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Normalization of the Cauchy Harish-Chandra Integral

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Abstract. This is the second of our three articles showing that the Cauchy Harish-Chandra integral maps invariant eigendistributions to invariant eigendistributions with the correct infinitesimal character. In this paper, we define a normalization of this integral for all real reductive dual pairs. Then we prove that the normalized Cauchy Harish-Chandra integral maps orbital integrals to orbital integrals for the pairs $(\mathrm{U}_{p,q}, \mathrm{U}_{1,1})$, $(\mathrm{O}_{p,q}, \mathrm{Sp}_2(\mathbb{R}))$ and $(\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{1,2})$.
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Introduction

Let $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}')$ be a real reductive dual pair in a metaplectic group $\widetilde{\mathrm{Sp}}$. Let Θ be the character of an oscillator representation of $\widetilde{\mathrm{Sp}}$. Assume $\widetilde{\mathrm{G}}'$ is compact and let (π, π') be two irreducible representations of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$ in Howe's correspondence. In these terms the First Fundamental Theorem of Classical Invariant Theory can be written as the following equality of distributions

$$\int_{\widetilde{\mathrm{G}}'} \Theta(gg')\overline{\Theta_{\pi'}(g')}dg' = \Theta_\pi(g)$$

where Θ_π and $\Theta_{\pi'}$ stand for the characters of π and π'. For a smooth compactly supported function ϕ on $\widetilde{\mathrm{G}}$, the formula

$$\phi'(g') = \int_{\widetilde{\mathrm{G}}} \phi(g)\Theta(gg')dg \tag{0.1}$$

defines a smooth compactly supported function on $\widetilde{\mathrm{G}}'$ and

$$\int_{\widetilde{\mathrm{G}}'} \overline{\Theta_{\pi'}(g')}\phi'(g')dg' = \int_{\widetilde{\mathrm{G}}} \Theta_\pi(g)\phi(g)dg.$$

The Cauchy Harish-Chandra integral (Chc) extends formula (0.1) to all dual pairs with rank of $\widetilde{\mathrm{G}}'$ less or equal to rank of $\widetilde{\mathrm{G}}$. One of the goals of this project is to

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prove that this analogue of formula (0.1) provides a smooth compactly supported function on $\widetilde{\mathrm{G}}'$. In this paper, we prove the existence of such function for the dual pairs

$$(\mathrm{U}_{p,q}, \mathrm{U}_{1,1}), (\mathrm{O}_{p,q}, \mathrm{Sp}_2(\mathbb{R})) \text{ and } (\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{1,2}).$$

Also, we obtain formulas for Chc both on the Lie algebra and the Lie group. These expressions become explicit when combine with the description of Chc done in [1].

Let W be a finite dimensional vector space over the reals, with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Let J be a positive compatible complex structure on W, $\mathrm{Sp}(\mathsf{W})$ (resp. $\mathfrak{sp}(\mathsf{W})$) the symplectic group (resp. the symplectic Lie algebra) associated to $\langle\ ,\ \rangle$ and let $\chi(r) = \exp(2\pi i r)$, $r \in \mathbb{R}$. Fix a Lebesgue measure dw on W so that

$$\int_{\mathsf{W}} \chi(\tfrac{i}{2}\langle Jw, w\rangle)\, dw = 1.$$

The conjugation by J is a Cartan involution θ on $\mathfrak{sp}(\mathsf{W})$ and the following formula defines a Killing form κ on $\mathrm{End}(\mathsf{W})$ and on $\mathfrak{sp}(\mathsf{W})$:

$$\kappa(x, y) = \mathrm{tr}(xy) \qquad (x, y \in \mathrm{End}(\mathsf{W})).$$

Let

$$\tilde{\kappa}(x, y) = -\kappa(\theta x, y) \qquad (x, y \in \mathfrak{sp}(\mathsf{W})).$$

This is a positive definite symmetric form. We shall denote by the same letter $\tilde{\kappa}$ the restriction of the form $\tilde{\kappa}$ to any subspace of $\mathfrak{sp}(\mathsf{W})$. The form $\tilde{\kappa}$ determines a normalization of the Lebesgue measure on that subspace as follows. Let e_1, e_2, $\cdots$, e_n be a basis of that subspace, and let $I = [0, 1] \subseteq \mathbb{R}$ denote the unit interval. Then

$$\mu(Ie_1 + Ie_2 + \cdots + Ie_n) = |\det(\tilde{\kappa}(e_j, e_k))|^{1/2}.$$

If we write $\mu = \mu_{\tilde{\kappa}}$, then

$$\mu_{t\tilde{\kappa}} = t^{n/2}\mu_{\tilde{\kappa}} \qquad (t > 0).$$

For any unimodular Lie subgroups $\mathrm{F} \subseteq \mathrm{E} \subseteq \mathrm{Sp}(\mathsf{W})$, the measure μ induces the left invariant Haar measure on E and a left invariant measure on the quotient E/F, assuming it does exist. We shall denote these induced measures also by μ.

Let $(\mathrm{G}, \mathrm{G}')$ be a reductive dual pair in $\mathrm{Sp}(\mathsf{W})$ (see [8] for the definition) and let $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$, $\mathfrak{g}' = \mathrm{Lie}(\mathrm{G}')$. We assume that $\mathfrak{g}$ and $\mathfrak{g}'$ are preserved by θ. Let H' be a Cartan subgroup of G' preserved by θ. We consider the Cartan decomposition of H' : $\mathrm{H}' = \mathrm{T}'\mathrm{A}'$ where T' (resp. A') is the compact (resp. split) part of H'. We consider the commutant A'' (resp. A''') of A' (resp. A'') in $\mathrm{Sp}(\mathsf{W})$. Then $(\mathrm{A}'', \mathrm{A}''')$ is a reductive dual pair of $\mathrm{Sp}(\mathsf{W})$, see [10]. Let $\mathsf{V}'_c = \{v \in \mathsf{V}' \,|\, a.v = v \ \forall\, a \in \mathrm{A}'\}$. There exists a unique complement V'_s of V'_c in V' such that the decomposition

$$\mathsf{V}' = \mathsf{V}'_c \oplus \mathsf{V}'_s$$

is preserved by H'. As $\mathrm{A}''' \subset \mathrm{H}'$, we may consider

$$\mathrm{A}'''_s = \{a \in \mathrm{A}''' \,|\, a_{|\mathsf{V}'_c} = \mathrm{id}\}.$$

Then $\mathrm{A}''' = \mathrm{A}_s'''$ if and only if $\mathsf{V}_c' = 0$ and $\mathrm{A}''' = \mathrm{A}_s''' \times \{\pm \mathrm{id}_{|\mathsf{V}_c'}\}$ otherwise. There exists a dense open subset $\mathsf{W}_{\mathrm{A}_s'''}$ of W such that the quotient $\mathrm{A}_s'''\backslash\mathsf{W}_{\mathrm{A}_s'''}$ is a smooth manifold. Define the measure μ on the quotient $\mathrm{A}_s''' \setminus \mathsf{W}_{\mathrm{A}_s'''}$ by

$$\int_{\mathsf{W}} f(w)\, dw = \int_{\mathrm{A}'''\backslash \mathsf{W}_{\mathrm{A}_s'''}} \int_{\mathrm{A}_s'''} f(aw)\, d\mu(a)\, d\mu(\mathrm{A}_s''' w).$$

Let

$$\chi_x(w) = \chi(\frac{1}{4}\langle xw, w\rangle), \qquad (x \in \mathfrak{sp}(\mathsf{W}),\ w \in \mathsf{W}).$$

Recall [10, p. 302] the Cauchy Harish-Chandra integral on the Lie algebra:

$$\widetilde{chc}_{\mathsf{W}}(x'+x) = \int_{\mathrm{A}_s'''\backslash \mathsf{W}_{\mathrm{A}_s'''}} \chi_{x'+x}(w)\, d\mu(\mathrm{A}_s''' w) \qquad (x' \in \mathfrak{h}'^{\mathrm{reg}},\ x \in \mathfrak{g}).$$

Definition 0.1. Define the normalized Cauchy Harish-Chandra integral by

$$chc_{\mathsf{W}}(x'+x) = \frac{1}{\mu(\mathrm{A}_s''' \setminus \mathrm{H}')}\widetilde{chc}_{\mathsf{W}}(x'+x) \qquad (x' \in \mathfrak{h}'^{\mathrm{reg}},\ x \in \mathfrak{g}).$$

Let $\widetilde{\mathrm{Sp}}(\mathsf{W})$ the connected two fold cover of $\mathrm{Sp}(\mathsf{W})$ and let $\mathcal{S}^*(\mathsf{W})$ the space of tempered distributions on W. Recall [10, Theorem 2.8] Howe's embedding

$$T : \widetilde{\mathrm{Sp}}(\mathsf{W}) \longrightarrow \mathcal{S}^*(\mathsf{W}).$$

For a subgroup P of $\mathrm{Sp}(\mathsf{W})$, let $\widetilde{\mathrm{P}}$ be the preimage of P in $\widetilde{\mathrm{Sp}}(\mathsf{W})$. We recall the definition of the Cauchy Harish-Chandra integral on the group and define an associated normalized integral.

Definition 0.2. [10, Definition 2.11] Let H' be a Cartan subgroup of G' and $x' \in \widetilde{\mathrm{H}}'^{\mathrm{reg}}$. Then

$$\widetilde{Chc}_{\mathsf{W}}(x'x) = \int_{\mathrm{A}_s'''\backslash \mathsf{W}_{\mathrm{A}_s'''}} T(x'x)(w) d\mu(\mathrm{A}_s''' w) \qquad (\, x \in \widetilde{\mathrm{G}}\,).$$

Define

$$Chc_{\mathsf{W}}(x'x) = \frac{1}{\mu(\mathrm{A}_s''' \setminus \mathrm{H}')}\widetilde{Chc}_{\mathsf{W}}(x'x) \qquad (\, x \in \widetilde{\mathrm{G}}\,).$$

Remark. Formally

$$\begin{aligned}
Chc_{\mathsf{W}}(x'x) &= \frac{1}{\mu(\mathrm{A}_s''' \setminus \mathrm{H}')} \int_{\mathrm{A}_s'''\setminus \mathsf{W}_{\mathrm{A}_s'''}} T(x'x)(w)\, d\mu(\mathrm{A}_s''' w) \\
&= \frac{1}{\mu(\mathrm{A}_s''' \setminus \mathrm{H}')} \int_{\mathrm{H}'\setminus \mathsf{W}_{\mathrm{A}_s'''}} \int_{\mathrm{A}_s'''\setminus \mathrm{H}'} T(x'x)(h'w)\, d\mu(\mathrm{A}_s''' h')\, d\mu(\mathrm{H}'w) \\
&= \int_{\mathrm{H}'\setminus \mathsf{W}_{\mathrm{A}_s'''}} T(x'x)(w)\, d\mu(\mathrm{H}'w) \\
&= \int_{\mathrm{G}'\setminus \mathsf{W}_{\mathrm{A}_s'''}} \int_{\mathrm{H}'\setminus \mathrm{G}'} T(x'x)(gw)\, d\mu(\mathrm{H}'g)\, d\mu(\mathrm{G}'w) \\
&= \int_{\mathrm{H}'\setminus \mathrm{G}'} \left(\int_{\mathrm{G}'\setminus \mathsf{W}_{\mathrm{A}_s'''}} T(g^{-1}x'gx)(w)\, d\mu(\mathrm{G}'w) \right) d\mu(\mathrm{H}'g).
\end{aligned}$$

Thus Chc_{W} looks like an orbital integral. We shall verify the corresponding precise statement in a forthcoming article.

Let

$$\widetilde{\mathrm{Sp}}(\mathsf{W}) \longrightarrow \mathrm{Sp}(\mathsf{W}), \quad x \longmapsto \underline{x} \tag{0.2}$$

be the canonical surjective map. Denote by $\tilde{1}$ the non-trivial element in the preimage of 1 in $\widetilde{\mathrm{Sp}}(\mathsf{W})$. Fix $\mathfrak{d} \in \widetilde{\mathrm{Sp}}(\mathsf{W})$ such that $\underline{\mathfrak{d}} = -1$.

Formulas for a dual pair of type II

Let $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$. Let V, V' be two finite dimensional left vector spaces over $\mathbb{D}$. On the real vector space $\mathsf{W} = \mathrm{Hom}(\mathsf{V}', \mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}, \mathsf{V}')$ define a symplectic form $\langle , \rangle$ by

$$\langle w, w' \rangle = \mathrm{tr}(xy') - \mathrm{tr}(yx') \ (w = (x,y),\, w' = (x', y') \in \mathsf{W}), \tag{0.3}$$

where $\mathrm{tr} = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}$. The groups $\mathrm{GL}(\mathsf{V})$ and $\mathrm{GL}(\mathsf{V}')$ act on W by the post-multiplication and pre-multiplication by the inverse, respectively. These actions preserve the symplectic form (0.3). The pair $(\mathrm{GL}(\mathsf{V}), \mathrm{GL}(\mathsf{V}'))$ is a dual pair in $\mathrm{Sp}(\mathsf{W})$ of type II, in the sense of Howe (see [8]). We assume that V' is a subspace of V and consider a direct sum decomposition

$$\mathsf{V} = \mathsf{V}' \oplus \mathsf{U}.$$

The above decomposition induces embeddings

$$\begin{aligned}
\mathrm{L} &= \mathrm{GL}(\mathsf{V}') \times \mathrm{GL}(\mathsf{U}) \subseteq \mathrm{GL}(\mathsf{V}), \\
\mathfrak{n} &= \mathrm{Hom}(\mathsf{U}, \mathsf{V}') \subseteq \mathfrak{gl}(\mathsf{V}).
\end{aligned}$$

Let $\mathfrak{l} = \mathrm{Lie}(\mathrm{L})$, and let K be the maximal compact subgroup of $\mathrm{GL}(\mathsf{V})$ fixed by θ. We assume that our Cartan involution θ preserves the subgroup $\mathrm{GL}(\mathsf{V}') \times \mathrm{GL}(\mathsf{U})$. Let H' be a Cartan subgroup of $\mathrm{GL}(\mathsf{V}')$. We may consider the group H' as a subgroup of $\mathrm{Sp}(\mathsf{W})$ in two different ways. We have:

$$\mathrm{H}' \subset \mathrm{GL}(\mathsf{V}') \subset^{\text{as a dual pair}} \mathrm{Sp}(\mathsf{W})$$

$$\mathrm{H}' \subset \mathrm{GL}(\mathsf{V}') \subset \mathrm{GL}(\mathsf{V}') \times \mathrm{GL}(\mathsf{U}) \subset \mathrm{GL}(\mathsf{V}) \subset^{\text{as a dual pair}} \mathrm{Sp}(\mathsf{W})$$

Let H'_1 (resp. H'_2) denote the group H' as a subgroup of $\mathrm{GL}(\mathsf{V}')$ with respect to the first (resp. second) inclusion above. Let $\mathfrak{h}'_i = \mathrm{Lie}(\mathrm{H}'_i)$. Denote by κ_i the restriction of the Killing form κ to $\mathfrak{h}'_i$. Then

$$\kappa_1(x,y) = 2\dim_{\mathbb{R}}(\mathsf{V})\ \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) \text{ for } x,y \in \mathfrak{h}_1,$$
$$\kappa_2(x,y) = 2\dim_{\mathbb{R}}(\mathsf{V}')\ \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) \text{ for } x,y \in \mathfrak{h}_2.$$

Let $\mathcal{S}(\mathfrak{gl}(\mathsf{V}))$ be the Schwartz space of $\mathfrak{gl}(\mathsf{V})$. For $\psi \in \mathcal{S}(\mathfrak{gl}(\mathsf{V}))$ define the following version of the Harish-Chandra transform of ψ

$$\psi^{\mathrm{K}}_{\mathfrak{n}}(z) = \frac{1}{\mu(\mathrm{K})}\int_{\mathrm{K}\times\mathfrak{n}} \psi(k.(z+n))d\mu(n)d\mu(k),$$

where $z \in \mathfrak{l}$ and $k.(z+n) = k(z+n)k^{-1}$.

Theorem 0.3. *Let* H' *be a Cartan subgroup of* $\mathrm{GL}(\mathsf{V}')$ *and let* $\psi \in \mathcal{S}(\mathfrak{gl}(\mathsf{V}))$*. Then, for any* $x' \in \mathfrak{h}'^{\mathrm{reg}}$*, we have*

$$\int_{\mathfrak{gl}(\mathsf{V})} \psi(x)\, chc_{\mathsf{W}}(x'+x)\, d\mu(x) = \frac{\sqrt{2}^{\dim_{\mathbb{R}}(\mathsf{W})}}{\sqrt{\dim_{\mathbb{D}}(\mathsf{V})}^{\dim_{\mathbb{R}}(\mathsf{V}')}}$$
$$\int_{\mathrm{GL}(\mathsf{V})/(\mathrm{H}'_2\times\mathrm{GL}(\mathsf{U}))}\int_{\mathfrak{gl}(\mathsf{U})} |\det(\mathrm{ad}(x'+y)_{\mathfrak{n}})|\psi(g.(x'+y))\, d\mu(y)\, d\mu(g(\mathrm{H}'_2\times\mathrm{GL}(\mathsf{U})).$$

Corollary 0.4. *We have the equality:*

$$\int_{\mathfrak{gl}(\mathsf{V})} \psi(x) chc_{\mathsf{W}}(x'+x) d\mu(x)$$
$$= \frac{\sqrt{2}^{\dim_{\mathbb{R}}(\mathsf{W})-\dim_{\mathbb{R}}(\mathfrak{n})}}{\sqrt{\dim_{\mathbb{D}}\mathsf{V}'}^{\dim_{\mathbb{R}}(\mathsf{V}')}\mu(\mathrm{K}\cap\mathrm{L})}\int_{\mathrm{GL}(\mathsf{V}')/\mathrm{H}'_1}\int_{\mathfrak{gl}(\mathsf{U})} \psi^{\mathrm{K}}_{\mathfrak{n}}(g.x'+y)d\mu(y)d\mu(g.\mathrm{H}'_1).$$

Let δ be the Dirac distribution on $\mathbb{R}$ supported at 0. For $s \neq 0$, we may consider the pull-back of δ by the function $\det(.)+s$. We denote this distribution by $\delta(\det(.)+s)$. We can then prove the existence of the limit $\lim_{s\to 0}\delta(\det(.)+s)$ in terms of distributions (cf. equality (4.6) of [10]). We denote this limit by $\delta\circ\det$. The proof of Theorem 0.3 will use the following lemma:

Lemma 0.5. *Let* $(\mathrm{G},\mathrm{G}') = (\mathrm{GL}_n(\mathbb{D}), \mathrm{GL}_1(\mathbb{D}))$ *with* $\mathbb{D} = \mathbb{R}$ *or* $\mathbb{C}$*. Then*

$$chc_{\mathsf{W}} = \frac{\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}}{\sqrt{\dim_{\mathbb{R}}\mathsf{W}}^{\dim_{\mathbb{R}}\mathsf{V}'}}\,\delta\circ\det.$$

Let $\mathcal{S}(\widetilde{\mathrm{GL}}(\mathsf{V}))$ be the Schwartz space of $\widetilde{\mathrm{GL}}(\mathsf{V})$ (see [9, p.450]). For $\psi \in \mathcal{S}(\widetilde{\mathrm{GL}}(\mathsf{V}))$ consider the following function of $x \in \widetilde{\mathrm{L}}$,

$$\psi^{\widetilde{\mathrm{L}}}(x) = |\det(\mathrm{Ad}(x)_{\mathfrak{n}})|\psi^{\widetilde{\mathrm{K}}}_{\widetilde{\mathrm{N}}}(x) \tag{0.4}$$

where

$$\psi_{\mathrm{N}}^{\widetilde{\mathrm{K}}}(x)=\int_{\widetilde{\mathrm{K}}\times\mathrm{N}}\psi(k.(xn))d\mu(n)d\mu(k).$$

The function $\psi^{\widetilde{\mathrm{L}}}$ is the Harish-Chandra transform of ψ (see [12] or Appendix A).

Theorem 0.6. *Let* H' *be a Cartan subgroup of* $\mathrm{GL}(\mathsf{V}')$, $x'\in\widetilde{\mathrm{H}}'^{\mathrm{reg}}$ *and* $\psi\in C_c^\infty(\widetilde{\mathrm{GL}}(\mathsf{V}))$. *Then*

$$\begin{aligned}&\int_{\widetilde{\mathrm{GL}}(\mathsf{V})}\psi(x)Chc_{\mathsf{W}}(x'x)d\mu(x)\\&=\frac{\sqrt{2}^{\dim_{\mathbb{R}}(\mathsf{W})-\dim_{\mathbb{R}}(\mathfrak{n})}}{\sqrt{\dim_{\mathbb{D}}\mathsf{V}'}^{\dim_{\mathbb{R}}(\mathsf{V}')}\mu(\mathrm{K}\cap\mathrm{L})}\epsilon(x')\int_{\mathrm{GL}(\mathsf{V}')/\mathrm{H}_1'}\int_{\widetilde{\mathrm{GL}}(\mathsf{U})}\epsilon(\mathfrak{d}x'y)\psi^{\widetilde{\mathrm{L}}}(g.(\mathfrak{d}x')y)d\mu(y)d\mu(g\mathrm{H}_1'),\end{aligned}$$

where the function ϵ *takes values in* $\{\pm1,\pm\mathrm{i}\}$ *and is defined in* [1], Section 2.

Formulas for a dual pair of type I

Let V,V' be two finite dimensional left vector spaces over $\mathbb{D}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ with non-degenerate forms $(,)$, $(,)'$- one hermitian and the other one skew-hermitian. Let $\mathsf{W}=\mathrm{Hom}(\mathsf{V}',\mathsf{V})$. Define a map

$$\begin{array}{ccc}\mathsf{W}&\longrightarrow&\mathrm{Hom}(\mathsf{V},\mathsf{V}')\\ w&\longmapsto&w^*\end{array}$$

by

$$(wv',v)=(v',w^*v)'\ \ (w\in\mathsf{W},\ v\in\mathsf{V},\ v'\in\mathsf{V}').$$

Define a symplectic form $\langle,\rangle$ on the real vector space W by

$$\langle w,w'\rangle=\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(w'^*w)\ \ (w,w'\in\mathsf{W}).$$

Let $\mathrm{G}\subset\mathrm{GL}(\mathsf{V})$ be the isometry group of the form $(\ ,\ )$ with the Lie algebra $\mathfrak{g}\subset\mathrm{End}(\mathsf{V})$. Similarly we have the isometry group $\mathrm{G}'\subset\mathrm{GL}(\mathsf{V}')$ of the form $(,)'$, with the Lie algebra $\mathfrak{g}'\subset\mathrm{End}(\mathsf{V}')$.

Recall the $\theta-$stable Cartan subgroup $\mathrm{H}'=\mathrm{T}'\mathrm{A}'\subseteq\mathrm{G}'$. Let $\mathsf{V}_c'\subseteq\mathsf{V}'$ be the subspace on which A' acts trivially, and let $\mathsf{V}_s'=\mathsf{V}_c'^{\perp}$ be the orthogonal complement of V_c' in V'. Then V_s' has a complete polarization

$$\mathsf{V}_s'=\mathsf{X}'\oplus\mathsf{Y}'\tag{0.5}$$

preserved by H'. We assume that V_s' is contained in V and that (0.5) is also a complete polarization with respect to the form $(\ ,\ )$. Then

$$\mathsf{V}=\mathsf{V}_s'\oplus\mathsf{U},\ \ \mathsf{U}=\mathsf{V}_s'^{\perp}.$$

The above decompositions induce embeddings:

$$\begin{aligned}&\mathrm{GL}(\mathsf{X}')\times\mathrm{G}(\mathsf{U})\subseteq\mathrm{G},\\&\mathfrak{n}'=\mathrm{Hom}(\mathsf{X}',\mathsf{V}_c')\oplus\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'\subseteq\mathfrak{g}',\\&\mathfrak{n}\ =\mathrm{Hom}(\mathsf{X}',\mathsf{U})\oplus\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}\subseteq\mathfrak{g}.\end{aligned}$$

Moreover, we have the following embeddings

$$\begin{aligned} &\mathrm{H}'_{|\mathsf{X}'} \subset \mathrm{G}' \subset^{\text{as a dual pair}} \mathrm{Sp}(\mathsf{W}),\\ &\mathrm{H}'_{|\mathsf{X}'} \subset \mathrm{GL}(\mathsf{X}') \times \mathrm{G}(\mathsf{U}) \subset \mathrm{G}. \end{aligned}$$

We denote $\mathrm{H}'_{1|\mathsf{X}'}$ (resp. $\mathrm{H}'_{2|\mathsf{X}'}$) the group $\mathrm{H}'_{|\mathsf{X}'}$ with respect to the first (resp. second) inclusion above. Let $\mathsf{W}_c = \mathrm{Hom}(\mathsf{V}'_c, \mathsf{U})$. We assume that the subgroup $\mathrm{GL}(\mathsf{X}') \times \mathrm{G}(\mathsf{U}) \subseteq \mathrm{G}$ is preserved by θ. It is convenient to introduce the following constants:

$$\begin{aligned} \gamma(\mathsf{V},\mathsf{V}',\mathsf{X}') &= \begin{cases} \dfrac{\sqrt{2}^{[\dim_{\mathbb{R}}(\mathsf{W})-\dim_{\mathbb{R}}(\mathsf{W}_c)+\dim_{\mathbb{R}}(\mathsf{X}')]}}{\sqrt{\dim_{\mathbb{D}}(\mathsf{V})}^{\dim_{\mathbb{R}}(\mathsf{X}')}} \sqrt{\dfrac{\dim_{\mathbb{D}}\mathsf{V}'}{\dim_{\mathbb{D}}\mathsf{V}'_s}}^{\dim_{\mathbb{R}}\mathrm{H}'} & \text{if } \mathfrak{h}' \text{ acts trivially on } \mathsf{W}_c,\\ \dfrac{\sqrt{2}^{[\dim_{\mathbb{R}}(\mathsf{W})-\dim_{\mathbb{R}}(\mathsf{W}_c)+\dim_{\mathbb{R}}(\mathsf{X}')]}}{\sqrt{\dim_{\mathbb{D}}(\mathsf{V})}^{\dim_{\mathbb{R}}(\mathsf{X}')}} \sqrt{\dfrac{\dim_{\mathbb{D}}(\mathsf{U})}{\dim_{\mathbb{D}}(\mathsf{V})}}^{\dim_{\mathbb{R}}(\mathrm{H}'|_{\mathsf{V}'_c})} & \text{otherwise.} \end{cases}\\ \eta(\mathsf{V},\mathsf{V}',\mathsf{X}') &= \begin{cases} \mu(\mathrm{K}\cap \mathrm{GL}(\mathsf{X}'))\sqrt{2}^{\dim_{\mathbb{R}}(\mathfrak{n})} & \text{if } \mathsf{U}=0,\\ \mu(\mathrm{K}\cap(\mathrm{GL}(\mathsf{X}')\times \mathrm{G}(\mathsf{U}))\sqrt{2}^{\dim_{\mathbb{R}}(\mathfrak{n})} & \text{otherwise.} \end{cases} \end{aligned} \tag{0.6}$$

Remark. Notice that $\mathfrak{h}'$ acts trivially on W_c if and only if $\mathsf{U} = 0$ or $\mathfrak{h}'$ acts trivially on V'_c. This second property is equivalent to $\mathrm{H}'_{|\mathsf{V}'_c}$ being finite.

Theorem 0.7. *Let $\psi \in \mathcal{S}(\mathfrak{g})$. If $\mathsf{V}'_c = (0)$ and $\mathsf{U} = (0)$, then*

$$\begin{aligned} &|\det(\mathrm{ad}\ x')_{\mathfrak{n}'}| \int_{\mathfrak{g}} \psi(x)\, chc_{\mathsf{W}}(x'+x)\, d\mu(x)\\ =\ &\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')|\det(\mathrm{ad}\ x')_{\mathfrak{n}}| \int_{\mathrm{G}/\mathrm{H}'_{2|\mathsf{X}'}} \psi(g\cdot x')\, d\mu(g\mathrm{H}'_{2|\mathsf{X}'}). \end{aligned}$$

If $\mathfrak{h}'$ acts trivially on W_c then

$$\begin{aligned} &|\det(\mathrm{ad}\, x')_{\mathfrak{n}'}| \int_{\mathfrak{g}} \psi(x)\, chc_{\mathsf{W}}(x'+x)\, d\mu(x)\\ &= \gamma(\mathsf{V},\mathsf{V}',\mathsf{X}') \int_{\mathrm{G}/(\mathrm{H}'_{2|\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))}\\ &\quad \int_{\mathfrak{g}(\mathsf{U})} |\det(\mathrm{ad}(x'+y))_{\mathfrak{n}}| \psi(g\cdot(x'_{|\mathsf{X}'}+y)) chc_{\mathsf{W}_c}(y)\, d\mu(y) d\mu(g(\mathrm{H}'_{2|\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))), \end{aligned}$$

where $chc_{\mathsf{W}_c} = 1$ if $\mathsf{W}_c = (0)$. Otherwise,

$$\begin{aligned} &|\det(\mathrm{ad}\, x')_{\mathfrak{n}'}| \int_{\mathfrak{g}} \psi(x)\, chc_{\mathsf{W}}(x'+x)\, d\mu(x)\\ &= \gamma(\mathsf{V},\mathsf{V}',\mathsf{X}') \int_{\mathrm{G}/(\mathrm{H}'_{2|\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))}\\ &\int_{\mathfrak{g}(\mathsf{U})} |\det(\mathrm{ad}(x'+y))_{\mathfrak{n}}| \psi(g\cdot(x'_{|\mathsf{X}'}+y))\, chc_{\mathsf{W}_c}(x'+y)\, d\mu(y) d\mu(g(\mathrm{H}'_{2|\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))). \end{aligned}$$

Corollary 0.8. *If* $\mathsf{V}'_c = (0)$ *and* $\mathsf{U} = (0)$, *then*

$$|\det(\operatorname{ad} x')_{\mathfrak{n}'}| \int_{\mathfrak{g}} \psi(x) chc_{\mathsf{W}}(x'+x) d\mu(x) = \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_{|\mathsf{X}'}} \psi_{\mathfrak{n}}^{\mathrm{K}}(g.x'_{|\mathsf{X}'}) d\mu(g\mathrm{H}'_{1|\mathsf{X}'}).$$

If $\mathfrak{h}'$ *acts trivially on* W_c, *then*

$$\begin{aligned} &|\det(\operatorname{ad} x')_{\mathfrak{n}'})| \int_{\mathfrak{g}} \psi(x) chc_{\mathsf{W}}(x'+x) d\mu(x) \\ &\quad = \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_{|\mathsf{X}'}} \int_{\mathfrak{g}(\mathsf{U})} \psi_{\mathfrak{n}}^{\mathrm{K}}(g.x'_{|\mathsf{X}'}+y) chc_{\mathsf{W}_c}(y) d\mu(y) d\mu(g\mathrm{H}'_{1|\mathsf{X}'}), \end{aligned}$$

where $chc_{\mathsf{W}_c} = 1$ *if* $\mathsf{W}_c = (0)$. *Otherwise*

$$\begin{aligned} &|\det(\operatorname{ad} x')_{\mathfrak{n}'})| \int_{\mathfrak{g}} \psi(x) chc_{\mathsf{W}}(x'+x) d\mu(x) \\ &= \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_{|\mathsf{X}'}} \int_{\mathfrak{g}(\mathsf{U})} \psi_{\mathfrak{n}}^{\mathrm{K}}(g.x'_{|\mathsf{X}'}+y) chc_{\mathsf{W}_c}(x'_{|\mathsf{V}'_c}+y) d\mu(y) d\mu(g\mathrm{H}'_{1|\mathsf{X}'}). \end{aligned}$$

Theorem 0.9. *Let* $\psi \in C_c^\infty(\widetilde{\mathrm{G}})$. *If* $\mathsf{V}'_c = (0)$ *and* $\mathsf{U} = (0)$, *then*

$$\begin{aligned} &|\det(\operatorname{Ad} \underline{x}'^{-1} - 1)_{\mathfrak{n}'}| \int_{\widetilde{\mathrm{G}}} \psi(x) Chc_{\mathsf{W}}(x'x) d\mu(x) \\ &\quad = \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \epsilon(\mathfrak{d}) |\det \operatorname{Ad}(\underline{x}')_{\mathfrak{n}'}|^{-\frac{1}{2}} \int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_{|\mathsf{X}'}} \psi^{\widetilde{\mathrm{L}}}(g.x'_{|\mathsf{X}'}) d\mu(g\mathrm{H}'_{1|\mathsf{X}'}) \end{aligned}$$

If $\mathfrak{h}'$ *acts trivially on* W_c, *then*

$$\begin{aligned} |\det(\operatorname{Ad} \underline{x}'^{-1} - 1)_{\mathfrak{n}'}| \int_{\widetilde{\mathrm{G}}} \psi(x) Chc_{\mathsf{W}}(x'x) d\mu(x) &= \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \epsilon(x'_s) |\det \operatorname{Ad}(\underline{x}')_{\mathfrak{n}'}|^{-\frac{1}{2}} \\ \int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_{|\mathsf{X}'}} \int_{\widetilde{\mathrm{G}}(\mathsf{U})} &\epsilon(\mathfrak{d} x'_s u_s) \psi^{\widetilde{\mathrm{L}}}(g.x'_{|\mathsf{X}'}+y) Chc_{\mathsf{W}_c}(y) d\mu(y) d\mu(g\mathrm{H}'_{1|\mathsf{X}'}), \end{aligned}$$

where $Chc_{\mathsf{W}_c} = 1$ *if* $\mathsf{W}_c = (0)$. *Otherwise,*

$$\begin{aligned} |\det(\operatorname{Ad} \underline{x}'^{-1} - 1)_{\mathfrak{n}'}| \int_{\widetilde{\mathrm{G}}} \psi(x) Chc_{\mathsf{W}}(x'x) d\mu(x) &= \frac{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \epsilon(x'_s) |\det \operatorname{Ad}(\underline{x}')_{\mathfrak{n}'}|^{-\frac{1}{2}} \\ \int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_{|\mathsf{X}'}} \int_{\widetilde{\mathrm{G}}(\mathsf{U})} &\epsilon(\mathfrak{d} x'_s y_s) \psi^{\widetilde{\mathrm{L}}}(\mathfrak{d} g.x'_{|\mathsf{X}'} y) Chc_{\mathsf{W}_c}(x'_{|\mathsf{V}'_c} y) d\mu(y) d\mu(g\mathrm{H}'_{1|\mathsf{X}'}), \end{aligned}$$

where $\psi^{\widetilde{\mathrm{L}}}$ *is the Harish-Chandra transform (cf. equality (0.4)).*

Properties of Chc for some dual pairs

We recall some notations used in [1]. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $\mathfrak{h} \subseteq \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Ψ be a positive root system for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let $\Psi^n \subseteq \Psi$ denote the subset of the non-compact roots and let $\Psi^c \subseteq \Psi$ denote the subset of the compact roots. Let Ψ^n_{st} be the family of strongly orthogonal subsets of Ψ^n. For each $S \in \Psi^n_{st}$ let $c(\mathcal{S}) \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ be the corresponding Cayley transform, [1, (1.6)], and let $\mathfrak{h}(\mathcal{S}) = \mathfrak{g} \cap c(\mathcal{S})(\mathfrak{h}_{\mathbb{C}})$ be the corresponding Cartan subalgebra. Let $\mathfrak{h}_{\mathcal{S}} = c(\mathcal{S})^{-1}\mathfrak{h}(\mathcal{S}) \subseteq \mathfrak{h}_{\mathbb{C}}$. Denote by $\Psi_{\mathcal{S},\mathbb{R}} \subseteq \Psi$ the set of real roots for $\mathfrak{h}(\mathcal{S})$, and let $\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}} \subseteq \Psi$ be the set of the non-compact imaginary roots.

If $\alpha \in \Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}$ and α is not strongly orthogonal to $\mathcal{S}$, then there is exactly one $\alpha' \in \mathcal{S}$ is not strongly orthogonal to α. Moreover, $\alpha + \alpha' \in \Psi^n$. Define

$$\mathcal{S} \vee \alpha = \begin{cases} (\mathcal{S} \setminus \alpha') \cup \{\alpha + \alpha'\} \cup \{\pm(\alpha - \alpha')\} \cap \Psi, \\ \text{if } \alpha \in \Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}} \text{ is not strongly orthogonal to } \mathcal{S}; \\ \mathcal{S} \cup \alpha, \\ \text{if } \alpha \in \Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}} \text{ is strongly orthogonal to } \mathcal{S}. \end{cases}$$

Here $\mathcal{S} \setminus \alpha' = \mathcal{S} \setminus \{\alpha'\}$ and $\mathcal{S} \cup \alpha = \mathcal{S} \cup \{\alpha\}$.

For $\alpha \in \Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}$ define the numbers $\epsilon(\Psi, \mathcal{S}, \alpha) = \pm 1$ and $d(\alpha) = 1$ or 2, as in Lemma 1.7 and Definition 1.8 in [1]. For a subset $A \subseteq \Psi \cup (-\Psi)$, let $\mathcal{A}(A) = \prod_{\alpha \in A} \frac{\alpha}{|\alpha|}$.

For $\alpha \in \Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}$ let $H_\alpha \in \mathfrak{h}_{\mathcal{S}\vee\alpha}$ be the corresponding coroot. If $\phi : \mathfrak{h}_{\mathcal{S}} \to \mathbb{C}$ is a function and $x \in \mathfrak{h}_{\mathcal{S}}$, let

$$\langle \phi \rangle_\alpha(x) = \langle \phi \rangle_{\mathrm{i}H_\alpha}(x) = \lim_{t \to 0+} \phi(x + \mathrm{i}tH_\alpha) - \lim_{t \to 0+} \phi(x - \mathrm{i}tH_\alpha),$$

whenever the limits exist.

For a function $\mathfrak{g}^{\mathrm{reg}} \to \mathbb{C}$ and a set $\mathcal{S} \in \Psi^n_{st}$ define

$$\mathcal{H}_{\mathcal{S}} f(x) = \mathcal{A}(\Psi_{\mathcal{S},\mathbb{R}})(x) \prod_{\alpha \in \Psi} \alpha(x) f(c_{\mathcal{S}}(x)) \qquad (x \in \mathfrak{h}_{\mathcal{S}}).$$

Let $\tilde{\mathcal{I}}(\mathfrak{g})$ denote the space of all the functions f satisfying the following three conditions:

(i) f is a smooth $\mathrm{G}-$invariant function on $\mathfrak{g}^{\mathrm{reg}}$,

(ii) All the derivatives of $\mathcal{H}_{\mathcal{S}} f$ are bounded,

(iii) For each $\alpha \in \Psi_{\mathcal{S},\mathbb{R}}$, each semiregular element $x \in \mathfrak{h}_{\mathcal{S}}$ with respect to α and each $w \in \mathrm{Sym}(\mathfrak{h}_{\emptyset,\mathbb{C}})$, we have

$$\langle \partial(w) \mathcal{H}_{\mathcal{S}} f \rangle_\alpha(x) = \mathrm{i}\epsilon(\Psi, \mathcal{S}, \alpha) d(\alpha) \partial(c(\mathcal{S} \vee \alpha)^{-1} c(\mathcal{S}) c(\alpha) w) \mathcal{H}_{\mathcal{S}\vee\alpha} f(x).$$

The space $\tilde{\mathcal{I}}(\mathfrak{g})$ contains the space of the (regular semisimple) orbital integrals of the Schwartz functions on $\mathfrak{g}$, [3, Section 3.1]. Let $\mathcal{I}(\widetilde{\mathrm{G}})$ be the space of the orbital integrals of the smooth compactly supported functions on $\widetilde{\mathrm{G}}$, [4, Section 3]. Similarly we have $\tilde{\mathcal{I}}(\mathfrak{g}')$ and $\mathcal{I}(\widetilde{\mathrm{G}}')$.

Theorem 0.10. *Let* $(\mathrm{G}, \mathrm{G}')$ *be one of the reductive dual pairs* $(\mathrm{U}_{p,q}, \mathrm{U}_{1,1})$, $(\mathrm{O}_{p,q}, \mathrm{Sp}_2(\mathbb{R}))$ *or* $(\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{1,2})$. *Then* $chc(\psi) \in \tilde{\mathcal{I}}(\mathfrak{g}')$ *for any* $\psi \in \mathcal{S}(\mathfrak{g})$.

Theorem 0.11. *Let* $(\mathrm{G}, \mathrm{G}')$ *be one of the reductive dual pairs* $(\mathrm{U}_{p,q}, \mathrm{U}_{1,1})$, $(\mathrm{O}_{p,q}, \mathrm{Sp}_2(\mathbb{R}))$ *or* $(\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{1,2})$. *Then* $Chc(\psi) \in \mathcal{I}(\widetilde{\mathrm{G}}')$ *for any* $\psi \in C_c^\infty(\widetilde{\mathrm{G}})$.

This is immediate from Theorem 0.10, because the support of $Chc(\psi)$ is compact module the compact part of any Cartan subgroup, and hence compact.

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Contents

1. Proof of Theorem 0.3 for the dual pairs $(\mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_1(\mathbb{R}))$

Here $\mathsf{W} = \mathbb{R}^n \oplus \mathbb{R}^{nT}$, $\langle (u,v),(u'v')\rangle = v'u - vu'$ and $J(u,v) = (v^T, -u^T)$, $(u, u' \in \mathbb{R}^n,\ v, v' \in \mathbb{R}^{nT})$. Furthermore, we have an embedding,

$$\mathbb{R} = \mathfrak{gl}_1(\mathbb{R}) \ni x \to \begin{pmatrix} xI_n & 0 \\ 0 & -xI_n \end{pmatrix} \in \mathfrak{sp}_{2n}(\mathbb{R}).$$

Thus,

$$\kappa(x,y) = 2nxy \qquad (x, y \in \mathbb{R} = \mathfrak{gl}_1(\mathbb{R})),$$

so that

$$d\mu(a) = \sqrt{2n}\, da/|a| \qquad (a \in \mathbb{R}^\times = \mathrm{GL}_1(\mathbb{R})),$$

where da is the usual Lebesgue measure on $\mathbb{R}$. Similarly, we have

$$\mathfrak{gl}_n(\mathbb{R}) \ni x \to \begin{pmatrix} x & 0 \\ 0 & -x^T \end{pmatrix} \in \mathfrak{sp}_{2n}(\mathbb{R}).$$

Thus,

$$\kappa(x, y) = 2\,\mathrm{tr}(xy) \qquad (x, y \in \mathfrak{gl}_n(\mathbb{R})),$$

so that

$$d\mu(a) = \sqrt{2}^{n^2}\, dg/|\det(g)|^n \qquad (g \in \mathrm{GL}_n(\mathbb{R})).$$

Let $e_1 = (1, 0, 0, 0, \cdots, 0)^T \in \mathbb{R}^n$. Then

$$\int_{\mathsf{W}} \phi(w)\, dw = \frac{|S^{n-1}|}{2\mu(\mathrm{O}_n)\sqrt{2n}} \int_{\mathrm{O}_n} \int_{\mathbb{R}^{nT}} \int_{\mathrm{GL}_1(\mathbb{R})} \phi(ke_1 a, a^{-1} v)\, dv\, d\mu(k)\, d\mu(a),$$

where $|S^{n-1}|$ is the Euclidean measure of the unit sphere in $\mathbb{R}^n$ and dv is the usual Lebesgue measure on $\mathbb{R}^{nT}$. Therefore,

$$\int_{\mathrm{GL}_1(\mathbb{R})\backslash \mathsf{W}} \phi(\mathrm{GL}_1(\mathbb{R})w)\, d\mu(\mathrm{GL}_1(\mathbb{R})w) = \frac{|S^{n-1}|}{2\mu(\mathrm{O}_n)\sqrt{2n}} \int_{\mathrm{O}_n} \int_{\mathbb{R}^{nT}} \phi(ke_1, v)\, dv\, d\mu(k).$$

For $x \in \mathfrak{gl}_n(\mathbb{R})$, we have

$$\begin{aligned}\chi_x(ke_1, v) &= \chi(\frac{1}{4}\langle x(ke_1, v), (ke_1, v)\rangle) \\ &= \chi(\frac{1}{4}\langle (xke_1, -vx), (ke_1, v)\rangle) = \chi(\frac{1}{2} vxke_1).\end{aligned}$$

Since $\mathrm{H}' = \mathrm{A}''' = \mathrm{A}_s''' = \mathrm{GL}_1(\mathbb{R})$, we have $chc_{\mathsf{W}}(\psi) = \widetilde{chc}_{\mathsf{W}}(\psi)$ and therefore

$$\begin{aligned} chc_{\mathsf{W}}(\psi) =& \frac{|S^{n-1}|\sqrt{2}^{n^2}}{2\mu(\mathrm{O}_n)\sqrt{2n}} \int_{\mathrm{O}_n} \int_{\mathbb{R}^{nT}} \int_{\mathfrak{gl}_n(\mathbb{R})} \psi(x)\chi(\frac{1}{2} vxke_1)\, dx\, dv\, d\mu(k) \\ =& \frac{|S^{n-1}|\sqrt{2}^{n^2} 2^n}{2\mu(\mathrm{O}_n)\sqrt{2n}} \int_{M_{n,n-1}(\mathbb{R})} \psi^{\mathrm{O}_n}(0, x)\, dx \\ =& \frac{|S^{n-1}|\sqrt{2}^{n^2 - n(n-1)} 2^n}{2\mu(\mathrm{O}_n)\sqrt{2n}} \int_{\mathbb{R}^{(n-1)T}} \int_{\mathfrak{gl}_{n-1}(\mathbb{R})} \psi^{\mathrm{O}_n}\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) |\det(x)|\, d\mu(x), \end{aligned}$$

where

$$\psi^{\mathrm{O}_n}(x) = \int_{\mathrm{O}_n} \psi(k.x)\, d\mu(k).$$

Let $e_2 = (0, 1, 0, 0, \cdots, 0)^T$, $e_3 = (0, 0, 1, 0, \cdots, 0)^T, \cdots$, $e_n = (0, 0, 0, \cdots, 0, 1)^T$. Set $\mathsf{V}' = \mathbb{R}e_1$, $\mathsf{U} = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \cdots \oplus \mathbb{R}e_n$. Then $\mathrm{Hom}(\mathsf{U}, \mathsf{V}') = \mathfrak{n} \subseteq \mathfrak{gl}_n(\mathbb{R})$. Let $\mathrm{K} = \mathrm{O}_n$ and let $\mathrm{N} = \exp(\mathfrak{n})$. Our computation above shows that

$$chc_{\mathsf{W}}(\psi) = \frac{|S^{n-1}|\sqrt{2}^{n^2 - n(n-1)} 2^n}{2\mu(\mathrm{O}_n)\sqrt{2n}} \int_{\mathrm{N}} \int_{\mathfrak{gl}(\mathsf{U})} \psi^{\mathrm{K}}(n.x)|\det(\mathrm{ad}(x)_{\mathfrak{n}})|\, d\mu(x)\, d\mu(n).$$

Notice that, since

$$\mathfrak{so}_n \ni z = \begin{pmatrix} 0 & -y^T \\ y & 0 \end{pmatrix} \to ze_1 = \begin{pmatrix} 0 \\ y \end{pmatrix} \ni \mathbb{R}^n,$$

we have

$$\kappa(z,z) = 2\,\mathrm{tr}(zz^T) = 4y^T y.$$

Hence,

$$\frac{\mu(\mathrm{O}_n)}{\frac{1}{2}\mu(\mathrm{O}_1 \times \mathrm{O}_{n-1})} = \mu(\mathrm{K}/Stab_{\mathrm{K}}(e_1)) = 2^{n-1}|S^{n-1}|.$$

We may compute $|S^{n-1}|$ from this formula, and obtain the following equation,

$$chc_{\mathsf{W}}(\psi) = \frac{2^n}{\sqrt{n}}\frac{1}{\mu(\mathrm{O}_1 \times \mathrm{O}_{n-1})\sqrt{2}^{(n-1)}}\int_{\mathrm{N}}\int_{\mathfrak{gl}(\mathsf{U})}\psi^{\mathrm{K}}(n.x)|\det(\mathrm{ad}(x)_{\mathfrak{n}})|\,d\mu(x)\,d\mu(n).$$

Let $\mathrm{L} = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_{n-1}(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$. Then, by Proposition A.1, we have

$$\int_{\mathrm{GL}_n(\mathbb{R})}\phi(g)\,d\mu(g) = \frac{1}{\mu(\mathrm{K}\cap\mathrm{L})\sqrt{2}^{(n-1)}}\int_{\mathrm{N}}\int_{\mathrm{K}}\int_{\mathrm{L}}\phi(knl)\,d\mu(l)\,d\mu(k)\,d\mu(n).$$

Thus,

$$chc_{\mathsf{W}}(\psi) = \frac{2^n}{\sqrt{n}}\int_{\mathrm{GL}_n(\mathbb{R})/\mathrm{L}}\int_{\mathfrak{gl}(\mathsf{U})}\psi(g\cdot y)|\det(\mathrm{ad}(y)_{\mathfrak{n}})|\,d\mu(y)\,d\mu(g\mathrm{L}).$$

Therefore, in terms of Theorem 0.3 and the identification $\mathbb{R} = \mathfrak{gl}(\mathsf{V}')$,

$$\begin{aligned}
&\int_{\mathfrak{gl}(V)}\psi(x)chc_{\mathsf{W}}(x'+x)\,d\mu(x) = \int_{\mathfrak{gl}(V)}\psi(x+x'id_{\mathsf{V}})chc_{\mathsf{W}}(x)\,d\mu(x)\\
= &\ \frac{2^n}{\sqrt{n}}\int_{\mathrm{GL}_n(\mathbb{R})/\mathrm{L}}\int_{\mathfrak{gl}(\mathsf{U})}\psi(g.y+x'id_V)|det(ad\,y)_{\mathfrak{n}}|\,d\mu(y)\,d\mu(g\mathrm{L})\\
= &\ \frac{2^n}{\sqrt{n}}\int_{\mathrm{GL}_n(\mathbb{R})/\mathrm{L}}\int_{\mathfrak{gl}(\mathsf{U})}\psi(g.(y+x'id_V))|det(ad\,y)_{\mathfrak{n}}|\,d\mu(y)\,d\mu(g\mathrm{L})\\
= &\ \frac{2^n}{\sqrt{n}}\int_{\mathrm{GL}_n(\mathbb{R})/\mathrm{L}}\int_{\mathfrak{gl}(\mathsf{U})}\psi(g.(y+x'id_V))|det(ad(y-x'id_{\mathsf{U}})_{\mathfrak{n}}|\,d\mu(y)\,d\mu(g\mathrm{L})\\
= &\ \frac{2^n}{\sqrt{n}}\int_{\mathrm{GL}_n(\mathbb{R})/\mathrm{L}}\int_{\mathfrak{gl}(\mathsf{U})}\psi(g.(x'+y))|det(ad(x'+y)_{\mathfrak{n}}|\,d\mu(y)\,d\mu(g\mathrm{L}).
\end{aligned}$$

This verifies Theorem 0.3 for our pair. ■

Proof. [Proof of Lemma 0.5 for $\mathbb{D} = \mathbb{R}$.] Let $\mathbb{R}$ be equipped with the bilinear form $(x,y) \to xy$, $(x,y \in \mathbb{R})$, and let dx be the corresponding Lebesgue measure, as above. We shall consider the Dirac delta δ as a generalized function by

$$\int_{\mathbb{R}} f(x)\delta(x)\,dx = f(0).$$

Let $\mathrm{G} = \mathrm{GL}_n(\mathbb{R})$ and let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$. For any $\psi \in \mathcal{D}(\mathfrak{g})$ supported in the set of $x \in \mathfrak{g}^{\mathrm{reg}}$ such that $\det'(x) \neq 0$, we have $\int_{\mathfrak{g}} \psi(x)\delta(\det(x))\, d\mu(x)$

$$= \sum \frac{1}{|W(\mathrm{H})|} \int_{\mathfrak{h}} |\det(\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}})| \delta(\det(x)) \int_{\mathrm{G/H}} \psi(g \cdot x)\, d\mu(g\mathrm{H})\, d\mu(x),$$

where the summation is a maximal family of mutually non-conjugate Cartan subgroups $\mathrm{H} \subseteq \mathrm{G}$, and $|W(\mathrm{H})|$ is the cardinality of the Weyl group $W(\mathrm{H})$. Up to conjugation, only the terms corresponding to the Cartan subgroups of the form $\mathrm{H} = \mathrm{GL}(\mathsf{V}') \times \mathrm{H}_{\mathsf{U}}$, where $\mathrm{H}_{\mathsf{U}} \subseteq \mathrm{GL}(\mathsf{U})$, may be non-zero. A term like that will occur $|W(\mathrm{H})|/|W(\mathrm{H}_{\mathsf{U}})|$ times. Hence,

$$\begin{aligned}
&\int_{\mathfrak{g}} \psi(x)\delta(\det(x))\, d\mu(x) \\
&= \sum \frac{1}{|W(\mathrm{H}_{\mathsf{U}})|} \int_{\mathfrak{gl}(\mathsf{V}')} \int_{\mathfrak{h}_{\mathsf{U}}} |\det(\mathrm{ad}(x_1 + y))_{\mathfrak{g}/\mathfrak{h}}| \delta(x_1 \det(y)) \\
&\qquad\qquad \int_{\mathrm{G/H}} \psi(g \cdot (x_1 + y))\, d\mu(g\mathrm{H})\, d\mu(y)\, d\mu(x_1) \\
&= \sum \frac{1}{|W(\mathrm{H}_{\mathsf{U}})|} \int_{\mathfrak{gl}(\mathsf{V}')} \delta(x_1) \int_{\mathfrak{h}_{\mathsf{U}}} |\det(\mathrm{ad}(x_1 + y))_{\mathfrak{g}/\mathfrak{h}}| |\det(y)|^{-1} \\
&\qquad\qquad \int_{\mathrm{G/H}} \psi(g \cdot (x_1 + y))\, d\mu(g\mathrm{H})\, d\mu(y)\, d\mu(x_1).
\end{aligned}$$

Since $d\mu(x_1) = \sqrt{2} dx_1$, the above is equal to

$$\begin{aligned}
&\sqrt{2} \sum \frac{1}{|W(\mathrm{H}_{\mathsf{U}})|} \int_{\mathfrak{h}_{\mathsf{U}}} |\det(\mathrm{ad}(y)_{\mathfrak{g}/\mathfrak{h}})| |\det(y)|^{-1} \int_{\mathrm{G/H}} \psi(g \cdot y)\, d\mu(g\mathrm{H})\, d\mu(y) \\
&= \sqrt{2} \int_{\mathrm{G/L}} \sum \frac{1}{|W(\mathrm{H}_{\mathsf{U}})|} \int_{\mathfrak{h}_{\mathsf{U}}} |\det(\mathrm{ad}(y)_{\mathfrak{g}/\mathfrak{h}})| |\det(y)| \\
&\qquad\qquad \int_{\mathrm{L}/\mathrm{GL}(\mathsf{V}') \times \mathrm{H}_{\mathsf{U}}} \psi(gl \cdot y)\, d\mu(l\, \mathrm{GL}(\mathsf{V}') \times \mathrm{H}_{\mathsf{U}})\, d\mu(y)\, d\mu(g\mathrm{L}) \\
&= \sqrt{2} \int_{\mathrm{G/L}} \int_{\mathfrak{gl}(\mathsf{U})} |\det(y)| \psi(g \cdot y)\, d\mu(y)\, d\mu(g\mathrm{L}),
\end{aligned}$$

where, as before $\mathrm{L} = \mathrm{GL}(\mathsf{V}') \times \mathrm{GL}(\mathsf{U})$. Thus

$$\int_{\mathfrak{g}} \psi(x)\delta(\det(x))\, d\mu(x) = \sqrt{2} \int_{\mathrm{G/L}} \int_{\mathfrak{gl}(\mathsf{U})} |\det(\mathrm{ad}(y))_{\mathfrak{n}}| \psi(g \cdot y)\, d\mu(y)\, d\mu(g\mathrm{L}).$$

We extend the measure $\delta \circ det$ by zero beyond the indicated subset of $\mathfrak{g}$. Then

$$chc_{\mathsf{W}} = \frac{2^n}{\sqrt{2n}}\, \delta \circ det = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\sqrt{\dim_{\mathbb{R}} \mathsf{W}}^{\dim_{\mathbb{R}} \mathsf{V}'}}\, \delta \circ \det . \tag{1.1}$$

■

2. Proof of Theorem 0.3 for the dual pairs $(\mathbf{GL}_n(\mathbb{C}), \mathbf{GL}_1(\mathbb{C}))$

Here $\mathsf{W} = \mathbb{C}^n \oplus \mathbb{C}^{nT}$, $\langle (u,v),(u'v')\rangle = Re(v'u - vu')$ and $J(u,v) = (\overline{v}^T, -\overline{u}^T)$, $(u, u' \in \mathbb{C}^n,\ v, v' \in \mathbb{C}^{nT})$. Furthermore, we have an embedding,

$$\mathbb{C} = \mathfrak{gl}_1(\mathbb{C}) \ni x \to \begin{pmatrix} xI_n & 0 \\ 0 & -xI_n \end{pmatrix} \in \mathfrak{sp}_{2n}(\mathbb{C}) \subseteq \mathfrak{sp}_{4n}(\mathbb{R}).$$

Thus,

$$\kappa(x,y) = 4n\, Re(xy) \qquad (x, y \in \mathbb{C} = \mathfrak{gl}_1(\mathbb{C})),$$

so that

$$d\mu(a) = \sqrt{4n}\, da/|a| \qquad (a \in \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})),$$

where da is the usual Lebesgue measure on $\mathbb{C}$. Similarly, we have

$$\mathfrak{gl}_n(\mathbb{C}) \ni x \to \begin{pmatrix} x & 0 \\ 0 & -x^T \end{pmatrix} \in \mathfrak{sp}_{2n}(\mathbb{C}) \subseteq \mathfrak{sp}_{4n}(\mathbb{R}).$$

Thus,

$$\kappa(x,y) = 4\, Re\, \mathrm{tr}(xy) \qquad (x, y \in \mathfrak{gl}_n(\mathbb{C})).$$

Since $\dim_{\mathbb{R}}(\mathfrak{gl}_n(\mathbb{C})) = 2n^2$,

$$d\mu(a) = 4^{n^2}\, dg/|\det(g)|^{2n} \qquad (g \in \mathrm{GL}_n(\mathbb{C})).$$

Let $e_1 = (1,0,0,0,\cdots,0)^T \in \mathbb{C}^n$. Then

$$\begin{aligned}\int_{\mathsf{W}} \phi(w)\, dw =& \frac{|S^{2n-1}|}{2\mu(\mathrm{U}_n)} \int_{\mathrm{U}_n} \int_{\mathbb{C}^{nT}} \int_{\mathbb{C}^\times} \phi(ke_1 a, a^{-1}v)\, dv\, d\mu(k)\, da/|a|,\\ =& \frac{|S^{2n-1}|}{2\mu(\mathrm{U}_n)\sqrt{4n}} \int_{U_n} \int_{\mathbb{C}^{nT}} \int_{\mathrm{GL}_1(\mathbb{C})} \phi(ke_1 a, a^{-1}v)\, dv\, d\mu(k)\, d\mu(a).\end{aligned}$$

where $|S^{2n-1}|$ is the Euclidean measure of the unit sphere in $\mathbb{C}^n$ and dv is the usual Lebesgue measure on $\mathbb{C}^{nT}$. Therefore,

$$\int_{\mathrm{GL}_1(\mathbb{R})\backslash \mathsf{W}} \phi(\mathrm{GL}_1(\mathbb{R})w)\, d\mu(\mathrm{GL}_1(\mathbb{R})w) = \frac{|S^{2n-1}|}{2\mu(\mathrm{U}_n)\sqrt{4n}} \int_{\mathrm{U}_n} \int_{\mathbb{C}^{nT}} \phi(ke_1, v)\, dv\, d\mu(k).$$

For $x \in \mathfrak{gl}_n(\mathbb{C})$, we have

$$\chi_x(ke_1, v) = \chi(\frac{1}{4}\langle x(ke_1,v),(ke_1,v)\rangle) = \chi(\frac{1}{2}\, Re(vxke_1)).$$

Hence,

$$\begin{aligned} ch c_{\mathsf{W}}(\psi) =& \frac{|S^{2n-1}| 4^{n^2}}{2\mu(\mathrm{U}_n)\sqrt{4n}} \int_{\mathrm{U}_n} \int_{\mathbb{C}^{nT}} \int_{\mathfrak{gl}_n(\mathbb{C})} \psi(x)\chi(\frac{1}{2}\, Re(vxke_1))\, dx\, dv\, d\mu(k)\\ =& \frac{|S^{2n-1}| 4^{n^2} 4^n}{2\mu(\mathrm{U}_n)\sqrt{4n}} \int_{M_{n,n-1}(\mathbb{C})} \psi^{\mathrm{U}_n}(0,x)\, dx\\ =& \frac{|S^{2n-1}| 4^{n^2 - n(n-1)} 4^n}{2\mu(\mathrm{U}_n)\sqrt{4n}} \int_{\mathbb{C}^{(n-1)T}} \int_{\mathfrak{gl}_{n-1}(\mathbb{C})} \psi^{\mathrm{U}_n}\left(\begin{pmatrix} 1 & y \\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) |\det(x)|^2\, d\mu(x),\end{aligned}$$

where

$$\psi^{\mathrm{U}_n}(x) = \int_{\mathrm{U}_n} \psi(k.x)\, d\mu(k).$$

Let $e_2 = (0,1,0,0,\cdots,0)^T$, $e_3 = (0,0,1,0,\cdots,0)^T, \cdots$, $e_n = (0,0,0,\cdots,0,1)^T$. Set $\mathsf{V}' = \mathbb{C}e_1$, $\mathsf{U} = \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \cdots \oplus \mathbb{C}e_n$. Then $\mathfrak{n} \subseteq \mathfrak{gl}_n(\mathbb{C})$. Let $\mathrm{K} = U_n$ and let $\mathrm{N} = \exp(\mathfrak{n})$. Our computation above shows that

$$\widetilde{chc}_{\mathsf{W}}(\psi) = \frac{|S^{2n-1}| 4^{n^2-n(n-1)} 4^n}{2\mu(\mathrm{U}_n)\sqrt{4n}} \int_{\mathrm{N}} \int_{\mathfrak{gl}(\mathsf{U})} \psi^{\mathrm{K}}(n.x) |\det(\mathrm{ad}(x)_{\mathfrak{n}})|^2\, d\mu(x)\, d\mu(n).$$

Notice that since

$$\mathfrak{u}_n \ni z = \begin{pmatrix} \mathrm{i}t & -\overline{y}^T \\ y & 0 \end{pmatrix} \to ze_1 = \begin{pmatrix} \mathrm{i}t \\ y \end{pmatrix} \ni \mathbb{C}^n,$$

we have

$$\tilde{\kappa}(z,z) = Re\,\mathrm{tr}(z\overline{z}^T) = 4\, Re\,\mathrm{tr} \begin{pmatrix} \mathrm{i}t & -\overline{y}^T \\ y & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{i}t & \overline{y}^T \\ -y & 0 \end{pmatrix} = 4(t^2 + 2\overline{y}^T y).$$

Hence,

$$\frac{\mu(\mathrm{U}_n)\mu(\mathrm{U}_1)}{\mu(\mathrm{U}_1 \times \mathrm{U}_{n-1})} = \mu(\mathrm{K}/Stab_{\mathrm{K}}(e_1)) = \sqrt{4}^{2n-1}\sqrt{2}^{2n-2}|S^{2n-1}| = 2^{2n-1}2^{n-1}|S^{2n-1}|.$$

We may compute $|S^{2n-1}|$ from this formula, and obtain the following equation,

$$\widetilde{chc}_{\mathsf{W}}(\psi) = \frac{2^n}{\sqrt{n}} \frac{\mu(\mathrm{U}_1)}{\mu(\mathrm{U}_1 \times \mathrm{U}_{n-1})} \int_{\mathrm{N}} \int_{\mathfrak{gl}(\mathsf{U})} \psi^{\mathrm{K}}(n.x) |\det(\mathrm{ad}(x)_{\mathfrak{n}})|^2\, d\mu(x)\, d\mu(n).$$

Let $\mathrm{L} = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_{n-1}(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$. Then according to Proposition A.1, we have

$$\int_{\mathrm{GL}_n(\mathbb{C})} \phi(g)\, d\mu(g) = \frac{1}{\mu(\mathrm{K} \cap \mathrm{L})\sqrt{2}^{2n-2}} \int_{\mathrm{K}} \int_{\mathrm{N}} \int_{\mathrm{L}} \phi(knl)\, d\mu(l)\, d\mu(n)\, d\mu(k).$$

Therefore,

$$\begin{aligned}\widetilde{chc}_{\mathsf{W}}(\psi) = {} & \frac{2^n}{\sqrt{n}} \frac{\mu(\mathrm{U}_1)}{\mu(\mathrm{U}_1 \times \mathrm{U}_{n-1})} \mu(\mathrm{K} \cap \mathrm{L})\sqrt{2}^{2n-2} \\ & \int_{\mathrm{GL}_n(\mathbb{C})/\mathrm{L}} \int_{\mathfrak{gl}(\mathsf{U})} \psi(g \cdot y) |\det(\mathrm{ad}(y)_{\mathfrak{n}})|^2\, d\mu(y)\, d\mu(g\mathrm{L}).\end{aligned}$$

In our case $\mathrm{H}' = \mathrm{G}' = \mathrm{GL}_1(\mathbb{C})$ and the group $\mathrm{A}_s''' = \mathbb{R}^*$. Hence, $\mu(\mathrm{H}'/\mathrm{A}''') = 2\sqrt{n}\pi$. Notice that U_1 in the preceding formula of $\widetilde{chc}_{\mathsf{W}}$ is considered as a subgroup of $\mathrm{U}_1 \times \mathrm{U}_{n-1} \subset \mathrm{GL}_n(\mathbb{C})$ thus $\kappa_{\mathfrak{u}_1 \times \mathfrak{u}_1}(x,y) = 4\, Re(xy)$ and $\mu(\mathrm{U}_1) = 4\pi$. Hence

$$\frac{1}{\mu(\mathrm{H}'/\mathrm{A}''')} \frac{2^n}{\sqrt{n}} \frac{\mu(\mathrm{U}_1)}{\mu(\mathrm{U}_1 \times \mathrm{U}_{n-1})} \mu(\mathrm{K} \cap \mathrm{L})\sqrt{2}^{2n-2} = \frac{2^{2n-1}2\pi}{\sqrt{n}\sqrt{n}\pi} = \frac{4^n}{n}.$$

Therefore,

$$chc_{\mathsf{W}}(\psi) = \frac{4^n}{n} \int_{\mathrm{GL}_n(\mathbb{C})/\mathrm{L}} \int_{\mathfrak{gl}(\mathsf{U})} \psi(g \cdot y) |\det(\mathrm{ad}\ y)_{\mathfrak{n}}|^2\, d\mu(y)\, d\mu(g\mathrm{L}).$$

This verifies Theorem 0.3 for our pair. ■

Proof. [Proof of Lemma 0.5 for $\mathbb{D}=\mathbb{C}$.] Let $\mathbb{C}$ be equipped with the bilinear form $(x,y)\to Re\, xy$, $(x,y\in\mathbb{C})$, and let dx be the corresponding Lebesgue measure, as above. We shall consider the Dirac delta δ as a generalized function by

$$\int_{\mathbb{C}} f(x)\delta(x)\,dx = f(0).$$

Let $\mathrm{G}=\mathrm{GL}_n(\mathbb{C})$ and let $\mathfrak{g}=\mathfrak{gl}_n(\mathbb{C})$. For any $\psi\in C_c(\mathfrak{g})$ supported in the set of $x\in\mathfrak{g}$ which are regular, semisimple and such that $\det'(x)\neq 0$, we have

$$\begin{aligned}&\int_{\mathfrak{g}}\psi(x)\delta(\det(x))\,d\mu(x)\\ &= \sum\frac{1}{|W(\mathrm{H})|}\int_{\mathfrak{h}}|\det(\mathrm{ad}\ x)_{\mathfrak{g}/\mathfrak{h}}|^2\delta(\det(x))\int_{\mathrm{G/H}}\psi(g\cdot x)\,d\mu(g\mathrm{H})\,d\mu(x),\end{aligned}$$

where $\mathfrak{g}/\mathfrak{h}$ is viewed as a real vector space, the summation is a maximal family of mutually non-conjugate Cartan subgroups $\mathrm{H}\subseteq\mathrm{G}$, and $|W(\mathrm{H})|$ is the cardinality of the Weyl group $W(\mathrm{H})$. Up to conjugation, only the terms corresponding to the Cartan subgroups of the form $\mathrm{H}=\mathrm{GL}(\mathsf{V}')\times\mathrm{H}_{\mathsf{U}}$, where $\mathrm{H}_{\mathsf{U}}\subseteq\mathrm{GL}(\mathsf{U})$, may be non-zero. A term like that will occur $|W(\mathrm{H})|/|W(\mathrm{H}_{\mathsf{U}})|$ times. Hence,

$$\begin{aligned}&\int_{\mathfrak{g}}\psi(x)\delta(\det(x))\,d\mu(x)\\ &=\sum\frac{1}{|W(\mathrm{H}_{\mathsf{U}})|}\int_{\mathfrak{gl}(\mathsf{V}')}\int_{\mathfrak{h}_{\mathsf{U}}}|\det(\mathrm{ad}(x_1+y)_{\mathfrak{g}/\mathfrak{h}})|^2\delta(x_1\det(y))\\ &\qquad\qquad\int_{\mathrm{G/H}}\psi(g\cdot(x_1+y))\,d\mu(g\mathrm{H})\,d\mu(y)\,d\mu(x_1)\\ &=\sum\frac{1}{|W(\mathrm{H}_{\mathsf{U}})|}\int_{\mathfrak{gl}(\mathsf{V}')}\delta(x_1)\int_{\mathfrak{h}_{\mathsf{U}}}|\det(\mathrm{ad}(x_1+y))_{\mathfrak{g}/\mathfrak{h}}|^2|\det(y)|^{-2}\\ &\qquad\qquad\int_{\mathrm{G/H}}\psi(g\cdot(x_1+y))\,d\mu(g\mathrm{H})\,d\mu(y)\,d\mu(x_1).\end{aligned}$$

Since $d\mu(x_1)=4\,dx_1$, the above is equal to

$$\begin{aligned}&4\sum\frac{1}{|W(\mathrm{H}_{\mathsf{U}})|}\int_{\mathfrak{h}_{\mathsf{U}}}|\det(\mathrm{ad}(y))_{\mathfrak{g}/\mathfrak{h}}|^2|\det(y)|^{-2}\int_{\mathrm{G/H}}\psi(g\cdot y)\,d\mu(g\mathrm{H})\,d\mu(y)\\ &= 4\int_{\mathrm{G/L}}\sum\frac{1}{|W(\mathrm{H}_{\mathsf{U}})|}\int_{\mathfrak{h}_{\mathsf{U}}}|\det(\mathrm{ad}(y))_{\mathfrak{g}/\mathfrak{h}}|^2|\det(y)|^2\\ &\quad\int_{\mathrm{L/GL}(\mathsf{V}')\times\mathrm{H}_{\mathsf{U}}}\psi(gl\cdot y)\,d\mu(l\,\mathrm{GL}(\mathsf{V}')\times\mathrm{H}_{\mathsf{U}})\,d\mu(y)\,d\mu(g\mathrm{L})\\ &= 4\int_{\mathrm{G/L}}\int_{\mathfrak{gl}(\mathsf{U})}|\det(y)|^2\psi(g\cdot y)\,d\mu(y)\,d\mu(g\mathrm{L}),\end{aligned}$$

where, as before $\mathrm{L}=\mathrm{GL}(\mathsf{V}')\times\mathrm{GL}(\mathsf{U})$. Thus

$$\int_{\mathfrak{g}}\psi(x)\delta(\det(x))\,d\mu(x)=4\int_{\mathrm{G/L}}\int_{\mathfrak{gl}(\mathsf{U})}|\det(\mathrm{ad}(y))_{\mathfrak{n}}|^2\psi(g\cdot y)\,d\mu(y)\,d\mu(g\mathrm{L}).$$

We extend the measure $\delta \circ det$ by zero beyond the indicated subset of $\mathfrak{g}$. Then, as in the case $\mathbb{D} = \mathbb{R}$,

$$chc_{\mathsf{W}} = \frac{4^n}{4n}\,\delta \circ \det = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\sqrt{\dim_{\mathbb{R}} \mathsf{W}}^{\dim_{\mathbb{R}} \mathsf{V}'}}\,\delta \circ \det\,. \tag{2.1}$$

■

3. Proof of Theorem 0.3 for the dual pairs $(\mathbf{GL}_n(\mathbb{D}), \mathbf{GL}_{n'}(\mathbb{D}))$, $\mathbb{D} = \mathbb{R}$ or $\mathbb{D} = \mathbb{C}$

An element $x \in \mathfrak{g}$, or rather the pair (x, V), is called decomposable if and only if there are two non-zero subspaces $U', U'' \subseteq V$, preserved by x such that $V = U' \oplus U''$. In this case we say that (x, V) is the direct sum of the elements (x, U') and (x, U''). The element x, or (x, V), is called indecomposable if and only if (x, V) is not decomposable. If x is semisimple then (x, V) decomposes into a direct sum of indecomposables. Two elements (x, V) and (y, U) are called similar if and only if there is an invertible linear map $g \in \mathrm{Hom}(U, V)$ such that $y = g^{-1}xg$. A direct sum of indecomposable elements is called isotypic if and only if all the indecomposable components are mutually similar. Two isotypic components are of the same type if and only if the corresponding indecomposable elements are similar.

Let x' be a regular element in a fixed Cartan subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}' = \mathfrak{gl}(\mathsf{V}')$. Let

$$\mathsf{V}' = \mathsf{V}'_1 \oplus \mathsf{V}'_2 \oplus \cdots \oplus \mathsf{V}'_m$$

be the decomposition into $x'-$isotypic components. Recall the symplectic space

$$\mathsf{W} = \mathrm{Hom}(\mathsf{V}', \mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}, \mathsf{V}').$$

For $j = 1, 2, \cdots, m$ let

$$\mathsf{W}_j = \mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}, \mathsf{V}'_j),$$
$$c_j = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_j}}{\sqrt{\dim_{\mathbb{R}} \mathsf{W}_j}^{\dim_{\mathbb{R}} \mathsf{V}'_j}}.$$

Since

$$\mathsf{W} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_m$$

is an orthogonal decomposition, the subspaces

$$\mathfrak{sp}(\mathsf{W}_j),\ \mathfrak{sp}(\mathsf{W}_k) \qquad (j \neq k)$$

are mutually orthogonal with respect to the form κ. Hence, the restriction of κ to each $\mathfrak{sp}(\mathsf{W}_j)$ coincides with the corresponding bilinear symmetric form given by the trace on $\mathfrak{sp}(\mathsf{W}_j)$. In particular the normalized measure on W is the product of the normalized measures on the W'_js and normalized measure on

$\sum_{j=1}^m(\mathfrak{gl}(\mathsf{V}'_j)+\mathfrak{gl}(\mathsf{V}))\subseteq\mathfrak{sp}(\mathsf{W})$ is the product of the normalized measures on the $\mathfrak{gl}(\mathsf{V}'_j)+\mathfrak{gl}(\mathsf{V}))\subseteq\mathfrak{sp}(\mathsf{W}_j)$. As shown in the previous two sections,

$$chc_{\mathsf{W}}(x'+x)=\prod_{j=1}^m c_j\,\delta\left(\det(x'+x)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V})}\right),$$

for $x\in\mathfrak{g}$. In particular an element $x\in\mathfrak{g}$ belongs to the support of $chc_{\mathsf{W}}(x'+\ )$ if and only if the semisimple part of x belongs to the support of $chc_{\mathsf{W}}(x'+\ )$. Suppose x is semisimple and belongs to the indicated support. Since the generalized function $chc_{\mathsf{W}}(x'+\ )$ is conjugation invariant, we may assume that $x|_{\mathsf{V}'}=x'$. Then

$$\mathsf{V}=\mathsf{V}'''\oplus\mathsf{U}_0,\quad \mathsf{V}'''=\mathsf{V}'''_1\oplus\mathsf{V}'''_2\oplus\cdots\oplus\mathsf{V}'''_m,$$

where each (x,V'''_j) is isotypic of the same type as (x,V'_j), and U_0 does not contain any $x-$ indecomposable components similar to any (x,V'_j), $j=1,2,\cdots,m$. Moreover,

$$\mathsf{V}'''_j=\mathsf{V}'_j\oplus\mathsf{V}''_j\qquad(j=1,2,\cdots,m),$$

where the decomposition is preserved by x. Define $x'''\in\mathfrak{g}$ by

$$x'''|_{\mathsf{V}'''}=x|_{\mathsf{V}'''},\quad x'''|_{\mathsf{U}_0}=0.$$

For a classical Lie algebra $\mathfrak{s}$, with the defining module V, and for a subset $S\subseteq\mathfrak{s}$, let S_ϵ denote the set of all elements $y\in S$ such that for any eigenvalue λ of y, acting on V, $|\lambda|<\epsilon$.

For $\epsilon>0$ let $\mathcal{U}_\epsilon$ be the set of all $y\in\mathfrak{g}^{x'''}$ such that

$$\begin{aligned}&y|_{\mathsf{V}'''}\in x'''|_{\mathsf{V}'''}+\mathfrak{gl}(\mathsf{V}''')^x_\epsilon,\ \text{and}\\ &d(eig(x'''|_{\mathsf{V}'''}),eig(y|_{\mathsf{U}_0}))>\epsilon.\end{aligned}$$

where $d(eig(x'''|_{\mathsf{V}'''}),eig(y|_{\mathsf{U}_0}))$ is the minimum of $|\lambda'-\lambda|$, where $\lambda'\in eig(x'''|_{\mathsf{V}'''})$ and $\lambda\in eig(y|_{\mathsf{U}_0})$.

Lemma 3.1. *For all $\epsilon>0$ small enough,*
(a) $x\in\mathcal{U}_\epsilon$;
(b) $\mathcal{U}_\epsilon$ is $\mathrm{G}^{x'''}-$stable;
(c) if $y_1,y_2\in\mathcal{U}_\epsilon$ and $h\in\mathrm{G}$ are such that $h\cdot y_1=y_2$, then $h\in\mathrm{G}^{x'''}$;
(d) $\det(\mathrm{ad}(y)_{\mathfrak{g}/\mathfrak{g}'''})\neq0$ for $y\in\mathcal{U}_\epsilon$.

Remark. Part (d) implies that the map

$$\begin{array}{rcl}\mathrm{G}\times\mathcal{U}_\epsilon&\longrightarrow&\mathfrak{g}\\ (g,y)&\longmapsto&g\cdot y\end{array}$$

is a submersion, and (c) shows that the fiber of the above map is equal to

$$\{(gh^{-1},h\cdot y);\ h\in\mathrm{G}^{x'''}\}.$$

Thus

$$\mathrm{G}\times_{\mathrm{G}^{x'''}}\mathcal{U}_\epsilon=\mathrm{G}\cdot\mathcal{U}_\epsilon.$$

Proof. Parts (a), (b), (c) are clear. We shall verify (d). Notice that

$$\mathfrak{g}^{x'''} = \bigoplus_{j=1}^{m} \mathfrak{gl}(\mathsf{V}_j''')^x \oplus \mathfrak{gl}(\mathsf{U}_0)$$

and

$$\mathfrak{g} = \mathfrak{g}^{x'''} \oplus \bigoplus_{j=1}^{m} {}^x\mathfrak{gl}(\mathsf{V}_j''') \oplus \bigoplus_{\substack{i,j=1 \\ i\neq j}}^{m} \mathrm{Hom}(\mathsf{V}_i''', \mathsf{V}_j''') \oplus \mathrm{Hom}(\mathsf{U}_0, \mathsf{V}''') \oplus \mathrm{Hom}(\mathsf{V}''', \mathsf{U}_0),$$

where the left superscript indicates the anticommutant:

$${}^xY = \{y \in Y;\ yx = -xy\}.$$

Hence

$$\begin{aligned}
&\left|\det\left(\mathrm{ad}(y)_{\mathfrak{g}/\mathfrak{g}'''}\right)\right| \\
&= \left|\det\left(\mathrm{ad}(y)_{\bigoplus_{j=1}^{m} {}^x\mathfrak{gl}(\mathsf{V}_j''') \oplus \bigoplus_{\substack{i,j=1 \\ i\neq j}}^{m} \mathrm{Hom}(\mathsf{V}_i''',\mathsf{V}_j''') \oplus \mathrm{Hom}(\mathsf{U}_0,\mathsf{V}''') \oplus \mathrm{Hom}(\mathsf{V}''',\mathsf{U}_0)}\right)\right| \\
&= \prod_{j=1}^{m} \left|\det\left(\mathrm{ad}(y)_{{}^x\mathfrak{gl}(\mathsf{V}_j''')}\right)\right| \prod_{\substack{i,j=1 \\ i\neq j}}^{m} \left|\det\left(\mathrm{ad}(y)_{\mathrm{Hom}(\mathsf{V}_i''',\mathsf{V}_j''')}\right)\right| \\
&\qquad \left|\det\left(\mathrm{ad}(y)_{\mathrm{Hom}(\mathsf{U}_0,\mathsf{V}''')}\right)\right| \left|\det\left(\mathrm{ad}(y)_{\mathrm{Hom}(\mathsf{V}''',\mathsf{U}_0)}\right)\right|.
\end{aligned}$$

The action of $\mathrm{ad}(y)$ on ${}^x\mathfrak{gl}(\mathsf{V}_j''')$ coincides with 2 times the left multiplication by $y|_{\mathsf{V}_j''}$. Hence,

$$\left|\det\left(\mathrm{ad}(y)_{{}^x\mathfrak{gl}(\mathsf{V}_j''')}\right)\right|$$

is a non-zero constant multiple of a power of

$$\left|\det\left(\mathrm{ad}(x')_{{}^{x'}\mathfrak{gl}(\mathsf{V}_j')}\right)\right|,$$

which is non-zero, by the regularity of x'.

Since

$$eig(y|_{\mathsf{V}_i'''}) \cap eig(y|_{\mathsf{V}_j'''}) = \emptyset \qquad (i \neq j),$$

we have

$$\det\left(\mathrm{ad}(y)_{\mathrm{Hom}(\mathsf{V}_i''',\mathsf{V}_j''')}\right) \neq 0 \qquad (i \neq j).$$

Similarly

$$\left|\det\left(\mathrm{ad}(y)_{\mathrm{Hom}(\mathsf{U}_0,\mathsf{V}''')}\right)\right| \left|\det\left(\mathrm{ad}(y)_{\mathrm{Hom}(\mathsf{V}''',\mathsf{U}_0)}\right)\right| \neq 0. \qquad \blacksquare$$

For $y \in \mathcal{U}_\epsilon$ we have

$$\begin{aligned}
&\prod_{j=1}^{m} \delta(\det(x'+y)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V})}) \\
&= \prod_{j=1}^{m} \delta\left(\det(x'+y)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_j)^x}\right) \prod_{j=1}^{m} \left|\det(x'+y)_{{}^x\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_j)}\right|^{-1} \\
&\qquad \prod_{\substack{j,k=1\\ j\neq k}}^{m} \left|\det(x'+y)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_k)}\right|^{-1} \left|\det(x'+y)_{\mathrm{Hom}(\mathsf{V}',\mathsf{U}_0)}\right|^{-1},
\end{aligned}$$

where all the vector spaces $\mathrm{Hom}(\ ,\ )$ are over $\mathbb{R}$. Thus

$$\begin{aligned}
\prod_{j=1}^{m} \delta\left(\det(x'+y)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V})}\right) &= \prod_{j=1}^{m} \delta\left(\det(x'+y)_{\mathrm{Hom}(V'_j,\mathsf{V}'''_j)^x}\right) \\
&\qquad \left|\det\left(\mathrm{ad}(y)_{\bigoplus_{j=1}^{m} {}^x\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_j)\ \bigoplus_{\substack{j,k=1\\ j\neq k}}^{m} \mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_k)\oplus\mathrm{Hom}(\mathsf{V}',\mathsf{U}_0)}\right)\right|^{-1}.
\end{aligned}$$

For each j, $\mathfrak{h}'|_{\mathsf{V}'_j} \subseteq \mathrm{End}(\mathsf{V}'_j)$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, as a field and of dimension $\dim_{\mathbb{R}} \mathsf{V}'_j$ over $\mathbb{R}$. Let

$$\mathsf{W}_j^x = \mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}'''_j)^x \oplus \mathrm{Hom}(\mathsf{V}'''_j, \mathsf{V}'_j)^x.$$

Then

$$\mathrm{GL}(\mathsf{V}'_j)^x = \mathrm{H}'|_{\mathsf{V}'_j}, \mathrm{GL}(\mathsf{V}'''_j)^x \subseteq \mathrm{Sp}(\mathsf{W}_j)^x$$

is a dual pair. Moreover, since

$$\mathsf{W} = \mathsf{W}_j^x \oplus \mathsf{W}_j^{x\perp},$$

our symmetric bilinear form on $\mathfrak{sp}(\mathsf{W}_j^x)$ coincides with the restriction of the form from $\mathfrak{sp}(\mathsf{W})$. Hence, for $y \in \mathcal{U}_\epsilon$, $d\mu(y|_{\mathsf{V}'''_j})$ is the same as the measure for the pair $(\mathrm{GL}(\mathsf{V}'_j)^x = \mathrm{H}'|_{\mathsf{V}'_j}, \mathrm{GL}(\mathsf{V}'''_j)^x)$. Let

$$c_j^x = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_j^x}}{\sqrt{\dim_{\mathbb{R}} \mathsf{W}_j^x}^{\dim_{\mathbb{R}} \mathsf{V}'_j}}.$$

Then according to (1.1) and (2.1), for $y \in \mathcal{U}_\epsilon$ and for all j,

$$\delta(\det(x'+y)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_j)}) = \frac{1}{c_j^x}\, chc_{\mathsf{W}_j^x}(x'+y) \left|\det(x'+y)_{{}^x\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_j)}\right|^{-1}.$$

Hence,

$$\begin{aligned}
chc_{\mathsf{W}}(x'+y) \quad &= \quad \prod_{j=1}^{m} c_j \frac{1}{c_j^x}\, chc_{\mathsf{W}_j^x}(x'+y) \\
&\qquad \left|\det\left(x'+y\right)_{\bigoplus_{j=1}^{m} {}^x\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_j)\ \bigoplus_{\substack{j,k=1\\ j\neq k}}^{m} \mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}'''_k)\oplus\mathrm{Hom}(\mathsf{V}',\mathsf{U}_0)}\right|^{-1}.
\end{aligned}$$

Define $x'' \in \mathfrak{g}$ by

$$x''|_{\mathsf{V}''} = x|_{\mathsf{V}''}, \quad x''|_{\mathsf{V}'\oplus\mathsf{U}_0} = 0.$$

Then

$$\mathfrak{g}^{x'''} = \bigoplus_{j=1}^{m} \mathfrak{gl}(\mathsf{V}_j''')^x \oplus \mathfrak{gl}(\mathsf{U}_0) \ \text{ and } \ \left(\mathfrak{g}^{x'''}\right)^{x''} = \bigoplus_{j=1}^{m} \left(\mathfrak{gl}(\mathsf{V}_j')^x \oplus \mathfrak{gl}(\mathsf{V}_j'')^x\right) \oplus \mathfrak{gl}(\mathsf{U}_0).$$

Also

$$\left(\mathrm{GL}(\mathsf{V}_j''')^x\right)^{x''} = \mathrm{GL}(\mathsf{V}_j')^x \times \mathrm{GL}(\mathsf{V}_j'')^x.$$

As we have shown in Sections 1 and 2,

$$\begin{aligned}
&\int_{\mathfrak{gl}(\mathsf{V}_j'')^x} \psi(y) chc_{\mathsf{W}_j^x}(x'+y)\, d\mu(y) \\
&= c_j^x \sqrt{2\,\dim_{\mathbb{R}} \mathsf{V}_j'}^{\dim_{\mathbb{R}} \mathsf{V}_j'} \int_{\mathrm{GL}(\mathsf{V}_j''')^x/(\mathrm{GL}(\mathsf{V}_j''')^x)^{x''}} \int_{\mathfrak{gl}(\mathsf{V}_j'')^x} \left|\det\left(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{V}_j',\mathsf{V}_j'')^x}\right)\right| \\
&\qquad \psi(g\cdot(x'|_{\mathsf{V}_j'}+y))\, d\mu(y)\, d\mu(g(\mathrm{GL}(\mathsf{V}_j''')^x)^{x''}).
\end{aligned}$$

Furthermore,

$$\prod_{j=1}^{m} c_j \sqrt{2\,\dim_{\mathbb{R}} \mathsf{V}_j'}^{\dim_{\mathbb{R}} \mathsf{V}_j'} = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\sqrt{\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{V}'}}.$$

Moreover,

$$\begin{aligned}
&\left|\det\left(\mathrm{ad}(x'+y)_{\bigoplus_{j=1}^{m} {}^x\mathrm{Hom}(\mathsf{V}_j',\mathsf{V}_j''') \bigoplus_{\substack{j,k=1\\ j\neq k}}^{m} \mathrm{Hom}(\mathsf{V}_j',\mathsf{V}_k''')\oplus \mathrm{Hom}(\mathsf{V}',\mathsf{U}_0)}\right)\right|^{-1} \\
&\qquad \left|\det\left(\mathrm{ad}(x'+y)_{\bigoplus_{j=1}^{m} \mathrm{Hom}(\mathsf{V}_j',\mathsf{V}_j''')^x}\right)\right| \left|\det\left(\mathrm{ad}(x'+y)_{\mathfrak{g}/\mathfrak{g}^{x'''}}\right)\right| \\
&= \left|\det\left(\mathrm{ad}(x'+y)_{\mathfrak{gl}(\mathsf{U})/\mathfrak{gl}(\mathsf{U})^{x''}}\right)\right| \left|\det\left(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{U},\mathsf{V}')}\right)\right|.
\end{aligned}$$

Let $\psi \in \mathcal{S}(\mathfrak{g})$ be supported in the set of the G$-$orbits passing through $\mathcal{U}_\epsilon$. Then our computations show that

$$\begin{aligned}
&\int_{\mathfrak{g}} \psi(x) chc_{\mathsf{W}}(x'+x)\, d\mu(x) \\
&= \int_{\mathrm{G}/\mathrm{G}^{x'''}} \int_{\mathfrak{g}^{x'''}} \psi(g\cdot y) chc_{\mathsf{W}}(x'+y) |\det(\mathrm{ad}(x'+y)_{\mathfrak{g}/\mathfrak{g}^{x'''}})|\, d\mu(y)\, d\mu(g\mathrm{G}^{x'''}) \\
&= \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\sqrt{\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{V}'}} \int_{\mathrm{G}/(\mathrm{G}^{x'''})^{x''}} \int_{(\mathfrak{g}^{x'''})^{x''}} \psi(g\cdot(x'+y)) \left|\det\left(\mathrm{ad}(x'+y)_{\mathfrak{gl}(\mathsf{U})/\mathfrak{gl}(\mathsf{U})^{x''}}\right)\right| \\
&\qquad \left|\det\left(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{U},\mathsf{V}')}\right)\right| d\mu(y)\, d\mu(g(\mathrm{G}^{x'''})^{x''})
\end{aligned}$$

$$= \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\sqrt{\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{V}'}} \int_{\mathrm{G/L}} \int_{\mathrm{GL}(\mathsf{V}')/\mathrm{GL}(\mathsf{V}')^{x'}} \int_{\mathfrak{gl}(\mathsf{U})} \psi(g \cdot (l \cdot x' + y))$$
$$\left| \det \left(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{U},\mathsf{V}')} \right) \right| d\mu(y)\, d\mu(l(\mathrm{GL}(\mathsf{V}')^{x'}))\, d\mu(g\mathrm{L}).$$

Since $\mathrm{GL}(\mathsf{V}')^{x'} = \mathrm{H}'$, we see that we have for every $\psi \in \mathcal{S}(\mathfrak{g})$,

$$\int_{\mathfrak{g}} \psi(x) chc_{\mathsf{W}}(x'+x)\, d\mu(x) = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\sqrt{\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{V}'}}$$
$$\int_{\mathrm{G/L}} \int_{\mathrm{GL}(\mathsf{V}')/\mathrm{H}'} \int_{\mathfrak{gl}(\mathsf{U})} \psi(g \cdot (l \cdot x' + y)) | \det(\mathrm{ad}(x'+y)_{\mathfrak{n}})|\, d\mu(y)\, d\mu(l\mathrm{H}_2')\, d\mu(g\mathrm{L}).$$

This verifies Theorem 0.3.

Proof. [Proof of Corollary 0.4] We denote by $D_{\mathrm{GL}(\mathsf{U})}$ the Weyl denominator of the Lie algebra $\mathfrak{gl}(\mathsf{U})$ (see equality (A.2) for the definition). The Weyl integration formula for the Lie algebra $\mathfrak{gl}(\mathsf{U})$ implies the following equality:

$$\int_{\mathrm{GL}(\mathsf{V})/\mathrm{H}'\times \mathrm{GL}(\mathsf{U})} \int_{\mathfrak{gl}(\mathsf{U})} | \det(\mathrm{ad}(x'+y)_{\mathfrak{n}'})| \psi(g.(x'+y)) d\mu(y) d\mu(g(\mathrm{H}_2' \times \mathrm{GL}(\mathsf{U})))$$
$$= \sum_{\mathrm{H}_{\mathsf{U}}} \frac{1}{|W_{\mathrm{GL}(\mathsf{U})}(\mathfrak{h}_{\mathsf{U}})|} \int_{\mathfrak{h}_{\mathsf{U}}} D_{\mathrm{GL}(\mathsf{U})}(y) | \det(\mathrm{ad}(x'+y)_{\mathfrak{n}'})|$$
$$\int_{\mathrm{GL}(\mathsf{V})/\mathrm{H}'\times \mathrm{H}_{\mathsf{U}}} \psi(g.(x'+y)) d\mu(g(\mathrm{H}_2' \times \mathrm{H}_{\mathsf{U}})) d\mu(y).$$

Corollary A.4 implies that the above is equal to

$$= \sum_{\mathrm{H}_{\mathsf{U}}} \frac{1}{|W_{\mathrm{GL}(\mathsf{U})}(\mathfrak{h}_{\mathsf{U}})|} \int_{\mathfrak{h}_{\mathsf{U}}} D_{\mathrm{GL}(\mathsf{U})}(y) \frac{1}{\sqrt{2}^{\dim_{\mathbb{R}}(\mathfrak{n}')} \mu(\mathrm{K} \cap \mathrm{L})}$$
$$\int_{\mathrm{GL}(\mathsf{V}')\times \mathrm{GL}(\mathsf{U})/\mathrm{H}'\times \mathrm{H}_{\mathsf{U}}} \psi_{\mathfrak{n}}^{\mathrm{K}}(g.(x'+y)) d\mu(g(\mathrm{H}_2' \times \mathrm{H}_{\mathsf{U}})) d\mu(y)$$
$$= \frac{1}{\sqrt{2}^{\dim_{\mathbb{R}}(\mathfrak{n}')} \mu(\mathrm{K} \cap \mathrm{L})} \int_{\mathrm{GL}(\mathsf{V}')/\mathrm{H}'} \int_{\mathfrak{gl}(\mathsf{U})} \psi_{\mathfrak{n}}^{\mathrm{K}}(g.x'+y) d\mu(y) \mu(g\mathrm{H}_2')$$
$$= \frac{1}{\sqrt{2}^{\dim_{\mathbb{R}}(\mathfrak{n}')} \mu(\mathrm{K} \cap \mathrm{L})} \sqrt{\frac{\dim_{\mathbb{D}}(\mathsf{V})}{\dim_{\mathbb{D}}(\mathsf{V}')}}^{\dim_{\mathbb{R}}(\mathfrak{h}')} \int_{\mathrm{GL}(\mathsf{V}')/\mathrm{H}'} \int_{\mathfrak{gl}(\mathsf{U})} \psi_{\mathfrak{n}}^{\mathrm{K}}(g.x'+y) d\mu(y) \mu(g\mathrm{H}_1').$$

If we notice that $\dim_{\mathbb{R}}(\mathfrak{h}') = \dim_{\mathbb{R}}(\mathsf{V}')$, then the formula of Corollary 0.4 follows from Theorem 0.3. ■

4. Proof of Theorem 0.7

In the proof of Theorem 0.7, we shall encounter a delicate point which we would like to explain here. Let $\mathfrak{h}_s' = \mathfrak{h}'|_{V_s'}$. There are two inclusions:

$$j_1 : \mathfrak{h}_s' \hookrightarrow \mathfrak{g} \text{ and } j_2 : \mathfrak{h}_s' \hookrightarrow \mathfrak{gl}(\mathsf{V})$$

defined by

$$\begin{array}{rcll} j_1(x')(v'+u) & = & x'v' & (x' \in \mathfrak{h}'_s,\ v' \in \mathsf{V}'_s,\ u \in \mathsf{U}),\\ j_2(x')(v'+u) & = & x'v' & (x' \in \mathfrak{h}'_s,\ v' \in \mathsf{X}',\ u \in \mathsf{U}+\mathsf{Y}'). \end{array}$$

Furthermore, there are the obvious inclusions:

$$\begin{array}{rcl} \mathfrak{g} & \hookrightarrow & \mathfrak{sp}(\mathsf{W}_s),\\ \mathfrak{gl}(\mathsf{V}) & \hookrightarrow & \mathfrak{sp}(\mathsf{W}_s), \end{array}$$

where $\mathsf{W}_s = \mathrm{Hom}(\mathsf{V}'_s, \mathsf{V})$. Let $\mathfrak{h}'_{s,i} = j_i(\mathfrak{h}'_s)$. Thus

$$\begin{array}{rcl} \mathfrak{h}'_{s,1} & \hookrightarrow & \mathfrak{sp}(\mathsf{W}_s),\\ \mathfrak{h}'_{s,2} & \hookrightarrow & \mathfrak{sp}(\mathsf{W}_s). \end{array}$$

We denote by $\mathrm{H}'_{s,i}$ the corresponding subgroup of $\mathrm{Sp}(\mathsf{W}_s)$. Let, for the moment, $\kappa_i(x,y) = \kappa(x,y)$ for $x, y \in \mathfrak{h}'_{s,i}$ with $i = 1, 2$. Then

$$\begin{aligned} \kappa_1(x,y) &= \mathrm{tr}\,(xy)_{\mathrm{Hom}(\mathsf{V}'_s,\mathsf{V}'_s)_{\mathbb{R}}} = 2\dim_{\mathbb{D}}(\mathsf{V}'_s)\ \mathrm{tr}\,(xy)_{\mathrm{End}(\mathsf{X}')_{\mathbb{R}}},\\ \kappa_2(x,y) &= \mathrm{tr}\,(xy)_{(\mathrm{Hom}(\mathsf{X}',\mathsf{X}')\oplus\mathrm{Hom}(\mathsf{Y}',\mathsf{Y}'))_{\mathbb{R}}} = \dim_{\mathbb{D}}(\mathsf{V}'_s)\,\mathrm{tr}\,(xy)_{\mathrm{End}(\mathsf{X}')_{\mathbb{R}}}. \end{aligned}$$

Thus

$$\kappa_2(x,y) = 2^{-1}\kappa_1(x,y) \qquad (x, y \in \mathfrak{h}'_s). \tag{4.1}$$

Let μ_1 and μ_2 be the corresponding measures on $\mathrm{H}'_{s,1}$ and $\mathrm{H}'_{s,2}$ respectively. Then

$$d\mu_2(x) = \frac{1}{\sqrt{2}^{\dim_{\mathbb{R}}(\mathsf{X}')}}d\mu_1(x) \qquad (x, y \in \mathfrak{h}'_s).$$

The pair of groups $(\mathrm{GL}(\mathsf{X}'), \mathrm{GL}(\mathsf{V}))$ is a dual pairs of type II in $\mathrm{Sp}(\mathsf{W}_s)$. In the third equality in (4.3) we shall apply Theorem 0.3 to this pair, but we have to be careful about the measures involved. We shall only need the case $\dim_{\mathbb{D}}(\mathsf{X}') = 1$, $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$. Let μ denote the measure on $\mathrm{GL}(\mathsf{V})$, the subgroups and the quotients of it, as in Theorem 0.3. Then

$$\begin{aligned} &\int_{\mathfrak{gl}(\mathsf{V})} \psi(x)\, chc_{\mathsf{W}}(x'+x)\, d\mu(x)\\ &= \frac{\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}_s}}{\sqrt{\dim_{\mathbb{R}}\mathsf{W}_s}^{\dim_{\mathbb{R}}\mathsf{X}'}} \int_{\mathfrak{gl}(\mathsf{V})} \delta(\det(x'+x))\psi(x)\, d\mu(x)\\ &= \frac{\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}_s}}{\sqrt{\dim_{\mathbb{D}}(\mathsf{V})}^{\dim_{\mathbb{R}}(\mathsf{X}')}} \int_{\mathrm{GL}(\mathsf{V})/(\mathrm{H}'_s\times\mathrm{GL}(\mathsf{U}))} \int_{\mathfrak{gl}(\mathsf{U})} F(y)\, d\mu(y)\, d\mu(g(\mathrm{H}'_s\times\mathrm{GL}(\mathsf{U}))),\\ &\text{where } F(y) = \left|\det\left(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{U},\mathsf{X}')}\right)\right|\psi(g\cdot(x'+y)). \end{aligned} \tag{4.2}$$

Let us equip the group $\mathrm{H}'_{s,1} \times \mathrm{GL}(\mathsf{U})$ with the measure $\mu_1 \otimes \mu$, keep the same measure μ on $\mathrm{GL}(\mathsf{V})$, and let $\tilde{\mu}$ denote the corresponding measure on the quotient $\mathrm{GL}(\mathsf{V})/(\mathrm{H}'_{s,1} \times \mathrm{GL}(\mathsf{U}))$. Then, by (4.1),

$$d\tilde{\mu}(g(\mathrm{H}'_{s,1} \times \mathrm{GL}(\mathsf{U}))) = \sqrt{2}^{\dim_{\mathbb{R}}(\mathsf{X}')} d\mu(g(\mathrm{H}'_{s,1} \times \mathrm{GL}(\mathsf{U}))).$$

Then (4.2) is equal to

$$\frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_s}}{\sqrt{\dim_{\mathbb{D}}(\mathsf{V})}^{\dim_{\mathbb{R}}(\mathsf{X}')}} \sqrt{2}^{\dim_{\mathbb{R}}(\mathsf{X}')} \int_{\mathrm{GL}(\mathsf{V})/(\mathrm{H}'_{s,1}\times \mathrm{GL}(\mathsf{U}))} \int_{\mathfrak{gl}(\mathsf{U})} \left| \det\left(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{U},\mathsf{X}')}\right)\right|$$
$$\psi(g\cdot(x'+y))\, d\mu(y)\, d\tilde{\mu}(g(\mathrm{H}'_{s,1}\times \mathrm{GL}(\mathsf{U}))).$$

Notice that the quotient measure in (4.3), denoted by μ there (for natural reasons) is $\tilde{\mu}$.

Recall the notation and the assumptions of Section 4. Let

$$\mathsf{V}'_s = \mathsf{V}'_1 \oplus \mathsf{V}'_2 \oplus \cdots \oplus \mathsf{V}'_m$$

be the isotypic decomposition with respect to x'. Then each V'_j has a complete polarization

$$\mathsf{V}'_j = \mathsf{X}'_j \oplus \mathsf{Y}'_j$$

preserved by H'. For $j = 1, 2, \cdots, m$ let

$$\mathsf{W}_j = \mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}, \mathsf{V}'_j),$$
$$c_j = \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_j}}{\sqrt{\dim_{\mathbb{R}} \mathsf{W}_j}^{\dim_{\mathbb{R}} \mathsf{V}'_j}}.$$

Notice that

$$\mathsf{W} = \mathsf{W}_s \oplus \mathrm{Hom}(\mathsf{V}'_c, \mathsf{V})$$

is an orthogonal decomposition. Since

$$\mathsf{W}_s = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_m$$

is an orthogonal decomposition, the subspaces

$$\mathfrak{sp}(\mathsf{W}_j), \mathfrak{sp}(\mathsf{W}_k) \qquad (j \neq k)$$

are mutually orthogonal with respect to the form κ. Hence, the restriction of κ to each $\mathfrak{sp}(\mathsf{W}_j)$ coincides with the corresponding bilinear symmetric form given by the trace on $\mathfrak{sp}(\mathsf{W}_j)$. In particular the normalized measure on W is the product of the normalized measures on the $\mathsf{W}'_j s$ and on $\mathrm{Hom}(\mathsf{V}'_c, \mathsf{V})$.

Similarly, the normalized measure on $\mathrm{A}'''_s \subseteq \mathrm{Sp}(\mathsf{W})$ is the product of the normalized measures on the $\mathrm{A}'''_s \cap \mathrm{Sp}(\mathsf{W}_j)$, and the normalized measure on $\mathrm{H}'|_{V'_s} \subseteq \mathrm{Sp}(\mathrm{Hom}(\mathsf{V}'_s, \mathsf{V}))$ is the product of the normalized measures on the $\mathrm{H}'|_{\mathsf{V}'_s} \cap \mathrm{Sp}(\mathsf{W}_j)$. Hence, for $x \in \mathfrak{g}$,

$$chc_{\mathsf{W}}(x'+x) = chc_{\mathsf{W}_s}(x'+x)\, chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V})}(x'+x),$$
$$chc_{\mathsf{W}_s}(x'+x) = \prod_{j=1}^{m} c_j\, \delta(\det(x'+x)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{V})}).$$

Notice that the measure $\mu(\mathrm{H}'|_{\mathsf{V}'_c})$ used to define the $chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V})}$ is different than the corresponding measure (denote it by $\mu_U(\mathrm{H}'|_{\mathsf{V}'_c})$) of $\mathrm{H}'|_{\mathsf{V}'_c}$ used to define chc_{W_c}. For $x, y \in \mathrm{End}(\mathsf{V}'_c)$ let

$$\kappa(x, y) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy)$$

where we view x and y as elements of $\mathrm{End}(\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}))$, and let

$$\kappa_{\mathsf{U}}(x,y)=\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy)$$

where we view x and y as elements of $\mathrm{End}(\mathsf{W}_c)$. Then

$$\kappa(x,y)=\frac{\dim_{\mathbb{D}}\mathsf{V}}{\dim_{\mathbb{D}}\mathsf{U}}\kappa_{\mathsf{U}}(x,y).$$

Hence,

$$\mu(\mathrm{H}'|_{\mathsf{V}'_c})=\sqrt{\frac{\dim_{\mathbb{D}}\mathsf{V}}{\dim_{\mathbb{D}}\mathsf{U}}}^{\dim_{\mathbb{R}}(\mathrm{H}'|_{\mathsf{V}'_c})}\mu_{\mathsf{U}}(\mathrm{H}'|_{\mathsf{V}'_c}).$$

An element $x\in\mathfrak{g}$ belongs to the support of $chc_{\mathsf{W}}(x'+\cdot)$ ($=supp(chc_{\mathsf{W}_s}(x'+\cdot))$) if and only if the semisimple part of x belongs to the support of $chc_{\mathsf{W}}(x'+\cdot)$. Suppose x is semisimple and belongs to the indicated support. Since the generalized function $chc_{\mathsf{W}}(x'+\cdot)$ is conjugation invariant, we may assume that $x|_{\mathsf{V}'}=x'$. Then

$$\mathsf{V}=\mathsf{V}'''\oplus\mathsf{U}_0,\quad \mathsf{V}'''=\mathsf{V}_1'''\oplus\mathsf{V}_2'''\oplus\cdots\oplus\mathsf{V}_m''',$$

where each (x,V_j''') is isotypic of the same type as (x,V'_j), and U_0 does not contain any x-indecomposable components similar to any (x,V'_j), $j=1,2,\cdots,m$. Moreover,

$$\mathsf{V}_j'''=\mathsf{V}'_j\oplus\mathsf{V}''_j\qquad (j=1,2,\cdots,m),$$

where V''_j is orthogonal to V'_j and the decomposition is preserved by x. For each j, let

$$\mathsf{V}''_j=\mathsf{X}''_j\oplus\mathsf{Y}''_j$$

be the complete polarization preserved by x. Let

$$\begin{aligned}&\mathsf{X}_j'''=\mathsf{X}'_j\oplus\mathsf{X}''_j\qquad (j=1,2,3,\cdots,m)\\&\mathsf{X}'''=\mathsf{X}_1'''+\mathsf{X}_2'''+\cdots+\mathsf{X}_m''',\quad \mathsf{Y}'''=\mathsf{Y}_1'''+\mathsf{Y}_2'''+\cdots+\mathsf{Y}_m'''.\end{aligned}$$

All these spaces are $x-$invariant, and the V''_j might be zero. Define $x'''\in\mathfrak{g}$ by

$$x'''|_{\mathsf{V}'''}=x|_{\mathsf{V}'''},\quad x'''|_{\mathsf{U}_0}=0.$$

For $\epsilon>0$ let $\mathcal{U}_\epsilon$ be the set of all $y\in\mathfrak{g}^{x'''}$ such that

$$\begin{aligned}&y|_{\mathsf{X}'''}\in x'''|_{\mathsf{X}'''}+\mathfrak{gl}(\mathsf{X}''')^x_\epsilon,\ \text{and}\\&d(eig(x'''|_{\mathsf{X}'''}),eig(y|_{\mathsf{U}_0}))>\epsilon.\end{aligned}$$

Hence, Lemma 3.1 holds. Notice that

$$\prod_{j=1}^m c_j\sqrt{2\,\dim_{\mathbb{R}}\mathsf{X}'_j}^{\dim_{\mathbb{R}}\mathsf{X}'_j}=\frac{\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}_s}}{\sqrt{\dim_{\mathbb{D}}\mathsf{V}}^{\dim_{\mathbb{R}}\mathsf{X}'}}.$$

As in Section 3, we check that

$$\begin{aligned}\prod_{j=1}^m\delta\Big(\det(x'+y)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{V})}\Big)&=\prod_{j=1}^m\delta\Big(\det(x'+y)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{X}_j''')^x}\Big)\\&\left|\det\Big(\mathrm{ad}(y)_{\bigoplus_{j=1}^m{}^x\mathrm{Hom}(\mathsf{X}'_j,\mathsf{X}_j''')\oplus\bigoplus_{\substack{j,k=1\\ j\neq k}}^m\mathrm{Hom}(\mathsf{X}'_j,\mathsf{X}_k''')\oplus\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0+\mathsf{Y}''')}\Big)\right|^{-1}.\end{aligned}$$

Notice that

$$\begin{aligned}
&\left|\det\left(\operatorname{ad}(y)_{\mathfrak{g}/(\sum_j \mathfrak{gl}(\mathsf{X}_j''')+\mathfrak{g}(\mathsf{U}_0))}\right)\right| \\
&= \left|\det\left(\operatorname{ad}(y)_{\mathrm{Hom}(\mathsf{X}''',\mathsf{U}_0)+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}'')+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}+\mathrm{Hom}(\mathsf{X}'',\mathsf{Y}'')\cap\mathfrak{g}}\right)\right|^2 \\
&\qquad\qquad \left|\det\left(\operatorname{ad}(y)_{\sum_{j,k=1}^m \mathrm{Hom}(\mathsf{X}_j''',\mathsf{X}_k''')}\right)\right|
\end{aligned}$$

and that

$$\left|\det\left(\operatorname{ad}(y)_{(\sum_j \mathfrak{gl}(\mathsf{X}_j''')+\mathfrak{g}(\mathsf{U}_0))/(\sum_j \mathfrak{gl}(\mathsf{X}_j''')^x+\mathfrak{g}(\mathsf{U}_0))}\right)\right| = \left|\det\left(\operatorname{ad}(y)_{\sum_j {}^x\mathfrak{gl}(\mathsf{X}_j''')}\right)\right|.$$

Hence, by a straightforward computation

$$\begin{aligned}
&\left|\det\left(\operatorname{ad}(x'+y)_{\mathfrak{g}/\mathfrak{g}^{x'''}}\right)\right| \\
&= \left|\det\left(\operatorname{ad}(x'+y)_{\bigoplus_{j=1}^m {}^x\mathrm{Hom}(\mathsf{X}_j',\mathsf{X}_j''')\oplus \bigoplus_{\substack{j,k=1\\ j\neq k}}^m \mathrm{Hom}(\mathsf{X}_j',\mathsf{X}_k''')\oplus\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0+\mathsf{Y}''')}\right)\right|^{-1} \\
&\qquad\qquad \left|\det\left(\operatorname{ad}(x'+y)_{\bigoplus_{j=1}^m \mathrm{Hom}(\mathsf{X}_j',\mathsf{X}_j''')^x}\right)\right|
\end{aligned}$$

$$\begin{aligned}
&= \left|\det\left(\operatorname{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}'',\mathsf{X}')+\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0+\mathsf{Y}'')+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}}\right)\right| \\
&\qquad \left|\det\left(\operatorname{ad}(x'+y)_{\mathfrak{g}(\mathsf{U})/\mathfrak{g}(\mathsf{U})^x}\right)\right| \left|\det\left(\operatorname{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'}\right)\right|^{-1} \\
&= \left|\det\left(\operatorname{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}}\right)\right| \\
&\qquad \left|\det\left(\operatorname{ad}(x'+y)_{\mathfrak{g}(\mathsf{U})/\mathfrak{g}(\mathsf{U})^x}\right)\right| \left|\det\left(\operatorname{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'}\right)\right|^{-1}.
\end{aligned}$$

Define $x'' \in \mathfrak{g}$ by

$$x''|_{\mathsf{V}''} = x|_{\mathsf{V}''}, \quad x''|_{\mathsf{V}''^\perp} = 0.$$

Then

$$\begin{aligned}
\mathfrak{g}^{x'''} &= \bigoplus_{j=1}^m \mathfrak{gl}(\mathsf{X}_j''')^x \oplus \mathfrak{gl}(\mathsf{U}_0), \text{ and} \\
\left(\mathfrak{g}^{x'''}\right)^{x''} &= \bigoplus_{j=1}^m (\mathfrak{gl}(\mathsf{X}_j')^x \oplus \mathfrak{gl}(\mathsf{X}_j'')^x) \oplus \mathfrak{gl}(\mathsf{U}_0).
\end{aligned}$$

Let $\psi \in \mathcal{S}(\mathfrak{g})$. Assume, $\mathrm{supp}(\psi) \subseteq \mathrm{G}\cdot\mathcal{U}_\epsilon$. Then

$$\begin{aligned}
&\int_{\mathfrak{g}} \psi(x) chc_{\mathsf{W}_s}\, d\mu(x) = \int_{\mathfrak{g}} \prod_{j=1}^{m} c_j \delta\Big(\det(x'+x)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{V})}\Big)\psi(x)\, d\mu(x)\\
&= \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_s}}{\sqrt{\frac{1}{2}\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{X}'}} \left|\det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'})\right|^{-1} \int_{\mathrm{G}/(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U})^{x''})} \int_{\mathfrak{g}(\mathsf{U})^{x''}}\\
&\qquad \left|\det(\mathrm{ad}(x'+y)_{\mathfrak{g}(\mathsf{U})/\mathfrak{g}(\mathsf{U})^{x''}})\right| \left|\det(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}})\right|\\
&\qquad\qquad \psi(g\cdot(x'+y))\, d\mu(y)\, d\mu(g(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U})^{x''}))\\
&= \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_s}}{\sqrt{\frac{1}{2}\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{X}'}} \left|\det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'})\right|^{-1} \int_{\mathrm{G}/(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))} \int_{\mathfrak{g}(\mathsf{U})}\\
&\left|\det(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}})\right|\psi(g\cdot(x'+y))\, d\mu(y)\, d\mu(g(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))).
\end{aligned} \tag{4.3}$$

This verifies the part of Theorem 0.7 corresponding to the case $\mathsf{V}'_c = 0$.

Notice that for $y \in \mathcal{U}_\epsilon$, such that y restricted to X' is conjugate to x' restricted to X', we have

$$\begin{aligned}
chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)}(x'+y) &= \sqrt{2}^{\dim_{\mathbb{R}} \mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)} \left| \det(x'+y)_{\mathrm{Hom}(\mathsf{V}_c,\mathsf{V}'_s)}\right|^{-1/2}\\
&= \sqrt{2}^{\dim_{\mathbb{R}} \mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)} \left| \det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{V}'_c)})\right|^{-1}.
\end{aligned}$$

Furthermore,

$$\mathsf{W} = \mathsf{W}_s \oplus \mathrm{Hom}(\mathsf{V}'_c, \mathsf{V}'_s) \oplus \mathsf{W}_c.$$

Assume that $\mathsf{U} \neq 0$ and $\mathrm{H}'|_{\mathsf{V}'_c}$ is not finite. Then,

$$\begin{aligned}
&\int_{\mathfrak{g}} chc_{\mathsf{W}}(x'+x)\psi(x)d\mu(x)\\
&= \int_{\mathfrak{g}} chc_{\mathsf{W}_s}(x'+x)\frac{1}{\mu(\mathrm{H}'|_{\mathsf{V}'_c})}\widetilde{chc}_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)}(x'+x)\\
&\qquad\qquad \widetilde{chc}_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U})}(x'+x)\psi(x)d\mu(x)\\
&= \int_{\mathfrak{g}} chc_{\mathsf{W}_s}(x'+x)\frac{\mu_{\mathsf{U}}(\mathrm{H}'|_{\mathsf{V}'_c})}{\mu(\mathrm{H}'|_{\mathsf{V}'_c})}\widetilde{chc}_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)}(x'+x) chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U})}(x'+x)\psi(x)d\mu(x)
\end{aligned}$$

$$= \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_s}}{\sqrt{\frac{1}{2}\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{X}'}} \left| \det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'}) \right|^{-1}$$

$$\int_{\mathrm{G}/(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))} \int_{\mathfrak{g}(\mathsf{U})} \left|\det(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}})\right| \frac{\mu_{\mathsf{U}}(\mathrm{H}'|_{\mathsf{V}'_c})}{\mu(\mathrm{H}'|_{\mathsf{V}'_c})} \widetilde{chc}_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)}(x'+y)$$
$$chc_{\mathsf{W}_c}(x'+y)\psi(g\cdot(x'+y))\, d\mu(y)\, d\mu(g(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U})))$$

$$= \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}_s+\dim_{\mathbb{R}}\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)}}{\sqrt{\frac{1}{2}\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{X}'}} \frac{\mu_{\mathsf{U}}(\mathrm{H}'|_{\mathsf{V}'_c})}{\mu(\mathrm{H}'|_{\mathsf{V}'_c})}$$
$$\left| \det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'})\right|^{-1} \left|\det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{V}'_c)})\right|^{-1}$$
$$\int_{\mathrm{G}/(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))} \int_{\mathfrak{g}(\mathsf{U})} \left|\det(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}})\right| \, chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U})}(x'+y)$$
$$\psi(g\cdot(x'+y))\, d\mu(y)\, d\mu(g(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U})))$$

$$= \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathrm{Hom}(\mathsf{V}'_s,\mathsf{V})+\dim_{\mathbb{R}}\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'_s)}}{\sqrt{\frac{1}{2}\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{X}'}} \sqrt{\frac{\dim_{\mathbb{D}}\mathsf{U}}{\dim_{\mathbb{D}}\mathsf{V}}}^{\dim_{\mathbb{R}}(\mathrm{H}'|_{\mathsf{V}'_c})}$$
$$\left| \det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'})\right|^{-1} \left|\det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{V}'_c)})\right|^{-1}$$
$$\int_{\mathrm{G}/(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))} \int_{\mathfrak{g}(\mathsf{U})} \left|\det(\mathrm{ad}(x'+y)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})+\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}})\right| \, \widetilde{chc}_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U})}(x'+y)$$
$$\psi(g\cdot(x'+y))\, d\mu(y)\, d\mu(g(\mathrm{H}'|_{\mathsf{X}'}\times \mathrm{G}(\mathsf{U}))).$$

Suppose $\mathfrak{h}'$ acts trivially on W_c. Then,

$$\begin{aligned}\int_{\mathfrak{g}} \widetilde{chc}_{\mathsf{W}}(x'+x)\psi(x)d\mu(x) &= \int_{\mathfrak{g}} \widetilde{chc}_{\mathsf{W}_s}(x'+x)\widetilde{chc}_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V})}(x'+x)\psi(x)d\mu(x)\\ &= \int_{\mathfrak{g}} \widetilde{chc}_{\mathsf{W}_s}(x'+x)chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V})}(x)\psi(x)d\mu(x)\end{aligned}$$

Hence, we are applying the case $\mathsf{V}'_c = 0$ to the function $chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V})}(x)\psi(x)$ and continue the computation as follows, keeping in mind that the measures on G, $\mathrm{H}'|_{\mathsf{X}'} \subseteq \mathrm{G}$ and $\mathrm{G}/(\mathrm{H}'|_{\mathsf{X}'})$ are different in these two cases. If $\mathsf{U} = 0$ then

$$\int_{\mathfrak{g}} chc_{\mathsf{W}}(x'+x)\psi(x)d\mu(x)$$
$$= \sqrt{\frac{\dim_{\mathbb{D}}\mathsf{V}'}{\dim_{\mathbb{D}}\mathsf{V}'_s}}^{\dim_{\mathbb{R}}\mathrm{H}'} \frac{\sqrt{2}^{\dim_{\mathbb{R}} \mathrm{Hom}(\mathsf{V}'_s,\mathsf{V})+\dim_{\mathbb{R}}\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V})}}{\sqrt{\frac{1}{2}\dim_{\mathbb{D}} \mathsf{V}}^{\dim_{\mathbb{R}} \mathsf{X}'}}$$
$$\left| \det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'})\right|^{-1} \left|\det(\mathrm{ad}(x')_{\mathrm{Hom}(\mathsf{X}',\mathsf{V}'_c)})\right|^{-1} \int_{\mathrm{G}/(\mathrm{H}'|_{\mathsf{X}'})} \psi(g\cdot(x'))\, d\mu(g(\mathrm{H}'|_{\mathsf{X}'})).$$

If $\mathsf{U} \neq 0$ then the same argument applies.

This verifies Theorem 0.7 for our function ψ with the support contained in $\mathrm{G}\cdot\mathcal{U}_\epsilon$. Since $\mathrm{G}\cdot\mathcal{U}_\epsilon$ is a completely invariant neighborhood of an arbitrary semisimple point in the support of our distribution, the formula holds for any function $\psi \in \mathcal{S}(\mathfrak{g})$.

The proof of Corollary 0.8 is the same as the proof of Corollary 0.4.

5. The conjugacy classes in an ordinary classical Lie group

Let G be an ordinary classical Lie group, with the defining module V and the Lie algebra $\mathfrak{g}$. An element $g \in \mathrm{G}$, or rather the pair (g, V), is called *decomposable* if and only if there are two non-zero subspaces $\mathsf{V}', \mathsf{V}'' \subseteq \mathsf{V}$, (which are orthogonal if G is of type I), preserved by g and such that $\mathsf{V} = \mathsf{V}' \oplus \mathsf{V}''$. In this case we say that (g, V) is the direct sum of the elements (g, V') and (g, V''). The element g, or (g, V), is called *indecomposable* if and only if (g, V) is not decomposable.

Let G' be another ordinary classical Lie group with the defining module V' and the Lie algebra $\mathfrak{g}'$. Let $g' \in \mathrm{G}'$ and let $g \in \mathrm{G}$. We shall say that the two elements (g, V) and (g', V') are similar ($(g, \mathsf{V}) \approx (g', \mathsf{V}')$) if and only if the two groups G, G' are of the same type and there is a linear bijection $g_0 : \mathsf{V} \to \mathsf{V}'$ (an isometry in the type I case) such that $g' = g_0 g g_0^{-1}$. In particular, if $\mathrm{G} = \mathrm{G}'$, (and therefore $\mathsf{V} = \mathsf{V}'$), then (g, V) is similar to (g', V) if and only if g and g' are in the same G-orbit. For simplicity we shall use the following notation:

$$\begin{aligned} &g.h = ghg^{-1} \qquad (g, h \in \mathrm{G}), \\ &\mathrm{G}.S = \{g.h \,|\, g \in \mathrm{G},\ h \in S\} \qquad (\mathcal{S} \subseteq \mathrm{G}). \end{aligned}$$

Let $g \in \mathrm{G}$ and let $g = g_s g_n$ be the Jordan decomposition of g, [11, II, page 26]. Then $g_n = \exp(x_n)$, where $x_n \in \mathfrak{g}$ is nilpotent.

Let $m = ht(x_n, \mathsf{V})$ denote the height of x_n. (This is the smallest non-negative integer k such that $x_n^k \neq 0$, but $x_n^{k+1} = 0$.). When convenient, by the height of (g, V) we shall understand the height of (x_n, V). One says that (g, V) is uniform if and only if $\ker(x_n^m) = x_n(\mathsf{V})$. In this case let $\overline{\mathsf{V}} = \mathsf{V}/\ker(x_n)$ and for $v \in \mathsf{V}$, let $\overline{v} = v + x_n(\mathsf{V}) \in \overline{\mathsf{V}}$. With this notation set $\overline{g}(\overline{v}) = \overline{g(v)}$, and if G is the isometry group of a non-degenerate form τ, let $\overline{\tau}(\overline{u}, \overline{v}) = \tau(u, x_n^m(v))$; $u, v \in \mathsf{V}$. Then $\overline{\tau}$ is a non-degenerate form on $\overline{\mathsf{V}}$, and $\overline{g}$ preserves this form. Furthermore, the element $(\overline{g}, \overline{\mathsf{V}})$ is semisimple. We shall refer to it as to the semisimple element attached to (g, V).

Theorem 5.1. *[2, Proposition 2 and 3] The similarity class of a uniform element (g, V) is uniquely determined by $ht(g, \mathsf{V})$ and by the similarity class of $(\overline{g}, \overline{\mathsf{V}})$. If (g, V) is indecomposable then it is uniform, and $(\overline{g}, \overline{\mathsf{V}})$ is indecomposable. Any element (g, V) is the direct sum of indecomposables.*

The element (g, V) is said to be isotypic (or V is isotypic with respect to g) if (g, V) is the direct sum of mutually similar indecomposable elements. The number of these elements is called the multiplicity of (g, V), and the similarity class of any one of them, the type of (g, V). We see from Theorem 5.1 that the similarity class of an isotypic element is determined by its height, its multiplicity and the type of the semisimple element attached to it.

Corollary 5.2. *[2] Let $g, g' \in \mathrm{G}$. Let*

$$\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus \cdots$$

be the decomposition of V into isotypic components with respect to g, and let

$$\mathsf{V} = \mathsf{V}'_1 \oplus \mathsf{V}'_2 \oplus \cdots$$

be the decomposition of V into isotypic components with respect to g'. Then g and g' are in the same G-orbit if and only if, up to a permutation of indecies, (g, V_j) is similar to (g, V'_j) for all j. In particular the G-orbit of g is determined by the number of isotypic components of (g, V) together with the height, the multiplicity and the type of the semisimple element attached to each of them.

Lemma 5.3. *[2, Table A, p.360] The following is a complete list of semisimple, indecomposable elements (g, V) for g in an ordinary classical Lie group of type* I.

(i) $\underline{\mathbb{D} = \mathbb{R}}$

(a) τ*-symmetric;* $\mathsf{V} = \mathbb{R}v$; $\epsilon - \pm 1 \in \mathbb{R}$; $g(v) = \epsilon v$; $eig(g) = \{\epsilon\}$.

(b) τ*-skew-symmetric;* $\mathsf{V} = \mathbb{R}u \oplus \mathbb{R}v$; $\epsilon - \pm 1 \in \mathbb{R}$; $g(u) = \epsilon u$, $g(v) = \epsilon v$; $eig(g) = \{\epsilon\}$.

(c) $\mathsf{V} = \mathbb{R}u \oplus \mathbb{R}v$; $\tau(u,u) = \tau(v,v) = 0$, $\tau(u,v) = 1$; $a \in \mathbb{R} \setminus \{0, \pm - 1\}$; $g(u) = au$, $g(v) = a^{-1}v$; $eig(g) = \{a, a^{-1}\}$.

(d) $\mathsf{V} = \mathbb{R}u \oplus \mathbb{R}v$; $\tau(u,v) = 0$, $\tau(u,u) = \tau(v,v) = \pm 1$; $\xi \in \mathbb{R} \setminus \pi\mathbb{Z}$; $g(u) = \cos(\xi)u - \sin(\xi)v$, $g(v) = \cos(\xi)u + \sin(\xi)v$; $eig(g) = \{e^{\pm \mathsf{i}\xi}\}$.

(e) $\mathsf{V} = \mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}v_1 \oplus \mathbb{R}v_2$; $\tau(u_i, u_j) = \tau(v_i, v_j) = 0, \quad (i,j = 1,2)$, $\tau(u_1, v_1) = \tau(u_2, v_2) = 1$; $\xi \in \mathbb{R} \setminus \pi\mathbb{Z}$, $a > 0$; $g(u_1) = a\cos(\xi)u_1 - a\sin(\xi)u_2$, $g(v_1) = a^{-1}\cos(\xi)v_1 - a^{-1}\sin(\xi)v_2$, $g(u_2) = a\cos(\xi)u_1 + a\sin(\xi)u_2$, $g(v_2) = a^{-1}\cos(\xi)v_1 + a^{-1}\sin(\xi)v_2$; $eig(g) = \{a^{\pm 1}e^{\pm \mathsf{i}\xi}\}$.

(f) $\mathsf{V} = \mathbb{R}u \oplus \mathbb{R}v$; $\tau(u,u) = \tau(v,v) = 0$, $\tau(u,v) = 1 = -\tau(v,u)$; $\xi \in \mathbb{R} \setminus \pi\mathbb{Z}$; $g(u) = \cos(\xi)u - \sin(\xi)v$, $g(v) = \cos(\xi)u + \sin(\xi)v$,; $eig(g) = \{e^{\pm \mathsf{i}\xi}\}$.

(ii) $\underline{\mathbb{D} = \mathbb{C}}$

(a) $\iota = 1$; τ*-symmetric;* $\mathsf{V} = \mathbb{C}v$; $\epsilon - \pm 1 \in \mathbb{R}$; $g(v) = \epsilon v$; $eig(g) = \{\epsilon\}$.

(b) $\iota = 1$; τ*-skew-symmetric;* $\mathsf{V} = \mathbb{C}u \oplus \mathbb{C}v$; $\epsilon - \pm 1 \in \mathbb{R}$; $g(u) = \epsilon u$, $g(v) = \epsilon v$; $eig(g) = \{\epsilon\}$.

(c) $\iota \neq 1$; τ*-hermitian;* $\mathsf{V} = \mathbb{C}v$; $a \in \mathbb{C}$, $|a| = 1$; $g(v) = av$; $eig(g) = \{a\}$.

(d) $\iota = 1$; $\mathsf{V} = \mathbb{C}u \oplus \mathbb{C}v$; $\tau(u,u) = \tau(v,v) = 0$, $\tau(u,v) = 1$; $a \in \mathbb{C} \setminus \{0, \pm 1\}$; $g(u) = au$, $g(v) = a^{-1}v$; $eig(g) = \{a^{\pm 1}\}$.

(e) $\iota \neq 1$; $\mathsf{V} = \mathbb{C}u \oplus \mathbb{C}v$; $\tau(u,u) = \tau(v,v) = 0$, $\tau(u,v) = 1 = \tau(v,u)$; $a \in \mathbb{C}$, $|a| \neq 1$; $g(u) = au$, $g(v) = \iota(a)^{-1}v$; $eig(g) = \{a, \iota(a)^{-1}\}$.

(iii) $\underline{\mathbb{D} = \mathbb{H}}$ *(here* $\iota \neq 1$ *preserves* $\mathbb{C} \subseteq \mathbb{H}$*)*

(a) $\mathsf{V}=\mathbb{H}v$; $\epsilon-\pm 1\in\mathbb{R}$; $g(v)=\epsilon v$; $eig(g)=\{\epsilon\}$.

(b) $\mathsf{V}=\mathbb{H}v$; $a\in\mathbb{C}$, $|a|=1$, $a\neq\pm 1$; $g(v)=av$; $eig(g)=\{a^{\pm 1}\}$.

(c) $\mathsf{V}=\mathbb{H}u\oplus\mathbb{H}v$; $\tau(u,u)=\tau(v,v)=0$, $\tau(u,v)=1$; $a\in\mathbb{C}$, $|a|\neq 1$; $g(u)=au$, $g(v)=\iota(a)^{-1}v$; $eig(g)=\{a^{\pm 1},\iota(a)^{\pm 1}\}$.

Lemma 5.4. *[2, Table A, p.360] The following is a complete list of semisimple, indecomposable elements (g,V) for g in an ordinary classical Lie group of type* II.

(i) $\underline{\mathbb{D}=\mathbb{R}}$

(a) $\mathsf{V}=\mathbb{R}v$; $\epsilon-\pm 1\in\mathbb{R}$; $g(v)=\epsilon v$; $eig(g)=\{\epsilon\}$.

(b) $\mathsf{V}=\mathbb{R}v$; $\epsilon\in\mathbb{R}\setminus\{0,\pm 1\}$; $g(v)=\epsilon v$; $eig(x)=\{\epsilon\}$.

(c) $\mathsf{V}=\mathbb{R}u\oplus\mathbb{R}v$; $\xi\in\mathbb{R}\setminus\pi\mathbb{Z}$, $a>0$;$g(u)=a\cos(\xi)u-a\sin(\xi)v$, $g(v)=a\cos(\xi)u+a\sin(\xi)v$; $eig(g)=\{ae^{\pm i\xi}\}$.

(ii) $\underline{\mathbb{D}=\mathbb{C}}$

(a) $\mathsf{V}=\mathbb{C}v$; $\epsilon-\pm 1\in\mathbb{R}$; $g(v)=\epsilon v$; $eig(g)=\{\epsilon\}$.

(b) $\mathsf{V}=\mathbb{C}v$; $a\in\mathbb{C}\setminus\{0,\pm 1\}$; $g(v)=av$; $eig(g)=\{a\}$.

(iii) $\underline{\mathbb{D}=\mathbb{H}}$ *(here* $\iota\neq 1$ *preserves* $\mathbb{C}\subseteq\mathbb{H}$*)*

(a) $\mathsf{V}=\mathbb{H}v$; $\epsilon-\pm 1\in\mathbb{R}$; $g(v)=\epsilon v$; $eig(g)=\{\epsilon\}$.

(b) $\mathsf{V}=\mathbb{H}v$; $a\in\mathbb{C}\setminus\{0,\pm-1\}$; $g(v)=av$; $eig(g)=\{a,\iota(a)\}$.

By inspecting the lists of Lemmas 5.3 and 5.4, we deduce the following corollary :

Corollary 5.5. *Let* G *be an ordinary classical Lie group with the defining module* U. *(If* G *is of type* I, *then* G *is the group of isometries of a form* τ *on* U.*) Let* $g,g'\in$ G, *and let* $\mathsf{V},\mathsf{V}'\subseteq\mathsf{U}$ *be two subspaces such that the elements* (g,V) *and* (g',V') *are indecomposable and semisimple.*

If (g,V) *and* (g',V') *are not similar, then either* $eig(g|_{\mathsf{V}})\cap eig(g'|_{\mathsf{V}'})=\emptyset$ *or* $eig(g|_{\mathsf{V}})=eig(g'|_{\mathsf{V}'})$ *and the group* G *is of type* I, *the form* τ *is hermitian and the restrictions of* τ *to* V *and* V' *are definite of opposite signatures.*

In particular, if g *preserves a non-zero isotropic subspaces of* V, *then* (g,V) *and* (g',V') *are similar if and only if the sets* $eig(g|_{\mathsf{V}})$, $eig(g'|_{\mathsf{V}'})$ *have a non-empty intersection:*

$$(g,\mathsf{V})\approx(g',\mathsf{V}')\ \ \textit{if and only if}\ \ eig(g|_{\mathsf{V}})\cap eig(g'|_{\mathsf{V}'})\neq\emptyset.$$

A slice through a semisimple point of G**.**

Let G be an ordinary classical Lie group, with the Lie algebra $\mathfrak{g}$. Fix a semisimple element $g\in\mathrm{G}$. Let $\mathfrak{z}\subseteq\mathfrak{g}$ denote the centralizer of g. Let $\mathrm{Z}\subseteq\mathrm{G}^g$ be the subgroup which acts trivially on the center of $\mathfrak{z}$. Let $r\subseteq\mathfrak{g}$ be the range of $1-\mathrm{Ad}(g)$. Set

$$\mathrm{Z}'=\{z\in\mathrm{Z}\,|\,\det(1-\mathrm{Ad}(g))_r\neq 0\}.$$

This is a G^g-invariant subset of Z, closed under taking the inverse. Let $\epsilon > 0$ and

$$\mathfrak{z}_\epsilon = \{y \in \mathfrak{z} \,|\, |\lambda| < \epsilon \text{ for all eigenvalues } \lambda \text{ of } y\}.$$

Define

$$\mathcal{Z}_\epsilon = g\,\exp(\mathfrak{z}_\epsilon), \quad \mathcal{G}_\epsilon = \mathrm{G}.\mathcal{Z}_\epsilon, \qquad (\epsilon > 0). \tag{5.1}$$

Theorem 5.6. *[11, II, p. 37] The sets $\mathcal{Z}_\epsilon$ are G^g-stable. For all sufficiently small $\epsilon > 0$, we have the following:*

(i) $\mathcal{Z}_\epsilon$ is an open subset of Z', and the map $\mathfrak{z}_\epsilon \ni y \to g\exp(y) \in \mathcal{Z}_\epsilon$ is an analytic diffeomorphism;

(ii) If $h \in \mathrm{G}$ and $h.[\mathrm{Z}_\epsilon] \cap \mathcal{Z}_\epsilon \neq \emptyset$ then $h \in \mathrm{G}^g$;

(iii) If $\mathcal{Z}_1 \subseteq \mathcal{Z}_\epsilon$ is G^g-invariant and closed in Z, then $\mathrm{G}.\mathcal{Z}_1$ is closed in G;

(iv) If $0 < \epsilon' < \epsilon$, then $Cl(\mathcal{Z}_{\epsilon'}) \subseteq \mathcal{Z}_\epsilon$;

(v) There is $\epsilon > 0$ such that the family $\mathcal{G}_{\epsilon'}$, $0 < \epsilon' < \epsilon$, is a basis for the completely invariant open neighborhoods of g in G.

Let G be an ordinary classical Lie group with the defining module V. Suppose

$$\mathsf{V} = \mathsf{V}' \oplus \mathsf{U}$$

is a direct sum decomposition, which is orthogonal in the type I case. The above decomposition induces the obvious embeddings

$$\mathrm{G}(\mathsf{V}') \times \mathrm{G}(\mathsf{U}) \subseteq \mathrm{G}(\mathsf{V}) = \mathrm{G}; \ \mathfrak{g}(\mathsf{V}') \oplus \mathfrak{g}(\mathsf{U}) \subseteq \mathfrak{g}(\mathsf{V}) = \mathfrak{g}.$$

Lemma 5.7. *Let $g \in \mathrm{G}$ be a semisimple element which preserves V' and U. If G is of type* I, *assume that V' has a complete polarization $\mathsf{V}' = \mathsf{X}' \oplus \mathsf{Y}'$ preserved by g. Then, with the notation (5.1), there is $\epsilon > 0$ such that for any $\epsilon' \in]0, \epsilon[$*

$$(g\mathrm{G}(\mathsf{U})) \cap \mathcal{G}_{\epsilon'} = \mathrm{G}(\mathsf{U})[(g\mathrm{G}(\mathsf{U})) \cap \mathcal{Z}_{\epsilon'}].$$

Proof. Obviously, for any $\epsilon' > 0$, the right hand side is contained in the left hand side. We shall find $\epsilon > 0$ such that the left hand side is contained in the right hand side.

Let

$$\mathsf{V}' = \mathsf{V}'_1 \oplus \mathsf{V}'_2 \oplus \cdots$$

be the decomposition of V' into isotypic components with respect to g. For $j \geq 1$ let $\mathsf{U}_j \subseteq \mathsf{U}$ be an isotypic component with respect to g, of the same type as V'_j, if it exists. If not, set $\mathsf{U}_j = 0$. Let $\mathsf{U}_0 \subseteq \mathsf{U}$ be the sum of all the other $g-$isotypic components of U. Set $\mathsf{V}_j = \mathsf{V}'_j \oplus \mathsf{U}_j$, $j \geq 1$. Then

$$\mathsf{V} = \mathsf{U}_0 \oplus \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus \cdots \tag{5.2}$$

Let ϵ be the minimum of all the numbers

$$\begin{aligned}
&|\ln(|\lambda_i\lambda_j^{-1}|)| \qquad (\lambda_i \in eig(g|_{\mathsf{V}_i}),\ \lambda_j \in eig(g|_{\mathsf{V}_j}),\ i \neq j;\ i,j \geq 1) \\
&|\ln(|\lambda_i\lambda_0^{-1}|)| \qquad (\lambda_i \in eig(g|_{\mathsf{V}_i}),\ \lambda_0 \in eig(g|_{\mathsf{U}_0});\ i \geq 1)
\end{aligned} \tag{5.3}$$

where the ln stands for the natural logarithm.

An arbitrary element of the set $(g\mathrm{G}(\mathsf{U})) \cap \mathcal{G}_{\epsilon'}$ looks as follows

$$gh = g_0[gz], \tag{5.4}$$

where $h \in \mathrm{G}(\mathsf{U})$, $z \in \exp(\mathfrak{z}_{\epsilon'})$ and $g_0 \in \mathrm{G}$. In particular, for each $j \geq 1$, V contains an gz-isotypic subspace W_j such that (gz, W_j) is similar to (g, V'_j). Since gz commutes with g we have

$$W_j = \bigoplus_{i \geq 1} W_j \cap \mathsf{V}_i \oplus W_j \cap \mathsf{U}_0.$$

Let $B_{\epsilon'} \subseteq \mathbb{C}$ be the disc of radius ϵ', centered at zero. For $i \geq 1$,

$$eig((gz)|_{W_j \cap \mathsf{V}_i}) \subseteq eig((gz)|_{\mathsf{V}_i}) \subseteq eig(g|_{\mathsf{V}_i}) \exp(B_{\epsilon'}) = eig(g|_{\mathsf{V}'_i})\, exp(B_{\epsilon'}).$$

Moreover,

$$eig((gz)|_{W_j \cap \mathsf{V}_i}) \subseteq eig((gz)|_{W_j}) = eig(g|_{\mathsf{V}'_j}).$$

Thus

$$eig((gz)|_{W_j \cap \mathsf{V}_i}) \subseteq (eig(g|_{\mathsf{V}'_i})\, exp(B_{\epsilon'})) \cap eig(g|_{\mathsf{V}'_j}). \tag{5.5}$$

Similarly

$$eig((gz)|_{W_j \cap \mathsf{U}_0}) \subseteq (eig(g|_{\mathsf{U}_0})\, exp(B_{\epsilon'})) \cap eig(g|_{\mathsf{V}'_j}). \tag{5.6}$$

By (5.3), the set (5.6) is empty, and the set (5.5) is non-empty if and only if $i = j$. Therefore, $W_j \subseteq \mathsf{V}_j$ for all $j \geq 1$.

Thus, after conjugating gz by an element of G, which preserves the decomposition (5.2), we may assume that gz preserves V'_j and U_j, and that $z|_{V'_j} = 1$, $j \geq 1$. Hence $z \in \mathrm{G}(\mathsf{U})$.

Let $\mathsf{U}(h) \subseteq \mathsf{U}$ be an gh-isotypic component. Suppose $(gh, \mathsf{U}(h))$ is of different type than any of the $(gh, \mathsf{V}'_j) = (g, \mathsf{V}'_j)$, $j \geq 1$. Then by (5.4) there is an $(gz)-$isotypic component $\mathsf{U}(z) \subseteq \mathsf{U}$ such that $(gh, \mathsf{U}(h))$ is similar to $(gz, \mathsf{U}(z))$. Suppose $(gh, \mathsf{U}(h))$ is of the same type as (gh, V'_j). Then, again by (5.4), there is an gz-isotypic component $\mathsf{U}(z) \subseteq \mathsf{U}$ such that $(gh, \mathsf{V}'_j + \mathsf{U}(h))$ is similar to $(gz, \mathsf{V}'_j + \mathsf{U}(z))$. Then, by Corollary 5.2, $(gh, \mathsf{U}(h))$ and $(gz, \mathsf{U}(z))$ are similar. Thus each $gh-$isotypic component of U is similar to some $gz-$isotypic component of U. Since the roles of h and z may be reversed, we see that $(gh)|_{\mathsf{U}}$ and $(gz)|_{\mathsf{U}}$ are in the same $\mathrm{G}(\mathsf{U})$-orbit. ∎

6. Proof of Theorem 0.6

The distribution $|Chc|$.

Let X be a finite dimensional vector space over the division algebra $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Consider a direct sum decomposition

$$\mathsf{X} = \mathsf{X}' \oplus \mathsf{X}'',$$

where $\dim_{\mathbb{D}}(\mathsf{X}') = 1$. Let $\mathrm{Q} \subseteq \mathrm{GL}(\mathsf{X})$ be the parabolic subgroup preserving X''. The Levi factor of Q is isomorphic to $\mathrm{GL}(\mathsf{X}') \times \mathrm{GL}(\mathsf{X}'')$. Let $\delta_{-1} \in \mathcal{D}'(\mathrm{GL}(\mathsf{X}'))$ be the Dirac delta at $-1 \in \mathrm{GL}(\mathsf{X}')$. Define the following distribution (generalized function) on $\mathrm{GL}(\mathsf{X})$:

$$|Chc(g)| = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}(\mathsf{X}')}(\delta_{-1} \otimes 1)(g) \qquad (g \in \mathrm{GL}(\mathsf{X})).$$

(See the Appendix A for the definition of an induced distribution.) This is a positive Borel measure. In terms of Lemma 13.1 in [10], we have

$$|Chc(g)| = |\det(g)|^{1/2}\, \delta(\det(g+1)) \qquad (g \in \mathrm{GL}(\mathsf{X})), \tag{6.1}$$

if $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$. If $\mathbb{D} = \mathbb{H}$, then the distribution on the left hand side of (6.1) is the restriction of the one on the right hand side via the embedding $\mathrm{GL}(\mathsf{X}) \subseteq \mathrm{GL}(\mathsf{X}|_{\mathbb{C}})$, where $\mathsf{X}|_{\mathbb{C}}$ stands for X viewed as a vector space over $\mathbb{C} \subseteq \mathbb{H}$. Sometimes it will be helpful to include the vector space X in our notation and write $|Chc_{\mathsf{X}}|$ for $|Chc|$.

Let V, V' be two finite dimensional vector spaces over $\mathbb{D}$ and let $\mathsf{X} = \mathrm{Hom}(\mathsf{V}', \mathsf{V})$. The groups $\mathrm{G} = \mathrm{GL}(\mathsf{V})$ and $\mathrm{G}' = \mathrm{GL}(\mathsf{V}')$ act on X as follows

$$g(x) = gx,\ g'(x) = xg'^{-1}, (g \in \mathrm{G},\ x \in \mathsf{X},\ g' \in \mathrm{G}'). \tag{6.2}$$

Hence we have the embeddings $\mathrm{G}, \mathrm{G}' \hookrightarrow \mathrm{GL}(\mathsf{X})$. Let θ be a Cartan involution on $\mathrm{GL}(\mathsf{X})$ which preserves both G and G'. Let $\mathrm{H}' = \mathrm{T}'\mathrm{A}' \subseteq \mathrm{G}'$ be a standard Cartan subgroup, with the compact part T' and the vector part A', as in [12]. Let

$$\mathsf{V}' = \mathsf{V}'_1 \oplus \mathsf{V}'_2 \oplus \cdots \tag{6.3}$$

be the decomposition into $\mathrm{H}'-$isotypic components.

Definition 6.1. For $h' \in \mathrm{H}'^{\mathrm{reg}}$, set

$$\begin{aligned} &|Chc_{h'}(g)| = |Chc_{\mathsf{X}}(h'g)| \\ = &\prod_{j \geq 1} |\det(h'g)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V})}|^{1/2}\, \delta\left(\det(h'g+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V})}\right)\ (g \in \mathrm{GL}(\mathsf{X})). \end{aligned}$$

By Proposition 1.8 in [10], the above product of distributions is well defined. The formulas (6.2) and the preceding formula imply that

$$\mathrm{supp}(|Chc_{h'}|) = \{g \in \mathrm{G} \mid eig(g) \cap eig(-h'|_{\mathsf{V}'_j}) \neq \emptyset \text{ for all } j \geq 1\}. \tag{6.4}$$

This set is empty if $\dim(\mathsf{V}') > \dim(\mathsf{V})$. Hence, from now we assume that $\dim(\mathsf{V}') \leq \dim(\mathsf{V})$. In particular V contains a subspace of the same dimension as V'. We shall identify such a subspace with V'. Let $\mathsf{U} \subseteq \mathsf{V}$ be a complementary subspace, so that

$$\mathsf{V} = \mathsf{V}' \oplus \mathsf{U}. \tag{6.5}$$

For $h' \in \mathrm{H}'^{\mathrm{reg}}$, let $I_{h'} \in \mathcal{D}'(\mathrm{G}')$ denote the orbital integral

$$I_{h'}(\Psi) = \int_{\mathrm{G}'/\mathrm{H}'} \Psi(g.h')\, d\mu(g\mathrm{H}'). \tag{6.6}$$

Let Q be the parabolic subgroup of G preserving U.

Theorem 6.2. *We have*

$$|Chc_{h'}| = \mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(I_{-h'} \otimes 1) \qquad (h' \in \mathrm{H}'^{\mathrm{reg}}).$$

Explicitly, for $\Psi \in \mathcal{D}(\mathrm{G})$ and $h' \in \mathrm{H}'^{\mathrm{reg}}$,

$$\int_{\mathrm{G}} |Chc(h'g)|\Psi(g)\, d\mu(g) = \int_{\mathrm{G}'/\mathrm{H}'} \int_{\mathrm{GL}(\mathsf{U})} \Psi^{\mathrm{L}}(h_1.(-h')h_2)\, d\mu(h_2)\, d\mu(h_1\mathrm{GL}(\mathsf{U})), \tag{6.7}$$

where $-h'$ is viewed as an element of $\mathrm{G}' \subseteq \mathrm{Q}$ preserving the decomposition (6.5) and acting as the identity on U.

Proof. Consider an element $g \in \mathrm{supp}(|Chc_{h'}|)$. Let

$$\mathsf{V} = \cdots \oplus \mathsf{V}_{-1} \oplus \mathsf{V}_0 \oplus \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus \cdots$$

be the decomposition V into isotypic components with respect to g, indexed in such a way that (in terms of Theorem 5.1) $(\overline{g}, \overline{\mathsf{V}}_j)$ is of the same type as $(-h', \mathsf{V}'_j)$, for each $j \geq 1$. By Corollary 5.5 and by (6.4), this is possible.

Let $g = g_s \exp(x_n)$ be the Jordan decomposition of g. For each $j \geq 1$ there is a subspace $W_j \subseteq \mathsf{V}_j$ such that

$$\mathsf{V}_j = W_j \oplus x_n W_j \oplus x_n^2 W_j \oplus \cdots$$

and $(\overline{g}, \overline{\mathsf{V}}_j)$ is similar to (g_s, W_j). In particular $\dim(W_j) = \dim(\overline{\mathsf{V}}_j)$. Let

$$\mathsf{U}' = \cdots + \mathsf{V}_{-1} + \mathsf{V}_0 + \sum_{j\geq 1} (x_n W_j + x_n^2 W_j + \cdots).$$

Then U' is g-invariant and $\dim(\mathsf{V}/\mathsf{U}') = \dim(\mathsf{V}')$. In particular there is $g_0 \in \mathrm{G}$ such that $\mathsf{U}' = g_0\mathsf{U}$. Hence, $g_0^{-1} g g_0$ preserves U and the resulting endomorphism of $\mathsf{V}' = \mathsf{V}/\mathsf{U}$ is conjugate to $-h'$.

We recall that the parabolic subgroup Q is equal to MAN, with $\mathrm{MA} = \mathrm{GL}(\mathsf{V}')\mathrm{GL}(\mathsf{U})$. We have just shown that

$$\mathrm{supp}(|Chc_{h'}|) = \mathrm{G}.[(-h')\mathrm{GL}(\mathsf{U})\mathrm{N}] = \mathrm{KGL}(\mathsf{V}').[-h']\mathrm{GL}(\mathsf{U})\mathrm{N}], \tag{6.8}$$

which, by Proposition A.5, coincides with the support of the distribution on the right hand side of (6.7). Since both distributions are positive Borel measures, and are G-invariant, this verifies the theorem 6.2 if $\mathsf{V}' = \mathsf{V}$.

By Harish-Chandra's Method of Descent, it will suffice to consider the distribution $|Chc_{h'}(x)| = |Chc(h'x)|$, $x \in \mathrm{G}$ in an arbitrary small completely invariant open neighborhood of a point in the support of $|Chc_{h'}|$. We shall need some additional notation.

Let $\mathfrak{h}''$ be the centralizer of $\mathfrak{h}'$ in $\mathfrak{sp}(\mathsf{W})$. Clearly $\mathfrak{h}'' \subset \mathfrak{a}''$.

From now on we assume that V' is a proper subspace of V. Let $\Psi \in C_c(\mathrm{G.L}')$. Then Proposition A.7 implies

$$\int_{\mathrm{G}} |Chc(h'g)|\Psi(g)\,d\mu(g) = \int_{\mathrm{G}'}\int_{\mathrm{GL}(\mathsf{U})} |Chc(h'h_1h_2)||\det(\mathrm{Ad}(h_1h_2)-1)_{\mathfrak{n}}| \\ |\det(\mathrm{Ad}(h_1h_2)_{\mathfrak{n}})|^{-1/2}\Psi^{\mathrm{L}}(h_1h_2)\,d\mu(h_2)\,d\mu(h_1). \quad (6.9)$$

Notice that for $h_1 \in \mathrm{GL}(\mathsf{V}')$ and $h_2 \in \mathrm{GL}(\mathsf{U})$, with $\det(\mathrm{Ad}(h_1h_2)-1)_{\mathfrak{n}} \neq 0$,

$$\begin{aligned} & |Chc(h'h_1h_2)| \qquad (6.10)\\ = & |Chc_{\mathrm{Hom}(\mathsf{V}',\mathsf{V})}(h'h_1h_2)| \\ = & \prod_{j\geq 1} |\det(h'h_1h_2)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V})}|^{1/2}\delta(\det(h'h_1h_2+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V})}) \\ = & \prod_{j\geq 1} \Big(|\det(h'h_1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}')}\det(h'h_2)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{U})}|^{1/2} \\ & \qquad \delta(\det(h'h_1+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}')}\det(h'h_2+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{U})})\Big) \\ = & \prod_{j\geq 1} |\det(h'h_1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}')}|^{1/2}\delta(\det(h'h_1+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}')}) \\ & \qquad \prod_{j\geq 1} |\det(h'h_2)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{U})}|^{1/2}|\det(h'h_2+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{U})}|^{-1} \\ = & |Chc_{\mathrm{Hom}(\mathsf{V}',\mathsf{V}')}(h'h_1)||\det(h'h_2)_{\mathrm{Hom}(\mathsf{V}',\mathsf{U})}|^{1/2}|\det(h'h_2+1)_{\mathrm{Hom}(\mathsf{V}',\mathsf{U})}|^{-1} \\ = & |Chc_{\mathrm{Hom}(\mathsf{V}',\mathsf{V}')}(h'h_1)||\det(\mathrm{Ad}(h'h_2)_{\mathfrak{n}}|^{1/2}|\det(\mathrm{Ad}(h'h_2)-1)_{\mathfrak{n}}|^{-1}, \end{aligned}$$

where $\mathfrak{n} = \mathrm{Hom}(\mathsf{V}',\mathsf{U})$ is the Lie algebra of the group N, and the last equation follows from our formula for the case $\mathsf{V}' = \mathsf{V}$, considered previously. Clearly, (6.9) and (6.10) imply Theorem 6.2.

We have shown so far (see Proposition A.5 and equality (6.8)) that the two positive invariant Borel measures which occur in (6.7) agree on $\mathrm{G}^{\mathrm{reg}}$. In order to complete the proof it will suffice to show that these measures vanish on $\mathrm{G}\setminus\mathrm{G}^{\mathrm{reg}}$. Proposition A.8 implies that this is indeed the case for the distribution $Ind_{\mathrm{L}}^{\mathrm{G}}(I_{-h'}\otimes 1)$. Hence we need to show that

$$\int_{\mathrm{G}} |Chc(h'g)|\Psi(g)\,d\mu(g) = \int_{\mathrm{G}^{\mathrm{reg}}} |Chc(h'g)|\Psi(g)\,d\mu(g),$$

for $\Psi \in \mathcal{D}(\mathrm{G})$. We shall use an argument parallel to the proof of [10, Lemma 7.10].

In order to complete the proof, we shall use an argument parallel to the proof of [10, Lemma 7.10]. As in [10, Section 7],

$$\mathrm{GL}(\mathsf{X})^{\mathrm{H}'} = \mathrm{GL}(\mathrm{Hom}(\mathsf{V}'_1,\mathsf{V}))^{\mathrm{H}'} \times \mathrm{GL}(\mathrm{Hom}(\mathsf{V}'_2,\mathsf{V}))^{\mathrm{H}'} \times \cdots$$

The restriction of $|Chc|$ from $\mathrm{GL}(\mathsf{X})^{\mathrm{A}'}$ to $\mathrm{GL}(\mathsf{X})^{\mathrm{H}'}$ exists and is given by

$$|Chc(g_1 g_2, \cdots)| = \prod_{j \geq 1} |\det(g_j)|^{1/2} \delta(\det(g_j + 1)), \tag{6.11}$$

where $g_j \in \mathrm{GL}(\mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}))^{\mathrm{H}'}$ and $\delta(\det \cdots)$ is as in [10, equality (5.10)] depending on the field $\mathfrak{h}'|_{\mathsf{V}'_j}$. Furthermore, the distribution $|Chc_{h'}|$ coincides with the pull-back of the distribution (6.11) via the embedding

$$\mathrm{G} \ni g \to h'g \in \mathrm{GL}(\mathsf{X})^{\mathrm{H}'}. \tag{6.12}$$

Thus the preceding formula holds with the determinant having values in the field $\mathfrak{h}'|_{V'_j}$. Let $g_0 \in \mathrm{G}$ be a semisimple element in the support of $|Chc_{h'}|$. Let

$$\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus \mathsf{V}_3 \oplus \cdots$$

be the decomposition of V into isotypic components with respect to g_0. Recall the decomposition (6.3). Each $\mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_k)$ is a vector space over the field $\mathfrak{h}'|_{\mathsf{V}'_j}$. As such it is either isotypic with respect to the action of g_0 or is the direct sum of two different isotypic components. The second possibility occurs if and only if $\mathbb{D} \neq \mathbb{C}$ and both fields $\mathfrak{h}'|_{\mathsf{V}'_j}$ and $\mathfrak{g}^{g_0}|_{\mathsf{V}_k}$ are isomorphic to $\mathbb{C}$. In any case we shall write

$$\mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_k) = \mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_k)_1 \oplus \mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_k)_2$$

keeping in mind the possibility that the second summand might be zero. Furthermore

$$\begin{aligned} \mathrm{G}^{g_0} &= \mathrm{GL}(V_1)^{g_0} \times \mathrm{GL}(\mathsf{V}_2)^{g_0} \times \cdots \\ \mathrm{GL}(\mathsf{X})^{\mathrm{H}' \cup \{g_0\}} &= \prod_{j,k,l} \mathrm{GL}(\mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_k)_l), \end{aligned}$$

and the embedding (6.12) restricts to

$$\mathrm{G}^{g_0} \ni g \to h'g \in \mathrm{GL}(\mathsf{X})^{\mathrm{H}' \cup \{g_0\}}.$$

We may arrange the indecies so that

$$\begin{aligned} \det(h'g_0 + 1)_{\mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_j)_1} &= 0 \quad \text{for all } j, \text{and} \\ \det(h'g_0 + 1)_{\mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_k)_l} &\neq 0 \quad \text{for } j \neq k, \text{ or } j = k \text{ and } l \neq 1. \end{aligned} \tag{6.13}$$

Let $\mathcal{Z}_\epsilon$ be a completely invariant open neighborhood of g_0 in the set of regular elements of G^{g_0}, so small that the second condition in (6.13) holds with the g_0 replaced by $g \in \mathcal{Z}_\epsilon$. Similarly, let $\mathcal{Z}'_{\epsilon'}$ be a completely invariant open neighborhood of $h'g_0$ in the set of regular elements of $\mathrm{GL}(\mathsf{X})^{\mathrm{H}' \cup \{g_0\}}$, so small that for each $g \in \mathcal{Z}'_{\epsilon'}$ and for each j at most one of the determinants $\det(g)_{\mathrm{Hom}(\mathsf{V}'_j, \mathsf{V}_k)_l}$ is zero. Then the restriction of the distribution (6.11) to $\mathcal{Z}'_{\epsilon'}$ may be written as

$$\prod_j \left(\prod_{k,l} |\det(g)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_k)_l}|^{1/2}\, \delta(\prod_{k,l} \det(g+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_k)_l}) \right)$$

$$= \prod_j \left(\sum_{k,l} |\det(g)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_k)_l}|^{1/2}\, \delta(\det(g+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_k)_l}) \right.$$

$$\left. \prod_{(k',l')\neq(k,l)} |\det(g)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_{k'})_{l'}}|^{1/2} |\det(g+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_{k'})_{l'}}|^{-1} \right).$$

Hence, assuming $\mathcal{Z}_\epsilon$ is small enough, we have for $g \in \mathcal{Z}_\epsilon$

$$|Chc(h'g)| = \prod_j \left(\sum_{k,l} |\det(h'g)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_k)_l}|^{1/2}\, \delta(\det(h'g+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_k)_l}) \right.$$

$$\left. \prod_{(k',l')\neq(k,l)} |\det(h'g)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_{k'})_{l'}}|^{1/2} |\det(h'g+1)_{\mathrm{Hom}(\mathsf{V}'_j,\mathsf{V}_{k'})_{l'}}|^{-1} \right).$$

As we can get the same formula for the left hand side of (6.7), we deduce Theorem 6.2. ∎

The Cauchy Harish-Chandra integral for a pair of type II, as an induced distribution.

Let W be a symplectic space over the reals. Let $\widetilde{\mathrm{Sp}}(\mathsf{W})$ be the metaplectic group. For any subgroup $\mathrm{E} \subseteq \mathrm{Sp}(\mathsf{W})$ let $\widetilde{\mathrm{E}}$ be the preimage of E under the covering map $\widetilde{\mathrm{Sp}}(\mathsf{W}) \ni g \to \underline{g} \in \mathrm{Sp}(\mathsf{W})$. (see (0.2)).

Let $\mathsf{W} = \mathsf{X} \oplus \mathsf{Y}$ be a complete polarization and let $\mathrm{Z} \subseteq \mathrm{Sp}(\mathsf{W})$ be the subgroup preserving both X and Y. Set

$$\widetilde{\mathrm{GL}}(\mathsf{X}) = \{(g,\eta) \in \mathrm{GL}(\mathsf{X}) \times \mathbb{C}^\times \mid \eta^2 = \det(g)\}.$$

This is a double cover of $\mathrm{GL}(\mathsf{X})$ under the map $(g,\eta) \to g$. Let ω denote the oscillator representation of $\widetilde{\mathrm{Sp}}(\mathsf{W})$, with the distribution character Θ, as in [5] or [10]. By considering the Schrödinger model of ω associated to the polarization $\mathsf{W} = \mathsf{X} \oplus \mathsf{Y}$, as in [5, Lemma 2.17], we deduce the following lemma:

Lemma 6.3. *The restriction map*

$$\mathrm{Z} \ni \underline{g} \to \underline{g}|_{\mathsf{X}} \in \mathrm{GL}(\mathsf{X}) \tag{6.14}$$

lifts to a group isomorphism

$$\widetilde{\mathrm{Z}} \ni g \to (\underline{g}|_{\mathsf{X}}, \eta) \in \widetilde{\mathrm{GL}}(\mathsf{X}), \tag{6.15}$$

where

$$\eta = \eta(g) = \frac{\Theta(g)}{|\Theta(g)|} |\det(\underline{g}|_{\mathsf{X}})|^{1/2} \quad \textit{if } \det(\underline{g}-1)_{\mathsf{W}} \neq 0.$$

In terms of Lemma 6.3 define

$$\epsilon(g) = \frac{\eta(g)}{|\eta(g)|} \qquad (g \in \widetilde{\mathrm{Z}}). \tag{6.16}$$

Then ϵ is a character of $\widetilde{\mathrm{Z}}$, with values in the group $\{z \in \mathbb{C} \,|\, z^4 = 1\}$. Moreover

$$\epsilon(g) = \frac{\Theta(g)}{|\Theta(g)|} \qquad (g \in \widetilde{\mathrm{Z}},\ \det(\underline{g} - 1)_{\mathsf{W}} \neq 0). \tag{6.17}$$

The function on the right hand side of (6.17) is well defined on $\widetilde{\mathrm{Sp}}^c(\mathsf{W}) \subseteq \widetilde{\mathrm{Sp}}(\mathsf{W})$, the domain of the Cayley transform c, and clearly does not depend on the polarization $\mathsf{W} = \mathsf{X} \oplus \mathsf{Y}$. Notice also that this function does not extend to a continuous function of the whole metaplectic group, because it is not constant.

We shall identify Z with $\mathrm{GL}(\mathsf{X})$ via (6.14), and $\widetilde{\mathrm{Z}}$ with $\widetilde{\mathrm{GL}}(\mathsf{X})$ via (6.15). In these terms, by [10, Lemma 13.1], the Cauchy Harish-Chandra integral for the group $\widetilde{\mathrm{GL}}(\mathsf{X})$ is given by the following formula

$$Chc(g) = Chc_{\mathsf{W}}(g) = \epsilon(g)|Chc_{\mathsf{X}}(\underline{g})| \qquad (g \in \widetilde{\mathrm{GL}}(\mathsf{X})). \tag{6.18}$$

Consider a dual pair of type II, $(\mathrm{G}, \mathrm{G}') \subseteq \mathrm{Sp}(\mathsf{W})$. We may assume that the groups G, G' preserve X and Y. Thus $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Z}$. The restriction (6.14) maps G, G' isomorphically onto a dual pair in $\mathrm{GL}(\mathsf{X})$. Recall the Cartan subgroup $\mathrm{H}' = \mathrm{T}'\mathrm{A}' \subseteq \mathrm{G}'$. The formula (6.18) implies

$$Chc_{\mathsf{W}}(g) = \epsilon(g)|Chc_{\mathsf{X}}(\underline{g})| \qquad (g \in \widetilde{\mathrm{A}}''),$$

where $\mathrm{A}'' \subseteq \mathrm{Z}$ is the centralizer of A' in the symplectic group $\mathrm{Sp}(\mathsf{W})$. Hence,

$$Chc_{\mathsf{W}}(h'g) = \epsilon(h'g)|Chc_{\mathsf{X}}(\underline{h'g})| \qquad (h' \in \widetilde{\mathrm{H}}'^{\mathrm{reg}},\ g \in \widetilde{\mathrm{G}}). \tag{6.19}$$

As we have seen in Section 6, the distribution (6.19) is zero if $\dim(\mathsf{V}') > \dim(\mathsf{V})$. Hence, from now on, we assume that $\dim(\mathsf{V}') \leq \dim(\mathsf{V})$ and use the notation of Theorem 6.2. As in (A.3) we have for $\Psi \in \mathcal{D}(\widetilde{\mathrm{G}})$, the Harish-Chandra transform

$$\Psi^{\widetilde{\mathrm{L}}}(l) = |\det(\mathrm{Ad}(\underline{h})_{\mathfrak{n}})|^{1/2}\Psi^{\mathrm{K}}_{\mathrm{N}_0}(l)$$

with

$$\Psi^{\mathrm{K}}_{\mathrm{N}_0}(l) = \int_{\widetilde{\mathrm{K}}}\int_{\widetilde{\mathrm{N}}_0} \Psi(k.[hn])\, d\mu(n)\, d\mu(k) \qquad (l \in \widetilde{\mathrm{L}}),$$

where $\widetilde{\mathrm{N}}_0$, the identity component of $\widetilde{\mathrm{N}}$, is the unipotent radical of $\widetilde{\mathrm{Q}}$.

The Levi factor of the group $\widetilde{\mathrm{Q}}$ is equal to $\widetilde{\mathrm{G}}'\widetilde{\mathrm{GL}}(\mathsf{U}) = \widetilde{\mathrm{L}}$. Furthermore, $\widetilde{\mathrm{G}}' \cap \widetilde{\mathrm{GL}}(\mathsf{U}) = \{1, \tilde{1}\}$ where $\tilde{1}$ is the nontrivial element in the preimage of the identity.

Lemma 6.4. *We have for any $\Psi \in \mathcal{D}(\widetilde{\mathrm{G}})$ and any $h' \in \widetilde{\mathrm{H}}'^{\mathrm{reg}}$,*

$$\int_{\widetilde{\mathrm{G}}} |Chc(\underline{h'g})|\Psi(g)\, d\mu(g) = \int_{\mathrm{G}'/\mathrm{H}'}\int_{\widetilde{\mathrm{GL}}(\mathsf{U})} \Psi^{\widetilde{\mathrm{L}}}(h_1.(\mathfrak{d}h')h_2)\, d\mu(h_2)\, d\mu(h_1\mathrm{H}'). \tag{6.20}$$

Proof. Let $\mathcal{U}$ an open set of G such that $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ are canonically isomorphic and $\Psi \in \mathcal{D}(\widetilde{\mathrm{G}})$ such that $\mathrm{supp}(\Psi) \subset \widetilde{\mathcal{U}}$. We consider than the function $\Phi \in \mathcal{D}(\mathrm{G})$ such that $\Psi(x) = \Phi(\underline{x})$ for $x \in \widetilde{\mathcal{U}}$ and $\Phi(x) = 0$ for $x \in \mathrm{G}\backslash\mathcal{U}$. We have the relations

$$\Psi^{\widetilde{\mathrm{L}}}(l) = 2\Phi^{\mathrm{L}}(\underline{l}) \quad \text{for } l \in \widetilde{\mathrm{L}}$$

and

$$\int_{\widetilde{\mathrm{K}}}\int_{\widetilde{\mathrm{N}}_0} \Psi^{\widetilde{\mathrm{L}}}(k.(hn))d\mu(k)d\mu(n) = \int_{\mathrm{K}}\int_{\mathrm{N}} \Phi^{\mathrm{L}}(k.(\underline{h}n))d\mu(k)d\mu(n) \quad \text{for } h \in \widetilde{\mathrm{L}}$$

We deduce that the formula (6.20) is verified for Ψ and with partition of unity, we deduce the formula for all functions. ■

By combining Lemma 6.4 with (6.19), we deduce the following theorem:

Theorem 6.5. *We have the equality*

$$Chc_{h'} = \epsilon(h')\,\mathrm{Ind}_{\widetilde{\mathrm{L}}}^{\widetilde{\mathrm{G}}}(\epsilon(\mathfrak{d}h'))I_{(\mathfrak{d}h')} \otimes \epsilon) \qquad (h' \in \widetilde{\mathrm{H}}'^{\mathrm{reg}}).$$

Here $I_{\mathfrak{d}h'} \in D'(\widetilde{\mathrm{G}}')$ is the orbital integral defined as in (6.6). Explicitly, for $\Psi \in \mathcal{D}(\widetilde{\mathrm{G}})$ and $h' \in \widetilde{\mathrm{H}}'^{\mathrm{reg}}$,

$$\int_{\widetilde{\mathrm{G}}} Chc(h'g)\Psi(g)\,d\mu(g) = \epsilon(h')\int_{\mathrm{G}'/\mathrm{H}'}\int_{\widetilde{\mathrm{GL}}(\mathsf{U})} \epsilon(\mathfrak{d}h'h_2)\Psi^{\widetilde{\mathrm{L}}}(h_1.(\mathfrak{d}h')h_2)\,d\mu(h_2)\,d\mu(h_1\mathrm{H}').$$

7. Proof of Theorem 0.9

Consider a dual pair $(\mathrm{G},\mathrm{G}') \subseteq \mathrm{Sp}(\mathsf{W})$ of type I. We consider the notations of Section 4. Let $\mathrm{Q}' \subseteq \mathrm{G}'$ and $\mathrm{Q} \subseteq \mathrm{G}$ be the parabolic subgroup with the Lie algebras $\mathfrak{q}'$ and $\mathfrak{q}$. Then $\mathrm{M}'\mathrm{A}' = \mathrm{GL}(\mathsf{X}')\mathrm{G}(\mathsf{V}'_c)$ and $\mathrm{MA} = \mathrm{GL}(\mathsf{X}')\mathrm{G}(\mathsf{U})$. Let

$$\begin{aligned}
\mathsf{X} &= \mathrm{Hom}(\mathsf{X}',\mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'),\\
\mathsf{Y} &= \mathrm{Hom}(\mathsf{Y}',\mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}'_c,\mathsf{Y}'),\\
\mathsf{W}_s &= \mathsf{X} \oplus \mathsf{Y}, \quad \mathsf{W}_c = \mathrm{Hom}(\mathsf{V}'_c,\mathsf{U}).
\end{aligned}$$

Then the restrictions of the symplectic form to both W_s and W_c are non-degenerate and X, Y are isotropic subspaces. Furthermore

$$\mathsf{W} = \mathsf{W}_s \oplus \mathsf{W}_c \tag{7.1}$$

is an orthogonal direct sum decomposition. We may and shall assume that the Cartan involution θ is such that,

$$\begin{aligned}
\theta\mathfrak{n} &= \mathrm{Hom}(\mathsf{V},\mathsf{X}') \oplus \mathrm{Hom}(\mathsf{Y}',\mathsf{X}') \cap \mathfrak{g},\\
\theta\mathfrak{n}' &= \mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}') \oplus \mathrm{Hom}(\mathsf{Y}',\mathsf{X}') \cap \mathfrak{g}'.
\end{aligned}$$

Any element $g \in \mathrm{Sp}(\mathsf{W})$, which preserves the decomposition (7.1), may be written as

$$g = g_s g_c,$$

where g_s acts trivially on W_c and g_c acts trivially on W_s. We identify g_s with its restriction to W_s and g_c with its restriction to W_c. In these terms, $g_s \in \mathrm{Sp}(\mathsf{W}_s) \subseteq \mathrm{Sp}(\mathsf{W})$ and $g_c \in \mathrm{Sp}(\mathsf{W}_c) \subseteq \mathrm{Sp}(\mathsf{W})$. Furthermore any element $g \in \widetilde{\mathrm{Sp}}(\mathsf{W})$ may be decomposed as

$$g = g_s g_c, \tag{7.2}$$

where $g_s \in \widetilde{\mathrm{Sp}}(\mathsf{W}_s)$ and $g_c \in \widetilde{\mathrm{Sp}}(\mathsf{W}_c)$. The decomposition (7.2) is unique up to simultaneous multiplication by $\tilde{1}$.

Since the groups L and H$'$ preserve W_s and W_c, we shall use the notation (7.2) for the elements of $\widetilde{\mathrm{L}}$ and $\widetilde{\mathrm{H}}'$. Recall the character ϵ of the stabilizer of X and Y in $\widetilde{\mathrm{Sp}}(\mathsf{W}_s)$ defined in (6.16). For $h' \in \widetilde{\mathrm{H}}'^{\mathrm{reg}}$ let $\underline{h}'_s = \underline{h}'|_{\mathsf{X}'}$. We view h'_s as an element of $\mathrm{GL}(\mathsf{X}') \subseteq \mathrm{Q}$ preserving the decomposition (6.4) acting as the identity on U. Set $\mathrm{H}'_s = \mathrm{GL}(\mathsf{X}') \cap \mathrm{H}'$. The generalized function

$$v_{h'}(mu) = \epsilon(h'_s m u_s) I_{\mathfrak{d}h'_s}(m) Chc_{\mathsf{W}_c}(h'_c u_c) \qquad (m \in \widetilde{\mathrm{GL}}(\mathsf{X}'),\ u \in \widetilde{\mathrm{G}}(\mathsf{U})) \tag{7.3}$$

does not depend on the decomposition (7.2) of m and u.

Theorem 7.1. *We have the equality*

$$Chc_{\mathsf{W},h'} = |\det(\mathrm{Ad}(\underline{h}')_{\mathfrak{n}'})|^{-1/2} |\det(\mathrm{Ad}(\underline{h}'^{-1}) - 1)_{\mathfrak{n}'})|^{-1} \mathrm{Ind}_{\widetilde{\mathrm{L}}}^{\widetilde{\mathrm{G}}}(v_{h'}),$$

where $h' \in \widetilde{\mathrm{H}}'^{\mathrm{reg}}$. Explicitly, for $\Psi \in \mathcal{D}(\widetilde{\mathrm{G}})$,

$$\int_{\widetilde{\mathrm{G}}} Chc_{\mathsf{W}}(h'g)\Psi(g)\, d\mu(g) = |\det(\mathrm{Ad}(\underline{h}')_{\mathfrak{n}'})|^{-1/2} |\det(\mathrm{Ad}(\underline{h}'^{-1}) - 1)_{\mathfrak{n}'})|^{-1} \epsilon(h'_s)$$
$$\int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_s} \int_{\widetilde{\mathrm{G}}(\mathsf{U})} \epsilon((-h'_s)u_s) Chc_{\mathsf{W}_c}(h'_c u_c) \Psi^{\widetilde{\mathrm{L}}}(h.(-h'_s)u)\, d\mu(u)\, d\mu(h\mathrm{H}'_s). \tag{7.4}$$

Proof. Let $\mathrm{L}' = \{l \in \mathrm{L} \mid \det((\mathrm{Ad}(l) - 1)_{\mathfrak{n}}) \neq 0\}$, as before. Consider first the case

$$\mathrm{supp}(\Psi) \subseteq \mathrm{G.L}'. \tag{7.5}$$

Then, by Proposition A.7,

$$\int_{\widetilde{\mathrm{G}}} Chc_{\mathsf{W}}(h'g)\Psi(g)\, d\mu(g) = \int_{\widetilde{\mathrm{L}}'} Chc_{\mathsf{W}}(h'l) |\det(\mathrm{Ad}(l) - 1)_{\mathfrak{n}})| |\det(\mathrm{Ad}(l)_{\mathfrak{n}})|^{-1/2} \Psi^{\widetilde{\mathrm{Q}}}(l)\, d\mu(l).$$

For $l \in \widetilde{\mathrm{L}}$, we have

$$Chc_{\mathsf{W}}(h'l) = Chc_{\mathsf{W}_s}(h'_s l_s) Chc_{\mathsf{W}_c}(h'_c l_c).$$

If $h = mu$, where $m \in \widetilde{\mathrm{GL}}(\mathsf{X}')$ and $u \in \widetilde{\mathrm{G}}(\mathsf{U})$, then $m_s = m$ and

$$\begin{aligned} Chc_{\mathsf{W}}(h'mu) &= Chc_{\mathsf{W}_s}(h'_s m u_s) Chc_{\mathsf{W}_c}(h'_c u_c) \\ &= \epsilon(h'_s m u_s)\, |Chc_{\mathsf{X}}(h'_s m u_s)|\, Chc_{\mathsf{W}_c}(h_c u_c) \end{aligned}$$

where u_s is the restriction of u to X'. Hence, we need to show that

$$|Chc_{\mathsf{X}}(h'_s m u_s)| = C|\det(\mathrm{Ad}(h')_{\mathfrak{n}'})|^{1/2}|\det(\mathrm{Ad}(\underline{h}')-1)_{\mathfrak{n}'})|^{-1}$$
$$I_{(-h'_s)}(m)|\det(\mathrm{Ad}(\underline{mu})-1)_{\mathfrak{n}})|^{-1}\,|\det(\mathrm{Ad}(\underline{mu})_{\mathfrak{n}})|^{1/2}. \quad (7.6)$$

But,

$$|Chc_{\mathsf{X}}(h'_s m u_s)| = |Chc_{\mathrm{Hom}(\mathsf{X}',\mathsf{V})}(h'mu)||Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}(h'm)|. \quad (7.7)$$

Furthermore, in terms of the following decomposition

$$\mathsf{V}'_s = \mathsf{V}'_1 \oplus \mathsf{V}'_2 \oplus \cdots,$$

where V'_i are the $\mathrm{H}'-$irreducible modules of V'_s, we get

$$|Chc_{\mathrm{Hom}(\mathsf{X}',\mathsf{V})}(h'mu)| = \prod_{j\geq 1}\Big(|\det(h'mu)_{\mathrm{Hom}(X'_j,V)}|^{1/2}\delta(\det(h'mu+1)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{V})})\Big)$$
$$= \prod_{j\geq 1}\Big(|\det(h'mu)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{X}')}|^{1/2}|\det(h'mu)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{U})}|^{1/2}|\det(h'mu)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{Y}')}|^{1/2}\times$$
$$\delta(\det(h'mu+1)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{X}')}\det(h'mu+1)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{U})}\det(h'mu+1)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{Y}')})\Big)$$
$$= \prod_{j\geq 1}\Big(|\det(h'm)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{X}')}|^{1/2}\delta(\det(h'm+1)_{\mathrm{Hom}(\mathsf{X}'_j,\mathsf{X}')})\Big)\,|\det(h')_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})}|^{1/2}\times$$
$$|\det(h'm)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')}|^{1/2}|\det(h'mu+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})}|^{-1}|\det(h'm+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')}|^{-1}$$
$$= I_{-h'_s}(m)|\det(h')_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})}|^{1/2}|\det(h'm)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')}|^{1/2}\times$$
$$|\det(h'mu+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})}|^{-1}|\det(h'm+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')}|^{-1}$$
$$= I_{-h'_s}(m)|\det(\mathrm{Ad}(mu))_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})}|^{1/2}|\det(\mathrm{Ad}(h'))_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'}|^{1/2}\times$$
$$|\det(\mathrm{Ad}(m))_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}}|^{1/2}|\det(\mathrm{Ad}(mu)-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})}|^{-1}\times$$
$$|\det(\mathrm{Ad}(h')-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}'}|^{-1}|\det(\mathrm{Ad}(m)-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}}|^{-1}, \quad (7.8)$$

where the fourth equality follows from Theorem 6.2. Moreover, by Lemma 6 (c) in [10],

$$|Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}(h'm)| = |\Theta(\mathfrak{d})\,\Theta(\mathfrak{d}h'm)|$$
$$= 2^{\dim(\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}|\det(\underline{h'm})_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}|^{1/2}|\det(\underline{h'm+1})_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}|^{-1}.$$

Hence

$$I_{-h'_s}(m)|Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}(h'm)| = I_{-h'_s}(m)2^{\dim(\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'))}$$
$$|\det(\mathrm{Ad}(h')_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')})|^{1/2}|\det(\mathrm{Ad}(h')-1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}|^{-1}. \quad (7.9)$$

Clearly, (7.7), (7.8) and (7.9) imply (7.6). Thus, the theorem holds for Ψ as in (7.5). We have the decomposition

$$\mathsf{V} = (\mathsf{X}'\oplus\mathsf{X}'')\oplus\mathsf{U}_0\oplus(\mathsf{Y}'\oplus\mathsf{Y}'').$$

Let $\mathrm{Q}_1 \subseteq \mathrm{G}$ be the parabolic subgroup with the Lie algebra $\mathfrak{q}_1$. Then $\mathrm{L}_1 = \mathrm{GL}(\mathsf{X}' + \mathsf{X}'')\mathrm{G}(\mathsf{U}_0)$. Let $\mathrm{L}_1' = \{l \in \mathrm{L}_1 \mid \det(\mathrm{Ad}(\underline{h}) - 1) \neq 0\}$. Suppose

$$\mathrm{supp}(\Psi) \subseteq \widetilde{\mathrm{L}}_1'. \tag{7.10}$$

Then, by Proposition A.7,

$$\begin{aligned}
&\int_{\widetilde{\mathrm{G}}} Chc_{\mathsf{W}}(h'g)\Psi(g)\, d\mu(g) &(7.11)\\
= &\int_{\widetilde{\mathrm{L}}_1'} Chc_{\mathsf{W}}(h'h)|\det(\mathrm{Ad}(\underline{h}) - 1)_{\mathfrak{n}_1}||\det(\mathrm{Ad}(\underline{h})_{\mathfrak{n}_1})|^{-1/2}\Psi^{\widetilde{\mathrm{L}}}(h)\, d\mu(h)
\end{aligned}$$

Let

$$\begin{aligned}
\mathsf{X}^1 &= \mathrm{Hom}(\mathsf{X}', \mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}_c', \mathsf{X}' + \mathsf{X}''),\\
\mathsf{Y}^1 &= \mathrm{Hom}(\mathsf{Y}', \mathsf{V}) \oplus \mathrm{Hom}(\mathsf{V}_c', \mathsf{Y}' + \mathsf{Y}''),\\
\mathsf{W}_s^1 &= \mathsf{X}^1 \oplus \mathsf{Y}^1, \quad \mathsf{W}_c^1 = \mathrm{Hom}(\mathsf{V}_c', \mathsf{U}_0),
\end{aligned} \tag{7.12}$$

and let ϵ^1 be the character of the stabilizer of X^1 and Y^1 in $\widetilde{\mathrm{Sp}}(\mathsf{W}_s^1)$, defined in (6.16). This is an extension of the character ϵ, which occurs in (7.3). Let $h \in \widetilde{\mathrm{L}}_1'$. Then

$$\begin{aligned}
Chc_{\mathsf{W}}(h'h) &= Chc_{\mathsf{W}_s^1}(h_{s^1}'h_{s^1})Chc_{\mathsf{W}_c^1}(h_{c^1}'h_{c^1})\\
&= \epsilon^1(h_{s^1}'h_{s^1})|Chc_{\mathsf{X}^1}(h_{s^1}'h_{s^1})|\, Chc_{\mathsf{W}_c^1}(h_{c^1}'h_{c^1}),
\end{aligned} \tag{7.13}$$

where $h = h_{s^1}h_{c^1}$ is the decomposition analogous to (7.2) for $\mathsf{W} = \mathsf{W}_s^1 \oplus \mathsf{W}_c^1$, and similarly for h'.

Let $\mathfrak{q}_2 = \mathfrak{m}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{n}_2 \subset \mathfrak{gl}(\mathsf{X}' + \mathsf{X}'')$ be the parabolic subalgebra preserving X''. Then

$$\begin{aligned}
\mathfrak{m}_2 \oplus \mathfrak{a}_2 &= \mathfrak{gl}(\mathsf{X}') \oplus \mathfrak{gl}(\mathsf{X}'') \text{ and}\\
\mathfrak{n}_2 &= \mathrm{Hom}(\mathsf{X}', \mathsf{X}'').
\end{aligned} \tag{7.14}$$

Let $\mathrm{Q}_2 \subseteq \mathrm{GL}(\mathsf{X}' + \mathsf{X}'')$ be the parabolic subgroup with the Lie algebra $\mathfrak{q}_2$ defined in (7.14). As in (7.7) we have

$$|Chc_{\mathsf{X}^1}(h_{s^1}'h_{s^1})| = |Chc_{\mathrm{Hom}(\mathsf{X}',\mathsf{V})}(h_{s^1}'h_{s^1})||Chc_{\mathrm{Hom}(\mathsf{V}_c',\mathsf{X}'+\mathsf{X}'')}(h_{s^1}'h_{s^1})|. \tag{7.15}$$

As in (7.8) we check that

$$\begin{aligned}
|Chc_{\mathrm{Hom}(\mathsf{X}',\mathsf{V})}(h_{s^1}'h_{s^1})| &= \mathrm{Ind}_{\mathrm{Q}_2}^{\mathrm{GL}(\mathsf{X}'+\mathsf{X}'')}(I_{-h_s'} \otimes 1)(h|_{\mathsf{X}'+\mathsf{X}''})\\
&\quad |\det(h'h)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0+\mathsf{Y}'+\mathsf{Y}'')}|^{1/2}|\det(h'h+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0+\mathsf{Y}'+\mathsf{Y}'')}|^{-1}.
\end{aligned} \tag{7.16}$$

Furthermore,

$$\begin{aligned}
|Chc_{\mathrm{Hom}(\mathsf{V}_c',\mathsf{X}'+\mathsf{X}'')}(h_{s^1}'h_{s^1})| &= C_2|\det(h'h)_{\mathrm{Hom}(\mathsf{V}_c',\mathsf{X}'+\mathsf{X}'')}|^{1/2}\\
&\quad |\det{}'(h'h+1)_{\mathrm{Hom}(\mathsf{V}_c',\mathsf{X}'+\mathsf{X}'')}|^{-1},
\end{aligned} \tag{7.17}$$

where $C_2 = 2^{\dim(\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'+\mathsf{X}''))}$. Hence,

$$
\begin{aligned}
&|Chc_{\mathrm{Hom}(\mathsf{X}',\mathsf{V})}(h'_{s^1}h_{s^1})|\,|Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'+\mathsf{X}'')}(h'_{s^1}h_{s^1})|\\
&= \mathrm{Ind}_{\mathrm{Q}_2}^{\mathrm{GL}(\mathsf{X}'+\mathsf{X}'')}(I_{-h'_s}\otimes 1)(h|_{\mathsf{X}'+\mathsf{X}''})\\
&\qquad |\det(h'h)_{\mathrm{Hom}"(\mathsf{X}',\mathsf{U}_0+\mathsf{Y}'+\mathsf{Y}'')}|^{1/2}|\det(h'h+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0+\mathsf{Y}'+\mathsf{Y}'')}|^{-1}\\
&\qquad\qquad |Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'+\mathsf{X}'')}(h'_{s^1}h_{s^1})|\\
&= \mathrm{Ind}_{\mathrm{Q}_2}^{\mathrm{GL}(\mathsf{X}'+\mathsf{X}'')}(I_{-h'_s}\otimes 1)(h|_{\mathsf{X}'+\mathsf{X}''})\\
&|\det(h'h)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}'+\mathsf{Y}'')}|^{1/2}|\det(h'h+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}'+\mathsf{Y}'')}|^{-1}|Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'+\mathsf{X}'')}(h'_{s^1}h_{s^1})|\\
&\qquad\qquad |\det(h'h)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0)}|^{1/2}|\det(h'h+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0)}|^{-1}\\
&= C_2\,\mathrm{Ind}_{\mathrm{Q}_2}^{\mathrm{GL}(\mathsf{X}'+\mathsf{X}'')}(I_{-h'_s}|\det(h')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\oplus\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}|^{1/2}\\
&|\det(h'+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\oplus\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}')}|^{-1}\otimes|\det(h')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}'')}|^{1/2}|\det(h'+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}'')}|^{-1}\\
&\qquad |\det(h')_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{1/2}|\det(h'+1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{-1})(h|_{\mathsf{X}'+\mathsf{X}''})\\
&\qquad\qquad |\det(h'h)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0)}|^{1/2}|\det(h'h+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0)}|^{-1}\\
&= C_2\,|\det(\mathrm{Ad}(h')_{\mathfrak{n}'})|^{1/2}|\det(\mathrm{Ad}(h')-1)_{\mathfrak{n}'}|^{-1}\\
&\qquad \mathrm{Ind}_{\mathrm{Q}_2}^{\mathrm{GL}(\mathsf{X}'+\mathsf{X}'')}(I_{-h'_s}|\det(h')_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}}|^{1/2}|\det(h'+1)_{\mathrm{Hom}(\mathsf{X}',Y')\cap\mathfrak{g}}|^{-1}\\
&\qquad\quad \otimes|\det(h')_{\mathrm{Hom}(\mathsf{X}',Y'')}|^{1/2}|\det(h'+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}'')}|^{-1}\\
&\qquad |\det(h')_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{1/2}|\det(h'+1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{-1})(h|_{\mathsf{X}'+\mathsf{X}''})\\
&\qquad\qquad |\det(h'h)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0)}|^{1/2}|\det(h'h+1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0)}|^{-1}\\
&= C_2\,|\det(\mathrm{Ad}(h')_{\mathfrak{n}'})|^{1/2}|\det(\mathrm{Ad}(h')-1)_{\mathfrak{n}'}|^{-1}\\
&\qquad \mathrm{Ind}_{\mathrm{Q}_2}^{\mathrm{GL}(\mathsf{X}'+\mathsf{X}'')}(I_{-h'_s}\otimes|\det(h'\ )_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{1/2}|\det(h'\ +1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{-1}\\
&\qquad\qquad |\det(\mathrm{Ad}(\ )_{\mathrm{Hom}(\mathsf{X}'',\mathsf{U}_0)\oplus\mathrm{Hom}(\mathsf{X}'',Y'')\cap\mathfrak{g}})|^{-1/2}\\
&\qquad\quad |\det(\mathrm{Ad}(\ )-1)_{\mathrm{Hom}(\mathsf{X}'',\mathsf{U}_0)\oplus\mathrm{Hom}(\mathsf{X}'',\mathsf{Y}'')\cap\mathfrak{g}}|)(h|_{\mathsf{X}'+\mathsf{X}''})\\
&\qquad\qquad |\det(\mathrm{Ad}(h)_{\mathfrak{n}_1}|^{1/2}|\det(\mathrm{Ad}(h)-1)_{\mathfrak{n}_1}|^{-1}.
\end{aligned}\tag{7.18}
$$

Let $\mathfrak{n}_3 = \mathrm{Hom}(\mathsf{X}'',\mathsf{U}_0)\oplus\mathrm{Hom}(\mathsf{X}'',\mathsf{Y}'')\cap\mathfrak{g}$. Then by (7.13)-(7.18),

$$
\begin{aligned}
Chc_{\mathsf{W}}(h'h) &= \epsilon^1(h'_{s^1}h_{s^1})C_2\,|\det(\mathrm{Ad}(h')_{\mathfrak{n}'})|^{1/2}\\
&\quad |\det(\mathrm{Ad}(h')-1)_{\mathfrak{n}'}|^{-1}|\det(\mathrm{Ad}(h)_{\mathfrak{n}_1}|^{1/2}|\det(\mathrm{Ad}(h)-1)_{\mathfrak{n}_1}|^{-1}\\
&\quad \mathrm{Ind}_{\mathrm{Q}_2}^{\mathrm{GL}(\mathsf{X}'+\mathsf{X}'')}\big(I_{-h'_s}\otimes|\det(h')_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{1/2}|\det(h'\ +1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{-1}\\
&\quad |\det(\mathrm{Ad}(\ )_{\mathfrak{n}_3})|^{-1/2}|\det(\mathrm{Ad}(\ )-1)_{\mathfrak{n}_3}|\big)\,(h|_{\mathsf{X}'+\mathsf{X}''})Chc_{\mathsf{W}^1_c}(h'_{c^1}h_{c^1}).
\end{aligned}\tag{7.19}
$$

The space $\mathfrak{n}_3$ is the nilradical of the parabolic subalgebra $\mathfrak{q}_3\subseteq\mathfrak{g}(\mathsf{U})$ preserving Y''. Let $\mathrm{Q}_3\subseteq\mathrm{G}(\mathsf{U})$ denote the corresponding parabolic subgroup. Then $\mathrm{Q}_3 = \mathrm{GL}(\mathsf{X}'')\mathrm{G}(\mathsf{U}_0)\mathrm{N}_3$, where N_3 is the unipotent radical.

By combining (7.11) and (7.19), we see that

$$\begin{aligned}\int_{\widetilde{\mathrm{G}}} Chc_{\mathsf{W}}(h'g)\Psi(g)\,d\mu(g) = C_2\,|\det(\mathrm{Ad}(h')_{\mathfrak{n}'})|^{1/2}|\det(\mathrm{Ad}(h')-1)_{\mathfrak{n}'}|^{-1}\\ \int_{\widetilde{\mathrm{GL}}(\mathsf{X}')}\int_{\widetilde{\mathrm{GL}}(\mathsf{X}'')}\int_{\widetilde{\mathrm{G}}(\mathsf{U}_0)} \epsilon^1(h'_{s^1}(mh_2h_0)_{s^1})I_{-h'_s}(m)\\ |\det(h'h_2)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{1/2}|\det(h'h_2+1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{-1}\\ |\det(\mathrm{Ad}(h_2h_0)_{\mathfrak{n}_3})|^{-1/2}|\det(\mathrm{Ad}(h_2h_0)-1)_{\mathfrak{n}_3}|\\ Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U}_0)}(h'_{c^1}h_0)\Psi^{\widetilde{\mathrm{Q}_1\mathrm{N}_3}}(mh_2h_0)\,d\mu(h_0)\,d\mu(h_2)\,d\mu(m). \quad (7.20)\end{aligned}$$

Now we consider the right hand side of the formula (7.4) of Theorem 7.1 in a neighborhood of a non-regular semisimple point g, which belongs to the support of the distribution in question. We may, and shall, assume that g preserves the decomposition

$$\mathsf{V} = \mathsf{X}' \oplus \mathsf{U} \oplus \mathsf{Y}',$$

and that the restriction of $\underline{g}$ to X'_s is equal to h'_s.

Since $\underline{g}$ is not a regular element of L, $\det(\mathrm{Ad}(\underline{g})-1)_{\mathfrak{n}} = 0$. However, by the regularity of h',

$$\det(\mathrm{Ad}(g)-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}} = \det(\mathrm{Ad}(h'_s)-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}} \neq 0.$$

Therefore, $\det(\mathrm{Ad}(g)-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U})} = 0$. Hence, U contains an g-indecomposable subspace similar to some g-indecomposable subspace of V'_s. Let $\mathsf{V}'' \subseteq \mathsf{U}$ be the sum of all such subspaces, and let $\mathsf{U}_0 = \mathsf{V}''^{\perp}$. Also, there is a complete polarization $\mathsf{V}'' = \mathsf{X}'' \oplus \mathsf{Y}''$ preserved by g, as in (7.12)-(7.20).

Let $\epsilon > 0$ be as in Lemma 5.7. For $\epsilon > \epsilon' > 0$, let $\mathcal{Z}_{\epsilon'} \subseteq \widetilde{\mathrm{G}}^g$ be an open neighborhood of g, as in (5.1). Then

$$\begin{aligned} & \{u \in \mathrm{G}(\mathsf{U}) \,|\, \text{there is } n \in \mathrm{N} \text{ such that } h'_s u\, n \in \mathrm{G}.\mathcal{Z}_{\epsilon'}\} \\ \subseteq\ & \{u \in \mathrm{G}(\mathsf{U}) \,|\, h'_s u \in Cl(\mathrm{G}.\mathcal{Z}_{\epsilon'})\} \\ \subseteq\ & \{u \in \mathrm{G}(\mathsf{U}) \,|\, h'_s u \in \mathrm{G}.\mathcal{Z}_{\epsilon}\} \\ =\ & \mathrm{G}(\mathsf{U}).(\mathrm{G}(\mathsf{U}) \cap \mathcal{Z}_{\epsilon}), \end{aligned}$$

where the last equality follows from lemma 5.7.

Notice that, by the construction,

$$\det(\mathrm{Ad}(g)-1)_{\mathrm{Hom}(\mathsf{X}'',\mathsf{U}_0)} \neq 0 \ \text{ and } \ \det(\mathrm{Ad}(g)-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{U}_0)} \neq 0.$$

By the regularity of h'_s,

$$\det(\mathrm{Ad}(g)-1)_{\mathrm{Hom}(\mathsf{X}'',\mathsf{Y}'')\cap\mathfrak{g}} \neq 0 \ \text{ and } \ \det(\mathrm{Ad}(g)-1)_{\mathrm{Hom}(\mathsf{X}',\mathsf{Y}')\cap\mathfrak{g}\oplus\mathrm{Hom}(\mathsf{X}',\mathsf{Y}'')} \neq 0.$$

Hence,

$$\det(\mathrm{Ad}(g)-1)_{\mathfrak{n}_3} \neq 0 \ \text{ and } \ \det(\mathrm{Ad}(g)-1)_{\mathfrak{n}_1} \neq 0,$$

and therefore, we may choose $\epsilon > 0$ small enough, so that

$$\begin{aligned} &\det(\mathrm{Ad}(u)-1)_{\mathfrak{n}_3} \neq 0 \qquad \text{for } u \in \mathrm{G}(\mathsf{U}) \cap \mathcal{Z}_{\epsilon} \text{ and} \\ &\det(\mathrm{Ad}(u)-1)_{\mathfrak{n}_1} \neq 0 \qquad \text{for } u \in \mathcal{Z}_{\epsilon}. \end{aligned} \quad (7.21)$$

Notice that, by Corollary 5.5, $\det(h'g+1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{Y}'')} \neq 0$. Since, $\mathrm{G}(\mathsf{U})\cap\mathcal{Z}_\epsilon \subseteq \mathrm{G}(\mathsf{U})^g$, every element $u \in \mathrm{G}(\mathsf{U}) \cap \mathcal{Z}_\epsilon$ preserves X''. Thus, again by Corollary 5.5,

$$\det(h'u+1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'')} \neq 0 \text{ for } u \in \mathrm{G}(\mathsf{U}) \cap \mathcal{Z}_\epsilon.$$

Suppose our function $\Psi \in \mathcal{D}(\widetilde{\mathrm{G}})$ is such that

$$\operatorname{supp} \Psi \subseteq \widetilde{\mathrm{G}}.\widetilde{\mathcal{Z}}_{\epsilon'}. \tag{7.22}$$

Then, by (7.21),

$$\{\underline{u} \in \mathrm{G}(\mathsf{U}) \,|\, h'_s u \in \operatorname{supp}(\Psi^{\widetilde{\mathrm{L}}})\} \subseteq \mathrm{G}(\mathsf{U}).(\mathrm{G}(\mathsf{U}) \cap \mathcal{Z}_\epsilon).$$

Hence, (7.21) and Proposition A.7 applied to the preimage of $\mathrm{Q}_3 = \mathrm{L}_3\mathrm{N}_3 \subseteq \mathrm{G}(\mathsf{U})$ imply that the integral on the right hand side of (7.4) is equal to

$$\begin{aligned}
&\int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_s} \int_{\widetilde{\mathrm{G}}(\mathsf{U})} \epsilon(\mathfrak{d}h'_s u_s) Chc_{\mathsf{W}_c}(h'_c u_c) \Psi^{\widetilde{\mathrm{Q}}}(h.(\mathfrak{d}h'_s)u)\, d\mu(u)\, d\mu(h\mathrm{H}'_s)\\
&= \int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_s} \int_{\widetilde{\mathrm{L}}'_3} \epsilon(\mathfrak{d}h'_s u_s) Chc_{\mathsf{W}_c}(h'_c u_c) |\det(\mathrm{Ad}(u)-1)_{\mathfrak{n}_3}|\\
&\qquad \int_{\widetilde{\mathrm{K}}\cap\widetilde{\mathrm{G}}(\mathsf{U})} \int_{\mathfrak{n}_3} \Psi^{\widetilde{\mathrm{L}}}(h.(\mathfrak{d}h'_s)l.(u\exp(z)))d\mu(l)\, d\mu(u)\, d\mu(h\mathrm{H}'_s) =\\
&\int_{\mathrm{GL}(\mathsf{X}')/\mathrm{H}'_s} \int_{\widetilde{\mathrm{L}}'_3} \epsilon(\mathfrak{d}h_s u_s) Chc_{\mathsf{W}_c}(h'_c u_c) |\det(\mathrm{Ad}(u)-1)_{\mathfrak{n}_3}| \Psi^{\widetilde{\mathrm{L}}_3}(h.(\mathfrak{d}h'_s)u)\, d\mu(u)\, d\mu(h\mathrm{H}'_s)
\end{aligned} \tag{7.23}$$

Furthermore, $\mathrm{L}_3 = \mathrm{GL}(\mathsf{X}'')\mathrm{G}(\mathsf{U}_0)$, and for $u = u_c = h_2 h_0$, with $h_2 \in \mathrm{GL}(\mathsf{X}'')$ and $h_0 \in \mathrm{G}(\mathsf{U}_0)$,

$$\begin{aligned}
&Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U})}(h'_c u)\\
&= Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{V}'')}(h'_c h_2) Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U}_0)}(h'_c h_0)\\
&= \frac{C_2}{C} |\det(h'h_2)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{1/2}\, |\det(h'h_2+1)_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{X}'')}|^{-1} Chc_{\mathrm{Hom}(\mathsf{V}'_c,\mathsf{U}_0)}(h'_c h_0).
\end{aligned}$$

Since $\mathfrak{n} + \mathfrak{n}_3 = \mathfrak{n}_1 + \mathfrak{n}_2$, and since by (7.21) the condition (7.22) implies (7.10), we see that as a consequence of (7.23), the right hand side of the equality (b) in Theorem 7.1 coincides with the expression (7.20). Thus, the formula (7.4) of the theorem holds for any test function Ψ which satisfies the condition (7.5) or (7.22). Hence, by localization, it holds for any $\Psi \in \mathcal{D}(\widetilde{\mathrm{G}})$. ■

8. Proof of Theorem 0.10

We recall some more notation from [1]. Let H' be a compact Cartan subgroup with the Lie algebra $\mathfrak{h}'$ of G', and let

$$\mathsf{V}' = \mathsf{V}'_0 \oplus \sum_{j=1}^{n'} \mathsf{V}'_j$$

be the decomposition into H'−irreducibles over $\mathbb{D}$. If $\mathrm{G}' = \mathrm{O}_{2p+1,2q}$ then H' acts trivially on V'_0 and $\dim(\mathsf{V}'_0) = 1$ otherwise V'_0 is zero. There is a complex structure J' on V' which belongs to $\mathfrak{h}'$. More precisely, the restriction of J' to V'_0 is zero and the restriction of J' to $\sum_{j=1}^{n'} \mathsf{V}'_j$ is a complex structure. Similarly, let H be a compact Cartan subgroup with the Lie algebra $\mathfrak{h}$ of G, and let

$$\mathsf{V} = \mathsf{V}_0 \oplus \sum_{j=1}^{n} \mathsf{V}_j$$

be the decomposition into H−irreducibles over $\mathbb{D}$. In particular, we exclude the case $\mathrm{G} = \mathrm{O}_{2p+1,2q+1}$. We are going to deal with this case at the end of this section. Here V_0 is trivial unless $\mathrm{G} = \mathrm{O}_{2p+1,2q}$, in which case H acts trivially on V_0 and $\dim(\mathsf{V}_0) = 1$. There is a complex structure J on V which belongs to $\mathfrak{h}$. More precisely, the restriction of J to V_0 is zero and the restriction of J to $\sum_{j=1}^{n} \mathsf{V}_j$ is a complex structure. We assume and identify:

$$\mathsf{V}'_j = \mathsf{V}_j, \ J'_j = J'|_{\mathsf{V}'_j} = J|_{\mathsf{V}_j} = J_j \qquad (1 \le j \le n'),$$

and let $J_j = J|_{\mathsf{V}_j}$ for all $1 \le j \le n$.

Let $W(\mathrm{H}_{\mathbb{C}}) = \mathrm{Normalizer}_{\mathrm{G}_{\mathbb{C}}}(\mathrm{H}_{\mathbb{C}})/\mathrm{H}_{\mathbb{C}}$. If $\mathbb{D} = \mathbb{C}$ then $W(\mathrm{H}_{\mathbb{C}})$ is identified with the permutation group Σ_n by

$$\sigma \cdot \sum_{j=1}^{n} x_j J_j = \sum_{j=1}^{n} x_j J_{\sigma(j)} \qquad (\sigma \in \Sigma_n).$$

If $\mathbb{D} \ne \mathbb{C}$, then $W(\mathrm{H}_{\mathbb{C}})$ is identified with the semidirect product of Σ_n and $\mathbb{Z}_2^n$, where $\mathbb{Z}_2 = \{0, 1\}$ with the addition modulo 2,

$$\epsilon \cdot \sum_{j=1}^{n} x_j J_j = \sum_{j=1}^{n} \hat{\epsilon}_j x_j J_j \qquad (\epsilon \in \mathbb{Z}_2^n),$$

where $\hat{\epsilon}_j = (-1)^{\epsilon_j}$. Thus, in any case, an element $s \in W(\mathrm{H}_{\mathbb{C}})$ may be written uniquely as $s = \sigma\epsilon$.

For $s \in W(\mathrm{H}_{\mathbb{C}})$ define $y_s \in \mathfrak{h}$ as in [1, Definition 3.4]. For a subset $A \subseteq \Psi \cup (-\Psi)$ let

$$\underline{A} = \{j \,|\, \text{there is } \alpha \in A \text{ such that } \alpha(J_j) \ne 0\}.$$

Also, we shall write $A(\mathrm{short})$ to indicate the subset of all the short roots in A, and $A(\mathrm{long})$ for the long roots. Furthermore, $\pi_{\mathfrak{g}'/\mathfrak{h}'}$ is the product of all the roots in a system of positive roots Ψ' for $(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{h}'_{\mathbb{C}})$, and Ψ' and Ψ are realized explicitly as in Appendix B of [1]. Fix a strongly orthogonal set $\mathcal{S} \in \Psi_{st}^n$ and an element $s \in W(\mathrm{H}_{\mathbb{C}})$. Let

$$\begin{aligned}
\mathfrak{h}'_{\mathcal{S},s} &= \sum_{k \notin \underline{\mathcal{S}}, \sigma^{-1}(k) \le n'} \mathbb{R} J_k + \mathfrak{h}_{\mathcal{S}} \cap \left(\sum_{k \in \underline{\mathcal{S} \setminus (s\mathfrak{h}')^{\perp}}} \mathbb{C} J_k \right), \\
\mathfrak{h}''_{\mathcal{S},s} &= \sum_{k \notin \underline{\mathcal{S}}, \sigma^{-1}(k) > n'} \mathbb{R} J_k + \mathfrak{h}_{\mathcal{S}} \cap \left(\sum_{k \in \underline{\mathcal{S} \cap (s\mathfrak{h}')^{\perp}}} \mathbb{C} J_k \right).
\end{aligned}$$

Then

$$\mathfrak{h}_{\mathcal{S}} = \mathfrak{h}'_{\mathcal{S},s} \oplus \mathfrak{h}''_{\mathcal{S},s}.$$

Let

$$\Gamma'_{s,\mathcal{S}} = \sum_{1 \le j \le n', \sigma(j) \notin \underline{\mathcal{S}}} (0,\infty) J^*_{\sigma(j)}(y_s) J_{\sigma(j)}.$$

This is the projection of $\Gamma_{s,\mathcal{S}}$ onto $\sum_{1 \le j \le n', \sigma(j) \notin \underline{\mathcal{S}}} \mathbb{R} J_{\sigma(j)}$. (See (7.6) in [1].) For convenience, we shall write

$$l'_{\mathcal{S},s} = \lim_{\Gamma'_{s,\mathcal{S}} \ni y \to 0}.$$

Let

$$\begin{aligned} R'_{\mathcal{S},s} &= \{\alpha \in \tilde{\Psi}_{\mathcal{S},\mathbb{R}} \,|\, \underline{\alpha} \subseteq \{\sigma(1), \cdots, \sigma(n')\}\}, \\ R''_{\mathcal{S},s} &= \{\alpha \in \tilde{\Psi}_{\mathcal{S},\mathbb{R}} \,|\, \underline{\alpha} \not\subseteq \{\sigma(1), \cdots, \sigma(n')\}\}. \end{aligned}$$

In these terms, Theorem 7.3 in [1] says that for $x' \in h'^{\mathrm{reg}}$,

$$\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \widetilde{chc}(\psi)(x') = \sum_{\mathcal{S},s} \mathcal{H}'_{\mathcal{S},s}\psi(x'), \tag{8.1}$$

where the summation is over all possible $\mathcal{S}$ and s,

$$\mathcal{H}'_{\mathcal{S},s}\psi(x') = l'_{\mathcal{S},s} \int_{\mathfrak{h}'_{\mathcal{S},s}} \frac{m_{\mathcal{S}}(s)}{\det(x' + x + \mathrm{i}y)_{s\mathsf{W}^{\mathfrak{h}'}}} \mathcal{A}(-R'_{\mathcal{S},s})(x) \mathcal{H}_{\mathcal{S},s}\psi(x)\, d\mu(x),$$

and for $x \in \mathfrak{h}'_{\mathcal{S},s}$,

$$\mathcal{H}_{\mathcal{S},s}\psi(x) = \int_{\mathfrak{h}''_{\mathcal{S},s}} \mathcal{A}(-R''_{\mathcal{S},s})(x + x'') \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1} \cdot (x + x'')) \mathcal{H}_{\mathcal{S}}\psi(x + x'')\, d\mu(x'').$$

Lemma B.1, with $V = \mathfrak{h}_{\mathcal{S}}$, $V' = \mathfrak{h}'_{\mathcal{S},s}$, $V'' = \mathfrak{h}''_{\mathcal{S},s}$, $A = \Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}$, $B = -R''_{\mathcal{S},s}$ and

$$\phi(x + x'') = \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1} \cdot (x + x'')) \mathcal{H}_{\mathcal{S}}\psi(x + x'') \qquad (x \in \mathfrak{h}'_{\mathcal{S},s}, x'' \in \mathfrak{h}''_{\mathcal{S},s}),$$

we see that $\mathcal{H}_{\mathcal{S},s}\psi$ is a Harish-Chandra Schwartz function on $\mathfrak{h}'_{\mathcal{S},s}$, with respect to $\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}} \cap (\mathfrak{h}''_{\mathcal{S},s})^{\perp}$:

$$\mathcal{H}_{\mathcal{S},s}\psi \in \mathcal{HCS}(\mathfrak{h}'_{\mathcal{S},s} \setminus \mathfrak{h}^{\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}} \cap (\mathfrak{h}''_{\mathcal{S},s})^{\perp}}). \tag{8.2}$$

Lemma 8.1. *Suppose one of the following conditions holds:*

(i) there is $\alpha \in \mathcal{S} \setminus (s\mathfrak{h}')^{\perp}$ *with* $\underline{\alpha} \cap \underline{\mathcal{S} \setminus \alpha} = \emptyset$ *and* $\underline{\alpha} \not\subseteq \{\sigma(1), \cdots, \sigma(n')\}$,

(ii) $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_{p,q}, \mathrm{U}_{1,1})$ *and* $\underline{\mathcal{S}} \cap \{\sigma(1), \cdots, \sigma(n')\} = \emptyset$,

(iii) $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2p+1,2q}, \mathrm{Sp}_2(\mathbb{R}))$, $\underline{\mathcal{S}} \cap \{\sigma(1), \cdots, \sigma(n')\} = \emptyset$ *and* $\sigma(1) \notin \underline{\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}(\mathrm{short})}$,

(iv) $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2p,2q}, \mathrm{Sp}_2(\mathbb{R}))$, *and* $\underline{\mathcal{S}} \cap \{\sigma(1), \cdots, \sigma(n')\} = \emptyset$.

Then $\mathcal{H}'_{\mathcal{S},s}\psi$ extends to a continuous function on $\mathfrak{h}'$.

Proof. Assume (i). Suppose the division algebra $\mathbb{D}\neq\mathbb{C}$. Then $\Gamma'_{s,S}=\emptyset$ and $n'=1$. Let $\underline{\alpha}=\{k,l\}$, with $k=\sigma(1)$. Then

$$\mathfrak{h}'_{\mathcal{S},s}=\{x=zJ_k+\overline{z}J_l\,|\,z\in\mathbb{C}\}$$

and for $x\in\mathfrak{h}'_{\mathcal{S},s}$, as above,

$$\det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}}=\mathsf{i}(x'_1-z),$$

where $x'=x'_1J'_1\in\mathfrak{h}'$. Hence, for any $x'\in\mathfrak{h}'$, the function

$$\mathfrak{h}'_{\mathcal{S},s}\ni x\to\frac{1}{\det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}}}\in\mathbb{C}$$

is locally integrable, with the integral over any compact subset of $\mathfrak{h}'_{\mathcal{S},s}$ defining a continuous function of x'. Since, by (8.2), the function

$$\mathfrak{h}'_{\mathcal{S},s}\ni x\to\mathcal{A}(-R'_{\mathcal{S},s})(x)\mathcal{H}'_{\mathcal{S},s}\psi(x)\in\mathbb{C}$$

is bounded and absolutely integrable, the claim follows.

Suppose $\mathbb{D}=\mathbb{C}$. Then $n'=2$. Let $\underline{\alpha}=\{k,l\}$, with $k\in\{\sigma(1),\sigma(2)\}$ and $l\notin\{\sigma(1),\sigma(2)\}$. Let $\{\sigma(1),\sigma(2)\}=\{j,k\}$. Then

$$\mathfrak{h}'_{\mathcal{S},s}=\{x=x_jJ_j+zJ_k+\overline{z}J_l\,|\,x_j\in\mathbb{R},\ z\in\mathbb{C}\}.$$

Moreover, there is $\epsilon=\pm1$ such that

$$\Gamma'_{s,S}=\epsilon(0,\infty)J_j.$$

Assume $\sigma(1)=j$. The case $\sigma(1)=k$ is analogous and we leave it to the reader. Let $y=y_jJ_j$ with $\epsilon y_j>0$. Then

$$\det(x'+x+y)_{s\mathsf{W}^{\mathfrak{h}'}}=\mathsf{i}(x'_1-x_j-\mathsf{i}y_j)\mathsf{i}(x'_2-z),$$

where $x'=x'_1J'_1-x'_2J_2\in\mathfrak{h}'$. We see from (8.2) that $\mathcal{H}_{\mathcal{S},s}\psi\in S(\mathfrak{h}'_{\mathcal{S},s})$. In particular, the function

$$\mathbb{C}\ni z\to\lim_{y_j\to0}\int_{\mathbb{R}}\frac{1}{x'_1-x_j-\mathsf{i}y_j}\mathcal{H}_{\mathcal{S},s}\psi(x_jJ_j+zJ_k+\overline{z}J_l)\,dx_j\in\mathbb{C}$$

is well defined for all $x'_1\in\mathbb{R}$ and belongs to the Schwartz space of $\mathbb{C}$, viewed as a vector space over $\mathbb{R}$. Furthermore, this function depends continuously on x'_1.

The function

$$\mathbb{C}\ni z\to\frac{1}{x'_2-z}\in\mathbb{C}$$

is locally integrable, with the integral over any compact subset of $\mathfrak{h}'_{\mathcal{S},s}$ defining a continuous function of $x'_2\in\mathbb{R}$. Since, $R'_{\mathcal{S},s}=\emptyset$, we see that $\mathcal{H}'_{\mathcal{S},s}\psi$ extends to a continuous function on $\mathfrak{h}'$.

Assume (ii). Then $n' = 2$,

$$\mathfrak{h}'_{\mathcal{S},s} = \mathbb{R}J_{\sigma(1)} + \mathbb{R}J_{\sigma(2)}$$

and $R'_{\mathcal{S},s} = \emptyset$. Let $\alpha \in \Psi$ be equal to plus or minus $\mathrm{i}J^*_{\sigma(1)} - \mathrm{i}J^*_{\sigma(2)}$. Suppose $\alpha \notin \Psi^n$. Then $\mathcal{H}_{\mathcal{S},s}\psi \in S(\mathfrak{h}'_{\mathcal{S},s})$. Let us write $x' = x'_1J'_1 - x'_2J'_2$ and $x = x_{\sigma(1)}J_{\sigma(1)} + x_{\sigma(2)}J_{\sigma(2)}$. In these terms $\mathcal{H}'_{\mathcal{S},s}\psi$ is a constant multiple of

$$l'_{\mathcal{S},s}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(x'_1 - x_{\sigma(1)} - \mathrm{i}y_{\sigma(1)})(x'_2 - x_{\sigma(2)} - \mathrm{i}y_{\sigma(2)})}\mathcal{H}_{\mathcal{S},s}\psi(x_{\sigma(1)}J_{\sigma(1)} + x_{\sigma(2)}J_{\sigma(2)})dx_{\sigma(1)}dx_{\sigma(2)}.$$

Since $\Gamma'_{\mathcal{S},s}$ is a proper open cone, it is easy to see that $\mathcal{H}_{\mathcal{S},s}\psi$ extends to a continuous function on $\mathfrak{h}'$.

Suppose $\alpha \in \Psi^n$. Then $\mathcal{H}_{\mathcal{S},s}\psi \in \mathcal{HCS}(\mathfrak{h}'_{\mathcal{S},s} \setminus \mathfrak{h}^\alpha$, where $\mathfrak{h}^\alpha = \{x \,|\, \alpha(x) = 0\}$. Moreover, there is $\epsilon = \pm 1$ such that $\Gamma'_{s,\mathcal{S}} \supseteq \epsilon(0,\infty)(J_{\sigma(1)} + J_{\sigma(2)})$. Hence, $\mathcal{H}'_{\mathcal{S},s}\psi$ is a constant multiple of

$$\begin{aligned}
&\lim_{\epsilon y \to 0+}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(x'_1 - x_{\sigma(1)} - \mathrm{i}y_{\sigma(1)})(x'_2 - x_{\sigma(2)} - \mathrm{i}y_{\sigma(2)})} \\
&\qquad\qquad \mathcal{H}_{\mathcal{S},s}\psi(x_{\sigma(1)}J_{\sigma(1)} + x_{\sigma(2)}J_{\sigma(2)})\,dx_{\sigma(1)}dx_{\sigma(2)} \\
&= \lim_{\epsilon y \to 0+}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(-x_{\sigma(1)} - \mathrm{i}y_{\sigma(1)})(-x_{\sigma(2)} - \mathrm{i}y_{\sigma(2)})} \\
&\qquad\qquad \mathcal{H}_{\mathcal{S},s}\psi((x'_1 + x_{\sigma(1)})J_{\sigma(1)} + (x'_2 + x_{\sigma(2)})J_{\sigma(2)})\,dx_{\sigma(1)}dx_{\sigma(2)},
\end{aligned}$$

and the conclusion follows.

Assume (iii). Then $n' = 1$, $\mathfrak{h}'_{\mathcal{S},s} = \mathbb{R}J_{\sigma(1)}$, $R'_{\mathcal{S},s} = \emptyset$ and $\mathcal{H}_{\mathcal{S},s}\psi \in S(\mathfrak{h}'_{\mathcal{S},s})$. Moreover, $\mathcal{H}'_{\mathcal{S},s}\psi$ is a constant multiple of

$$l'_{\mathcal{S},s}\int_{\mathbb{R}}\frac{1}{x'_1 - x_{\sigma(1)} - \mathrm{i}y_{\sigma(1)}}\mathcal{H}_{\mathcal{S},s}\psi(x_{\sigma(1)}J_{\sigma(1)})\,dx_{\sigma(1)}$$

where $\Gamma'_{s,\mathcal{S}}$ is a proper open convex cone. Hence it is easy to see that $\mathcal{H}'_{\mathcal{S},s}\psi$ extends to a continuous function of $x' \in \mathfrak{h}'$.

The proof in the case (iv) is the same as in the case (iii). ∎

Lemma 8.2. *(a) Suppose* $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_{p,q}, \mathrm{U}_{1,1})$, $\epsilon = \pm 1$ *and* $\alpha = \epsilon(e_{\sigma(1)} - e_{\sigma(2)}) \in \mathcal{S}$. *Then*

$$\begin{aligned}
&\langle \mathcal{H}'_{\mathcal{S},s}\psi + \mathcal{H}'_{\mathcal{S},s_\alpha s}\psi\rangle_{J'_1 - J'_2}(u'(J'_1 + J'_2)) \\
&= \lim_{y \to 0}\mathrm{i}4\pi^2\tilde{\kappa}(J_1, J_1)|m_{\mathcal{S}}(s)|\int_{\mathfrak{h}''_{\mathcal{S},s}}\prod_{\beta \in \Psi, \underline{\beta} \cap \underline{\alpha} = \emptyset}|\beta(x)|^2 \cdot \prod_{\beta \in \Psi, \underline{\beta} \cap \underline{\alpha} \neq \emptyset}|\beta(x)| \\
&\qquad\qquad \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g \cdot c(\mathcal{S})x)\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''),
\end{aligned}$$

where $x = u'(J_{\sigma(1)} + J_{\sigma(2)}) + \mathrm{i}y(J_{\sigma(1)} - J_{\sigma(2)}) + x''$.

(b) Suppose $(\mathrm{G},\mathrm{G}')=(\mathrm{Sp}_{2n}(\mathbb{R}),\mathrm{O}_{1,2})$ *and* $\sigma(1)\notin\underline{\mathcal{S}}$. *Let* $\alpha=2e_{\sigma(1)}$. *Then*

$$\begin{aligned}&\langle\mathcal{H}'_{\mathcal{S},s}\psi+\mathcal{H}'_{\mathcal{S},s_\alpha s}\psi\rangle_{J'}(0)\\&=\lim_{y\to 0}\ \mathrm{i}2\pi\tilde{\kappa}(J_1,J_1)^{1/2}|m_{\mathcal{S}}(s)|4^{-2n+2}\int_{\mathfrak{h}''_{\mathcal{S},s}} chc_{\mathrm{Hom}(V^{\perp}_{\sigma(1)},\mathsf{V}'_0)}(c(\mathcal{S})x)\\&\qquad\prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(yJ_{\sigma(1)})|\cdot\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\\&\qquad\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot c(\mathcal{S})(yJ_{\sigma(1)}+x))\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha))d\mu(x),\end{aligned}$$

(c) Suppose $(\mathrm{G},\mathrm{G}')=(\mathrm{Sp}_{2n}(\mathbb{R}),\mathrm{O}_{1,2})$ *and* $\sigma(1)\in\underline{\mathcal{S}}$. *Let* $\alpha=2e_{\sigma(1)}$. *Then* $\alpha\in\mathcal{S}$ *and*

$$\begin{aligned}&\langle\mathcal{H}'_{\mathcal{S},s}\psi+\mathcal{H}'_{\mathcal{S},s_\alpha s}\psi\rangle_{J'}(0)\\&=\lim_{y\to 0}\ \mathrm{i}2\pi\tilde{\kappa}(J_1,J_1)^{1/2}|2m_{\mathcal{S}}(s)|4^{-2n+2}\int_{\mathfrak{h}''_{\mathcal{S},s}} chc_{\mathrm{Hom}(\mathsf{V}^{\perp}_{\sigma(1)},\mathsf{V}'_0)}(c(\mathcal{S})x)\\&\qquad\prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(yJ_{\sigma(1)})|\cdot\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\\&\qquad\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot c(\mathcal{S})(yJ_{\sigma(1)}+x))\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha))d\mu(x),\end{aligned}$$

(d) Let $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_{2p+1,2q},\mathrm{Sp}_2(\mathbb{R}))$ *and let* $\alpha=e_{\sigma(1)}$. *Suppose* $\sigma(1)\notin\underline{S}$ *and* $\alpha\in\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}$. *Then*

$$\begin{aligned}&\langle\mathcal{H}'_{\mathcal{S},s}\psi+\mathcal{H}'_{\mathcal{S},s_\alpha s}\psi+\frac{m_{\mathcal{S}}(s)}{m_{\mathcal{S}\vee\alpha}(s)}(\mathcal{H}'_{\mathcal{S}\vee\alpha,s}\psi+\mathcal{H}'_{\mathcal{S}\vee\alpha,s_\alpha s}\psi)\rangle_{J'}(0)=\lim_{y\to 0}\\&\mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|m_{\mathcal{S}}(s)|\int_{\mathfrak{h}_{\mathcal{S}\vee\alpha}\cap\mathfrak{h}^{\alpha}}\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(yJ_{\sigma(1)})|\\&\qquad\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot c(\mathcal{S})(yiJ_{\sigma(1)}+x))\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha))d\mu(x).\end{aligned}$$

(e) Let $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_{2p,2q},\mathrm{Sp}_2(\mathbb{R}))$ *and let* $\alpha\in\mathcal{S}$. *Suppose* $\sigma(1)\in\underline{\alpha}\subseteq\underline{\mathcal{S}\setminus\alpha}$. *Let* $w\in W(\mathrm{H}_{\mathcal{S}})$ *be the reflection with respect to* $J_{\sigma(1)}$. *Then*

$$\begin{aligned}&\langle\mathcal{H}'_{\mathcal{S},s}\psi+\mathcal{H}'_{\mathcal{S},ws}\psi\rangle_{J'}(0)=\mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|m_{\mathcal{S}}(s)|\\&\qquad\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_{\sigma(1)}}}\prod_{\beta\in\Psi,\sigma(1)\notin\underline{\beta}}|\beta(x)|^2\cdot\prod_{j\neq\sigma(1)}|e_j(x)|^2\\&\qquad\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g\cdot c(\mathcal{S})x)\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x).\end{aligned}$$

Proof. Consider the case (a). Here $R'_{\mathcal{S},s}=\{\alpha\}$, $R''_{\mathcal{S},s}=S\setminus\alpha$ and $\Gamma'_{\mathcal{S},s}=\emptyset$.

Hence, for $x' \in \mathfrak{h}'^{\mathrm{reg}}$,

$$\mathcal{H}'_{\mathcal{S},s}\psi(x') + \mathcal{H}'_{\mathcal{S},s_\alpha s}\psi(x') = m_{\mathcal{S}}(s) \int_{\mathfrak{h}'_{\mathcal{S},s}} \left(\frac{1}{\det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}}} - \frac{1}{\det(x'+x)_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\right)\left(-\frac{\alpha(x)}{|\alpha(x)|}\right)\mathcal{H}_{\mathcal{S},s}\psi(x)\,d\mu(x). \quad (8.3)$$

For $v' \in \mathbb{R} \setminus 0$ and $z \in \mathbb{C} \setminus \{v', -v'\}$ let

$$f_{v'}(z) = \frac{1}{2\mathsf{i}}\left(\frac{1}{(z-v')(\overline{z}+v')} - \frac{1}{(z+v')(\overline{z}-v')}\right).$$

Let us write

$$\begin{aligned} x' &= u'(J'_1 + J'_2) + v'(J'_1 - J'_2), \\ x &= u(J_{\sigma(1)} + J_{\sigma(2)}) + \mathsf{i}v(J_{\sigma(1)} - J_{\sigma(2)}). \end{aligned}$$

Then,

$$\begin{aligned} \det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}} &= \mathsf{i}(u'+v'-u-\mathsf{i}v)\mathsf{i}(u'-v'-u+\mathsf{i}v), \\ \det(x'+x)_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}} &= \mathsf{i}(u'+v'-u+\mathsf{i}v)\mathsf{i}(u'-v'-u-iv). \end{aligned}$$

Hence,

$$\frac{1}{\det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}}} - \frac{1}{\det(x'+x)_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}} = -2\mathsf{i}f_{v'}(u-u'+\mathsf{i}v).$$

Let

$$\phi(u+\mathsf{i}v) = -2\mathsf{i}\epsilon\tilde{\kappa}(J_1,J_1)m_{\mathcal{S}}(s)\mathcal{H}_{\mathcal{S},s}\psi((u+u')(J_{\sigma(1)}+J_{\sigma(2)}) + \mathsf{i}v(J_{\sigma(1)} - J_{\sigma(2)})).$$

Then ϕ is a Schwartz function on $\mathbb{C}$, viewed as a vector space over $\mathbb{R}$, and $\phi(u-\mathsf{i}v) = \phi(u+\mathsf{i}v)$. Since $\frac{\alpha(x)}{|\alpha(x)|} = \epsilon\frac{v}{|v|}$, the function (8.3) coincides with

$$\tilde{\phi}(v') = \int_{\mathbb{R}}\int_{\mathbb{R}} f_{v'}(u+\mathsf{i}v)\frac{v}{|v|}\phi(u+\mathsf{i}v)\,dudv.$$

Corollary A.4 in [1] shows that

$$\langle\tilde{\phi}\rangle(0) = -2\pi^2\phi(o) = \mathsf{i}4\pi^2\tilde{\kappa}(J_1,J_1)m_{\mathcal{S}}(s)\mathcal{H}_{\mathcal{S},s}\psi(u'(J_{\sigma(1)}+J_{\sigma(2)})).$$

Let $w = w(v') = u'(J_{\sigma(1)}+J_{\sigma(2)}) + \mathsf{i}v'(J_{\sigma(1)} - J_{\sigma(2)})$. Then,

$$\begin{aligned} &\epsilon m_{\mathcal{S}}(s)\mathcal{H}_{\mathcal{S},s}\psi(u'(J_{\sigma(1)}+J_{\sigma(2)})) \\ &= \lim_{v'\to 0}\epsilon m_{\mathcal{S}}(\sigma)\int_{\mathfrak{h}''_{\mathcal{S},s}} \pi_{\mathfrak{z}/\mathfrak{h}}(\sigma^{-1}\cdot x'')\mathcal{A}(-(\mathcal{S}\setminus\alpha))(x'')\mathcal{A}(\mathcal{S}\setminus\alpha)(x'') \\ &\qquad \frac{\alpha(w)}{|\alpha(w)|}\prod_{\beta\in\Psi}\beta(w+x'')\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g\cdot c(\mathcal{S})(w+x''))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''). \end{aligned}$$

For a regular element $x \in \mathfrak{h}_{\mathcal{S}}$, we have

$$\epsilon\ \mathrm{sgn}(\sigma)\pi_{\mathfrak{z}/\mathfrak{h}}(\sigma^{-1}\cdot x)\mathcal{A}(-(\mathcal{S}\setminus\alpha))(x)\mathcal{A}(\mathcal{S}\setminus\alpha)(x)\frac{\alpha(w)}{|\alpha(w)|}\prod_{\beta\in\Psi}\beta(x) \tag{8.4}$$

$$\begin{aligned}
&= \prod_{\beta\in\Psi,\underline{\beta}\cap\{1,2\}=\emptyset}(-\beta(\sigma^{-1}\cdot x))\cdot(-1)^{|\mathcal{S}\setminus\alpha|}\cdot\epsilon\frac{\alpha(w)}{|\alpha(w)|}\cdot\prod_{\beta\in\Psi}\beta(\sigma^{-1}\cdot x)\\
&= \prod_{\beta\in\Psi,\underline{\beta}\cap\{1,2\}=\emptyset}(-\beta(\sigma^{-1}\cdot x)^2)\cdot(-1)^{|\mathcal{S}\setminus\alpha|}\cdot|\alpha(x)|\\
&\quad \prod_{\substack{\beta\in\Psi\setminus\epsilon\sigma^{-1}\alpha,\underline{\beta}\cap\{1,2\}\neq\emptyset,\\ \underline{\sigma\alpha}\cap\underline{\mathcal{S}\setminus\alpha}\neq\emptyset}}\beta(\sigma^{-1}\cdot x)\prod_{\substack{\beta\in\Psi\setminus\epsilon\sigma^{-1}\alpha,\underline{\beta}\cap\{1,2\}\neq\emptyset,\\ \underline{\sigma\alpha}\cap\underline{\mathcal{S}\setminus\alpha}=\emptyset}}\beta(\sigma^{-1}\cdot x).
\end{aligned}$$

Notice that $\sigma\Psi|_{\mathfrak{h}''_{\mathcal{S},s}}$ is a positive root system for the restriction of $\mathfrak{g}$ to $\sum_{j\notin\underline{\alpha}}\mathsf{V}_j$. Also, there are numbers $\epsilon_\beta = \pm 1$ such that $\{\epsilon_\beta\beta \,|\, \beta\in S\setminus\alpha\}$ is a strongly orthogonal set of non-compact imaginary roots in $\sigma\Psi|_{\mathfrak{h}''_{\mathcal{S},s}}$. Furthermore, $\mathfrak{h}''_{\mathcal{S},s}$ is the corresponding Cartan subalgebra. Therefore [1, (1.11)] implies that

$$\begin{aligned}
&\prod_{\beta\in\Psi,\underline{\beta}\cap\{1,2\}=\emptyset}(-\beta(\sigma^{-1}\cdot x)^2)\cdot(-1)^{|\mathcal{S}\setminus\alpha|}\\
&= \prod_{\beta\in\sigma\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}(-\beta(x))\cdot(-1)^{|\mathcal{S}\setminus\alpha|}\cdot\prod_{\beta\in\sigma\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}\beta(x)\\
&= \prod_{\beta\in\sigma\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}\overline{\beta(x)}\cdot\prod_{\beta\in\sigma\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}\beta(x)\geq 0.
\end{aligned}$$

The roots $\beta\in\sigma\Psi\setminus\epsilon\alpha$, with $\underline{\beta}\cap\underline{\alpha}\neq\emptyset$ and $\underline{\beta}\cap\underline{\mathcal{S}\setminus\alpha}\neq\emptyset$ may be partitioned into groups of four

$$\begin{aligned}
&e_{\sigma(1)}-e_{\sigma(k)},\ e_{\sigma(1)}-e_{\sigma(l)},\\
&e_{\sigma(2)}-e_{\sigma(k)},\ e_{\sigma(2)}-e_{\sigma(l)},
\end{aligned}$$

so that $e_{\sigma(2)}(x) = -\overline{e_{\sigma(1)}(x)}$, $e_{\sigma(l)}(x) = -\overline{e_{\sigma(k)}(x)}$. Clearly the product of these four roots evaluated at x is non-negative. Therefore

$$\prod_{\substack{\beta\in\sigma\Psi\setminus\epsilon\alpha,\underline{\beta}\cap\underline{\alpha}\neq\emptyset,\\ \underline{\beta}\cap\underline{\mathcal{S}\setminus\alpha}\neq\emptyset}}\beta(x)\geq 0.$$

The roots $\beta\in\sigma\Psi\setminus\epsilon\alpha$, with $\underline{\beta}\cap\underline{\alpha}\neq\emptyset$ and $\underline{\beta}\cap\underline{\mathcal{S}\setminus\alpha}\neq\emptyset$, may be combined into $n-|\underline{\mathcal{S}}|$ pairs

$$e_{\sigma(1)}-e_{\sigma(k)},\ e_{\sigma(2)}-e_{\sigma(k)}$$

with $e_{\sigma(2)}(x) = -\overline{e_{\sigma(1)}(x)}$ and $e_{\sigma(k)}(x) = -\overline{e_{\sigma(k)}(x)}$. Hence,

$$\prod_{\substack{\beta\in\sigma\Psi\setminus\epsilon\alpha,\underline{\beta}\cap\underline{\alpha}\neq\emptyset,\\ \underline{\beta}\cap\underline{\mathcal{S}\setminus\alpha}=\emptyset}}\beta(x) = (-1)^{n-|\underline{\mathcal{S}}|}\prod_{\substack{\beta\in\sigma\Psi\setminus\epsilon\alpha,\underline{\beta}\cap\underline{\alpha}\neq\emptyset,\\ \underline{\beta}\cap\underline{\mathcal{S}\setminus\alpha}=\emptyset}}|\beta(x)|.$$

Since $|\underline{\mathcal{S}}|$ is even, $(-1)^{n-|\underline{\mathcal{S}}|}=(-1)^n$ and therefore (8.4) coincides with

$$(-1)^n \prod_{\beta\in\sigma\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot \prod_{\beta\in\sigma\Psi\setminus\epsilon\alpha,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)|\cdot|\alpha(x)|.$$

Also, (see Appendix B in [1]), $2p_+=2n$ and $u''=(-1)^{2-2n}$, so that

$$u=(-1)^{p_+}u''=(-1)^n.$$

Therefore,

$$\langle\tilde{\phi}\rangle(0)=\mathrm{i}4\pi^2\tilde{\kappa}(J_1,J_1)|m_{\mathcal{S}}(s)|\int_{\mathfrak{h}''_{\mathcal{S},s}} \prod_{\beta\in\sigma\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot \prod_{\beta\in\sigma\Psi,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)|$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot(u'(J_{\sigma(1)}+J_{\sigma(2)})+x''))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'').$$

Consider the case (b). Here $R'_{\mathcal{S},s}=\emptyset$, $R''_{\mathcal{S},s}=\tilde{\Psi}_{\mathcal{S},\mathbb{R}}$ and $\mathfrak{h}'_{\mathcal{S},s}=\mathbb{R}J_{\sigma(1)}$. Furthermore, consistently with (7.12) in [1],

$$\langle J\ ,\ \rangle_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)},,\mathsf{V}'_1)}>0$$

so that the restriction of $y_{s,\mathcal{S}}$ to $s\mathsf{W}^{\mathfrak{h}'}$ is equal to $-J_{\sigma(1)}$. Moreover, by Lemma B.1,

$$\mathcal{H}_{\mathcal{S},s}\psi\in\mathcal{HCS}(\mathfrak{h}'_{\mathcal{S},s}\setminus 0).$$

Let us write $x'=x'_1J'_1$, $x=x_{\sigma(1)}J_{\sigma(1)}$ and let $y_0>0$. Then,

$$\begin{aligned}
&\det(x'+x+\mathrm{i}y_0y_{s,\mathcal{S}})_{s\mathsf{W}^{\mathfrak{h}'}}=-\mathrm{i}\hat{\epsilon}_1(-\hat{\epsilon}_1x'_1+x_{\sigma(1)}-\mathrm{i}y_0),\\
&\det(x'+x+\mathrm{i}y_0y_{s_\alpha s,\mathcal{S}})_{s\mathsf{W}^{\mathfrak{h}'}}=\mathrm{i}\hat{\epsilon}_1(\hat{\epsilon}_1x'_1+x_{\sigma(1)}-\mathrm{i}y_0),\\
&m_{\mathcal{S}}(s_\alpha s)=m_{\mathcal{S}}(s),
\end{aligned}$$

and therefore,

$$\mathcal{H}'_{\mathcal{S},s}\psi(x')+\mathcal{H}'_{\mathcal{S},s_\alpha s}\psi(x')=\lim_{y_0\to0+}\frac{m_{\mathcal{S}}(s)}{i\hat{\epsilon}_1}\tilde{\kappa}(J_{\sigma(1)},J_{\sigma(1)})^{1/2}$$
$$\int_{\mathbb{R}}\left(\frac{1}{\hat{\epsilon}_1x'_1+x_{\sigma(1)}-\mathrm{i}y_0}-\frac{1}{-\hat{\epsilon}_1x'_1+x_{\sigma(1)}-\mathrm{i}y_0}\right)\mathcal{H}_{\mathcal{S},s}\psi(x_{\sigma(1)}J_{\sigma(1)}))\,dx_{\sigma(1)}. \quad (8.5)$$

By Lemma B.1, (8.5) is equal to a continuous function of $x'_1\in\mathbb{R}$ plus

$$\frac{m_{\mathcal{S}}(s)}{\hat{\epsilon}_1}\tilde{\kappa}(J_{\sigma(1)},J_{\sigma(1)})^{1/2}\int_0^{-1}\left(\frac{1}{\hat{\epsilon}_1x'_1+\mathrm{i}y}+\frac{1}{\hat{\epsilon}_1x'_1-\mathrm{i}y}\right)\langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle_{J_{\sigma(1)}}(\mathrm{i}yJ_{\sigma(1)})\,dy.$$

Hence, by Lemma B.2,

$$\langle\mathcal{H}'_{\mathcal{S},s}\psi+\mathcal{H}'_{\mathcal{S},s_\alpha s}\psi\rangle_{J'}(0)=m_{\mathcal{S}}(s)\tilde{\kappa}(J_{\sigma(1)},J_{\sigma(1)})^{1/2}2\pi(-1)\langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle_{J_{\sigma(1)}}(0).$$

But

$$\begin{aligned}
\langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle_{J_{\sigma(1)}}(0) &= \langle\mathcal{H}_{\mathcal{S},s}\psi\rangle_{J_{\sigma(1)}}(0) \qquad (8.6)\\
&= \int_{\mathfrak{h}''_{\mathcal{S},s}}\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x'')\mathcal{A}(-\tilde{\Psi}_{\mathcal{S},\mathbb{R}})(x'')\langle\mathcal{H}_{\mathcal{S},s}\psi\rangle_{J_{\sigma(1)}}(x'')\,d\mu(x'').
\end{aligned}$$

By a theorem of Harish-Chandra (see Theorem 2.1 in [3])

$$\langle \mathcal{H}_{\mathcal{S},s}\psi\rangle_{J_{\sigma(1)}}(x'') = \langle \mathcal{H}_{\mathcal{S},s}\psi\rangle_{\mathrm{i}H_\alpha}(x'') = \epsilon(\Psi, S, \alpha)\mathrm{i}d(\alpha)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x'').$$

However, as we computed in (C.18) in [1],

$$\mathcal{A}(-\tilde{\Psi}_{\mathcal{S},\mathbb{R}})(x'')\epsilon(\Psi, S, \alpha) = \mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x''),$$

and by Lemma 1.9 in [1], $d(\alpha) = 1$. Hence, (8.6) coincides with

$$\begin{aligned}
&\mathrm{i}\int_{\mathfrak{h}''_{\mathcal{S},s}} \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x'')\mathcal{A}(-\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x'')\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x'')\,d\mu(x'')\\
&= \lim_{x_{\sigma(1)}\to 0}\mathrm{i}\int_{\mathfrak{h}''_{\mathcal{S},s}} \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot(x_{\sigma(1)}J_{\sigma(1)}+x''))\mathcal{A}(-\tilde{\Psi}_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x_{\sigma(1)}J_{\sigma(1)}+x'')\\
&\qquad\qquad \mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x_{\sigma(1)}J_{\sigma(1)}+x'')\,d\mu(x''). \qquad (8.7)
\end{aligned}$$

Let $x\in\mathfrak{h}^{\mathrm{reg}}_{\mathcal{S}}$. Then,

$$\begin{aligned}
&\mathrm{sgn}(\text{short})(s)\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x)\mathcal{A}(-\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x)\\
&= \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x)\mathcal{A}(-\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x)\mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}})(x)\\
&\qquad \prod_{\beta\in\Psi(\text{short})}\beta(s^{-1}\cdot x)\cdot\prod_{\beta\in\Psi(\text{long})}\beta(x)\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot x)\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha)). \qquad (8.8)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x)\mathcal{A}(-\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x)\mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}})(x)\prod_{\beta\in\Psi(\text{short})}\beta(s^{-1}\cdot x)\cdot\prod_{\beta\in\Psi(\text{long})}\beta(x)\\
&= \prod_{\beta\in\Psi(\text{short}),1\notin\underline{\alpha}}(-\beta(\sigma^{-1}\cdot x))\mathcal{A}(-\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x)\mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}})(x)\\
&\qquad \prod_{\beta\in\Psi(\text{short}),1\notin\underline{\alpha}}\beta(\sigma^{-1}\cdot x)\cdot\prod_{\beta\in\Psi(\text{short}),1\in\underline{\alpha}}\beta(\sigma^{-1}\cdot x)\cdot\prod_{\beta\in\Psi(\text{long})\setminus\alpha}\beta(x)\cdot\alpha(x). \qquad (8.9)
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{A}(-\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\text{short}))(x)\mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}})(x) &= (-1)^{|\mathcal{S}(\text{short})|}\mathcal{A}(\Psi_{\mathcal{S},\mathbb{R}}(\text{long}))(x)\frac{\alpha(x)}{|\alpha(x)|}\\
&= (-1)^{|\mathcal{S}(\text{short})|}\mathcal{A}(\mathcal{S}(\text{long}))(x)\frac{\alpha(x)}{|\alpha(x)|},
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{\beta\in\Psi(\text{short}),1\notin\underline{\alpha}}(-\beta(\sigma^{-1}\cdot x))\cdot\prod_{\beta\in\Psi(\text{short}),1\notin\underline{\alpha}}\beta(\sigma^{-1}\cdot x)\cdot\prod_{\beta\in\Psi(\text{long})\setminus\alpha}\beta(x)\\
&= \prod_{\beta\in\Psi(\text{short}),1\notin\underline{\alpha}}(-\beta(\sigma^{-1}\cdot x)^2)\cdot\prod_{\beta\in\Psi(\text{long}),1\notin\underline{\alpha}}\beta(\sigma^{-1}\cdot x) \qquad (8.10)\\
&= \prod_{\beta\in\Psi,1\notin\underline{\alpha}}(-\beta(\sigma^{-1}\cdot x)^2)\cdot\prod_{\beta\in\Psi(\text{long}),1\notin\underline{\alpha}}(-\beta(\sigma^{-1}\cdot x)^{-1})\\
&= (-1)^{|\Psi_{\mathcal{S},\mathbb{R}}|}\prod_{\beta\in\Psi,1\notin\underline{\alpha}}|\beta(\sigma^{-1}\cdot x)|^2\cdot\prod_{\beta\in\Psi(\text{long}),1\notin\underline{\alpha}}(-\beta(\sigma^{-1}\cdot x)^{-1}),
\end{aligned}$$

where the last equality follows from [1, (1.11)]. But, by [1, (7.17)],

$$\begin{aligned}\prod_{\beta\in\Psi(\mathrm{long}),1\notin\underline{\alpha}}(-\beta(\sigma^{-1}\cdot x)^{-1}) &= \prod_{\beta\in\Psi(\mathrm{long}),\beta\neq\alpha}(-\beta(x)^{-1})\\ &= 4^{-n+1}chc_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)}^{\perp},\mathsf{V}_0')}\mathcal{A}(-\mathcal{S}(\mathrm{long}))(x).\end{aligned}$$

Hence, (8.10) is equal to

$$(-1)^{|\Psi_{\mathcal{S},\mathbb{R}}|}\prod_{\beta\in\Psi,1\notin\underline{\alpha}}|\beta(\sigma^{-1}\cdot x)|^2\cdot 4^{-n+1}chc_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)}^{\perp},\mathsf{V}_0')}\mathcal{A}(-\mathcal{S}(\mathrm{long}))(x).$$

Therefore, (8.9) coincides with

$$\begin{aligned}&\mathcal{A}(\mathcal{S}(\mathrm{long}))(x)\frac{\alpha(x)}{|\alpha(x)|}(-1)^{|\Psi_{\mathcal{S},\mathbb{R}}|+|\mathcal{S}(\mathrm{short})|}\cdot\prod_{\beta\in\Psi,1\notin\underline{\alpha}}|\beta(\sigma^{-1}\cdot x)|^2\\ &4^{-n+1}chc_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)}^{\perp},\mathsf{V}_0')}\mathcal{A}(-\mathcal{S}(\mathrm{long}))(x)\cdot\prod_{\beta\in\Psi(\mathrm{short}),1\in\underline{\alpha}}\beta(\sigma^{-1}\cdot x)\cdot\alpha(x)\\ = &|\alpha(x)|\prod_{\beta\in\Psi,1\notin\underline{\alpha}}|\beta(\sigma^{-1}\cdot x)|^2\\ &4^{-n+1}chc_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)}^{\perp},\mathsf{V}_0')}(c(\mathcal{S})\cdot x)\cdot\prod_{\beta\in\Psi(\mathrm{short}),1\in\underline{\alpha}}\beta(\sigma^{-1}\cdot x)\cdot(-1)^{|\mathcal{S}(\mathrm{short})|}.\end{aligned}$$

Notice that if $\alpha(x)=0$, then

$$\begin{aligned}&\prod_{\beta\in\Psi(\mathrm{short}),1\in\underline{\alpha}}\beta(\sigma^{-1}\cdot x)=\prod_{k=2}^{n}(e_1(\sigma^{-1}\cdot x)^2-e_k(\sigma^{-1}\cdot x)^2) \qquad (8.11)\\ = &\prod_{k=2}^{n}(e_{\sigma(1)}(x)^2-e_{\sigma(k)}(x)^2)=\prod_{k=2}^{n}(-e_{\sigma(k)}(x)^2)\\ = &\prod_{\beta\in\Psi(\mathrm{long}),\beta\neq\alpha}(-\frac{1}{4}\beta(x)^2)=\frac{1}{4^{n-1}}(-1)^{|\mathcal{S}(\mathrm{short})|}\prod_{\beta\in\Psi(\mathrm{long}),\beta\neq\alpha}|\beta(x)|^2,\end{aligned}$$

because the set of numbers $\{\beta(x)\,|\,\beta\in\Psi(\mathrm{long}),\beta\neq\alpha\}\subseteq\mathbb{C}$ is the union of $|\mathcal{S}(\mathrm{short})|-$pairs of the form $\{z,\overline{z}\}$ and purely imaginary numbers. Hence, (8.7), when multiplied by $\mathrm{sgn}(\mathrm{short})(s)$, is equal to

$$\begin{aligned}&\lim_{x_{\sigma(1)}\to 0}\mathsf{i}4^{-2n+2}\int_{\mathfrak{h}''_{\mathcal{S},s}}|\alpha(x_{\sigma(1)}J_{\sigma(1)}+x'')|\\ &\prod_{\beta\in\Psi(\mathrm{short}),1\in\underline{\alpha}}|\beta(\sigma^{-1}\cdot x'')|^2\cdot chc_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)}^{\perp},\mathsf{V}_0)}(c(\mathcal{S})\cdot x'')\cdot\prod_{\beta\in\Psi(\mathrm{long}),\beta\neq\alpha}|\beta(x)|^2\\ &\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot(x_{\sigma(1)}J_{\sigma(1)}+x''))\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha))d\mu(x''). \qquad (8.12)\end{aligned}$$

Furthermore, $p_+=2n$ and $u''=(-1)^{1(1+1)/2}=-1$. Thus

$$u=(-1)^{p_+}u''=-1. \qquad (8.13)$$

By combining (8.6), (8.11) and (8.12), we see that (b) follows.

Consider the case (c). Here $R'_{\mathcal{S},s} = \emptyset$, $R''_{\mathcal{S},s} = \tilde{\Psi}_{\mathcal{S},\mathbb{R}}$, $\Gamma'_{\mathcal{S},s} = \emptyset$, $\mathfrak{h}'_{\mathcal{S},s} = \mathbb{R}J_{\sigma(1)}$ and, by Lemma B.1,

$$\mathcal{H}_{\mathcal{S},s}\psi \in S(\mathfrak{h}'_{\mathcal{S},s}).$$

Let us write $x = \mathsf{i}y_{\sigma(1)}J_{\sigma(1)}$. Then

$$\begin{aligned}
&\det(x' + x)_{s\mathsf{W}^{\mathfrak{h}'}} = \mathsf{i}(x'_1 - \mathsf{i}\hat{\epsilon}_1 y_{\sigma(1)}),\\
&\det(x' + x)_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}} = \mathsf{i}(x'_1 + \mathsf{i}\hat{\epsilon}_1 y_{\sigma(1)}),\\
&m_{\mathcal{S}}(s_\alpha s) = m_{\mathcal{S}}(s).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{H}'_{\mathcal{S},s}\psi(x') + \mathcal{H}'_{\mathcal{S},s_\alpha s}\psi(x') &= -\mathsf{i}m_{\mathcal{S}}(s)\tilde{\kappa}(J_{\sigma(1)}, J_{\sigma(1)})^{1/2}\\
&\int_{\mathbb{R}} \left(\frac{1}{x'_1 - \mathsf{i}\hat{\epsilon}_1 y_{\sigma(1)}} + \frac{1}{x'_1 + i\hat{\epsilon}_1 y_{\sigma(1)}} \right) \mathcal{H}_{\mathcal{S},s}\psi(iy_{\sigma(1)}J_{\sigma(1)}))\, dy_{\sigma(1)}.
\end{aligned}$$

The above, Lemma B.2 and (8.13) imply

$$\langle \mathcal{H}'_{\mathcal{S},s}\psi + \mathcal{H}'_{\mathcal{S},s_\alpha s}\psi\rangle_{J'}(0) = \mathsf{i}|m_{\mathcal{S}}(s)|\tilde{\kappa}(J_{\sigma(1)}, J_{\sigma(1)})^{1/2} 2 \cdot 2\pi \operatorname{sgn}(\text{short})(s)\mathcal{H}_{\mathcal{S},s}\psi(0).$$

But $\operatorname{sgn}(\text{short})(s)\mathcal{H}_{\mathcal{S},s}\psi(0)$ coincides with (8.8) if the $\mathcal{S} \vee \alpha$ is replaced by $\mathcal{S}$ and $x_{\sigma(1)} = 0$. Thus, the formula follows as in the previous case.

Consider the case (d). Here $R'_{\mathcal{S},s} = \emptyset$, $R''_{\mathcal{S},s} = \Psi_{\mathcal{S},\mathbb{R}}$, $\mathfrak{h}'_{\mathcal{S},s} = \mathbb{R}J_{\sigma(1)}$ and $\mathcal{H}_{\mathcal{S},s}\psi \in \mathcal{HCS}(\mathfrak{h}'_{\mathcal{S},s} \setminus 0)$. Suppose $\alpha \in \Psi^n$. Then

$$\langle J' \ , \ \rangle_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)},\mathsf{V}')} > 0 \ \text{ and } \ \langle J' \ , \ \rangle_{\mathrm{Hom}(\mathsf{V}_0,\mathsf{V}')} < 0. \tag{8.14}$$

Therefore, $y_{s,\mathcal{S}}|_{s\mathsf{W}^{\mathfrak{h}'}} = -J^*_{\sigma(1)}(y_s)J_{\sigma(1)} = -\hat{\epsilon}_1 J_{\sigma(1)}$. Thus,

$$\mathcal{H}'_{\mathcal{S},s}\psi(x') = \lim_{y_0 \to 0+} \int_{\mathfrak{h}'_{\mathcal{S},s}} \frac{m_{\mathcal{S}}(s)}{\det(x' + x - \mathsf{i}y_0\hat{\epsilon}_1 J_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}} \mathcal{H}_{\mathcal{S},s}\psi(x)\, d\mu(x). \tag{8.15}$$

Since $\det(x' + x - \mathsf{i}y_0\hat{\epsilon}_1 J_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}} = \mathsf{i}(x'_1 - x_{\sigma(1)} + \mathsf{i}y_0\hat{\epsilon}_1) = -\mathsf{i}(-x'_1 + x_{\sigma(1)} - \mathsf{i}y_0\hat{\epsilon}_1)$, Lemma B.2 implies that (8.15) is equal to

$$\mathsf{i}\int_0^{-\hat{\epsilon}_1} \frac{m_{\mathcal{S}}(s)}{\det(x' + \mathsf{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}} \langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle(\mathsf{i}yJ_{\sigma(1)})\, d\mu(yJ_{\sigma(1)}) \tag{8.16}$$

plus a continuous function of $x' \in \mathfrak{h}'$. If we replace s by $s_\alpha s$, then (8.16) transforms to

$$-\mathsf{i}\int_0^{\hat{\epsilon}_1} \frac{m_{\mathcal{S}}(s)}{\det(x' + \mathsf{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}} \langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle(\mathsf{i}yJ_{\sigma(1)})\, d\mu(yJ_{\sigma(1)}).$$

Since the sum

$$\mathcal{H}'_{\mathcal{S},s}\psi + \mathcal{H}'_{\mathcal{S},s_\alpha s}\psi \tag{8.17}$$

does not depend on the transposition $s \to s_\alpha s$, we may assume that $\hat{\epsilon}_1 = 1$. Then (8.17) is equal to

$$\mathrm{i}m_{\mathcal{S}}(s)\left(\int_0^{-1} \frac{1}{\det(x' + \mathrm{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}} \langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle_{J_{\sigma(1)}}(\mathrm{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}) - \int_0^1 \frac{1}{\det(x' + \mathrm{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}} \langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle_{J_{\sigma(1)}}(\mathrm{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)})\right) \quad (8.18)$$

plus a continuous function of $x' \in \mathfrak{h}'$.

Notice that

$$\langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle_{J_{\sigma(1)}}(\mathrm{i}yJ_{\sigma(1)}) = \langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle_{\mathrm{i}H_\alpha}(\mathrm{i}yJ_{\sigma(1)}) \quad (8.19)$$
$$= \int_{\mathfrak{h}''_{\mathcal{S},s}} \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x'')\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x'')\epsilon(\Psi,\mathcal{S},\alpha)\mathrm{i}d(\alpha)\sum_{p=0}^{N}\frac{\partial(\mathrm{i}yJ_{\sigma(1)})^p}{p!}\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x'')\,d\mu(x'')$$
$$= \int_{\mathfrak{h}''_{\mathcal{S},s}} \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x'')\mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}}\setminus\alpha))(x'')\mathrm{i}2\sum_{p=0}^{N}\frac{\partial(\mathrm{i}yJ_{\sigma(1)})^p}{p!}\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x'')\,d\mu(x'').$$

Furthermore, (8.19) is equal to

$$\int_{\mathfrak{h}''_{\mathcal{S},s}} \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x'')\mathcal{A}(-(\Psi_{\mathcal{S},\mathbb{R}}\setminus\alpha))(x'')\mathrm{i}2\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(iyJ_{\sigma(1)} + x'')\,d\mu(x'') \quad (8.20)$$

plus a function which vanishes at $y = 0$ like y^{N+1}.

Notice that $R'_{\mathcal{S}\vee\alpha,s} = \{\alpha\}$, $R''_{\mathcal{S}\vee\alpha,s} = \Psi_{\mathcal{S}\vee\alpha}\setminus\alpha$, $\mathfrak{h}'_{\mathcal{S}\vee\alpha,s} = \mathbb{R}\mathrm{i}J_{\sigma(1)}$ and $\mathcal{H}_{\mathcal{S}\vee\alpha}\psi \in S(h'_{\mathcal{S}\vee\alpha,s})$. In particular, (8.20) coincides with

$$\mathrm{i}2\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(\mathrm{i}yJ_{\sigma(1)}).$$

Thus, if we replace (8.19) by (8.20) in (8.18), we obtain the following function

$$m_{\mathcal{S}}(s)\left(\int_0^{-1} \frac{-2}{\det(x' + \mathrm{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathrm{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}) + \int_0^1 \frac{2}{\det(x' + \mathrm{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathrm{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)})\right) \quad (8.21)$$

plus a continuous function of $x' \in \mathfrak{h}'$. Clearly, (8.21) coincides with

$$m_{\mathcal{S}}(s)\int_{-1}^1\left(\frac{2}{\det(x' + \mathrm{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}I_{[-1,0]}(y) + \frac{2}{\det(x' + \mathrm{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}I_{[0,1]}(y)\right)\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathrm{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}), \quad (8.22)$$

where $I_{[a,b]}$ is the indicator function of the interval $[a,b]$.

On the other hand

$$\mathcal{H}'_{\mathcal{S}\vee\alpha,s}\psi(x') + \mathcal{H}'_{\mathcal{S}\vee\alpha,s_\alpha s}\psi(x') = m_{\mathcal{S}\vee\alpha}(s) \int_{\mathfrak{h}'_{\mathcal{S}\vee\alpha,s}}\left(\frac{1}{\det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}}} - \frac{1}{\det(x'+x)_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\right)\left(-\frac{\alpha(x)}{|\alpha(x)|}\right)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x)\,d\mu(x). \quad (8.23)$$

Let $x = \mathrm{i}yJ_{\sigma(1)}$. Then

$$-\frac{\alpha(x)}{|\alpha(x)|} = -\frac{\mathrm{i}J^*_{\sigma(1)}(\mathrm{i}yJ_{\sigma(1)})}{|\mathrm{i}J^*_{\sigma(1)}(\mathrm{i}yJ_{\sigma(1)})|} = \frac{y}{|y|}.$$

Hence, (8.23) coincides with

$$m_{\mathcal{S}\vee\alpha}(s)\int_{-1}^{1}\left(\frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}} - \frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\right) \\ \frac{y}{|y|}\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathrm{i}yJ_{\sigma(1)})\,d\mu(\mathrm{i}yJ_{\sigma(1)}). \quad (8.24)$$

plus a continuous function of $x' \in \mathfrak{h}'$. Clearly (8.24) is equal to

$$m_{\mathcal{S}\vee\alpha}(s)\int_{-1}^{1}\left(\frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}I_{[0.1]}(y) - \frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}I_{[0.1]}(y) \right. \\ \left. - \frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}I_{[-1,0]}(y) + \frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}I_{[-1,0]}(y)\right) \\ \mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathrm{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}). \quad (8.25)$$

Now we multiply (8.25) by $\frac{m_{\mathcal{S}}(s)}{m_{\mathcal{S}\vee\alpha}(s)}$, add to (8.22) and obtain

$$m_{\mathcal{S}}(s)\int_{-1}^{1}\left(\frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}} + \frac{1}{\det(x'+\mathrm{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\right) \\ \mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathrm{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}).$$

Hence, Lemma B.2 implies

$$\langle \mathcal{H}'_{\mathcal{S},s}\psi + \mathcal{H}'_{\mathcal{S},s_\alpha s}\psi + \frac{m_{\mathcal{S}}(s)}{m_{\mathcal{S}\vee\alpha}(s)}(\mathcal{H}'_{\mathcal{S}\vee\alpha,s}\psi + \mathcal{H}'_{\mathcal{S}\vee\alpha,s_\alpha s}\psi)\rangle_{J'}(0) \\ = -\mathrm{i}m_{\mathcal{S}}(s)\tilde{\kappa}(J_{\sigma(1)},J_{\sigma(1)})^{1/2}4\pi\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(0). \quad (8.26)$$

Also, $p_+ = 2q$ and $u'' = -1$, so that

$$u = (-1)^{p_+}u'' = -1. \quad (8.27)$$

Furthermore,

$$\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(0) \\ = \int_{\mathfrak{h}''_{\mathcal{S}\vee\alpha,s}} \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x'')\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(x'')\,\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x'')\,d\mu(x'') \\ = \lim_{y\to 0}\int_{\mathfrak{h}''_{\mathcal{S}\vee\alpha,s}} \tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot(\mathrm{i}yJ_{\sigma(1)}+x''))\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(\mathrm{i}yJ_{\sigma(1)}+x'') \\ \mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(\mathrm{i}yJ_{\sigma(1)}+x'')\,d\mu(x''). \quad (8.28)$$

For $x \in \mathfrak{h}_{\mathcal{S}\vee\alpha}^{\mathrm{reg}}$,

$$\begin{aligned}&\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x)\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(x)\,\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x)\\ &=\prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x))\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(x)\\ &\qquad\prod_{\beta\in\Psi}\beta(s^{-1}\cdot x)\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot x)\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha)).\end{aligned}\tag{8.29}$$

Notice that $\{s\beta\,|\,\beta\in\Psi(\mathrm{long}),1\notin\underline{\beta}\}$ is a positive root system for $\mathrm{O}_{2p,2q}$ and that $\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{long})$ is the corresponding system of real roots for the Cartan subalgebra $\mathfrak{h}''_{\mathcal{S},s}$. Thus ([1, (1.11)] implies

$$\prod_{\beta\in\Psi(\mathrm{long}),1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x)^2)\cdot(-1)^{|\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{long})|}=\prod_{\beta\in\Psi(\mathrm{long}),1\notin\underline{\beta}}|\beta(s^{-1}\cdot x)|^2.$$

Furthermore,

$$\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(x)\mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}})(x)\alpha(x)=(-1)^{|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha|}|\alpha(x)|.$$

Notice that if $\alpha(x)=0$, then

$$\begin{aligned}\prod_{\beta\in\Psi\setminus e_1,1\in\underline{\beta}}\beta(\sigma^{-1}\cdot x)&=\prod_{k=2}^{n}(e_1(s^{-1}\cdot x)^2-e_k(s^{-1}\cdot x)^2)\\ &=\prod_{k=2}^{n}(e_{\sigma(1)}(x)^2-e_{\sigma(k)}(x)^2)=\prod_{k=2}^{n}(-e_{\sigma(k)}(x)^2)\\ &=\prod_{\beta\in\Psi(\mathrm{short}),1\notin\underline{\beta}}(-\beta(x)^2).\end{aligned}$$

Hence, for such x

$$\begin{aligned}&\prod_{\beta\in\Psi(\mathrm{short}),1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x))\cdot\prod_{\beta\in\Psi(\mathrm{short}),1\notin\underline{\beta}}\beta(s^{-1}\cdot x)\prod_{\beta\in\Psi\setminus e_1,1\in\underline{\beta}}\beta(s^{-1}\cdot x)\\ &=\prod_{\beta\in\Psi(\mathrm{short}),1\notin\underline{\beta}}\beta(s^{-1}\cdot x)^4=\prod_{\beta\in\Psi(\mathrm{short})\setminus\alpha}\beta(s^{-1}\cdot x)^4=\prod_{\beta\in\Psi(\mathrm{short})\setminus\alpha}|\beta(s^{-1}\cdot x)|^4,\end{aligned}$$

where the last equality follows from the fact that the set $\{\beta(x)\,|\,\beta\in\Psi(\mathrm{short})\setminus\alpha\}\subseteq\mathbb{C}$ may be partitioned into pairs $\{z,-\overline{z}\}$, single real numbers and single imaginary numbers. Furthermore,

$$\begin{aligned}&|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha|-|\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{long})|=|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus(\alpha\cup\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{long}))|\\ =\;&|(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\mathrm{long})\cup\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\mathrm{short}))\setminus(\alpha\cup\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{long}))|\\ =\;&|\{\beta\in\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\mathrm{long})\,|\,\underline{\beta}\supseteq\underline{\alpha}\}\cup\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})|\\ =\;&|\{\beta\in\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\mathrm{long})\,|\,\underline{\beta}\supseteq\underline{\alpha}\}|+|\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})|=2|\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})|,\end{aligned}$$

so that

$$(-1)^{|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha|}(-1)^{|\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{long})|}=1.$$

Therefore, (8.29) is equal to

$$\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi(\mathrm{short})\setminus\alpha}|\beta(x)|^2\cdot|\alpha(x)| \tag{8.30}$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot x)\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha)).$$

By combining (8.26), (8.27), (8.28) and (8.30), we see that (c) holds for $\alpha\in\Psi^n$.

Suppose $\alpha\in\Psi^c$. Then there is a unique non-compact short root $\nu\in\mathcal{S}$. Moreover, (8.14) implies that

$$\langle J'\ ,\ \rangle_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)},\mathsf{V}')}<0.$$

Therefore, $y_{s,\mathcal{S}}|_{s\mathsf{W}^{\mathfrak{h}'}}=\hat{\epsilon}_1 J_{\sigma(1)}$. Thus $\mathcal{H}'_{\mathcal{S},s}\psi(x')$ is equal to

$$\mathsf{i}\int_0^{\hat{\epsilon}_1}\frac{m_{\mathcal{S}}(s)}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}\langle(\mathcal{H}_{\mathcal{S},s}\psi)_N\rangle(\mathsf{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)})$$

plus a continuous function of $x'\in\mathfrak{h}'$. Hence, as before, assuming that $\hat{\epsilon}_1=1$, $\mathcal{H}'_{\mathcal{S},s}\psi+\mathcal{H}'_{\mathcal{S},s_\alpha s}\psi$ is equal to

$$m_{\mathcal{S}}(s)\left(\int_0^1\frac{-2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathsf{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)})\right.$$
$$\left.+\int_0^{-1}\frac{2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathsf{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)})\right)$$

plus a continuous function of $x'\in\mathfrak{h}'$. On the other hand, $\mathcal{H}'_{\mathcal{S}\vee\alpha,s}\psi+\mathcal{H}'_{\mathcal{S}\vee\alpha,s_\alpha s}\psi$ is equal to

$$m_{\mathcal{S}\vee\alpha}(s)\int_{-1}^1\left(\frac{1}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}-\frac{1}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\right)\frac{y}{|y|}$$
$$\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathsf{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}), \tag{8.31}$$

plus a continuous function of $x'\in\mathfrak{h}'$. By symmetry, (8.31) is equal to

$$m_{\mathcal{S}\vee\alpha}(s)\int_0^1\left(\frac{2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}-\frac{2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\right)$$
$$\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathsf{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}). \tag{8.32}$$

If we multiply (8.32) by $\frac{m_{\mathcal{S}}(s)}{m_{\mathcal{S}\vee\alpha}(s)}$, we get

$$m_{\mathcal{S}}(s)\left(\int_0^{-1}\frac{2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathsf{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)})\right.$$
$$\left.-\int_0^1\frac{2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(iyJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)})\right)$$
$$=-m_{\mathcal{S}}(s)\int_0^1\left(\frac{2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s\mathsf{W}^{\mathfrak{h}'}}}+\frac{2}{\det(x'+\mathsf{i}yJ_{\sigma(1)})_{s_\alpha s\mathsf{W}^{\mathfrak{h}'}}}\right)$$
$$\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(\mathsf{i}yJ_{\sigma(1)})\,d\mu(yJ_{\sigma(1)}).$$

Hence, Lemma B.2 and (8.29) show that

$$\langle \mathcal{H}'_{\mathcal{S},s}\psi + \mathcal{H}'_{\mathcal{S},s_\alpha s}\psi + \frac{m_{\mathcal{S}}(s)}{m_{\mathcal{S}\vee\alpha}(s)}(\mathcal{H}'_{\mathcal{S}\vee\alpha,s}\psi + \mathcal{H}'_{\mathcal{S}\vee\alpha,s_\alpha s}\psi)\rangle_{J'}(0)$$
$$= -\mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|m_{\mathcal{S}}(s)|\,\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(0). \quad (8.33)$$

As before,

$$\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(0) = \lim_{y\to 0}\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha,s}\psi(yJ_{\sigma(1)})$$
$$= \lim_{y\to 0}\int_{\mathfrak{h}''_{\mathcal{S}\vee\alpha,s}}\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot(\mathrm{i}yJ_{\sigma(1)}+x''))\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(\mathrm{i}yJ_{\sigma(1)}+x'')$$
$$\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(\mathrm{i}yJ_{\sigma(1)}+x'')\,d\mu(x''). \quad (8.34)$$

Also, for $x\in\mathfrak{h}^{\mathrm{reg}}_{\mathcal{S}\vee\alpha}$,

$$\tilde{\pi}_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot(x))\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(x)\,\mathrm{sgn}(s)\mathcal{H}_{\mathcal{S}\vee\alpha}\psi(x)$$
$$= \prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x))\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(x)\mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}})(x)\cdot\prod_{\beta\in\Psi}\beta(s^{-1}\cdot x)$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\alpha)}\psi(g\cdot x)\,d\mu(g\mathrm{H}(\mathcal{S}\vee\alpha)). \quad (8.35)$$

Notice that,

$$\prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x))\mathcal{A}(-(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha))(x)\mathcal{A}(\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha)(x)\frac{\alpha(x)}{|\alpha(x)|}\prod_{\beta\in\Psi}\beta(s^{-1}\cdot x)$$
$$= \prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x)^2)\prod_{\beta\in\Psi\setminus e_1,1\in\underline{\beta}}\beta(s^{-1}\cdot x)\cdot|\alpha(x)|(-1)^{|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha|}. \quad (8.36)$$

If we divide (8.36) by $|\alpha(x)|$ and take x with $\alpha(x)=0$, we obtain

$$\begin{aligned}
&\prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x)^2)\cdot\prod_{\beta\in\Psi\setminus e_1,1\in\underline{\beta}}\beta(s^{-1}\cdot x)\cdot(-1)^{|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha|}\\
&= \prod_{\beta\in\Psi(\mathrm{long}),1\notin\underline{\beta}}(-\beta(s^{-1}\cdot x)^2)\cdot\left(\prod_{\beta\in\Psi(\mathrm{short})\setminus\alpha}(-\beta(x)^2)\right)^2\cdot(-1)^{|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha|}\\
&= \prod_{\beta\in\Psi(\mathrm{long}),1\notin\underline{\beta}}|\beta(x)|^2\cdot\prod_{\beta\in\Psi(\mathrm{short})\setminus\alpha}|\beta(x)|^4\cdot(-1)^{|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\alpha|-|\Psi_{\mathcal{S}\setminus\alpha,\mathbb{R}}(\mathrm{long})|}\\
&= -\prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi(\mathrm{short})\setminus\alpha}|\beta(x)|^4\cdot(-1)^{|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\Psi_{\mathcal{S}\setminus\nu,\mathbb{R}}(\mathrm{long})|}
\end{aligned} \quad (8.37)$$

Notice that

$$\begin{aligned}
|\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}\setminus\Psi_{\mathcal{S}\setminus\nu,\mathbb{R}}(\mathrm{long})| &= |\Psi_{(\alpha\vee\nu)\cup(\mathcal{S}\setminus\nu),\mathbb{R}}\setminus\Psi_{\mathcal{S}\setminus\nu,\mathbb{R}}(\mathrm{long})| \quad (8.38)\\
&= |\Psi_{\mathcal{S}\vee\alpha,\mathbb{R}}(\mathrm{short})| + |\Psi_{(\alpha\vee\nu)\cup(\mathcal{S}\setminus\nu),\mathbb{R}}(\mathrm{long})\setminus\Psi_{\mathcal{S}\setminus\nu,\mathbb{R}}(\mathrm{long})|,
\end{aligned}$$

where both summands are even numbers. Therefore, (8.38) is equal to

$$-\prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi(\mathrm{short})\setminus\alpha}|\beta(x)|^4. \tag{8.39}$$

We combine (8.33)-(8.39) and deduce (d).

Consider the case (e). Let $\underline{\alpha}=\{k,l\}$ and let $\sigma(1)=k$. Then $\mathfrak{h}'_{\mathcal{S},s}=\mathbb{R}\mathsf{i}J_k+\mathbb{R}\mathsf{i}J_l$. Moreover, $\Gamma'_{s,\mathcal{S}}=\emptyset$, $R'_{\mathcal{S},s}=\emptyset$, $R''_{\mathcal{S},s}=\Psi_{\mathcal{S},\mathbb{R}}$ and $\mathcal{H}_{\mathcal{S},s}\psi\in S(\mathfrak{h}'_{\mathcal{S},s})$. Furthermore, since $m_{\mathcal{S}}(ws)=m_{\mathcal{S}}(s)$,

$$\begin{aligned}\mathcal{H}'_{\mathcal{S},s}\psi(x')+\mathcal{H}'_{\mathcal{S},ws}\psi(x')=m_{\mathcal{S}}(s)\int_{\mathbb{R}}\left(\frac{1}{\mathsf{i}(x'_1\pm x_k i)}+\frac{1}{\mathsf{i}(x'_1\mp x_k i)}\right)&\\ \int_{\mathbb{R}}\mathcal{H}_{\mathcal{S},s}\psi(x_k\mathsf{i}J_k+x_l\mathsf{i}J_l)\,d\mu(x_k\mathsf{i}J_k)\,d\mu(x_l\mathsf{i}J_l).&\end{aligned}$$

Therefore, Lemma B.2 implies that

$$\langle\mathcal{H}'_{\mathcal{S},s}\psi+\mathcal{H}'_{\mathcal{S},ws}\psi\rangle_{J'}(0)=-\mathsf{i}4\pi m_{\mathcal{S}}(s)\tilde{\kappa}(J_1,J_1)^{1/2}\int_{\mathbb{R}}\mathcal{H}_{\mathcal{S},s}\psi(x_l\mathsf{i}J_l)\,d\mu(x_l\mathsf{i}J_l).$$

Also, $p_+=2q$ and $u''=1$, so that

$$u=(-1)^{p_+}u''=1.$$

Notice that

$$\begin{aligned}&\mathrm{sgn}(s)\int_{\mathbb{R}}\mathcal{H}_{\mathcal{S},s}\psi(x_l\mathsf{i}J_l)\,d\mu(x_l\mathsf{i}J_l)\\ &=\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_k^*}}\pi_{\mathfrak{z}/\mathfrak{h}}(s^{-1}.x)\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x)\mathcal{A}(\Psi_{\mathcal{S},\mathbb{R}})(x)\prod_{\beta\in\Psi}\beta(s^{-1}.x)\\ &\qquad\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g.c(\mathcal{S})x)\,d\mu(g\mathrm{H}(\mathcal{S}))\,d\mu(x).\end{aligned}$$

Furthermore, for $x\in\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_k^*}$,

$$\begin{aligned}&\pi_{\mathfrak{z}/\mathfrak{h}}(s^{-1}.x)\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x)\mathcal{A}(\Psi_{\mathcal{S},\mathbb{R}})(x)\prod_{\beta\in\Psi}\beta(s^{-1}.x) \qquad (8.40)\\ &=\prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}.x)^2)\cdot(-1)^{|\Psi_{\mathcal{S},\mathbb{R}}|}\cdot\prod_{\beta\in\Psi,1\in\underline{\beta}}\beta(s^{-1}.x)\\ &=\prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}.x)^2)\cdot(-1)^{|\{\beta\in\Psi_{\mathcal{S},\mathbb{R}}\,|\,\sigma(1)\notin\underline{\beta}\}|}\cdot\prod_{\beta\in\Psi,1\in\underline{\beta}}\beta(s^{-1}.x),\end{aligned}$$

because $|\{\beta\in\Psi_{\mathcal{S},\mathbb{R}}\,|\,\sigma(1)\in\underline{\beta}\}|$ is even. By [1, (1.11)],

$$\prod_{\beta\in\Psi,1\notin\underline{\beta}}(-\beta(s^{-1}.x)^2)\cdot(-1)^{|\{\beta\in\Psi_{\mathcal{S},\mathbb{R}};\ \sigma(1)\notin\underline{\beta}\}|}=\prod_{\beta\in\Psi,1\notin\underline{\beta}}|\beta(s^{-1}.x)|^2=\prod_{\beta\in\Psi,\sigma(1)\notin\underline{\beta}}|\beta(x)|^2.$$

Let $\mathcal{S}' = \{\beta \in \mathcal{S} \,|\, \underline{\beta} \subseteq \underline{\mathcal{S}}\}$. Then $|\underline{\mathcal{S}'}|$ is even and therefore,

$$\begin{aligned}
\prod_{\beta\in\Psi, 1\notin\underline{\beta}} \beta(s^{-1}.x) &= \prod_{j=2}^{n}(-e_j(x)^2) = \prod_{j\in\underline{\mathcal{S}'}\setminus\sigma(1)} (-e_j(x)^2)\cdot \prod_{j\in\underline{\mathcal{S}}\setminus\underline{\mathcal{S}'}} (-e_j(x)^2)\cdot \prod_{j\notin\underline{\mathcal{S}}}(-e_j(x)^2)\\
&= \prod_{j\in\underline{\mathcal{S}'}\setminus\sigma(1)} (-e_j(x)^2)\cdot \prod_{j\in\underline{\mathcal{S}}\setminus\underline{\mathcal{S}'}} |e_j(x)|^2\cdot \prod_{j\notin\underline{\mathcal{S}}}|e_j(x)|^2\\
&= (-1)^{|\underline{\mathcal{S}'}\setminus\sigma(1)|} \prod_{j\in\underline{\mathcal{S}'}\setminus\sigma(1)} |e_j(x)|^2\cdot \prod_{j\in\underline{\mathcal{S}}\setminus\underline{\mathcal{S}'}} |e_j(x)|^2\cdot \prod_{j\notin\underline{\mathcal{S}}}|e_j(x)|^2\\
&= -\prod_{j\neq\sigma(1)} |e_j(x)|^2.
\end{aligned}$$

Thus (8.40) is equal to

$$-\prod_{\beta\in\Psi,\sigma(1)\notin\underline{\beta}} |\beta(x)|^2\cdot \prod_{j\neq\sigma(1)} |e_j(x)|^2,$$

and (e) follows. ■

Lemma 8.3. *Fix $\alpha\in\Psi^n$. If $(\mathrm{G},\mathrm{G}') = (\mathrm{U}_{p,q},\mathrm{U}_{1,1})$ then*

$$\begin{aligned}
&\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J_1'-J_2'}(u'(J_1'+J_2'))\\
&=\sum_{\alpha\in\mathcal{S}} \lim_{y\to 0} \mathrm{i}4\pi^2\tilde{\kappa}(J_1,J_1)m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}|\Psi^n| \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{e_j\,;\,j\in\underline{\alpha}}} \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2\\
&\qquad \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)| \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''), \qquad (8.41)
\end{aligned}$$

where $x = yH_\alpha + u'\sum_{j\in\underline{\alpha}} J_j + x''$.

If $(\mathrm{G},\mathrm{G}') = (\mathrm{Sp}_{2n}(\mathbb{R}),\mathrm{O}_{1,2})$, assume that α is long. Then,

$$\begin{aligned}
&\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J'}(0)\\
&=\sum_{\alpha\in\Psi_{\mathcal{S},\mathbb{R}}} \lim_{y\to 0} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}2\frac{m_{\mathcal{S}}}{2^{1-n}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}8^{-n+1}|\Psi^n(\mathrm{long})|\\
&\qquad \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\alpha}} chc_{\mathrm{Hom}(\mathsf{V}_{\underline{\alpha}}^{\perp},\mathsf{V}_0')}(c(\mathcal{S})x'')\cdot \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2\\
&\qquad \prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2\cdot|\alpha(x)| \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''), \qquad (8.42)
\end{aligned}$$

where $x = yH_\alpha + x''$.

If $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2p+1,2q},\mathrm{Sp}_2(\mathbb{R}))$, assume that α is short. Then,

$$\begin{aligned}
&\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J'}(0) = \sum_{\alpha\in\Psi_{\mathcal{S},\mathbb{R}}} \lim_{y\to 0} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}|\Psi^n(\mathrm{short})|\\
&\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\alpha}} \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2\cdot|\alpha(x)| \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''),
\end{aligned}$$

(8.43)

where $x = yH_\alpha + x''$.

If $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2p,2q}, \mathrm{Sp}_2(\mathbb{R}))$, *then*

$$\begin{aligned}&\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J'}(0)\\ &= \sum_{\alpha\in\mathcal{S}, k\in\underline{\alpha}\subseteq\underline{\mathcal{S}\setminus\alpha}} \mathsf{i}2\pi\tilde{\kappa}(J_1, J_1)^{1/2} m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}|\Psi^n|\\ &\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_k^*}} \prod_{\beta\in\Psi, k\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq k} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x).\end{aligned} \tag{8.44}$$

Proof. By Lemmas 8.1 and 8.2, the left hand side of (8.41) is equal to

$$\begin{aligned}&\sum_{\mathcal{S},s;\ \zeta=e_{\sigma(1)}-e_{\sigma(2)}\in\mathcal{S}} \langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J_1'-J_2'}(u'(J_1'+J_2'))\\ &= \sum_{\mathcal{S},s;\ \zeta=e_{\sigma(1)}-e_{\sigma(2)}\in\mathcal{S}} \lim_{y\to 0} \mathsf{i}4\pi^2\tilde{\kappa}(J_1, J_1)|m_{\mathcal{S}}(s)| \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{e_{\sigma(1)},e_{\sigma(2)}}} \prod_{\beta\in\Psi, \underline{\beta}\cap\underline{\zeta}=\emptyset} |\beta(x)|^2\\ &\qquad \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\zeta}\neq\emptyset} |\beta(x)| \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'')\\ &= \sum_{\alpha\in\mathcal{S}} \lim_{y\to 0} \mathsf{i}4\pi^2\tilde{\kappa}(J_1, J_1)|m_{\mathcal{S}}(s)||W(\mathrm{H}_{\mathbb{C}}, \mathrm{Z}_{\mathbb{C}}))||\Psi^n| \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\{e_j;j\in\underline{\alpha}\}}} \prod_{\beta\in\Psi, \underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2\\ &\qquad \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)| \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''),\end{aligned}$$

which coincides with the right hand side because $|m_{\mathcal{S}}(s)||W(\mathrm{H}_{\mathbb{C}}, \mathrm{Z}_{\mathbb{C}}))| = m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}$.

By Lemmas 8.1 and 8.2, the left hand side of (8.42) is equal to

$$\begin{aligned}&\sum_{\mathcal{S},s;\sigma(1)\notin\underline{\mathcal{S}},\hat{\epsilon}_1=1} \lim_{y\to 0} \mathsf{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}|m_{\mathcal{S}}(s)|8^{-n+1}\\ &\quad \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{e_{\sigma(1)}}} chc_{\mathrm{Hom}(\mathsf{V}_{\sigma(1)}^{\perp},\mathsf{V}_0')}(c(\mathcal{S})x'') \cdot \prod_{\beta\in\Psi,\beta\neq 2e_{\sigma(1)}} |\beta(yH_{2e_{\sigma(1)}} + x'')|^2\\ &\qquad \prod_{\beta\in\Psi(\mathrm{long}),\beta\neq 2e_{\sigma(1)}} |\beta(yH_{2e_{\sigma(1)}} + x'')|^2 \cdot |\alpha(yH_{2e_{\sigma(1)}} + x'')|\\ &\qquad\quad \int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee 2e_{\sigma(1)})} \psi(g\cdot c(\mathcal{S})(yH_{2e_{\sigma(1)}} + x''))\, d\mu(g\mathrm{H}(\mathcal{S}\vee 2e_{\sigma(1)}))d\mu(x'')\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathcal{S},s;\sigma(1)\notin\underline{S},\hat{\epsilon}_1=1} \lim_{y\to 0} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}2|m_{\mathcal{S}\vee 2e_{\sigma(1)}}(s)|8^{-n+1}n \\
&\quad \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{e_1}} chc_{\mathrm{Hom}(V_1^{\perp},\mathsf{V}_0')}(c(\mathcal{S})x'')\cdot \prod_{\beta\in\Psi,\beta\neq 2e_1} |\beta(yH_{2e_1}+x'')|^2 \\
&\quad \prod_{\beta\in\Psi(\mathrm{long}),\beta\neq 2e_1} |\beta(yH_{2e_1}+x'')|^2\cdot|\alpha(yH_{2e_1}+x'')| \\
&\quad \int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee 2e_1)} \psi(g\cdot c(\mathcal{S})(yH_{2e_1}+x''))\,d\mu(g\mathrm{H}(\mathcal{S}\vee 2e_1))d\mu(x'') \\
&= \sum_{2e_1\in S} \lim_{y\to 0} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}2|m_{\mathcal{S}}(s)||W(\mathrm{H}_{\mathbb{C}},\mathrm{Z}_{\mathbb{C}})|8^{-n+1}n \\
&\quad \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{e_1}} chc_{\mathrm{Hom}(\mathsf{V}_1^{\perp},\mathsf{V}_0')}(c(\mathcal{S})x'')\cdot \prod_{\beta\in\Psi,\beta\neq 2e_1} |\beta(yH_{2e_1}+x'')|^2 \\
&\quad \prod_{\beta\in\Psi(\mathrm{long}),\beta\neq 2e_1} |\beta(yH_{2e_1}+x'')|^2\cdot|\alpha(yH_{2e_1}+x'')| \\
&\quad \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})(yH_{2e_1}+x''))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''),
\end{aligned}
$$

which coincides with the right hand side (see [1, (7.11)]).

We need to verify (8.43). Notice that if $\Gamma'_{s,S}=\emptyset$ and if $w\in W(\Delta_{\mathcal{S},\mathbb{R}})$, then

$$
\begin{aligned}
\mathcal{H}'_{\mathcal{S},ws}\psi(x') &= \int_{\mathfrak{h}_{\mathcal{S}}} \frac{m_{\mathcal{S}}(ws)}{\det(x'+x)_{ws\mathsf{W}^{\mathfrak{h}'}}} \pi_{\mathfrak{z}/\mathfrak{h}}(s^{-1}w^{-1}\cdot x)\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x)\mathcal{H}_{\mathcal{S}}\psi(x)\,d\mu(x) \\
&= \int_{\mathfrak{h}_{\mathcal{S}}} \frac{m_{\mathcal{S}}(s)}{\det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}}} \pi_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x)\,\mathrm{sgn}(w)\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(w\cdot x)\mathcal{H}_{\mathcal{S}}\psi(x)\,d\mu(x) \\
&= \int_{\mathfrak{h}_{\mathcal{S}}} \frac{m_{\mathcal{S}}(s)}{\det(x'+x)_{s\mathsf{W}^{\mathfrak{h}'}}} \pi_{\mathfrak{z}/\mathfrak{h}}(s^{-1}\cdot x)\mathcal{A}(-\Psi_{\mathcal{S},\mathbb{R}})(x)\mathcal{H}_{\mathcal{S}}\psi(x)\,d\mu(x).
\end{aligned}
$$

Let $\zeta\in\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}$, $\underline{\zeta}\cap\underline{\mathcal{S}}=\emptyset$. Suppose $\zeta\in\Psi^n$. There are $w_i\in W(\Delta_{\mathcal{S}\vee\zeta,\mathbb{R}})$, $1\le i\le m$ (with $w_1=1$) such that

$$
\Psi_{\mathcal{S}\vee\zeta,\mathbb{R}}(\mathrm{short})=\{w_1\zeta,w_2\zeta,\cdots,w_m\zeta\}.
$$

Moreover, by [1, (8.3)],

$$
\frac{|m_{\mathcal{S}}(s)|}{|m_{\mathcal{S}\vee\zeta}(s)|}=m.
$$

Hence,

$$
\frac{|m_{\mathcal{S}}(s)|}{|m_{\mathcal{S}\vee\zeta}(s)|}\mathcal{H}'_{\mathcal{S}\vee\zeta,s}\psi(x')=\sum_{i=1}^{m}\mathcal{H}'_{\mathcal{S}\vee\zeta,w_is}\psi(x').
$$

Suppose $\zeta\in\Psi^c$. Then there is a unique element $\nu\in\mathcal{S}\cap\Psi^n(\mathrm{short})$. Moreover, $\zeta-\nu\in\mathcal{S}\vee\zeta$. Let $w_1=1$ and let $w_2=s_\zeta$. Then $w_1,w_2\in W(\Delta_{\mathcal{S}\vee\zeta,\mathbb{R}})$ and, by [1, (8.3)],

$$
\frac{|m_{\mathcal{S}}(s)|}{|m_{\mathcal{S}\vee\zeta}(s)|}=2.
$$

Hence,

$$\frac{|m_{\mathcal{S}}(s)|}{|m_{\mathcal{S}\vee\zeta}(s)|}\mathcal{H}'_{\mathcal{S}\vee\zeta,s}\psi(x') = \sum_{i=1}^{2}\mathcal{H}'_{\mathcal{S}\vee\zeta,w_is}\psi(x').$$

For two functions, $f, g : \mathfrak{h}'\setminus 0 \to \mathbb{C}$ we shall write $f =_{jump} g$ if $f-g$ extends to a continuous function on $\mathfrak{h}'$.

Lemma 8.1 implies

$$\begin{aligned}
\pi_{\mathfrak{z}/\mathfrak{h}}\widetilde{chc}(\psi) &=_{jump} \sum_{\mathcal{S},s;\zeta\in\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}},\underline{\zeta}\cap\underline{S}=\emptyset,\underline{\zeta}=\{\sigma(1)\}}\mathcal{H}'_{\mathcal{S},s}\psi + \sum_{\mathcal{S},s;\sigma(1)\in\underline{\Psi_{\mathcal{S},\mathbb{R}}}}\mathcal{H}'_{\mathcal{S},s}\psi\\
&= \sum_{\substack{\mathcal{S},s;\zeta\in\Psi^n(\mathrm{short}),\underline{\zeta}\cap\underline{S}=\emptyset,\\ \sigma(1)\in\underline{\zeta}}}\mathcal{H}'_{\mathcal{S},s}\psi + \sum_{\substack{\mathcal{S},s;\zeta\in\Psi^c(\mathrm{short}),\underline{\zeta}\cap\underline{S}=\emptyset,\\ \sigma(1)\in\underline{\zeta}}}\mathcal{H}'_{\mathcal{S},s}\psi\\
&\quad + \sum_{\substack{\mathcal{S},s;\zeta\in\Psi^n(\mathrm{short}),\underline{\zeta}\cap\underline{S}=\emptyset,\\ \sigma(1)\in\underline{\Psi_{\mathcal{S}\vee\zeta,\mathbb{R}}(\mathrm{short})}}}\mathcal{H}'_{\mathcal{S}\vee\zeta,s}\psi + \sum_{\substack{\mathcal{S},s;\zeta\in\Psi^c(\mathrm{short}),\underline{\zeta}\cap\underline{S}=\emptyset,\\ \sigma(1)\in\underline{\zeta}\cup\underline{\nu}}}\mathcal{H}'_{\mathcal{S}\vee\zeta,s}\psi\\
&= \sum_{\substack{\mathcal{S},s;\zeta\in\Psi^n(\mathrm{short})\cap\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}},\\ \sigma(1)\in\underline{\zeta}}}\left(\mathcal{H}'_{\mathcal{S},s}\psi + m\mathcal{H}'_{\mathcal{S}\vee\zeta,s}\psi\right) + \sum_{\substack{\mathcal{S},s;\zeta\in\Psi^c(\mathrm{short})\cap\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}},\\ \sigma(1)\in\underline{\zeta}}}\left(\mathcal{H}'_{\mathcal{S},s}\psi + 2\mathcal{H}'_{\mathcal{S}\vee\zeta,s}\psi\right)\\
&= \sum_{\mathcal{S},s;\zeta\in\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}},\sigma(1)\in\underline{\zeta}}\left(\mathcal{H}'_{\mathcal{S},s}\psi + \frac{|m_{\mathcal{S}}(s)|}{|m_{\mathcal{S}\vee\zeta}(s)|}\mathcal{H}'_{\mathcal{S}\vee\zeta,s}\psi\right).
\end{aligned}$$

Hence, Lemma 8.2 implies that the left hand side of (8.43) is equal to

$$\begin{aligned}
&\sum_{\substack{\mathcal{S},s;\zeta\in\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}(\mathrm{short}),\\ \sigma(1)\in\underline{\zeta},\hat{\epsilon}_1=1}}\lim_{y\to 0}\mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|m_{\mathcal{S}}(s)|\\
&\qquad\int_{\mathfrak{h}_{\mathcal{S}\vee\zeta}\cap\mathfrak{h}^{\zeta}}\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\zeta}=\emptyset}|\beta(x'')|^2\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\zeta}=\emptyset}|\beta(x'')|^2\cdot|\alpha(yH_\zeta+x'')|\\
&\qquad\qquad\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\zeta)}\psi(g\cdot c(\mathcal{S})(yH_\zeta+x''))\,d\mu(g\mathrm{H}(\mathcal{S}\vee\zeta))d\mu(x'')\\
=&\sum_{\mathcal{S};\zeta\in\Psi^n_{\mathcal{S},\mathrm{i}\mathbb{R}}(\mathrm{short})}\lim_{y\to 0}\mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|W(\mathrm{H}_{\mathbb{C}},\mathrm{Z}_{\mathbb{C}})||m_{\mathcal{S}}(s)|\\
&\qquad\int_{\mathfrak{h}_{\mathcal{S}\vee\zeta}\cap\mathfrak{h}^{\zeta}}\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\zeta}=\emptyset}|\beta(x'')|^2\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\zeta}=\emptyset}|\beta(x'')|^2\cdot|\alpha(yH_\zeta+x'')|\\
&\qquad\qquad\int_{\mathrm{G}/\mathrm{H}(\mathcal{S}\vee\zeta)}\psi(g\cdot c(\mathcal{S})(yH_\zeta+x''))\,d\mu(g\mathrm{H}(\mathcal{S}\vee\zeta))d\mu(x'')
\end{aligned}$$

$$= \sum_{\mathcal{S};\zeta\in\mathcal{S}(\mathrm{short})} \lim_{y\to 0} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})| m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}$$
$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\zeta}} \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\zeta}=\emptyset} |\beta(x'')|^2 \prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\zeta}=\emptyset} |\beta(x'')|^2\cdot|\alpha(yH_\zeta+x'')|$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})(yH_\zeta+x''))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'')$$
$$+ \sum_{\mathcal{S}=\mathcal{S}(\mathrm{long});\zeta\in\Psi^c\cap\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})} \lim_{y\to 0} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2} 2 m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}$$
$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\zeta}} \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\zeta}=\emptyset} |\beta(x'')|^2 \prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\zeta}=\emptyset} |\beta(x'')|^2\cdot|\alpha(yH_\zeta+x'')|$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})(yH_\zeta+x''))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'')$$
$$= \sum_{\mathcal{S};\zeta\in\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})} \lim_{y\to 0} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2} m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}$$
$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\zeta}} \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\zeta}=\emptyset} |\beta(x'')|^2 \prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\zeta}=\emptyset} |\beta(x'')|^2\cdot|\alpha(yH_\zeta+x'')|$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})(yH_\zeta+x''))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x''),$$

which coincides with the right hand side.

By Lemmas 8.1 and 8.2, the left hand side of (8.44) is equal to

$$\langle\sum_{\mathcal{S},s}\mathcal{H}_{\mathcal{S},s}\psi\rangle_{J'}(0) = \sum_{\substack{\mathcal{S},s;\ \alpha\in\mathcal{S},\sigma(1)\in\underline{\alpha}\subseteq\underline{\mathcal{S}\setminus\alpha},\\ \hat{\epsilon}_{\sigma(1)}=1}} \langle\mathcal{H}_{\mathcal{S},s}\psi+\mathcal{H}_{\mathcal{S},ws}\psi\rangle_{J'}(0)\cdot\frac{|\Psi^n|}{2}$$
$$= \sum_{\substack{\mathcal{S},s;\ \alpha\in\mathcal{S},\sigma(1)\in\underline{\alpha}\subseteq\underline{\mathcal{S}\setminus\alpha},\\ \hat{\epsilon}_{\sigma(1)}=1}} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|m_{\mathcal{S}}(s)|\frac{|\Psi^n|}{2}$$
$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_{\sigma(1)}}} \prod_{\beta\in\Psi,\sigma(1)\notin\underline{\beta}} |\beta(x)|^2\cdot\prod_{j\neq\sigma(1)}|e_j(x)|^2\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g\cdot c(\mathcal{S})x)\,d\mu(g\mathrm{H}(\mathcal{S}))\,d\mu(x)$$
$$= \sum_{\mathcal{S};\ \alpha\in\mathcal{S},k\in\underline{\alpha}\subseteq\underline{\mathcal{S}\setminus\alpha}} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}|m_{\mathcal{S}}(s)||W(\mathrm{H}_{\mathbb{C}},\mathrm{Z}_{\mathbb{C}}))|\frac{|\Psi^n|}{2}$$
$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_k}} \prod_{\beta\in\Psi,k\notin\underline{\beta}} |\beta(x)|^2\cdot\prod_{j\neq k}|e_j(x)|^2\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g\cdot c(\mathcal{S})x)\,d\mu(g\mathrm{H}(\mathcal{S}))\,d\mu(x),$$

which coincides with the right hand side. ∎

Lemma 8.4. *With the notations of Corollary 0.8, we have*

$$\begin{aligned}&\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J_1'-J_2'}(u'(J_1'+J_2')) \qquad (8.45)\\ = \;& \frac{\mathsf{i}2\pi^2\tilde{\kappa}(J_1,J_1)\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\int_{\mathfrak{g}(\mathsf{U})}\psi_{\mathfrak{n}}^K(u'\sum_{j\in\underline{\alpha}}J_j+x)\,d\mu(x),\end{aligned}$$

if $(\mathrm{G},\mathrm{G}')=(\mathrm{U}_{p,q},\mathrm{U}_{1,1})$*;*

$$\begin{aligned}&\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J'}(0) \qquad (8.46)\\ =\;&\begin{cases}\dfrac{\mathsf{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}2^{-n+1}}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\displaystyle\int_{\mathfrak{g}(\mathsf{U})}chc_{\mathrm{Hom}(\mathsf{U},\mathsf{V}_0')}(x)\psi_{\mathfrak{n}}^K(x)\,d\mu(x) \\ \qquad\qquad \textit{if } (\mathrm{G},\mathrm{G}')=(\mathrm{Sp}_{2n}(\mathbb{R}),\mathrm{O}_{1,2})\\ \dfrac{\mathsf{i}2\pi\tilde{\kappa}(J_1,J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\displaystyle\int_{\mathfrak{g}(\mathsf{U})}\psi_{\mathfrak{n}}^K(x)\,d\mu(x) \\ \qquad\qquad \textit{if } \mathrm{G}=\mathrm{O}_{2p+1,2q} \textit{ or } \mathrm{O}_{2p,2q} \textit{ and } \mathrm{G}'=\mathrm{Sp}_2(\mathbb{R})).\end{cases}\end{aligned}$$

Proof. Consider the case $(\mathrm{G},\mathrm{G}')=(\mathrm{U}_{p,q},\mathrm{U}_{1,1})$. We may assume that $\alpha=e_1-e_{p+1}$. Then, the identification (0.5) is

$$\mathsf{V}'=\mathsf{V}_s'=\mathsf{V}_1+\mathsf{V}_{p+1}.$$

For $\beta\in\mathcal{S}$ let

$$\tilde{c}(\beta)=\exp(-\mathsf{i}\frac{\pi}{4}(X_\beta+X_{-\beta}))\in\mathrm{End}(\mathsf{V}),$$

and let

$$\tilde{c}(\mathcal{S})=\prod_{\beta\in\mathcal{S}}\tilde{c}(\beta).$$

Then the Cayley transform $c(\mathcal{S})$ coincides with the conjugation by $\tilde{c}(\mathcal{S})$. Thus if $x\in\mathfrak{h}_{\mathcal{S}}$, then $c(\mathcal{S})x$ acts on $\tilde{c}(\mathcal{S})V_j$ via the multiplication by $e_j(x)$. Let

$$X=\tilde{c}(\mathcal{S})\mathsf{V}_1,\ \mathsf{Y}=\tilde{c}(\mathcal{S})\mathsf{V}_{p+1},\ \mathsf{U}=\sum_{j\notin\underline{\alpha}}\mathsf{V}_j.$$

Then $H_\alpha=-\mathsf{i}(J_1-J_{p+1})$ acts via the multiplication by 1 on X and -1 on Y. Hence,

$$\mathsf{V}_1+\mathsf{V}_{p+1}=\mathsf{X}\oplus\mathsf{Y}$$

is a complete polarization. Moreover,

$$\mathsf{V}=(\mathsf{X}\oplus\mathsf{Y})\oplus\mathsf{U},$$

and $\mathsf{U} = \tilde{c}(\mathcal{S})\mathsf{U}$. Hence, for $x \in \mathfrak{h}_{\mathcal{S}}$,

$$\begin{aligned}
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}}| &= \prod_{j\notin\underline{\alpha}} |e_1(x)-e_j(x)|^2 \cdot |e_1(x)-e_{p+1}(x)|,\\
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})}| &= \prod_{j<k;j,k\notin\underline{\alpha}} |e_j(x)-e_k(x)|^2,\\
\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)| &= \prod_{j\notin\underline{\alpha}} (|e_1(x)-e_j(x)||e_{p+1}(x)-e_j(x)|)\cdot|e_1(x)-e_{p+1}(x)|,\\
\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 &= \prod_{j<k;j,k\notin\underline{\alpha}} |e_j(x)-e_k(x)|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)|\\
&= \prod_{j\notin\underline{\alpha}} \frac{|e_1(x)-e_j(x)||e_{p+1}(x)-e_j(x)|}{|e_1(x)-e_j(x)|^2} |\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}}|\cdot|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})}|,
\end{aligned} \tag{8.47}$$

and

$$\lim_{\alpha(x)\to 0} \prod_{j\notin\underline{\alpha}} \frac{|e_1(x)-e_j(x)||e_{p+1}(x)-e_j(x)|}{|e_1(x)-e_j(x)|^2} = 1. \tag{8.48}$$

In terms of Lemma 8.3, $\mathfrak{h}_{\mathcal{S}} \cap \mathfrak{h}^{\{e_j, j\in\underline{\alpha}\}} = \mathfrak{h}_{\mathcal{S}} \cap \mathfrak{g}(\mathsf{U})_{\mathbb{C}}$. Moreover, the set of all $\mathcal{S}\setminus\alpha$, where $\mathcal{S}\in\Psi^n_{st}$, coincides with the set of all the sets of strongly orthogonal non-compact imaginary roots for $\mathfrak{g}(\mathsf{U})$. Furthermore,

$$m_{\mathcal{S}} = 2^{-|\mathcal{S}|}\frac{1}{p!q!}$$

so that

$$m_{\mathcal{S}} = \frac{1}{2q} m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}\setminus\alpha}.$$

Thus, by (8.47), the Weyl integration formula for $\mathfrak{g}(\mathsf{U})$ and Corollary A.3,

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{S}} m_{\mathcal{S}} \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\{e_j,j\in\underline{\alpha}\}}} \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)|\\
&\qquad \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(gH(\mathcal{S}))d\mu(x'')\\
&= \sum_{\alpha\in\mathcal{S}} \frac{1}{2q} m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}\setminus\alpha} \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{g}(\mathsf{U})_{\mathbb{C}}} \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)| \frac{1}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\\
&\qquad \frac{1}{\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}}} \int_{\mathrm{G}(\mathsf{U})/\mathrm{H}(\mathsf{U})(\mathcal{S}\setminus\alpha)} \psi^K_{\mathfrak{n}}(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathsf{U})(\mathcal{S}\setminus\alpha))d\mu(x'')\\
&= \frac{1}{pq\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \int_{\mathfrak{g}(\mathsf{U})} \prod_{j\notin\underline{\alpha}} \frac{|e_1(x)-e_j(x)||e_{p+1}(x)-e_j(x)|}{|e_1(x)-e_j(x)|^2} \psi^K_{\mathfrak{n}}(c(\mathcal{S})x)\, d\mu(x''),
\end{aligned} \tag{8.49}$$

where $x = y + u' \sum_{j\in\underline{\alpha}} J_j + x''$. Clearly (8.45) follows from (8.48) and (8.49).

Consider the case $(\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{1,2})$. We may assume that $\alpha = 2e_1$, so that $\mathrm{i}H_\alpha = J_1$. Let $\mathsf{U} = \sum_{j\notin\underline{\alpha}} \mathsf{V}_j$. Define $\tilde{c}(\mathcal{S}) \in \mathrm{End}(\mathsf{V}_\mathbb{C})$ as in the previous case. Then $\tilde{c}(\mathcal{S})$ preserves $\mathsf{V}_{1,\mathbb{C}}$ and $\mathsf{U}_\mathbb{C}$, and $\tilde{c}(\mathcal{S})|_{\mathsf{V}_{1,\mathbb{C}}} = \tilde{c}(\alpha)$.

Let $\mathsf{V}_{1,\mathbb{C},H_\alpha=\pm1} \subseteq \mathsf{V}_{1,\mathbb{C}}$ be the subspace on which H_α acts via the multiplication by ±1. Let

$$\mathsf{X} = (\tilde{c}(\mathcal{S})\mathsf{V}_{1,\mathbb{C},H_\alpha=1}) \cap \mathsf{V}_1, \quad \mathsf{Y} = (\tilde{c}(\mathcal{S})\mathsf{V}_{1,\mathbb{C},H_\alpha=-1}) \cap \mathsf{V}_1.$$

Then

$$\mathsf{V}_1 = \mathsf{X} \oplus \mathsf{Y}$$

is a complete polarization, and we identify with $\mathsf{V}'_s = \mathsf{X} \oplus \mathsf{Y}$. It is easy to check that for each $1 \le j \le n$

$$\tilde{c}(\mathcal{S})\mathsf{V}_{j,\mathbb{C}} = (\tilde{c}(\mathcal{S})\mathsf{V}_{j,\mathbb{C}})_{e_j\circ c(\mathcal{S})^{-1}} \oplus (\tilde{c}(\mathcal{S})\mathsf{V}_{j,\mathbb{C}})_{-e_j\circ c(\mathcal{S})^{-1}},$$

where the subscript indicates the weight by which $\mathfrak{h}(\mathcal{S})_\mathbb{C}$ acts on the indicated space. Hence, for $x \in \mathfrak{h}_\mathcal{S}$,

$$\begin{aligned}
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_\mathbb{C}}| &= \prod_{j\notin\underline{\alpha}} |e_1(x)^2 - e_j(x)^2| \cdot |2e_1(x)|,\\
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_\mathbb{C}}| &= \prod_{j<k;j,k\notin\underline{\alpha}} |e_1(x)^2 - e_j(x)^2|^2 \cdot \prod_{j\notin\underline{\alpha}} |2e_j(x)|^2,\\
\prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot |\alpha(x)| &= \prod_{j\notin\underline{\alpha}} |2e_j(x)|^2 \cdot |2e_1(x)|,\\
\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 &= \prod_{j<k;j,k\notin\underline{\alpha}} |e_j(x)^2 - e_k(x)^2|^2 \cdot \prod_{j\notin\underline{\alpha}} |2e_j(x)|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot \prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(x)|^2 \cdot |\alpha(x)|\\
&\quad = \prod_{j\notin\underline{\alpha}} \frac{|2e_j(x)|^2}{|e_1(x)^2 - e_j(x)^2|} \cdot |\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_\mathbb{C}}| \cdot |\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_\mathbb{C}}|, \qquad (8.50)
\end{aligned}$$

and

$$\lim_{\alpha(x)\to0} \prod_{j\notin\underline{\alpha}} \frac{|2e_j(x)|^2}{|e_1(x)^2 - e_j(x)^2|} = 4^{n-1}. \qquad (8.51)$$

In this case,

$$m_\mathcal{S} = \frac{1}{2^{|\mathcal{S}|}n!}$$

so that

$$m_\mathcal{S} = \frac{1}{2n} m_{\mathcal{S}\setminus\alpha}^{\mathfrak{g}(\mathsf{U})}.$$

Thus, as before,

$$\sum_{\alpha\in\mathcal{S}} m_{\mathcal{S}} \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\alpha} chc_{\mathrm{Hom}(\mathsf{V}_{\underline{\alpha}}^\perp,\mathsf{V}'_0)}(c(\mathcal{S})x) \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(yH_\alpha+x)|^2$$
$$\prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(yH_\alpha+x)|^2\cdot|\alpha(yH_\alpha+x)|$$
$$\int_{\mathrm{G/H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})(yH_\alpha+x))\,d\mu(gH(\mathcal{S}))d\mu(x)$$
$$=\sum_{\alpha\in\mathcal{S}} \frac{1}{2n} m_{\mathcal{S}\setminus\alpha}^{\mathfrak{g}(\mathsf{U})} \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{g}(\mathsf{U})_{\mathbb{C}}} chc_{\mathrm{Hom}(\mathsf{V}_{\underline{\alpha}}^\perp,\mathsf{V}'_0)}(c(\mathcal{S})x) \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(yH_\alpha+x)|^2$$
$$\prod_{\beta\in\Psi(\mathrm{long}),\underline{\beta}\cap\underline{\alpha}\neq\emptyset} |\beta(yH_\alpha+x)|^2\cdot|\alpha(yH_\alpha+x)| \frac{1}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}$$
$$\frac{1}{\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}}} \int_{\mathsf{G}(\mathsf{U})/\mathsf{H}(\mathsf{U})(\mathcal{S}\setminus\alpha)} \psi_{\mathfrak{n}}^K(g\cdot c(\mathcal{S})(yH_\alpha+x))\,d\mu(g\mathrm{H}(\mathsf{U})(\mathcal{S}\setminus\alpha))d\mu(x)=$$
$$\frac{1}{2n\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\int_{\mathfrak{g}(\mathsf{U})} chc_{\mathrm{Hom}(\mathsf{V}_{\underline{\alpha}}^\perp,\mathsf{V}'_0)}(c(\mathcal{S})x)\prod_{j\notin\underline{\alpha}} \frac{|2e_j(x)|^2}{|e_1(x)^2-e_j(x)^2|}\cdot\psi_{\mathfrak{n}}^K(c(\mathcal{S})(yH_\alpha+x))\,d\mu(x),$$

and thus (8.46) follows from (8.51) and (8.42).

Consider the case $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_{2p+1,2q},\mathrm{Sp}_2(\mathbb{R}))$. Suppose that $\alpha\in\mathcal{S}(\mathrm{short})$ We may assume that $\alpha=e_{p+1}$, so that $\mathrm{i}H_\alpha=2J_{p+1}$.
Define $\tilde{c}(\alpha)\in\mathrm{End}((\mathsf{V}_0+\mathsf{V}_{p+1})_{\mathbb{C}})$ as before. Let

$$\begin{aligned}
&\mathsf{X}=(\mathsf{V}_0+\mathsf{V}_{p+1})\cap\tilde{c}(\alpha)\mathsf{V}_{p+1,\mathbb{C},H_\alpha=2},\\
&\mathsf{Y}=(\mathsf{V}_0+\mathsf{V}_{p+1})\cap\tilde{c}(\alpha)\mathsf{V}_{p+1,\mathbb{C},H_\alpha=-2},\\
&\mathsf{V}_{(c)}=(\mathsf{X}+\mathsf{Y})^\perp\cap(\mathsf{V}_0+\mathsf{V}_{p+1}),\\
&\mathsf{U}=\mathsf{V}_{(c)}+\sum_{1\leq j,j\notin\underline{\alpha}}\mathsf{V}_j.
\end{aligned}$$

Then

$$\mathsf{V}_0+\mathsf{V}_{p+1}=\mathsf{X}\oplus\mathsf{Y}\oplus\mathsf{U},$$

the spaces X, Y are isotropic and the space $\mathsf{V}_{(c)}$ is anisotropic. Moreover,

$$\mathsf{V}=\mathsf{X}\oplus\mathsf{Y}\oplus\mathsf{U},$$

and we identify with $\mathsf{V}'_s=\mathsf{X}\oplus\mathsf{Y}$. Define $\tilde{c}(\mathcal{S})\in\mathrm{End}(\mathsf{V}_{\mathbb{C}})$ as before. Then $\tilde{c}(\mathcal{S})$ preserves $(\mathsf{V}_0+\mathsf{V}_{p+1})_{\mathbb{C}}$, $\mathsf{U}_{\mathbb{C}}$, and the restriction of $\tilde{c}(\mathcal{S})$ to $(\mathsf{V}_0+\mathsf{V}_{p+1})_{\mathbb{C}}$ is equal to $\tilde{c}(\alpha)$. Furthermore, $\mathrm{Hom}(\mathsf{X},\mathsf{Y})\cap\mathfrak{g}=0$. Hence, for $x\in\mathfrak{h}_{\mathcal{S}}$,

$$\begin{aligned}
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_{\mathbb{C}}}| &= \prod_{1\leq j, j\notin\underline{\alpha}} |e_{p+1}(x)^2-e_j(x)^2|\cdot|e_{p+1}(x)|,\\
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_{\mathbb{C}}}| &= \prod_{1\leq j<k; j,k\notin\underline{\alpha}} |e_1(x)^2-e_j(x)^2|^2\cdot\prod_{1\leq j\notin\underline{\alpha}}|e_j(x)|^2,\\
\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(x)| &= \prod_{1\leq j\notin\underline{\alpha}}|e_j(x)|^2\cdot|e_{p+1}(x)|,\\
\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2 &= \prod_{1\leq j<k; j,k\notin\underline{\alpha}}|e_j(x)^2-e_k(x)^2|^2\cdot\prod_{1\leq j\notin\underline{\alpha}}|e_j(x)|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(x)|\\
&\qquad=\prod_{1\leq j\notin\underline{\alpha}}\frac{|e_j(x)|^2}{|e_{p+1}(x)^2-e_j(x)^2|}\cdot|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_{\mathbb{C}}}|\cdot|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_{\mathbb{C}}}|,
\end{aligned}$$

and

$$\lim_{\alpha(x)\to 0}\prod_{1\leq j\notin\underline{\alpha}}\frac{|e_j(x)|^2}{|e_{p+1}(x)^2-e_j(x)^2|}=1.$$

Suppose $\mathcal{S}=\mathcal{S}(\mathrm{long})$. We may assume that $\alpha=e_1$, so that $\mathrm{i}H_\alpha=2J_1$. Also, we may assume that $e_1\pm e_{p+1}\in S$. Define $\tilde{c}(\mathcal{S})\in\mathrm{End}(V_{\mathbb{C}})$ as before. Let

$$\begin{aligned}
\mathsf{X}&=(\mathsf{V}_1+\mathsf{V}_{p+1})\cap\tilde{c}(\mathcal{S})\mathsf{V}_{p+1,\mathbb{C},H_\alpha=2},\\
\mathsf{Y}&=(\mathsf{V}_1+\mathsf{V}_{p+1})\cap\tilde{c}(\mathcal{S})\mathsf{V}_{p+1,\mathbb{C},H_\alpha=-2},\\
\mathcal{U}&=(\mathsf{V}_1+\mathsf{V}_{p+1})\cap(\mathsf{X}+\mathsf{Y})^\perp+\sum_{0\leq j\neq p+1, j\neq 1}\mathsf{V}_j.
\end{aligned}$$

The spaces X, Y are isotropic,

$$\mathsf{V}=\mathsf{X}\oplus\mathsf{Y}\oplus\mathsf{U},$$

and we identify $\mathsf{V}'=\mathsf{X}\oplus\mathsf{Y}$. Let $x\in\mathfrak{h}_{\mathcal{S}}$. Then,

$$\begin{aligned}
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_{\mathbb{C}}}| &= \prod_{1\leq j\neq p+1}|e_1(x)^2-e_j(x)^2|\cdot|e_1(x)|,\\
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_{\mathbb{C}}}| &= \prod_{2\leq j<k}|e_1(x)^2-e_j(x)^2|^2\cdot\prod_{2\leq j}|e_j(x)|^2,\\
\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(x)| &= \prod_{2\leq j}|e_j(x)|^2\cdot|e_1(x)|,\\
\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2 &= \prod_{2\leq j<k}|e_j(x)^2-e_k(x)^2|^2\cdot\prod_{2\leq j}|e_j(x)|^2.
\end{aligned}$$

Hence,

$$\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(x)|$$
$$=\prod_{2\le j}\frac{|e_j(x)|^2}{|e_1(x)^2-e_j(x)^2|}\cdot|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_{\mathbb{C}}}|\cdot|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_{\mathbb{C}}}|,$$

and

$$\lim_{\alpha(x)\to 0}\prod_{2\le j}\frac{|e_j(x)|^2}{|e_1(x)^2-e_j(x)^2|}=1.$$

Let

$$a(\mathcal{S})=|\{\beta\in\mathcal{S}(\mathrm{long})\,|\,\underline{\beta}\subseteq\underline{\mathcal{S}\setminus\beta}\}|,$$
$$b(\mathcal{S})=|\{\beta\in\mathcal{S}(\mathrm{long})\,|\,\underline{\beta}\cap\underline{\mathcal{S}\setminus\beta}=\emptyset\}|.$$

Then

$$m_{\mathcal{S}}=\frac{1}{2^{a(\mathcal{S})/2}\cdot 2^{b(\mathcal{S})}\cdot 1\cdot 3\cdot 5\cdots(a(\mathcal{S})+2|\mathcal{S}(\mathrm{short})|-1)}\cdot\frac{1}{2^p p!2^q q!}.$$

In particular, if $\nu\in\mathcal{S}$ is short, then

$$m_{\mathcal{S}}=\frac{1}{a(\mathcal{S})+1}\cdot\frac{1}{2q}\cdot m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}\setminus\nu}.$$

Let $\mathcal{S}=\mathcal{S}(\mathrm{long})$. Let $\zeta\in\Psi^c(\mathrm{short})$, $\nu\in\Psi^n(\mathrm{short})$, $\zeta\vee\nu\cup\mathcal{S}''=\mathcal{S}$. Then,

$$m_{\mathcal{S}}=\frac{1}{2q}\cdot m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}''\vee\nu}.$$

Furthermore, $a(\mathcal{S})+1=|\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})|$. Thus, with $x=yH_\alpha+x''$, we have

$$\sum_{\alpha\in\Psi_{\mathcal{S},\mathbb{R}}}m_{\mathcal{S}}\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\alpha}\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(x)|$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g\cdot c(\mathcal{S})(x))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'')$$
$$=\sum_{\alpha\in\mathcal{S}(\mathrm{short})}(a(\mathcal{S})+1)m_{\mathcal{S}}\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\alpha}\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(x)|$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g\cdot c(\mathcal{S})(x))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'')$$
$$+\sum_{\mathcal{S}=\mathcal{S}(\mathrm{long}),\alpha\in\Psi^c\cap\Psi_{\mathcal{S},\mathbb{R}}(\mathrm{short})}2m_{\mathcal{S}}\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^\alpha}\prod_{\beta\in\Psi(\mathrm{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2$$
$$\prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset}|\beta(x)|^2\cdot|\alpha(x)|\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g\cdot c(\mathcal{S})(x))\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'')$$

$$
\begin{aligned}
&= \sum_{\alpha\in\mathcal{S}(\text{short})} \frac{1}{2q} m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}\setminus\alpha} \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\alpha}} \prod_{\beta\in\Psi(\text{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \cdot \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2\cdot|\alpha(x)| \\
&\qquad \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})(x))\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'') \\
&+ \sum_{\mathcal{S}=\mathcal{S}(\text{long}),\alpha\in\Psi^c\cap\Psi_{\mathcal{S},\mathbb{R}}(\text{short})} \frac{1}{2q} m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}''\vee\nu} \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{\alpha}} \prod_{\beta\in\Psi(\text{short}),\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2 \\
&\qquad \prod_{\beta\in\Psi,\underline{\beta}\cap\underline{\alpha}=\emptyset} |\beta(x)|^2\cdot|\alpha(x)| \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})(x))\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x'') \\
&= \frac{1}{2q\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \int_{\mathfrak{g}(\mathsf{U})} \prod_{2\le j} \frac{|e_j(x)|^2}{|e_1(x)^2-e_j(x)^2|}\cdot \psi^K_{\mathfrak{n}}(c(\mathcal{S})(x))\, d\mu(x''),
\end{aligned}
$$

Now we take the limit if $y\to 0$ and deduce (8.46).

Consider the pair $(\mathrm{G},\mathrm{G}') = (\mathrm{O}_{2p,2q},\mathrm{Sp}_2(\mathbb{R}))$. In terms of (8.44) let us fix an element $k\in\underline{\alpha}$. Then

$$
\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J'}(0) = 2 \sum_{\alpha\in\mathcal{S},\underline{\alpha}\subseteq\underline{\mathcal{S}\setminus\alpha}} \mathsf{i}2\pi\tilde{\kappa}(J_1,J_1)^{1/2} m_{\mathcal{S}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}|\Psi^n|
$$

$$
\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_k^*}} \prod_{\beta\in\Psi,k\notin\underline{\beta}} |\beta(x)|^2\cdot\prod_{j\ne k}|e_j(x)|^2 \cdot \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x). \quad (8.52)
$$

Let $\underline{\alpha}=\{k,l\}$ and let

$$
\begin{aligned}
\mathsf{X} &= (\mathsf{V}_k+\mathsf{V}_l)\cap\tilde{c}(\mathcal{S})\mathsf{V}_{k,\mathbb{C},\mathsf{i}J_k=1},\\
\mathsf{Y} &= (\mathsf{V}_k+\mathsf{V}_l)\cap\tilde{c}(\mathcal{S})\mathsf{V}_{k,\mathbb{C},\mathsf{i}J_k=-1},\\
\mathsf{U} &= (\mathsf{V}_k+\mathsf{V}_l)\cap(\mathsf{X}+\mathsf{Y})^{\perp}+\mathsf{U}'',
\end{aligned}
$$

where $\mathsf{U}''=\sum_{j\notin\underline{\alpha}}\mathsf{V}_j$. Then,

$$
\mathsf{V}=\mathsf{X}\oplus\mathsf{Y}\oplus\mathsf{U},
$$

and for $x\in\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_k^*}$,

$$
\begin{aligned}
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_{\mathbb{C}}}| &= \prod_{j\ne k}|e_j(x)|^2,\\
|\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_{\mathbb{C}}}| &= \prod_{\beta\in\Psi,k\notin\underline{\beta}}|\beta(x)|^2.
\end{aligned}
$$

Moreover, we see from the formula (7.9) in [1] that if we denote by α^c the unique element of $\mathcal{S}$ such that $\alpha^c\ne\alpha$ and $\underline{\alpha}=\underline{\alpha^c}$, then

$$
\begin{aligned}
m_{\mathcal{S}} &= \frac{1}{2^{(|\mathcal{S}''|/2)}2^{|\mathcal{S}'|}\cdot 1\cdot 3\cdot 5\cdots(|\mathcal{S}''|-1)}\frac{1}{2^p p!2^q q!}\\
&= \frac{1}{(|\mathcal{S}''|-1)8pq} m^{\mathfrak{g}(\mathsf{U}'')}_{\mathcal{S}}\\
&= \frac{2(|\mathcal{S}''\setminus\{\alpha,\alpha^c\}|+1)}{(|\mathcal{S}''|-1)8pq} m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}} = \frac{1}{4pq} m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}\setminus\{\alpha,\alpha^c\}},
\end{aligned}
$$

Thus (8.52) may be rewritten as

$$\sum_{\alpha\in\mathcal{S},\underline{\alpha}\subseteq\underline{\mathcal{S}\setminus\alpha}} \mathrm{i}4\pi\tilde{\kappa}(J_1,J_1)^{1/2}\frac{1}{4pq}m^{\mathfrak{g}(\mathsf{U})}_{\mathcal{S}\setminus\{\alpha,\alpha^c\}}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}|\Psi^n|$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_k^*}} |\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{n}_{\mathbb{C}}}||\det(\mathrm{ad}(c(\mathcal{S})x))_{\mathfrak{g}(\mathsf{U})_{\mathbb{C}}}|\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})}\psi(g{\cdot}c(\mathcal{S})x)\,d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x),$$

which coincides with the right hand side of (8.46). ∎

It is clear from Theorem 7.4 in [1] that, in order to verify Theorem 0.10, we may assume that the differential operator $\partial(w)=1$. In view of Corollary 0.8 and Lemma 8.4, we shall be done as soon as we show that

$$\frac{2\pi^2\tilde{\kappa}(J_1,J_1)\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}}{\mu(H')\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}=1, \quad \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{U}_{p,q},\mathrm{U}_{1,1}) \tag{8.53}$$

$$\frac{4\pi\tilde{\kappa}(J_1,J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}2^{-n+1}}{\mu(H')\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}=2, \quad \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{Sp}_{2n}(\mathbb{R}),\mathrm{O}_{1,2}) \tag{8.54}$$

$$\frac{2\pi\tilde{\kappa}(J_1,J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}}{\mu(H')\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}=1, \quad \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{O}_{2p+1,2q},\mathrm{Sp}_2(\mathbb{R})) \tag{8.55}$$

$$\frac{2\pi\tilde{\kappa}(J_1,J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}}}{\mu(H')\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}=1, \quad \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{O}_{2p,2q},\mathrm{Sp}_2(\mathbb{R})) \tag{8.56}$$

The left hand side of (8.53) is equal to

$$\frac{2\pi^2\cdot 8\cdot\sqrt{2}^{2\cdot 2\cdot n}}{8n\pi^2\cdot\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{2^{2n+1}\cdot n^{-1}}=1.$$

The left hand side of (8.54) is equal to

$$\frac{4\pi\cdot\sqrt{6}\cdot\sqrt{2}^{2n\cdot 3}2^{-n+1}}{4\sqrt{n}\pi\cdot\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{2^{2n+1}\cdot\sqrt{\frac{3}{2n}}}=2.$$

The left hand side of (8.55) is equal to

$$\frac{2\pi\cdot 2\cdot\sqrt{2}^{2(2n+1)}}{2\sqrt{4n+2}\,\pi\cdot\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\frac{2^{2n+1+\frac{1}{2}}}{\sqrt{2n+1}}}=1.$$

The left hand side of (8.56) is equal to

$$\frac{2\pi\cdot 2\cdot\sqrt{2}^{2\cdot 2n}}{4\pi\sqrt{n}\cdot\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}\cdot\frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\frac{\sqrt{2}^{(2\cdot 2n+1)}}{\sqrt{2n}}}=1.$$

Consider the pair $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_{2p+1,2q+1},\mathrm{Sp}_2(\mathbb{R}))$. In this case the defining module V for G has the following orthogonal direct sum decomposition

$$\mathsf{V}=\mathsf{V}_1\oplus\mathsf{V}'', \tag{8.57}$$

where $G(\mathsf{V}_1)$ is isomorphic to $O_{1,1}$ and $G(\mathsf{V}'')$ is isomorphic to $O_{2p,2q}$. Let $\mathfrak{h}''$ be an elliptic Cartan subalgebra of $\mathfrak{g}(\mathsf{V}'')$. Let $\mathfrak{h} = \mathfrak{g}(\mathsf{V}_1) \oplus \mathfrak{h}''$. This is a fundamental Cartan subalgebra of $\mathfrak{g}$.

Every Cartan subalgebra of $\mathfrak{g}$ is conjugate to one which preserves the decomposition (8.57), and two such are G−conjugate if and only if they are $G(\mathsf{V}'')$−conjugate.

Let $\Psi(\mathsf{V}'')$ be a system of positive roots of $\mathfrak{h}''$ in $\mathfrak{g}(\mathsf{V}'')_{\mathbb{C}}$, $\Psi^n(\mathsf{V}'') \subseteq \Psi(\mathsf{V}'')$ be the subset of the non-compact roots and $\Psi^n_{st}(\mathsf{V}'')$ be the set of the strongly orthogonal subsets of $\Psi^n(\mathsf{V}'')$. Let Ψ be a system of positive roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ such that, if we extends each root in $\Psi(\mathsf{V}'')$ by zero to $\mathfrak{g}(\mathsf{V}_1)$, then $\Psi(\mathsf{V}'') \subseteq \Psi$. In order to simplify the notation we shall write Ψ^n for $\Psi^n(\mathsf{V}'')$ and Ψ^n_{st} for $\Psi^n_{st}(\mathsf{V}'')$.

For $\mathcal{S} \in \Psi^n_{st}$ there is an element $\tilde{c}(\mathcal{S}) \in \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ such that

$$\begin{aligned}
&\tilde{c}(\mathcal{S})\mathsf{V}_{1,\mathbb{C}} = \mathsf{V}_{1,\mathbb{C}},\ \tilde{c}(\mathcal{S})\mathsf{V}''_{\mathbb{C}} = \mathsf{V}''_{\mathbb{C}},\\
&\mathfrak{g}(\mathsf{V}_1) = \mathbb{R}\tilde{c}(\mathcal{S})\mathrm{i}J_1\tilde{c}(\mathcal{S})^{-1},\\
&Ad(\tilde{c}(\mathcal{S})|_{\mathfrak{g}(\mathsf{V}'')_{\mathbb{C}}}) \text{ is the Cayley transform for } \mathcal{S} \text{ and } \mathfrak{g}(\mathsf{V}''),
\end{aligned}$$

Let $c(\mathcal{S}) = \mathrm{Ad}(\tilde{c}(\mathcal{S})) \in \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ and let $\mathfrak{h}(\mathcal{S}) = \mathfrak{g}(\mathsf{V}_1) \oplus \mathfrak{h}''(\mathcal{S})$. Define

$$\mathfrak{h}_{\mathcal{S}} = c(\mathcal{S})^{-1}(\mathfrak{h}(\mathcal{S})_{\mathbb{C}}) \cap \mathfrak{g}.$$

Then

$$\mathfrak{h}_{\mathcal{S}} = \mathbb{R}\mathrm{i}J_1 \oplus \mathfrak{h}''_{\mathcal{S}}$$

and, as we have seen in [1, (11.2)],

$$m_{\mathcal{S}} = \frac{1}{2(|\mathcal{S}''|+1)} m_{\mathcal{S}}^{\mathfrak{g}(\mathsf{V}'')} \tag{8.58}$$

where $\mathcal{S}'' = \{\alpha \in \mathcal{S} \,|\, \underline{\alpha} \subseteq \overline{\mathcal{S} \setminus \alpha}\}$ and $\mathcal{S}' = \mathcal{S} \setminus \mathcal{S}''$.

For $\mathcal{S} \in \Psi^n_{st}$ and $s \in W(\mathrm{H}_{\mathbb{C}})$ let

$$\begin{aligned}
\mathfrak{h}'_{\mathcal{S},s} &= \sum_{k \notin \underline{\mathcal{S}}, k \geq 2, \sigma^{-1}(k)=1} \mathbb{R}J_k + \sum_{k=1,\sigma(k)=1} \mathbb{R}\mathrm{i}J_1 + \mathfrak{h}_{\mathcal{S}} \cap \left(\sum_{k \in \underline{\mathcal{S} \setminus (s\mathfrak{h}')^{\perp}}, k \geq 2} \mathbb{C}J_k \right),\\
\mathfrak{h}''_{\mathcal{S},s} &= \sum_{k \notin \underline{\mathcal{S}}, \sigma^{-1}(k) \neq 1} \mathbb{R}J_k + \sum_{k=1,\sigma(k)\neq 1} \mathbb{R}\mathrm{i}J_1 + \mathfrak{h}_{\mathcal{S}} \cap \left(\sum_{k \in \underline{\mathcal{S} \cap (s\mathfrak{h}')^{\perp}}, k \geq 2} \mathbb{C}J_k \right).\\
\Gamma'_{s,\mathcal{S}} &= \sum_{\sigma(1) \notin \underline{\mathcal{S}}} (0,\infty) J_1^*(y_s) J_1,
\end{aligned}$$

where y_s is defined as in [1, Definition 3.4]. Then, the formula (8.1) holds. Furthermore, the following statements are true, with the proofs almost identical to the proofs of the corresponding statements in the $(O_{2p,2q}, \mathrm{Sp}_2(\mathbb{R}))$ case.

Lemma 8.5. *Suppose there is $\alpha \in \mathcal{S} \setminus (s\mathfrak{h}')^{\perp}$ with $\underline{\alpha} \cap \overline{\mathcal{S} \setminus \alpha} = \emptyset$ and $\sigma(1) \notin \underline{\mathcal{S}} \cup \{1\}$. Then $\mathcal{H}'_{\mathcal{S},s}\psi$ extends to a continuous function on $\mathfrak{h}'$.*

Lemma 8.6. *Suppose* $\alpha \in \mathcal{S}$, $\sigma(1) \in \underline{\alpha} \subseteq \underline{\mathcal{S} \setminus \alpha}$ *or* $\sigma(1) = 1$. *Let* $w \in W(\mathrm{H}_{\mathcal{S}})$ *be the reflection with respect to* $J_{\sigma(1)}$. *Then*

$$\langle \mathcal{H}'_{\mathcal{S},s}\psi + \mathcal{H}'_{\mathcal{S},ws}\psi\rangle_{J'}(0) = \mathrm{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}|m_{\mathcal{S}}(s)| \int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_{\sigma(1)}}} \prod_{\beta\in\Psi,\sigma(1)\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq\sigma(1)} |e_j(x)|^2$$
$$\int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))d\mu(x).$$

Hence, $\langle \pi_{\mathfrak{g}'/\mathfrak{h}'}\widetilde{chc}(\psi)\rangle_{J'}(0) = \langle \sum_{\mathcal{S},s} \mathcal{H}_{\mathcal{S},s}\psi\rangle_{J'}(0)$

$$= \sum_{\mathcal{S},s;\ \alpha\in\mathcal{S}'',\sigma(1)\in\underline{\alpha},\hat{\epsilon}_{\sigma(1)}=1} \langle \mathcal{H}_{\mathcal{S},s}\psi + \mathcal{H}_{\mathcal{S},ws}\psi\rangle_{J'}(0) + \sum_{\mathcal{S},s;\ \sigma(1)=1,\hat{\epsilon}_{\sigma(1)}=1} \langle \mathcal{H}_{\mathcal{S},s}\psi + \mathcal{H}_{\mathcal{S},ws}\psi\rangle_{J'}(0)$$

$$= \sum_{\mathcal{S},s;\ \alpha\in\mathcal{S}'',\sigma(1)\in\underline{\alpha},\hat{\epsilon}_{\sigma(1)}=1} \mathrm{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}|m_{\mathcal{S}}(s)|$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_{\sigma(1)}}} \prod_{\beta\in\Psi,\sigma(1)\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq\sigma(1)} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$+ \sum_{\mathcal{S},s;\ \sigma(1)=1,\hat{\epsilon}_{\sigma(1)}=1} \mathrm{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}|m_{\mathcal{S}}(s)|$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_1}} \prod_{\beta\in\Psi,1\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq1} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$= \sum_{\mathcal{S};\ \alpha\in\mathcal{S}'',k\in\underline{\alpha}} \mathrm{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}} m_{\mathcal{S}}$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_k}} \prod_{\beta\in\Psi,k\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq k} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$+ \sum_{\mathcal{S}} \mathrm{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}} m_{\mathcal{S}}$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_1}} \prod_{\beta\in\Psi,1\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq1} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$= \sum_{\mathcal{S};\ \mathcal{S}''\neq\emptyset} \mathrm{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}} |\mathcal{S}''| m_{\mathcal{S}}$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_1}} \prod_{\beta\in\Psi,1\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq1} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$+ \sum_{\mathcal{S}} \mathrm{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}}\mathsf{W}} m_{\mathcal{S}}$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J^*_1}} \prod_{\beta\in\Psi,1\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq1} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$= \sum_{\mathcal{S}} \mathsf{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}(|\mathcal{S}''|+1)m_{\mathcal{S}}$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_1^*}} \prod_{\beta\in\Psi,1\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq 1} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$= \sum_{\mathcal{S}} \mathsf{i}4\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}(|\mathcal{S}''|+1)\frac{1}{2(|\mathcal{S}''|+1)} m_{\mathcal{S}}^{\mathfrak{g}(\mathsf{V}'')}$$

$$\int_{\mathfrak{h}_{\mathcal{S}}\cap\mathfrak{h}^{J_1^*}} \prod_{\beta\in\Psi,1\notin\underline{\beta}} |\beta(x)|^2 \cdot \prod_{j\neq 1} |e_j(x)|^2 \int_{\mathrm{G}/\mathrm{H}(\mathcal{S})} \psi(g\cdot c(\mathcal{S})x)\, d\mu(g\mathrm{H}(\mathcal{S}))\, d\mu(x)$$

$$= \frac{\mathsf{i}2\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \int_{\mathfrak{g}(\mathsf{U})} \psi_{\mathfrak{n}}^{K}(x)\, d\mu(x),$$

where the sixth equation follows from (8.58). Furthermore

$$\frac{2\pi\tilde{\kappa}(J_1, J_1)^{1/2}\sqrt{2}^{\dim_{\mathbb{R}} \mathsf{W}}}{\mu(\mathrm{H}')\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \cdot \frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\gamma(\mathsf{V},\mathsf{V}',\mathsf{X}')}$$

$$= \frac{2\pi\cdot 2\cdot\sqrt{2}^{2\cdot 2(n+1)}}{4\pi\sqrt{n+1}\cdot\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')} \cdot \frac{\eta(\mathsf{V},\mathsf{V}',\mathsf{X}')}{\frac{\sqrt{2}^{(2\cdot 2(n+1)+1)}}{\sqrt{2(n+1)}}} = 1.$$

This completes the proof of Theorem 0.10.

Appendix A.

Let G be a reductive Lie group ($\mathrm{Lie}(\mathrm{G}) = \mathfrak{g}$) and θ a Cartan involution on $\mathfrak{g}$. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ associated to θ and an invariant non degenerate bilinear form κ such that the form

$$\begin{array}{ccc} \mathfrak{g}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & -\kappa(\theta(x), y) \end{array}$$

is positive definite. We denote this form $\tilde{\kappa}$. Let V be a subspace of $\mathfrak{g}$. As the restriction of $\tilde{\kappa}$ to V is non-degenerate this induces a volume form and a measure on V. We denote the volume form λ_V and the measure μ_V. Let $\mathfrak{s}$ an abelian subalgebra of $\mathfrak{g}$ included in $\mathfrak{p}$ and $\mathfrak{m}$ the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. Denote by Q the parabolic Lie subgroup of G such that $\mathrm{Lie}(\mathrm{Q}) = \mathfrak{m}\oplus\mathfrak{n}$ and let K the maximal compact subgroup of G such that $\mathrm{Lie}(\mathrm{K}) = \mathfrak{k}$. Let H be a subgroup of G. Consider the volume form λ_{H} on H defined by $\lambda_{\mathrm{H},g}(x) = \lambda_{\mathrm{Lie}(\mathrm{H})}(g^{-1}\cdot x)$ for $x \in T_g(\mathrm{H})$, the tangent space of H for $g\in\mathrm{G}$. We denote the Haar measure induced by λ_{H}, by μ_{H}.

Proposition A.1. *We have the following equality*

$$\mu(\mathrm{K}\cap\mathrm{Q})\sqrt{2}^{\dim(\mathfrak{n})}\int_{\mathrm{G}}\phi(g)d\mu_{\mathrm{G}}(g) = \int_{\mathrm{Q}}\int_{\mathrm{K}}\phi(kq)d\mu_{\mathrm{K}}(k)d\mu_{\mathrm{Q}}(q)$$

for all $\phi\in L^1(\mathrm{G})$.

Proof. Consider the map

$$\begin{array}{rcl} \psi : \mathrm{K} \times \mathrm{Q} & \longrightarrow & \mathrm{G} \\ (k,q) & \longmapsto & kq. \end{array}$$

This map is a surjective submersion. For $(k,q) \in \mathrm{K} \times \mathrm{Q}$, let $\psi_{k,q}$ be the derivative of ψ in the point (k,q). Let $g = kq$ with $k \in \mathrm{K}$ and $q \in \mathrm{Q}$ Then the fiber $\psi^{-1}(g) = \{(ku, u^{-1}q) \,|\, u \in \mathrm{K} \cap \mathrm{Q}\}$. In particular, we see that this set is a compact smooth manifold. We may define a natural measure on this set. The tangent space of $\psi^{-1}(\mathfrak{g})$ is:

$$T_{(ku,u^{-1}q)}\psi^{-1}(g) = \{(ku \cdot v, -v \cdot u^{-1}q) \,|\, v \in \mathfrak{k} \cap \mathfrak{q}\}.$$

Define a volume form on $\psi^{-1}(g)$:

$$\lambda_{\psi^{-1}(g),(ku,u^{-1}q)}(x_1, \cdots, x_n) = \frac{\lambda_{\mathrm{K},ku} \times \lambda_{\mathrm{Q},u^{-1}q}(x_1, \cdots, x_n, y_1, \cdots, y_q)}{\lambda_{\mathrm{G},g}(\psi_{ku,u^{-1}p}(y_1), \cdots, \psi_{ku,u^{-1}p}(y_q))}, \qquad \text{(A.1)}$$

where $(x_1, \ldots, x_n)$ (resp., $(x_1, \ldots, x_n, y_1, \ldots, y_q)$) denote a basis of $T_{(ku,u^{-1}q)}\psi^{-1}(g)$ (resp. $T_{(ku,u^{-1}q)}(\mathrm{K} \times \mathrm{Q})$). It's immediate that $\lambda_{\psi^{-1}(g),(ku,u^{-1}q)}(x_1, \cdots, x_n)$ is well defined and $\lambda_{\psi^{-1}(g)}$ is a volume form on $\psi^{-1}(\mathfrak{g})$. We denote the measure attached to this form by $d_{\psi^{-1}(g)}$. We have the integration formula

$$\int_{\mathrm{Q}} \int_{\mathrm{K}} \psi(k,q) d\mu_{\mathrm{K}}(k) d\mu_{\mathrm{Q}}(q) = \int_{\mathrm{G}} \int_{\psi^{-1}(g)} \psi(k,q) d\mu_{\psi^{-1}(g)} d\mu_{\mathrm{G}}(g)$$

for $\phi \in L^1(\mathrm{K} \times \mathrm{Q})$. We want now to simplify the expression $(A.1)$. Let $p_{\mathfrak{p}}$ (resp $p_{\mathfrak{k}}$) the orthogonal projection on $\mathfrak{p}$ (resp. $\mathfrak{k}$). We observe that we have the following equalities :

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n} \oplus \theta(\mathfrak{n}) = \mathfrak{m} \overset{\perp}{\oplus} p_{\mathfrak{k}}(\mathfrak{n}) \overset{\perp}{\oplus} p_{\mathfrak{p}}(\mathfrak{n})$$

Let $\mathfrak{k}' = \{x + \theta(x) \,|\, x \in \mathfrak{n}\}$. Then $\mathfrak{k}'$ is a subspace of $\mathfrak{k}$ and

$$\mathfrak{k} = \mathfrak{k}' \overset{\perp}{\oplus} (\mathfrak{m} \oplus \mathfrak{n}) \cap \mathfrak{k}.$$

Since the following is an isomorphism

$$\begin{array}{rcl} \mathfrak{m} \oplus \mathfrak{n} \oplus \theta(\mathfrak{n}) & \xrightarrow{\simeq} & \mathfrak{m} \oplus \mathfrak{k}' \oplus \mathfrak{n}, \\ (x_m, x_n, x_n') & \longmapsto & (x_m, x_{n'} + \theta(x_{n'}), x_n - \theta(x_{n'})), \end{array}$$

we see that

$$\begin{array}{rcl} \lambda_{\mathfrak{g}} & = & \lambda_{\mathfrak{m}} \otimes \lambda_{p_{\mathfrak{k}}(\mathfrak{n})} \otimes \lambda_{p_{\mathfrak{p}}(\mathfrak{n})}, \\ \lambda_{\mathfrak{k}} \otimes \lambda_{\mathfrak{q}} & = & \lambda_{\mathfrak{k} \cap \mathfrak{m}} \otimes \lambda_{\mathfrak{k}'} \otimes \lambda_{\mathfrak{m}} \otimes \lambda_{\mathfrak{n}}. \end{array}$$

Let $w = (w_1, \cdots, w_l)$, $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_n)$ and $z = (z_1, \cdots, z_p)$ be the basis of $\mathfrak{k} \cap \mathfrak{m}$, $\mathfrak{k}'$, $\mathfrak{m}$ and q respectively. Then

$$\begin{array}{rcl} \lambda_{\mathfrak{k}} \otimes \lambda_{\mathfrak{q}}(w + x, y + z) & = & \lambda_{\mathfrak{k} \cap \mathfrak{m}}(w) \times \lambda_{\mathfrak{k}'}(x) \times \lambda_{\mathfrak{m}}(y) \times \lambda_{\mathfrak{n}}(z), \\ \lambda_{\mathfrak{g}}(x + y + z) & = & \lambda_{\mathfrak{m}}(y) \times \lambda_{\mathfrak{n} \oplus \theta(\mathfrak{n})}(x + z) \end{array}$$

and

$$\lambda_{\mathfrak{n}\oplus\theta(\mathfrak{n})}(x+z) = \lambda_{\mathfrak{k}'}(x) \times \lambda_{p_{\mathfrak{q}}(\mathfrak{n})}(p_{\mathfrak{q}}(z)) = \sqrt{2}^{-\dim(\mathfrak{n})} \lambda_{\mathfrak{k}'}(x) \times \lambda_{\mathfrak{n}}(z).$$

We deduce

$$\frac{\lambda_{\mathfrak{k}} \otimes \lambda_{\mathfrak{q}}(w+x, y+z)}{\lambda_{\mathfrak{g}}(x+y+z)} = \sqrt{2}^{\dim(\mathfrak{n})} \lambda_{\mathfrak{k}\cap\mathfrak{m}}(w).$$

The equality (A.1) is equivalent to

$$\lambda_{\psi^{-1}(g),(ku,u^{-1}q)} = \sqrt{2}^{\dim(\mathfrak{n})} \lambda_{\mathrm{K}\cap\mathrm{M},u}.$$

Let $\phi \in L^1(\mathrm{G})$. We consider the function η defined on $\mathrm{K}\times\mathrm{Q}$ by $\eta(k,q) = \phi(kq)$. This function belongs to $L^1(\mathrm{K}\times\mathrm{Q})$ and is constant on all fibers of ψ thus we get

$$\int_{\mathrm{Q}} \int_{\mathrm{K}} \phi(kq) d\mu_{\mathrm{K}}(k) d\mu_{\mathrm{Q}}(q) = \mu(\mathrm{K}\cap\mathrm{Q}) \sqrt{2}^{\dim(\mathfrak{n})} \int_{\mathrm{G}} \phi(g) d\mu_{\mathrm{G}}(g). \qquad \blacksquare$$

Let M the Levi factor of Q such that $\mathrm{Lie}(\mathrm{M}) = \mathfrak{m}$. We denote by $Car(\mathrm{G})$ (resp. $Car(\mathrm{M})$) the set of Cartan subgroups of G (resp. M). We have the inclusion $Car(\mathrm{M}) \subset Car(\mathrm{G})$.

Corollary A.2. *Let* $\mathrm{H} \in Car(\mathrm{M})$ *and* $\phi \in L^1(\mathrm{G}/\mathrm{H})$. *Then*

$$\mu(\mathrm{K}\cap\mathrm{Q}) \sqrt{2}^{\dim(\mathfrak{n})} \int_{\mathrm{G}/\mathrm{H}} \phi(g\mathrm{H}) d\mu_{\mathrm{G}/\mathrm{H}}(g\mathrm{H}) = \int_{\mathrm{K}\times\mathrm{Q}/\mathrm{H}} \phi(kq\mathrm{H}) d\mu_{\mathrm{K}} d\mu_{\mathrm{Q}/\mathrm{H}}(m\mathrm{H}).$$

Let $\psi \in C_c(\mathfrak{g})$. We denote by N the unipotent radical of Q. Consider the function

$$\psi^{\mathrm{K}}_{\mathfrak{n}}(y) = \int_{\mathrm{K}\times\mathfrak{n}} \phi(k.(y+n)) d\mu(k) d\mu(n) \qquad (y \in \mathfrak{m}).$$

Recall the Weyl denominators:

$$\begin{aligned} D_{\mathrm{G}}(x) &= |\det(\mathrm{ad}(x))_{\mathfrak{g}/\mathfrak{g}^x}|^{1/2} \text{ for } x \in \mathfrak{g}^{\mathrm{reg}}, \\ D_{\mathrm{M}}(x) &= |\det(\mathrm{ad}(x))_{\mathfrak{m}/\mathfrak{m}^x}|^{1/2} \text{ for } x \in \mathfrak{m}^{\mathrm{reg}}. \end{aligned} \tag{A.2}$$

We have $\mathrm{K}\cap\mathrm{Q} = \mathrm{K}\cap\mathrm{M}$.

Corollary A.3. *Let* $\mathrm{H} \in Car(\mathrm{M})$, $x \in \mathfrak{h}^{\mathrm{reg}}$ *and* $\psi \in C_c(\mathfrak{g})$. *Then*

$$\mu(\mathrm{K}\cap\mathrm{M}) \sqrt{2}^{\dim(\mathfrak{n})} \int_{\mathrm{G}/\mathrm{H}} \psi(g.x) d\mu_{\mathrm{G}/\mathrm{H}}(g\mathrm{H}) = \frac{D_{\mathrm{M}}(x)}{D_{\mathrm{G}}(x)} \int_{\mathrm{M}/\mathrm{H}} \psi^{\mathrm{K}}_{\mathfrak{n}}(m.x) d\mu_{\mathrm{M}/\mathrm{H}}(m\mathrm{H}).$$

Let $\psi \in C_c(\mathrm{G})$. We note N the nilpotent radical of Q. We consider the function

$$\psi^{\mathrm{K}}_{\mathrm{N}}(y) = \int_{\mathrm{K}\times\mathrm{N}} \phi(k.(yn)) d\mu(k) d\mu(n) \qquad (y \in \mathrm{M})$$

Consider the Weyl denominators:

$$\begin{aligned} D_{\mathrm{G}}(x) &= |\det\left(1 - \mathrm{Ad}(x^{-1})\right)_{\mathfrak{g}/\mathfrak{g}^x}|^{1/2} \text{ for } x \in \mathrm{G}^{\mathrm{reg}}, \\ D_{\mathrm{M}}(x) &= |\det\left(1 - \mathrm{Ad}(x^{-1})\right)_{\mathfrak{m}/\mathfrak{m}^x}|^{1/2} \text{ for } x \in \mathrm{M}^{\mathrm{reg}}. \end{aligned}$$

Corollary A.4. *Let* $\mathrm{H} \in Car(\mathrm{M})$, $x \in \mathrm{H}^{\mathrm{reg}}$ *and* $\psi \in C_c(\mathrm{G})$. *Then*

$$\mu(\mathrm{K} \cap \mathrm{M})\sqrt{2}^{\dim(\mathfrak{n})} \int_{\mathrm{G/H}} \psi(g.x) d\mu_{\mathrm{G/H}}(g\mathrm{H}) = \frac{D_{\mathrm{M}}(x)}{D_{\mathrm{G}}(x)} \int_{\mathrm{M/H}} \psi_{\mathrm{N}}^{\mathrm{K}}(m.x) d\mu_{\mathrm{M/H}}(m\mathrm{H}).$$

Induced distributions.

Let $\mathrm{L} = \mathrm{MA}$ a Levi factor of G. For any function $\Psi \in \mathcal{D}(\mathrm{G})$ define

$$\begin{aligned}
\Psi^{\mathrm{K}}(g) &= \int_{\mathrm{K}} \Psi(k.g)\, d\mu(k), \qquad\qquad (\mathrm{A}.3)\\
\Psi_{\mathrm{N}}^{\mathrm{K}}(g) &= \int_{\mathrm{N}} \Psi^{\mathrm{K}}(gn)\, d\mu(n) = \int_{\mathfrak{n}} \Psi^{\mathrm{K}}(g \exp(z))\, d\mu(z),\\
\Psi^{\mathrm{L}}(ma) &= |\det(\mathrm{Ad}(ma)_{\mathfrak{n}})|^{1/2} \Psi_{\mathrm{N}}^{\mathrm{K}}(ma) \qquad (g \in \mathrm{G},\ m \in \mathrm{M},\ a \in \mathrm{A}).
\end{aligned}$$

Clearly

$$\mathcal{D}(\mathrm{G}) \ni \Psi \to \Psi^{\mathrm{L}}|_{\mathrm{L}} \in \mathcal{D}(\mathrm{L})$$

is a well defined continuous linear map. For a distribution $u \in \mathcal{D}'(\mathrm{MA})$ define a distribution $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u) \in \mathcal{D}'(\mathrm{G})$ by

$$\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u)(\Psi) = u(\Psi^{\mathrm{L}}) \qquad (\Psi \in \mathcal{D}(\mathrm{G})).$$

Proposition A.5. *If* $u \in \mathcal{D}'(\mathrm{L})^{\mathrm{L}}$, *then* $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u) \in \mathcal{D}'(\mathrm{G})^{\mathrm{G}}$ *does not depend on* θ. *Furthermore,* $\mathrm{supp}\left(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u)\right) = \mathrm{K}.(\mathrm{supp}(u)\,\mathrm{N})$, *and the set of semisimple elements in the support of* $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u)$ *is equal to the union of all the* G*-orbits passing through the semisimple points of* $\mathrm{supp}(u)$.

Proof. The invariance is well known, see [11, part II, Proposition 31]. Since all Cartan involutions are conjugated to each other, the independence of θ is clear. The last statement is obvious. ∎

Let

$$\mathrm{L}^{\mathrm{reg}} = \{l \in \mathrm{L} \mid \det(\mathrm{Ad}(l) - 1)_{\mathfrak{n}} \neq 0\}$$

denote the set of regular elements in L.

Restriction of a distribution.

Let $\mathcal{V}$ be a completely invariant open set of $\mathfrak{m}$ and $\mathcal{U} = \mathrm{G}.\mathcal{V}$. For $Y \in \mathcal{V}$, we consider

$$\nu_{\mathfrak{g}/\mathfrak{m}}(Y) = |\det(\mathrm{ad}\, Y)_{\mathfrak{g}/\mathfrak{m}}|^{1/2}.$$

Let

$$\begin{aligned}
\pi_* : \mathcal{D}(\mathrm{G} \times \mathcal{V}) &\longrightarrow \mathcal{D}(\mathcal{U}),\\
\pi_*(\phi)(g.X) &= \nu_{\mathfrak{g}/\mathfrak{m}}(X)^{-1} \int_{\mathrm{M}} \phi(gm^{-1}, m.X) d\mu(m);\\
p_* : \mathcal{D}(\mathrm{G} \times \mathcal{V}) &\longrightarrow \mathcal{D}(\mathcal{V}),\\
p_*(\phi)(X) &= \int_{\mathrm{G}} \phi(g, X) d\mu(g).
\end{aligned}$$

According to Harish-Chandra's descent, we have a bijective map $\mathrm{Res} : \Theta \mapsto \theta$ from $\mathcal{D}'(\mathcal{U})^{\mathrm{G}}$ onto $\mathcal{D}'(\mathcal{V})^{\mathrm{M}}$ such that $\Theta \circ \pi_* = \theta \circ p_*$.

Proposition A.6. *Let $u \in \mathcal{D}'(\mathfrak{g})^{\mathrm{G}}$ and $\phi \in \mathcal{D}(\mathcal{U})$ then we have*

$$u(\phi) = \frac{1}{\sqrt{2}^{\dim(\mathfrak{n})}\mu(\mathrm{K}\cap\mathrm{M})}\,\mathrm{Res}(u)(\phi_{\mathfrak{n}}^{\mathrm{K}}).$$

Proof. Let $x \in \mathfrak{g}^{\mathrm{reg}}$ and H the Cartan subgroup of G such that $\mathrm{Lie}(\mathrm{H}) = \mathfrak{h}$ and $x \in \mathfrak{h}$.

$$J_{\mathfrak{g},x}(\phi) = |\det\left(\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}\right)|^{1/2}\int_{\mathrm{G}/\mathrm{H}}\phi(g.x)d\mu(g\mathrm{H})$$

we consider also $J_{\mathfrak{m},x}(\phi)$. Recall that we have the equality

$$\mathcal{V}^{\mathrm{reg}} = \mathcal{V}\cap\mathfrak{g}^{\mathrm{reg}}.$$

According to the lemma 5.1.3 of [3], we have for $x \in \mathcal{V}^{\mathrm{reg}}$ $\mathrm{Res}(J_{\mathfrak{g},x}) = J_{\mathfrak{m},x}$. According to the Corollary A.3, we have

$$J_{\mathfrak{g},x}(\phi) = \frac{1}{\sqrt{2}^{\dim(\mathfrak{n})}\mu(\mathrm{K}\cap\mathrm{M})}J_{\mathfrak{m},x}(\phi_{\mathfrak{n}}^{\mathrm{K}})$$

for $\phi \in \mathcal{D}(\mathfrak{g})$ and $x \in \mathcal{V}^{\mathrm{reg}}$. Thus for $x \in \mathcal{V}^{\mathrm{reg}}$ and $\phi \in \mathcal{D}(\mathcal{U})$, we have

$$J_{\mathfrak{g},x}(\phi) = \frac{1}{\sqrt{2}^{\dim(\mathfrak{n})}\mu(\mathrm{K}\cap\mathrm{M})}\,\mathrm{Res}(J_{\mathfrak{g},x})(\phi_{\mathfrak{n}}^{\mathrm{K}})$$

As the space $\mathrm{Vect}\{J_{\mathfrak{g},x}\,|\,x \in \mathcal{V}^{\mathrm{reg}}\}$ is weakly dense in $\mathcal{D}'(\mathcal{U})^{\mathrm{G}}$ according to the Corollary 4.1.3 of [3] and the map $\phi \mapsto \phi_{\mathfrak{n}}^{\mathrm{K}}$ is continuous, we deduce that result. ∎

We consider now a completely invariant open set $\mathcal{V}$ of M and let $\mathcal{U} = \mathrm{G}.\mathcal{V}$. For $\phi \in \mathcal{D}(\mathcal{U})$, we consider the function $\phi_{\mathrm{N}}^{\mathrm{K}} \in \mathcal{D}(\mathcal{V})$ such that

$$\phi_{\mathrm{N}}^{\mathrm{K}}(m) = \int_{\mathrm{N}}\int_{\mathrm{K}}\phi(k.(mn))d\mu(k)d\mu(n).$$

For a Cartan subgroup $\mathrm{H} \subseteq \mathrm{G}$ and $x \in \mathrm{H}^{\mathrm{reg}}$, the orbital integral of ϕ at $x \in \mathrm{H}^{\mathrm{reg}}$ is

$$J_{\mathrm{G},x}(\phi) = D_{\mathrm{M}}(x)\int_{\mathrm{G}/\mathrm{H}}\phi(g.x)\,d\mu(g\mathrm{H}).$$

We consider also $J_{\mathrm{M},x}$ for $x \in \mathrm{H}^{\mathrm{reg}}$ and H a Cartan subgroup of M. According to the Harish-Chandra's descent, there is a map

$$\mathrm{Res} : \mathcal{D}'(\mathcal{U})^{\mathrm{G}} \xrightarrow{\simeq} \mathcal{D}'(\mathcal{V})^{\mathrm{M}}.$$

Moreover, we know that

$$\mathrm{Res}(J_{\mathrm{G},x}) = J_{\mathrm{M},x},$$

for any $x \in \mathcal{V}^{\mathrm{reg}}$. Notice that $\mathcal{V}^{\mathrm{reg}} = \mathcal{V}\cap\mathrm{G}^{\mathrm{reg}}$. We deduce from the lemma A.4, that for $\phi \in \mathcal{D}(\mathcal{U})$ and $x \in \mathcal{V}^{\mathrm{reg}}$:

$$J_{\mathrm{G},x}(\phi) = \frac{1}{\sqrt{2}^{\dim(\mathfrak{n})}\mu(\mathrm{K}\cap\mathrm{M})}\frac{D_{\mathrm{M}}(x)}{D_{\mathrm{G}}(x)}J_{\mathrm{M},x}(\phi_{\mathrm{N}}^{\mathrm{K}}).$$

As the space $\mathrm{Vect}\{J_{x,\mathrm{G}}\}$ is weakly dense in $\mathcal{D}'(\mathcal{U})^{\mathrm{G}}$ (Corollary 3.3.2 of [4]), we deduce the following proposition :

Proposition A.7. *Let $u \in \mathcal{D}'(\mathrm{G})^{\mathrm{G}}$ and $\phi \in \mathcal{D}(\mathcal{U})$. Then*

$$u(\phi) = \frac{1}{\sqrt{2}^{\dim(\mathfrak{n})}\mu(\mathrm{K}\cap\mathrm{M})}\,\mathrm{Res}(u)(\phi^{\mathrm{K}}_{\mathrm{N}}).$$

Proposition A.8. *Suppose $u \in \mathcal{D}'(\mathrm{L})$ is an $\mathrm{L}-$invariant Borel measure concentrated on $\mathrm{M}\cap\mathrm{G}^{\mathrm{reg}}$. Then the induced distribution $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u)$ is a $\mathrm{G}-$invariant Borel measure concentrated on $\mathrm{G}^{\mathrm{reg}}$.*

Proof. Notice that $\mathrm{L}\cap\mathrm{G}^{\mathrm{reg}} \subseteq \mathrm{L}^{\mathrm{reg}}$. Hence, for $\Psi \in \mathbb{C}_c(\mathrm{G})$,

$$\begin{aligned}
&\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u)(\Psi) = u(\Psi^{\mathrm{L}}) = u(\Psi^{\mathrm{L}}|_{\mathrm{L}\cap\mathrm{G}^{\mathrm{reg}}})\\
&= \int_{\mathrm{L}'} u(l)\Psi^{\mathrm{L}}(l)\,dl = \int_{\mathrm{L}'} u(l)|\det(\mathrm{Ad}(l)_{\mathfrak{n}})|^{1/2}\int_{\mathrm{N}}\Psi^{\mathrm{K}}(ln)\,d\mu(n)\,d\mu(l)\\
&= \int_{\mathrm{L}'} u(h)|\det(\mathrm{Ad}(h^{-1}-1)_{\mathfrak{n}})||\det(\mathrm{Ad}(l)_{\mathfrak{n}})|^{1/2}\int_{\mathrm{N}}\Psi^{\mathrm{K}}(nln^{-1})\,d\mu(n)\,d\mu(l)\\
&= \int_{\mathrm{L}\cap\mathrm{G}^{\mathrm{reg}}} u(l)|\det(\mathrm{Ad}(l^{-1}-1)_{\mathfrak{n}})||\det(\mathrm{Ad}(l)_{\mathfrak{n}})|^{1/2}\int_{\mathrm{N}}\Psi^{\mathrm{K}}(nln^{-1})\,d\mu(n)\,d\mu(l)\\
&= \int_{\mathrm{L}\cap\mathrm{G}^{\mathrm{reg}}} u(l)|\det(\mathrm{Ad}(l^{-1}-1)_{\mathfrak{n}})||\det(\mathrm{Ad}(l)_{\mathfrak{n}})|^{1/2}\int_{\mathrm{N}}\Psi^{\mathrm{K}}|_{\mathrm{G}^{\mathrm{reg}}}(nln^{-1})\,d\mu(n)\,d\mu(l)\\
&= \int_{\mathrm{L}'} u(l)(\Psi|_{\mathrm{G}^{\mathrm{reg}}})^{\mathrm{L}}(h)\,d\mu(l)\\
&= \mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}(u)(\Psi|_{\mathrm{G}^{\mathrm{reg}}}).
\end{aligned}$$

■

Appendix B.

Let V be a finite dimensional vector space over $\mathbb{R}$ and let $A \subseteq V^*$ be a finite set such that no two elements of A are proportional. Let

$$V^A = \{x \in V|\ \text{there exists } \alpha \in A \text{ such that } \alpha(x) = 0\}.$$

We shall say that a function $\phi \in C^\infty(V \setminus V^A)$ is a Harish-Chandra Schwartz function with respect to A if and only if for every constant coefficient differential operator D on V and for every polynomial function P on V,

$$\sup_{x\in V\setminus V^A} |P(x)D\phi(x)| < \infty, \tag{B.1}$$

and for every connected component $C \subseteq V \setminus V^A$ the restriction of $D\phi$ to C extends to a continuous function on $\overline{C}$, the closure of C in V. (If V is a Cartan subalgebra in a real semisimple Lie algebra and C is a connected component of the set of the regular semisimple elements, then as shown in [6, sections 7 and 12], (B.1) implies the existence of the extension.) Notice that this extension is a rapidly decreasing function on $\overline{C}$. We shall denote by $\mathcal{HCS}(V \setminus V^A)$ the space of all the Harish-Chandra Schwartz functions with respect to A and equip this space with the topology induced by the seminorms (B.1). Our definition is motivated by a theorem of Harish-Chandra concerning his orbital integrals, see Theorem 23 on page 23 and the proof of Proposition 10 in the Appendix of part I of [11] or section 14 in [7].

Suppose we have a direct sum decomposition

$$V = V' \oplus V''.$$

Let $A' = \{\alpha \in A \,|\, \alpha(V'') = 0\}$. Let $B \subseteq V^*$ be a finite set such that

$$V^\alpha \neq V^\beta \qquad (\alpha \in A,\ \beta \in B).$$

Fix a Lebesgue measure μ on V''.

Lemma B.1. *The formula*

$$\phi'(x) = \int_{V''} \prod_{\beta \in B} \frac{\beta(x+x'')}{|\beta(x+x'')|} \cdot \phi(x+x'')\, d\mu(x'') \qquad (\phi \in \mathcal{HCS}(V \backslash V^A),\ x \in V' \backslash V'^{,A'})$$

defines a continuous map

$$\mathcal{HCS}(V \setminus V^A) \ni \phi \to \phi' \in \mathcal{HCS}(V' \setminus V'^{,A'}).$$

Proof. Since ϕ is rapidly decreasing, ϕ' is well defined and is also rapidly decreasing. Let $A'' = \{\alpha \in A \,|\, \alpha(V'') \neq 0\} \cup B$. Then

$$\dim(V'' \cap (V^A - x)) < \dim V'' \qquad (x \in V). \tag{B.2}$$

Indeed, if not then there is an element $\alpha \in A''$ such that

$$\dim(V'' \cap (V^\alpha - x)) = \dim V'' \qquad (x \in V). \tag{B.3}$$

Since

$$(V'' \cap (V^\alpha - x)) - (V'' \cap (V^\alpha - x)) \subseteq V(\alpha),$$

(B.3) implies that V^α contains an non-empty open subset of V''. Therefore $V'' \subseteq V^\alpha$, a contradiction.

Let $C'(V') \subseteq V' \setminus V'^{,A'}$ be a connected component, with the closure $\overline{C'(V')} \subseteq V'$. Let $x \in \overline{C'(V')}$ and let $x_n \in \overline{C'(V')}$ be a sequence with $\lim_{n\to\infty} x_n = x$. Let $C' = C'(V') + V''$. This is a connected component of $V \setminus V'^{,A'}$. We see from (B.2) that

$$V'' \setminus (V''^{A''} - x) \subseteq V''$$

is an open dense subset. Let $x'' \in V'' \setminus (V''^{A''} - x)$. Then $x + x'' \in V \setminus V^{A''}$. Let $C'' \subseteq V \setminus V^{A''}$ be the connected component containing $x + x''$. Then $C = C' \cap C''$ is a connected component of $V \setminus V^{A \cup B}$ and

$$x + x'' \in \overline{C'} \cap C'' \subseteq \overline{C}.$$

Since $\overline{C'}$ is invariant under translations by the elements of V'', $x_n + x'' \in \overline{C'}$ for all n. Since C'' is open, there is N such that $x_n + x'' \in C''$ for all $n \geq N$. Therefore,

$$x_n + x'' \in \overline{C} \qquad (n \geq N).$$

Since the restriction of ϕ to $\overline{C}$ is continuous,

$$\lim_{n\to\infty}\phi(x_n+x'')=\phi(x+x'').$$

Thus by Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_{V''}\phi(x_n+x'')d\mu(x'')=\int_{V''}\phi(x+x'')d\mu(x'').$$

Therefore ϕ' extends to a continuous function on $\overline{C'(V')}$.

Since the above argument applies to any derivative of ϕ, we see that $\phi'\in\mathcal{HCS}(V'\setminus V'^{A'})$. The proof of the continuity of the map $\phi\to\phi'$ is easy and we leave it to the reader. ■

For a function $\phi:\mathbb{R}\to\mathbb{C}$ let

$$\langle\phi\rangle(0)=\lim_{x\to 0+}\phi(x)-\lim_{x\to 0+}\phi(-x),$$

whenever the indicated limits exist. Also, for a non-negative integer N, let

$$\phi_N(x+\mathsf{i}y)=\sum_{p=0}^{N}\frac{(\mathsf{i}y)^p}{p!}\phi^{(p)}(x)\qquad(x\in\mathbb{R}\setminus 0,\ y\in\mathbb{R}).$$

Lemma B.2. *Let $\phi\in\mathcal{HCS}(\mathbb{R}\setminus 0)$ and let $\epsilon=\pm 1$. Then, for $N\geq 1$,*

$$\lim_{y\to 0+}\int_{\mathbb{R}\setminus 0}\frac{1}{x'+x+\mathsf{i}\epsilon y}\phi(x)\,dx=\int_0^{\epsilon}\frac{1}{x'+\mathsf{i}y}\langle\phi_N\rangle(\mathsf{i}y)\,d(iy)+\tilde{\phi}(x')\qquad(x'\in\mathbb{R}),$$

where $\tilde{\phi}$ extends to a continuous function on $\mathbb{R}$.

Proof. Suppose $\epsilon=-1$. (The case $\epsilon=1$ is entirely analogous.) Let $z=x+\mathsf{i}y$ and let $f(z)=\frac{1}{x'+z}$.

Fix $\delta>0$ and let

$$\mathcal{C}_\delta^-=\{z\,|\,x<-\delta,\ -1<y<0\},$$
$$\mathcal{C}_\delta^+=\{z\,|\,x>\delta,\ -1<y<0\},$$

with the boundaries oriented counter-clockwise. Then Stokes' Theorem implies that for any $y_0>0$.

$$\int_{\mathcal{C}_\delta^+}d(f(z-\mathsf{i}y_0)\phi_N(z)dz)=\int_{\infty}^{\delta}f(x-\mathsf{i}y_0)\phi(x)\,dx$$
$$+\int_0^{-1}f(\delta+\mathsf{i}y-\mathsf{i}y_0)\phi_N(\delta+\mathsf{i}y)\,d(iy)+\int_{\delta}^{\infty}f(x-\mathsf{i}-\mathsf{i}y_0)\phi_N(x-\mathsf{i})\,dx$$

and

$$\int_{\mathcal{C}_\delta^-}d(f(z-\mathsf{i}y_0)\phi_N(z)dz)=\int_{-\delta}^{-\infty}f(x-\mathsf{i}y_0)\phi(x)\,dx$$
$$+\int_{-1}^{0}f(\delta+\mathsf{i}y-\mathsf{i}y_0)\phi_N(\delta+\mathsf{i}y)\,d(iy)+\int_{-\infty}^{-\delta}f(x-\mathsf{i}-\mathsf{i}y_0)\phi_N(x-\mathsf{i})\,dx.$$

Hence,

$$\left(\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}\right) f(x-\mathrm{i}-\mathrm{i}y_0)\phi_N(x-\mathrm{i})\,dx \tag{B.4}$$
$$= \int_0^{-1} \left(f(\delta+\mathrm{i}y-\mathrm{i}y_0)\phi_N(\delta+\mathrm{i}y) - f(-\delta+\mathrm{i}y-\mathrm{i}y_0)\phi_N(-\delta+\mathrm{i}y)\right)\, d(\mathrm{i}y)$$
$$+ \left(\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}\right) f(x-\mathrm{i}-\mathrm{i}y_0)\phi_N(x-\mathrm{i})\,dx - \int_{\mathcal{C}_\delta^-\cup\mathcal{C}_\delta^+} d(f(z-\mathrm{i}y_0)\phi_N(z)dz).$$

Since, it is well known and easy to check,

$$d(f(z-\mathrm{i}y_0)\phi_N(z)dz) = f(x+\mathrm{i}y-\mathrm{i}y_0)\frac{(y\mathrm{i})^N}{N!}\phi^{(N+1)}(x)\,dx\,\mathrm{i}dy,$$

we may take the limit as δ goes to zero of both sides of (B.4) and obtain the following equality

$$\int_{\mathbb{R}\setminus 0} f(x-\mathrm{i}y_0)\phi(x)\,dx = \int_0^{-1} f(\mathrm{i}y-\mathrm{i}y_0)\langle\phi_N\rangle(\mathrm{i}y)\,d(\mathrm{i}y)$$
$$+\int_{\mathbb{R}\setminus 0} f(x-\mathrm{i}-\mathrm{i}y_0)\phi_N(x-\mathrm{i})\,dx - \int_{-1}^{0}\int_{\mathbb{R}\setminus 0} f(x+\mathrm{i}y-\mathrm{i}y_0)\frac{(y\mathrm{i})^N}{N!}\phi^{(N+1)}(x)\,dx\;d(\mathrm{i}y). \tag{B.5}$$

Now we take the limit of both sides of (B.5) as y_0 goes to zero and deduce the equation of our Lemma, with

$$\tilde\phi(x') = \int_{\mathbb{R}\setminus 0}\frac{1}{x'+x-\mathrm{i}}\phi_N(x-\mathrm{i})\,dx - \int_{-1}^{0}\int_{\mathbb{R}\setminus 0}\frac{1}{x'+x+\mathrm{i}y}\frac{(y\mathrm{i})^N}{N!}\phi^{(N+1)}(x)\,dx\;d(\mathrm{i}y).$$

Since $N\geq 1$ and since $\phi^{(N+1)}$ is absolutely integrable, $\tilde\phi$ extends to a continuous function on $\mathbb{R}$. ■

Lemma B.3. *Let $\phi : [-1,1]\to\mathbb{C}$ be a continuous function and let $\epsilon = \pm 1$. For $x'\in\mathbb{R}\setminus 0$ define*

$$\tilde\phi(x') = \int_0^{\epsilon}\left(\frac{1}{x'+\mathrm{i}y}+\frac{1}{x'-\mathrm{i}y}\right)\phi(y)\,dy.$$

Then

$$\langle\phi\rangle(0) = 2\pi\epsilon\phi(0).$$

Proof. Since

$$\tilde\phi(x') = \int_0^{\epsilon}\frac{2x'}{x'^2+y^2}\phi(y)\,dy,$$

is an odd function, $\langle\phi\rangle(0) = 2\lim_{x'\to 0+}\tilde\phi(x') =$

$$2\lim_{x'\to 0+}\int_0^{\epsilon/x'}\frac{2}{1+y^2}\phi(x'y)\,dy = 2\epsilon\int_0^{\infty}\frac{2}{1+y^2}\,dy\,\phi(0) = 2\epsilon\pi\phi(0).$$ ■

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