an integral to be positive. In fact, this is true as long as the group is amenable (compact, commutative,); see [G]. In contrast, for any semisimple group one can find an absolutely integrable positive definite function on it, whose integral is negative [G]. It is however not easy to produce specific counterexamples. It seems even harder to produce positive definite functions, one in $L^p(\tilde{G})$ and the other in $L^q(\tilde{G})$ $\left(\text{with } \frac{1}{p} + \frac{1}{q} = 1, p > 1, q > 1\right)$, such that the product of them has negative integral over $\tilde{G}$. Notice that this is the situation we encounter in 3.2(b′).

The main difference between our approach and that of [Li1] is that we deal with the hermitian form 3.1(*) defined on H^∞ rather than with a form defined on the tensor product $H^\infty \otimes H_\Pi^\infty$ [Li1, (4)]. This allows us to use Howe's duality theorem [H1, Theorem 2.1] to employ the positivity 3.2(b′) to produce unitary representations of $\tilde{G}'$.

In [Li1] Howe's duality theorem is not used. Instead, Li combines Mackey theory with such a positivity result to construct an irreducible unitary representation of $\tilde{G}'$ (the contragredient of Π'). His argument is parallel to the one used by

Howe in [H6] and does not seem to work beyond the stable range. Then one needs additional work [Li1, sec. 6] to show that Π' corresponds to Π via Howe's correspondence.

Needless to say, the idea of Theorem 3.1 is motivated by [Li1]. In fact, it may be thought of as "the trace of Li's construction".

4. Estimates for matrix coefficients. In this section G stands for a member of a reductive dual pair, as long as G is of inner type. If G is not of inner type, we shall use the same letter G to denote the Zariski identity component G_1 of G (which is of inner type).

Fix a norm $\| \ \|$ on $\tilde{G}$ [W, 2.A.2]. Let $\Xi_{\tilde{G}}$ be the Harish-Chandra Ξ-function for $\tilde{G}$ [W, 4.5.3]. Denote by $d_{\tilde{G}}$ the Weyl denominator of $\tilde{G}$ [W, 2.4.4]. Harish-Chandra has shown that for some $m \geqslant 0$

$$\int_{\tilde{G}} |d_{\tilde{G}}(g)|^{-1/2} \Xi_{\tilde{G}}(g)(1 + \log \|g\|)^{-m} \, dg < \infty \tag{4.1}$$

(see [Wa, 8.3.7.6]). We shall exploit this fact to see when condition 3.1(a) is satisfied and to obtain some rough estimates for the matrix coefficients of the representation Π' constructed in 3.1.

Let Π be an irreducible unitary representation of $\tilde{G}$. We identify the distribution character Θ_Π of Π with the corresponding function on $\tilde{G}$, as explained in the discussion preceding 3.1. Recall [M, page 63] that there is a notion of the rate of growth of the function Θ_Π. In particular, Θ_Π has the rate of growth $\gamma \geqslant 0$ if there are constants $C, C' \geqslant 0$ such that

$$|\Theta_\Pi(g)| \leqslant C |d_{\tilde{G}}(g)|^{-1/2} \Xi_{\tilde{G}}^{-\gamma}(g)(1 + \log \|g\|)^{C'} \qquad (g \in \tilde{G}^{rs}); \tag{4.2}$$

see [M, Theorem 1, page 69]. By [M, Theorem 1, page 79], Θ_Π has the rate of growth $\gamma \in \mathbf{R}$ if and only if for any $\tilde{K}$-finite matrix coefficient μ of Π there are constants $C, C' \geqslant 0$ such that

$$|\mu(g)| \leqslant C \Xi_{\tilde{G}}^{1-\gamma}(g)(1 + \log \|g\|)^{C'} \qquad (g \in \tilde{G}). \tag{4.3}$$

Since Π is unitary, μ is bounded and $\gamma \leqslant 1$; see [BW, 4.5.1]. If $\gamma = 0$, then Π is tempered [W, 5.1.1].

Recall that there is a constant $C' \geqslant 0$ such that

$$\int_{\tilde{G}} \Xi_{\tilde{G}}^{2}(g)(1 + \log \|g\|)^{-C'} \, dg < \infty . \tag{4.4}$$

Hence by (4.3), if Θ_Π has the rate of growth $\gamma \in \mathbf{R}$, then for any $\tilde{K}$-finite matrix coefficient μ of Π

$$(4.5) \qquad \mu \in L^p(\tilde{G}/center(\tilde{G})), \qquad \text{where } p = \frac{2}{1-\gamma} + \varepsilon, \quad \text{for any } \varepsilon > 0.$$

Let ${}^0\tilde{G}'$ be the maximal subgroup of $\tilde{G}'$ with the Lie algebra $[\mathfrak{g}', \mathfrak{g}']$. Denote by $\Lambda_{\tilde{G},\tilde{G}'}$ the set of pairs $(\lambda, \lambda') \in \mathbf{R}^2$ such that

$$(4.6) \qquad \Omega(gg') \leqslant const_m\, \Xi_{\tilde{G}}^{\lambda}(g)(1 + log\,\|g\|)^{-m}\Xi_{\tilde{G}'}^{\lambda'}(g')$$

for $g \in \tilde{G}$, $g' \in {}^0\tilde{G}'$ and $m = 0, 1, 2, \ldots$. The point of introducing ${}^0\tilde{G}'$ here is that $\tilde{G} \cap {}^0\tilde{G}'$ is compact, while $\tilde{G} \cap \tilde{G}'$ does not have to be compact, and in that case (4.6) might not make sense. By applying (4.2), (4.6), (4.1), Proposition 1.13, and (4.3) respectively, we obtain the following result.

PROPOSITION 4.7. *Suppose the character Θ_Π has the rate of growth $\gamma \geqslant 0$ and $(1 + \gamma, 1 - \gamma') \in \Lambda_{\tilde{G},\tilde{G}'}$. Then*

$$\int_{\tilde{G}} |\Theta_\Pi(g)|\,\Omega(gg')\,dg \leqslant const\ \Xi_{\tilde{G}'}^{1-\gamma'}(g') \qquad (g' \in {}^0\tilde{G}').$$

Thus in this case condition 3.1(a) *holds. Moreover, if condition* 3.1(b) *is satisfied, then for any $\tilde{K}'$-finite matrix coefficient μ' of the representation* Π', *corresponding to* Π (*see* 3.1 *and* 3.3),

$$|\mu'(g')| \leqslant const\ \Xi_{\tilde{G}'}^{1-\gamma'}(g') \qquad (g' \in \tilde{G}').$$

Hence the character $\Theta_{\Pi'}$ has the rate of growth γ'.

We shall describe the set $\Lambda_{G,G'}$ (4.6) geometrically. Let $\mathfrak{u} \subseteq sp$ denote the Lie algebra of the maximal compact subgroup $U \subseteq Sp$. Then we have a Cartan decomposition $sp = \mathfrak{u} \oplus \mathfrak{q}$, which induces Cartan decompositions for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and for $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$, by restriction. Choose maximal abelian Lie subalgebras $\mathfrak{a} \subseteq \mathfrak{g}$ and $\mathfrak{a}' \subseteq \mathfrak{g}'$ such that $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{a}' \subset \mathfrak{p}'$. Let $\mathfrak{s} = \mathfrak{a} \cap \mathfrak{a}'$. Then $\mathfrak{s}$ is a standard split component of $\mathfrak{g}$ and of $\mathfrak{g}'$ [W, 2.2.1]. Set ${}^0\mathfrak{a} = \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$ and ${}^0\mathfrak{a}' = \mathfrak{a}' \cap [\mathfrak{g}', \mathfrak{g}']$. From the structure of dual pairs [H1], we know that

$$(4.8) \qquad \mathfrak{a} \oplus \mathfrak{a}' = {}^0\mathfrak{a} \oplus \mathfrak{s} \oplus {}^0\mathfrak{a}', \qquad \mathfrak{a} = {}^0\mathfrak{a} \oplus \mathfrak{s}, \qquad \mathfrak{a}' = \mathfrak{s} \oplus {}^0\mathfrak{a}'.$$

Choose Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{a}' \oplus \mathfrak{n}'$ [W, 2.1.7]. Let $\rho_\mathfrak{n}(x) = tr(ad(x)|_\mathfrak{n})$, $x \in \mathfrak{a}$. Then $\rho_\mathfrak{n} \in \mathfrak{a}^*$ and the restriction of $\rho_\mathfrak{n}$ to ${}^0\mathfrak{a}$ is nonzero, unless ${}^0\mathfrak{a} = 0$. Similarily, we have $\rho_{\mathfrak{n}'} \in \mathfrak{a}'^*$. Set

$$|x|_\mathfrak{a} = \max_{w \in W(\tilde{G}, A)} \rho_\mathfrak{n}(w(x)), \qquad |x'|_{\mathfrak{a}'} = \max_{w \in W(\tilde{G}', A')} \rho_{\mathfrak{n}'}(w(x')),$$

where $x \in \mathfrak{a}$ and $x' \in \mathfrak{a}'$. Then $|\ |_{\mathfrak{a}}$ and $|\ |_{\mathfrak{a}'}$ are norms on ${}^0\mathfrak{a}$ and ${}^0\mathfrak{a}'$ respectively. Since $\mathfrak{a} + \mathfrak{a}' \subseteq \mathfrak{q}$ is a commutative Lie algebra, we can choose a maximal abelian Lie subalgebra $\mathfrak{a}_{sp} \subseteq sp$ such that $\mathfrak{a} \oplus \mathfrak{a}' \subseteq \mathfrak{a}_{sp} \subseteq \mathfrak{q}$. Recall the norm $|\ |_{\Omega}$ on $\mathfrak{a}_{sp}$, (1.12). Denote by $\Lambda^+_{G,G'}$ the subset of the interior of $\Lambda_{G,G'}$ consisting of pairs (λ, λ') of positive numbers. This is the essential part of $\Lambda_{G,G'}$.

PROPOSITION 4.9. *A pair (λ, λ') is in $\Lambda^{+\,+}_{G,G'}$ if and only if the image of the unit ball in $\mathfrak{a} + \mathfrak{a}'$, with respect to the norm $|\ |_{\Omega}$, under the projection map ${}^0\mathfrak{a} \oplus \mathfrak{s} \oplus {}^0\mathfrak{a}' \to {}^0\mathfrak{a} \oplus {}^0\mathfrak{a}'$ along $\mathfrak{s}$, is contained in the interior of the unit ball defined by the norm $\lambda|x|_{\mathfrak{a}} + \lambda'|x'|_{\mathfrak{a}'}$, $x \in {}^0\mathfrak{a}$, $x' \in {}^0\mathfrak{a}'$.*

Proof. Fix a norm $|\ |_{\mathfrak{s}}$ on $\mathfrak{s}$. It follows from (1.12) and from Harish-Chandra's estimate for the Ξ-function [W4.5.3], that $(\lambda, \lambda') \in \Lambda^+_{G,G'}$ if and only if there is $\varepsilon > 0$ such that, for all $x \in {}^0\mathfrak{a}$, $s \in \mathfrak{s}$, $x' \in {}^0\mathfrak{a}'$,

$$e^{(-|x+s+x'|_{\Omega})} \leqslant const_m\, e^{(-(\lambda+\varepsilon)|x|_{\mathfrak{a}})}(1 + |s|_{\mathfrak{s}})^{-m} e^{(-(\lambda'+\varepsilon)|x'|_{\mathfrak{a}'})},$$

or equivalently

$$|x + s + x'|_{\Omega} \geqslant (\lambda + \varepsilon)|x|_{\mathfrak{a}} + (\lambda' + \varepsilon)|x'|_{\mathfrak{a}'} + m\, log(1 + |s|_{\mathfrak{s}}) - log(const_m). \quad \blacksquare$$

Suppose G, G' is an irreducible dual pair [H7]. Then it is either a pair of type I or of type II. In the first case there is a finite-dimensional division algebra $\mathbf{D}$ over $\mathbf{R}$ with a (possibly trivial) involution, two finite-dimensional vector spaces V and V' over $\mathbf{D}$, with nondegenerate forms $(\ ,\ )$ and $(\ ,\ )$, one hermitian and the other skew-hermitian, such that G is isomorphic to the group of isometries of $(\ ,\ )$ and G' to the group of isometries of $(\ ,\ )'$. In the second case we have the vector spaces V, V' but without the forms $(\ ,\ )$, $(\ ,\ )'$ and G, G' are isomorphic to $GL_{\mathbf{D}}(V)$, $GL_{\mathbf{D}}(V')$, respectively. Let

$$r = \begin{cases} 2\, dim_{\mathbf{R}}\, \mathfrak{g}/dim_{\mathbf{R}}\, V & \text{if } G \text{ is an isometry group} \\ dim_{\mathbf{R}}\, \mathfrak{g}/dim_{\mathbf{R}}\, V & \text{if } G \text{ is a general linear group.} \end{cases} \tag{4.10}$$

Set

$$\lambda_{max} = sup\{\lambda;\, |x + s|_{\Omega} \geqslant |x|_{\mathfrak{a}},\, x \in {}^0\mathfrak{a},\, s \in \mathfrak{s}\},$$

$$\lambda'_{max} = sup\{\lambda';\, |x' + s|_{\Omega} \geqslant |x'|_{\mathfrak{a}},\, x' \in {}^0\mathfrak{a}',\, s \in \mathfrak{s}\}.$$

Thus λ_{max} is the supremum of the projection of $\Lambda^+_{G,G'}$ on the second coördinate.

PROPOSITION 4.11. *With the above notation, we have*

(a) $\lambda_{max} = dim_{\mathbf{D}} V'/(r - 1)$ *and* $\lambda'_{max} = dim_{\mathbf{D}}\, V/(r' - 1)$.

(b) *Condition* 3.1(a) *holds if Θ_{Π} has the rate of growth $\gamma < \gamma_{max} = \lambda_{max} - 1$.*

(c) *If* $\gamma < \gamma_{max}$ *and if condition* 3.1(b) *is satisfied, then* $\Theta_{\Pi'}$ *has the rate of growth* $\gamma' = 1 - \lambda'$, *where* $\lambda' = \left(1 - \dfrac{1+\gamma}{\lambda_{max}}\right)\lambda'_{max}$. *In particular, the* $\tilde{K}'$*-finite matrix coefficients of* Π' *are in* $L^p(\tilde{G}'/center(\tilde{G}'))$, $p = \dfrac{2}{\lambda'} + \varepsilon$, *for any* $\varepsilon > 0$.

For a reader familiar with the notion of the stable range (see [Li1] or (8.1)) it might be interesting to notice that condition 4.11(b) is independent of that notion. For example, if G is an orthogonal group and G' is a symplectic group (over $\mathbf{D} = \mathbf{R}$ or $\mathbf{C}$), then it makes sense if $dim_{\mathbf{D}}\, V' \geqslant dim_{\mathbf{D}}\, V - 1$, while stable range (with G the smaller member) requires $dim_{\mathbf{D}}\, V' \geqslant \frac{1}{2}\, dim_{\mathbf{D}}\, V$. Thus with G' fixed we have much more flexibility in changing G.

Proof. Due to the symmetric nature of the pair G, G', it will suffice to verify the first formula in (a). Choose a maximal isotropic subspace $Y \subseteq V$ and a basis $y_1, y_2, \ldots, y_n$ of Y over $\mathbf{D}$. Let Y_j be the span of $y_1, y_2, \ldots, y_j$, $1 \leqslant j \leqslant n$. Let $\mathfrak{n}$ be the nilradical of the parabolic Lie subalgebra of $\mathfrak{g}$ which preserves the flag $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_n$. Denote by $\mathfrak{a}$ the vector part of the maximally split Cartan subalgebra of $\mathfrak{g}$, which acts diagonally on Y with respect to this basis

$$x(y_i) = x_i y_i \qquad (x \in \mathfrak{a},\ 1 \leqslant i \leqslant n).$$

Then $Y \otimes V'$ is an isotropic subspace of the symplectic space $W = V \otimes V'$ (here $\otimes = \otimes_{\mathbf{R}}$) [H7] and (see (1.12))

$$|x|_{\Omega} = dim_{\mathbf{R}}\, V' \cdot \frac{1}{2} \sum_{i=1}^{n} |x_i| \qquad (x \in \mathfrak{a}). \tag{1}$$

Set $\rho_i = \frac{1}{2}(r + 1 - 2i)$, $1 \leqslant i \leqslant n$. Then, as is well known,

$$|x|_{\mathfrak{a}} = \begin{cases} dim_{\mathbf{R}}\, \mathbf{D} \cdot \max\limits_{\sigma} \sum\limits_{i=1}^{n} |\rho_i x_{\sigma(i)}| & \text{for type I pairs} \\ dim_{\mathbf{R}}\, \mathbf{D} \cdot \max\limits_{\sigma} \left| \sum\limits_{i=1}^{n} \rho_i x_{\sigma(i)} \right| & \text{for type II pairs,} \end{cases} \tag{2}$$

where $x \in \mathfrak{a}$ and σ varies over all possible permutations of n elements. Since in any case

$$|x|_{\mathfrak{a}} \leqslant dim_{\mathbf{R}}\, \mathbf{D} \cdot max_{\sigma} \sum_{i=1}^{n} |\rho_i x_{\sigma(i)}|,$$

(a) follows. Statement (b) follows from 4.9.

For (c) one checks that

$$|x + x'|_{\Omega} \geqslant |x'|_{\Omega} \qquad (x \in \mathfrak{a},\ x' \in {}^0\mathfrak{a}')$$

and

$$|x + x'|_\Omega \geqslant |x'|_\Omega \qquad (x \in {}^0\mathfrak{a}, x' \in \mathfrak{a}').$$

Hence for $0 \leqslant t \leqslant 1$

$$|x + x'|_\Omega \geqslant t\lambda_{max}|x|_\mathfrak{a} + (1 - t)\lambda'_{max}|x'|_{\mathfrak{a}'} \qquad (x \in \mathfrak{a}, x' \in {}^0\mathfrak{a}').$$

Thus (c) follows from 4.9, 4.7, and (4.5). ■

Suppose $\gamma_{max} > 1$ (see 4.11(b)) so that 3.1(a) holds for any unitary representation Π of $\tilde{G}$. Then 4.11(c), with $\gamma = 1$, yields

$$\lambda' = \frac{dim_\mathbf{D} V' - 2(r - 1)}{r' - 1} \cdot \frac{dim_\mathbf{D} V}{dim_\mathbf{D} V'}.$$

Thus statement 4.11(c) proves a conjecture of Scaramuzzi [Sc, 3.3.4] if $dim_\mathbf{D} V' \geqslant 2\, dim_\mathbf{D} V(dim_\mathbf{D} V - 1) + 1$ and improves the estimates of Howe [H3, Theorem 8.4] for the pair $O(p, q)$, $Sp(m, \mathbf{R})$ if $2m > 2(p + q)(p + q - 5/2) + 4$. In fact, a more careful investigation of the set $\Lambda^+_{O(p,q),Sp(m,\mathbf{R})}$ improves all the estimates [H3, Theorem 8.4], but since this method does not lead to sharp results, we shall not pursue it further. Of course, here we use the fact (see [Li2]) that the representations considered in [Sc] and [H3] occur in Howe's correspondence.

5. Lifting of characters via the Cayley transform. Let $\mathbf{D}$ be a finite-dimensional division algebra over $\mathbf{R}$, with an involution, and let V be a finite-dimensional vector space over $\mathbf{D}$ equipped with a hermitian or skew-hermitian form (,), which is either nondegenerate or zero. In this section

$$G = \{g \in GL_\mathbf{D}(V); (gv, gv') = (v, v'), v, v' \in V\}$$

stands for the isometry group of the form (,) and

$$\mathfrak{g} = \{x \in End_\mathbf{D}(V); (xv, v') = -(v, xv'), v, v' \in V\}$$

for the Lie algebra of G. Clearly, if the form (,) is zero, then $G = GL_\mathbf{D}(V)$ and $\mathfrak{g} = End_\mathbf{D}(V)$.

For $x \in End_\mathbf{D}(V)$ with $x - 1$ invertible, define the Cayley transform

$$c(x) = (x + 1)(x - 1)^{-1} \qquad (x \in \mathfrak{g}^c). \tag{5.1}$$

This notation is consistent with the one used in Theorem 1.7.

Let $det_\mathbf{R}(x)$ denote the determinant of $x \in End_\mathbf{D}(V)$ viewed as an element of

$x \in End_{\mathbf{R}}(V)$. Motivated by Theorem 1.7(b), we introduce the function on $\mathfrak{g}$

$$(5.2)\quad ch(x) = \begin{cases} |det_{\mathbf{R}}(1-x)|^{1/2} & \text{if } G \text{ is an isometry group} \\ |det_{\mathbf{R}}(1-x)|^{1/2}\,|det_{\mathbf{R}}(1+x)|^{1/2} & \text{if } G \text{ is a general linear group,} \end{cases}$$

where $x \in \mathfrak{g}$. Let $\mathfrak{g}^{ch} = \{x \in \mathfrak{g};\ ch(x) \neq 0\}$. Then $\mathfrak{g}^{ch} \subseteq \mathfrak{g}^{c}$.

Let $\tilde{G}$ be a finite central extension of G, not necessarily related to the setting of dual pairs and the metaplectic group. Notice that $\tilde{G}$ is a real reductive group in the sense of Wallach [W, 2.1.1]. Choose a real analytic lifting $\tilde{c}_-$ of the Cayley transform $c_- = -c$ (5.1) so that $\tilde{c}_-(0)$ is the identity of $\tilde{G}$. Thus we have the commutative diagram

$$(5.3)\quad \begin{array}{ccc} \mathfrak{g}^c & \xrightarrow{\tilde{c}_-} & \tilde{G} \\ \| & & \downarrow \\ \mathfrak{g}^c & \xrightarrow{c_-} & G \end{array}$$

where the vertical arrow stands for the covering map.

Let Θ be a distribution on $\tilde{G}$. Since $\tilde{c}_-$ is a diffeomorphism onto its image, the pullback $\tilde{c}_-^*\Theta$ is a well-defined distribution $\mathfrak{g}^{ch}$ [Hö, 6.1.2]. Hence for any real number s we have a distribution

$$(5.4)\qquad ch^s \cdot \tilde{c}_-^*\Theta \in C_c^\infty{}^*(\mathfrak{g}^{ch})$$

on $\mathfrak{g}^{ch}$. From now until the end of this section, we assume that $\Theta = \Theta_\Pi$ is the distribution character of an irreducible admissible representation Π of $\tilde{G}$. We identify Θ with the corresponding function on $\tilde{G}$, as explained previously. Thus the distribution (5.4) is given by the formula

$$(5.5)\qquad ch^s \cdot \tilde{c}_-^*\Theta(\psi) = \int_{\mathfrak{g}} ch^s(x)\Theta(\tilde{c}_-(x))\psi(x)\,dx \qquad (\psi \in C_c^\infty(\mathfrak{g}^{ch}).$$

We would like to extend the distribution (5.5) from $\mathfrak{g}^{ch}$ to $\mathfrak{g}$. In order to do so, we need some preparation.

Let D denote the Weyl denominator on $\mathfrak{g}$ [W, 2.3.1]. Fix a norm $|\ |$ on $\mathfrak{g}$ and a Cartan subgroup $H \subseteq \tilde{G}$ with the Lie algebra $\mathfrak{h}$. Let $\mathfrak{h}^r = \{x \in \mathfrak{h};\ D(x) \neq 0\}$ be the set of regular elements in $\mathfrak{h}$. The proof of Harish-Chandra's theorem on semisimple orbital integrals [W, 7.3] verifies the following statement.

THEOREM 5.6. *For any $j \geqslant 0$ there is $l \geqslant 0$ such that*

$$\sup_{x \in \mathfrak{h}^r} (1+|x|)^j |D(x)|^{1/2} \int_{\tilde{G}/H} (1+|Ad\, g(x)|)^{-l} d(gH) < \infty.$$

Here Ad denotes the adjoint representation of $\tilde{G}$ on $\mathfrak{g}$ and $d(gH)$ a (nonzero) $\tilde{G}$-invariant measure on the quotient space $\tilde{G}/H$.

Let $d = d_{\tilde{G}}$ denote the Weyl denominator on $\tilde{G}$ [W, 2.4.4]. Recall the number r (4.10). By a straightforward calculation we obtain the following lemma.

LEMMA 5.7. *Set $\iota = 1$ if $\mathbf{D} \neq \mathbf{H}$ (the quaternions) and $\iota = 1/2$ if $\mathbf{D} = \mathbf{H}$. Then*

$$|D(x)/d(\tilde{c}_-(x))|^{1/2} = const \cdot ch^{r-\iota}(x) \qquad (x \in \mathfrak{g}).$$

Fix an Iwasawa decomposition $\tilde{G} = \tilde{K}AN$. Let Δ_0 be the corresponding set of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For a subset $F \subseteq \Delta_0$ let (P_F, A_F) be the standard parabolic pair as defined in [W, 2.2]. Suppose P_F is cuspidal. Let $H_F = T_F A_F$ be a fundamental Cartan subgroup of $M_F = {}^0M_F A_F$, as in [W, 2.1–2.2]. We identify $\mathfrak{a}_F^*$, the real dual vector space to the Lie algebra $\mathfrak{a}_F$ of A_F, with the subspace of elements of $\mathfrak{a}^*$ that vanish on ${}^*\mathfrak{a}_F = {}^0\mathfrak{m}_F \cap \mathfrak{a}$. For $\nu \in \mathfrak{a}_F^*$ set

$$\xi_\nu(a) = \sum_{w \in W(\tilde{G}, A)} a^\nu \qquad (a \in A).$$

We extend the ξ_ν to a function on H_F by the formula

$$\xi_\nu(ta) = \xi_\nu(a) \qquad (t \in T_F,\ a \in A).$$

Let Y be a maximal isotropic subspace of V stabilized by A. Denote the weights of $\mathfrak{a}$ in Y by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathfrak{a}^*$. For $\nu = \nu_1\varepsilon_1 + \nu_2\varepsilon_2 + \cdots + \nu_n\varepsilon_n \in \mathfrak{a}^*$, set

$$|\nu| = max\{|\nu_1|, |\nu_2|, \ldots, |\nu_n|\}/dim_{\mathbf{R}}\, \mathbf{D}.$$

A straightforward case by case analysis implies the following statement.

LEMMA 5.8. *Suppose $\nu \in \mathfrak{a}^*$ and $s \geqslant 2|\nu|$. Then there is $m \geqslant 0$ such that, for any Cartan subalgebra $\mathfrak{h}_F$, $F \subseteq \Delta_0$,*

$$ch^s(x)\xi_\nu(\tilde{c}_-(x)) \leqslant const(1 + |x|)^m \qquad (x \in \mathfrak{h}_F).$$

By combining a well-known formula for the characters of standard induced representations [Wo, 4.3.8.3] with [W, 5.5.3] and Lemma 5.7 and 5.8, we obtain the following lemma.

LEMMA 5.9. *Let $\nu \in \mathfrak{a}^*$ denote the Langlands parameter of* Π [W, 5.5.3]. *Suppose $s \geqslant 2|\nu| - r + \iota$ (see (4.10), 5.7). Then there is $m \geqslant 0$ such that, for any Cartan subalgebra $\mathfrak{h}_F$, $F \subset \Delta_0$,*

$$|D(x)|^{1/2} ch^s(x)|\Theta(\tilde{c}_-(x))| \leqslant const(1 + |x|)^m \qquad (x \in \mathfrak{h}_F).$$

Let us denote by κ the Gelfand-Kirillov dimension of the representation Π [BV].

Then as was shown in [BV], $dim\, Ass(I_\Pi) = 2\kappa$, and there is a function $u_\kappa(x)$, $x \in \mathfrak{g}^{rs}$, homogeneous of degree $-\kappa$, such that

$$\lim_{t \to 0} t^\kappa \Theta(exp(tx)) = u_\kappa(x) \qquad (x \in \mathfrak{g}^{rs}). \tag{5.10}$$

Since the functions $\tilde{c}_-(x)$ and $exp(2x)$, $x \in \mathfrak{g}$, have the same derivative at $x = 0$, (5.10) implies that

$$\lim_{t \to 0} t^\kappa ch^s(tx)\Theta(\tilde{c}_-(tx)) = const\; u_\kappa(x) \qquad (x \in \mathfrak{g}^{rs}). \tag{5.11}$$

THEOREM 5.12. *Let $\nu \in \mathfrak{a}^*$ denote the Langlands parameter of Π [W, 5.5.3]. Suppose*

$$s > 2|\nu| - r - \iota. \tag{*}$$

Then there is $l \geqslant 0$ such that

$$\int_{\mathfrak{g}} ch^s(x)|\Theta(\tilde{c}_-(x))|(1 + |x|)^{-l}\, dx < \infty, \tag{a}$$

so that the integral (5.5), with $\psi \in S(\mathfrak{g})$, defines a tempered distribution $ch^s \cdot \tilde{c}_-^* \Theta \in S^*(\mathfrak{g})$.

Moreover, if Π has the Gelfand-Kirillov dimension κ, then the function u_κ (5.9) is locally integrable and defines a tempered distribution via integration against the Haar measure (without the assumption (*)), and for any $\psi \in S(\mathfrak{g})$,

$$\lim_{t \to 0} t^\kappa \int_{\mathfrak{g}} ch^s(tx)\Theta(\tilde{c}_-(tx))\psi(x)\, dx = \int_{\mathfrak{g}} u_\kappa(x)\psi(x)\, dx. \tag{b}$$

Proof. Since the function u_κ is equal to $|D|^{-1/2}$ times a locally bounded function and since the $|D|^{-1/2}$ is locally integrable [W, 7.3.9], so is u_κ. Hence the integral

$$\int_{\mathfrak{g}} u_\kappa(x)\psi(x)\, dx \qquad (\psi \in S(\mathfrak{g}))$$

converges absolutely and defines a tempered distribution; see [Hö, 7.1.18].

The integral formula [W, 2.4.3], expressing the integral over $\mathfrak{g}$ as a sum of integrals over Cartan subalgebras, together with Theorem 5.6 imply that the statements (a) and (b) will follow if we show that for any Cartan subalgebra $\mathfrak{h}_F$

(a′) there is $j \geqslant 0$ such that $\displaystyle\int_{\mathfrak{h}_F} |D(x)|^{1/2} ch^s(x)|\Theta(\tilde{c}_-(x))|(1 + |x|)^{-j}\, dx < \infty$, and for any $\psi \in S(\mathfrak{h}_F)$,

$$\text{(b}'\text{)} \quad \lim_{t \to 0} t^{\kappa} \int_{\mathfrak{h}_F} |D(x)|^{1/2} ch^s(tx)\Theta(\tilde{c}_-(tx))\psi(x)\, dx = \int_{\mathfrak{h}_F} |D(x)|^{1/2} u_{\kappa}(x)\psi(x)\, dx.$$

By the assumption (*) and Lemma 5.9, we have

$$(1) \qquad |D(x)|^{1/2} ch^s(x)|\Theta(\tilde{c}_-(x))| \leqslant const \cdot ch^{s'}(x)(1 + |x|)^m \qquad (x \in \mathfrak{h}_F)$$

for some $m \geqslant 0$ and $s' > -2\iota$. The condition on s' is chosen so that the function $ch^{s'}$ is locally integrable on $\mathfrak{h}_F$, and hence (a′) follows.

Recall from [BV] that for each connnected component $\mathscr{C}(\mathfrak{h}_F^r)$ of $\mathfrak{h}_F^r$ there is $\varepsilon > 0$ such that the function on the left-hand side of the inequality (1), when restricted to the region of $x \in \mathscr{C}(\mathfrak{h}_F^r)$ with $|x| < \varepsilon$, has a real analytic extension to an open neighborhood of the closure of this region. Hence Taylor's formula [Hö, (1.1.7)′] reveals that for $\psi \in S(\mathfrak{h}_F)$

$$\lim_{t \to 0} t^{\kappa} \int_{|tx| < \varepsilon} |D(x)|^{1/2} ch^s(tx)\Theta(\tilde{c}_-(tx))\psi(x)\, dx = \int_{\mathfrak{h}_F} |D(x)|^{1/2} u_{\kappa}(x)\psi(x)\, dx.$$

Denote the homogeneity degree of the function $|D(x)|^{1/2}$ by m'. Set $\kappa' = m' - \kappa$. Then $\kappa' \geqslant 0$ and by (1)

$$\begin{aligned} & t^{\kappa} \int_{|tx| \geqslant \varepsilon} |D(x)|^{1/2} ch^s(tx)|\Theta(\tilde{c}_-(tx))\psi(x)|\, dx \\ &= t^{-\kappa'} \int_{|tx| \geqslant e} |D(tx)|^{1/2} ch^s(tx)|\Theta(\tilde{c}_-(tx))\psi(x)|\, dx \\ &\leqslant const \cdot t^{-\kappa'} \int_{|x| \geqslant t^{-1}\varepsilon} ch^{s'}(tx)(1 + |tx|)^m |\psi(x)|\, dx. \end{aligned}$$

An elementary argument shows that the last integral tends to zero if $t \to 0$. ∎

Let $\rho_{\mathfrak{n}}(x) = \frac{1}{2}\, tr(ad(x)|_{\mathfrak{n}})$, $x \in \mathfrak{a}$, as in Section 4. Then $|\rho_{\mathfrak{n}}| = (r - 1)/2$ (see (4.3)) and the character Θ (or $\Theta|_{\tilde{G}_1}$ if G is an orthogonal group with $dim_{\mathbf{D}}\, V$ even (see 3.3) in the Theorem 5.12 has the rate of growth $\gamma = |\nu|/|\rho_{\mathfrak{n}}|$, if Π is unitary. Thus (if Π is unitary) the condition 5.12(*) may be rewritten as

$$(5.12{*}') \qquad s > \gamma(r - 1) - r - \iota.$$

6. Asymptotic behaviour of the integral intertwining distributions. Consider an irreducible dual pair $G, G' \subseteq Sp$, as in Section 3. Let Π be an irreducible unitary representation of $\tilde{G} \subseteq \widetilde{Sp}$. Then, under the assumption 3.1(a), the formula

$$(6.1) \qquad f = \int_{\tilde{G}} \overline{\Theta}_{\Pi}(g)\rho^{-1}\omega(g)\,dg$$

defines a tempered distribution on W. We shall always assume that the restriction of Π to the kernel of the covering map $\tilde{G} \to G$ is a multiple of the unique nontrivial character of this kernel, so that the function under the integral sign (6.1) is constant on the fibers of the covering map. The title of this section refers to the distribution f (6.1). If the condition 3.1(b) holds too, then the representation of $\tilde{G} \cdot \tilde{G}'$ on the image of the map $\rho(f)$: $H^\infty \to H^{\infty *}$ is (infinitesimally) irreducible, and f is indeed the intertwining distribution corresponding to Π, as defined in Section 2, or in [P1, Sec. 5].

Recall that G_1 denotes the Zariski-identity component of G. As we have already noticed, the complement $G_0 = G \backslash G_1$ is empty unless G is an orthogonal group over **R** or **C**. Set

$$(6.2) \qquad \tilde{f} = \int_{\tilde{G}_1} \overline{\Theta}_{\Pi}(g)\rho^{-1}\omega(g)\,dg, \qquad \tilde{\tilde{f}} = \int_{\tilde{G}_0} \overline{\Theta}_{\Pi}(g)\rho^{-1}\omega(g)\,dg.$$

Then obviously $f = \tilde{f} + \tilde{\tilde{f}}$. We shall study the asymptotic behaviour of these distributions using Howe's Theorem 1.7.

Let us fix a real analytic lifting $\tilde{c}$: $sp^c \to \widetilde{Sp}$ as in 1.7(b). Set $\tilde{c}_-(x) = \tilde{c}(x)\tilde{c}(0)^{-1}$, $x \in sp^c$. Then $\tilde{c}_-$ is a lifting of the Cayley transform c_-, as in 5.3. Let $j(x)$, $x \in \mathfrak{g}^c$, denote the Jacobian of $\tilde{c}_-$. Notice that $\tilde{c}(0)$ is in the center of $\widetilde{Sp}$ and hence in the center of $\tilde{G}$. Hence

$$(6.3) \qquad \tilde{f} = 2\int_{im\,\tilde{c}} \overline{\Theta}_{\Pi}(g)\rho^{-1}\omega(g)\,dg = 2\int_{\mathfrak{g}} \overline{\Theta}_{\Pi}(\tilde{c}(x))j(x)\rho^{-1}\omega(\tilde{c}(x))\,dx$$
$$= const \int_{\mathfrak{g}} \overline{\Theta}_{\Pi}(\tilde{c}_-(x))j(x)\rho^{-1}\omega(\tilde{c}(x))\,dx.$$

Since $im\,\tilde{c} \subseteq \tilde{G}_1$, the distribution $\tilde{f}$ (6.3) is well defined under the condition 3.1(a) with G replaced by G_1. We shall study the $\tilde{f}$ distribution under this possibly more general assumption.

Recall (Sec. 4) that the groups G, G' act on the defining modules V, V'. Thus we may identify G with a subgroup of $GL_{\mathbf{D}}(V)$, $\mathfrak{g}$ with a Lie subalgebra of $End_{\mathbf{D}}(V)$, and likewise for G', $\mathfrak{g}'$. Set $d' = dim_{\mathbf{D}}\, V'$. Recall the number r (4.10) and the function ch (5.2). One can normalize the Lebesgue measure on $\mathfrak{g}$ so that the Jacobian $j(x) = ch^{-2r}(x)$, $x \in \mathfrak{g}^c$; see [P1, (3.11)]. From the structure of dual pairs and from 1.7(b) we see that, for $x \in \mathfrak{g}$, $\Theta_\omega(\tilde{c}(x)) = \Theta_\omega(\tilde{c}(0))ch^{d'}(x)$. For $x \in \mathfrak{g}$ set $\sigma(x) =$

$|det(i - Jx)|^{1/2d'}$ (see (1.10)) so that

$$|\Omega(\tilde{c}(x))| = const\ ch^{d'}(x)\sigma^{-d'}(x) \qquad (x \in \mathfrak{g}^c). \tag{6.4}$$

LEMMA 6.5. *There is a seminorm q on $S(W)$ such that*

(a) $$\left|\int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\,dw\right| \leqslant q(\phi)\sigma^{-d'}(x) \qquad (\phi \in S(W),\ x \in \mathfrak{g}),$$

(b) $$\int_{\mathfrak{g}} |\Theta_\Pi(\tilde{c}_-(x))|\ ch^{d'-2r}(x)\sigma^{-d'}(x)\,dx < \infty.$$

The distribution (6.3) *is given by the explicit formula*

(c) $$\tilde{f}(\phi) = const \int_{\mathfrak{g}} \overline{\Theta}_\Pi(\tilde{c}_-(x))\ ch^{d'-2r}(x) \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\,dw\,dx,$$

where $\phi \in S(W)$ and the integral over $\mathfrak{g}$ is absolutely convergent. Moreover, if G is Zariski-connected, then

(d) $$f = 2\tilde{f}$$

and

(e) $$f = \tilde{f}, \qquad \text{if } G \neq G_1 \text{ and } \Pi|_{\tilde{G}_1} \text{ is reducible}.$$

Proof. By substituting $g = \tilde{c}(x)$ in 1.13(a) and comparing the result with 1.7(c), (5.2), and (6.4), we obtain 6.5(a). Since we work under the condition

$$\int_{\tilde{G}_1} |\Theta_\Pi(g)|\Omega(g)\,dg < \infty,$$

the change of variables $g = \tilde{c}(x)$ implies the finiteness 6.5(b). Statement (c) follows directly from (a), (b), and (6.3). For (d) see (6.2) and (6.3). Finally, if $\Pi|_{\tilde{G}_1}$ is reducible, then the character Θ_Π is zero on $\tilde{G}_0$, so that $\tilde{f} = 0$ and (e) follows. ■

For a function ϕ on W and for $t > 0$ let $\phi_t(w) = t^{-2n}\phi(t^{-1}w)$, $w \in W$, $2n = dim\ W$, as usual [Hö, (3.2.18)].

THEOREM 6.6. *Let G, G' be an irreducible dual pair, with $2r \leqslant d'$. Then there is a seminorm q on $S(W)$ such that*

(a) $$\int_{\mathfrak{g}} \left|\int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\,dw\right| dx \leqslant q(\phi) \qquad (\phi \in S(W)).$$

Suppose $\dim \Pi < \infty$ *and set* $m = 2 \dim_{\mathbf{R}} \mathfrak{g}$. *Then the condition* 3.1(a) *holds, and the distribution* (6.3) *satisfies*

$$\text{(b)} \qquad \lim_{t\to\infty} t^m \tilde{f}(\phi_t) = C \int_{\mathfrak{g}} \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\, dw\, dx$$

for all $\phi \in S(W)$, *where* C *is a nonzero constant.*

Proof. It follows from 4.11 and (4.4) that

$$\text{(1)} \qquad \int_{\tilde{G}} \Omega^{(2r-1)/d'}(g)\, dg < \infty .$$

Hence (see (6.4))

$$\text{(2)} \quad \int_{\mathfrak{g}} \sigma^{-2r}(x)\, dx = \int_{\mathfrak{g}} (ch(x)\sigma^{-1}(x))^{2r} ch^{-2r}(x)\, dx$$

$$= const \int_{im\, \tilde{c}} \Omega^{2r/d'}(g)\, dg \leqslant const \int_{im\, \tilde{c}} \Omega^{(2r-1)/d'}(g)\, dg < \infty .$$

Since the function $\det(iI - Jx)$ (1.10) is a product of terms like $1 + a^2$, where a is an eigenvalue of the symmetric map $Jx \in End(W)$, we see that

$$\text{(3)} \qquad 1 \leqslant \sigma(\varepsilon x) \leqslant \sigma(x) \qquad (0 \leqslant \varepsilon \leqslant 1,\ x \in \mathfrak{g}).$$

In particular, $\int_{\mathfrak{g}} \sigma^{-d'}(x)\, dx \leqslant \int_{\mathfrak{g}} \sigma^{-2r}(x)\, dx < \infty$, and (a) follows from 6.5(a).

If $\dim \Pi < \infty$, then the function $\Theta_\Pi(g)$, $g \in \tilde{G}$, is bounded, and (1) implies that the condition 3.1(a) holds.

Further (3), (6.4), and the fact that Ω is bounded imply that for $0 \leqslant \varepsilon \leqslant 1$

$$\text{(6.7)} \quad |\Theta_\Pi(\tilde{c}_-(\varepsilon x))|\, ch^{d'-2r}(\varepsilon x) \leqslant const\ ch^{d'-2r}(\varepsilon x) \leqslant const\ \sigma^{d'-2r}(x) \qquad (x \in \mathfrak{g}).$$

A straightforward change of variables shows that for $t > 0$ and $\phi \in S(W)$

$$t^m \tilde{f}(\phi_t) = const \int_{\mathfrak{g}} \overline{\Theta}_\Pi(\tilde{c}_-(t^{-2}x)) ch^{d'-2r}(t^{-2}x) \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\, dw\, dx .$$

By combining this formula with (6.7), 6.5(a), and (2), we obtain (b). ■

THEOREM 6.8. *Under the assumptions of Theorem* 6.6, *suppose that the function*

$$\overline{\Theta}_\Pi(\tilde{c}_-(x)) ch^{d'-2r}(x) = \sum_j a_j(x) \qquad (x \in \mathfrak{g}^c),$$

where the sum is finite and each function $a_j(x)$, $x \in \mathfrak{g}$, is homogeneous of degree j. Then there is a seminorm q on $S(W)$ such that for all j

$$\text{(a)} \qquad \int_{\mathfrak{g}} |a_j(x)| \left| \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\, dw \right| dx \leqslant q(\phi) \qquad (\phi \in S(W)),$$

so that the formula

$$\tilde{f}_j(\phi) = \int_{\mathfrak{g}} a_j(x) \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\, dw\, dx \qquad (\phi \in S(W))$$

defines a tempered distribution $\tilde{f}_j$ on W, homogeneous of degree $-2\dim_{\mathbf{R}} \mathfrak{g} - 2j$, and the distribution (6.3)

$$\text{(c)} \qquad \tilde{f} = \sum_j \tilde{f}_j$$

is a finite sum of homogeneous distributions.

Proof. Notice that there are continuous functions p_j: $[1/2, 1] \to \mathbf{C}$ such that the integral

$$\int_{1/2}^{1} p_j(\varepsilon)\varepsilon^{j'}\, d\varepsilon$$

is equal to 1 if $j = j'$ and is equal to 0 if $j \neq j'$ for all $a_j \neq 0$, $a_{j'} \neq 0$. Thus

$$a_j(x) = \int_{1/2}^{1} p_j(\varepsilon)\overline{\Theta}_{\Pi}(\tilde{c}_-(\varepsilon x))ch^{d'-2r}(\varepsilon x)\, d\varepsilon \qquad (x \in \mathfrak{g}^c).$$

Therefore, (6.7) implies that

$$|a_j(x)| \leqslant const\ \sigma^{d'-2r}(x) \qquad (x \in \mathfrak{g}),$$

and (a) follows from 6.5(a) and 6.6(2). The rest is clear. ■

If the representation Π is infinite-dimensional, then the estimates 6.5(a) and (b) do not seem to be sufficient to obtain limit formulas like 6.6(b). Our main tool in this case is Theorem 5.12. In order to be able to use it, we need to know that the function on the left-hand side of 6.6(a) is rapidly decreasing. For that reason we have to impose some additional conditions on the function ϕ.

Each element $w \in W$ defines a linear map

$$\text{(6.9)} \qquad \mathfrak{g} \ni x \to x(w) \in W.$$

Let $W_{\mathfrak{g}} = \{w \in W;$ the map (6.9) is injective$\}$. The fundamental estimate of the method of stationary phase [Hö, 7.7.1], i.e. integration by parts, yields the following lemma.

Lemma 6.10. *Suppose $W_{\mathfrak{g}} \neq \emptyset$ and $\phi \in C_c^\infty(W_{\mathfrak{g}})$. Then for any $l \geqslant 0$ there is a constant $C_l < \infty$ such that*

$$\left| \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\, dw \right| \leqslant C_l(1 + |x|^2)^{-l} \qquad (x \in \mathfrak{g}).$$

Proof. For a fixed $x \in \mathfrak{g}$, the derivative of the function

$$W \ni w \to \tau_{\mathfrak{g}}(w)x \in \mathbf{R},$$

at $w \in W$, coincides with the linear map

$$W \ni w' \to \frac{1}{2}\langle x(w), w' \rangle \in \mathbf{R}.$$

Let $\phi \in C_c^\infty(W_{\mathfrak{g}})$. Fix a norm $|\ |$ on W and on $\mathfrak{g}$. Then

$$\inf_{|x|=1} \inf_{\phi(w) \neq 0} |x(w)| > 0.$$

Thus the lemma follows from [Hö, 7.7.1]. ∎

Denote by κ the Gelfand-Kirillov dimension of Π and let u_κ be the lowest term in the asymptotic expansion of Θ_Π (5.10). Then the Fourier transform $(\mathscr{F}_{\mathfrak{g}}^{-1})^*(u_\kappa)$ is a finite sum of nonzero G-invariant measures, supported on closures of nilpotent coadjoint orbits, of the same dimension 2κ, [BV].

Since the restriction of the moment map $\tau_{\mathfrak{g}}$ to $W_{\mathfrak{g}}$ is a submersion, it defines a continuous pullback

$$\tau_{\mathfrak{g}}^*\colon C_c^\infty{}^*(\tau_{\mathfrak{g}}(W_{\mathfrak{g}})) \to C_c^\infty{}^*(W_{\mathfrak{g}}); \tag{6.11}$$

see [Hö, 6.1.2].

Theorem 6.12. *Let G, G' be an irreducible dual pair. Suppose the character Θ_Π has the rate of growth $\gamma \geqslant 0$ such that $d' > \gamma(r-1) + r - \iota$. Then the condition* 3.1(a) *holds. (Here we replace G by G_1 and Π by $\Pi|_{\tilde{G}_1}$ in the case explained in* 3.3.) *Let $m = 2\, dim_{\mathbf{R}}\, \mathfrak{g} - 2\kappa$. Then*

$$\begin{aligned} \lim_{t\to\infty} t^m \tilde{f}(\phi_t) &= \tau_{\mathfrak{g}}^*(\mu_\kappa|_{\tau_{\mathfrak{g}}(W_{\mathfrak{g}})})(\phi) \\ &= const \int_{\mathfrak{g}} \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\, dw\, u_\kappa(x)\, dx \qquad (\phi \in C_c^\infty(W_{\mathfrak{g}})). \end{aligned}$$

Proof. Since $\iota \leqslant 1$, the rate of growth $\gamma < \gamma_{max}$ (4.11(b)), and therefore Θ_Π satisfies 3.1(a). Notice that

$$\tilde{f}(\phi) = \int_{\mathfrak{g}} ch^{d'-2r}(x)\overline{\Theta}_\Pi(\tilde{c}_-(x))\psi(x)\,dx,$$

where

$$\psi(x) = \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\,dw \qquad (x \in \mathfrak{g})$$

is a rapidly decreasing function, by 6.10. Since $d' - 2r > \gamma(r-1) - r - \iota$, all the assumptions of 5.12 are satisfied (see (5.11∗′)), and our theorem follows from 5.12(b). ■

Remark 6.13. Theorems 6.8 and 6.12 prove Theorem (5.9) in [P1] without any use of deep microlocal results like [Hö, 8.2.4]. Notice also that if the pair G, G' is in the stable range with G the smaller member, then the distribution $\tilde{f}_0$ (6.8) is a measure. This was conjectured in [P1, (5.23)].

The purpose of Theorem 6.8 is to distinguish a class of irreducible unitary representations of G' which are attached to nilpotent coadjoint orbits via a "Cayley-Kirillov-Rossmann" character formula. In fact, if G is compact, then we have the character formula [P1, (6.7)]

$$(*) \qquad ch_{\mathfrak{g}'}^{-d} \cdot \tilde{c}_-^* \Theta_{\Pi'} = const_\Pi \cdot \mathscr{F}_{\mathfrak{g}'}^* \circ \tau_{\mathfrak{g}'}(f)$$

where $\tau_{\mathfrak{g}'}(f)(\psi) = f(\psi \circ \tau_{\mathfrak{g}'})$, $\psi \in S(\mathfrak{g}'^*)$. Suppose further that the pair G, G' is in the stable range with G the smaller member and that the representation Π satisfies the conditions of 6.8. Then $\tau_{\mathfrak{g}'}(f)$ is a finite sum of homogeneous distributions with the support equal to the closure of a single nilpotent coadjoint orbit; see 7.10. Thus in such a case, (∗) is the formula we are looking for.

Suppose G is not Zariski-connected. Then $\mathbf{D} = \mathbf{R}$ or $\mathbf{C}$ and G is an orthogonal group. Let V_1 be a one-dimensonal anisotropic subspace of V. Set $V_2 = V_1^\perp$. Then $V = V_1 \oplus V_2$. Define an element $b \in G$ by $b|_{V_1} = -1, b|_{V_2} = 1$. Then the determinant of b is -1, so that $b \in G_0$. Denote the -1, 1 eigenspaces of $Ad\, b$ in $\mathfrak{g}$ by $\mathfrak{g}_1$, $\mathfrak{g}_2$, respectively. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The space $\mathfrak{g}_2$ coincides with the Lie subalgebra of $\mathfrak{g}$ preserving the subspace $V_2 \subseteq V$. The elements of $\mathfrak{g}_1$ map V_2 into V_1. Recall that the symplectic space $W = Hom_{\mathbf{D}}(V', V)$. Let $W_i = \{w \in W; im\, w \subseteq V_i\}$, $i = 1, 2$. Then $W = W_1 \oplus W_2$ and $W_2 = W_1^\perp$.

LEMMA 6.14. *There is a seminorm q on $S(W)$ such that*

$$\text{(a)} \qquad \left| \int_{W_2} \chi(\tau_{\mathfrak{g}_2}(w_2)x_2)\phi(x_1(w_2) + w_2)\,dw_2 \right| \leqslant q(\phi)\sigma^{-d'}(x_1 + x_2),$$

for $x_1 \in \mathfrak{g}_1$, $x_2 \in \mathfrak{g}_2$, *and* $\phi \in S(W)$. *Fix an element* $\tilde{b} \in \tilde{G}$ *in the preimage of* b. *Then* (*recall that we work under the assumption* 3.1(a))

(b) $$\int_{\mathfrak{g}} |\Theta_\Pi(\tilde{c}_-(x)\tilde{b})|\, ch^{d'-2r}(x)\sigma^{-d'}(x)\, dx < \infty .$$

The distribution $\tilde{\tilde{f}}$ (6.2) *is given by the explicit formula*

(c) $$\tilde{\tilde{f}}(\phi) = const \int_{\mathfrak{g}_1}\int_{\mathfrak{g}_2} \overline{\Theta}_\Pi(\tilde{c}_-(x_1+x_2)\tilde{b})ch^{d'-2r}(x_1+x_2) \int_{W_2} \chi(\tau_{\mathfrak{g}_2}(w_2)x_2)\phi(x_1(w_2)+w_2)\, dw_2\, dx_2\, dx_1 ,$$

where $\phi \in S(W)$.

Proof. For $x \in sp$ and $\phi \in S(W)$ set

$$\gamma_x(\phi) = \int_W \chi\left(\frac{1}{4}\langle x(w), w\rangle\right)\phi(w)\, dw .$$

Let $x_1 \in \mathfrak{g}_1$, $x_2 \in \mathfrak{g}_2$, and $x = x_1 + x_2$. Then

(1) $$\begin{aligned} &\gamma_x \natural \rho^{-1}\omega(\tilde{b})(\phi) \\ &= const \int_{W_1}\int_{W_1}\int_{W_2} \gamma_x(w_1'+w_2)\chi\left(\frac{1}{2}\langle w_1', w_1\rangle\right)\phi(w_1+w_2)\, dw_2\, dw_1\, dw_1' \\ &= const \int_{W_1}\int_{W_1}\int_{W_2} \gamma_{x_2}(w_2)\chi\left(\frac{1}{2}\langle w_1', w_1 - x_1(w_2)\rangle\right)\phi(w_1+w_2)\, dw_2\, dw_1\, dw_1' \\ &= const \int_{W_2} \gamma_{x_2}(w_2)\phi(x_1(w_2)+w_2)\, dw_2 . \end{aligned}$$

Hence, via replacing ϕ by $\rho^{-1}\omega\natural\phi$ in 6.5(a), we obtain (a). To check (b) we use the fact that $\tilde{b} \in \tilde{K} \subseteq \tilde{U}$ and that Ω is $\tilde{U}$-bi-invariant:

$$\begin{aligned} \int_{\mathfrak{g}} |\Theta_\Pi(\tilde{c}_-(x)\tilde{b})|\, ch^{d'-2r}(x)\sigma^{-d'}(x)\, dx &= \int_{\mathfrak{g}} |\Theta_\Pi(\tilde{c}_-(x)\tilde{b})|\,\Omega(\tilde{c}(x))ch^{-2r}(x)\, dx \\ &\leqslant \int_{\tilde{G}} |\Theta_\Pi(g\tilde{b})|\,\Omega(g)\, dg = \int_{\tilde{G}} |\Theta_\Pi(g\tilde{b})|\,\Omega(g\tilde{b})\, dg = \int_{\tilde{G}} |\Theta_\Pi(g)|\,\Omega(g)\, dg < \infty \end{aligned}$$

where the finiteness follows from 3.1(a). Finally, (c) follows from (1) because, by the definition (6.2) and 1.7,

$$\tilde{\tilde{f}} = const \int_{\mathfrak{g}} \overline{\Theta}_\Pi(\tilde{c}_-(x)\tilde{b}) ch^{d'-2r}(x)\gamma_x \natural \rho^{-1}\omega(\tilde{b})\, dx. \qquad \blacksquare$$

LEMMA 6.15. *Let $W_{\mathfrak{g}\mathfrak{g}} = \{w \in W;\ im\ w = V\}$. Then for any function $\phi \in C_c^\infty(W_{\mathfrak{g}\mathfrak{g}})$ and for any $l \geqslant 0$, there is a constant $C < \infty$ such that*

$$\text{(a)} \qquad \left| \int_{W_2} \chi(\tau_{\mathfrak{g}_2}(w_2)x_2)\phi(x_1(w_2) + w_2)\, dw_2 \right| \leqslant C(1 + |x_1 + x_2|^2)^{-l}$$

for all $x_1 \in \mathfrak{g}$ and $x_2 \in \mathfrak{g}_2$.

Suppose $\dim \Pi < \infty$ and $d' > 2r$. Let $m = 2 \dim_{\mathbf{R}} \mathfrak{g}$. Then

$$\text{(b)} \qquad \lim_{t \to \infty} t^m \tilde{\tilde{f}}(\phi_t) = 0 \qquad (\phi \in C_c^\infty(W_{\mathfrak{g}\mathfrak{g}})).$$

Proof. Fix $\phi \in C_c^\infty(W_{\mathfrak{g}\mathfrak{g}})$. Since $inf\{|x_2(w)|;\ |x_2| = 1,\ w_1 + w_2 \in supp\ \phi\} > 0$ and $sup\{|x_1|;\ x_1(w_2) + w_2 \in supp\ \phi\} < \infty$, (a) follows by the argument used in the proof of 6.10. Set $m' = (d' - 2r) \dim_{\mathbf{R}} \mathbf{D}$. A straightforward change of variables yields

$$t^m \tilde{\tilde{f}}(\phi_t) = t^{-m'} const \int_{\mathfrak{g}_1} \int_{\mathfrak{g}_2} \overline{\Theta}_\Pi(\tilde{c}_-(x_1 + t^{-2}x_2)\tilde{b}) ch^{d'-2r}(x_1 + t^{-2}x_2)$$
$$\int_{W_2} \chi(\tau_{\mathfrak{g}_2}(w_2)x_2)\phi(x_1(w_2) + w_2)\, dw_2\, dx_2\, dx_1;$$

hence (b) follows from (a). $\blacksquare$

CONJECTURE 6.16. *The statement* 6.15(b), *with $m = 2 \dim_{\mathbf{R}} \mathfrak{g} - 2\kappa$, holds under the assumptions of* 6.12.

We summarize the main results of this section in the following statement.

THEOREM 6.17. *Let G, G' be an irreducible dual pair and let Π be an irreducible unitary representation of $\tilde{G}$ satisfying the assumptions explained in the definition of the distribution f (6.1). Denote by κ the Gelfand-Kirillov dimension of Π and let u_κ be the lowest term in the asymptotic expansion of the character Θ_Π (5.9). Set $m = 2 \dim_{\mathbf{R}} \mathfrak{g} - 2\kappa$. Then under any of the assumptions*

(a) *$G = G_1$, $\dim \Pi < \infty$, $d' \geqslant 2r$, and $\phi \in S(W)$; or*
(b) *$G = G_1$, $\dim \Pi \leqslant \infty$, Θ_Π has the rate of growth $\gamma \geqslant 0$, $d' > \gamma(r - 1) + r - \iota$, and $\phi \in C_c^\infty(W_{\mathfrak{g}})$; or*
(c) *$G \neq G_1$, $\Pi|_{\tilde{G}_1}$ is irreducible, $\dim \Pi < \infty$, $d' > 2r$, and $\phi \in C_c^\infty(W_{\mathfrak{g}\mathfrak{g}})$; or*
(d) *$G \neq G_1$, $\Pi|_{\tilde{G}_1}$ is reducible, $\dim \Pi < \infty$, $d' \geqslant 2r$, and $\phi \in S(W)$; or*
(e) *$G \neq G_1$, $\Pi|_{\tilde{G}_1}$ is reducible, $\dim \Pi \leqslant \infty$, $\Theta_{\Pi|_{\tilde{G}_1}}$ has the rate of growth $\gamma \geqslant 0$, $d' > \gamma(r - 1) + r - \iota$, and $\phi \in C_c^\infty(W_{\mathfrak{g}})$,*

there is a constant $C \neq 0$ such that formula

$$(*) \qquad \lim_{t\to\infty} t^m f(\phi_t) = C \int_{\mathfrak{g}} \int_W \chi(\tau_{\mathfrak{g}}(w)x)\phi(w)\, dw\, u_\kappa(x)\, dx.$$

holds.

In all cases (a)–(e), *the integral over* $\mathfrak{g}$ *in* $(*)$ *is absolutely convergent and defines a distribution on* W, $W_{\mathfrak{g}}$, $W_{\mathfrak{g}\mathfrak{g}}$ *respectively, supported on* $\tau_{\mathfrak{g}}^{-1}(\text{supp}\ \mu_\kappa)$, *where* $\mu_\kappa = (\mathscr{F}_{\mathfrak{g}}^{-1})^*(u_\kappa)$.

Proof. Suppose $G = G_1$. Then $f = 2\tilde{f}$, by 6.5(d). Hence the theorem follows from 6.6 and 6.12. This takes care of cases (a) and (b).

Suppose $G \neq G_1$. In case (c) the theorem follows from 6.6 and 6.15(b). In cases (d) and (e) $f = \tilde{f}$, by 6.5(e); hence the theorem follows from 6.6 and 6.12. ■

7. Associated varieties and wave-front sets. Let us choose some linear coördinates $w_1, w_2, \ldots, w_{2n}$ on W. Then a polynomial coefficient differential operator P on W may be written as a finite sum

$$P = P(w, \partial) = \sum_{\alpha,\beta} a_{\alpha\beta} w^\alpha \partial_w^\beta$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{2n})$ are multi-indices. Set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{2n}$. Let $k = max\{|\alpha| - |\beta|;\ a_{\alpha\beta} \neq 0\}$. Put

$$(7.1) \qquad P_{max} = P_{max}(w, \partial) = \sum_{|\alpha|-|\beta|=k} a_{\alpha\beta} w^\alpha \partial_w^\beta.$$

LEMMA 7.2. *Suppose W_0 is an open cone in W, f, and f_0 are distributions on W_0 and suppose m is an integer such that*

$$\lim_{t\to\infty} t^m f(\phi_t) = f_0(\phi) \qquad (\phi \in C_c^\infty(W_0)).$$

Then for any polynomial coefficient differential operator P on W, $f \circ P = 0$ implies $f_0 \circ P_{max} = 0$, on W_0.

Proof. We calculate

$$0 = t^{m-k} f \circ P(\phi_t) = \sum_{\alpha\beta} a_{\alpha\beta} t^{m-k+|\alpha|-|\beta|} f((w^\alpha \partial_w^\beta \phi)_t)$$

$$= t^m f((P_{max}\phi)_t) + \sum_{-k+|\alpha|-|\beta|<0} a_{\alpha\beta} t^{-k+|\alpha|-|\beta|} t^m f((w^\alpha \partial_w^\beta \phi)_t)$$

and take the limit with $t \to \infty$. ■

Let $W_{\mathbf{C}} = W \otimes_{\mathbf{R}} \mathbf{C}$ denote the complexification of W. The form $\langle\ ,\ \rangle$ extends to

$W_{\mathbf{C}}$. Let $\mathbf{W}$ be the Weyl algebra associated to the form $2\pi i\langle\ ,\ \rangle$ on $W_{\mathbf{C}}$, i.e. $\mathbf{W}$ is the quotient of the tensor algebra of $W_{\mathbf{C}}$ by the ideal generated by the elements

$$w \otimes w' - w' \otimes w - 2\pi i\langle w, w'\rangle \qquad (w, w' \in W_{\mathbf{C}}).$$

Let $\mathbf{W}_k$ be the subspace of $\mathbf{W}$ spanned by the identity and products of at most $k \geqslant 0$ elements of $W_{\mathbf{C}}$. Then $\mathbf{W}_0 \subseteq \mathbf{W}_1 \subseteq \mathbf{W}_2 \subseteq \cdots$ is an exhaustive filtration of $\mathbf{W}$. The associated graded algebra

$$gr\ \mathbf{W} = \bigoplus_{k=0}^{\infty} \mathbf{W}_k/\mathbf{W}_{k-1} \qquad (\mathbf{W}_{-1} = 0)$$

may be identified with the algebra $\mathscr{P}(W_{\mathbf{C}})$ of polynomial functions on $W_{\mathbf{C}}$ by the formula

$$(7.3)\qquad gr(\mathbf{w})(w) = \frac{1}{k!}[\cdots[[\mathbf{w}, \underbrace{w]w]\cdots w]}_{k\text{ times}} \qquad (\mathbf{w} \in \mathbf{W}_k\backslash\mathbf{W}_{k-1},\ w \in W_{\mathbf{C}}).$$

Let $\mathscr{P}\mathscr{D}(W)$ denote the algebra of polynomial coefficient differential operators on W. For $w \in W$ define $\tilde{\partial}(w) \in \mathscr{P}\mathscr{D}(W)$ by

(7.4)

$$\tilde{\partial}(w)\phi(w') = \lim_{t\to 0} t^{-1}(\delta_{tw}\natural\phi(w') - \phi(w')) = \lim_{t\to 0} t^{-1}(e^{\pi i\langle w, w'\rangle t}\phi(w' - tw) - \phi(w'))$$

$$= \pi i\langle w, w'\rangle\phi(w') + \partial_w * \phi(w') \qquad (\phi \in S(W),\ w' \in W),$$

where $\partial_w * \phi$ indicates the usual abelian directional derivative of ϕ in the direction of $-w$ [H2, (19.2.2)]. Then $\tilde{\partial}$ extends to an injective algebra homomorphism

$$(7.5)\qquad \tilde{\partial}\colon \mathbf{W} \to \mathscr{P}\mathscr{D}(W).$$

Recall the symplectic Fourier transform [H4, 2.1]

$$(7.6)\quad \hat{\phi}(w') = 2^{-n}\int_W \phi(w)\chi\left(\frac{1}{2}\langle w, w'\rangle\right)dw, \quad \hat{f}(\phi) = f(\hat{\phi})$$

$$(\phi \in S(W),\ w' \in W,\ 2n = \dim W,\ f \in S^*(W)).$$

LEMMA 7.7. *Suppose W_0 is an open cone in W, $f \in S^*(W)$, $f_0 \in C_c^\infty{}^*(W_0)$ and there is $m \in \mathbf{R}$ such that*

$$\lim_{t\to\infty} t^m f(\phi_t) = f_0(\phi) \qquad (\phi \in C_c^\infty(W_0)).$$

Then for any $\mathbf{w} \in \mathbf{W}$, if $\hat{f} \circ \tilde{\partial}(\mathbf{w}) = 0$, then $gr\ \mathbf{w}|_{supp\, f_0} = 0$.

Proof. Let $\tilde{\tilde{\partial}}(\mathbf{w}) \in \mathscr{P}\mathscr{D}(W)$ denote the operator obtained via conjugation by the symplectic Fourier transform (7.6), i.e.,

$$f \circ \tilde{\tilde{\partial}}(\mathbf{w}) = (\hat{f} \circ \tilde{\partial}(\mathbf{w}))\hat{} \qquad (f \in S^*(W), \mathbf{w} \in \mathbf{W}).$$

Then (7.1), (7.3), and (7.4) imply that for $\mathbf{w} \in \mathbf{W}$

$$\tilde{\tilde{\partial}}(\mathbf{w})_{max}(w, \partial) = \tilde{\tilde{\partial}}(\mathbf{w})_{max}(w) = const\ gr\ \mathbf{w}(w) \qquad (w \in W).$$

Thus the lemma follows from 7.2. ∎

Let G, G' be an irreducible dual pair in $Sp(W)$. Let Π be an irreducible unitary representation of $\tilde{G}$ and let $\mu_\kappa \in S^*(\mathfrak{g}^*)$ be the Fourier transform of the lowest term in the asymptotic expansion of the character Θ_Π (see (5.9)). Denote by *max sup* μ_κ the union of the coadjoint orbits of maximal dimension ($= 2\kappa$) in *supp* μ_κ. Let $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{g}'_{\mathbf{C}}$ denote the complexifications of $\mathfrak{g}$, $\mathfrak{g}'$ respectively, and let $Ass(I_{\Pi'}) \subseteq \mathfrak{g}^*_{\mathbf{C}}$ be the associated variety of $I_{\Pi'}$, the annihilator of the Harish-Chandra module of Π' in the universal enveloping algebra of $\mathfrak{g}'$, [Ma].

PROPOSITION 7.8. *Assume that the positivity condition* 3.1(∗) *holds.*

(a) *If G and Π satisfy the assumptions of* 6.17, *then the irreducible unitary representation Π' of $\tilde{G}'$ constructed in* 3.1 *satisfies*

$$(*) \qquad Ass(I_{\Pi'}) \supseteq \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(max\, supp\ \mu_\kappa) \cap W_{\mathfrak{g}})$$

in all the cases 6.17(a)–(e) *except* (*possibly*?) 6.17(c). *In this last case,* (∗) *holds with $W_{\mathfrak{g}}$ replaced by $W_{\mathfrak{g}\mathfrak{g}}$.*

(b) *If $G \neq G_1$, $\Pi|_{\tilde{G}_1}$ is irreducible and Θ_Π has the rate of growth $\gamma \geqslant 0$ such that $d' > \gamma(r-1) + r - \iota$ (as in* 6.12), *then* (∗) *holds with $Ass(I_{\Pi'})$ replaced by $Ass(I_{\Pi'}) \cup Ass(I_{\Pi \otimes det)'})$; see* 3.3.

Proof. The positivity assumptions ensures that the formula (6.1) does indeed define the intertwining distribution f corresponding to the representation $\Pi \otimes \Pi'$; see 3.1.

Let us work under the assumptions of 6.17 first. Denote by f_0 the limit distribution defined by the right-hand side of 6.17(∗). This is a distribution on W, $W_{\mathfrak{g}}$, or $W_{\mathfrak{g}\mathfrak{g}}$ depending on the assumption 6.17(a)–6.17(e) we are considering. In fact, $W_{\mathfrak{g}\mathfrak{g}}$ occurs only in the case 6.17(c). Each of these sets is an open cone in W and the support of f_0 coincides with the preimage under $\tau_{\mathfrak{g}}$ of the support of μ_κ.

Suppose $\mathbf{w} \in \mathbf{W}$ and $f \circ \tilde{\partial}(\mathbf{w}) = 0$. Then since $\hat{f}$ is a nonzero constant multiple of f, we have $\hat{f} \circ \tilde{\partial}(\mathbf{w}) = 0$. Therefore, $gr\ \mathbf{w}|_{supp\, f_0} = 0$ by 7.7.

On the other hand, if $A(\mathfrak{g}', f) = \{\mathbf{w} \in \rho^{-1}\omega(\mathscr{U}(\mathfrak{g}');\ \mathbf{w}\natural f = 0\}$, then $Ass(I_{\Pi'}) = \tau_{\mathfrak{g}'}\{w \in W_{\mathbf{C}};\ gr(\mathbf{w})(w) = 0$ for all $\mathbf{w} \in \rho^{-1}\omega(\mathscr{U}(\mathfrak{g}'))\}$; see [P1, (7.26)]. Hence part (a) of the theorem follows.

The difficulty with part (b) is that Lemma 6.15 does not apply. Thus we do not have full control of the asymptotic behaviour of the intertwining distribution f, still given by the formula (6.1). However, we do know the asymptotic behaviour of the distribution $\tilde{f}$ (6.2); see 6.12. By 3.3 the image of $\rho(\tilde{f})$ is a realization of $\Pi \otimes \Pi' \oplus (\Pi \otimes det) \otimes (\Pi \otimes det)'$. Hence part (b) follows by an argument parallel to the one used to verify (a). ■

If the pair G, G' is of type I and $W = Hom_{\mathbf{D}}(V', V)$, set $W_{\mathfrak{g}\mathfrak{g}} = \{w \in W; im\, w = V\}$. (We have defined this set previously (6.15) only for the orthogonal groups.) If the pair G, G' is of type II and $W = Hom_{\mathbf{D}}(V, V') \oplus Hom_{\mathbf{D}}(V', V)$, set $W_{\mathfrak{g}\mathfrak{g}} = \{w = (S, T); ker\, S = 0 \text{ and } im\, T = V\}$. In all cases $W_{\mathfrak{g}\mathfrak{g}} \subseteq W_{\mathfrak{g}}$, see (6.9).

The polynomial maps $\tau_{\mathfrak{g}}, \tau_{\mathfrak{g}'}$ extend canonically to the complexification $W_{\mathbf{C}}$ of W. We denote these extended maps by the same letters.

THEOREM 7.9. *Suppose that*

$$(*) \qquad max\, supp\, \mu_{\kappa} \cap \tau_{\mathfrak{g}}(W_{\mathfrak{g}\mathfrak{g}}) \neq \emptyset.$$

Then

(a) *under the assumptions of* 7.8(a),

$$Ass(I_{\Pi'}) = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(Ass(I_{\Pi})));$$

(b) *under the assumptions of* 7.8(b),

$$Ass(I_{\Pi'}) \cup Ass((I_{(\Pi \otimes det)'}) = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(Ass(I_{\Pi}))).$$

Proof. Consider the case 7.9(a). As was shown in [P1, (7.1)], there is an inclusion

$$(1) \qquad Ass(I_{\Pi'}) \subseteq \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(Ass(I_{\Pi}))).$$

By a theorem of Borho, Brylinski, and Joseph [V2, Cor. 4.7] and [BV, Theorem 4.1], $Ass(I_{\Pi})$ coincides with the closure $\bar{O}_{\Pi}$ of the complex coadjoint orbit $O_{\Pi} \subseteq \mathfrak{g}^*$ of any element of $max\, supp\, \mu_{\kappa}$. Thus from 7.9(*) and 7.8(a) we see that

$$(2) \qquad Ass(I_{\Pi'}) \supseteq \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(O_{\Pi})).$$

The complexified algebraic groups $G_{\mathbf{C}}, G'_{\mathbf{C}}$ form a complex reductive dual pair, which is either irreducible or is a direct sum of two such pairs [H7]. In either case, we may define $W_{\mathbf{C}_{\mathfrak{g}_{\mathbf{C}}\mathfrak{g}_{\mathbf{C}}}}$. Since, by 7.9(*), $\tau_{\mathfrak{g}}^{-1}(O_{\Pi}) \cap W_{\mathbf{C}_{\mathfrak{g}_{\mathbf{C}}\mathfrak{g}_{\mathbf{C}}}} \neq \emptyset$, some analysis of the $G_{\mathbf{C}}, G'_{\mathbf{C}}$-orbits in $W_{\mathbf{C}}$ [KP2, 4.3] shows that

(3) $\tau_{\mathfrak{g}}^{-1}(O_{\Pi}) \cap W_{\mathbf{C}_{\mathfrak{g}_{\mathbf{C}}\mathfrak{g}_{\mathbf{C}}}}$ contains a unique orbit $O_{\Pi \otimes \Pi'}$ of maximal dimension

and

$$(4) \qquad \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(\bar{O}_{\Pi}))) = \bar{O}_{\Pi'}, \qquad \text{where} \quad O_{\Pi'} = \tau_{\mathfrak{g}'}(O_{\Pi \otimes \Pi'}).$$

Hence

$$\tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(\bar{O}_{\Pi})) \supseteq Ass(I_{\Pi'}) \supseteq \overline{\tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(O_{\Pi}))} = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(\bar{O}_{\Pi})) = \bar{O}_{\Pi'}.$$

This completes the proof of 7.9(a). The case 7.8(b) is analogous. ■

Notice that the above proof of 7.9(a) shows that $Ass(I_{\Pi'})$ is the closure of a single nilpotent orbit. Thus knowing the Borho-Brylinski-Joseph theorem for I_{Π}, we get it automatically for $I_{\Pi'}$.

COROLLARY 7.10. *Suppose the pair G, G' is in the stable range, with G the smaller member* (*see* 8.1), $\Pi \otimes \Pi' \in \mathscr{R}(G \cdot G', \omega)$, *dim* $\Pi < \infty$, *and* Π *is unitary. Then*

$$WF(\Pi') = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(WF(\Pi)))\,(= \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(0))).$$

Here the "WF" stands for the wave-front set of a unitary representation [H5].

Proof. We shall see in 8.6 that in this case the positivity condition 3.1(*) holds. Further, if $G \neq G_1$, then G is an orthogonal group and $d' > 2r$; see (4.10). Another nice property of the stable range is that $0 \in \tau_{\mathfrak{g}}(W_{\mathfrak{g}\mathfrak{g}})$; see [P1, (2.14)]. Thus we may apply 7.9(a):

$$Ass(I_{\Pi'}) = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(0)). \tag{1}$$

Some simple linear algebra shows that the intersection of the subset (1) of $\mathfrak{g}_{\mathbf{C}}^{*}$ with $\tau_{\mathfrak{g}'}(W) \subseteq \mathfrak{g}^{*}$ coincides with $\tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(0))$. Since the wave-front set is contained in the associated variety, (1) and 2.8 imply the inclusion

$$WF(\Pi') \subseteq \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(0)). \tag{2}$$

The set on the right-hand side of (2) is the closure of a single G'-orbit in $\mathfrak{g}'^{*}$ of real dimension equal to the complex dimension of the variety $Ass(I_{\Pi'})$, which in turn is equal to the real dimension of any maximal orbit in $WF(\Pi')$, by [BV]. Thus the inclusion (2) is an equality. ■

8. Positivity for a pair G, G' of type I. In this and in the next section, we take a closer look at the condition 3.1(b), or rather 3.2(b′). We do not have any final results describing it. We show that it holds in, and in many cases well beyond, the stable range [H3]; see 8.6 and 8.7.

In this section, $\mathbf{D} = (\mathbf{R}, \mathbf{C}, \mathbf{H})$ is a finite-dimensional division algebra, with an involution, over $\mathbf{R}$; V and V' are finite-dimensional vector spaces over $\mathbf{D}$ equipped with nondegenerate forms $(\ ,\ )$ and $(\ ,\ )'$, respectively—one hermitian and the other skew-hermitian. The groups G, G' are the isometry groups of the forms $(\ ,\ )$, $(\ ,\ )'$ respectively. The symplectic space $W = Hom_{\mathbf{D}}(V', V)$. The groups G and G' act on W via postmultiplication and premultiplication by the inverse, respectively. These actions embed G and G' into the symplectic group $Sp(W)$, [H7].

Here is a slight generalization of the notion of the stable range introduced in [H8].

Definition 8.1. The pair G, G' is in the almost stable range, with G the smaller member, if either of the following two conditions holds:

(a) if the form $(\ ,\ )$ is hermitian, then there is an isotropic subspace $X' \subseteq V'$ such that $dim_{\mathbf{D}}\, X' = dim_{\mathbf{D}}\, V - 1$;

(b) if the form $(\ ,\ )$ is skew-hermitian, then there is an isotropic subspace $X' \subseteq V'$ such that $dim_{\mathbf{D}}\, X' = dim_{\mathbf{D}}\, V$.

Recall [H8] that the stable range requires $dim_{\mathbf{D}}\, X' = dim_{\mathbf{D}}\, V$ in both cases 8.1(a) and 8.1(b).

For X' as in 8.1 let $\mathscr{B}(X')$ denote the space of forms on X' of the same type as $(\ ,\ )$. Denote by $\mathscr{B}(X')^{max}$ the Zariski-open set of nonsingular forms in $\mathscr{B}(X')$. Set $X = Hom_{\mathbf{D}}(X', V)$ and let

$$\beta: X \ni x \to (\ ,\ ) \circ x \in \mathscr{B}(X'). \tag{8.2}$$

LEMMA 8.3. *The subset $X^{max} = \beta^{-1}(\mathscr{B}(X')^{max}) \subseteq X$ is Zariski-open and dense. The stabilizers*

$$G(x) = \{g \in G; gx = x\} \qquad (x \in X^{max})$$

are compact.

Proof. The first statement is obvious. For the second one, we notice that each $x \in X^{max}$ defines a direct sum decomposition

$$V = im\, x \oplus (im\, x)^{\perp}.$$

The restriction to the second summand is an isomorphism from $G(x)$ onto the group of isometries of the form $(\ ,\ )$ restricted to $(im\, x)^{\perp}$. By 8.1, if $(im\, x)^{\perp} \neq 0$, then $dim_{\mathbf{D}}(im\, x)^{\perp} = 1$ and the restricted form is hermitian and nondegenerate. Hence the corresponding group of isometries is compact. ■

Choose an isotropic complement Y' to X' in V' and let V'_0 be the orthogonal complement to $X' + Y'$ in V', so that

$$V' = X' \oplus V'_0 \oplus Y'.$$

Set $Y = Hom_{\mathbf{D}}(Y', V)$ and $W_0 = Hom_{\mathbf{D}}(V'_0, V)$. Then, with the obvious identifications,

$$W = X \oplus W_0 \oplus Y, \tag{8.4}$$

where X, Y are isotropic subspaces of W and $W_0 = (X + Y)^{\perp}$ is a symplectic space, or zero.

We consider the mixed model of the oscillator representation ω attached to the decomposition (8.4). Suppose $W_0 \neq 0$. Then there is an oscillator representation ω_0 of $\widetilde{Sp}(W_0)$ on a Hilbert space H_{ω_0} such that the Hilbert space of ω, $H_\omega = L^2(X, H_{\omega_0})$, and the group $\tilde{G} \subseteq \widetilde{Sp}(W)$ act according to the formula

$$\omega(\tilde{g})v(x) = \omega_0(\tilde{g})(v(g^{-1}x)), \tag{8.5}$$

where $\tilde{g} \in \tilde{G}$ is in the preimage of $g \in G$, $v \in H_\omega$, and $x \in X$, [H8]. If $W_0 = 0$, then (8.5) holds with ω_0, a unitary character of $\tilde{G}$, and $H_{\omega_0} = \mathbf{C}$. From now on let X' be an isotropic subspace of V'', maximal with $\dim_{\mathbf{D}} X' \leqslant \dim_{\mathbf{D}} V$.

In the next lemma we reproduce an argument of Jian-Shu Li [Li1, (50)], in our slightly more general situation.

LEMMA 8.6. *Suppose the pair G, G' is in the almost stable range, with G the smaller member. Let Π be an irreducible unitary representation of $\tilde{G}$, whose restriction to the kernel of the covering map $\tilde{G} \to G$ is a multiple of the unique nontrivial character of this kernel. Denote by H_Π the Hilbert space of Π. Suppose $0 \neq u \in H_\Pi^\infty$ is such that*

$$\int_{\tilde{G}} |(u, \Pi(g)u)|\,\Omega(g)\,dg < \infty. \tag{a}$$

Then

$$0 \leqslant \int_{\tilde{G}} (u, \Pi(g)u)(\omega(g)v, v)\,dg < \infty \qquad (v \in H_\omega^\infty). \tag{b}$$

There is a v for which the integral (b) *is nonzero if either $\dim_{\mathbf{D}} X' = \dim_{\mathbf{D}} V$, or $\dim_{\mathbf{D}} X' = \dim_{\mathbf{D}} V - 1$ and the trivial representation of the stabilizer of an anisotropic line in V, occurs in $\Pi \otimes \omega_0$.*

Proof. It follows from (a) and 1.13(a) that the integral (b) is a continuous function of $v \in H_\omega^\infty$. Hence it will suffice to verify the nonnegativity in (b) for all $v \in C_c^\infty(X^{max}, H_{\omega_0}^\infty)$. Lemma 8.3 shows that, for such a v, the support of the function

$$\tilde{G} \times X^{max} \ni (\tilde{g}, x) \to (\omega_0(\tilde{g})(v(g^{-1}x)), v(x)) \in \mathbf{C}$$

is compact. Hence we may interchange the order of integration:

$$\int_{\tilde{G}} (u, \Pi(\tilde{g})u)(\omega(\tilde{g})v, v)\,d\tilde{g} = \int_X \int_{\tilde{G}} (u, \Pi(\tilde{g})u)(\omega_0(\tilde{g})(v(g^{-1}x)), v(x))\,d\tilde{g}\,dx. \tag{1}$$

Let db denote a Lebesgue measure on $\mathscr{B}(X')$. As a consequence of 8.1, there is a nonnegative function $j_{\mathscr{B}}$ on $\mathscr{B}(X')^{max}$, such that

$$\int_{X^{max}} \phi(x)\,dx = \int_{\mathscr{B}(X')^{max}} \int_G \phi(h^{-1}x)\,dh\,j_{\mathscr{B}}(\beta(x))\,d\beta(x)$$

for $\phi \in C_c(X^{max})$. Hence we may rewrite the integral (1) as one-half times

$$\int_{\mathscr{B}(X')^{max}} \int_{\tilde{G}} \int_{\tilde{G}} (u, \Pi(\tilde{g})u)(\omega_0(\tilde{g})(v(g^{-1}h^{-1}x), v(h^{-1}x))\, d\tilde{g}\, d\tilde{h}\, j_{\mathscr{B}}(\beta(x))\, d\beta(x)$$

$$= \int_{\mathscr{B}(X')^{max}} \int_{\tilde{G}} \int_{\tilde{G}} (\Pi(\tilde{g})u, \Pi(\tilde{h})u)(\omega_0(\tilde{g})(v(g^{-1}x)), \omega_0(\tilde{h})(v(h^{-1}x))\, d\tilde{g}\, d\tilde{h}\, j_{\mathscr{B}}(\beta(x))\, d\beta(x)$$

$$= \int_{\mathscr{B}(X')^{max}} \left\| \int_{\tilde{G}} \Pi(\tilde{g})u \otimes \omega_0(\tilde{g})(v(g^{-1}x))\, d\tilde{g} \right\|^2 j_{\mathscr{B}}(\beta(x))\, d\beta(x) \geqslant 0.$$

This proves (b).

Suppose the last integral vanishes for all v. Then it also vanishes for all u and all v (see 3.2). Thus

$$\text{(2)} \qquad \int_{\tilde{G}} \Pi(\tilde{g})u \otimes \omega_0(\tilde{g})(v(g^{-1}x))\, d\tilde{g} = 0 \qquad (x \in X^{max}, u \in H_\Pi^\infty, v \in C_c^\infty(X^{max}, H_{\omega_0}^\infty)).$$

Fix $x \in X^{max}$ and a vector $w \in H_{\omega_0}^\infty$. Let $\phi_n \in C_c^\infty(X^{max}, \mathbf{C})$ $(n = 1, 2, 3, \ldots)$ be a Dirac sequence converging to the Dirac delta at x. Set $v_n = \phi_n \cdot w$. By substituting v_n for v in (2) and taking limit if $n \to \infty$, we get (for $G(x)$, see 8.3)

$$\text{(3)} \qquad \int_{\tilde{G}(x)} \Pi(\tilde{g})u \otimes \omega_0(\tilde{g})w\, d\tilde{g} = 0 \qquad (u \in H_\Pi^\infty, w \in H_{\omega_0}^\infty).$$

By our assumption $\Pi(\tilde{g}) \otimes \omega_0$ factorizes to a representation of $G(x)$. Thus (3) means that the trivial representation of $G(x)$ does not occur in $\Pi(\tilde{g}) \otimes \omega_0$. This is possible only if $G(x)$ is nontrivial. Thus $dim_{\mathbf{D}}\, X' = dim_{\mathbf{D}}\, V - 1$. Since the $x \in X^{max}$ is arbitrary, we are done. ■

Consider another irreducible dual pair G, G_2', where G_2' is the isometry group of a form $(\ ,\ )_2'$ on a finite-dimensional vector space V_2' over $\mathbf{D}$, of the same type as the form $(\ ,\ )'$. Set $W_2 = Hom_{\mathbf{D}}(V_2', V)$ and $W_3 = Hom_{\mathbf{D}}(V' \oplus V_2', V)$. Here $V' \oplus V_2'$ is equipped with the form $(\ ,\ )_3' = (\ ,\ )' \oplus (\ ,\ )_2'$. Let G_3' denote the isometry group of this form. Then G, G_3' is again an irreducible dual pair in $Sp(W_3)$. We shall assume that the form $(\ ,\ )_3'$ is split. Then the preimages of G in $\widetilde{Sp}(W)$ and in $\widetilde{Sp}(W_2)$ are isomorphic. We shall identify them and denote both of them by the same symbol $\tilde{G}$.

Let ω_2 and ω_3 denote the oscillator representations of the metaplectic groups $\widetilde{Sp}(W_2)$ and $\widetilde{Sp}(W_3)$ associated to the same character χ (1.1) of $\mathbf{R}$ [H3], respectively.

LEMMA 8.7. *With the above notation, assume that*
(a) *the pair G, G_3' is in the almost stable range, with G the smaller member,*
(b) $dim_{\mathbf{D}}\, V' + dim_{\mathbf{D}}\, V_2' \geqslant 2r - 1$ (*see* (4.10)),
(c) *the form $(\ ,\ )_3'$ is split,*

(d) *the form* $(\ ,\ )'_2$ *is anisotropic, and*

(e) Π *is an irreducible unitary representation of* $\tilde{G}$ *such that the contragradient* Π^c *occurs as a subrepresentation of the Hilbert space* H_{ω_2} *of* $\omega_2|_{\tilde{G}}$.

Then for any $\tilde{K}$*-finite vector u in the Hilbert space* H_Π *of* Π,

$$(*)\qquad 0 \leqslant \int_{\tilde{G}} (u, \Pi(g)u)(\omega(g)v, v)\, dg < \infty \qquad (v \in H_\omega^\infty),$$

and there are vectors u, v such that the above integral is nonzero.

Proof. Let $\tilde{\tilde{G}}$ denote the preimage of G in $\widetilde{Sp}(W_3)$. It follows from (c) that the covering $\tilde{\tilde{G}} \to G$ splits. Let $\varepsilon\colon \tilde{\tilde{G}} \to \{+1, -1\}$ be the character whose kernel is G. The condition (b) together with 6.6(1) imply that the Gaussian function on $\widetilde{Sp}(W_3)$ is integrable over $\tilde{G}$. Consequently, 8.6 shows that

$$(1)\qquad 0 \leqslant \int_{\tilde{\tilde{G}}} \varepsilon(g)(\omega_3(g)v_3, v_3)\, dg < \infty \qquad (v_3 \in H_{\omega_3}^\infty).$$

Since taking direct sums of symplectic spaces results in tensoring of the corresponding oscillator representations [H3], the second equation in the following calculation is justified:

$$(2)\qquad \int_{\tilde{G}} (u, \Pi(g)u)(\omega(g)v, v)\, dg = \int_{\tilde{G}} (\omega_2(g)u, u)(\omega(g)v, v)\, dg$$

$$= \int_{\tilde{\tilde{G}}} \varepsilon(g)(\omega_3(g)(u \otimes v), u \otimes v)\, dg \geqslant 0$$

by (1). This proves the nonnegativity in $(*)$.

Assume for simplicity that the form $(\ ,\ )'_2$ is positive definite. Then by (c), there is a subspace V'_{-2} in V' of the same dimension as V'_2, such that the restriction of the form $(\ ,\ )'$ to V'_{-2} is negative definite. The orthogonal complement of V'_{-2} in V' can be written as $V'_+ \oplus V'_-$, where the restriction of the form $(\ ,\ )'$ to V'_+ is positive definite, the restriction to V'_- is negative definite, and these spaces are perpendicular to each other. Set $W_+ = Hom_{\mathbf{D}}(V'_+, V)$, $W_- = Hom_{\mathbf{D}}(V'_-, V)$, and $W_{-2} = Hom_{\mathbf{D}}(V'_{-2}, V)$. Then $W_3 = W_2 \oplus W = W_2 \oplus W_{-2} \oplus W_+ \oplus W_-$. Let $\omega_{-2}, \omega_+, \omega_-$ be the oscillator representations of $\widetilde{Sp}(W_{-2}), \widetilde{Sp}(W_+), \widetilde{Sp}(W_-)$ realized on Hilbert spaces $H_{\omega_{-2}}, H_{\omega_+}, H_{\omega_-}$ respectively. Then ω_3 may be realized on the space $H_{\omega_3} = H_{\omega_2} \otimes H_{\omega_{-2}} \otimes H_{\omega_+} \otimes H_{\omega_-}$. Since ω_{-2} is isomorphic to ω_2^*, we may identify $H_{\omega_{-2}}$ with H_{ω_2}. Similarly, ω_- is isomorphic to ω_+^*, and we identify H_{ω_-} with H_{ω_+}. Let $u \in H_\Pi \subseteq H_{\omega_2} = H_{\omega_{-2}}$ be a nonzero $\tilde{K}$ finite vector and let $v_+ \in H_{\omega_+}^\infty = H_{\omega_-}^\infty$ be a nonzero vector. Then for $v = v_+ \otimes v_+$, $\tilde{g} \in \tilde{G}$, and $\tilde{\tilde{g}} \in \tilde{\tilde{G}}$ in the corresponding preimage of

$g \in G$,

(3) $(u, \Pi(\tilde{g})u)(\omega(\tilde{g})v, v) = (\omega_2(\tilde{\tilde{g}})u, u)(u, \omega_2(\tilde{\tilde{g}})u)(\omega_+(\tilde{\tilde{g}})v_+, v_+)(v_+, \omega_+(\tilde{\tilde{g}})v_+) \geqslant 0,$

and for almost all $g \in G$ this number is nonzero. Hence the integral (2) is nonzero. ■

Remark 8.8. It seems plausible that 8.7 holds without the assumption (d).

THEOREM 8.9. *Suppose the groups G, G' and the irreducible unitary representation Π of $\tilde{G}$ satisfy the conditions of Lemma* 8.6 *or Lemma* 8.7. *Then $\Pi \in \mathscr{R}(G, \omega)$ and the representation Π' of $\tilde{G}'$ corresponding to Π via Howe's correspondence is unitary.*

Proof. Lemmas 8.6 and 8.7 ensure that the condition 3.2(b′) is satisfied. Hence in 3.1(b) positivity holds, and consequently our theorem follows from Theorem 3.1. ■

Example 8.10. Consider the pair $G = Sp(n, \mathbf{R})$, $G' = O(p + q, q)$. Suppose for simplicity that the numbers n, p are even. This allows us to forget about all the coverings. The conditions of Lemma 8.7 can be expressed as follows:

(a) $p + q \geqslant 2n$;
(b) $2p + 2q \geqslant 2(2n + 1) - 1$;
(c) and (d) hold by the choice of the dual pair, because the forms $(\ ,\ )'$, $(\ ,\ )'_2$, $(\ ,\ )'_3$ are symmetric with signatures $(p + q, q)$, $(0, q)$, $(p + q, p + q)$ respectively;
(e) $\Pi^c \otimes \Pi'_2 \in \mathscr{R}(Sp(n, \mathbf{R}) \cdot O(p), \omega_2)$ for some Π'_2.

The first two conditions, (a) and (b), may be put together as $p + q \geqslant 2n + 1$. Theorem 8.9 says that there is an irreducible unitary representation Π' of $O(p + q, q)$ such that $\Pi \otimes \Pi' \in \mathscr{R}(Sp(n, \mathbf{R}) \cdot O(p + q, q), \omega)$. Notice that this dual pair is not in the stable range (with $Sp(n, \mathbf{R})$ the smaller member) if $2n > q$. Thus in this case the unitarity of Π' does not follow from [Li1].

Further it is clear from the proof of 8.7 that the trivial representation of $Sp(n, \mathbf{R})$ corresponds to some irreducible unitary representation Π'_3 of $O(p + q, p + q)$ and that the representation $\Pi' \otimes \Pi'_2$ occurs in the restriction of Π'_3 to the subgroup $O(p + q, q) \times O(q)$. (Compare the formula (2) in the proof of 8.7 with 3.1(*).) This seems to be of independent interest.

9. Positivity for a pair G, G' of type II. Here $\mathbf{D}$, V, V' are as in the previous section, but $G = GL_{\mathbf{D}}(V)$ and $G' = GL_{\mathbf{D}}(V')$. The symplectic space

$$W = X \oplus Y \tag{9.1}$$

where $X = Hom_{\mathbf{D}}(V', V)$ and $Y = Hom_{\mathbf{D}}(V, V')$ are maximal isotropic subspaces. The groups G, G' act on W in the obvious fashion, via the post- and premultiplication by the inverse.

We work in the Schrödinger model of ω attached to the decomposition (9.1),

[H3]. Then the Hilbert space $H_\omega = L^2(X)$ and

$$\omega(\tilde{g})v(x) = \xi(\tilde{g})^{-1} v(g^{-1}x)$$

where $\tilde{g} \in \tilde{G}$ is in the preimage of $g \in G$, $v \in H_\omega$, $x \in X$, $\xi(\tilde{g})^2 = (det_{\mathbf{R}}\, g)^{d'}$, $d' = dim_{\mathbf{D}}\, V'$.

The argument used in the proof of 8.6, with $X^{max} = \{x;\, im\, x = V\}$, verifies the following lemma.

LEMMA 9.2. *Let Π be an irreducible unitary representation of $\tilde{G}$, whose restriction to the kernel of the covering map $\tilde{G} \to G$ is a multiple of the unique nontrivial character of this kernel. Suppose $0 \neq u \in H_\Pi$ is such that*

(a) $$\int_{\tilde{G}} |(u, \Pi(g)u)|\, \Omega(g)\, dg < \infty .$$

Then

(b) $$0 \leqslant \int_{\tilde{G}} (u, \Pi(g)u)(\omega(g)v, v)\, dg < \infty \qquad (v \in H_\omega^\infty),$$

and there is a vector v for which this integral (b) *is nonzero.*

THEOREM 9.3. *Suppose the groups G, G' and the irreducible unitary representation Π of $\tilde{G}$ satisfy the conditions of Lemma* 9.2. *Then $\Pi \in \mathscr{R}(G, \omega)$ and the representation Π' of $\tilde{G}'$ corresponding to Π via Howe's correspondence is unitary.*

Proof. This is of course well known. However, the argument verifying 8.9 works too. Lemma 9.2 ensures that condition 3.2(b′) is satisfied. Hence in 3.1(b) positivity holds, and consequently our theorem follows from Theorem 3.1. ∎

10. Proof of Theorem 0.9. The nonvanishing of the integral 0.9(b) for some $v \in H_\omega^\infty$ forces Π to be a genuine representation of $\tilde{G}$, in the sense that the restriction of Π to the kernel of the covering map $\tilde{G} \to G$ (a two element group) is a multiple of the unique nontrivial character of this kernel.

The assumtion 0.9(a) ensures that we can use Theorem 6.12. The first part of 6.12 shows that the condition 3.1(a) is satisfied. Further, 0.9(b) and 3.2 show that 3.1(b) holds. Hence Theorem 3.1 says that $\Pi \otimes \Pi' \in \mathscr{R}(G \cdot G', \omega)$ and that Π' is unitary.

The assumption 0.9(c) is the same as 7.9(∗). By 0.9(a) G and Π satisfy 6.17(b) or 6.17(e). Thus the statement about the associated varieties follows from Proposition 7.9. ∎

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