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Howe correspondence and Springer correspondence for real reductive dual pairs

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Abstract. We consider a real reductive dual pair $(\mathrm{G}', \mathrm{G})$ of type I, with $\mathrm{rank}(\mathrm{G}') \leq \mathrm{rank}(\mathrm{G})$. Given a nilpotent coadjoint orbit $\mathcal{O}' \subseteq \mathfrak{g}'^*$, let $\mathcal{O}'_{\mathbb{C}} \subseteq \mathfrak{g}'^*_{\mathbb{C}}$ denote the complex orbit containing $\mathcal{O}'$. Under some condition on the partition λ' parametrizing $\mathcal{O}'$, we prove that, if λ is the partition obtained from λ by adding a column on the very left, and $\mathcal{O}$ is the nilpotent coadjoint orbit parametrized by λ, then $\mathcal{O}_{\mathbb{C}} = \tau(\tau'^{-1}(\mathcal{O}'_{\mathbb{C}}))$, where τ, τ' are the moment maps. Moreover, if $chc(\hat{\mu}_{\mathcal{O}'}) \neq 0$, where chc is the infinitesimal version of the Cauchy-Harish-Chandra integral, then the Weyl group representation attached by Wallach to $\mu_{\mathcal{O}'}$ with corresponds to $\mathcal{O}_{\mathbb{C}}$ via the Springer correspondence.

1. Introduction

Consider a real reductive group G, as defined in [40]. Let Π be an irreducible admissible representation of G with the distribution character Θ_Π, [12]. Denote by u_Π the lowest term in the asymptotic expansion of Θ_Π, [2]. This is a finite linear combination of Fourier transforms of nilpotent coadjoint orbits, $u_\Pi = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}$. As shown by Rossmann [34], the closure of the union of the nilpotent orbits which occur in this sum is equal to $\mathrm{WF}(\Pi)$, the wave front set of the representation Π, defined in [20]. Furthermore there is a unique nilpotent coadjoint orbit $\mathcal{O}_\Pi$ in the complexification $\mathfrak{g}^*_{\mathbb{C}}$ of the dual Lie algebra $\mathfrak{g}$ of G such that the associated variety of the annihilator of the Harish-Chandra module of Π in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is equal to the closure of $\mathcal{O}_\Pi$, [6]. Moreover, the closure of $\mathcal{O}_\Pi$ coincides with the complexification of $\mathrm{WF}(\Pi)$, see [2, Theorem 4.1] and [34].

Let us be more specific and consider a real reductive dual pair $(\mathrm{G}', \mathrm{G})$ in a symplectic group Sp(W). We shall always assume that the rank of G' is less than or equal to the rank of G. Let Π' be an irreducible admissible representation of $\widetilde{\mathrm{G}}'$, the metaplectic cover of G', and let Π be the irreducible admissible representation

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of $\widetilde{\mathrm{G}}$ which corresponds to Π' via Howe correspondence for the pair $(\mathrm{G}', \mathrm{G})$, [21]. Howe correspondence is governed by a Capelli-Harish-Chandra homomorphism

$$\mathsf{C}\colon \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}} \to \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{\mathrm{G}'}$$

which has the property that if $\gamma_{\Pi'} : \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{\mathrm{G}'} \to \mathbb{C}$ is the infinitesimal character of Π' then $\gamma_{\Pi} = \gamma_{\Pi'} \circ \mathsf{C}$ is the infinitesimal character of Π, see (25) below. Let $\mathcal{P}(\mathfrak{g}^*_{\mathbb{C}})^{\mathrm{G}}$ be the algebra of the G-invariant complex valued polynomials on the dual of the complexification of $\mathfrak{g}$. The homomorphism C may be thought of as a "smooth deformation" of another homomorphism

$$\mathsf{c}\colon \mathcal{P}(\mathfrak{g}^*_{\mathbb{C}})^{\mathrm{G}} \to \mathcal{P}(\mathfrak{g}'^*_{\mathbb{C}})^{\mathrm{G}'}$$

defined by the correspondence of the semisimple coadjoint orbits induced by the moment maps

$$\begin{aligned}&\tau\colon \mathrm{W} \to \mathfrak{g}^* \quad \tau'\colon \mathrm{W} \to \mathfrak{g}'^*,\\ &\tau(w)(x) = \langle x(w), w\rangle,\ \ \tau(w)(x') = \langle x'(w), w\rangle,\ \ w \in \mathrm{W},\ \ x \in \mathfrak{g},\ \ x' \in \mathfrak{g}',\end{aligned}$$

see Lemma 10 below.

Here are some natural problems in this context. Express the character Θ_Π in terms of $\Theta_{\Pi'}$, u_Π in terms of $u_{\Pi'}$, $\mathrm{WF}(\Pi)$ in terms of $\mathrm{WF}(\Pi')$, $\mathcal{O}_\Pi$ in terms of $\mathcal{O}_{\Pi'}$. Not much is known about them in general, though the following equality holds for pairs in the stable range with Π' unitary

$$\mathcal{O}_\Pi = \tau(\tau'^{-1}(\mathcal{O}_{\Pi'})), \tag{1}$$

see [27]. (Under some strong assumptions the above equality holds for the wave front sets, see [16].)

As an attempt to solve the first problem the third author constructed an integral kernel operator Chc which maps invariant distributions on $\widetilde{\mathrm{G}}'$ to invariant distributions on $\widetilde{\mathrm{G}}$, [29, Def. 2.17]. In fact, with an appropriate normalization of all the measures involved, this operator maps invariant eigendistributions with the infinitesimal character γ' to invariant eigendistributions with the infinitesimal character $\gamma' \circ \mathsf{C}$,

$$Chc\colon D'(\widetilde{\mathrm{G}}')^{\mathrm{G}'}_{\gamma'} \to D'(\widetilde{\mathrm{G}})^{\mathrm{G}}_{\gamma'\circ\mathsf{C}},$$

see [5, (7) and Theorem 4]. There are reasons to believe that $Chc(\Theta_{\Pi'})$ is a non-zero constant multiple of $\Theta_{\Pi'_1}$, where Π'_1 is the quasisimple admissible representation of $\widetilde{\mathrm{G}}$ whose unique irreducible quotient is Π, as in [21]. Often $\Pi'_1 = \Pi$. (For instance this equality holds when the dual pair is in the stable range and Π' is unitary.) By a limiting process, parallel to that one which leads from C to c, one obtains a Cauchy-Harish-Chandra integral on the Lie algebra

$$chc\colon \mathcal{S}^*(\mathfrak{g}')^{\mathrm{G}'}_0 \to \mathcal{S}^*(\mathfrak{g})^{\mathrm{G}}_0,$$

see (62) and (48) below. Here the subscript 0 indicates the 0-eigensubspace for the action of non-constant invariant constant coefficient differential operators, see

Theorem 12 below. Also, the domain of chc is not always the whole space $\mathcal{S}^*(\mathfrak{g}')_0^{\mathrm{G}'}$. The limiting process is such that if $Chc(\Theta_{\Pi'})$ is a non-zero constant multiple of Θ_{Π}, then $chc(u_{\Pi'})$ is a non-zero constant multiple of u_{Π}.

In this context it makes sense to ask if chc maps the Fourier transform of a nilpotent orbital integral $\hat{\mu}_{\mathcal{O}'}$ to a non-zero constant multiple of the Fourier transform of a nilpotent orbital integral $\hat{\mu}_{\mathcal{O}}$, and how are the orbits $\mathcal{O}'$, $\mathcal{O}$ related. We don't know the answer in general, but offer a sort of "diplomatic solution" which involves Springer correspondence, see Theorem 18 (or Theorem 1-a simplified version) below.

More precisely, given a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ we have the corresponding Weyl group W. The Springer correspondence associates an irreducible representation of W to each complex nilpotent coadjoint orbit, assuming the group is connected. See [33] for a convenient geometric construction. We shall use this construction in Appendix A to extend the notion of Springer correspondence to cover the case when the reductive group is an orthogonal group (which is disconnected) and refer to this extended version as the "combinatorial Springer correspondence", denote it by CSC, see (18) below. Thus $\mathrm{CSC}(\mathcal{O}_{\Pi})$ is an irreducible representation of W corresponding to the complex nilpotent coadjoint orbit $\mathcal{O}_{\Pi}$.

As explained above, $\hat{\mu}_{\mathcal{O}'}$ is "harmonic" with respect to the non-constant invariant constant coefficient differential operators. Hence the product of the restriction of $\hat{\mu}_{\mathcal{O}'}$ to the regular set of any Cartan subalgebra of $\mathfrak{g}$, when multiplied by the product of the positive roots, is a harmonic function, see Corollary 13 below. Hence the action of the Weyl group W on the polynomial functions defined on that Cartan subalgebra generates a representation of W. We shall refer to it as to the Weyl group representation generated by the restriction of $chc(\hat{\mu}_{\mathcal{O}'})$ to the Cartan subalgebra.

For simplicity of exposition, let us assume that $(\mathrm{G}', \mathrm{G})$ is not a complex dual pair. The groups G', G come with the defining modules V', V respectively, see [19]. The complexified groups $(\mathrm{G}'_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$ also form a dual pair with the defining modules $\mathsf{V}'(\mathbb{C})$ and $\mathsf{V}(\mathbb{C})$ respectively (which don't need to be the complexifications of V', V). Given a nilpotent coadjoint orbit $\mathcal{O}' \subseteq \mathfrak{g}'^*$ let $\mathcal{O}'_{\mathbb{C}} \subseteq \mathfrak{g}'^*_{\mathbb{C}}$ denote the complex orbit containing $\mathcal{O}'$. Then $\mathcal{O}' = \mathcal{O}'(\lambda')$ corresponds to a partition λ', see [9]. Denote by $\mathrm{ht}(\lambda')$ the height of the partition λ' (see Sect. 3 below).

Theorem 1. *Assume that $chc(\hat{\mu}_{\mathcal{O}'}) \neq 0$. If the pair $(\mathrm{G}', \mathrm{G})$ is of type I, assume that*

$$\mathrm{ht}(\lambda') < \dim \mathsf{V}(\mathbb{C}) - \dim \mathsf{V}'(\mathbb{C}).$$

Let $r = \dim(\mathsf{V}(\mathbb{C})) - \dim(\mathsf{V}'(\mathbb{C}))$ and let $\lambda = (1^r) \oplus \lambda'$ be the partition obtained by adding a column of length r to λ' on the very left. Denote by $\mathcal{O}_{\mathbb{C}}(\lambda) \subseteq \mathfrak{g}^*_{\mathbb{C}}$ the corresponding nilpotent coadjoint orbit. Then

$$\mathcal{O}_{\mathbb{C}}(\lambda) = \tau(\tau'^{-1}(\mathcal{O}_{\mathbb{C}}(\lambda'))$$

and the Weyl group representation generated by the restriction of $chc(\hat{\mu}_{\mathcal{O}'})$ to any Cartan subalgebra of $\mathfrak{g}$ is equal to $\mathrm{CSC}(\mathcal{O}_{\mathbb{C}}(\lambda))$. In other words, in the context of Wallach's approach to the Springer correspondence (see Sect. 2)*, $chc(\hat{\mu}_{\mathcal{O}'})$ behaves like $\hat{\mu}_{\mathcal{O}}$, where the complexification of $\mathcal{O}$ is equal to $\mathcal{O}_{\mathbb{C}}(\lambda)$.*

In order to prove Theorem 1 we actually compute $chc(\hat{\mu}_{\mathcal{O}'})$, see Theorem 14 below. We also show (Theorem 11) that chc intertwines the action of the algebra of the invariant constant coefficient differential operators on $\mathfrak{g}$ with the action of the algebra of the invariant constant coefficient differential operators on $\mathfrak{g}'$ via the canonical algebra homomorphism C (24). As a consequence, chc maps eigendistributions to eigendistribution (Theorem 12), so that $chc(\hat{\mu}_{\mathcal{O}'})$ does indeed provide the harmonic polynomials needed in the Springer correspondence. This may be seen independently from Theorem 14 and results of Sect. 5.

The idea behind Chc is to make sense out of the following, often divergent, integral

$$\int_{\mathrm{G}'} \Theta(g'g)\,\Theta_{\Pi'^c}(g')\,dg' \qquad (g \in \widetilde{\mathrm{G}}),$$

where Θ is the character of the oscillator representation and Π'^c is the representation contragredient to Π'.

In case of a dual pair defined over a finite field the above integral, with the $\widetilde{\mathrm{G}}$ replaced by G, is a finite sum which obviously converges and defines a class function on G. This class function decomposes into a sum of several irreducible characters Θ_Π. In other words Howe's correspondence often does not associate a single irreducible representation to Π' and the situation is quite complex. We study it in a separate article, [1].

2. Wallach's approach to Springer's correspondence

Let $\mathfrak{g}$ be a semisimple Lie algebra over the reals, $\mathrm{G} \subseteq \mathrm{End}(\mathfrak{g})$ the corresponding adjoint group, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra and $\mathsf{W} = \mathsf{W}(\mathfrak{h}_\mathbb{C}, G_\mathbb{C})$, the Weyl group of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $\mathcal{D}(\mathfrak{g})^{\mathrm{G}}$ denote the algebra of the G-invariant polynomial coefficient differential operators on $\mathfrak{g}$, and let $\mathcal{D}(\mathfrak{h})^{\mathsf{W}}$ denote the algebra of the W-invariant polynomial coefficient differential operators on $\mathfrak{h}$. Let $a(x)f(y) = \frac{d}{dt}f(y + t[x, y])|_{t=0}$, $x, y \in \mathfrak{g}$. Set $\mathcal{I} = \mathcal{D}(\mathfrak{g})^{\mathrm{G}} \cap (\mathcal{D}(\mathfrak{g})a(\mathfrak{g}))$. As a culmination of the works of [14,24,41], and [25] we have the following short exact sequence

$$0 \to \mathcal{I} \to \mathcal{D}(\mathfrak{g})^{\mathrm{G}} \to \mathcal{D}(\mathfrak{h})^{\mathsf{W}} \to 0, \tag{2}$$

where $\mathcal{I} \to \mathcal{D}(\mathfrak{g})^{\mathrm{G}}$ is the inclusion and $\delta : \mathcal{D}(\mathfrak{g})^{\mathrm{G}} \to \mathcal{D}(\mathfrak{h})^{\mathsf{W}}$ stands for the Harish-Chandra homomorphism.

Consider an invariant tempered distribution $u \in S^*(\mathfrak{g})^{\mathrm{G}}$. Then, $a(\mathfrak{g})u = 0$ and therefore the $\mathcal{D}(\mathfrak{g})^{\mathrm{G}}$-module generated by u, $\mathcal{D}(\mathfrak{g})^{\mathrm{G}}u$, satisfies $\mathcal{I}\mathcal{D}(\mathfrak{g})^{\mathrm{G}}u = 0$. Hence, by (2), $\mathcal{D}(\mathfrak{g})^{\mathrm{G}}u$ may be viewed as a $\mathcal{D}(\mathfrak{h})^{\mathsf{W}}$-module.

Let $\mathcal{P}(\mathfrak{h})$ denote the space of the complex valued polynomial functions on $\mathfrak{h}$ and let $\widehat{\mathsf{W}}$ be the set of the (equivalence classes of) irreducible representations of W. The algebra $\mathcal{D}(\mathfrak{h})^{\mathsf{W}}$ acts on $\mathcal{P}(\mathfrak{h})$, as usual, and so does the group W. Furthermore these actions commute. Wallach showed that

$$\mathcal{P}(\mathfrak{h}) = \sum_{\rho \in \widehat{\mathsf{W}}} \rho' \otimes \rho, \tag{3}$$

as a $(\mathcal{D}(\mathfrak{h})^W, \mathsf{W})$-module, where ρ' stands for a simple $\mathcal{D}(\mathfrak{h})^{\mathsf{W}}$-module and the function $\rho \to \rho'$, defined on $\hat{\mathsf{W}}$, is injective, see [41]. (Here $\rho' \otimes \rho = \mathcal{P}(\mathfrak{h})_\rho$ is the ρ-isotypic component of $\mathcal{P}(\mathfrak{h})$ under the action of W.) By taking a Fourier transform on $\mathfrak{h}$, the symmetric algebra $\mathcal{S}(\mathfrak{h})$ becomes a $\mathcal{D}(\mathfrak{h})^{\mathsf{W}}$-module and (3) is transformed to

$$\mathcal{S}(\mathfrak{h}) = \sum_{\rho \in \widehat{\mathsf{W}}} \hat{\rho}' \otimes \rho, \tag{4}$$

a $(\mathcal{D}(\mathfrak{h})^W, \mathsf{W})$-module, where $\hat{\rho}'$ is a simple $\mathcal{D}(\mathfrak{h})^{\mathsf{W}}$-module equal to the Fourier transform of ρ'.

Let $\pi \in \mathcal{P}(\mathfrak{h})$ be the product of all the positive roots for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, with respect to some fixed order of the roots, $\mathfrak{h}^{reg} = \{x \in \mathfrak{h};\ \pi(x) \neq 0\}$ the subset of the regular elements and let $C \subseteq \mathfrak{h}^{reg}$ a connected component. Let $\mathcal{O} \subseteq \mathfrak{g}$ be a nilpotent G-orbit and let $\mu_{\mathcal{O}} \in S^*(\mathfrak{g})$ be the corresponding invariant measure, as in [32]. Let $\hat{\mu}_{\mathcal{O}}$ be the Fourier transform of $\mu_{\mathcal{O}}$ defined with respect to a non-degenerate G-invariant symmetric bilinear form on $\mathfrak{g}$. According to Harish-Chandra Regularity Theorem, [15], $\hat{\mu}_{\mathcal{O}}$ is a function on $\mathfrak{g}$. Wallach showed that there is a unique $\rho \in \hat{\mathsf{W}}$ such that, in terms of (4), $\mathcal{D}(\mathfrak{g})^{\mathrm{G}}\mu_{\mathcal{O}} = \hat{\rho}'$ and that

$$(\hat{\mu}_{\mathcal{O}}\pi)|_C \in \mathcal{P}(\mathfrak{h})_\rho|_C, \tag{5}$$

where $\hat{\mu}_{\mathcal{O}}$ is viewed as a function on $\mathfrak{h}^{reg}$ and $|_C$ stands for the restriction of functions from $\mathfrak{h}^{reg}$ to C. Wallach deduced form [18] that ρ corresponds to the complex orbit $\mathcal{O}_{\mathbb{C}} = \mathrm{G}_{\mathbb{C}}\mathcal{O} \subseteq \mathfrak{g}_{\mathbb{C}}$ via the Springer correspondence, [38].

3. Representations of the Weyl groups

Partitions, tableaux, tabloids, etc.: Let n be an integer. A *partition* of n is a finite sequence $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0]$ of integers λ_i such that $\sum_{i=1}^k \lambda_i = n$. Let $\mathrm{ht}(\lambda)$ denote the *height* of the partition λ (that is, the largest i with $\lambda_i \neq 0$). Flipping a Young diagram of a partition λ of n over its main diagonal (from upper left to lower right), we obtain the Young diagram of another partition ${}^t\lambda$ of n, which is called the *conjugate partition* of λ. Thus, for $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k]$, we have ${}^t\lambda = [{}^t\lambda_1 \geq {}^t\lambda_2 \geq \cdots \geq {}^t\lambda_l]$, where $l = \lambda_1$ and ${}^t\lambda_j = |\{i\ :\ 1 \leq i \leq k, \lambda_i \geq j\}|$ for $1 \leq j \leq l$.

First we consider the group of permutations of n letters, $\mathsf{W} = \mathfrak{S}_n$. The irreducible representations of W are parameterized by partitions $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k]$ of n as follows. A *tabloid* of the shape λ is the Cartesian product $A_1 \times A_2 \times \cdots \times A_k$, where $A_1 \cup A_2 \cup \cdots \cup A_k = \{1, 2, \ldots, n\}$ and $|A_i| = \lambda_i$. The group $\mathfrak{S}_n$ acts on the set of all the tabloids of shape λ in the obvious way. Let V_λ be the complex vector space spanned by all the tabloids of shape λ.

A *tableau T of shape* λ is defined to be the Young diagram of λ filled labels $1, 2, \ldots, n$, which are increasing (from left to right) in rows and (from top to bottom) in columns:

$$T = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array}$$

To every tableau T we associate a tabloid $\{T\}$ equal to the Cartesian product of the sets made of the rows of T:

$$\{T\} = \{1, 4, 6\} \times \{2, 5, 7\} \times \{3\}.$$

Let $C(T) \subseteq \mathfrak{S}_n$ be the subgroup preserving the columns, $T_1, T_2, \ldots, T_\ell$ of T.

A *polytabloid* of T is the element of V_λ defined by

$$\sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\sigma\{T\}. \tag{6}$$

The subspace of V_λ spanned by all the polytabloids is irreducible under the action of $\mathfrak{S}_n$, is denoted by $\mathcal{S}_\lambda$ and is called the Specht module corresponding to λ. This establishes a one to one correspondence between the partitions of n and the (equivalence classes of the) irreducible representations of the group $\mathfrak{S}_n$ (see for example [22]). The trivial representation and the sign representation correspond to the partitions (n), (1^n), respectively.

We would like to realize each $\mathcal{S}_\lambda$ in the space of the polynomials $\mathbb{C}[x_1, x_2, \ldots, x_n]$. This is done as follows. Let $t_{i1}, t_{i2}, \ldots, t_{i\lambda_i}$ be the ith row of a tableau T. We associate to T the product

$$\prod_{i=1}^{\mathrm{ht}(\lambda)} x_{t_{i1}}^{i-1} x_{t_{i2}}^{i-1} \cdots x_{t_{i\lambda_i}}^{i-1}.$$

For the above example we get

$$\begin{array}{l} x_1^0\, x_4^0\, x_6^0 \\ x_2^1\, x_5^1\, x_7^1\,. \\ x_3^2 \end{array}$$

This extends to a $\mathfrak{S}_n$-intertwining map from V_λ to $\mathbb{C}[x_1, x_2, \ldots, x_n]$. The restriction of this map to $\mathcal{S}_\lambda$ is not zero. In fact, the image of (6) is equal to the product of the Vandermonde determinants made of the variables which are in the columns of T:

$$\Delta_T = \Delta_{T_1} \Delta_{T_2} \cdots \Delta_{T_\ell}.$$

Here

$$\Delta_{T_j} := (x_m - x_h)(x_m - x_j)(x_h - x_j) \quad \text{if} \quad T_j = \begin{array}{|c|}\hline x_j \\ \hline x_h \\ \hline x_m \\ \hline\end{array}.$$

The polynomial Δ_T where $T_1 = (1, 2, 3, \ldots, {}^t\lambda_1)$, $T_2 = ({}^t\lambda_1 + 1, {}^t\lambda_1 + 2, {}^t\lambda_1 + 3, \ldots, {}^t\lambda_1 + {}^t\lambda_2)$, …, shall be denoted by Δ_λ. The $\mathfrak{S}_n$-submodule of

$\mathbb{C}[x_1, x_2, \ldots, x_n]$ generated by Δ_λ is isomorphic to $\mathcal{S}_\lambda$ and shall be denoted by ρ_λ. It occurs in the space of the homogeneous polynomials of degree $n(\lambda)$, the degree of the polynomial Δ_λ, which is the sum of the degrees of the corresponding Vandermonde determinants. This number may be visualized as follows. Fill the first row of Young diagram λ with zeros, the second row with 1s, the third row with 2s and so on. Then add all these numbers. The result is $n(\lambda)$. This is the lowest degree in which (an isomorphic module to) ρ_λ occurs in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ and it occurs in this degree with multiplicity one (see [26] or [11, 5.4.4]).

Now we consider the hyperoctahedral group $W_n = W(B_n)$, which is equal to the semidirect product of $\mathfrak{S}_n$ acting on $(\mathbb{Z}/2\mathbb{Z})^n$. There is a unique character $\mathrm{sgn}_{\mathrm{CD},n} : W_n \to \{\pm 1\}$ whose restriction to the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ is the product of the sign characters and that is trivial on the subgroup $\mathfrak{S}_n$. The kernel of $\mathrm{sgn}_{\mathrm{CD},n}$ is isomorphic to the Weyl group $W(D_n)$. The restriction of the character $\mathrm{sgn}_{\mathrm{CD},n}$ to the subgroup W_{n-1} of W_n equals the character $\mathrm{sgn}_{\mathrm{CD},n-1}$. Because of this, we will denote $\mathrm{sgn}_{\mathrm{CD},n}$ simply by $\mathrm{sgn}_{\mathrm{CD}}$.

Let (ξ, η) be a pair of partitions with $|\xi| = s$, $|\eta| = t$ where $s + t = n$. We extend the action of the group $\mathfrak{S}_s$ on the Specht module $\mathcal{S}_\xi$ to an action of the W_s by letting $(\mathbb{Z}/2\mathbb{Z})^s$ act trivially. Similarly, we extend the action of the group $\mathfrak{S}_t$ on the Specht module $\mathcal{S}_\eta$ to an action of the W_t by letting each $\mathbb{Z}/2\mathbb{Z}$ act via the non-trivial character. The induced representation $\mathrm{Ind}_{W_s \times W_t}^{W_n}(\mathcal{S}_\xi \otimes \mathcal{S}_\eta)$ is irreducible. This way the irreducible representations of W_n are parameterized by the pairs of partitions of n (see [26]). The trivial representation of W_n corresponds to $((n), \emptyset)$ while the sign representation corresponds to $(\emptyset, (1^n))$ and the representation afforded by the character $\mathrm{sgn}_{\mathrm{CD}} = \mathrm{sgn}_{\mathrm{CD},n}$ corresponds to $(\emptyset, (n))$.

Let $\Delta_{\xi,\eta}$ be the product of Δ_ξ (in the variables $x_1^2, x_2^2, \ldots, x_s^2$), Δ_η (in the variables $x_{s+1}^2, x_{s+2}^2, \ldots, x_n^2$) and the monomial $x_{s+1}x_{s+2}\ldots x_n$. The W_n-submodule of $\mathbb{C}[x_1, x_2, \ldots, x_n]$ generated by $\Delta_{\xi,\eta}$ is isomorphic to the irreducible representation corresponding to (ξ, η). We denote it by $\rho_{(\xi,\eta)}$. It occurs in the degree

$$2n(\xi) + 2n(\eta) + |\eta|. \tag{7}$$

As in the previous case, this is the lowest degree in which (an isomorphic module to) $\rho_{(\xi,\eta)}$ occurs in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ and it occurs in this degree with multiplicity one (see [26] or [11, 5.5.3]).

Notice the following formulas

$$\begin{aligned} &\max_{1 \le j \le n} \deg_{x_j} \Delta_\lambda(x) = \mathrm{ht}(\lambda) - 1, \qquad (8)\\ &\max_{1 \le j \le n} \deg_{x_j} \Delta_{\xi,\eta}(x) = \max\{2\,\mathrm{ht}(\xi) - 1, 2\,\mathrm{ht}(\eta)\} - 1. \end{aligned}$$

Definition 2. A partition λ is called *symplectic* (resp. *orthogonal*) if each odd (resp. even) row occurs with even multiplicity.

For N a given integer, let $\mathcal{P}^{\mathrm{sp}}(N)$ (resp. $\mathcal{P}^{\mathrm{or}}(N)$) denote the set of symplectic (resp. orthogonal) partitions of N.

Though this article concerns real reductive dual pairs, the combinatorics involved carries over to a study of dual pairs over a finite field. Therefore, for future reference, [1], we consider $\mathbb{K}$, an algebraically closed field of characteristic $p \geq 0$, with $p \neq 2$. (The field $\mathbb{K}$ could be equal to $\mathbb{C}$, of course.) For each group $G = \mathrm{GL}_n(\mathbb{K})$, $\mathrm{Sp}_{2n}(\mathbb{K})$, $\mathrm{O}_{2n+1}(\mathbb{K})$ or $\mathrm{O}_{2n}(\mathbb{K})$ we have the defining module $\mathbb{K}^n$, $\mathbb{K}^{2n}$, $\mathbb{K}^{2n+1}$, $\mathbb{K}^{2n}$ respectively. The nilpotent orbits in the corresponding Lie algebra are parameterized by partitions λ of the dimension of the defining module, as in [9]. The partition λ is symplectic (resp. orthogonal) if $G = \mathrm{Sp}_{2n}(\mathbb{K})$ [resp. $\mathrm{O}_{2n}(\mathbb{K})$ or $\mathrm{O}_{2n+1}(\mathbb{K})$].

We will now recall the algorithm described in [8, 13.3]. To each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ we attach the *sequence of β-numbers*

$$\lambda^* = (\lambda_1^* < \lambda_2^* < \cdots < \lambda_k^*), \text{ defined by } \lambda_j^* := \lambda_{k-j+1} + j - 1, \text{ for } 1 \leq j \leq k. \tag{9}$$

For instance, we have

$$(n)^* = (n), \quad (n-1,1)^* = (1,n),$$
$$(n-2,2)^* = (2,n-1), \quad (n-2,1,1)^* = (1,2,n).$$

(In the proof of Proposition 5 below it will be simpler to work with partitions with increasing terms, $\lambda = (\bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \cdots \leq \bar{\lambda}_k)$ where $\bar{\lambda}_i := \lambda_{k-i+1}$).

Consider a symplectic or orthogonal partition λ and the corresponding group G. We ensure that the number of parts of λ has same parity as the defining module of G, by calling the last part 0 if necessary. Thus $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2k}$ (resp. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2k+1}$) if $G = \mathrm{Sp}_{2n}(\mathbb{K})$ or $\mathrm{O}_{2n}(\mathbb{K})$ [resp. $G = \mathrm{O}_{2n+1}(\mathbb{K})$]. We then divide λ^* into its odd and even parts. Let the odd parts and the even parts of λ^* be

$$2\xi_1^* + 1 < 2\xi_2^* + 1 < \cdots < 2\xi_k^* + 1 \ \ (\text{resp. } 2\xi_{k+1}^* + 1) \quad \text{and}$$
$$2\eta_1^* < 2\eta_2^* < \cdots < 2\eta_k^*,$$

respectively. Then we have

$$0 \leq \xi_1^* < \xi_2^* < \cdots < \xi_k^* \ \ (\text{resp. } \xi_{k+1}^*) \quad \text{and} \quad 0 \leq \eta_1^* < \eta_2^* < \cdots < \eta_k^*.$$

Next we define $\xi_i := \xi_{k-i+1}^* - (k-i)$ and $\eta_i := \eta_{k-i+1}^* - (k-i)$ for each i. We then have $\xi_i \geq \xi_{i+1} \geq 0$, $\eta_i \geq \eta_{i+1} \geq 0$, and $|\xi| + |\eta| = n$.

Thus we obtain an injective map

$$\varphi \colon \lambda \mapsto (\xi, \eta) \tag{10}$$

from $\mathcal{P}^{\mathrm{sp}}(2n)$ or $\mathcal{P}^{\mathrm{or}}(2n)$ [resp. $\mathcal{P}^{\mathrm{or}}(2n+1)$] to the set of pairs of partitions of n.

The following lemma and corollary are going to use them in the proof of Theorem 18.

Lemma 3. *Let λ be either a symplectic or an orthogonal partition and let (ξ, η) be the corresponding pair of partitions. Then*

$$\mathrm{ht}(\lambda) = \begin{cases} \max\{2\,\mathrm{ht}(\xi) - 1, 2\,\mathrm{ht}(\eta)\} & \text{if } |\lambda| \text{ is even,} \\ \max\{2\,\mathrm{ht}(\xi) - 1, 2\,\mathrm{ht}(\eta) + 1\} & \text{if } |\lambda| \text{ is odd.} \end{cases} \tag{11}$$

Proof. Let λ be a symplectic partition.

Suppose $\mathrm{ht}(\lambda) = 2k$. If λ_{2k} is even then $\eta_k \neq 0$. Thus $\mathrm{ht}(\lambda) = 2\,\mathrm{ht}(\eta)$. Since $\mathrm{ht}(\xi) \leq k$, (11) follows. If λ_{2k} is odd then $\lambda_{2k-1} = \lambda_{2k}$ is odd. Hence, $\lambda_2^* \geq 2$ is even. This is the smallest even part of λ^*. Therefore $\mathrm{ht}(\eta) = k$ and (11) follows.

Suppose $\mathrm{ht}(\lambda) = 2k - 1$. Since $\lambda_{2k} = 0$, $\mathrm{ht}(\eta) < k$. If λ_{2k-1} is even then λ_2^* is odd and greater or equal to 3. Therefore $\mathrm{ht}(\xi) = k$ and (11) follows. If λ_{2k-1} is odd then $\lambda_{2k-2} = \lambda_{2k-1}$ and therefore $\lambda_3^* = \lambda_{2k-2} + 2 \geq 3$ is odd. This is the smallest odd part of λ^*. Thus $\mathrm{ht}(\xi) = k$ and (11) follows.

Let λ be an orthogonal partition with $|\lambda|$ even. Then $\mathrm{ht}(\lambda) = 2k$.

If λ_{2k} is even then $\mathrm{ht}(\eta) = k$ and (11) follows. If λ_{2k} is odd and greater than 1 then $\lambda_1^* > 1$. Hence the smallest even part of λ^* is positive. Thus $\mathrm{ht}(\eta) = k$, which implies (11). If $\lambda_{2k} = 1$, then $\lambda_2^* > 1$, which implies $\mathrm{ht}(\eta) = k$. Again (11) follows.

Let λ be an orthogonal partition with $|\lambda|$ odd. Then $\mathrm{ht}(\lambda) = 2k + 1$.

If λ_{2k+1} is even, then $\mathrm{ht}(\xi) = k + 1$ and $\mathrm{ht}(\eta) = k$. Thus (11) follows. If $\lambda_{2k+1} > 1$ is odd, then $\mathrm{ht}(\xi) = k + 1$ and $\mathrm{ht}(\eta) \leq k$, which implies (11). If $\lambda_{2k} = 1$, then $\mathrm{ht}(\xi) = \mathrm{ht}(\eta) = k$, which implies (11). □

By combining (8) with Lemma 3 we deduce the following Corollary.

Corollary 4. *For any partition λ,*

$$\max_{1 \leq j \leq n} \deg_{x_j} \Delta_\lambda(x) = \mathrm{ht}(\lambda) - 1. \tag{12}$$

For a symplectic or orthogonal partition λ and the corresponding pair of partitions (ξ, η)

$$\max_{1 \leq j \leq n} \deg_{x_j} \Delta_{\xi,\eta}(x) = \mathrm{ht}(\lambda) - 1, \tag{13}$$

unless λ is orthogonal with $|\lambda|$ odd and $2\mathrm{ht}(\xi) - 1 < 2\mathrm{ht}(\eta) + 1$. (Equivalently the smallest part of λ is 1.) In this case

$$\max_{1 \leq j \leq n} \deg_{x_j} \Delta_{\xi,\eta}(x) = \mathrm{ht}(\lambda) - 2. \tag{14}$$

4. Springer correspondence

For an irreducible complex dual pair (G, G'), let τ, τ' be the corresponding moment maps. We assume that the rank of G' is smaller than the rank of G. Let $\mathcal{O}(\lambda)$ denote the nilpotent orbit in the Lie algebra of G parameterized by the partition λ and let $\overline{\mathcal{O}(\lambda)}$ denote the closure of $\mathcal{O}(\lambda)$.

Given a nilpotent orbit $\mathcal{O}(\lambda')$ there is a unique partition λ such that

$$\tau(\tau'^{-1}(\overline{\mathcal{O}(\lambda')})) = \overline{\mathcal{O}(\lambda)}, \tag{15}$$

see [10]. Let V be the defining module for G and V' for G$'$. If

$$r := \dim \mathsf{V} - \dim \mathsf{V}' \geq \mathrm{ht}(\lambda'), \tag{16}$$

then [10, Theorems 5.2, 5.6] show that λ is obtained from λ' by adding a column of length r on the very left. [The condition (16) implies, that in the notation of [10, Definitions 5.1, 5.5], we are in the case where $i_0 = r + 1$ and $r_{i_0} = 0$.] In other words,

$$\lambda = (1^r) \oplus \lambda', \tag{17}$$

that is, if $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{k'})$ (here $k' = \mathrm{ht}(\lambda')$) then $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$, where $\lambda_i = \lambda'_i + 1$ for $1 \leq i \leq k$, with the convention that $\lambda'_i = 0$ for $k' + 1 \leq i \leq r$. It is clear that the partition λ defined by (17) has the right type. Indeed, if λ' is symplectic, the condition that each odd row of λ' occurs with even multiplicity implies that each even row of λ has the same property. Hence λ is orthogonal. If λ' is orthogonal, each odd λ_i with $1 \leq i \leq k'$ occurs with even multiplicity. In particular, it implies that the height of λ has same parity as r, that is, $r - k'$ is even. Hence λ is symplectic.

Springer constructed irreducible representations of the Weyl group in cohomology groups of varieties associated to nilpotent orbits, [38]. This is known as Springer correspondence. We shall use a combinatorial description of this correspondence, which may be found in [8], Section 13.3. (The description in [9] for the group $\mathrm{Sp}_{2n}(\mathbb{C})$ is incorrect and for the group $\mathrm{O}_{2n}(\mathbb{C})$ is rather sketchy.) Since the orthogonal group $\mathrm{O}_{2n}(\mathbb{C})$ occurs as a member of dual pair, we need a "Springer correspondence" in this case, which seems to be unavailable in the literature. In order to include this case we provide an extension of Rossmann's approach to the Springer correspondence, [33], in Appendix A.

If $G = \mathrm{GL}_n(\mathbb{K})$ then the Weyl group coincides with $\mathfrak{S}_n$ and the Springer correspondence associates to an orbit $\mathcal{O}(\lambda)$ the representation ρ_λ.

In all remaining cases the Weyl group is the semidirect product of $\mathfrak{S}_n$ acting on $(\mathbb{Z}/2\mathbb{Z})^n$. For the groups $\mathrm{Sp}_{2n}(\mathbb{K})$ and $\mathrm{O}_{2n+1}(\mathbb{K})$, we associate the representation $\rho_{\xi,\eta}$ to the orbit $\mathcal{O}(\lambda)$, where $(\xi, \eta) := \varphi(\lambda)$, with φ defined by (10).

Consider the group $\mathrm{O}_{2n}(\mathbb{K})$. In this case the nilpotent orbits are parameterized by partitions λ of $2n$ where the even rows occur with even multiplicities. We attach to such a partition λ the ordered pair of partitions (ξ, η) defined by $(\xi, \eta) := \varphi(\lambda)$. Then we associate to $\mathcal{O}(\lambda)$ the representation $\rho_{\eta,\xi}$.

Thus in any case we have a "combinatorial Springer correspondence"

$$\mathrm{CSC} \colon \mathcal{O} \mapsto \rho, \tag{18}$$

which is compatible with our extension of Rossmann's work, [33], as explained in A.2.

The following proposition shows what happens to the Weyl group representations if we map a nilpotent orbit from one Lie algebra to a nilpotent orbit in the other Lie algebra via the moment maps, as in (15), and apply our "combinatorial Springer correspondence".

Proposition 5. *Let* $(\mathbf{G}, \mathbf{G}')$ *be an irreducible reductive dual pair of type I over* $\mathbb{K}$*, and let* $\mathcal{O}(\lambda')$ *be a nilpotent orbit in the Lie algebra of* $\mathbf{G}'$ *satisfying the condition* (16).

Let $\mathcal{O}(\lambda)$ be the corresponding orbit in the Lie algebra of $\mathbf{G}$*, as in* (15)*, or equivalently in* (17)*. Let* $\mathrm{CSC}(\mathcal{O}(\lambda')) = \rho_{\xi',\eta'}$ *and* $\mathrm{CSC}(\mathcal{O}(\lambda)) = \rho_{\xi,\eta}$*. Let ℓ be the difference of the rank of* $\mathbf{G}$ *and* $\mathbf{G}'$*. Then*

$$(\xi,\eta) = \begin{cases} ((1^{\ell})\oplus\xi',\eta') & \text{if} \quad \mathbf{G}' = \mathrm{Sp}_{2n'}(\mathbb{K}) \quad \text{and} \quad \mathbf{G} = \mathrm{O}_{2n}(\mathbb{K}) \\ (\xi',(1^{\ell})\oplus\eta') & \text{if} \quad \mathbf{G}' = \mathrm{O}_{2n'}(\mathbb{K}) \quad \text{and} \quad \mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K}) \\ ((1^{\ell})\oplus\eta',\xi') & \text{if} \quad \mathbf{G}' = \mathrm{Sp}_{2n'}(\mathbb{K}) \quad \text{and} \quad \mathbf{G} = \mathrm{O}_{2n+1}(\mathbb{K}) \\ ((1^{\ell})\oplus\xi',\eta') & \text{if} \quad \mathbf{G}' = \mathrm{O}_{2n'+1}(\mathbb{K}) \quad \text{and} \quad \mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K}). \end{cases} \tag{19}$$

Proof. We proceed via a case by case analysis and write the partitions in the increasing order ($\lambda_1 \le \lambda_2 \le \cdots$) as in [8].

Case $\mathbf{G}' = \mathrm{Sp}_{2n'}(\mathbb{K})$ and $\mathbf{G} = \mathrm{O}_{2n}(\mathbb{K})$.

Here $\lambda'_1 \le \lambda'_2 \le \cdots \le \lambda'_{2k}$ and $\lambda = (1^{2n-2n'}) \oplus \lambda$. Therefore,

$$\lambda^*_j = \lambda_j + j - 1 = \begin{cases} j & \text{if} \quad 1 \le j \le 2n-2n'-2k, \\ \lambda'_{j-2n+2n'+2k} + j & \text{if} \quad 2n-2n'-2k < j \le 2n-2n', \end{cases}$$

where $\lambda'_{j-2n+2n'+2k} + j = \lambda'^*_{j-2n+2n'+2k} + 2n - 2n' - 2k + 1$. Therefore,

$$\begin{array}{ll} 2\eta^*_\nu = 2\nu & \text{if} \quad 2 \le 2\nu \le 2n-2n'-2k, \\ 2\eta^*_\nu = 2\xi'^*_{\nu'} + 1 + 2n - 2n' - 2k + 1 & \text{if} \quad \nu = n-n'-k+\nu',\ 1 \le \nu' \le k. \end{array}$$

Thus,

$$\begin{array}{ll} \eta^*_\nu = \nu & \text{if} \quad 1 \le \nu \le n-n'-k, \\ \eta^*_\nu = \xi'^*_{\nu'} + n - n' - k + 1 & \text{if} \quad \nu = n-n'-k+\nu',\ 1 \le \nu' \le k. \end{array}$$

If $1 \le \nu \le n-n'-k$ then $\eta_\nu = \eta^*_\nu - \nu + 1 = 1$. If $\nu = n-n'-k+\nu'$ and $1 \le \nu' \le k$ then

$$\begin{aligned} \eta_\nu &= \eta^*_\nu - \nu + 1 = \xi'^*_{\nu'} + n - n' - k + 1 - n + n' + k - \nu' + 1 \\ &= \xi'^*_{\nu'} - \nu + 1 + 1 = \xi'_{\nu'}. \end{aligned}$$

Thus $\eta = (1^{\ell}) \oplus \xi'$.

Similarly,

$$\begin{array}{ll} 2\xi^*_\nu + 1 = 2\nu - 1 & \text{if} \quad 1 \le 2\nu - 1 \le 2n-2n'-2k-1, \\ 2\xi^*_\nu + 1 = 2\eta'^*_{\nu'} + 2n - 2n' - 2k + 1 & \text{if} \quad \nu = n-n'-k+\nu',\ 1 \le \nu' \le k. \end{array}$$

If $1 \le \nu \le n-n'-k$, then

$$\xi_\nu = \xi^*_\nu - \nu + 1 = \nu - 1 - \nu + 1 = 0.$$

If $\nu = n-n'-k+\nu'$ and $1 \le \nu' \le k$ then

$$\begin{aligned} \xi_\nu = \xi^*_\nu - \nu + 1 = \eta'^*_{\nu'} + n - n' - k - \nu + 1 = \eta'^*_{\nu'} + n - n' - k - n + n' \\ + k - \nu' + 1 = \eta'_{\nu'}. \end{aligned}$$

Hence, $\xi = \eta'$. Recall that in this case we associate to λ the pair (η, ξ), which (as we just computed) is equal to $((1^{\ell}) \oplus \xi', \eta')$.

Case $\mathbf{G}' = \mathrm{O}_{2n'}(\mathbb{K})$ and $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K})$.

Here, λ', λ, ξ and η are exactly as in the previous case except that we associate to λ the pair $(\xi, \eta) = (\eta', (1^\ell) \oplus \xi')$.

Case $\mathbf{G}' = \mathrm{Sp}_{2n'}(\mathbb{K})$ and $\mathbf{G} = \mathrm{O}_{2n+1}(\mathbb{K})$.

Here, $\lambda'_1 \leq \lambda'_2 \leq \cdots \leq \lambda'_{2k}$ and $\lambda = (1^{2n-2n'+1}) \oplus \lambda'$. Therefore,

$$\lambda^*_j = \begin{cases} j & \text{if} \quad 1 \leq j \leq 2n - 2n' + 1 - 2k, \\ \lambda'_{j-2n+2n'-1+2k} + j & \text{if} \quad 2n - 2n' + 1 - 2k < j \leq 2n - 2n' + 1, \end{cases}$$

where $\lambda'_{j-2n+2n'-1+2k} + j = \lambda'^*_{j-2n+2n'-1+2k} + 2n - 2n' + 1 - 2k + 1$
Therefore,

$$\begin{array}{ll} 2\xi^*_\nu + 1 = 2\nu - 1 & \text{if} \quad 1 \leq \nu \leq n - n' + 1 - k, \\ 2\xi^*_\nu + 1 = 2\xi'^*_{\nu'} + 1 + 2n - 2n' + 1 - 2k + 1 & \text{if} \quad \nu = n - n' + 1 - k + \nu', \\ \quad 1 \leq \nu' \leq k. & \end{array}$$

Thus, for $1 \leq \nu \leq n - n' + 1 - k$,

$$\xi_\nu = \xi^*_\nu - \nu + 1 = 0$$

and for the remaining ν,

$$\xi_\nu = \xi^*_\nu - \nu + 1 = \xi'^*_{\nu'} + n - n' + 1 - k - \nu + 1 = \xi'^*_{\nu'} - \nu' + 1 = \xi'_{\nu'}.$$

Hence, $\xi = \xi'$. Also,

$$\begin{array}{ll} 2\eta^*_\nu = 2\nu & \text{if} \quad 1 \leq \nu \leq n - n' - k, \\ 2\eta^*_\nu = 2\eta'^*_{\nu'} + 2n - 2n' + 1 - 2k + 1 & \text{if} \quad \nu = n - n' - k + \nu',\ 1 \leq \nu' \leq k. \end{array}$$

Therefore $\eta_\nu = 1$, if $1 \leq \nu \leq n - n' - k$. If $\nu = n - n' - k + \nu'$ and $1 \leq \nu' \leq k$, then

$$\eta_\nu = \eta^*_\nu - \nu + 1 = \eta'^*_{\nu'} + n - n' + 1 - k - \nu + 1 = \eta'_{\nu'} + 1.$$

Hence, $\eta = (1^\ell) \oplus \eta'$. Thus $(\xi, \eta) = (\xi', (1^\ell) \oplus \eta')$.

Case $G' = \mathrm{O}_{2n'+1}(\mathbb{K})$ and $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{K})$.

Here, $\lambda'_1 \leq \lambda'_2 \leq \cdots \leq \lambda'_{2k-1}$ and $\lambda = (1^{2n-2n'-1}) \oplus \lambda'$. Since the number of the parts (rows) of λ is odd, we introduce artificially $\lambda_1 = 0$, as required by Lusztig's algorithm [8]. Then

$$\begin{array}{l} \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 1, \ldots,\ \lambda_{2n-2n'-2k+1} = 1, \\ \lambda_{2n-2n'-2k+2} = \lambda'_1 + 1,\ \lambda_{2n-2n'-2k+3} = \lambda'_2 + 1, \ldots,\ \lambda_{2n-2n'} = \lambda'_{2k-1} + 1. \end{array}$$

Hence,

$$\begin{array}{l} \lambda^*_1 = 0,\ \lambda^*_2 = 2,\ \lambda^*_3 = 3, \ldots,\ \lambda^*_{2n-2n'-2k+1} = 2n - 2n' - 2k + 1, \\ \lambda^*_{2n-2n'-2k+2} = \lambda'^*_1 + 2n - 2n' - 2k + 2,\ \lambda^*_{2n-2n'-2k+3} = \lambda'^*_2 + 2n - 2n' - 2k + 2, \ldots, \\ \lambda^*_{2n-2n'} = \lambda'^*_{2k-1} + 2n - 2n' - 2k + 2. \end{array}$$

Therefore,

$$\xi_1^* = 1,\ \xi_2^* = 2,\ \xi_3^* = 3, \ldots,\ \xi_{n-n'-k}^* = n - n' - k,$$
$$\xi_\nu^* = \xi'^*_{\nu'} + n - n' - k,\ \nu = n - n' - k + \nu'.$$

Thus,

$$\xi_1 = 1,\ \xi_2 = 1,\ \xi_3 = 1, \ldots,\ \xi_{n-n'-k} = 1,$$
$$\xi_\nu = \xi'^*_{\nu'} + n - n' - k - \nu + 1 = \xi'_{\nu'} + 1.$$

Hence, $\xi = (1^\ell) \oplus \xi'$. Therefore,

$$\eta_1^* = 0,\ \eta_2^* = 1,\ \eta_3^* = 2, \ldots,\ \eta_{n-n'-k}^* = n - n' - k - 1,$$
$$\eta_\nu^* = \eta'^*_{\nu'} + n - n' - k,\ \nu = n - n' - k + \nu'.$$

Thus,

$$\eta_1 = 0,\ \eta_2 = 0,\ \eta_3 = 0, \ldots,\ \eta_{n-n'-k} = 0,$$
$$\eta_\nu = \eta'^*_{\nu'} + n - n' - k - \nu + 1 = \eta'_{\nu'}.$$

Hence, $\eta = \eta'$. □

5. W-harmonic polynomials

For a Weyl group W, as before, let us define the type of W, say $[\mathsf{W}]$, to be A if $\mathsf{W} = \mathfrak{S}_n$ and B otherwise.

Let $\mathrm{H}^{\mathsf{W}}\mathbb{C}[x_1, x_2, \ldots, x_n] \subseteq \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the space of the W-harmonic polynomials. Let

$$\Delta^{\mathrm{A}} = \Delta_{(1^n)} \tag{20}$$
$$\Delta^{\mathrm{B}} = \Delta_{\emptyset,(1^n)} \tag{21}$$

Then

$$\mathrm{H}^{\mathsf{W}}\mathbb{C}[x_1, x_2, \ldots, x_n] = \mathbb{C}[\partial_1, \partial_2, \ldots, \partial_n]\Delta^{[\mathsf{W}]}(x_1, x_2, \ldots, x_n). \tag{22}$$

Lemma 6. *If $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ transforms under the sign representation of W then it is divisible by $\Delta^{[\mathsf{W}]}$.*

For both (22) and Lemma 6, see [17].

Corollary 7. *For any irreducible representation ρ of the Weyl group W, the corresponding polynomial Δ_ρ is W-harmonic. (In this notation we identify ρ with the corresponding partition or a pair of partitions.)*

Proof. Let $D \in \mathbb{C}[\partial_1, \partial_2, \ldots, \partial_n]$ be W-invariant of positive degree. Suppose $\mathsf{W} = \mathfrak{S}_n$. Then $\Delta_\rho = \Delta_\lambda = \Delta_T$, where T is the standard tableau, as in Sect. 3. Δ_T is skew-symmetric with respect to the group $C(T) \subseteq \mathfrak{S}_n$. Hence, so is $D\Delta_T$. Lemma 6 implies that it is divisible by Δ_T. However the degree of $D\Delta_T$ is smaller than the degree of Δ_T. Therefore, $D\Delta_T = 0$.

The case $\mathsf{W} \neq \mathfrak{S}_n$ is analogous. □

Let $(\mathrm{G}, \mathrm{G}')$ be a complex dual pair with the rank of G equal n and the rank of G' equal $n' < n$, and let $\ell = n - n'$. In these terms define the following map.

$$\begin{aligned} &\mathbb{C}[x_1, x_2, \dots, x_{n'}] \ni P \qquad \mapsto Q \in \mathbb{C}[x_1, x_2, \dots, x_n], \\ &Q(x_1, x_2, \dots, x_{n'}, \dots, x_n) = P(x_1, x_2, \dots, x_{n'})\Delta_\rho(x_{n'+1}, x_{n'+2}, \dots, x_n), \end{aligned} \tag{23}$$

where

$$\rho = \begin{cases} (1^\ell) & \text{if} \quad (G, G') = (\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_{n'}(\mathbb{C})), \\ (1^\ell, \emptyset) & \text{if} \quad (G, G') = (\mathrm{O}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n'}(\mathbb{C})), \\ (\emptyset, 1^\ell) & \text{if} \quad (G, G') = (\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{O}_{2n'}(\mathbb{C})), \\ (\emptyset, 1^\ell) & \text{if} \quad (G, G') = (\mathrm{O}_{2n+1}(\mathbb{C}), \mathrm{Sp}_{2n'}(\mathbb{C})), \\ (1^\ell, \emptyset) & \text{if} \quad (G, G') = (\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{O}_{2n'+1}(\mathbb{C})). \end{cases}$$

Lemma 8. *The map* (23) *sends* W'*-harmonic polynomials to* W*-harmonic polynomials.*

Proof. We see from (22) that there is a differential operator D' such that $P = D'\Delta^{[\mathsf{W}']}$. Notice that $\Delta^{[\mathsf{W}']}(x_1, x_2, \dots, x_{n'})\Delta_\rho(x_{n'+1}, x_{n'+2}, \dots, x_n)$ is W-harmonic (by the argument used in the proof of Corollary 7). Hence, by (22), Q is W-harmonic. □

(Lemma 8 also follows from Corollary 13 below.) As an obvious consequence of Proposition 5 we obtain the following Theorem.

Theorem 9. *Let* (G, G') *be an irreducible complex dual pair and let* $\mathcal{O}(\lambda')$ *be a nilpotent orbit in the Lie algebra of* G' *satisfying the condition* (16)*. Let* $\mathcal{O}(\lambda)$ *be the corresponding orbit in the Lie algebra of G, as in* (15)*. We identify* $\mathrm{CSC}(\mathcal{O}(\lambda'))$ *and* $\mathrm{CSC}(\mathcal{O}(\lambda)$ *with their realizations in harmonic polynomials as in Sect. 5. Then the image of* $\mathrm{CSC}(\mathcal{O}(\lambda'))$ *under the map* (23) *is equal to* $\mathrm{CSC}(\mathcal{O}(\lambda))$.

6. Differential operators and *chc*

Let G, G' be a real reductive pair acting on the symplectic space W, with $\mathrm{rk}(\mathrm{G}) \geq \mathrm{rk}(\mathrm{G}')$, as before. Let $\tau : \mathrm{W} \to \mathfrak{g}^*$, $\tau' : \mathrm{W} \to \mathfrak{g}'^*$ be the corresponding moment maps. Let $\mathcal{P}(\mathfrak{g}^*_\mathbb{C})^\mathrm{G}$ be the algebra of the G-invariant complex valued polynomials on the dual of the complexification of $\mathfrak{g}$. There is an algebra homomorphism

$$\mathsf{c} : \mathcal{P}(\mathfrak{g}^*_\mathbb{C})^\mathrm{G} \to \mathcal{P}(\mathfrak{g}'^*_\mathbb{C})^{\mathrm{G}'}, \tag{24}$$

determined by $f \circ \tau = \mathsf{c}(f) \circ \tau'$, $f \in \mathcal{P}(\mathfrak{g}_\mathbb{C})^G$. (See the beginning of the proof of Lemma 10 below for an explanation).

Similarly, if $\mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathrm{G}$ denotes the algebra of the G-invariants in the universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{C}$, then we have the Capelli Harish-Chandra homomorphism

$$\mathsf{C} : \mathcal{U}(\mathfrak{g}_\mathbb{C})^\mathrm{G} \to \mathcal{U}(\mathfrak{g}'_\mathbb{C})^{\mathrm{G}'}, \tag{25}$$

which determines the relation between the infinitesimal characters for representations which occur in Howe's correspondence, [28] or [30]. (Specifically, if a representation Π with the infinitesimal character $\gamma_\Pi : \mathcal{U}(\mathfrak{g}_\mathbb{C})^G \to \mathbb{C}$ corresponds to a representation Π' with the infinitesimal character $\gamma_{\Pi'} : \mathcal{U}(\mathfrak{g}'_\mathbb{C})^{\mathrm{G}'} \to \mathbb{C}$, then $\gamma_\Pi = \gamma_{\Pi'} \circ \mathsf{C}$.) Recall the symmetrization map, from the symmetric algebra $\mathcal{S}(\mathfrak{g}_\mathbb{C})$ onto the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$

$$\mathsf{s} : \mathcal{S}(\mathfrak{g}_\mathbb{C}) \to \mathcal{U}(\mathfrak{g}_\mathbb{C}), \tag{26}$$

[14]. The action of $\mathbb{C}^\times$ on $\mathfrak{g}_\mathbb{C}$, $\mathfrak{g}_\mathbb{C} \ni x \to tx \in \mathfrak{g}_\mathbb{C}$, extends to an action on $\mathcal{S}(\mathfrak{g}_\mathbb{C})$, denoted by

$$\mathcal{S}(\mathfrak{g}_\mathbb{C}) \ni u \to t.u \in \mathcal{S}(\mathfrak{g}_\mathbb{C}). \tag{27}$$

We shall identify the symmetric algebra $\mathcal{S}(\mathfrak{g}_\mathbb{C})$ with the polynomial algebra on the dual $\mathcal{P}(\mathfrak{g}^*_\mathbb{C})$. Then $t.u(\xi) = u(t\xi)$, $\xi \in \mathfrak{g}^*_\mathbb{C}$. Also, since the Lie algebra $\mathfrak{h}$ is commutative, $\mathcal{U}(\mathfrak{h}_\mathbb{C}) = \mathcal{S}(\mathfrak{h}_\mathbb{C}) = \mathcal{P}(\mathfrak{h}^*_\mathbb{C})$.

The following Lemma points to a known fact that C is a "smooth deformation/quantization" of c.

Lemma 10. *The following formula holds*

$$\lim_{t\to 0} t^{-1}.\mathsf{s}^{-1}(\mathsf{C}(\mathsf{s}(t.u))) = \mathsf{c}(u) \qquad (u \in \mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}}).$$

Proof. The map (24) may be explained in more detail as follows. Let us view the symplectic space W as the odd part of the Lie superalgebra corresponding to our dual pair. Then we may talk about the semisimple elements in W. Every semisimple GG'-orbit in W passes through a Cartan subspace $\mathfrak{h}_1 \subseteq \mathrm{W}$, [31]. Let us identify $\mathfrak{g}$ with the dual $\mathfrak{g}^*$ via a G-invariant bilinear symmetric non-degenerate form on $\mathfrak{g}$, and similarly for $\mathfrak{g}'$. Then the moment maps τ and τ' take values in $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. The linear span, $\mathrm{span}(\tau'(\mathfrak{h}_1)) \subseteq \mathfrak{g}'$, is a Cartan subalgebra of $\mathfrak{g}'$. Also, the subset

$$\{(\tau'(w), \tau(w));\ w \in \mathfrak{h}_1\} \subseteq \mathrm{span}(\tau'(\mathfrak{h}_1)) \times \mathrm{span}(\tau(\mathfrak{h}_1))$$

extends to a linear bijection

$$\mathrm{span}(\tau'(\mathfrak{h}_1)) \to \mathrm{span}(\tau(\mathfrak{h}_1)). \tag{28}$$

We shall use (28) to identify

$$\mathrm{span}(\tau'(\mathfrak{h}_1)) = \mathrm{span}(\tau(\mathfrak{h}_1)) \tag{29}$$

and denote both by $\mathfrak{h}'$. Thus $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}'$ and a commutative subalgebra of $\mathfrak{g}$, consisting of semisimple elements. In these terms, $f \in \mathcal{P}(\mathfrak{g}_\mathbb{C})$ and $\mathsf{c}(f) \in \mathcal{P}(\mathfrak{g}'_\mathbb{C})$ have the same restriction to $\mathfrak{h}'$, and this determines the map c.

Next we recall the definition of the homomorphism (25), [30, (5.5)]. Let $\mathfrak{z} \subseteq \mathfrak{g}$ be the centralizer of $\mathfrak{h}'$ and let $\mathfrak{z}'\prime$ be the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{z}$ so that $\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{z}''$. Denote by Z, Z$''$ $\subseteq$ G the corresponding subgroups. Let

$$\gamma_{\mathfrak{g}/\mathfrak{h}} : \mathcal{U}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}} \to \mathcal{U}(\mathfrak{h})^{\mathsf{W}} \tag{30}$$

be the Harish-Chandra isomorphism and let

$$\epsilon_{\mathfrak{z}''} : \mathcal{U}(\mathfrak{z}''_{\mathbb{C}})^{\mathrm{Z}} \to \mathbb{C} \tag{31}$$

be the augmentation homomorphism, if G' is not an orthogonal group of type B, i.e. $\mathrm{G}' \neq \mathrm{O}_{odd}$. If G' is an orthogonal group of type B, then $\mathfrak{z}''$ is a symplectic Lie algebra and we denote by $\epsilon_{\mathfrak{z}''}$ the infinitesimal character of the oscillator representation of $\mathfrak{z}''$. (Unfortunately, this case is misrepresented in [30], but the necessary correction is easy (see *http : //crystal.ou.edu/ tprzebin/chch_corrected.pdf*). The corresponding statement in [28] is correct.)

Fix a Cartan subalgebra $h \subseteq \mathfrak{z}$. Then $\mathfrak{h}' \subseteq \mathfrak{h}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$\mathsf{C} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}} \underset{\gamma_{\mathfrak{g}/\mathfrak{h}}}{\to} \mathcal{U}(\mathfrak{h})^{\mathsf{W}} \underset{\gamma_{\mathfrak{z}/\mathfrak{h}}^{-1}}{\to} \mathcal{U}(\mathfrak{z}_{\mathbb{C}})^{\mathrm{Z}} \underset{1\otimes\epsilon_{\mathfrak{z}''}}{\to} \mathcal{U}(\mathfrak{h}')^{\mathsf{W}'} \underset{\gamma_{\mathfrak{g}'/\mathfrak{h}'}^{-1}}{\to} \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{\mathrm{G}'}. \tag{32}$$

The isomorphism (30) is constructed as follows, [40, sec. 3.2]. Fix a system of positive roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ and let $\mathfrak{n}^+$ denote the sum of the corresponding positive root subspaces of $\mathfrak{g}_{\mathbb{C}}$. Similarly, $\mathfrak{n}^-$ is the sum of the negative root subspaces so that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}^+$. Then

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \oplus (\mathfrak{n}^-\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) + \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}^+). \tag{33}$$

Denote by

$$P_{\mathfrak{g}/\mathfrak{h}} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \tag{34}$$

the projection onto the first summand. Similarly we have

$$\mathcal{S}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{S}(\mathfrak{h}_{\mathbb{C}}) \oplus (\mathfrak{n}^-\mathcal{S}(\mathfrak{g}_{\mathbb{C}}) + \mathcal{S}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}^+). \tag{35}$$

Let

$$R_{\mathfrak{g}/\mathfrak{h}} : \mathcal{S}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{S}(\mathfrak{h}_{\mathbb{C}}) \tag{36}$$

be the projection onto the first summand. If we identify the symmetric algebra with the algebra of the polynomials, as above, then (36) coincides with the restriction from $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{h}_{\mathbb{C}}$.

Since $\mathfrak{h}$ is commutative, $\mathcal{U}(\mathfrak{h}_{\mathbb{C}}) = \mathcal{S}(\mathfrak{h}_{\mathbb{C}})$. Therefore

$$P_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s} : \mathcal{S}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{S}(\mathfrak{h}_{\mathbb{C}}).$$

Furthermore,

$$\lim_{t\to 0} t^{-1}.P_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s}(t.u) = R_{\mathfrak{g}/\mathfrak{h}}(u) \qquad (u \in \mathcal{S}(\mathfrak{g}_{\mathbb{C}})). \tag{37}$$

Indeed, if $u \in \mathcal{S}(\mathfrak{h}_{\mathbb{C}})$ then

$$t^{-1}.P_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s}(t.u) = t^{-1}.P_{\mathfrak{g}/\mathfrak{h}}(t.u) = t^{-1}.(t.u) = u$$

and (37) follows. Suppose $u \in (\mathfrak{n}^-\mathcal{S}(\mathfrak{g}_{\mathbb{C}}) + \mathcal{S}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}^+)$ is homogeneous of degree $d \geq 1$. Then $P_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s}(u) \subseteq \mathcal{S}(\mathfrak{h}_{\mathbb{C}})$ is a sum of terms of degrees smaller than d. Since $t.u = t^d u$, the left hand side of (37) is zero. In this case the right hand side of (37) is also zero. Since a general element of the symmetric algebra is the sum of the two elements just considered (37) follows.

For $\rho \in \mathfrak{h}_{\mathbb{C}}^*$ let

$$\mathrm{Tr}_\rho : \mathcal{P}(\mathfrak{h}_{\mathbb{C}}^*) \to \mathcal{P}(\mathfrak{h}_{\mathbb{C}}^*) \tag{38}$$

denote the translation by ρ, that is the linear map which transforms $f(\xi)$ to $f(\xi-\rho)$. If we identify the polynomial algebra with the symmetric algebra then (38) coincides with the unique linear map

$$\mathrm{Tr}_\rho : \mathcal{S}(\mathfrak{h}_{\mathbb{C}}) \to \mathcal{S}(\mathfrak{h}_{\mathbb{C}}) \tag{39}$$

which is obtained from the linear transformation

$$\mathfrak{h}_{\mathbb{C}} \ni X \to X - \rho(X) \in \mathcal{S}(\mathfrak{h}_{\mathbb{C}})$$

via the universal property of $\mathcal{S}$. We see from the definition that

$$\mathrm{Tr}_\rho^{-1} = \mathrm{Tr}_{-\rho} \text{ and } \mathrm{Tr}_\rho(t.u) = t.\mathrm{Tr}_{t\rho}(u). \tag{40}$$

If $\rho = \rho_{\mathfrak{g}}$ is equal to one half times the sum of all the positive roots, then the restriction of $\mathrm{Tr}_\rho \circ P_{\mathfrak{g}/\mathfrak{h}}$ to $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ is the Harish-Chandra isomorphism $\gamma_{\mathfrak{g}/\mathfrak{h}}$. Notice that, by (40) and (37),

$$\begin{aligned} & t^{-1}. \circ \mathrm{Tr}_\rho \circ \mathcal{P}_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s} \circ t. = t^{-1}. \circ \mathrm{Tr}_\rho \circ t. \circ (t^{-1}. \circ \mathcal{P}_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s} \circ t.) \\ & = \mathrm{Tr}_{t\rho} \circ (t^{-1}. \circ \mathcal{P}_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s} \circ t.) \underset{t \to 0}{\to} \mathrm{Tr}_0 \circ R_{\mathfrak{g}/\mathfrak{h}} = R_{\mathfrak{g}/\mathfrak{h}}. \end{aligned} \tag{41}$$

Furthermore,

$$t.(\mathcal{S}(\mathfrak{g}_{\mathbb{C}})^G) = \mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}} \text{ and } \mathsf{s}(\mathcal{S}(\mathfrak{g}_{\mathbb{C}})^G) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}.$$

Therefore,

$$\lim_{t \to 0} t^{-1}. \circ \gamma_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s} \circ t.(u) = R_{\mathfrak{g}/\mathfrak{h}}(u) \qquad (u \in \mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}), \tag{42}$$

and by taking the inverse,

$$\lim_{t \to 0} t^{-1}. \circ \mathsf{s}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}}^{-1} \circ t.(u) = R_{\mathfrak{g}/\mathfrak{h}}^{-1}(u) \qquad (u \in \mathcal{S}(\mathfrak{h}_{\mathbb{C}})^{\mathsf{W}}). \tag{43}$$

Moreover, if $\epsilon_{\mathfrak{z}''}$ is the augmentation map then clearly

$$\lim_{t \to 0} (t^{-1}. \circ (1 \otimes \epsilon_{\mathfrak{z}''}) \circ \mathsf{s} \circ t.) = R_{\mathfrak{z}/\mathfrak{h}'} \tag{44}$$

is the restriction from $\mathfrak{z}_{\mathbb{C}}$ to $\mathfrak{h}'_{\mathbb{C}}$. Suppose $\epsilon_{\mathfrak{z}''}$ is the infinitesimal character of the oscillator representation of $\mathfrak{z}''$, or in fact any algebra homomorphism from $\mathcal{U}(\mathfrak{z}''_{\mathbb{C}})^{\mathrm{Z}}$ to $\mathbb{C}^\times$. Let $\mathfrak{h}'' \subseteq \mathfrak{z}''$ be a Cartan subalgebra. Then there is an element $\lambda \in \mathfrak{h}''^*_{\mathbb{C}}$ such that $\epsilon_{\mathfrak{z}''}(z) = \gamma_{\mathfrak{z}''/\mathfrak{h}''}(z)(\lambda)$ for $z \in \mathcal{U}(\mathfrak{z}''_{\mathbb{C}})^{\mathrm{Z}}$. Hence, (43) shows that for any $u \in \mathcal{S}(\mathfrak{z}''_{\mathbb{C}})^{\mathrm{Z}}$,

$$\begin{aligned} \epsilon_{\mathfrak{z}''}(t.u) &= \gamma_{\mathfrak{z}''/\mathfrak{h}''}(\mathsf{s}(t.u))(\lambda) = (t.t^{-1}.\gamma_{\mathfrak{z}''/\mathfrak{h}''}(\mathsf{s}(t.u)))(\lambda) \\ &= (t^{-1}.\gamma_{\mathfrak{z}''/\mathfrak{h}''}(\mathsf{s}(t.u)))(t.\lambda) \underset{t \to 0}{\to} R_{\mathfrak{z}''/\mathfrak{h}''}(u)(0) = u(0). \end{aligned}$$

Therefore the equation (44) still holds when both sides are applied to an element of $\mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}$.

Furthermore,

$$\begin{aligned} & t^{-1}. \circ \mathsf{s}^{-1} \circ \mathsf{C} \circ \mathsf{s} \circ t. \\ = & (t^{-1}. \circ \mathsf{s}^{-1} \circ \gamma_{\mathfrak{g}'/\mathfrak{h}'}^{-1} \circ t.) \circ (t^{-1}. \circ (1 \otimes \epsilon_{\mathfrak{z}''}) \circ \mathsf{s} \circ t. \\ & \circ (t^{-1}. \circ \mathsf{s}^{-1} \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ t.) \circ (t^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}} \circ \mathsf{s} \circ t.). \end{aligned}$$

Hence, (37), (42) and (44) show that

$$\lim_{t \to 0} t^{-1}. \circ \mathsf{s}^{-1} \circ \mathsf{C} \circ \mathsf{s} \circ t. = R_{\mathfrak{g}'/\mathfrak{h}'}^{-1} \circ R_{\mathfrak{z}/\mathfrak{h}'} \circ R_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ R_{\mathfrak{g}/\mathfrak{h}} = \mathsf{c}.$$

□

Let

$$\partial(x)\psi(y) = \frac{d}{dt}\psi(y + tx)|_{t=0} \qquad (x, y \in \mathfrak{g}, \psi \in C^{\infty}(\mathfrak{g})).$$

The map ∂ extends to an isomorphism from $\mathcal{S}(\mathfrak{g}_{\mathbb{C}})$ onto the algebra of the constant coefficient differential operators on $\mathfrak{g}$. Recall the Cauchy Harish-Chandra integral

$$chc : C_c^{\infty}(\mathfrak{g}) \to C^{\infty}(\mathfrak{g}'^{reg})^{\mathrm{G}'}, \tag{45}$$

[5,29], where $\mathfrak{g}'^{reg}$ is the set of the regular semisimple elements in $\mathfrak{g}'$.

Theorem 11. *The following formula holds,*

$$\partial(\mathsf{c}(u)) \circ chc = chc \circ \partial((-1).u) \qquad (u \in \mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}).$$

Proof. Recall that $\widetilde{\mathrm{G}}'$ denotes the preimage of G' in the metaplectic group, $\widetilde{\mathrm{G}}'^{reg}$ is the set of the regular semisimple elements in $\widetilde{\mathrm{G}}'$ and let L be the left regular representation. We shall use analogous notation for $\widetilde{\mathrm{G}}$. Recall the Cauchy Harish-Chandra integral

$$Chc : C_c^{\infty}(\widetilde{\mathrm{G}}) \to C_c^{\infty}(\widetilde{\mathrm{G}}'^{reg})^{\mathrm{G}'}, \tag{46}$$

and the formula

$$\mathrm{L}(\mathsf{C}(u)) \circ Chc = Chc \circ \mathrm{L}(\check{u}) \qquad (u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}), \tag{47}$$

where $u \to \check{u}$ be the involution on the universal enveloping algebra, extending the map $\mathfrak{g}_{\mathbb{C}} \ni u \to -u \in \mathfrak{g}_{\mathbb{C}}$, [5, Theorem 3]. Furthermore, the following equation holds,

$$\begin{aligned} & \lim_{t \to 0+} t^m \int_{\mathfrak{g}} Chc(-\tilde{c}(tx')\tilde{c}(tx))\, \psi(x)\, dx \\ & = \Theta(-\tilde{1}) \int_{\mathfrak{g}} chc(x' + x)\, \psi(x)\, dx \quad (x' \in \mathfrak{g}'^{reg}, \psi \in C_c^{\infty}(\mathfrak{g})), \end{aligned} \tag{48}$$

where $c : \mathfrak{g} \ni x \to (x+1)(x-1)^{-1} \in \mathrm{G}$ is the Cayley transform (defined where $x-1$ is Invertible), $\tilde{c} : \mathfrak{g} \to \widetilde{\mathrm{G}}$ is a lift of c, m is one half of the dimension of the symplectic space W and $\Theta(-\tilde{1}) \in \mathbb{C}$ is the value of the character Θ of the underlying oscillator representation at the preimage of minus identity in the metaplectic group, [29, Theorem 2.13]. Our Theorem follows from (46) and (48), as explained below.

Let $c_-(x) = -c(x)$, so that $c_-(0) = 1$. For $y \in \mathfrak{g}$ we have the differential operator $\mathrm{L}(y)$ on the group and its pullback to the Lie algebra, $c_-^*(\mathrm{L}(y))$ defined by

$$c_-^*(\mathrm{L}(y))\psi(x) = (\mathrm{L}(y)(\psi \circ c_-^{-1})) \circ c_-(x).$$

Recall the following formula

$$c_-^*(\mathrm{L}(y))\psi(x) = \partial\left(\frac{1}{2}(x-1)y(x+1)\right)\psi(x), \tag{49}$$

[30, (2.3)]. We also have

$$c^*(\mathrm{L}(y))\psi(x) = \partial\left(\frac{1}{2}(x-1)y(x+1)\right)\psi(x). \tag{50}$$

Indeed, by definition, the left hand side of (50) is equal to

$$\frac{d}{dt}(\psi \circ c)(exp(-ty)c(x))|_{t=0} = \frac{1}{2}\frac{d}{dt}\psi(c(c_-(ty)^{-1}c(x)))|_{t=0},$$

because $c = c^{-1}$ and the derivative of c_- at zero is two times the identity. Notice that $c_-^{-1}(g) = c(-g)$. Hence,

$$\begin{aligned} c(c_-(y)^{-1}c(x)) &= c(-c(y)^{-1}c(x)) = c_-^{-1}(c(y)^{-1}c(x)) = c_-^{-1}(c(-y)c(x)) \\ &= c_-^{-1}(c_-(-y)c_-(x)) = c_-^{-1}(c_-(y)^{-1}c_-(x)). \end{aligned}$$

Thus (50) follows from (49). Let

$$\delta_t\psi(x) = \psi(t^{-1}x) \qquad (t \in \mathbb{R}, x \in \mathfrak{g}).$$

Then

$$(\delta_{t^{-1}}c_-^*(\mathrm{L}(-2ty)\delta_t)\psi(x) = \partial((1-tx)y(tx+1))\psi(x) \tag{51}$$

and

$$(\delta_{t^{-1}}c^*(\mathrm{L}(2ty)\delta_t)\psi(x) = \partial((1-tx)(-y)(tx+1))\psi(x). \tag{52}$$

Indeed, the left hand side of (51) is equal to

$$\begin{aligned} (c_-^*(\mathrm{L}(-2ty))\delta_t)\psi(tx) &= c_-^*(\mathrm{L}(-2ty))(\delta_t\psi)(tx) \\ &= \partial\left(\frac{1}{2}(tx-1)(-2ty)(tx+1)\right)(\delta_t\psi)(tx) \\ &= \partial\left(\frac{1}{2}(tx-1)(-2y)(tx+1)t^{-1}\right)\psi(x), \end{aligned}$$

which coincides with the right hand side. Similarly, the left hand side of (52) is equal to

$$\begin{aligned}(c^*(\mathrm{L}(2ty))\delta_t)\psi(tx) &= c^*(\mathrm{L}(2ty))(\delta_t\psi)(tx)\\ &= \partial\left(\frac{1}{2}(tx-1)(2ty)(tx+1)\right)(\delta_t\psi)(tx)\\ &= \partial\left(\frac{1}{2}(tx-1)(2y)(tx+1)t^{-1}\right)\psi(x),\end{aligned}$$

which coincides with the right hand side.

In particular, (51) and (52) imply

$$\lim_{t\to 0}\delta_{t^{-1}}\circ c_-^*(\mathrm{L}(\mathsf{s}((-2t).u)))\circ\delta_t = \partial(u) \qquad (u\in\mathcal{S}(\mathfrak{g}_{\mathbb{C}})), \tag{53}$$

and

$$\lim_{t\to 0}\delta_{t^{-1}}\circ c^*(\mathrm{L}(\mathsf{s}((2t).u)))\circ\delta_t = \partial((-1).u) \qquad (u\in\mathcal{S}(\mathfrak{g}_{\mathbb{C}})). \tag{54}$$

Let $\psi_t = t^{-\dim\mathfrak{g}}\delta_t\psi$. In these terms, (48) shows that for $t>0$,

$$\begin{aligned} &t^m\int_{\widetilde{\mathrm{G}}} Chc(\tilde{c}(tx')g)(\psi_t\circ\tilde{c}_-^{-1})(g)\,dg = t^m\int_{\mathfrak{g}} Chc(\tilde{c}(tx')\tilde{c}_-(x))\psi_t(x)j(x)\,dx\\ &= t^m\int_{\mathfrak{g}} Chc(-\tilde{c}(tx')\tilde{c}(tx))\psi(x)j(tx)\,dx \underset{t\to 0}{\to} \Theta(-\tilde{1})j(0)\\ &\times\int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx\end{aligned} \tag{55}$$

where $j(x)$ is the Jacobian of c_-. On the other hand, since $\mathsf{s}(u)\check{} = \mathsf{s}((-1).u)$ for $u\in\mathcal{S}(\mathfrak{g}_{\mathbb{C}})^G$, (46) shows that

$$\begin{aligned}&\mathrm{L}(\mathsf{C}(\mathsf{s}((2t).u)))\int_{\widetilde{\mathrm{G}}} Chc(g'g)(\psi_t\circ\tilde{c}_-^{-1})(g)\,dg\\ &= \int_{\widetilde{\mathrm{G}}} Chc(g'g)\mathrm{L}(\mathsf{s}((-2t).u))(\psi_t\circ\tilde{c}_-^{-1})(g)\,dg.\end{aligned} \tag{56}$$

By (53) and (55),

$$
\begin{aligned}
& t^m \int_{\widetilde{\mathrm{G}}} Chc(\tilde{c}(tx')g)\mathrm{L}(\mathsf{s}((-2t).u))(\psi_t \circ \tilde{c}_-^{-1})(g)\,dg \\
&= t^m \int_{\mathfrak{g}} Chc(\tilde{c}(tx')\tilde{c}_-(x))\mathrm{L}(\mathsf{s}((-2t).u))(\psi_t \circ \tilde{c}_-^{-1})(\tilde{c}_-(x))\,j(x)\,dx \\
&= t^m \int_{\mathfrak{g}} Chc(\tilde{c}(tx')\tilde{c}_-(x))\tilde{c}_-^*(\mathrm{L}(\mathsf{s}((-2t).u)))\psi_t(x)\,j(x)\,dx \\
&= t^m \int_{\mathfrak{g}} Chc(\tilde{c}(tx')\tilde{c}_-(tx))(\delta_{t^{-1}}\tilde{c}_-^*(\mathrm{L}(\mathsf{s}((-2t).u)))\delta_t)\psi(x)\,j(tx)\,dx \\
&\underset{t\to 0+}{\to} \Theta(-\tilde{1})\,j(0)\int_{\mathfrak{g}} chc(x'+x)(\partial(u)\psi)(x)\,dx.
\end{aligned} \tag{57}
$$

Moreover, if $g' = \tilde{c}(tx')$, then

$$
\begin{aligned}
& \mathrm{L}(\mathsf{C}(\mathsf{s}((2t).u)))t^m \int_{\widetilde{\mathrm{G}}} Chc(g'g)(\psi_t \circ \tilde{c}_-^{-1})(g)\,dg \\
&= (\delta_{t^{-1}}\tilde{c}^*(\mathrm{L}(\mathsf{C}(\mathsf{s}((2t).u))))\delta_t)t^m \int_{\widetilde{\mathrm{G}}} Chc(\tilde{c}(tx')g)(\psi_t \circ \tilde{c}_-^{-1})(g)\,dg
\end{aligned} \tag{58}
$$

We see from Lemma 10 and (54) that

$$
\begin{aligned}
& (\delta_{t^{-1}}\tilde{c}^*(\mathrm{L}(\mathsf{C}(\mathsf{s}((2t).u))))\delta_t) \\
&= (\delta_{t^{-1}}\tilde{c}^*(\mathrm{L}(\mathsf{s}((2t).(2t)^{-1}.\mathsf{s}^{-1}C(\mathsf{s}((2t).u))))\delta_t) \\
&\underset{t\to 0+}{\to} \partial(\mathsf{c}((-1).u)).
\end{aligned} \tag{59}
$$

Therefore,

$$
\begin{aligned}
& t^m\mathrm{L}(\mathsf{C}(\mathsf{s}((2t).u))) \int_{\widetilde{\mathrm{G}}} Chc(\tilde{c}(tx')g)(\psi_t \circ \tilde{c}_-^{-1})(g)\,dg \\
&\underset{t\to 0+}{\to} \Theta(-\tilde{1})\,j(0)\partial(\mathsf{c}((-1).u))\int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx.
\end{aligned} \tag{60}
$$

The theorem follows from (56), (57) and (60). $\square$

A G-invariant distribution f on $\mathfrak{g}$ is called an $\mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}$-eigendistribution if there is an algebra homomorphism $\gamma : \mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}} \to \mathbb{C}$ such that

$$
f \circ \partial(u) = \gamma(u) f \qquad (u \in \mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}). \tag{61}
$$

By Harish-Chandra's Regularity Theorem on a semisimple Lie algebra, [15, Theorem 1], any such distribution coincides with a locally integrable function which is real analytic on $\mathfrak{g}^{reg}$.

For a G'-invariant, $\mathcal{S}(\mathfrak{g}'_{\mathbb{C}})^{\mathrm{G}'}$-eigendistribution f' on $\mathfrak{g}'$ define a G-invariant distribution $chc(f')$ on $\mathfrak{g}$ by the formula

$$chc(f')(\psi) = \sum \frac{1}{|\mathsf{W}(\mathrm{H}')|} \int\limits_{\mathfrak{h}'^{reg}} f'(x')|\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|^2 chc(\psi)(x')\, dx', \qquad (62)$$

where $\psi \in C_c^\infty(\mathfrak{g})$, the summation is over a maximal family of mutually non-conjugate Cartan subgroups $\mathrm{H}' \subseteq \mathrm{G}'$, $\mathsf{W}(\mathrm{H}') = \mathsf{W}(\mathrm{H}', \mathrm{G}')$ is the Weyl group of H' in G', $\pi_{\mathfrak{g}'/\mathfrak{h}'}$ is the product of all the positive roots of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ under some fixed order of roots, and we assume that all the integrals in (62) are absolutely convergent. We shall quantify this last assumption later, in (94), for the case when f' is the Fourier transform of a nilpotent orbital integral.

Theorem 12. *If f' is a G'-invariant $\mathcal{S}(\mathfrak{g}'_{\mathbb{C}})^{\mathrm{G}'}$-eigendistribution corresponding to a homomorphism $\gamma' : \mathcal{S}(\mathfrak{g}'_{\mathbb{C}})^{\mathrm{G}'} \to \mathbb{C}$ as in (61), then $chc(f')$ is a G-invariant $\mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}$-eigendistribution corresponding to the homomorphism $\gamma : \mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}} \ni u \to \gamma' \circ \mathsf{C}((-1).u) \in \mathbb{C}$.*

Proof. Let $\psi \in C_c^\infty(\mathfrak{g})$. Fix a completely G'-invariant open set $\mathcal{U}' \subseteq \mathfrak{g}'$ and a subset $K' \subseteq \mathcal{U}'$ which is compact, modulo the conjugation by G'. Assume that the closure of $\mathcal{U}'$ is also compact modulo the conjugation by G'. Then, by [7, Corollary 2.3..2] there is a smooth G'-invariant function χ supported in $\mathcal{U}'$ which has values between 0 and 1, and is equal to 1 on K'. Theorem 1 in [5] says that the function $chc(\psi)$ satisfies the conditions $\mathcal{I}_1(\mathfrak{g}')$, $\mathcal{I}_2(\mathfrak{g}')$ and $\mathcal{I}_3(\mathfrak{g}')$ in [7, page 171]. Hence the product $\chi\, chc(\psi)$ satisfies $\mathcal{I}_1(\mathfrak{g}')$, $\mathcal{I}_2(\mathfrak{g}')$, $\mathcal{I}_3(\mathfrak{g}')$ and $\mathcal{I}_4(\mathfrak{g}')$. Therefore, by [7, Theorem 4.1.1 (i)], there is a function $\psi_\chi \in C_c^\infty(\mathfrak{g}')$ whose orbital integrals are equal to $\chi\, chc(\psi)$:

$$\mathcal{I}(\psi_\chi)(x') = \int\limits_{\mathrm{G}'/\mathrm{G}'^{x'}} \psi_\chi(g.x')\, d(g\mathrm{G}'^{x'}) = \chi(x')\, chc(\psi)(x') \qquad (x' \in \mathfrak{g}'^{reg}). \qquad (63)$$

(Here $\mathrm{G}'^{x'}$ is the centralizer of x' in G'.)

If the sets $K' \subseteq \mathcal{U}'$ increase to fill up $\mathfrak{g}'$, i.e. $\chi \to 1$, then by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} chc(f')(\psi) &= \lim_{\chi \to 1} \sum \frac{1}{|\mathsf{W}(\mathrm{H}')|} \int\limits_{\mathfrak{h}'^{reg}} f'(x')|\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|^2 \chi(x') chc(\psi)(x')\, dx' \\ &= \lim_{\chi \to 1} \sum \frac{1}{|\mathsf{W}(\mathrm{H}')|} \int\limits_{\mathfrak{h}'^{reg}} f'(x')|\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|^2 \mathcal{I}(\psi_\chi)(x')\, dx' \\ &= \lim_{\chi \to 1} f'(\psi_\chi). \end{aligned} \qquad (64)$$

Furthermore, by (63) and Theorem 10, for $u \in \mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}$,

$$\begin{aligned} \mathcal{I}((\partial(u)\psi)_\chi) &= \chi\, chc(\partial(u)\psi) = \chi\, \partial(\mathsf{C}((-1).u)) chc(\psi) \\ &= \partial(\mathsf{C}((-1).u))(\chi\, chc(\psi)) \\ &= \partial(\mathsf{C}((-1).u))\mathcal{I}(\psi_\chi) = \mathcal{I}(\partial(\mathsf{C}((-1).u))\psi_\chi). \end{aligned}$$

Therefore,

$$f'((\partial(u)\psi)_\chi) = f'(\partial(\mathsf{c}((-1).u))\psi_\chi). \tag{65}$$

We see from (64) and (65) that

$$\begin{aligned} chc(f')(\partial(u)\psi) &= \lim_{\chi\to 1} f'((\partial(u)\psi)_\chi) = \lim_{\chi\to 1} f'(\partial(\mathsf{c}((-1).u))\psi_\chi) \\ &= \lim_{\chi\to 1} \gamma'(\mathsf{c}((-1).u)) f'(\psi_\chi) \\ &= \gamma'(\mathsf{c}((-1).u)) chc(f')(\psi). \end{aligned}$$

□

Harish-Chandra homomorphism (2), when restricted to $\partial(\mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}})$ is an isomorphism of algebras

$$\delta_{\mathfrak{g}/\mathfrak{h}} : \partial(\mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}}) \to \partial(\mathcal{S}(\mathfrak{h}_\mathbb{C})^{\mathsf{W}}) \tag{66}$$

given explicitly by

$$\delta_{\mathfrak{g}/\mathfrak{h}}(\partial(u)) = \partial(R_{\mathfrak{g}/\mathfrak{h}}(u)) \qquad (u \in \mathcal{S}(\mathfrak{h}_\mathbb{C})^{\mathsf{W}}). \tag{67}$$

(Here W is the complex Weyl group.) Thus, for a G-invariant $\mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}}$-eigendistribution f corresponding to a homomorphism $\gamma : \mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}} \to \mathbb{C}$,

$$\partial(R_{\mathfrak{g}/\mathfrak{h}}(u))(\pi_{\mathfrak{g}/\mathfrak{h}} f|_{\mathfrak{h}^{reg}}) = \pi_{\mathfrak{g}/\mathfrak{h}}(\partial(u) f)|_{\mathfrak{h}^{reg}} \qquad (u \in \mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}}). \tag{68}$$

By a theorem of Chevalley, [40, sec. 3.1.2], the restriction map $R_{\mathfrak{g}/\mathfrak{h}}$ is bijective. Hence, $\pi_{\mathfrak{g}/\mathfrak{h}} f|_{\mathfrak{h}^{reg}}$ corresponds to the homomorphism $\gamma \circ R_{\mathfrak{g}/\mathfrak{h}}^{-1} : \mathcal{S}(\mathfrak{h}_\mathbb{C})^{\mathsf{W}} \to \mathbb{C}$:

$$\partial(u)(\pi_{\mathfrak{g}/\mathfrak{h}} f|_{\mathfrak{h}^{reg}}) = \gamma \circ R_{\mathfrak{g}/\mathfrak{h}}^{-1}(u)(\pi_{\mathfrak{g}/\mathfrak{h}} f|_{\mathfrak{h}^{reg}}) \qquad (u \in \mathcal{S}(\mathfrak{h}_\mathbb{C})^{\mathsf{W}}). \tag{69}$$

A G-invariant $\mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}}$-eigendistribution f corresponding to a homomorphism $\gamma : \mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}} \to \mathbb{C}$ is called $\mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}}$-harmonic if the homomorphism γ annihilates all the elements of positive degree. In this case (69) shows that $\pi_{\mathfrak{g}/\mathfrak{h}} f|_{\mathfrak{h}^{reg}}$ is $\mathcal{S}(\mathfrak{h}_\mathbb{C})^{\mathsf{W}}$-harmonic in the same sense. Furthermore, an argument of Harish-Chandra, [13, pages 130–133] shows that the restriction of $\pi_{\mathfrak{g}/\mathfrak{h}} f|_{\mathfrak{h}^{reg}}$ to any connected component $C(\mathfrak{h}) \subseteq \mathfrak{h}^{reg}$ is a polynomial. Moreover, by [15, Theorem 2, page 19], $\pi_{\mathfrak{g}/\mathfrak{h}} f|_{\mathfrak{h}^{reg}}$ coincides with an analytic function on each connected component $C_r(\mathfrak{h})$ in the complement of the union of the zeros of the real roots for $\mathfrak{h}$. Thus, for every such component, $\pi_{\mathfrak{g}/\mathfrak{h}} f|_{C_r(\mathfrak{h})}$ is a $\mathcal{S}(\mathfrak{h}_\mathbb{C})^{\mathsf{W}}$-harmonic polynomial.

Corollary 13. *For any nilpotent orbit $\mathcal{O}' \subseteq \mathfrak{g}'$ satisfying the condition (94) below,*

$$chc(\hat{\mu}_{\mathcal{O}'}) \text{ is } \mathcal{S}(\mathfrak{g}_\mathbb{C})^{\mathrm{G}}\text{-harmonic}, \tag{70}$$

and for each component $C_r(\mathfrak{h})$,

$$\pi_{\mathfrak{g}/\mathfrak{h}}\, chc(\hat{\mu}_{\mathcal{O}'})|_{C_r(\mathfrak{h})} \text{ is a } \mathcal{S}(\mathfrak{h}_\mathbb{C})^{\mathsf{W}}\text{-harmonic polynomial}. \tag{71}$$

Proof. The condition (94) ensures that the integrals in (62) are absolutely convergent, so that $chc(\hat{\mu}_{\mathcal{O}'})$ is well defined. As is well known, the Fourier transform of an invariant measure supported on a nilpotent orbit is harmonic. Therefore Theorem 12 implies (70). The statement (71) follows from (69). □

7. A closer look at Cartan subalgebras and the orbit correspondence

In this section we describe representatives of the conjugacy classes of Cartan subalgebras in a way suitable for the explicit computation of the action of *chc* on the Fourier transform of a nilpotent orbital integral. This description will be used in the proof of Theorem 14 in Sect. 8 below.

Fix a Cartan involution θ on G and consider a θ-stable Cartan subgroup $\mathrm{H} \subseteq \mathrm{G}$. Then $\mathrm{H} = \mathrm{TA}$, where T is the compact part and A is the "vector part", as in [40, 2.3.6]. [Explicitly, $\mathrm{A} = \mathrm{H} \cap \exp(\mathfrak{p})$, where $\mathfrak{p} \subseteq \mathfrak{g}$ is the (-1) - eigenspace for θ.] Let V be the defining module for G. (This is a finite dimensional left vector space over a division algebra $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.) Then $\mathsf{V} = \mathsf{V}_s \oplus \mathsf{V}_c$, where V_c is the trivial component for the action of A and both summands are preserved by H. It could very well happen that $\mathsf{V}_c = 0$. In fact this is always the case if G is a general linear group or a complex group other than $\mathrm{O}_{odd}(\mathbb{C})$ or a group isomorphic to $\mathrm{O}_{1,1}$.

Denote by H_s the restriction of H to V_s and by H_c the restriction of H to V_c. Then $\mathrm{H} = \mathrm{H}_s\mathrm{H}_c$ is isomorphic to the direct product $\mathrm{H}_s \times \mathrm{H}_c$. As before, let $\mathsf{W}(\mathrm{H}) = \mathsf{W}(\mathrm{H}, \mathrm{G})$ be the Weyl group equal to the normalizer of H in G divided by H. Let $\mathfrak{h}$ be the Lie algebra of H. Then

$$\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{h}_c, \tag{72}$$

where $\mathfrak{h}_s$, $\mathfrak{h}_c$ are the Lie algebras of H_s, H_c respectively. Fix a positive root system $\Psi = \Psi(\mathfrak{h})$ for the roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ and let $\pi_{\mathfrak{g}/\mathfrak{h}} = \prod_{\alpha\in\Psi} \alpha$, as before. Then we have the Weyl integration formula

$$\int_{\mathfrak{g}} \psi(x)\, dx = \sum \frac{1}{|\mathsf{W}(\mathrm{H})|} \int_{\mathfrak{h}} |\pi_{g/\mathfrak{h}}(x)|^2 \int_{\mathrm{G}/\mathrm{H}} \psi(g.x)\, d(g\mathrm{H})\, dx, \tag{73}$$

where ψ is a test function and the summation is over a maximal family of mutually non-conjugate Cartan subgroups $\mathrm{H} \subseteq \mathrm{G}$ and, unlike in Sect. 6, $g.x$ denotes the adjoint action of G on $\mathfrak{g}$.

The group H_c is compact (and is contained in T). Let

$$\mathsf{V}_c = \mathsf{V}_{c,0} \oplus \sum_{j=1}^{n(\mathfrak{h}_c)} \mathsf{V}_{c,j} \tag{74}$$

be the decomposition into H_c-irreducibles over $\mathbb{D}$. [Here $\mathsf{V}_{c,0} = 0$ unless $\mathrm{G}(\mathsf{V}_c)$ is isomorphic to the real orthogonal group $\mathrm{O}_{odd,even}$, in which case H_c acts trivially on $\mathsf{V}_{c,0}$ and $\dim\, \mathsf{V}_{c,0} = 1$.] There is an element $J \in \mathfrak{h}_c$ such that the restriction of J to $\sum_{j=0}^{n(\mathfrak{h}_c)} \mathsf{V}_{c,j}$ is a complex structure on that space (i.e. the square of it equals minus the identity). Let J_j denote the restriction of J to $\mathsf{V}_{c,j}$. Then every element $x \in \mathfrak{h}_c$ may be written uniquely as

$$x = \sum_{j=1}^{n(\mathfrak{h}_c)} x_j J_j, \tag{75}$$

where $x_j \in \mathbb{R}$. According to [35], the Cartan subalgebras of $\mathfrak{g}(\mathsf{V}_c)$ are parameterized by certain equivalence classes $[S]$ of sets S of positive non-compact roots of $\mathfrak{h}_c$ in $\mathfrak{g}(\mathsf{V}_c)_\mathbb{C}$. Let us pick an S in each $[S]$ and write $\mathfrak{h}_c(S)$ for the corresponding Cartan subalgebra. Then there is a Cayley transform

$$c(S) : \mathfrak{h}_{c\mathbb{C}} \to \mathfrak{h}_c(S)_\mathbb{C}. \tag{76}$$

Furthermore, $c(S)$ extends to

$$c_{\mathfrak{h},\mathfrak{h}(S)} : \mathfrak{h}_\mathbb{C} \to \mathfrak{h}(S)_\mathbb{C}, \tag{77}$$

where $\mathfrak{h}(S) = \mathfrak{h}_s \oplus \mathfrak{h}_c(S)$, the restriction of $c_{\mathfrak{h},\mathfrak{h}(S)}$ to $\mathfrak{h}_s$ is the identity and the restriction of $c_{\mathfrak{h},\mathfrak{h}(S)}$ to $\mathfrak{h}_c(S)$ is equal $c(S)$. As in (72) we have

$$\mathfrak{h}(S) = \mathfrak{h}(S)_s \oplus \mathfrak{h}(S)_c, \tag{78}$$

where $\mathfrak{h}(S)_s = \mathfrak{h}_s \oplus \mathfrak{h}_c(S)_s$ and $\mathfrak{h}(S)_c = \mathfrak{h}_c(S)_c$.

We shall select a representative $\mathfrak{h}$ from each conjugacy class of the Cartan subalgebras of $\mathfrak{g}$. For that selection (72) holds. Then we choose the strongly orthogonal sets S so that (78) is consistent with (72). For two Cartan subalgebras $\mathfrak{h}_1 \neq \mathfrak{h}_2$ in that selection we have the Cayley transform

$$c_{\mathfrak{h}_1,\mathfrak{h}_2} : \mathfrak{h}_{1\mathbb{C}} \to \mathfrak{h}_{2\mathbb{C}}, \tag{79}$$

if and only if $\mathfrak{h}_{1c} \supseteq \mathfrak{h}_{2c}$ (and $\mathfrak{h}_{1s} \subseteq \mathfrak{h}_{2s}$). We shall also assume that our systems of positive roots $\Psi(\mathfrak{h})$ for each Cartan subalgebra $\mathfrak{h}$ are chosen so that they coincide via the Cayley transform (79):

$$\Psi(\mathfrak{h}_2) \circ c_{\mathfrak{h}_1,\mathfrak{h}_2} = \Psi(\mathfrak{h}_1). \tag{80}$$

If G is a general linear group or a complex group or a group isomorphic to $\mathrm{O}_{1,1}$, then $\mathfrak{h} = \mathfrak{h}_s$ for all $\mathfrak{h}$ and we don't need any Cayley transforms.

Consider a dual pair G, G' with the defining modules V, V'. We shall always assume that the rank of G is greater or equal to the rank of G'. The embedding

$$\mathfrak{h}' \subseteq \mathfrak{g}, \tag{81}$$

introduced in (29) may be realized as follows. As in the case of the group G we have a selection of Cartan subgroups $\mathrm{H}' = \mathrm{H}'_s\mathrm{H}'_c \subseteq \mathrm{G}'$ and the corresponding decompositions $\mathsf{V}' = \mathsf{V}'_s \oplus \mathsf{V}'_c$. Because of our assumption on the ranks, $\dim \mathsf{V}'_s \leq \dim V$. Hence, we may assume that

$$\mathsf{V} = \mathsf{V}'_s \oplus \mathsf{U}, \tag{82}$$

where $\mathsf{U} = \mathsf{V}'^{\perp}_s$ in the type I case. This leads to an embedding

$$\mathfrak{h}'_s \subseteq \mathfrak{g}(\mathsf{V}'_s) \subseteq \mathfrak{g}. \tag{83}$$

If $\mathfrak{h}' = \mathfrak{h}'_s$ then (72) is the embedding (81). Suppose $\mathfrak{h}'_c \neq 0$. Then our pair is of type I. Assume that $\mathrm{G}(\mathsf{U})$ is not isomorphic to any real orthogonal group of the form $\mathrm{O}_{odd,odd}$. Then $\mathrm{G}(\mathsf{U})$ has a compact Cartan subgroup $\mathrm{H}(\mathsf{U})$. Let

$$\mathsf{U} = \mathsf{U}_0 \oplus \sum_{j=1}^{n(\mathsf{U})} \mathsf{U}_j \tag{84}$$

be the decomposition into $\mathrm{H}(\mathsf{U})$-irreducibles over $\mathbb{D}$. Similarly, we have

$$\mathsf{V}'_c = \mathsf{V}'_{c,0} \oplus \sum_{j=1}^{n(\mathfrak{h}'_c)} \mathsf{V}'_{c,j} \tag{85}$$

with respect to H'_c. We shall identify

$$\mathsf{U}_j = \mathsf{V}'_{c,j} \qquad (1 \leq j \leq n(\mathfrak{h}'_c)), \tag{86}$$

which is possible, again because of our assumption on the ranks. Hence,

$$\mathfrak{h}'_c \subseteq \mathfrak{h}(\mathsf{U}) \subseteq \mathfrak{g}(\mathsf{U}) \subseteq \mathfrak{g}. \tag{87}$$

The combination of (83) and (87) gives the embedding (81).

Suppose $\mathrm{G}(\mathsf{U})$ is isomorphic to $\mathrm{O}_{odd,odd}$. Let $\mathrm{H}(\mathsf{U})$ be a fundamental Cartan subgroup of $\mathrm{G}(\mathsf{U})$. Then again we have the decomposition (84) with $\mathsf{U} = 0$ and the identifications (86). We may assume that $\mathrm{H}(\mathsf{U})|_{\mathsf{U}_j}$ is compact for $1 \leq j \leq n(\mathsf{U})-1$. If $n(\mathfrak{h}'_c) \leq n(\mathsf{U}) - 1$, then again we have the embedding (87). If $n(\mathfrak{h}'_c) = n(\mathsf{U})$, then there is no such embedding and (81) doesn't happen. However, in this case we shall encounter an inclusion

$$\mathfrak{h}' \subseteq \mathfrak{g}_{\mathbb{C}}, \tag{88}$$

constructed as follows. Let $\tilde{\mathfrak{h}}' \subseteq \mathfrak{g}'$ be another Cartan subalgebra defined by

$$\begin{aligned} &\tilde{\mathfrak{h}}'|_{\mathsf{V}_s} = \mathfrak{h}'|_{\mathsf{V}_s} = \mathfrak{h}'_s, \ \ \tilde{\mathfrak{h}}'|_{\mathsf{V}'_{c,j}} = \mathfrak{h}'|_{\mathsf{V}'_{c,j}} \ \text{ for } 1 \leq j \leq n(\mathfrak{h}'_c) - 1, \ \text{ and} \\ &\tilde{\mathfrak{h}}'|_{\mathsf{V}'_{cn(\mathfrak{h}'_c)}} \ \text{ is a split Cartan subalgebra of } \ \mathfrak{g}'(\mathsf{V}'_{cn(\mathfrak{h}'_c)})(= sl_2(\mathbb{R})). \end{aligned} \tag{89}$$

There is an $\mathbb{R}$-linear injection

$$\mathfrak{h}'|_{\mathsf{V}'_{cn(\mathfrak{h}'_c)}} \to (\tilde{\mathfrak{h}}'|_{\mathsf{V}'_{cn(\mathfrak{h}'_c)}})_{\mathbb{C}}, \tag{90}$$

and (81) holds for $\tilde{\mathfrak{h}}'$. This leads to (88).

8. An explicit formula for $chc(\hat{\mu}_{\mathcal{O}'})$

Let $\mathcal{O}' \subseteq \mathfrak{g}'$ be a nilpotent G'-orbit and let $\mu_{\mathcal{O}'}$ be a positive G'-invariant measure supported on $\mathcal{O}'$ and viewed as a tempered distribution on $\mathfrak{g}'$. Recall that for any Cartan subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}'$,

$$\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')} = (-1)^a \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \qquad (x' \in \mathfrak{h}'),$$

where a is the number of the positive imaginary roots. Therefore, as explained at the end of Sect. 6, the restriction of

$$\hat{\mu}_{\mathcal{O}'}(x')\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')} \qquad (x' \in \mathfrak{h}')$$

to any connected component $C_r(\mathfrak{h}') \subseteq \mathfrak{h}'$ of the complement of the union of the kernels of the real roots is a $\mathcal{S}(\mathfrak{h}'_{\mathbb{C}})^{\mathsf{W}'}$-harmonic polynomial.

Recall that for a non-complex dual pair of type I there is a number $\underline{p}$, [29, (1.12)] which plays a role in the estimates for the Cauchy Harish-Chandra integral. Explicitly,

Dual pair G, G′	division algebra $\mathbb{D}$	$\underline{p}$
$\mathrm{O}_{p,q},\ \mathrm{Sp}_{2n}(\mathbb{R})$	$\mathbb{R}$	$p+q-2n$
$\mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{O}_{p,q}$	$\mathbb{R}$	$2n-p-q+1$
$\mathrm{U}_{p,q},\ \mathrm{U}_{r,s}$	$\mathbb{C}$	$p+q-r-s$
$\mathrm{Sp}_{p,q},\ \mathrm{O}^*_{2n}$	$\mathbb{H}$	$2p+2q-2n+1$
$\mathrm{O}^*_{2n},\ \mathrm{Sp}_{p,q}$	$\mathbb{H}$	$2n-2p-2q$

(91)

Let $\mathsf{V}(\mathbb{C})$ be the defining module for the complexification $\mathrm{G}_{\mathbb{C}}$ of G and let $\mathsf{V}'(\mathbb{C})$ be the defining module for the complexification $\mathrm{G}'_{\mathbb{C}}$. (If $\mathbb{D} = \mathbb{R}$, then $\mathsf{V}(\mathbb{C})$ is the complexification of V, if $\mathbb{D} = \mathbb{C}$, then $\mathsf{V}(\mathbb{C}) = \mathsf{V}$ and if $\mathbb{D} = \mathbb{H}$ then $\mathsf{V}(\mathbb{C})$ is the space V, considered as a vector space over $\mathbb{C}$.) We see from (91) that

$$\underline{p} = \begin{cases} \dim \mathsf{V}(\mathbb{C}) - \dim \mathsf{V}'(\mathbb{C}) & \text{if} \quad \mathrm{G}'_{\mathbb{C}} = \mathrm{Sp}_{2n}(\mathbb{C}) \quad \text{or} \quad \mathrm{GL}_n(\mathbb{C}), \\ \dim \mathsf{V}(\mathbb{C}) - \dim \mathsf{V}'(\mathbb{C}) + 1 & \text{if} \quad \mathrm{G}'_{\mathbb{C}} = \mathrm{O}_p(\mathbb{C}). \end{cases} \tag{92}$$

Let $p = \underline{p}$ if $\mathbb{D} = \mathbb{C}$ or if $\underline{p}$ is even. If $\mathbb{D} \neq \mathbb{C}$ and $\underline{p}$ is odd, let $p = \underline{p} - 1$. Then a simple case by case verification shows that

$$p = \begin{cases} \dim \mathsf{V}(\mathbb{C}) - \dim \mathsf{V}'(\mathbb{C}) - 1 & \text{if} \quad \mathrm{G}' = \mathrm{Sp}_{2n}(\mathbb{R}) \quad \text{and } \mathrm{G} = \mathrm{O}_{r,s} \quad \text{with} \quad r+s \text{ odd}, \\ \dim \mathsf{V}(\mathbb{C}) - \dim \mathsf{V}'(\mathbb{C}) & \text{otherwise.} \end{cases} \tag{93}$$

If G' is not a general linear group, a complex group, or a group isomorphic to $\mathrm{O}_{1,1}$, then we shall assume that for all the Cartan subalgebras $\mathfrak{h}' \subseteq \mathfrak{g}'$, in terms of (75),

$$\max_{1 \le j \le n(\mathfrak{h}'_c)} \deg_{x'_j} \hat{\mu}_{\mathcal{O}'}(x')\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')} < p. \tag{94}$$

Then, [3, Theorem 1] shows that the integrals (62), with $f' = \hat{\mu}_{\mathcal{O}'}$, are absolutely convergent and therefore $chc(\hat{\mu}_{\mathcal{O}'})$ is well defined. Furthermore, we know from (70) that $chc(\hat{\mu}_{\mathcal{O}'})$ is a G-invariant $\mathcal{S}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{G}}$-harmonic distribution on $\mathfrak{g}$ and from

(71) that the restriction of $\pi_{\mathfrak{g}/h} chc(\hat{\mu}_{\mathcal{O}'})$ to any connected component $C_r(\mathfrak{h})$ is a $\mathcal{S}(\mathfrak{h}_{\mathbb{C}})^{\mathsf{W}}$-harmonic polynomial. We shall give a formula for that polynomial below.

Given Cartan subalgebras $\mathfrak{h}' \subseteq \mathfrak{g}'$ and $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\mathfrak{h}'_s = \mathfrak{h}|_{\mathsf{V}'_s}$, let $\mathfrak{h}_{00} \subseteq \mathfrak{g}^{\mathfrak{h}'_s} \cap \mathfrak{h}'^{\perp}_s$ be a fundamental Cartan subalgebra containing $\mathfrak{h}'_c$ [see (87)] and let

$$\mathfrak{h}_0(\mathfrak{h}') = \mathfrak{h}'_s + \mathfrak{h}_{00}. \tag{95}$$

This is a Cartan subalgebra of $\mathfrak{g}$ and, as explained in (77), there is a Cayley transform

$$c_{\mathfrak{h}_0(\mathfrak{h}'),\mathfrak{h}} \colon \mathfrak{h}_0(\mathfrak{h}')_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}.$$

Furthermore, $\mathfrak{h}_0(\mathfrak{h}') = \mathfrak{h}' \oplus \mathfrak{h}_0(\mathfrak{h}') \cap \mathfrak{h}'^{\perp}$, so that any function defined on $\mathfrak{h}'_{\mathbb{C}}$ may be extended to a function defined on $\mathfrak{h}_0(\mathfrak{h}')_{\mathbb{C}}$ by the composition with the projection $\mathfrak{h}'_{\mathbb{C}} \oplus \mathfrak{h}_0(\mathfrak{h}')_{\mathbb{C}} \cap \mathfrak{h}'^{\perp} \to \mathfrak{h}'_{\mathbb{C}}$. Moreover, the condition $\mathfrak{h}'_s = \mathfrak{h}|_{\mathsf{V}'_s}$ implies that $\mathfrak{h} = \mathfrak{h}'_s \oplus \mathfrak{h} \cap \mathfrak{h}'^{\perp}_s$. Hence each real root of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ may be first restricted to $\mathfrak{h}'_s$ and then extended to $\mathfrak{h}$ via the composition with the projection onto the first summand. We shall use these conventions in Theorem 14 below.

Let $\mathfrak{z} = \mathfrak{g}^{\mathfrak{h}'} \subseteq \mathfrak{g}$ be the centralizer of $\mathfrak{h}'$, as in (32). Let $\pi_{\mathfrak{z}/\mathfrak{h}}$ denote the product of all the positive roots of $\mathfrak{h}$ in $\mathfrak{z}_{\mathbb{C}}$, as usual. If G' is not isomorphic to $\mathrm{O}_{odd,even}$, let $\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}} = \pi_{\mathfrak{z}/\mathfrak{h}}$. If G' is isomorphic to $\mathrm{O}_{odd,even}$, let $\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}$ be the product of the short roots only. (In this case G is a symplectic group.) Similarly, for the purpose of the proof of the Theorem 14, we define $\tilde{\pi}_{\mathfrak{g}^{\mathfrak{h}'_s}/\mathfrak{h}_0(\mathfrak{h}')}$ and a character

$$\begin{aligned} &\widetilde{\mathrm{sgn}} : \mathsf{W}(\mathrm{H}_0(\mathfrak{h}')_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}}^{\mathfrak{h}'_s}) \to \mathbb{C}^{\times}, \\ &\tilde{\pi}_{\mathfrak{g}^{\mathfrak{h}'_s}/\mathfrak{h}_0(\mathfrak{h}')}(s.x) = \widetilde{\mathrm{sgn}}(s)\tilde{\pi}_{\mathfrak{g}^{\mathfrak{h}'_s}/\mathfrak{h}_0(\mathfrak{h}')}(x) \qquad (x \in \mathfrak{h}_0(\mathfrak{h}')) \end{aligned} \tag{96}$$

and notice that the group $\mathsf{W}(\mathrm{H}_0(\mathfrak{h}')_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}}^{\mathfrak{h}'_s})$ may be identified with the stabilizer of $\mathfrak{h}'_s$ in $\mathsf{W}(\mathrm{H}_0(\mathfrak{h}')_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$ and, via the Cayley transform, with the stabilizer of $\mathfrak{h}'_s$ in $\mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$.

Theorem 14. *Under the assumption (94), for any Cartan subalgebras $\mathfrak{h}' \subseteq \mathfrak{g}'$, $\mathfrak{h} \subseteq \mathfrak{g}$ and $s \in \mathrm{Stab}_{\mathsf{W}(\mathrm{H}_{\mathbb{C}},\mathrm{G}_{\mathbb{C}})}(\mathfrak{h}'_s)$ there are (explicitly computed in the proof below) functions $F_{\mathfrak{h}',\mathfrak{h},s} : \mathfrak{h} \to \mathbb{C}$, constant on each connected component of the complement of the union of the kernels of the real roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ and the kernels of the real roots of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$, such that, for any regular element $x \in \mathfrak{h}$,*

$$\begin{aligned} &(\pi_{\mathfrak{g}/\mathfrak{h}} chc(\hat{\mu}_{\mathcal{O}'}))(x) \\ &= \sum_{\mathfrak{h}',\, \mathfrak{h}'_s = \mathfrak{h}|_{\mathsf{V}'_s}} \ \sum_{s \in \mathrm{Stab}_{\mathsf{W}(\mathrm{H}_{\mathbb{C}},\mathrm{G}_{\mathbb{C}})}(\mathfrak{h}'_s)} (\hat{\mu}_{\mathcal{O}'}\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}})(c^{-1}_{\mathfrak{h}_0(\mathfrak{h}'),\mathfrak{h}} s^{-1}.x)\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}.x)F_{\mathfrak{h}',\mathfrak{h},s}(x). \end{aligned} \tag{97}$$

Here, the first sum is over all the Cartan subalgebras $\mathfrak{h}' \subseteq \mathfrak{g}'$ such that $\mathfrak{h}'_s = \mathfrak{h}|_{\mathsf{V}'_s}$. If $\mathfrak{h}' = \mathfrak{h}'_s$, then the summation over $\mathrm{Stab}_{\mathsf{W}(\mathrm{H}_{\mathbb{C}},\mathrm{G}_{\mathbb{C}})}$ *in (97) is equal to one term*

$$(\hat{\mu}_{\mathcal{O}'}\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}})(x)\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(x)F_{\mathfrak{h}',\mathfrak{h},1}(x). \tag{98}$$

Proof. We shall proceed via a case by case analysis, following [4, Theorems 3 and 7] and [3, Theorem 7.3]. Also, since we already know that the distribution in question coincides with a function, we may assume that the test function ψ we are going to use is compactly supported in the set of the regular semisimple elements of $\mathfrak{g}$. Then all the orbital integrals of ψ define smooth compactly supported functions on the corresponding Cartan subalgebras.

Let $\mathrm{G} = \mathrm{GL}(\mathsf{V})$, $\mathrm{G}' = \mathrm{GL}(\mathsf{V}')$, $\mathfrak{n} = \mathrm{Hom}(\mathsf{U}, \mathsf{V}') \subseteq \mathfrak{g}$ and $\lambda = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathrm{W}}}{\sqrt{\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{V}'}}$. Theorem 3 in [4] says that

$$\begin{aligned}
&\int\limits_{\mathfrak{g}} chc(x' + x)\psi(x)\, dx \\
&= \lambda \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{G}(\mathsf{U})} \int\limits_{\mathfrak{g}(\mathsf{U})} |\det\, \mathrm{ad}(x' + y)_{\mathfrak{n}}| \psi(g.(x' + y))\, dy\, d(g(\mathrm{H}'\mathrm{G}(\mathsf{U}))) \\
&= \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{G}(\mathsf{U})} \sum_{\mathfrak{h}(\mathsf{U})} \frac{\lambda}{|\mathsf{W}(\mathrm{H}(\mathsf{U}), \mathrm{G}(\mathsf{U}))|} \int\limits_{\mathfrak{h}(\mathsf{U})} |\det\, \mathrm{ad}(x'+y)_{\mathfrak{n}}||\det\, \mathrm{ad}(x'+y)_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})}| \\
&\qquad \int\limits_{\mathrm{G}(\mathsf{U})/\mathrm{H}(\mathsf{U})} \psi(g.(x' + k.y))\, d(k(\mathrm{H}(\mathsf{U}))\, dy\, d(g(\mathrm{H}'\mathrm{G}(\mathsf{U}))) \\
&= \sum_{\mathfrak{h}(\mathsf{U})} \frac{\lambda}{|\mathsf{W}(\mathrm{H}(\mathsf{U}), \mathrm{G}(\mathsf{U}))|} \int\limits_{\mathfrak{h}(\mathsf{U})} |\det \mathrm{ad}(x' + y)_{\mathfrak{g}'/\mathfrak{h}'}|^{-\frac{1}{2}} |\det\, \mathrm{ad}(x' + y)_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})}|^{\frac{1}{2}} \\
&\qquad |\det\, \mathrm{ad}(x' + y)_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}(\mathsf{U}))}|^{\frac{1}{2}} \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{H}(\mathsf{U})} \psi(g.(x' + y))\, d(g(\mathrm{H}'\mathrm{H}(\mathsf{U}))\, dy.
\end{aligned} \tag{99}$$

Notice that in terms of (29)–(30), $\mathfrak{g}(\mathsf{U}) = \mathfrak{z}''$. Hence (99) may be rewritten as

$$\begin{aligned}
&|\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')| \int\limits_{\mathfrak{g}} chc(x' + x)\psi(x)\, dx = \sum_{\mathfrak{h}''} \frac{\lambda}{|\mathsf{W}(\mathrm{H}'', \mathrm{Z}'')|} \int\limits_{\mathfrak{h}''} |\pi_{\mathfrak{z}''/\mathfrak{h}''}(x'')| \\
&|\pi_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}'')}(x' + x'')| \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{H}''} \psi(g.(x' + y))\, d(g(\mathrm{H}'\mathrm{H}''))\, dx''.
\end{aligned} \tag{100}$$

Therefore,

$$\begin{aligned}
&\int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x') |\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|^2 \int\limits_{\mathfrak{g}} chc(x' + x)\psi(x)\, dx \\
&= \sum_{\mathfrak{h}''} \frac{\lambda}{|\mathsf{W}(\mathrm{H}'', \mathrm{Z}'')|} \int\limits_{\mathfrak{h}'} \int\limits_{\mathfrak{h}''} \hat{\mu}_{\mathcal{O}'}(x') |\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')||\pi_{\mathfrak{z}''/\mathfrak{h}''}(x'')| \\
&\quad |\pi_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}'')}(x' + x'')| \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{H}''} \psi(g.(x' + y))\, d(g(\mathrm{H}'\mathrm{H}''))\, dx''\, dx'.
\end{aligned} \tag{101}$$

$$= \sum_{\mathfrak{h}''} \frac{\lambda}{|\mathsf{W}(\mathrm{H}'', \mathrm{Z}'')|} \int_{\mathfrak{h}'} \int_{\mathfrak{h}''} \hat{\mu}_{\mathcal{O}'}(x') \overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x') \pi_{\mathfrak{z}''/\mathfrak{h}''}(x'')$$

$$\left(\frac{|\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|}{\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')}} \frac{|\pi_{\mathfrak{z}''/\mathfrak{h}''}(x'')|}{\pi_{\mathfrak{z}''/\mathfrak{h}''}(x'')} \frac{|\pi_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}'')}(x'+x'')|}{\pi_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}'')}(x'+x'')} \right)$$

$$\pi_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}'')}(x'+x'') \int_{\mathrm{G}/\mathrm{H}'\mathrm{H}''} \psi(g.(x'+y))\, d(g(\mathrm{H}'\mathrm{H}''))\, dx''\, dx'.$$

The term in the large parenthesis is equal to a constant multiple of

$$\frac{\det(x'+x'')_{\mathfrak{n}}}{|\det(x'+x'')_{\mathfrak{n}}|} \tag{102}$$

which is smooth, except for the zeros of some real roots. Also, $\mathfrak{h}' = \mathfrak{h}'_s$. We see that with $F_{\mathfrak{h}', \mathfrak{h}'+\mathfrak{h}'', 1}(x'+x'')$ equal to an appropriate constant multiple of (102), (97) follows from (101).

From now on we consider dual pairs of type I. Let

$$\mathsf{V}'_s = \mathsf{X}'_s + \mathsf{Y}'_s \tag{103}$$

be a complete polarization. Define

$$\begin{aligned} \mathfrak{n}' &= \mathrm{Hom}(\mathsf{X}', \mathsf{V}'_c) \oplus \mathrm{Hom}(\mathsf{X}', \mathsf{Y}') \cap \mathfrak{g}' \subseteq \mathfrak{g}', \\ \mathfrak{n} &= \mathrm{Hom}(\mathsf{X}', \mathsf{U}) \oplus \mathrm{Hom}(\mathsf{X}', \mathsf{Y}') \cap \mathfrak{g} \subseteq \mathfrak{g}. \end{aligned} \tag{104}$$

(These are nilradicals of some parabolic subalgebras.) Recall the number $\gamma(\mathsf{V}, \mathsf{V}', \mathsf{X}')$, [4, (0.5)].

Suppose $\mathsf{V}'_c = 0$ and $\mathsf{U} = 0$. Then, according to [4, Theorem 7],

$$|\det(ad\, x')_{\mathfrak{n}'}| \int_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx = \gamma(\mathsf{V}, \mathsf{V}', \mathsf{X}') |\det(ad\, x')_{\mathfrak{n}}|$$

$$\int_{\mathrm{G}/\mathrm{H}'} \psi(g.x')\, d(g\mathrm{H}'). \tag{105}$$

Hence,

$$\int_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x') |\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|^2 \int_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx$$

$$= \int_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x') |\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|^2 |\det(ad\, x')_{\mathfrak{n}'}|^{-1} \gamma(\mathsf{V}, \mathsf{V}', \mathsf{X}')$$

$$|\det(ad\, x')_{\mathfrak{n}}| \int_{\mathrm{G}/\mathrm{H}'} \psi(g.x')\, d(g\mathrm{H}') \tag{106}$$

$$= \gamma(\mathsf{V}, \mathsf{V}', \mathsf{X}') \int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x') |\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')| |\pi_{\mathfrak{g}/\mathfrak{h}'}(x')| \int\limits_{\mathrm{G}/\mathrm{H}'} \psi(g.x')\, d(g\mathrm{H}')$$

$$= \gamma(\mathsf{V}, \mathsf{V}', \mathsf{X}') \int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x') \overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x') \left(\frac{|\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|}{\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')}} \frac{|\pi_{\mathfrak{g}/\mathfrak{h}'}(x')|}{\pi_{\mathfrak{g}/\mathfrak{h}'}(x')} \right)$$

$$\pi_{\mathfrak{g}/\mathfrak{h}'}(x') \int\limits_{\mathrm{G}/\mathrm{H}'} \psi(g.x')\, d(g\mathrm{H}')$$

The term in the large parenthesis is equal to 1 unless $(\mathrm{G}'_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$ is isomorphic to $(\mathrm{O}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C}))$ or $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{O}_{2n}(\mathbb{C}))$. In these cases, it is equal to the product of the signs of the long real roots for the symplectic Lie algebra. Thus this case gives the contribution to (97), with $\mathfrak{h} = \mathfrak{h}' = \mathfrak{h}'_s$ and $F_{\mathfrak{h}',\mathfrak{h},1}(x)$ is equal to a constant multiple of the term in the parenthesis. [There might be additional summands for this $\mathfrak{h}$ coming from different $\mathfrak{h}'$, see (62)].

Suppose $\mathfrak{h}'$ acts trivially on V'_c and $\mathsf{U} \neq 0$. Recall the symplectic space $\mathrm{W}_c = \mathrm{Hom}(\mathsf{V}'_c, \mathsf{U})$. Then, by [4, Theorem 7],

$$\frac{1}{\gamma(\mathsf{V}, \mathsf{V}', \mathsf{X}')} |\det(ad\, x')_{\mathfrak{n}'}| \int\limits_{\mathfrak{g}} chc(x' + x)\psi(x)\, dx \tag{107}$$

$$= \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{G}(\mathsf{U})} \int\limits_{\mathfrak{g}(\mathsf{U})} |\det \mathrm{ad}(x' + y)_{\mathfrak{n}}| chc_{\mathrm{W}_c}(y) \psi(g.(x' + y))\, dy\, d(g(\mathrm{H}'\mathrm{G}(\mathsf{U})))$$

$$= \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{G}(\mathsf{U})} \sum_{\mathfrak{h}(\mathsf{U})} \frac{1}{|\mathsf{W}(\mathrm{H}(\mathsf{U}), \mathrm{G}(\mathsf{U}))|} \int\limits_{\mathfrak{h}(\mathsf{U})} |\det \mathrm{ad}(x' + y)_{\mathfrak{n}}| chc_{\mathrm{W}_c}(y) |\pi_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})}(y)|^2$$

$$\int\limits_{\mathrm{G}(\mathsf{U})/\mathrm{H}(\mathsf{U})} \psi(g.(x' + k.y))\, d(k\mathrm{H}(\mathsf{U}))\, dy\, d(g(\mathrm{H}'\mathrm{G}(\mathsf{U})))$$

$$= \sum_{\mathfrak{h}(\mathsf{U})} \frac{1}{|\mathsf{W}(\mathrm{H}(\mathsf{U}), \mathrm{G}(\mathsf{U}))|} \int\limits_{\mathfrak{h}(\mathsf{U})} |\det \mathrm{ad}(x' + y)_{\mathfrak{n}}| chc_{\mathrm{W}_c}(y)$$

$$\times |\pi_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})}(y)|^2 \pi_{\mathfrak{g}/(\mathfrak{h}' + \mathfrak{h}(\mathsf{U}))}(x' + y)^{-1}$$

$$\pi_{\mathfrak{g}/(\mathfrak{h}' + \mathfrak{h}(\mathsf{U}))}(x' + y) \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{H}(\mathsf{U})} \psi(g.(x' + y))\, d(g(\mathrm{H}'\mathrm{H}(\mathsf{U})))\, dy$$

Therefore,

$$\int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x') |\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|^2 \int\limits_{\mathfrak{g}} chc(x' + x)\psi(x)\, dx$$

$$= \int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x') \overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x') \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') |\det(ad\, x')_{\mathfrak{n}'}|^{-1}$$

$$
\begin{aligned}
&|\det(ad\, x')_{\mathfrak{n}'}| \int_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx\, dx' \\
&= \sum_{\mathfrak{h}(\mathsf{U})} \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{|\mathsf{W}(\mathrm{H}(\mathsf{U}),\mathrm{G}(\mathsf{U}))|} \int_{\mathfrak{h}'}\int_{\mathfrak{h}(\mathsf{U})} \hat{\mu}_{\mathcal{O}'}(x')\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x') \qquad (108)\\
&\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')|\det(ad\, x')_{\mathfrak{n}'}|^{-1}|\det(\mathrm{ad}(x'+y)_{\mathfrak{n}})|chc_{\mathsf{W}_c}(y)|\pi_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})}(y)|^2\right.\\
&\left.\pi_{\mathfrak{g}/(\mathfrak{h}'+h(\mathsf{U}))}(x'+y)^{-1}\right)\\
&\pi_{\mathfrak{g}/(\mathfrak{h}'+h(\mathsf{U}))}(x'+y) \int_{\mathrm{G}/\mathrm{H}'\mathrm{H}(\mathsf{U})} \psi(g.(x'+y))\, d(g(\mathrm{H}'\mathrm{H}(\mathsf{U})))\, dy\, dx'.
\end{aligned}
$$

Here if $\mathsf{V}'_c = 0$ then $\mathrm{W}_c = 0$ and $chc_{\mathsf{W}_c} = 1$. If $\mathsf{V}'_c \neq 0$, then $\mathrm{G}' = \mathrm{O}_{odd,even}$ and $chc_{\mathsf{W}_c}(y)$ is a constant multiple of $\det(y)^{-1}_{\mathsf{W}_c}$, which is a constant multiple of the reciprocal of the product of the long roots for the symplectic Lie algebra. Also,

$$
\pi_{\mathfrak{g}'/\mathfrak{h}'}(x'h)\det(ad\, x')^{-1}_{\mathfrak{n}'} = \pi_{\mathfrak{gl}(\mathsf{X}')/\mathfrak{h}'|_{\mathsf{X}'}}(x')
$$

and

$$
\det(\mathrm{ad}(x'+y)_{\mathfrak{n}})\pi_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})}(y)\pi_{\mathfrak{g}/(\mathfrak{h}'+h(\mathsf{U}))}(x'+y)^{-1} = \pi_{\mathfrak{gl}(\mathsf{X}')/\mathfrak{h}'|_{\mathsf{X}'}}(x')^{-1}.
$$

Therefore the term in the parenthesis is equal to a constant multiple of

$$
\mathrm{sgn}(\det\, ad\, x')_{\mathfrak{n}'} \cdot \mathrm{sgn}(\det\, \mathrm{ad}(x'+y)_{\mathfrak{n}'}) \cdot \tilde{\pi}_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})}(y)
$$

and (97) follows, with $F_{\mathfrak{h}',\mathfrak{h}'+\mathfrak{h}(\mathsf{U}),1}(x'+y)$ equal to a constant multiple of $\mathrm{sgn}(\det\, ad\, x')_{\mathfrak{n}'} \cdot \mathrm{sgn}(\det\, \mathrm{ad}(x'+y)_{\mathfrak{n}'})$.

Now we consider the remaining case when $\mathfrak{h}'$ does not act trivially on W_c. We assume first that $\mathfrak{h}' = \mathfrak{h}'_c$ is elliptic and begin with the following two well known lemmas, included for reader's convenience.

Lemma 15. *For any non-zero $y \in \mathbb{R}$,*

$$
\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x-iy} = \pi\, i\, \mathrm{sgn}(y).
$$

Proof. By taking the complex conjugate we may assume that $y > 0$. Then, for R large enough,

$$
2\pi\, i = \int_{-R}^{R} \frac{dx}{x-iy} + \int_0^{\pi} \frac{Re^{i\theta} i\, d\theta}{Re^{i\theta}-iy}
$$

and

$$
\lim_{R\to\infty} \int_0^{\pi} \frac{Re^{i\theta} i\, d\theta}{Re^{i\theta}-iy} = \int_0^{\pi} i\, d\theta = \pi\, i.
$$

□

Lemma 16. *As a generalized function of $x \in \mathbb{R}$, i.e. in terms of distributions,*

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x' - x - \epsilon i0}\, dx' = \epsilon i\pi \qquad (\epsilon = \pm 1).$$

Proof. Let $ln(z) = ln(|z|) + i\, Arg(z)$, $z \in \mathbb{C} \setminus 0$, so that

$$ln(x + i0) = \begin{cases} ln(|x|) & \text{if} \quad x > 0, \\ ln(|x|) + i\pi & \text{if} \quad x < 0. \end{cases}$$

Thus, for a test function $\psi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \frac{1}{x + i0} \psi(x)\, dx = -\int_{\mathbb{R}} ln(x + i0) \frac{d}{dx} \psi(x)\, dx = -i\pi\, \psi(0) \\ - \int_{\mathbb{R}} ln(|x|) \frac{d}{dx} \psi(x)\, dx.$$

Hence, with $\psi'(t) = \frac{d}{dt}\psi(t)$,

$$\int_{\mathbb{R}} \frac{1}{x' - x - \epsilon i0} \psi(x)\, dx = \epsilon i\pi\, \psi(x') + \int_{\mathbb{R}} ln(|x|) \psi'(x' - x)\, dx. \tag{109}$$

Furthermore, if ψ is supported in the bounded interval $[-A, A]$, with $A + Z > 1$, then the integral

$$\int_{\mathbb{R}} |ln(|x|)||\psi'(x' - x)|\, dx$$

is dominated by

$$\int_{-R}^{R} \int_{-A}^{A} |ln(|x' - x|)|\, dx\, dx' \le \int_{-R}^{R} \left(\int_{-1}^{1} |ln(|x|)|\, dx + 2A\, ln(A + R) \right) dx',$$

which is finite. Therefore,

$$\begin{aligned} \textstyle\int_{-R}^{R} \int_{\mathbb{R}} ln(|x|) \psi'(x' - x)\, dx\, dx' &= \textstyle\int_{\mathbb{R}} \int_{-R}^{R} ln(|x|) \psi'(x' - x)\, dx'\, dx \\ = \textstyle\int_{\mathbb{R}} ln(|x|)(\psi(R - x) - \psi(-R - x))\, dx &= \textstyle\int_{\mathbb{R}} (ln(|x - R|) - ln(|x + R|)) \psi(x)\, dx \\ &= \textstyle\int_{\mathbb{R}} ln\left(\frac{|x-R|}{|x+R|}\right) \psi(x)\, dx. \end{aligned} \tag{110}$$

We may assume that R is so large comparing to A that

$$2 < R - A \leq |x \pm R| \leq R + A \qquad (|x| \leq A).$$

Then

$$\int_{-A}^{A} \left| ln\left(\frac{|x-R|}{|x+R|}\right)\right| dx \leq 2A\, ln\left(\frac{R+A}{R-A}\right) \underset{R\to\infty}{\to} 0.$$

Therefore (110) tends to zero if $R \to \infty$. Hence, by (109),

$$\int_{-R}^{R}\int_{\mathbb{R}} \frac{1}{x'-x-\epsilon i0}\psi(x)\,dx\,dx' \underset{R\to\infty}{\to} \epsilon i\pi \int_{\mathbb{R}} \psi(x')\,dx'.$$

□

In order to formulate Corollary 17 below we need to recall some notation from [3]. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an elliptic Cartan subalgebra. For a strongly orthogonal set S of no-compact positive roots α of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ let $\mathfrak{h}_S = c(S)^{-1}\mathfrak{h}(S) \subseteq \mathfrak{h}_{\mathbb{C}}$. Define the support of S, $\underline{S}$ to be the set of the integers j between 1 and $n = \dim\mathfrak{h}$ such that there is $\alpha \in S$ with $\alpha(J_j) \neq 0$. (We shall denote the dimension of $\mathfrak{h}'$ by n'.) The Weyl group $\mathsf{W}(\mathrm{H}_{\mathbb{C}}) = \mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$ is isomorphic either to the permutation group on n letters $\mathfrak{S}_n$ or to the semidirect product $\mathfrak{S}_n \ltimes \mathbb{Z}_2^n$, where $\mathbb{Z}_2 = \{0, 1\}$. We shall denote by σ the elements of $\mathfrak{S}_n$ and by ϵ the elements of $\mathbb{Z}_2^n$. Thus any element of the Weyl group may be written uniquely as $\sigma\epsilon$. For $\epsilon \in \mathbb{Z}_2$ let $\hat{\epsilon} = (-1)^{\epsilon}$. Recall the embedding $\mathfrak{h}' \subseteq \mathfrak{h}$ induced by (86). Let $\mathrm{W}^{\mathfrak{h}'} \subseteq \mathrm{W}$ be the subspace of the elements which commute with $\mathfrak{h}'$.

As explained in [3, Sec.3], the Weyl group $\mathsf{W}(\mathrm{H}_{\mathbb{C}})$ acts on the symplectic space W. In particular, for $s \in \mathsf{W}(\mathrm{H}_{\mathbb{C}})$, $s\mathrm{W}^{\mathfrak{h}'}$ is image of $\mathrm{W}^{\mathfrak{h}'}$ under this action. Furthermore, given $s = \sigma\epsilon \in \mathsf{W}(\mathrm{H}_{\mathbb{C}})$ there is $y_s \in \mathfrak{h}$ [3, Def. 3.4] a convex cone $\Gamma_{s,S} \subseteq \mathfrak{h}$ [3, Lemma 7.1] and a positive definite symmetric bi-linear form $\tilde{\kappa}$ on $\mathfrak{sp}(\mathrm{W})$ [4, Page 1].

Corollary 17. *For $R > 0$ let $B_R = \{x \in \mathfrak{h}';\ |x'_j| \leq R,\ 1 \leq j \leq n'\}$. Then, as a generalized function of $x \in \mathfrak{h}_S$,*

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{y\in\Gamma_{s,S},\, y\to 0} \int_{B_R} \frac{1}{\det(x'+x+iy)_{s\mathsf{W}^{h'}}}\,dx' \\
&= \prod_{j=1}^{n'}(\pi\tilde{\kappa}(J_j, J_j)^{\frac{1}{2}}\hat{\epsilon}_j) \cdot \prod_{1\leq j\leq n',\ \sigma(j)\notin\underline{S}} \mathrm{sgn}(J^*_{\sigma(j)}(y_s)) \qquad (111)\\
&\quad \cdot \prod_{\alpha\in S,\ 1\leq j\leq n',\ \sigma(j)\in\underline{\alpha}} \mathrm{sgn}(\alpha(iJ_{\sigma(j)}))\mathrm{sgn}(\alpha(x)).
\end{aligned}$$

Proof. Recall, that if we view W as $\mathrm{Hom}(\mathsf{V}, \mathsf{V}')$, as we may, then [3, Appendix B]

$$\det(x'+x+iy)_{s\mathsf{W}^{h'}} = \prod_{j=1}^{n'} i(x'_j - \hat{\epsilon}_j(x_{\sigma(j)} + iy_{\sigma(j)})).$$

Hence, up to the constant multiple $\prod_{j=1}^{n'} \tilde{\kappa}(J_j, J_j)^{\frac{1}{2}}$, coming from the normalization of all the measures involved [4], the limit (111) is equal to

$$\lim_{R\to\infty} \lim_{y\in\Gamma_{s,S},\, y\to 0} \prod_{j=1}^{n'} \frac{1}{i} \int_{-R}^{R} \frac{dx'}{x'_j - \hat{\epsilon}_j x_{\sigma(j)} - \hat{\epsilon}_j i y_{\sigma(j)}}. \tag{112}$$

Notice that for $\alpha \in S$ and $k \in \underline{\alpha}$,

$$\mathrm{sgn}(Im\, x_k) = \mathrm{sgn}(\alpha(i J_k))\, \mathrm{sgn}(\alpha(x)). \tag{113}$$

Indeed,

$$\alpha = \alpha(J_k) J_k^* + \alpha(J_l) J_l^*,\ \alpha(J_k) \in i\mathbb{R}, \quad \text{and} \quad \alpha(x) \in \mathbb{R},$$

and either $\alpha(J_l) = 0$ or $\alpha(J_l) \neq 0$. In the first case,

$$\begin{aligned}\alpha(x) &= \alpha(J_k) J_k^*(x) = \alpha(J_k) x_k = \alpha(i J_k)(-i x_k)\\ &= Re(\alpha(i J_k)(-i x_k)) = \alpha(i J_k) Re(-i x_k) = \alpha(i J_k) Im\, x_k.\end{aligned}$$

In the second case, $\alpha(J_l) = -\alpha(J_k)$ and

$$\alpha(x) = \alpha(J_k)(x_k - x_l) = \alpha(J_k) 2i\, Im\, x_k,$$

and (113) follows.

We see from (113) and Lemma 15 that

$$\begin{aligned}&\lim_{R\to\infty} \lim_{y\in\Gamma_{s,S},\, y\to 0} \prod_{(\alpha,j)\in S\times[1,n'],\, \sigma(j)\in\underline{\alpha}} \frac{1}{i} \int_{-R}^{R} \frac{dx'}{x'_j - \hat{\epsilon}_j x_{\sigma(j)} - \hat{\epsilon}_j i y_{\sigma(j)}}\\ &= \lim_{R\to\infty} \prod_{(\alpha,j)\in S\times[1,n'],\, \sigma(j)\in\underline{\alpha}} \frac{1}{i} \int_{-R}^{R} \frac{dx'}{x'_j - \hat{\epsilon}_j x_{\sigma(j)}}\\ &= \prod_{(\alpha,j)\in S\times[1,n'],\, \sigma(j)\in\underline{\alpha}} \pi \hat{\epsilon}_j\, \mathrm{sgn}(Im\, x_{\sigma(j)}).\end{aligned} \tag{114}$$

Also, Lemma 16 implies that

$$\begin{aligned}&\lim_{R\to\infty} \lim_{y\in\Gamma_{s,S},\, y\to 0} \prod_{1\le j\le n',\, \sigma(j)\notin\underline{S}} \frac{1}{i} \int_{-R}^{R} \frac{dx'}{x'_j - \hat{\epsilon}_j x_{\sigma(j)} - \hat{\epsilon}_j i y_{\sigma(j)}}\\ &= \prod_{1\le j\le n',\, \sigma(j)\notin\underline{S}} \pi \hat{\epsilon}_j\, \mathrm{sgn}(J^*_{\sigma(j)}(y_s)).\end{aligned} \tag{115}$$

The corollary follows from (112), (114) and (115). □

We shall need some more notation from [3]. Let $\Psi_{S,\mathbb{R}}$ denote the set of the positive roots for $\mathfrak{h}$ which are real on $\mathfrak{h}_S$. For a set of roots A, let $\mathcal{A}(A) = \prod_{\alpha\in A}\frac{\alpha}{|\alpha|}$. Given our test function ψ defines the pull-back of the Harish-Chandra orbital integral of ψ, from $\mathfrak{h}(S)$ to $\mathfrak{h}_S$ via $c(S)$ by

$$\mathcal{H}_S\psi(x) = \mathcal{A}(\Psi_{S,\mathbb{R}})(x)\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int\limits_{\mathrm{G}/\mathrm{H}(S)} \psi(g.c(S)x)\, d(gH(S)) \qquad (x \in \mathfrak{h}_S). \tag{116}$$

Let $\widetilde{\Psi}_{S,\mathbb{R}} = \Psi_{S,\mathbb{R}}$ unless $\mathrm{G}' = \mathrm{O}_{odd,even}$. In this case we let $\widetilde{\Psi}_{S,\mathbb{R}}$ be the product of the short roots only. For S as above define

$$m_{[S]} = \frac{u\sqrt{2}^{\dim \mathrm{W}}}{|\mathsf{W}(\mathrm{H}(S))|\,|\mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{Z}_{\mathbb{C}})|} \cdot \begin{cases} 1 & \text{if} \quad \mathrm{G}' \neq \mathrm{O}_{odd,even}, \\ 2^{n-n'} & \text{if} \quad \mathrm{G}' = \mathrm{O}_{odd,even}, \end{cases} \tag{117}$$

where $u = 1, -1, i, -i$ depends on our choice of the positive roots for $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ and for $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$. Recall also the normalizing factor

$$\mu(\mathrm{H}') = \prod_{j=1}^{n'} (\tilde{\kappa}(J_j, J_j)^{\frac{1}{2}}\, 2\pi), \tag{118}$$

which is used to pass from the un-normalized chc in [3] to the normalize chc in [4] and [5]. The function $\hat{\mu}_{\mathcal{O}'}(x')\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x')$, $x' \in \mathfrak{h}'$, is a polynomial, which we shall denote by $P(x')$ below. Then,

$$\begin{aligned} &\int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x')\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x')\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int\limits_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx\, dx' \\ &= \lim_{R\to\infty} \int\limits_{B_R} P(x')\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int\limits_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx\, dx' \\ &= \lim_{R\to\infty} \sum_{[S]} \sum_{s\in \mathsf{W}(\mathrm{H}_{\mathbb{C}})} \frac{m_{[S]}}{\mu(\mathrm{H}')} \widetilde{\mathrm{sgn}}(s) \\ &\qquad \int\limits_{B_R} \lim_{y\in\Gamma_{s,S},\, y\to 0} \int\limits_{\mathfrak{h}_S} \frac{P(s^{-1}.x)\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}.x)}{\det(x'+x+iy)_{s\mathrm{W}^{\mathfrak{h}'}}} \mathcal{A}(-\widetilde{\Psi}_{S,\mathbb{R}})(x)\mathcal{H}_S\psi(x)\, dx\, dx', \end{aligned} \tag{119}$$

where the first equality holds because we have absolute convergence in (62) and the second one follows from [3, Theorem 7.3]. Furthermore, since ψ is supported in the set of the regular semisimple elements, $\mathcal{H}_S\psi$ is smooth and compactly supported, and therefore Corollary 17 applies. Hence, (119) is equal to

$$
\begin{aligned}
&\sum_{[S]}\sum_{s\in \mathsf{W}(\mathrm{H}_{\mathbb{C}})} \frac{m_{[S]}}{\mu(\mathrm{H}')}\widetilde{\operatorname{sgn}}(s)\int\limits_{\mathfrak{h}_S} P(s^{-1}.x)\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}.x)\\
&\left(\prod_{j=1}^{n'}(\pi\tilde{\kappa}(J_j,J_j)^{\frac{1}{2}}\hat{\epsilon}_j)\cdot \prod_{1\le j\le n',\ \sigma(j)\notin \underline{S}} \operatorname{sgn}(J^*_{\sigma(j)}(y_s))\right.\\
&\left.\prod_{\alpha\in S,\ 1\le j\le n',\ \sigma(j)\in\underline{\alpha}} \operatorname{sgn}(\alpha(iJ_{\sigma(j)}))\operatorname{sgn}(\alpha(x))\right)\cdot\\
&\mathcal{A}(-\widetilde{\Psi}_{S,\mathbb{R}})(x)\mathcal{H}_S\psi(x)\,dx.
\end{aligned}
\tag{120}
$$

Furthermore, by [3, Definition 3.4],

$$
\operatorname{sgn}(J^*_{\sigma(j)}(y_s)) = \operatorname{sgn}\langle J\ ,\ \rangle_{s(\mathrm{Hom}(\mathsf{V}_j,\mathsf{V}_j)^J)} \qquad (1\le j\le n),
$$

where the right hand side is 1 if the symmetric form $\langle J\ ,\ \rangle_{s(\mathrm{Hom}(\mathsf{V}_j,\mathsf{V}_j)^J)}$ on the two-dimensional real vector space $s(\mathrm{Hom}(\mathsf{V}_j,\mathsf{V}_j)^J)$ is positive definite and -1 if it is negative definite. The indefinite case doesn't occur. Therefore,

$$
\begin{aligned}
&\int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x')\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x')\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')\int\limits_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx\,dx'\\
&=\sum_{[S]}\frac{1}{|\mathsf{W}(\mathrm{H}(S))|}\int\limits_{\mathfrak{h}(S)}\sum_{s\in\mathsf{W}(\mathrm{H}(S)_{\mathbb{C}})} P(c(S)^{-1}s^{-1}.x)\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}(S)}(s^{-1}.x)\\
&\left(\widetilde{\operatorname{sgn}}(s)|\mathsf{W}(\mathrm{H}(S))|m_{[S]}2^{-n'}\prod_{j=1}^{n'}(\hat{\epsilon}_j)\cdot\prod_{1\le j\le n',\ \sigma(j)\notin\underline{S}}\operatorname{sgn}\langle J\ ,\ \rangle_{s(\mathrm{Hom}(\mathsf{V}_j,\mathsf{V}_j)^J)}\right.\\
&\prod_{\alpha\in S,\ 1\le j\le n',\ \sigma(j)\in\underline{\alpha}} \operatorname{sgn}(\alpha(iJ_{\sigma(j)}))\operatorname{sgn}(\alpha(c(S)^{-1}x))\mathcal{A}(-\widetilde{\Psi}_{S,\mathbb{R}})(c(S)^{-1}x)\cdot\\
&\left.\mathcal{A}(\Psi_{S,\mathbb{R}})(c(S)^{-1}x)\right)\\
&\pi_{\mathfrak{g}/\mathfrak{h}(S)}(x)\int\limits_{\mathrm{G}/\mathrm{H}(S)}\psi(g.x)\,d(g\mathrm{H}(S))\,dx,
\end{aligned}
\tag{121}
$$

where $\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}(S)}(x)=\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(c(S)^{-1}x)$ and $\pi_{\mathfrak{g}/\mathfrak{h}(S)}(x)=\pi_{\mathfrak{g}/\mathfrak{h}}(c(S)^{-1}x)$. In this case $F_{\mathfrak{h}',\mathfrak{h}(S),s}(x)$ is a constant multiple of the term in the parenthesis. This is the contribution to (97) associated to the Cartan subalgebras $\mathfrak{h}(S)\subseteq\mathfrak{g}$ and $\mathfrak{h}'\subseteq\mathfrak{g}'$.

Suppose $\mathfrak{h}' = \mathfrak{h}'_s + \mathfrak{h}'_c$ with both $\mathfrak{h}'_s \neq 0$ and $\mathfrak{h}'_c \neq 0$. Then, by [4, Theorem 7],

$$\begin{aligned} &|\det(ad\, x')_{\mathfrak{n}'}| \int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx \\ &= \gamma(\mathsf{V},\mathsf{V}',\mathsf{X}') \int_{\mathrm{G}/\mathrm{H}'\mathrm{G}(\mathsf{U})} \int_{\mathfrak{g}(\mathsf{U})} |\det(\mathrm{ad}(x'+y)_{\mathfrak{n}})| chc_{\mathrm{W}_c}(x'+y) \\ &\quad \psi(g.(x'+y))\,dy\,d(g(\mathrm{H}'\mathrm{G}(\mathsf{U}))). \end{aligned} \tag{122}$$

Let $C' \subseteq \mathfrak{h}'$ be a connected component in the complement of the union of the kernels of the real roots. Then $C' = C' \cap \mathfrak{h}'_s + \mathfrak{h}'_c$. Hence, for a fixed $x'_s \in C' \cap \mathfrak{h}'_s$, the function

$$\mathfrak{h}'_c \ni x'_c \to \hat{\mu}_{\mathcal{O}'}(x'_s + x'_c)\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x'_s + x'_c) \in \mathbb{C} \tag{123}$$

is a polynomial, which shall be denoted by $P(x'_s + x'_c)$, (see [39, Theorem 3, page 93]). We don't include the C' in the notation, because this information is encoded in x'_s. For a fixed x'_s and $g \in \mathrm{G}$ consider the integral

$$\int_{\mathfrak{h}'_c} P(x'_s + x'_c)\pi_{g'(\mathsf{V}'_c)/h'_c}(x'_c) \int_{\mathfrak{g}(\mathsf{U})} chc_{\mathrm{W}_c}(x'_c + y)\psi(g.(x'+y))\,dy\,dx'_c, \tag{124}$$

where $x' = x'_s + x'_c$. Notice that $\mathfrak{h}'_c \subseteq \mathfrak{g}'(\mathsf{V}'_c)$ is an elliptic Cartan subalgebra, that $(\mathrm{G}'(\mathsf{V}'_c), \mathrm{G}(\mathsf{U}))$ is a dual pair in $\mathrm{Sp}(\mathrm{W}_c)$ and that the number $\underline{p}$, [29, (1.12)] for this dual pair is the same as for the pair $(\mathrm{G}', \mathrm{G})$. Hence, we may apply the argument leading to (121). Let $\mathfrak{h}(\mathsf{U}) \subseteq \mathfrak{g}(\mathsf{U})$ be as in Sect. 7 and let $\Psi^n_{st}(\mathsf{U})$ denote the family of the sets S of positive strongly orthogonal non-compact imaginary roots of $\mathfrak{h}(\mathsf{U})$ in $\mathfrak{g}(\mathsf{U})_{\mathbb{C}}$. Let $\mathfrak{z}(\mathsf{U}) = \mathfrak{g}(\mathsf{U})^{\mathfrak{h}'_c}$. Then (124) is equal to

$$\begin{aligned} &\sum_{[S]\subseteq \Psi^n_{st}(\mathsf{U})} \frac{1}{|\mathsf{W}(\mathrm{H}(\mathsf{U})(S), \mathrm{G}(\mathsf{U}))|} \int_{\mathfrak{h}(\mathsf{U})(S)} \sum_{s\in \mathsf{W}(\mathrm{H}(\mathsf{U})(S)_{\mathbb{C}}, \mathrm{G}(\mathsf{U})_{\mathbb{C}})} \\ &P(x'_s + c(S)^{-1}s^{-1}.x)\tilde{\pi}_{\mathfrak{z}(\mathsf{U})/\mathfrak{h}(\mathsf{U})(S)}(s^{-1}.x)\,\widetilde{\mathrm{sgn}}(s) \\ &\Bigg(|\mathsf{W}(\mathrm{H}(\mathsf{U})(S))| m_{[S]}(\mathsf{U}) 2^{-n'(\mathsf{U})} \prod_{j=1}^{n'(\mathsf{U})} (\hat{\epsilon}_j) \cdot \prod_{1\le j\le n'(\mathsf{U}),\ \sigma(j)\notin \underline{S}} \mathrm{sgn}\langle J\ ,\ \rangle_{s(\mathrm{Hom}(\mathsf{V}_{c,j},\mathsf{V}_{c,j})^J)} \cdot \\ &\prod_{\alpha\in S,\ 1\le j\le n'(\mathsf{U}),\ \sigma(j)\in\underline{\alpha}} \mathrm{sgn}(\alpha(iJ_{\sigma(j)}))\mathrm{sgn}(\alpha(c(S)^{-1}x))\ \mathcal{A}(-\widetilde{\Psi}_{S,\mathbb{R}}(\mathsf{U}))(c(S)^{-1}x)\mathcal{A}(\Psi_{S,\mathbb{R}}(\mathsf{U}))(c(S)^{-1}x) \Bigg) \\ &\pi_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})(S)}(x) \int_{\mathrm{G}(\mathsf{U})/\mathrm{H}(\mathsf{U})(S)} \psi(g.(x'_s + k.x))\,d(k\mathrm{H}(\mathsf{U})(S))\,dx. \end{aligned} \tag{125}$$

Notice that the function

$$\mathfrak{h}'_c \ni x'_c \to \det\ \mathrm{ad}(x'_s + x'_c)_{\mathrm{Hom}(\mathsf{X}',\mathsf{V}'_c)} \in \mathbb{R}$$

has no zeros. Furthermore,

$$\mathrm{sgn}(\det(ad\,x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'})$$

is constant on C'. Therefore,

$$|\det(ad\,x')_{\mathfrak{n}'}| = \epsilon'(x')\det(ad\,x')_{\mathfrak{n}'} \qquad (x' \in \mathfrak{h}'), \tag{126}$$

where $\epsilon'(x') = \pm 1$ is constant on each C'. Similarly,

$$|\det(ad(x'+y))_{\mathfrak{n}}| = \epsilon(x')\det(ad(x'+y))_{\mathfrak{n}} \qquad (x' \in \mathfrak{h}', y \in \mathfrak{g}(\mathsf{U})), \tag{127}$$

where $\epsilon(x') = \pm 1$. Furthermore,

$$\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') = \pi_{\mathfrak{gl}(\mathsf{X}')/\mathfrak{h}'_s}(x') \cdot \det(ad\,x')_{\mathfrak{n}'} \cdot \pi_{\mathfrak{g}'(\mathsf{V}'_c)/\mathfrak{h}'_c}(x') \qquad (x' \in \mathfrak{h}') \tag{128}$$

and

$$\pi_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}(\mathsf{U})(S))}(x) = \pi_{\mathfrak{gl}(\mathsf{X}')/\mathfrak{h}'_s}(x) \cdot \det(ad\,x)_{\mathfrak{n}} \cdot \pi_{\mathfrak{g}(\mathsf{U})/\mathfrak{h}(\mathsf{U})(S)}(x)$$
$$(x \in \mathfrak{h}' + \mathfrak{h}(\mathsf{U})(S)). \tag{129}$$

Moreover,

$$\mathfrak{z} = \mathfrak{g}^{\mathfrak{h}'} = \mathfrak{h}'_s + \mathfrak{g}(\mathsf{U})^{\mathfrak{h}'_c} = \mathfrak{h}'_s + \mathfrak{z}(\mathsf{U}). \tag{130}$$

By combining (122)–(130) we obtain the following formula,

$$\begin{aligned}
&\int\limits_{\mathfrak{h}'} \hat{\mu}_{\mathcal{O}'}(x')\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}}(x')\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int\limits_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx\,dx' \\
&= \sum_{[S]\subseteq\Psi^n_{st}(\mathsf{U})} \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{|\mathrm{W}(\mathrm{H}'\mathrm{H}(\mathsf{U})(S),\mathrm{G})|} \int\limits_{\mathfrak{h}'_s} \int\limits_{\mathfrak{h}(\mathsf{U})(S)} \sum_{s\in\mathrm{W}(\mathrm{H}(\mathsf{U})(S)_{\mathbb{C}},\mathrm{G}(\mathsf{U})_{\mathbb{C}})} \\
&\overline{\hat{\mu}_{\mathcal{O}'}(x'_s + c(S)^{-1}s^{-1}.x)\pi_{\mathfrak{g}'/\mathfrak{h}'}(x'_s + c(S)^{-1}s^{-1}.x)}\tilde{\pi}_{\mathfrak{z}/(\mathfrak{h}'_s+\mathfrak{h}(\mathsf{U})(S))}(x'_s + s^{-1}.x) \\
&\Bigg(\widetilde{\mathrm{sgn}}(s)\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')\epsilon(x')\epsilon'(x')|\mathrm{W}(\mathrm{H}'\mathrm{H}(\mathsf{U})(S),\mathrm{G})|m_{[S]}(\mathsf{U})2^{-n'(\mathsf{U})}\prod_{j=1}^{n'(\mathsf{U})}(\hat{\epsilon}_j)\cdot \\
&\prod_{1\leq j\leq n'(\mathsf{U}),\ \sigma(j)\notin \underline{S}} \mathrm{sgn}\langle J\ ,\ \rangle_{s(\mathrm{Hom}(\mathsf{V}_{c,j},\mathsf{V}_{c,j})^J)} \qquad\qquad (131)\\
&\prod_{\alpha\in S,\ 1\leq j\leq n'(\mathsf{U}),\ \sigma(j)\in\underline{\alpha}} \mathrm{sgn}(\alpha(iJ_{\sigma(j)}))\mathrm{sgn}(\alpha(c(S)^{-1}x))\mathcal{A}(-\widetilde{\Psi}_{S,\mathbb{R}}(\mathsf{U}))(c(S)^{-1}x) \\
&\mathcal{A}(\Psi_{S,\mathbb{R}}(\mathsf{U}))(c(S)^{-1}x)\Bigg) \cdot \pi_{\mathfrak{g}/(\mathfrak{h}'+\mathfrak{h}(\mathsf{U})(S))}(x) \int\limits_{\mathrm{G}/\mathrm{H}'\mathrm{H}(\mathsf{U})(S)} \psi(g.(x'_s + x)) \\
&d(g(\mathrm{H}'\mathrm{H}(\mathsf{U})(S)))\,dx\,dx'_s.
\end{aligned}$$

Here we see the contribution to (97) on the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}'_s + \mathfrak{h}(\mathsf{U})(S)$. The function $F_{\mathfrak{h}',\mathfrak{h},s}(x)$ is a constant multiple of the term in the parenthesis.

This ends the proof of Theorem 14. □

Here is our main result.

Theorem 18. *Suppose* G, G' *is not a complex dual pair, so that the complexification* $\mathrm{G}_{\mathbb{C}}, \mathrm{G}'_{\mathbb{C}}$ *is an irreducible dual pair over* $\mathbb{C}$.

Let $\mathcal{O}' \subseteq \mathfrak{g}'$ *be a nilpotent* G'*-orbit. Let* $\mathfrak{h}' \subseteq \mathfrak{g}'$ *be a Cartan subalgebra and let* $\mathrm{H}' \subseteq \mathrm{G}'$ *be the Cartan subgroup with the Lie algebra* $\mathfrak{h}'$. *Let* $C' \subseteq \mathfrak{h}'$ *be a connected component of the complement of the union of the kernels of the real roots of* $\mathfrak{h}'$ *in* $\mathfrak{g}'_{\mathbb{C}}$. *Denote by* ρ' *the irreducible representation of the Weyl group* $\mathsf{W}(\mathrm{H}'_{\mathbb{C}}, \mathrm{G}'_{\mathbb{C}})$ *generated by the harmonic polynomial equal to* $\hat{\mu}_{\mathcal{O}'}\pi_{\mathfrak{g}'/\mathfrak{h}'}$ *on* C'. *Let* λ' *be the partition associated to the nilpotent* $\mathrm{G}'_{\mathbb{C}}$*-orbit in* $\mathfrak{g}'_{\mathbb{C}}$ *attached to* ρ' *via the Springer correspondence. Assume* (16). *Recall the number* p (93). *If the pair* G, G' *is of type I, assume that*

$$\mathrm{ht}(\lambda') \le p. \tag{132}$$

Then the integrals defining $chc(\hat{\mu}_{\mathcal{O}'})$ *are absolutely convergent. Assume* $chc(\hat{\mu}_{\mathcal{O}'}) \neq 0$.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ *be a Cartan subalgebra and let* $\mathrm{H} \subseteq \mathrm{G}$ *be the Cartan subgroup with the Lie algebra* $\mathfrak{h}$. *Let* $C \subseteq \mathfrak{h}$ *be a connected component of the complement of the union of the kernels of the real roots of* $\mathfrak{h}$ *in* $\mathfrak{g}_{\mathbb{C}}$. *Then* $(\pi_{\mathfrak{g}/\mathfrak{h}}chc(\hat{\mu}_{\mathcal{O}'}))|_C$ *is* $\mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$*-harmonic and generates an irreducible representation* ρ *of this Weyl group. Let* λ *be the partition associated to the nilpotent* $\mathrm{G}_{\mathbb{C}}$*-orbit in* $\mathfrak{g}_{\mathbb{C}}$ *attached to* ρ *via the Springer correspondence. Then* λ *is obtained from* λ' *by adding a column of the appropriate length, as in* (17).

If the irreducible dual pair G, G' *is complex, then the complexification* $\mathrm{G}_{\mathbb{C}}, \mathrm{G}'_{\mathbb{C}}$ *is the direct sum of two copies of* G, G'. *Assume* $chc(\hat{\mu}_{\mathcal{O}'}) \neq 0$. *Let* $\rho' = \rho'_1 \otimes \rho'_2$ *denote the irreducible representation of the Weyl group* $\mathsf{W}(\mathrm{H}'_{\mathbb{C}}, \mathrm{G}'_{\mathbb{C}}) = \mathsf{W}(\mathrm{H}', \mathrm{G}') \times \mathsf{W}(\mathrm{H}, \mathrm{G}')$ *generated by the harmonic polynomial* $\hat{\mu}_{\mathcal{O}'}\pi_{\mathfrak{g}'/\mathfrak{h}'}$ *on* $\mathfrak{h}$. *Denote by* λ'_1, λ'_2 *the pair of partitions associated to the nilpotent* $\mathrm{G}'_{\mathbb{C}}$*-orbit in* $\mathfrak{g}'_{\mathbb{C}}$ *attached to* $\rho' = \rho'_1 \otimes \rho'_2$ *via the Springer correspondence. Assume* (16) *for both* λ'_1 *and* λ'_2. *Then* $\pi_{\mathfrak{g}/\mathfrak{h}}chc(\hat{\mu}_{\mathcal{O}'})$ *is* $\mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}}) = \mathsf{W}(\mathrm{H}, \mathrm{G}) \times \mathsf{W}(\mathrm{H}, \mathrm{G})$*-harmonic and generates an irreducible representation* $\rho = \rho_1 \otimes \rho_2$ *of this Weyl group. Let* λ_1, λ_2 *be the pair of partitions associated to the nilpotent* $\mathrm{G}_{\mathbb{C}}$*-orbit in* $\mathfrak{g}_{\mathbb{C}}$ *attached to* $\rho = \rho_1 \otimes \rho_2$ *via the Springer correspondence. Then* λ_i *is obtained from* λ'_i $(i = 1, 2)$ *by adding the same column of the appropriate length, as in* (17).

Proof. By Corollary 4, (132) implies (94). Therefore, according to [3, Theorem 1], the integrals defining $chc(\hat{\mu}_{\mathcal{O}'})$ are absolutely convergent. Also, the formulas (97) and (98) of Theorem 14 holds.

Assume first that G, G' is not a complex dual pair. The polynomial $\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}$ does not depend on the real form G, G' of the complexification $\mathrm{G}_{\mathbb{C}}, \mathrm{G}'_{\mathbb{C}}$. If we choose the real form to be compact and introduce the coordinates on the Cartan subalgebras as in (75), then we obtain the following identifications

$$\begin{aligned} \mathfrak{h}_{\mathbb{C}} &= \{x = (x_1, x_2, \ldots, x_n);\ x_j \in \mathbb{C}\}, \\ \mathfrak{h}'_{\mathbb{C}} &= \{x' = (x_1, x_2, \ldots, x_{n'});\ x_j \in \mathbb{C}\}, \end{aligned} \tag{133}$$

so that the complex Weyl groups act via the permutations of the coordinates if the dual pair is of type II, and the permutations and all the sign changes if the dual pair is of type I. Recall that we assume $n' \le n$. If $n' < n$, then (for an appropriate choice of the positive root system) $\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(x)$, is equal to

$$\prod_{n'+1\le i<j\le n} (x_i - x_j) \text{ if } \mathrm{G}_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C}), \mathrm{G}'_{\mathbb{C}} = \mathrm{GL}_{n'}(\mathbb{C}) \tag{134}$$
$$\prod_{n'+1\le i<j\le n} (x_i^2 - x_j^2) \text{ if } \mathrm{G}_{\mathbb{C}} = \mathrm{O}_{2n}(\mathbb{C}), \mathrm{G}'_{\mathbb{C}} = \mathrm{Sp}_{2n'}(\mathbb{C}),$$
$$\prod_{n'+1\le i<j\le n} (x_i^2 - x_j^2) \prod_{n'+1\le j\le n} 2x_j \text{ if } \mathrm{G}_{\mathbb{C}} = \mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{G}'_{\mathbb{C}} = \mathrm{O}_{2n'}(\mathbb{C}),$$
$$\prod_{n'+1\le i<j\le n} (x_i^2 - x_j^2) \prod_{n'+1\le j\le n} x_j \text{ if } \mathrm{G}_{\mathbb{C}} = \mathrm{O}_{2n+1}(\mathbb{C}), \mathrm{G}'_{\mathbb{C}} = \mathrm{Sp}_{2n'}(\mathbb{C}),$$
$$\prod_{n'+1\le i<j\le n} (x_i^2 - x_j^2) \text{ if } \mathrm{G}_{\mathbb{C}} = \mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{G}'_{\mathbb{C}} = \mathrm{O}_{2n'+1}(\mathbb{C}).$$

If $n = n'$, then $\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(x) = 1$.

Recall the Cartan subalgebra $\mathfrak{h}_0(\mathfrak{h}') \subseteq \mathfrak{g}$, (95). By the construction

$$\mathfrak{h}_0(\mathfrak{h}') = \mathfrak{h}' \oplus \mathfrak{h}'^{\perp} \cap \mathfrak{h}_0(\mathfrak{h}').$$

Any function $f : \mathfrak{h}'_{\mathbb{C}} \to \mathbb{C}$ may be extended to $f : \mathfrak{h}_0(\mathfrak{h}')_{\mathbb{C}} \to \mathbb{C}$ via the composition with the projection $\mathfrak{h}_0(\mathfrak{h}')_{\mathbb{C}} \to \mathfrak{h}'_{\mathbb{C}}$. Then (134) implies that the map

$$\mathcal{P}(\mathfrak{h}'_{\mathbb{C}}) \ni f \to f\,\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}} \in \mathcal{P}(\mathfrak{h}_{\mathbb{C}}) \tag{135}$$

coincides with (23).

By assumption, $(\hat{\mu}_{\mathcal{O}'}\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}})(x)$ generates the irreducible representation ρ' of the Weyl group $\mathsf{W}(\mathrm{H}'_{\mathbb{C}}, \mathrm{G}'_{\mathbb{C}})$. Hence, every non-zero term

$$(\hat{\mu}_{\mathcal{O}'}\overline{\pi_{\mathfrak{g}'/\mathfrak{h}'}})(c^{-1}_{\mathfrak{h}_0(\mathfrak{h}'),\mathfrak{h}} s^{-1}.x)\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}.x) \qquad (s \in \mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})),$$

in Theorem 14, generates the representation ρ of $\mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$, constructed in Theorem 9. In particular the statement about the partitions follows.

If G, G' is a complex dual pair, then (135) coincides with (23) on each of the two copies of $\mathfrak{h}$ in $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \oplus \mathfrak{h}$. Hence, the term (98) in Theorem 14 generates the representation $\rho = \rho_1 \otimes \rho_2$ of $\mathsf{W}(\mathrm{H}_{\mathbb{C}}, \mathrm{G}_{\mathbb{C}})$, where each ρ_i $(i = 1, 2)$ is constructed in Theorem 9 and the statement about the pairs of the partitions follows. □

Appendix A

We begin by recalling Rossmann's construction of Springer's correspondence, [33]. At this point it seems fair to mention that the first construction of the Springer correspondence independent of étale cohomology, [38], was done in [23]. In fact on the level cohomology (see below) Rossmann's construction coincides with that of [23], as explained in [33, Appendix]. We prefer to use [33] mainly because of the connection with the Weyl group action on harmonic polynomials on a Cartan subalgebra described explicitly in [33].

Rossmann considers the adjoint group, but everything he does is valid for any connected semisimple complex group. For our applications $\mathrm{G} = \mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$ or $\mathrm{SO}_{2n}(\mathbb{C})$. Also, in these cases the adjoint group and G have the same Weyl group and the same nilpotent orbits.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathcal{B}$ be the flag manifold realized as the variety of the Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$. Denote by $\mathcal{B}^*$ the cotangent bundle of $\mathcal{B}$. Explicitly, $\mathcal{B}^* = \{(\mathfrak{b}, \nu);\ \mathfrak{b} \in \mathcal{B},\ \nu \in \mathfrak{b}^{\perp} \subseteq \mathfrak{g}^*\}$. When convenient, we shall identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via a Killing form. Then the ν belongs to the nilradical of $\mathfrak{b}$.

Let us fix a Borel subalgebra $\mathfrak{b}_1 \subseteq \mathfrak{g}$ and let $\mathfrak{h} \subseteq \mathfrak{b}_1$ be a Cartan subalgebra of $\mathfrak{g}$. Denote by W the Weyl group, equal to the normalizer of $\mathfrak{h}$ in G divided by the centralizer of $\mathfrak{h}$ in G.

Fix a regular element $\lambda \in \mathfrak{h}^*$ and let $\Omega_\lambda \subseteq \mathfrak{g}^*$ be the G-orbit through λ. Let $\mathrm{U} \subseteq \mathrm{G}$ be a maximal compact subgroup. Then U acts transitively on $\mathcal{B}$ and we have a bijection

$$p_\lambda \colon \mathcal{B}^* \ni u.(\mathfrak{b}_1, \nu) \to u.(\lambda + \nu) \in \Omega_\lambda \qquad (u \in \mathrm{U},\ \nu \in \mathfrak{b}_1^{\perp}). \tag{A.1}$$

Since for any $w \in \mathsf{W}$, $\Omega_{w\lambda} = \Omega_\lambda$, the following formula defines a transformation of $\mathcal{B}^*$:

$$a_\lambda(w) = p_{w\lambda}^{-1} \circ p_\lambda \colon \mathcal{B}^* \to \mathcal{B}^* \qquad (w \in \mathsf{W}). \tag{A.2}$$

Then

$$a_\lambda(w_1 w_2) = a_{w_2\lambda}(w_1) a_\lambda(w_2) \qquad (w_1, w_2 \in \mathsf{W}). \tag{A.3}$$

Let $\mathcal{N} \subseteq \mathfrak{g}$ be the nilpotent cone. Fix an element $\nu \in \mathcal{N}$. Define

$$\mathcal{B}^*(\nu) = \{(\mathfrak{b}, \nu) \in \mathcal{B}^*\} = \{(\mathfrak{b}, \nu') \in \mathcal{B}^*,\ \nu' = \nu\}. \tag{A.4}$$

Let $|\ |$ denote an Euclidean norm on $\mathfrak{g}$. For $\epsilon > 0$ let $U_\epsilon = \{\nu' \in \mathcal{N};\ |\nu' - \nu| < \epsilon\}$ and let

$$\mathcal{B}^*(U_\epsilon) = \{(\mathfrak{b}, \nu') \in \mathcal{B}^*,\ \nu' \in U_\epsilon\} = \{(\mathfrak{b}, \nu') \in \mathcal{B}^*,\ |\nu' - \nu| < \epsilon\}. \tag{A.5}$$

According to Rossmann, for any sufficiently small ϵ, the inclusion $\iota \colon \mathcal{B}^*(\nu) \to \mathcal{B}^*(U_\epsilon)$ has a proper homotopy inverse $p \colon \mathcal{B}^*(U_\epsilon) \to \mathcal{B}^*(\nu)$:

$$p \circ \iota \sim 1 \text{ on } \mathcal{B}^*(\nu) \text{ and } \iota \circ p \sim 1 \text{ on } \mathcal{B}^*(U_\epsilon). \tag{A.6}$$

Rossmann shows that for all λ small enough,

$$a_\lambda(w)(\mathcal{B}^*(\nu)) \subseteq \mathcal{B}^*(U_\epsilon) \qquad (w \in \mathsf{W}). \tag{A.7}$$

The transformations

$$a_\lambda^\nu(w) = p \circ a_\lambda(w) \circ \iota \colon \mathcal{B}^*(\nu) \to \mathcal{B}^*(\nu) \qquad (w \in \mathsf{W}) \tag{A.8}$$

are well defined for all regular $\lambda \in \mathfrak{h}^*$ in a small ball about zero. Since these λ form a connected set, the proper homotopy class of $a^\nu(w)$ of $a_\lambda^\nu(w)$ is independent of λ and the Equation (A.3) implies

$$a^\nu(w_1 w_2) = a^\nu(w_1) a^\nu(w_2) \qquad (w_1, w_2 \in \mathsf{W}). \tag{A.9}$$

This way a^ν gives a proper homotopy action of W on $\mathcal{B}^*(\nu)$. As a consequence, we have a representation of W on the relative homology

$$\mathrm{H}_*(\mathcal{B}^*(\nu)) := \mathrm{H}_*(\overline{\mathcal{B}^*(\nu)}, \partial\mathcal{B}^*(\nu);\ \mathbb{C}). \tag{A.10}$$

The group U acts on $\mathcal{B}^*$ by

$$\mathcal{B}^* \ni u.(\mathfrak{b}_1, \nu) \to vu.(\mathfrak{b}_1, \nu) \in \mathcal{B}^* \qquad (u, v \in \mathrm{U},\ \nu \in \mathfrak{b}_1^\perp) \tag{A.11}$$

Since the maps p_λ, (A.1), are U-equivariant, the operators $a_\lambda(w) = p_{w\lambda}^{-1} \circ p_\lambda$, (A.2), commute with the action of U. Let $\mathrm{A}(\nu)$ denote the component group of the stabilizer of ν in G. This group acts on $\mathrm{H}_*(\mathcal{B}^*(\nu))$ via (A.11). Hence, the actions of $\mathrm{A}(\nu)$ and W on $\mathrm{H}_*(\mathcal{B}^*(\nu))$ commute. Denote by $\mathrm{H}_*(\mathcal{B}^*(\nu))^{\mathrm{A}(\nu)}$ the subspace of the $\mathrm{A}(\nu)$-invariants in $\mathrm{H}_*(\mathcal{B}^*(\nu))$. Let $e(\nu) = \dim_{\mathbb{C}} \mathrm{H}_*(\mathcal{B}^*(\nu))$. Rossmann proved (a theorem of Springer) that

$$\mathrm{H}_{2e(\nu)}(\mathcal{B}^*(\nu))^{\mathrm{A}(\nu)} \text{ is an irreducible } \mathsf{W} \text{ module.} \tag{A.12}$$

Let $\mathcal{O}(\nu) \subseteq \mathcal{N}$ be the G-orbit through ν. Since G is connected, this representation does not depend on the choice of ν in the orbit. Thus each nilpotent orbit provides an irreducible representation of W. This is known as the Springer correspondence for G. Let $\mathcal{B}^\nu = \{\mathfrak{b} \in \mathcal{B};\ \nu \in \mathfrak{b}^\perp\}$. Obviously the projection $\mathcal{B}^* \to \mathcal{B}$ restricts to a bijection

$$\mathcal{B}^*(\nu) \ni (\mathfrak{b}, \nu) \to \mathfrak{b} \in \mathcal{B}^\nu \tag{A.13}$$

and $e(\nu) = \dim_{\mathbb{C}} \mathcal{B}^\nu$. We may use (A.13) to transfer the actions of the groups W and $\mathrm{A}(\nu)$ from $\mathrm{H}_{2e(\nu)}(\mathcal{B}^*(\nu))$ to $\mathrm{H}_{2e(\nu)}(\mathcal{B}^\nu)$. In these terms, the Springer correspondence attaches the orbit $\mathcal{O}(\nu)$ to the irreducible representation of W on $\mathrm{H}_{2e(\nu)}(\mathcal{B}^\nu)^{\mathrm{A}(\nu)}$.

The Weyl group acts on the flag manifold $\mathcal{B}$ by

$$\mathcal{B} \ni u.\mathfrak{b}_1 \to uw.\mathfrak{b}_1 \in \mathcal{B} \qquad (u \in \mathrm{U},\ w \in \mathsf{W}). \tag{A.14}$$

(Here, for any $w \in \mathsf{W}$, we choose a representative of w in U.) Rossmann showed that the inclusion $\mathcal{B}^\nu \subseteq \mathcal{B}$ induces a W-equivariant injection

$$\mathrm{H}_{2e(\nu)}(\mathcal{B}^\nu)^{\mathrm{A}(\nu)} \to \mathrm{H}_{2e(\nu)}(\mathcal{B}). \tag{A.15}$$

[On the left W acts via (A.8) and on the right by (A.14)]. Furthermore, we have Borel's isomorphism

$$\mathrm{H}_*(\mathcal{B}) \to \mathcal{H}(\mathfrak{h}^*), \tag{A.16}$$

where $\mathcal{H}(\mathfrak{h}^*)$ is the space of the W-harmonic polynomials on $\mathfrak{h}^*$. The map (A.16) is W-equivariant. (For the obvious action of W on the polynomials.) By composing (A.15) and (A.16) we obtain a realization of the Springer representation in a subspace $\mathcal{H}(\mathfrak{h}^*)_\nu \subseteq \mathcal{H}(\mathfrak{h}^*)$ of the harmonics, which is contained in $\mathcal{H}_{e(\nu)}(\mathfrak{h}^*)$ - the subspace of the harmonic polynomials homogeneous of degree $e(\nu)$.

The groups $SO_{2n+1}(\mathbb{C})$ and $O_{2n+1}(\mathbb{C})$ have the same Weyl group and the same co-adjoint orbits. Hence, we have (the obvious) Springer correspondence for the group $O_{2n+1}(\mathbb{C})$.

From now on let $G = SO_{2n}(\mathbb{C})$ and let $G'' = O_{2n}(\mathbb{C})$. Then $G \subseteq G''$. Let W'' be the normalizer of $\mathfrak{h}$ in G'' divided by the centralizer of $\mathfrak{h}$ in G'' (which is equal to the centralizer of $\mathfrak{h}$ in G). Then W is a subgroup of W'' of index 2. Fix an element $s \in \mathsf{W}'' \setminus \mathsf{W}$. When convenient we'll think of s as of an element of $G'' \setminus G$.

As before, we have a fixed nilpotent $\nu \in \mathcal{N}$ and the G-orbit $\mathcal{O}(\nu)$ through ν. Let $\mathcal{O}''(\nu)$ be the G''-orbit through ν. There are two possibilities: either

$$\mathcal{O}''(\nu) = \mathcal{O}(\nu) \text{ i.e. there is } g \in G \text{ such that } g.\nu = s.\nu, \tag{A.17}$$

or

$$\mathcal{O}''(\nu) = \mathcal{O}(\nu) \cup \mathcal{O}(s.\nu) \text{ (disjoint union).} \tag{A.18}$$

Lemma A.1. *The representation of* W'' *on the subspace of* $\mathcal{H}_{e(\nu)}(\mathfrak{h}^*)$ *generated by* $\mathcal{H}_{e(\nu)}(\mathfrak{h}^*)_\nu + \mathcal{H}_{e(\nu)}(\mathfrak{h}^*)_{s.\nu}$ *is irreducible.*

We shall refer to this representation as to "Springer representation attached to the orbit $\mathcal{O}''(\nu)$".

Proof. We may assume that $s.\mathfrak{b}_1 = \mathfrak{b}_1$. Recall, [33] the following U-invariant two-form on $\mathcal{B}$:

$$\tau_\lambda(x\mathfrak{b}_1, y\mathfrak{b}_1) = \lambda([x, y]) \qquad (x, y \in \mathfrak{u}).$$

Then

$$\tau_{s^{-1}.\lambda}(x\mathfrak{b}_1, y\mathfrak{b}_1) = \lambda([sx, sy]) \qquad (x, y \in \mathfrak{u}).$$

Hence, the map

$$\mathcal{B} \ni u.\mathfrak{b}_1 \to sus^{-1}.\mathfrak{b}_1 \in \mathcal{B} \tag{A.19}$$

intertwines the action of s on $H_*(\mathcal{B})$ with the action of s on $\mathcal{H}(\mathfrak{h}^*)$ via Borel isomorphism (A.16).

We need to construct an action of s on $H_*(\mathcal{B}^\nu)$ compatible with (A.15). Notice that

$$\mathcal{B}^* \ni u.(\mathfrak{b}_1, \nu') \underset{p_\lambda}{\to} u.(\lambda + \nu') \underset{s}{\to} su.(\lambda + \nu') = sus^{-1}.(s\lambda + s\nu') \in \Omega_{s\lambda} \tag{A.20}$$

and

$$\mathcal{B}^* \ni u.(\mathfrak{b}_1, \nu') \underset{p_{s\lambda}}{\to} u.(s\lambda + \nu') \in \Omega_{s\lambda}.$$

Set $a_\lambda(s) = p_{s\lambda}^{-1} \circ p_\lambda$. Then

$$a_\lambda(s)\colon \mathcal{B}^* \ni u.(\mathfrak{b}_1, \nu') \to sus^{-1}.(\mathfrak{b}_1, s\nu') = su.(\mathfrak{b}_1, \nu') \in \mathcal{B}^*. \tag{A.21}$$

In particular $a_\lambda(s) = a(s)$ does not depend on λ. Furthermore, the action of s on $\mathcal{B}^*$ defined by (A.21) coincides with the action induced by the adjoint action on the Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$.

Notice that for any $w \in \mathsf{W}$ and any regular $\lambda \in \mathfrak{h}^*$,

$$a(s)a_\lambda(w)a(s)^{-1} = a_{s\lambda}(sws^{-1}). \tag{A.22}$$

Indeed, suppose

$$a(s)a_\lambda(w)a(s)^{-1}(u.(\mathfrak{b}_1, \nu')) = u''.(\mathfrak{b}_1, \nu'').$$

Then

$$p_\lambda a(s)^{-1}(u.(\mathfrak{b}_1, \nu')) = p_{w\lambda} a(s)^{-1}(u''.(\mathfrak{b}_1, \nu'')).$$

Equivalently,

$$p_\lambda(s^{-1}us.(\mathfrak{b}_1, s^{-1}\nu')) = p_{w\lambda}(s^{-1}u''s.(\mathfrak{b}_1, s^{-1}\nu'')),$$

which means that

$$s^{-1}us.(\lambda + s^{-1}\nu') = s^{-1}u''s.(w\lambda + s^{-1}\nu'').$$

Hence,

$$u.(s\lambda + \nu') = u''.(sws^{-1}(s\lambda) + \nu'') = p_{sws^{-1}(s\lambda)}(u''.(\mathfrak{b}_1, \nu'')).$$

Therefore,

$$u''.(\mathfrak{b}_1, \nu'') = p^{-1}_{sws^{-1}(s\lambda)} \circ p_\lambda(u.(\mathfrak{b}_1, \nu')),$$

which verifies (A.22).

Clearly, $a(s)\colon \mathcal{B}^*(\nu) \to \mathcal{B}^*(s\nu)$ and the resulting map

$$\tilde{a}(s)\colon \mathrm{H}_*(\mathcal{B}^*(\nu)) \to \mathrm{H}_*(\mathcal{B}^*(s\nu)) \tag{A.23}$$

intertwines the action of $\mathrm{A}(\nu)$ with the action of $\mathrm{A}(s\nu) = s\mathrm{A}(\nu)s^{-1}$.

Suppose $\mathcal{O}(\nu) = \mathcal{O}(s\nu)$ as in (A.17). Since G is connected, there is a homotopy equivalence $\mathcal{B}^*(\nu) \sim \mathcal{B}^*(s\nu)$. Hence, (A.23) gives

$$\tilde{s}\colon \mathrm{H}_*(\mathcal{B}^*(\nu)) \to \mathrm{H}_*(\mathcal{B}^*(\nu)). \tag{A.24}$$

Also, the action (A.24) commutes with the action of $\mathrm{A}(\nu)$. Thus

$$\tilde{s}\colon \mathrm{H}_{2e(\nu)}(\mathcal{B}^*(\nu))^{\mathrm{A}(\nu)} \to \mathrm{H}_{2e(\nu)}(\mathcal{B}^*(\nu))^{\mathrm{A}(\nu)}. \tag{A.25}$$

This way, $\mathrm{H}_{2e(\nu)}(\mathcal{B}^*(\nu))^{\mathrm{A}(\nu)} = \mathrm{H}_{2e(\nu)}(\mathcal{B}^\nu)^{\mathrm{A}(\nu)}$ becomes a representation of W'' which restricts to an irreducible representation of W.

Suppose $\mathcal{O}(\nu) \neq \mathcal{O}(s\nu)$ as in (A.18). Then $\mathrm{H}_{2e(\nu)}(\mathcal{B}^\nu)^{\mathrm{A}(\nu)}$ and $\mathrm{H}_{2e(s\nu)}(\mathcal{B}^{s\nu})^{\mathrm{A}(s\nu)}$ are two non-isomorphic W-modules. As representations of W, the second one is isomorphic to the first one transformed via the composition with the automorphism of W equal to the conjugation by s. Furthermore, $e(s\nu) = e(\nu)$. Thus

$$\mathrm{H}_{2e(\nu)}(\mathcal{B}^\nu)^{\mathrm{A}(\nu)} \oplus \mathrm{H}_{2e(\nu)}(\mathcal{B}^{s\nu})^{\mathrm{A}(s\nu)} \tag{A.26}$$

is a representation of W'' which restricts to the sum of the two inequivalent representations of W corresponding to $\mathcal{O}(\nu)$ and $\mathcal{O}(s\nu)$ via Springer. The representation (A.26) is irreducible, because (as is easy to check) the only endomorphism which commutes with the action of W'' is a constant multiple of the identity. □

We still need to explain the combinatorial description of the representation described in Lemma A.1.

Let λ be the partition of $2n$ associated to the orbit $\mathcal{O}''(\nu)$. Suppose $\mathcal{O}''(\nu) = \mathcal{O}(\nu) \cup \mathcal{O}(s\nu)$ as in (A.18). Then all the parts of λ are even and Lusztig's algorithm associates to λ a pair of identical partitions (ξ, η), $\xi = \eta$. The corresponding representation of W'', $\rho_{(\xi,\eta)}$, has the property that its restriction to W splits into the direct sum of two inequivalent representations. These representations occur in the harmonics of degree $e(\nu)$ and not in any lower degree. Thus this is the representation constructed in Lemma A.1.

Suppose $\mathcal{O}''(\nu) = \mathcal{O}(\nu)$ as in (A.17). Then there are two representations of W'' which restrict to the same irreducible representation of W and we have to make the correct choice in our combinatorial description, (18). We'll show that one of these representations occurs in the correct degree $e(\nu)$ among the harmonics and the other one does not.

Lemma A.2. *Let λ be the partition of $2n$ corresponding to the orbit $\mathcal{O}''(\nu)$ and let (ξ, η) be the ordered pair of partitions obtained from λ via the modified Lusztig algorithm, as described in Proposition 5. Then*

$$\deg\ \Delta_{\xi,\eta} = e(\nu) \tag{A.27}$$

and

$$\deg\ \Delta_{\eta,\xi} = e(\nu) \textit{ if} \quad \textit{and} \quad \textit{only if } \xi = \eta. \tag{A.28}$$

Proof. We shall transfer the problem from the orthogonal group $\mathrm{O}_{2n}(\mathbb{C})$ to the symplectic group $\mathrm{Sp}_{2m}(\mathbb{C})$, $m > n$, using Proposition 5 which is purely combinatorial in nature.

In order to use this proposition we change the notation. Let λ' be the orthogonal partition of $2n'$ and let (ξ', η') be the ordered pair of partitions obtained from λ' via the modified Lusztig algorithm. Let $\lambda = (1^{2\ell}) \oplus \lambda'$ be the symplectic partition obtained from λ' by adding a column of height 2ℓ. Denote by (ξ, η) the ordered pair of partitions obtained from λ via Lusztig's algorithm [8, 13.3]. Let $\mathrm{G}' = \mathrm{O}_{2n'}(\mathbb{C})$ and let $\mathrm{G} = \mathrm{Sp}_{2(n'+l)}(\mathbb{C})$. Denote by $\nu' \in \mathfrak{g}'$ a nilpotent element in the G'-orbit described by λ' and let $\nu \in \mathfrak{g}$ a nilpotent element in the G-orbit corresponding to λ. Let $C(\nu') \subseteq \mathrm{G}'$ be the centralizer of ν' and let $\mathrm{C}(\nu) \subseteq \mathrm{G}$ be the centralizer of ν. Then, [8, Theorem 5.10.2(a)],

$$2e(\nu') + \operatorname{rank} \mathrm{G}' = \dim\ \mathrm{C}(\nu') \tag{A.29}$$

and

$$2e(\nu) + \operatorname{rank} \mathrm{G} = \dim\ \mathrm{C}(\nu). \tag{A.30}$$

We would like to compare deg $\Delta_{\xi,\eta}$ and deg $\Delta_{\eta,\xi}$ to $e(\nu')$. Since Lusztig's algorithm does describe the Springer correspondence for the symplectic group, (A.30) translates to

$$2 \text{ deg } \Delta_{\xi,\eta} + \text{rank G} = \dim \mathrm{C}(\nu). \tag{A.31}$$

We see from (A.29) that in order to verify (A.27) we need to show

$$2 \text{ deg } \Delta_{\xi',\eta'} + \text{rank G}' = \dim \mathrm{C}(\nu'). \tag{A.32}$$

Proposition 5 shows that $(\xi, \eta) = (\xi', (1^{\ell}) \oplus \eta')$. Notice that $\text{rankG} = \text{rank G}' + \ell$ and that

$$\begin{aligned} \text{deg } \Delta_{\xi,\eta} &= 2n(\xi) + 2n(\eta) + |\eta| = 2n(\xi') + 2n(\eta') + \ell(\ell-1) + \ell + |\eta'| \\ &= \text{deg } \Delta_{\xi',\eta'} + \ell^2. \end{aligned}$$

Also, in terms of the notation in [9, page 89]

$$\begin{aligned} \dim \mathrm{C}(\nu) &= \frac{1}{2}\left(\sum_{i\geq 1} s_i^2 + \sum_{i\ odd} r_i\right) = \frac{1}{2}\left((2\ell)^2 + \sum_{i\geq 2} s_i^2 + \sum_{i\ odd} r_i\right) \\ &= 2\ell^2 + \dim \mathrm{C}(\nu') + \frac{1}{2}\left(\sum_{i\ odd} r_i' + \sum_{i\ odd} r_i\right) \\ &= 2\ell^2 + \dim \mathrm{C}(\nu') + \frac{1}{2}\left(\sum_{i\ odd} r_i' + \sum_{i\ even} r_i' + 2\ell - \text{ht}(\lambda')\right) \\ &= 2\ell^2 + \dim \mathrm{C}(\nu') + \ell. \end{aligned}$$

Hence, (A.32) follows. This completes the proof of (A.27).

In order to verify (A.28) we choose $\ell > \text{ht}(\xi)$, greater than the number of parts in $\xi = \xi'$. Then, as in [8, page 420] we extend ξ to $\tilde{\xi}$ by adding zeros so that $\tilde{\xi}$ has $\ell + 1$ parts, which is one more than the number of parts in η. Then $\tilde{\xi}_1 = 0$. As shown by Lusztig [8, page 420], the pair $(\tilde{\xi}, \eta)$ satisfies the following inequalities:

$$\begin{aligned} &\tilde{\xi}_1 \leq \eta_1 + 1 \leq \tilde{\xi}_2 + 2 \leq \eta_2 + 3 \leq \tilde{\xi}_3 + 4 \leq \eta_3 + 5 \leq \tilde{\xi}_4 + 6 \leq \dots \\ &\quad \leq \eta_\ell + 2\ell + 1 \leq \tilde{\xi}_{\ell+1} + 2\ell + 2. \end{aligned}$$

Since, $\eta_i = \eta_i' + 1$, we have

$$\begin{aligned} &\tilde{\xi}_1 \leq \eta_1' + 2 \leq \tilde{\xi}_2 + 2 \leq \eta_2' + 4 \leq \tilde{\xi}_3 + 4 \leq \eta_3' + 6 \leq \tilde{\xi}_4 + 6 \leq \dots \\ &\quad \leq \eta_\ell' + 2\ell + 2 \leq \tilde{\xi}_{\ell+1} + 2\ell + 2. \end{aligned}$$

Therefore

$$\eta_1' \leq \tilde{\xi}_2,\ \eta_2' \leq \tilde{\xi}_3, \dots, \eta_\ell' \leq \tilde{\xi}_{\ell+1}. \tag{A.33}$$

Since $\tilde{\xi}_1 = 0$, we have $|\tilde{\xi}| = |\xi| = |\xi'|$. Thus (A.33) implies

$$|\eta'| \leq |\xi'|. \tag{A.34}$$

Also, equality in (A.34) implies equalities in (A.33). Thus

$$|\eta'| \leq |\xi'| \text{ if and only if } \eta' = \xi'. \tag{A.35}$$

Since, $\deg \Delta_{\xi',\eta'} - \deg \Delta_{\xi,\eta} = |\eta'| - |\xi'|$, (A.35) implies (A.28). □

The aim of the last part this section is to extend the above results to the case of an orthogonal group $\mathrm{O}_{2n}(\mathbb{K})$ where $\mathbb{K}$ is the algebraic closure of a finite field $\mathbb{F}_q$ of odd characteristic. We shall need them in [1].

Let $\mathbf{T}_{\mathbb{C}} \subset \mathrm{SO}_{2n}(\mathbb{C})$ and $\mathbf{T} \subset \mathrm{SO}_{2n}(\mathbb{K})$ be maximal tori such that the root data of $\mathrm{SO}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n}(\mathbb{K})$ with respect to $\mathbf{T}_{\mathbb{C}}$ and $\mathbf{T}$, respectively, are isomorphic. We also fix a Borel subgroup $\mathbf{B} \subset \mathrm{SO}_{2n}(\mathbb{K})$ containing $\mathbf{T}$. Let W be the Weyl group of $\mathrm{SO}_{2n}(\mathbb{K})$ with respect to $\mathbf{T}$ and let $S \subset \mathsf{W}$ denote the set of simple reflections determined by $\mathbf{B}$. Then the pair (W, S) can be canonically identified with the corresponding pair in $\mathrm{SO}_{2n}(\mathbb{C})$ defined with respect to $\mathbf{T}_{\mathbb{C}} \subset \mathbf{B}_{\mathbb{C}}$.

Let $\mathcal{N}(\mathrm{SO}_{2n}(\mathbb{C})), \mathcal{N}(\mathrm{SO}_{2n}(\mathbb{K}))$ denote the set of unipotent classes of $\mathrm{SO}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{K})$, respectively. Let $\Xi\colon \mathcal{N}(\mathrm{SO}_{2n}(\mathbb{C})) \to \mathcal{N}(\mathrm{SO}_{2n}(\mathbb{K}))$ be Spaltenstein's map, see [37, Théorème III 5.2]. This map is uniquely characterized by the following three properties:

(1) It preserves the usual partial orderings $\leq$ on $\mathcal{N}(\mathrm{SO}_{2n}(\mathbb{C}))$ and on $\mathcal{N}(\mathrm{SO}_{2n}(\mathbb{K}))$;
(2) it preserves the dimensions of classes;
(3) it satisfies certain compatibility conditions with respect to parabolic subgroups in $\mathrm{SO}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n}(\mathbb{K})$ containing $\mathbf{B}_{\mathbb{C}}$ and $\mathbf{B}$, respectively.

Moreover, since q is assumed to be odd, Ξ is an isomorphism of partially ordered sets. We can also canonically identify the component group

$$A(u) := \pi_0(\mathrm{C}_{\mathrm{SO}_{2n}(\mathbb{K})}(u))$$

of the centralizer of u in $\mathrm{SO}_{2n}(\mathbb{K})$ with the component group $\pi_0(\mathrm{C}_{\mathrm{SO}_{2n}(\mathbb{C})}(u'))$ where $u \in \mathcal{O}$ for some $\mathcal{O} \in \mathcal{N}(\mathrm{SO}_{2n}(\mathbb{K}))$ and $u' \in \Xi(\mathcal{O})$. Then it follows from its explicit description that the Springer correspondence coincides for $\mathrm{SO}_{2n}(\mathbb{K})$ and for $\mathrm{SO}_{2n}(\mathbb{C})$ (see [36] and the references there).

Let u be a unipotent element in $\mathrm{SO}_{2n}(\mathbb{K})$, let $\mathcal{O}(u) \in \mathcal{N}(\mathrm{SO}_{2n}(\mathbb{K}))$ denote the conjugacy class of u in $\mathrm{SO}_{2n}(\mathbb{K})$, and let $\mathcal{O}''(u)$ denote the conjugacy class of u in $\mathrm{O}_{2n}(\mathbb{K})$. As before, there are two possibilities: either $\mathcal{O}''(u) = \mathcal{O}(u)$, or $\mathcal{O}''(u) = \mathcal{O}(u) \cup \mathcal{O}(sus^{-1})$ (disjoint union), where we think of $s \in \mathsf{W}''\backslash\mathsf{W}$ as an element of $\mathrm{O}_{2n}(\mathbb{K})\backslash\mathrm{SO}_{2n}(\mathbb{K})$.

We assume first that $\mathcal{O}''(u) = \mathcal{O}(u) \cup \mathcal{O}(sus^{-1})$ (disjoint union). Then the inverse images by Ξ of $\mathcal{O}(u)$ and $\mathcal{O}(sus^{-1})$ are also disjoint, and similarly for $\mathcal{O}(\nu)$ and $\mathcal{O}(s\nu)$, with $u = \exp \nu$. It follows from the above discussion that

$$\mathrm{H}_{2e(\nu)}(\mathcal{B}^{\nu})^{A(\nu)} \oplus \mathrm{H}_{2e(\nu)}(\mathcal{B}^{s\nu})^{A(s\nu)} \tag{A.36}$$

is an irreducible representation of W'' which restricts to the sum of two inequivalent representations of W corresponding to $\mathcal{O}(u)$ and $\mathcal{O}(sus^{-1})$ by the Springer correspondence for the group $\mathrm{SO}_{2n}(\mathbb{K})$. Let λ be the partition of $2n$ associated to $\mathcal{O}''(u)$. All the parts of λ are even and Lusztig's algorithm associates to λ a pair of identical partitions (ξ, η), $\xi = \eta$. The restriction to W of the corresponding representation of W'', $\rho_{\xi,\eta}$, splits into the direct sum of two inequivalent representations. The b-invariant of these representations [as defined in (7)] equals

$$e(\nu) = \frac{1}{2} \dim \mathcal{B}_u,$$

where $\mathcal{B}_u$ is the Springer fiber of all the Borel subgroups of $\mathrm{SO}_{2n}(\mathbb{K})$ which contain u. Thus $\rho_{\xi,\eta}$ is the representation given in (A.36).

We suppose now that $\mathcal{O}''(u) = \mathcal{O}(u)$. As previously, we have then two representations of W'' which restrict to the same irreducible representation of W (and we have made the correct choice). Formulas (A.29) and (A.30) are still valid with $\mathbb{K}$ instead of $\mathbb{C}$. Then the same proof as that of Lemma A.2 gives the result.

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