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HOLOMORPHICITY OF A CLASS OF SEMIGROUPS OF MEASURES OPERATING ON $L^p(G/H)$

TOMASZ PRZEBINDA

ABSTRACT. In the present paper we consider the class of stable semigroups of measures on a Lie group G. This class contains the Gaussian semigroups. We prove that under certain strongly continuous representations of G acting in $L^p(G/H)$, $1 \leqslant p < \infty$, these semigroups are holomorphic and uniformly bounded.

Introduction. For a fixed Lie group G, let (S) denote the smallest family of semigroups of measures in G that contains all Gaussian semigroups (i.e., those semigroups whose infinitesimal generators are sub-Laplacians) and is closed with respect to taking sums of generators and subordination. Hulanicki [**2**] has posed the problem of determining if semigroups in (S) are holomorphic. It is known that Gaussian semigroups are holomorphic [**5**], but beyond this, additional assumptions are needed. For example, if G is a class two nilpotent group, any semigroup in (S) with L^2 densities is holomorphic [**3**]. In this paper we consider semigroups in the image of (S) under strongly continuous representations of G, and show that, for a certain class of representations, these semigroups are holomorphic.

Preliminaries. We identify the Lie algebra of G, $\mathfrak{g}$, with left-invariant differential operators by setting

$$Xf(x) = \frac{d}{dt} f(\exp tX)\,|_{t=0}.$$

For fixed basis $\{X_1,\ldots,X_n\}$ of $\mathfrak{g}$ and multi-index $\mathbf{z} = (z_1,\ldots,z_n)$ we set $|\mathbf{z}| = z_1 + \cdots + z_n$ and

$$X^{\mathbf{z}}f = X_1^{z_1}\cdots X_n^{z_n}f.$$

We denote by $C_0(G)$ the space of continuous real-valued functions on G that vanish at infinity, and for a positive integer k we set

$$C_0^k = \{ f \in C_0(G) \mid X^{\mathbf{z}}f \in C_0 \text{ for } |\mathbf{z}| \leqslant k \}.$$

$C_0(G)$ is given the topology of uniform convergence, and $C_1^k(G)$, the topology of uniform convergence for derivatives of order k.

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We identify $C_0(G)^*$ with $M(G)$, the space of bounded, regular Borel measures on G. $M(G)$ is a Banach *-algebra with respect to convolution $\mu * \nu$, defined by

$$\langle \mu * \nu, F\rangle = \int_G \int_G F(xy)\, d\mu(x)\, d\nu(y),$$

and involution μ^*, defined by

$$\langle \mu^*, F\rangle = \int_G F(x^{-1})\, d\mu(x).$$

We let $P(G)$ denote the subalgebra of nonnegative measures with norm ≤ 1. A subset $\{\mu_t\}_{t>0} \subseteq P(G)$ is said to be a *semigroup of measures on* G if $\mu_s * \mu_t = \mu_{s+t}$ and

$$\lim_{t\to 0} \|\mu_t * f - f\|_\infty = 0, \qquad f \in C_0(G).$$

We denote by $C_c^\infty(G)$ the space of test functions on G, and by $\mathcal{D}'(G)$ the space of distributions on G. A distribution D is said to be *dissipative* if $\langle D, f\rangle \leq 0$ whenever $f \in C_c^\infty(G)$ and $f(e) = \|f\|_\infty$.

Given a semigroup of measures on G, $\{\mu_t\}_{t>0}$, the *infinitesimal generator* of $\{\mu_t\}_{t>0}$, A, is the closed operator on $C_0(G)$ defined by

$$Af = \lim_{t\downarrow 0} \frac{1}{t}(\mu_t * f - f),$$

where the domain $\mathrm{D}(A)$ of A is the subset of $C_0(G)$ for which the right-hand side exists. One has that $C_c^\infty(G) \subseteq \mathrm{D}(A)$ and that the mapping $f \to Af(e)$ is a dissipative distribution on G. Conversely, given a dissipative distribution D on G, define the operator A_D on $C_c^\infty(G)$ by $A_D f(x) = \langle D, f_x\rangle$. Then A_D is closable in $C_0(G)$ and is the infinitesimal generator of a unique semigroup of measures on G. (These results are essentially due to Hunt [4], with more modern treatments found in [**1** and **2**].)

Let $\{\mu_t\}_{t>0}$ and $\{\nu_t\}_{t>0}$ be semigroups of measures on G with infinitesimal generators A and B, respectively. Clearly, $f \to (A + B)f(e)$ is a dissipative distribution, and hence there is a unique semigroup whose infinitesimal generator on $C_c^\infty(G)$ is $A + B$. Also, given $0 < a < 1$, the distribution

$$f \to -\Gamma(-a)^{-1} \int_0^\infty t^{-a-1}(\mu_t * f - f)(e)\, dt$$

is dissipative. We denote the corresponding semigroup of measures by $\{\mu_t^{(a)}\}_{t>0}$ and its infinitesimal generator by $|A|^a$. $\{\mu_t^{(a)}\}_{t>0}$ is said to be *subordinate* to $\{\mu_t\}_{t>0}$.

Given a subset $\{X_1, \ldots, X_k\} \subseteq \mathfrak{g}$, $f \to (X_1^2 + \cdots + X_k^2)f(e)$ is a dissipative distribution. Such distributions generate the *Gaussian semigroups*. The family of *stable* semigroups, (S), consists of those semigroups of measures whose corresponding dissipative distribution belongs to $S(\mathfrak{g}) = \bigcup_{k=0}^\infty S_k(\mathfrak{g})$, where

$$S_0(\mathfrak{g}) = \{f \to X^2 f(e) \mid X \in \mathfrak{g}\},$$
$$S_{2k}(\mathfrak{g}) = \{D_1 + D_2 \mid D_i \in S_{2k-1}(\mathfrak{g})\} \cup S_{2k-1}(\mathfrak{g})$$

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and

$$S_{2k+1}(\mathfrak{g}) = \{f \to -1\,|\,A\,|^a f(e)\,|\,f \to Af(e) \in S_{2k}(\mathfrak{g})\} \cup S_{2k}(\mathfrak{g})$$

for $k = 1, 2, \ldots$. For $D \in S(\mathfrak{g})$, the *hull* of D is the smallest Lie subalgebra of $\mathfrak{g}$, $\mathfrak{h}$, such that $D \in S(\mathfrak{h})$.

Holomorphicity of represented semigroups.

THEOREM 1. *Let* $\{\mu_t\}_{t>0}$ *be a stable semigroup of measures on G whose corresponding dissipative distribution has hull* $\mathfrak{g}$ *as above. There exists a* $0 < \theta \leqslant \pi/2$, *a positive integer p, and a family of holomorphic functions* $\{\omega_{\mathbf{n}}\,|\,\mathbf{n} \in (\mathbf{Z}^+)^p\}$ *defined on* $\Omega_\theta = \{Z \in \mathbf{C}\,|\,|\arg Z| < \theta\}$ *with values in* $M(G)$ *satisfying*

(i) $\omega_{\mathbf{n}}(t) \geqslant 0$ *and* $\|\omega_{\mathbf{n}}(t)\| = 1$, *for* $t > 0$,

(ii) $\lim_{n_1\to\infty} \cdots \lim_{n_p\to\infty} \omega_{\mathbf{n}}(t) * f = \mu_t * f$ *for* $t > 0, f \in L^1(G)$,

(iii) *there is an* $n_0 \in \mathbf{Z}^+$ *such that for* $n_i \geqslant n_0$, $1 \leqslant i \leqslant p$, *and* $z \in \Omega_\theta$, $\omega_{(n_1,\ldots,n_p)}(z) \in L^1(G)$.

PROOF. The proof is by induction on the smallest integer k such that dissipative distribution D corresponding to $\{\mu_t\}_{t>0}$ is in $S_k(\mathfrak{g})$. If $k = 0$ then $\dim(\mathfrak{g}) = 1$ and the theorem follows from well-known facts about the Gaussian semigroup on $\mathbf{R}$ or $\mathbf{T}$. For example, on $\mathbf{R}$ we take $\theta = \pi/2$, $p = 1$ and for $z \in \Omega_\theta$, $n \in \mathbf{Z}^+$, we have $\omega_n(z) = \omega_1(z) = (4\pi z)^{1/2} e^{-x^2/4z}\,dx$, where dx is Lebesgue measure on $\mathbf{R}$.

We assume the theorem for $j < 2k+1$ and let $D \in S_{2k+1}(\mathfrak{g})$. There is a $D_1 \in S_{2k}(\mathfrak{g})$ and a $0 < a < 1$ so that $\mu_t = \nu_t^{(a)}$ where $\{\nu_t\}_{t>0}$ is generated by D_1. Now, the hull of D_1 is $\mathfrak{g}$, and hence, by induction there exist, θ_1, p_1 and $\{\omega_{\mathbf{n}}^1\,|\,\mathbf{n} \in (\mathbf{Z}^+)^{p_1}\}$ satisfying the conditions of the theorem. We let $\theta = a\theta_1$, $p = p_1$, and for $z \in \Omega_\theta$, $\mathbf{n} \in (\mathbf{Z}^+)^p$, we define

$$\omega_{\mathbf{n}}(z) = \int_0^\infty f_a(\lambda)\,\omega_{\mathbf{n}}^1(\lambda z^{1/a})\,d\lambda,$$

where

$$f_a(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - z^a}\,dz, \qquad \sigma > 0, \lambda \geqslant 0.$$

Conditions (i) and (ii) then follow immediately from the properties of f_a and for (ii) we have, for $t > 0$ and $f \in L^1(G)$,

$$\lim_{n_1\to\infty} \cdots \lim_{n_p\to\infty} \omega_{\mathbf{n}}(t) * f = \lim_{n_1\to\infty} \cdots \lim_{n_p\to\infty} \int_0^\infty f_a(\lambda)\,\omega_{\mathbf{n}}^1(\lambda t^{1/a}) * f\,d\lambda$$
$$= \int_0^\infty f_a(\lambda)\,\nu_{\lambda t^{1/a}} * f\,d\lambda = \lambda_t^{(a)} * f.$$

(The reader is referred to **[8]**, for properties of f_a.)

Finally, we assume the theorem for $j < 2k$ and let $D \in S_{2k}(\mathfrak{g})$. Let D_1, $D_2 \in S_{2k-1}(\mathfrak{g})$ such that $D = D_1 + D_2$. Let $\mathfrak{g}_i$ be the hull of D_i and let G_i be the

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corresponding connected Lie subgroups of G. (Note that $\mathfrak{g}$ is generated by $\mathfrak{g}_1$ and $\mathfrak{g}_2$.) There exist θ_i, p_i, and $\omega^i_{\mathbf{n}}$ for $\mathbf{n} \in (\mathbf{Z}^+)^{p_i}$ satisfying the conditions of the theorem with respect to G_i. Let $\theta = \min\{\theta_1, \theta_2\}$, $p = p_1 + p_2 + 1$, and for $z \in \Omega_\theta$, $\mathbf{n} \in (\mathbf{Z}^+)^p$ let

$$\omega_{\mathbf{n}}(z) = \left(\omega^1_{n^1}(z/n_p) * \omega^2_{n^2}(z/n_p)\right)^{n_p},$$

where $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2, n_p)$. $\omega_{\mathbf{n}}$ is holomorphic on Ω_θ and satisfies (i). Furthermore, if $\{\mu^i_t\}_{t>0}$ is the semigroup generated by D_i, then, for $f \in L^1(G)$, $t > 0$, one has, by Chernoff's theorem, that

$$\begin{aligned}\lim_{n_1 \to \infty} \cdots \lim_{n_p \to \infty} \omega_{\mathbf{n}}(t) * f &= \lim_{n_1 \to \infty} \cdots \lim_{n_p \to \infty} \left(\omega^1_{\mathbf{n}^1}(t/n_p) * \omega^2_{\mathbf{n}^2}(t/n_p)\right)^{n_p} * f \\ &= \lim_{n_p \to \infty} \left(\mu^1_{t/n_p} * \mu^2_{t/n_p}\right)^{n_p} * f = \mu_t * f.\end{aligned}$$

Condition (iii) is an immediate consequence of a known theorem:[1] Let $G_1, \ldots, G_k$ be connected Lie subgroups of the Lie group G whose Lie algebras generate the Lie algebra of G. There is an integer N such that

$$\left(L^1(G_1) * L^1(G_2) * \cdots * L^1(G_k)\right)^N \subseteq L^1(G).$$

Let H be a closed subgroup of the connected Lie group G and suppose there is a measure on G/H that is invariant with respect to the action of G. Suppose further that γ: $G \times G/H \to \mathbf{C}$ is a continuous function with $\|\gamma\|_\infty \leq 1$ such that, for $1 \leq p < \infty$, the mapping $y \to \Gamma^p_y$ of G into the bounded operators on $L^p(G/H)$ given by

$$\Gamma^p_y F(\dot{x}) = \gamma(y, \dot{x}) F(y^{-1}\dot{x})$$

for $F \in L^p(G/H)$ and a.e. $\dot{x} \in G/H$ defines a strongly continuous representation of G that is unitary for $p = 2$.

THEOREM 2. *Let G be a connected Lie group, let $\{\mu_t\}_{t>0} \in (S)$, and suppose that the hull of the distribution generating $\{\mu_t\}_{t>0}$ is the Lie algebra of G. Let H and γ be as above and suppose that $\Gamma^p(L^1(G))$ is contained in the space of compact operators on $L^p(G/H)$ for $1 \leq p < \infty$. There is a $0 < \theta \leq \pi/2$ and a holomorphic T^p: $\Omega_\theta \to B(L^p(G/H))$ such that*

(i) $\sup\{\|T^p_z\| \mid z \in \Omega_\theta\} \leq 1$,

(ii) *for* $t > 0$, $T^p_t = \Gamma^p(\mu_t)$.

PROOF. Let θ, p and ω_n be as in Theorem 1. Notice that $\omega_1(t)$ is the semigroup of probabilistic measures generated by X_1^2, whose image $\Gamma^2(X_1^2) = A_1$ is an essentially selfadjoint operator [**1**, Theorem 12, Example 4] and generates the semigroup of contractions $\Gamma^2(\omega_1(t))$, so that

$$\Gamma^2(\omega_1(z)) = \int_{-\infty}^0 e^{\lambda z}\, dP(\lambda)$$

[1]See remarks.

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where P is the spectral measure of A_1, and hence $\|\Gamma^2(\omega_1(z))\| \leqslant 1$ for $\mathrm{Re}(z) > 0$. Now by an induction argument analogous to that used in the proof of Theorem 1, one can show that for, $f \in L^p(G/H)$, $1 \leqslant p$ and $t > 0$,

$$\lim_{n_1 \to \infty} \cdots \lim_{n_p \to \infty} \Gamma^p(\omega_{\mathbf{n}}(t))f = \Gamma^p(\mu_t)f$$

and that $\|\Gamma^2(\omega_{\mathbf{n}}(z))\| \leqslant 1$ for $z \in \Omega_\theta$.

Let n_0 be as in Theorem 1, and set $\mathbf{Z}_0 = \{k \in \mathbf{Z}, k > n_0\}$. Then if $\mathbf{n} \in (\mathbf{Z}_0)^p$, $\omega_n(z) \in L^1(G)$ for $z \in \Omega_\theta$, and so $\Gamma^p(\omega_{\mathbf{n}}(z))$ and $\Gamma^2(\omega_{\mathbf{n}}(z))$ are compact operators. By **[6]**, their spectra coincide, and hence also their norms. Thus $\|\Gamma^p(\omega_{\mathbf{n}}(z))\| \leqslant 1$. Therefore, $\{\Gamma^p(\omega_{\mathbf{n}}) \mid \mathbf{n} \in (\mathbf{Z}_0)^p\}$ is a family of holomorphic functions on Ω_θ that is uniformly bounded and convergent on $\mathbf{R}^+$. Thus, by Vitali's theorem,

$$\lim_{n_1 \to \infty} \cdots \lim_{n_p \to \infty} \Gamma^p(\omega_{\mathbf{n}}(z)) = T_z^p$$

exists for $z \in \Omega_\theta$, and $z \to T_z^p$ is holomorphic.

Remarks. 1. I have learned from Joe W. Jenkins and Andrzej Hulanicki that the theorem mentioned at the end of the proof of Theorem 1 was known but I could not find it in the literature. Here is a rough outline of a proof.

Assume for simplicity that $k = 2$. Using the fact that the zero set of a nontrivial real analytic function defined on an open subset of $\mathbf{R}^n$ has Lebesgue measure zero, one can prove the following:

LEMMA. *Let M and G be two real analytic manifolds of dimensions N and n respectively ($N \geqslant n$), with Lebesgue measures μ and ν. Suppose that M is connected. If $F\colon M \to G$ is a real analytic function whose derivative has rank n at a point in M, then $F(\mu)$ is absolutely continuous with respect to ν.*

Let $\mathfrak{g}_i$ be the Lie algebra of G_i with basis $\{X_{i1}, X_{i2}, \ldots, X_{ip_i}\}$ $(i = 1, 2)$. Let $\mathfrak{M}^0 = \{X_{11}, \ldots, X_{1p_1}, X_{21}, \ldots, X_{2p_2}\}$ and let $\mathfrak{M}^p = \mathfrak{M}^{p-1} \cup [\mathfrak{M}^{p-1}, \mathfrak{M}^{p-1}]$ for $p = 1, 2, 3, \ldots$. The assumption that $\mathfrak{g}_1 \cup \mathfrak{g}_2$ generates $\mathfrak{g}$ implies that there is such a p that $\mathfrak{M}^p$ contains the basis $\{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$. The Baker-Campbell-Hausdorff formula [7, Theorem 2.15.4] implies that there exists an m such that for every $X \in \mathfrak{M}^p$ there is a function $S_X\colon \mathbf{R} \to \mathbf{R}^{m(p_1+p_2)}$ such that for $|t|$ small enough we have $f_m(S_X(t)) = \exp(tX + r_X(t))$, where

$$\begin{aligned} f_m(&u_{111}, u_{112}, \ldots, u_{11p_1}, u_{121}, u_{122}, \ldots, u_{12p_2}, \ldots, u_{m11}, \\ &\qquad u_{m12}, \ldots, u_{m1p_1}, u_{m21}, u_{m22}, \ldots, u_{m2p_2}) \\ &= \exp\left(\sum_{j=1}^{p_1} u_{11j}X_{1j}\right)\exp\left(\sum_{j=1}^{p_2} u_{12j}X_{2j}\right)\cdots\exp\left(\sum_{j=1}^{p_1} u_{m1j}X_{1j}\right)\exp\left(\sum_{j=1}^{p_2} u_{m2j}X_{2j}\right)\end{aligned}$$

and

$$r_X(t) = \sum_{\alpha+\beta>2^p} |t|^{(\alpha+\beta)/2^p}(\mathrm{sign}(t))^{\varepsilon_{\alpha\beta}}X_{\alpha\beta} \qquad (X_{\alpha\beta} \in \mathfrak{g}, \varepsilon_{\alpha\beta} \in \{0, 1\}).$$

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For example, if $p_1 = p_2 = 1$ and $X = [X_{11}, X_{21}]$ then $m = 2$ and

$$S_X(t) = \left(|t|^{1/2}, |t|^{1/2}\operatorname{sign}(t), -|t|^{1/2}, -|t|^{1/2}\operatorname{sign}(t)\right).$$

Choose S_{X_i} as above for every vector X_i from the basis of $\mathfrak{g}$, and put $S(\mathbf{t}) = S(t_1, t_2, \ldots, t_n) = (S_{X_1}(t_1), S_{X_2}(t_2), \ldots, S_{X_n}(t_n)) \in \mathbf{R}^{N(p_1+p_2)}$ where $N = nm$. Define $M = G_1 \times G_2 \times \cdots \times G_1 \times G_2$ (N times),

$$M \ni (x_{11}, x_{12}, \ldots, x_{N1}, x_{N2}) \to F(x_{11}, x_{12}, \ldots, x_{N1}, x_{N2}) = x_{11}x_{12}\cdots x_{N1}x_{N2} \in G$$

and $\Psi\colon \mathbf{R}^{N(p_1+p_2)} \to M$ by the formula

$$\begin{aligned}
\Psi(u_{111}, u_{112}, \ldots, u_{11p_1}, u_{121}, u_{122}, \ldots, u_{12p_2}, \ldots, u_{N11},&\\
u_{N12}, \ldots, u_{N1p_1}, u_{N21}, u_{N22}, \ldots, u_{N2p_2})&\\
= \left(\exp\left(\sum_{j=1}^{p_1} u_{11j}X_{1j}\right), \exp\left(\sum_{j=1}^{p_2} u_{12j}X_{2j}\right), \ldots,\right.&\\
\left.\exp\left(\sum_{j=1}^{p_1} u_{N1j}X_{1j}\right), \exp\left(\sum_{j=1}^{p_2} u_{N2j}X_{2j}\right)\right)&.
\end{aligned}$$

Thus

$$\begin{aligned}
F \circ \Psi \circ S(\mathbf{t}) &= f_m\left(S_{X_1}(t_1)\right) f_m\left(S_{X_2}(t_2)\right) \cdots f_m\left(S_{X_n}(t_n)\right)\\
&= \exp\left(t_1 X_1 + r_{X_1}(t_1)\right)\exp\left(t_2 X_2 + r_{X_2}(t_2)\right) \cdots \exp\left(t_n X_n + r_{X_n}(t_n)\right)
\end{aligned}$$

in a neighborhood of the origin in $\mathbf{R}^n$, which implies that the Jacobian J of the map $F \circ \Psi \circ S$ is equal to the identity at $\mathbf{t} = 0$ and that there exists a neighborhood U of the origin in $\mathbf{R}^n$ such that for $\mathbf{t} \in U$, $J(\mathbf{t}) \neq 0$. Take $\mathbf{t}$ from U such that $t_i \neq 0$ for each $i = 1, 2, 3, \ldots, n$. Then

$$\begin{aligned}
n = \operatorname{rank}(T_{\mathbf{t}}(F \circ \Psi \circ S)) &= \operatorname{rank}(T(F)_{\Psi(S(\mathbf{t}))} \circ T_{\mathbf{t}}(\Psi \circ S))\\
&\leqslant \operatorname{rank}(T(F)_{\Psi(S(\mathbf{t}))}) \leqslant n,
\end{aligned}$$

so $\operatorname{rank}(T(F)_{\Psi(S(\mathbf{t}))}) = n$. By the lemma $F(\mu_1 \times \mu_2 \times \cdots \times \mu_1 \times \mu_2)$ is absolutely continuous with respect to the Haar measure ν on G, i.e.

$$(\mu_1 * \mu_2)^N(E) = \mu_1 \times \mu_2 \times \cdots \times \mu_1 \times \mu_2(F^{-1}(E)) = 0$$

for every set E with $\nu(E) = 0$.

2. Theorem 2 gives only a partial answer to Hulanicki's question: determine if semigroups in (S) are holomorphic. The proof and the boundedness of $\|T_z^p\|$ for $z \in \Omega_\theta$ depend very much on the assumption that those operators are compact. Suppose G is a locally compact group and H is a subgroup of G such that there exists an invariant measure on G/H. Moreover assume that $\gamma = 1$ and G/H is not compact. Then one can show [6] that if $f \in L^1(G)$ and $\int_G f(y)\,dy \neq 0$ it follows that $\Gamma^p(f)$ is not compact in any $L^p(G/H)$, $1 \leqslant p < \infty$. The reader who is interested in the importance of having the semigroups holomorphic on all L^p spaces, $1 \leqslant p < \infty$, is referred to [2].

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