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Platonic Orthonormal Wavelets[1]

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We classify all orthonormal wavelets which occur in the L^2 space of the faces of a platonic solid. © 1997 Academic Press

0. INTRODUCTION

Let G be the ‘‘$ax + b$’’ group. Thus $G = \mathbb{R}^x \times \mathbb{R}$ with the group structure given by

$$(a, b)(a', b') = (aa', ab' + b), (a, b)^{-1} = (a^{-1}, -a^{-1}b) \quad ((a, b), (a', b') \in G).$$

This group acts on the real line $\mathbb{R}$ by the formula

$$(a, b)x = ax + b \quad ((a, b) \in G, x \in \mathbb{R})$$

and on the space $L^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$:

$$\sigma(a, b)v(x) = |a|^{-1/2}v((a, b)^{-1}x) = |a|^{-1/2}v(a^{-1}(x - b)), \quad v \in L^2(\mathbb{R}). \tag{0.1}$$

It is easy to see that the formula (0.1) defines a unitary representation $(\sigma, L^2(\mathbb{R}))$ of G. It is well known, and easy to check, that this representation is irreducible. Indeed, suppose T is a bounded operator on $L^2(\mathbb{R})$ which commutes with $\sigma(G)$. Then it commutes with all translations. Therefore $F \circ T \circ F^{-1}$, the conjugate of T by a Fourier transform F, coincides with multiplication by a function. Since this multiplication commutes with all dilatations, the function is constant. Thus T is a constant multiple of the identity.

Let H be the dyadic ‘‘$ax + b$’’ subgroup of G, i.e., $H = \{(2^j, k2^l); j, k, l \in \mathbb{Z}\}$. This subgroup is neither discrete nor dense in G. Recall [3] that an orthonormal wavelet is a unit vector $v \in L^2(\mathbb{R})$ such that the orbit $\sigma(H)v$ contains an orthonormal basis of $L^2(\mathbb{R})$. In fact, this orthonormal basis can be written as $\sigma(D) \cdot \sigma(T)v$, where

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$D = \{(2^j, 0); j \in \mathbb{Z}\} \subseteq H$ is the subgroup of dilations, and $T = \{(1, k); k \in \mathbb{Z}\} \subseteq H$ is the subgroup of translations. Note that the subset $D \cdot T \subseteq H$ is not a subgroup.

In this article we take a closer look at this aspect of the theory of wavelets when G is a finite group, where there seems to be no good analog of D, and all analytic problems are absent. Nevertheless some applications to coding theory seem to be in sight. Specifically we are interested in the following situation.

Let G be a finite group and let (σ, V) be an irreducible unitary (or orthogonal, if the ground field is $\mathbb{R}$) representation of G. Let H be a subgroup of G.

DEFINITION 0.2. An H-wavelet in V is a unit vector $v \in V$ whose stabilizer in H is trivial, and such that $\sigma(H)v$ is an orthonormal basis of V.

Let $(\lambda, L^2(H))$ denote the left regular representation of H:

$$\lambda(g)f(h) = f(g^{-1}h), \quad h, g \in H, f \in L^2(H).$$

Let $v \in V$ be an H-wavelet. Then the map $L^2(H) \to L^2(V)$

$$f \to \sum_{h \in H} f(h)\sigma(h)v$$

is a bijective isometry, which intertwines the actions of H on both spaces. Thus the restriction of σ to H is equivalent to λ: $\sigma|_H \cong \lambda$.

Suppose $u \in V$ is another H-wavelet. Then the map $V \to V$

$$\sum_{h \in H} c_h \sigma(h)u \to \sum_{h \in H} c_h \sigma(h)v$$

is a unitary bijection which commutes with the action of H.

Let $U(V)$ denote the group of unitary operators on the Hilbert space V. The above argument shows that $U(V)^H$, the centralizer of H in $U(V)$, acts simply transitively on the set of orthonormal wavelets. Let $\chi_\sigma = \operatorname{tr} \sigma(g)$, $g \in G$ be the character of σ. This is a conjugation invariant function on G, which determines the representation σ. The following lemma essentially reduces our problem to character theory and is fundamental for the rest of the paper.

LEMMA 0.3. *The following conditions are equivalent:*

(*a*) *the representation* (σ, V) *has an* H*-wavelet;*
(*b*) $(\sigma|_H, V)$ *is equivalent to the left regular representation* $(\lambda, L^2(H))$*;*
(*c*)

$$\chi_\sigma(h) = \begin{cases} |H|, & \text{if } h = 1 \\ 0, & \text{if } h \in H \backslash \{1\}. \end{cases}$$

Moreover, the group $U(V)^H$ *acts simply transitively on the set of all* H*-wavelets in* V*. In particular this set is homeomorphic to the group* $U(V)^H$*.*

Proof. All the assertions of the lemma with the exception of "(c) implies (a)" follow immediately from the preceding discussion.

Assuming (c) σ restricted to H is the left regular representation, because it is determined by its character. A technical problem is that the given inner product on V may not agree with the standard one on $L^2(H)$. Let $V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be the decomposition of V into isotypic components. Then the V_j's are orthogonal with respect to both inner products, by Schur's lemma; see [4]. Each V_j can be written as $U_j \otimes W_j$, where H acts irreducibly on U_j, W_j is an auxiliary space with trivial H-action. On U_j there is, up to a positive scalar multiple, a unique H-invariant inner product $(\ , \ )_j$. The restriction of the inner product from V to V_j is of the form $(\ , \ )_j \otimes (\ , \ )'_j$, where $(\ , \ )'_j$ is an inner product on W_j. The group $GL(W_j)$ acts transitively on inner products on W_j. Therefore we can choose $g_j \in GL(W_j)$ so that $\Sigma_j I_{U_j} \otimes g_j$ is an isometry between the two inner product spaces $L^2(H)$ and V. Clearly the left regular representation $L^2(H)$ has an H-wavelet, say v. Then $(\Sigma_j I_j \otimes g_j)v$ is an H-wavelet for V. ■

Corollary 0.4. *If the representation* $(\sigma|_H, V)$ *has* H*-wavelets then* $\dim V^H = 1$. (*Here* V^H *stands for the space of* H*-invariants in* V).

Trivial Examples.

(a) If $\dim V = 1$, then any unit vector in V is an H-wavelet for $H = \{1\}$, and this is the only subgroup of G admitting wavelets.

(b) Let G be the cyclic group of order 3. This group acts on $\mathbb{R}^2$ via rotations by $2\pi/3$ and the resulting representation is irreducible (over $\mathbb{R}$). Since G has no subgroups of order 2, there are no wavelets in this representation.

Let Symm(n) denote the symmetric group of all permutations on n objects and let Alt(n) denote the alternating group of even permutations. These groups act on $\mathbb{C}^n$ (or $\mathbb{R}^n$) by permuting the coordinates. If $n \geqslant 4$, they act irreducibly on the subspace V consisting of vectors whose coordinates add up to zero (i.e., the orthogonal complement of the diagonal). This is the (so-called) standard representation of Symm(n) and Alt(n), which shall be denoted, from now on, by (ρ, V). The standard inner product induced from $\mathbb{R}^n$ is invariant, and, up to a scalar multiple, this is the only invariant inner product, because the representation is irreducible over $\mathbb{C}$.

Theorem 0.5. *The standard representation of* Alt(n) *has an* H*-wavelet if and only if* n *is not congruent to* 3 *modulo* 4.

Proof. If n is congruent to 3 modulo 4, then the order of $H = \dim V = n - 1$ is of the form $2(2k + 1)$ for some natural number k. So there must be an element of order 2 in H. As a permutation this element has to be a product of $2k + 1$ disjoint transpositions, by Lemma 0.3(b). But this element is not an even permutation, contradicting that H was a subgroup of Alt(n).

Conversely, if n is not congruent to 3 modulo 4, then $n - 1$ is not congruent to 2 modulo 4. Let $n - 1 = 2^k m$ with m odd, so $k = 0$ or $k \geqslant 2$. Consider the abstract group $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, where there are k two-cyclic factors $\mathbb{Z}/2\mathbb{Z}$, so the order of H is $n - 1$. The embedding of H into Symm($n - 1$) via the left action of H on itself

$$\phi: H \to \mathrm{Symm}(n - 1)$$

has its image contained in $\mathrm{Alt}(n-1)$. Indeed, for any nonidentity element $h \in H$, $\phi(h)$ has a disjoint cycle decomposition consisting of cycles with length equal to the order of h, which is odd when $n-1$ is odd ($k=0$), or there is an even number of such cycles ($k \geqslant 2$). Now $\phi(H)$ can be regarded as a subgroup of $\mathrm{Alt}(n)$, stabilizing the nth object. The character of the standard representation of $\mathrm{Alt}(n)$ restricted to $\phi(H)$ is as in Lemma 0.3(c), therefore there is a $\phi(H)$-wavelet. ■

Remark. The proof above shows that there exist H-wavelets for the standard representation of $\mathrm{Symm}(n)$ for every n.

In the rest of this article we consider the group G of rotational symmetries of a platonic solid S, a polyhedron in $\mathbb{R}^3$ whose faces are regular polygons. An elementary reference for these objects is [**1**]. Complete information on the representations of these groups is in [**2**]. The best reference for our purposes is [**4**, 1.6], which unfortunately is not yet available. We shall show that every irreducible unitary representation (σ, V) of G has an H-wavelet and we shall describe each one.

We may assume that $\dim V \geqslant 2$. Then each (σ, V) is a complexification of an irreducible orthogonal representation. Thus we may work over the reals, $\mathbb{R}$.

Let F_S denote the set of faces of S and let $L^2(F_S)$ be the real L^2 space of F_S equipped with the counting measure. Let π denote the permutation representation of G on $L^2(F_S)$:

$$\pi(g)f(\phi) = f(g^{-1}\phi) \quad (g \in G,\ f \in L^2(F_S),\ \phi \in F_S).$$

We shall describe the decomposition of π into irreducible components and exhibit the wavelets as elements of $L^2(F_S)$. In other words, we figure out all possible ways of assigning real numbers to the faces of a platonic solid so that for some subgroup H of rotations, moving the solid by elements of H gives an orthonormal basis of an irreducible subrepresentation (whose dimension coincides with the order of H) of the space of functions on the faces.

We begin with the following.

THEOREM 0.6. *Let G be the group of rotational symmetries of a platonic solid and let (σ, V) be an irreducible unitary (or orthogonal) representation of G. Then, up to conjugation, there is only one subgroup $H \subseteq G$ for which the H-wavelets exist.*

Proof. The group $G = \mathrm{Alt}(4)$ or $\mathrm{Symm}(4)$ or $\mathrm{Alt}(5)$.

(i) $G = \mathrm{Alt}(4)$. The only representation of dimension greater than 1 is the standard representation (ρ, V), of dimension 3. The character is given by the following table:

Conjugacy classes	1	(123)	(132)	(12)(34)
χ_ρ	3	0	0	−1.

All subgroups of order 3 are conjugate in G to $H = \langle(123)\rangle$, the subgroup generated by the cycle (123). Since the character χ_ρ vanishes on $H\backslash\{1\}$ the statement follows from Lemma 0.3.

(ii) $G = \mathrm{Symm}(4)$. This group has three irreducible representations of dimension greater than one: ρ-the standard one, $\rho' = \det \otimes \rho$—both of dimension 3 and δ of dimension 2. Here is the character table:

Conjugacy classes	1	(12)	(123)	(12)(34)	(1234)
χ_ρ	3	1	0	-1	-1
$\chi_{\rho'}$	3	-1	0	-1	1
χ_δ	2	0	-1	2	0.

Since all subgroups of G of order 3 are conjugate to $\langle(123)\rangle$, previous argument suffices for the representations ρ and ρ'. There are two conjugacy classes of elements of order 2 in G: (12) and (12)(34). But (12)(34) is in the kernel of δ, so its orbits are trivial. On the other hand the character χ_δ vanishes on $H = \langle(12)\rangle$, so the statement follows from Lemma 0.3.

(iii) $G = \mathrm{Alt}(5)$. This group has 4 irreducible representations of dimension greater than 1. Let $\mu = (1 + \sqrt{5})/2$ be the golden mean, and let $\bar{\mu} = (1 - \sqrt{5})/2$. (This is the conjugate of μ in $\mathbb{Q}(\sqrt{5})$ over $\mathbb{Q}$). Here is the character table:

Conjugacy classes	1	(12)(34)	(123)	(12345)	(13524)
χ_α	3	-1	0	μ	$\bar{\mu}$
χ_β	3	-1	0	$\bar{\mu}$	μ
χ_ρ	4	0	1	-1	-1
χ_γ	5	1	-1	0	0.

As before $H = \langle(123)\rangle$ for the representations α and β, $H = \langle(12345)\rangle$ for γ and $H = \{1, (12)(34), (13)(24), (14)(23)\}$ for ρ, up to conjugacy. (In fact the last group is the only subgroup of order 4, because it is a Sylow subgroup). ■

1. THE TETRAHEDRON

Let $G = \mathrm{Alt}(4)$ and let (ρ, V) be the standard representation of G, over $\mathbb{R}$. Let $e_1 = (-1, 1, 1, -1)$, $e_2 = (1, -1, 1, -1)$, $e_3 = (1, 1, -1, -1)$. Then $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$. Define a scalar product on V by declaring e_1, e_2, e_3 to be an orthonormal basis.

LEMMA 1.1. *Let $H = \langle(123)\rangle$. The H-wavelets for the representation (ρ, V) are given by the formula*

$$\frac{\epsilon}{3}((1 + 2\cos(x))e_1 + (1 - \cos(x) + \sqrt{3}\sin(x))e_2 + (1 - \cos(x) - \sqrt{3}\sin(x))e_3),$$

where $\epsilon = \pm 1$ and $x \in \mathbb{R}$. The group $O(V)^H$ is isomorphic to $SO(2) \times O(1)$, a disjoint union of two circles.

Proof. Clearly $\{e_1, e_2, e_3\}$ is a single H-orbit. Thus e_1 is an H-wavelet. The group $O(V)^H$ preserves the direct sum decomposition

$$V = \mathbb{R}(e_1 + e_2 + e_3)^\perp \oplus \mathbb{R}(e_1 + e_2 + e_3)$$

and, hence, is isomorphic to $SO(2) \times O(1)$. In particular one can calculate that the $O(V)^H$-orbit of e_1 consists of the vectors listed in the lemma. ■

Let T be the tetrahedron whose edges have midpoints $\pm e_i$, for $i = 1, 2, 3$. The group G preserves T and this way may be identified with the group of rotational symmetries of T.

There is a G-intertwining isometry

$$V \to L^2(F_T) \tag{1.2}$$

which assigns to $v \in V$ a function $f \in L^2(F_T)$ whose value on a face ϕ is equal to the scalar product of v with the vector of length $\sqrt{3}/2$ perpendicular to ϕ and pointing outward. Here we list these vectors:

$$\frac{1}{2}(-e_1 - e_2 - e_3), \frac{1}{2}(-e_1 + e_2 + e_3), \frac{1}{2}(e_1 - e_2 + e_3), \frac{1}{2}(e_1 + e_2 - e_3). \tag{1.3}$$

Since the sum of the vectors (1.3) is zero, the image $L^2(F_T)_\rho$ of the map (1.2) consists of functions whose values sum up to zero. The orthogonal complement $L^2(F_T)_1$ consists of constant functions. Thus

$$L^2(F_T) = L^2(F_T)_1 \oplus L^2(F_T)_\rho. \tag{1.4}$$

PROPOSITION 1.5. *The wavelets which occur in* $L^2(F_T)$ *are:*

(a) *the constant functions* $= \pm 1/2$ *in* $L^2(F_T)_1$, *for* $H = \{1\}$;
(b) *the function* $f \in L^2(F_T)_\rho$ *which takes the values*

$$-\frac{\epsilon}{2}, \frac{\epsilon}{6}(1 - 4\cos(x)), \frac{\epsilon}{6}(1 + 2\cos(x) - 2\sqrt{3}\sin(x)),$$
$$\frac{\epsilon}{6}(1 + 2\cos(x) + 2\sqrt{3}\sin(x))$$

on the faces perpendicular to the vectors (1.3), *respectively, for* $H = \langle(123)\rangle$, $\epsilon = \pm 1$, $x \in \mathbb{R}$.

Proof. Part (a) is obvious. For part (b) we apply the map (1.2) to the vectors of Lemma 1.1. ■

2. THE CUBE

Here $G = \text{Symm}(4)$ and (ρ, V) is the obvious extension of the standard representation of Alt(4) constructed in the previous section. Let sgn denote the sign representation of G (composition of ρ with the determinant on End(V)). Let $\rho' = \text{sgn} \otimes \rho$. This notation is consistent with the one used in the proof of Theorem 0.6.

Let C be the cube in V whose faces have centers at $\pm e_i$, for $i = 1, 2, 3$, so the vertices are $\pm e_1 \pm e_2 \pm e_3$. Under the action ρ' the group G preserves C and may be identified with the group of rotational symmetries of C.

There is a G-intertwining isometry

$$V \to L^2(F_C) \tag{2.1}$$

which assigns to $v \in V$ a function $f \in L^2(F_C)$ whose value on a face ϕ is equal to the scalar product of v with the vector of length $\sqrt{2}/2$ perpendicular to ϕ and pointing outward. Here we list these vectors:

$$\frac{\sqrt{2}}{2}\, e_1,\ \frac{\sqrt{2}}{2}\, e_2,\ \frac{\sqrt{2}}{2}\, e_3,\ -\frac{\sqrt{2}}{2}\, e_1,\ -\frac{\sqrt{2}}{2}\, e_2,\ -\frac{\sqrt{2}}{2}\, e_3. \tag{2.2}$$

It is clear that the image of the map (2.1), $L^2(F_C)_{\rho'}$, consists of all functions which take opposite values on antipodal faces. The trivial representation occurs as the subspace $L^2(F_C)_1$ of constant functions. The orthogonal complement, $L^2(F_C)_\delta$, of $L^2(F_C)_{\rho'} \oplus L^2(F_C)_1$ consists of functions which take the same values on antipodal faces and all of whose values sum to zero. It is easy to see that $\dim L^2(F_C)_\delta = 2$ and that G acts on it irreducibly. Thus,

$$L^2(F_C) = L^2(F_C)_1 \oplus L^2(F_C)_\delta \oplus L^2(F_C)_{\rho'} \tag{2.3}$$

is the decomposition into irreducible components.

PROPOSITION 2.4. *The wavelets which occur in* $L^2(F_C)$ *are:*

(a) *the constant functions* $= \pm\sqrt{6}/6$ *in* $L^2(F_C)_1$, *for* $H = \{1\}$;
(b) *the function* $f \in L^2(F_C)_{\rho'}$ *which takes the values*

$$\frac{\epsilon\sqrt{2}}{6}(1 + 2\cos(x)),\ \frac{\epsilon\sqrt{2}}{6}(1 - \cos(x) + \sqrt{3}\sin(x)),\ \frac{\epsilon\sqrt{2}}{6}(1 - \cos(x) - \sqrt{3}\sin(x)),$$
$$-\frac{\epsilon\sqrt{2}}{6}(1 + 2\cos(x)),\ -\frac{\epsilon\sqrt{2}}{6}(1 - \cos(x) + \sqrt{3}\sin(x)),$$
$$-\frac{\epsilon\sqrt{2}}{6}(1 - \cos(x) - \sqrt{3}\sin(x))$$

on the faces corresponding to the vectors (2.2), *respectively, for* $H = \langle(123)\rangle$, $\epsilon = \pm 1$, $x \in \mathbb{R}$;
(c) *the function* $f \in L^2(F_C)_\delta$ *which takes the values*

$$\frac{\sqrt{3}\epsilon_1}{6}(\sqrt{3}\epsilon_2 - 1),\ \frac{\sqrt{3}\epsilon_1}{6}(-\sqrt{3}\epsilon_2 - 1),\ \frac{\sqrt{3}\epsilon_1}{3},$$
$$\frac{\sqrt{3}\epsilon_1}{6}(\sqrt{3}\epsilon_2 - 1),\ \frac{\sqrt{3}\epsilon_1}{6}(-\sqrt{3}\epsilon_2 - 1),\ \frac{\sqrt{3}\epsilon_1}{3}$$

on the faces corresponding to the vectors (2.2) *respectively, for* $H = \langle(12)\rangle$, $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$.

Proof. Part (a) is obvious. Part (b) follows by applying the map (2.1) to vectors of Lemma 1.1. For part (c) we proceed as follows.

Let $f \in L^2(F_C)_\delta$ be an H-wavelet ($H = \langle(12)\rangle$). Let x, y, z be the three possible values of the function f. Then

$$x + y + z = 0, \quad x^2 + y^2 + z^2 = 1, \quad y^2 + 2xz = 0. \tag{2.5}$$

The solutions of the system (2.5) are

$$(x, y, z) = \pm\left(\frac{\sqrt{3}}{6}(\sqrt{3} - 1), -\frac{\sqrt{3}}{6}(\sqrt{3} + 1), \frac{\sqrt{3}}{3}\right). \quad \blacksquare$$

Since $\rho = \mathrm{sgn} \otimes \rho'$, the wavelets for ρ coincide with the wavelets for ρ'.

3. THE OCTAHEDRON

Here $G = \mathrm{Symm}(4)$ acts on V via ρ' as in Section 2. Let O be the regular octahedron whose faces are centered on the vertices of the cube C.

There is a G-intertwining isometry

$$L^2(F_C) \to L^2(F_O) \tag{3.1}$$

which assigns to a function $f_C \in L^2(F_C)$ a function $f_O \in L^2(F_O)$ whose value on a face $\phi \in F_O$ is equal to $\sqrt{3}/6$ times the sum of the values of f_C on the three faces of C adjacent to the vertex of C corresponding to the face ϕ. The images of $L^2(F_C)_1$ and $L^2(F_C)_{\rho'}$ are nonzero and, hence, irreducible. We denote them by $L^2(F_O)_1$ and $L^2(F_O)_{\rho'}$, respectively. The image of $L^2(F_C)_\delta$ is zero.

The functions in $L^2(F_O)_{\rho'}$ have opposite values on antipodal faces. The dimension of the space of all such functions is 4. Hence the orthogonal complement of $L^2(F_O)_{\rho'}$ in this space has dimension 1. Denote it by $L^2(F_O)_{\mathrm{sgn}}$. Multiplication by a nonzero function from $L^2(F_O)_{\mathrm{sgn}}$ maps $L^2(F_O)_{\rho'}$ to $L^2(F_O)_\rho$, the ρ-isotypic component of $L^2(F_O)$. The elements of $L^2(F_O)_\rho$ have the same values on antipodal faces. Altogether we have the decomposition into irreducible components

$$L^2(F_O) = L^2(F_O)_1 \oplus L^2(F_O)_{\mathrm{sgn}} \oplus L^2(F_O)_\rho \oplus L^2(F_O)_{\rho'}. \tag{3.2}$$

By composing the map (3.1) with the map (2.1) we get

$$V \to L^2(F_O) \tag{3.2}$$

which assigns to $v \in V$ a function $f \in L^2(F_O)$ whose value on a face ϕ is equal to the scalar product of v with the vector of length $\sqrt{2}/4$ perpendicular to ϕ and pointing outward. Here we list these vectors:

$$\frac{\sqrt{2}}{4}(e_1+e_2+e_3),\ \frac{\sqrt{2}}{4}(-e_1+e_2+e_3),\ \frac{\sqrt{2}}{4}(e_1-e_2+e_3),$$
$$\frac{\sqrt{2}}{4}(-e_1-e_2+e_3),\ -\frac{\sqrt{2}}{4}(e_1+e_2+e_3),\ -\frac{\sqrt{2}}{4}(-e_1+e_2+e_3),$$
$$-\frac{\sqrt{2}}{4}(e_1-e_2+e_3),\ -\frac{\sqrt{2}}{4}(-e_1-e_2+e_3). \quad (3.4)$$

PROPOSITION 3.5. *The wavelets which occur in* $L^2(F_O)$ *are:*

(a) *the constant functions* $= \pm\sqrt{2}/4$ *in* $L^2(F_O)_1$, *for* $H = \{1\}$;
(b) *the function* $f \in L^2(F_O)_{\text{sgn}}$ *which takes the values*

$$\frac{\epsilon\sqrt{2}}{4},\ -\frac{\epsilon\sqrt{2}}{4},\ \frac{\epsilon\sqrt{2}}{4},\ -\frac{\epsilon\sqrt{2}}{4},\ -\frac{\epsilon\sqrt{2}}{4},\ \frac{\epsilon\sqrt{2}}{4},\ -\frac{\epsilon\sqrt{2}}{4},\ \frac{\epsilon\sqrt{2}}{4}$$

on the faces corresponding to the vectors (3.4), *respectively, for* $H = \{1\}$, $\epsilon = \pm 1$;
(c) *the function* $f \in L^2(F_O)_{\rho'}$ *which takes the values*

$$\frac{\sqrt{2}}{4}\epsilon,\ \frac{\sqrt{2}}{12}\epsilon(1-4\cos(x)),\ \frac{\sqrt{2}}{12}\epsilon(1+2\cos(x)-2\sqrt{3}\sin(x)),$$
$$\frac{\sqrt{2}}{12}\epsilon(-1-2\cos(x)-2\sqrt{3}\sin(x)),\ -\frac{\sqrt{2}}{4}\epsilon,\ -\frac{\sqrt{2}}{12}\epsilon(1-4\cos(x)),$$
$$-\frac{\sqrt{2}}{12}\epsilon(1+2\cos(x)-2\sqrt{3}\sin(x)),\ \frac{\sqrt{2}}{12}\epsilon(1+2\cos(x)+2\sqrt{3}\sin(x))$$

on the faces corresponding to the vectors (3.4), *respectively, for* $H = \langle(123)\rangle$, $\epsilon = \pm 1$, $x \in \mathbb{R}$.
(d) *the function* $f \in L^2(F_O)_{\rho}$ *which takes the values*

$$\frac{\sqrt{2}}{4}\epsilon,\ \frac{\sqrt{2}}{12}\epsilon(1-4\cos(x)),\ \frac{\sqrt{2}}{12}\epsilon(1+2\cos(x)-2\sqrt{3}\sin(x)),$$
$$\frac{\sqrt{2}}{12}\epsilon(-1-2\cos(x)-2\sqrt{3}\sin(x)),\ \frac{\sqrt{2}}{4}\epsilon,\ -\frac{\sqrt{2}}{12}\epsilon(1-4\cos(x)),$$
$$\frac{\sqrt{2}}{12}\epsilon(1+2\cos(x)-2\sqrt{3}\sin(x)),\ \frac{\sqrt{2}}{12}\epsilon(1+2\cos(x)+2\sqrt{3}\sin(x))$$

on the faces corresponding to the vectors (3.4), *respectively, for* $H = \langle(123)\rangle$, $\epsilon = \pm 1$, $x \in \mathbb{R}$.

Proof. Parts (a) and (b) are obvious. Part (c) is obtained by applying map (3.3) to the vectors of Lemma 1.1. Part (d) follows from (c) via multiplication by an element of $L^2(F_O)_{\text{sgn}}$. ■

4. THE DODECAHEDRON

Here $G = \mathrm{Alt}(5)$. Recall the golden mean $\mu = (1 + \sqrt{5})/2$ and the conjugate $\bar{\mu} = (1 - \sqrt{5})/2$. In the three-dimensional space V defined in Section 2, let D be the dodecahedron whose faces are centered on the vectors

$$-\mu e_2 + e_3,\ e_1 + \mu e_3,\ -e_1 + \mu e_3,\ -\mu e_1 - e_2,\ -\mu e_2 - e_3,\ \mu e_1 - e_2 \tag{4.1}$$

and on the antipodal vectors.

The group Alt(4) is contained in G as the stabilizer of the number 5. Let $s = (12345)$. Then Alt(4) and s generate G. Recall the standard representation (ρ, V) of Alt(4). One can extend this representation to a representation α of G on V by letting s act via rotations by $2\pi/5$ about the axis passing through midpoints of some two opposite faces. Specifically,

$$\alpha(s)\colon -\mu e_2 + e_3 \to -\mu e_2 + e_3$$
$$\alpha(s)\colon e_1 + \mu e_3 \to -e_1 + \mu e_3 \to -\mu e_1 - e_2 \to -\mu e_2 - e_3 \to \mu e_1 - e_2 \to e_1 + \mu e_3. \tag{4.2}$$

Thus the matrix of $\alpha(s)$ with respect to the basis $\{e_1, e_2, e_3\}$ is

$$\frac{1}{2}\begin{pmatrix} -\bar{\mu} & -1 & -\mu \\ 1 & \mu & \bar{\mu} \\ \mu & \bar{\mu} & 1 \end{pmatrix}. \tag{4.3}$$

We use the representation α to identify G with the group of rotational symmetries of D and to get the representation π on $L^2(F_D)$.

There is a G-intertwining isometry

$$V \to L^2(F_D) \tag{4.4}$$

which assigns to $v \in V$ a function $f \in L^2(F_D)$ whose value on a face ϕ is equal to the scalar product of v with the vector of length $\frac{1}{2}$ perpendicular to ϕ and pointing outward. We obtain these vectors by multiplying the vectors (4.1) by $1/2\sqrt{1 + \mu^2}$.

If we replace μ by $\bar{\mu}$ in the formulas (4.2) (or (4.3)) we get another extension of ρ to a representation β of G on V. Since $\mathrm{tr}\ \beta(s) = \bar{\mu} \neq \mu = \mathrm{tr}\ \alpha(s)$, these two representations are not equivalent. Under the action β, G does not permute the vectors (4.1). In other words it does not preserve the dodecahedron D. However it does preserve a different dodecahedron $\bar{D}$ (of the same size) whose faces are centered on the conjugates (μ replaced by $\bar{\mu}$) of the vectors (4.1):

$$-\bar{\mu} e_2 + e_3,\ e_1 + \bar{\mu} e_3,\ -e_1 + \bar{\mu} e_3,\ -\bar{\mu} e_1 - e_2,\ -\bar{\mu} e_2 - e_3,\ \bar{\mu} e_1 - e_2 \tag{4.5}$$

and the antipodal vectors. This defines a bijection of faces: $F_D \ni \phi \leftrightarrow \bar{\phi} \in F_D$.

Let G act on V by β. Let

$$V \to L^2(F_D) \tag{4.6}$$

be the map which assigns to $v \in V$ a function $f \in L^2(F_D)$ whose value on a face $\phi \in F_D$ is equal to the scalar product of v with the vector of length $\frac{1}{2}$ perpendicular to the corresponding face $\bar{\phi} \in F_{\bar{D}}$ and pointing outward of $\bar{D}$.

The images of (4.4) and (4.6) are both nonzero and, hence, irreducible. Moreover, they fill up the subspace of $L^2(F_D)$ consisting of functions having opposite values on antipodal faces. The orthogonal complement consists of functions having the same values on antipodal faces. This last subspace splits into an orthogonal direct sum of the trivial representation (constant functions) and the five-dimensional vertex representation γ. Thus, altogether,

$$L^2(F_D) = L^2(F_D)_\alpha \oplus L^2(F_D)_\beta \oplus L^2(F_D)_1 \oplus L^2(F_D)_\gamma. \tag{4.7}$$

PROPOSITION 4.8. *The wavelets which occur in* $L^2(F_D)$ *are:*

(a) *the constant functions* $= \pm\sqrt{3}/6$ in $\mathrm{L}^2(F_D)_1$, *for* $H = \{1\}$*;*
(b) *the function* $f \in L^2(F_D)_\gamma$ *which takes the values*

$$-\frac{\sqrt{3}}{6}\epsilon,$$

$$\frac{\sqrt{2}}{10}\epsilon\left(\frac{\sqrt{6}}{6} - 2\cos(x) - 2\cos(y)\right),$$

$$\frac{\sqrt{2}}{10}\epsilon\left(\frac{\sqrt{6}}{6} + \bar{\mu}\cos(x) + \mu\cos(y) + 5^{1/4}\mu^{1/2}\sin(x) + 5^{1/4}(-\bar{\mu})^{1/2}\sin(y)\right),$$

$$\frac{\sqrt{2}}{10}\epsilon\left(\frac{\sqrt{6}}{6} + \mu\cos(x) + \bar{\mu}\cos(y) + 5^{1/4}\mu^{-1/2}\sin(x) - 5^{1/4}(-\bar{\mu})^{-1/2}\sin(y)\right),$$

$$\frac{\sqrt{2}}{10}\epsilon\left(\frac{\sqrt{6}}{6} + \mu\cos(x) + \bar{\mu}\cos(y) - 5^{1/4}\mu^{-1/2}\sin(x) + 5^{1/4}(-\bar{\mu})^{-1/2}\sin(y)\right),$$

$$\frac{\sqrt{2}}{10}\epsilon\left(\frac{\sqrt{6}}{6} + \bar{\mu}\cos(x) + \mu\cos(y) - 5^{1/4}\mu^{1/2}\sin(x) - 5^{1/4}(-\bar{\mu})^{1/2}\sin(y)\right)$$

on the faces corresponding to the vectors (4.1), *respectively and the same values on antipodal faces, for* $H = \langle(12345)\rangle$, $\epsilon = \pm 1$, $x, y \in \mathbb{R}$. *The group* $O(L^2(F_D)_\gamma)^H$, *and hence the variety of all these wavelets, is isomorphic to* $SO(2) \times SO(2) \times O(1)$, *i.e., the surface of a disjoint union of two donuts.*

(c) *the function* $f \in L^2(F_D)_\alpha$ *which takes the values*

$$\frac{\epsilon}{6\sqrt{\mu + 2}}\left(-\mu + 1 + (\mu - 1)\cos(x) - \sqrt{3}(\mu + 1)\sin(x)\right),$$

$$\frac{\epsilon}{6\sqrt{\mu + 2}}\left(\mu + 1 + (2 - \mu)\cos(x) - \sqrt{3}\mu\sin(x)\right),$$

$$\frac{\epsilon}{6\sqrt{\mu+2}}\left(\mu - 1 - (\mu+2)\cos(x) - \sqrt{3}\mu\sin(x)\right),$$

$$\frac{\epsilon}{6\sqrt{\mu+2}}\left(-\mu - 1 + (1-2\mu)\cos(x) - \sqrt{3}\sin(x)\right),$$

$$\frac{\epsilon}{6\sqrt{\mu+2}}\left(-\mu - 1 + (\mu+1)\cos(x) + \sqrt{3}(1-\mu)\sin(x)\right),$$

$$\frac{\epsilon}{6\sqrt{\mu+2}}\left(\mu - 1 + (2\mu+1)\cos(x) - \sqrt{3}\sin(x)\right)$$

on the faces corresponding to the vectors (4.1), *respectively and the opposite values on antipodal faces, for* $H = \langle(123)\rangle$, $\epsilon = \pm 1$, $x \in \mathbb{R}$.

(d) *The function* $f \in L^2(F_D)_\beta$ *obtained by replacing* μ *by* $\bar{\mu}$ *in* (c).

Proof. Part (a) is trivial. Part (c) is obtained by applying map (4.4) to the vectors of Lemma 1.1. Similarly part (d) follows by applying map (4.6). For (b) we proceed as follows.

Denote by $x_0/\sqrt{2}$, $x_1/\sqrt{2}$, $x_2/\sqrt{2}$, $x_3/\sqrt{2}$, $x_4/\sqrt{2}$, $x_5/\sqrt{2}$ the values of a function $f \in L^2(F_D)_\gamma$ on faces corresponding to the vectors (4.1). Then $x_0 = -x_1 - x_2 - \cdots - x_5$. Moreover, f is an H-wavelet if and only if

$$x_0^2 + x_1^2 + \cdots + x_5^2 = 1, \text{ and } x_0^2 + x_1x_{\sigma(1)} + x_2x_{\sigma(2)} + \cdots + x_5x_{\sigma(5)} = 0$$

for any cyclic rotation σ of the indices 1, 2, 3, 4, 5. These equations are equivalent to

$$x_1^2 + x_2^2 + \cdots + x_5^2 = 5/6$$
$$(x_1 + x_2 + \cdots + x_5)^2 = 1/6$$
$$x_1(x_3 - x_2) + x_2(x_4 - x_3) + x_3(x_5 - x_4) + x_4(x_1 - x_5) + x_5(x_2 - x_1) = 0. \quad (4.9)$$

One solution of (4.9) is

$$x_1 = \frac{1 - 4\sqrt{6}}{5\sqrt{6}}, \quad x_2 = x_3 = x_4 = x_5 = \frac{1+\sqrt{6}}{5\sqrt{6}}.$$

This gives one wavelet $f_1 \in L^2(F_D)_\gamma$.

Define a representation of the group $H = \langle s\rangle$ on $\mathbb{R}^5$ by identifying s with the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the column vector $\epsilon_1 = \text{column}(1, 0, 0, 0, 0) \in \mathbb{R}^5$ is an H-wavelet and we have an isometry

$$\mathbb{R}^5 \to L^2(F_D)_\gamma \tag{4.10}$$

sending ϵ_1 to f_1 (see the Introduction). The matrix s has five distinct eigenvalues. Hence the group $O(5)^H$ is isomorphic to $SO(2) \times SO(2) \times O(1)$. This group can be calculated explicitly. This gives the variety of H-wavelets in $\mathbb{R}^5$: $O(5)^H \epsilon_1$. The image of this orbit under the map (4.10) is described in (b). ■

5. THE ICOSAHEDRON

Let $G = \text{Alt}(5)$ and let I be the icosahedron whose faces are centered on the vertices of the dodecahedron D. The only representation of G which does not occur in $L^2(F_D)$ is the standard representation (ρ, V).

Let $\mathbb{R}^5$ be equipped with the usual scalar product. Then V is the subspace consisting of vectors whose coordinates add up to zero. Let

$$v_1 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0\right), \quad v_2 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0\right),$$

$$v_3 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0\right), \quad v_4 = \left(\frac{\sqrt{5}}{10}, \frac{\sqrt{5}}{10}, \frac{\sqrt{5}}{10}, \frac{\sqrt{5}}{10}, -\frac{4\sqrt{5}}{10}\right). \tag{5.1}$$

The above vectors form an orthonormal basis of V. Let $h_1 = (12)(34)$, $h_2 = (13)(24)$, $h_3 = (14)(23)$ and let $H = \{1, h_1, h_2, h_3\}$. We identify V with $\mathbb{R}^4$ using the basis (5.1). Then the elements h_1, h_2, h_3 are identified with the matrices:

$$h_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

Hence the group $O(4)^H$ consists of all matrices with ± 1 on the diagonal. This is a maximal commutative 2-group in $O(4)$.

A vector $v = (x, y, z, w) \in \mathbb{R}^4$ is an H-wavelet if and only if

$$x^2 + y^2 + z^2 + w^2 = 1, \quad -x^2 - y^2 + z^2 + w^2 = 0, \quad -x^2 + y^2 - z^2 + w^2 = 0,$$
$$x^2 - y^2 - z^2 + w^2 = 0. \tag{5.3}$$

The solutions of (5.3) are

$$x = \pm\tfrac{1}{2}, \quad y = \pm\tfrac{1}{2}, \quad z = \pm\tfrac{1}{2}, \quad w = \pm\tfrac{1}{2}.$$

Hence the H-wavelets in V are $xv_1 + yv_2 + zv_3 + wv_4$:

$$\pm\frac{1}{2\sqrt{5}}(\mu, \mu, \mu, 3\bar{\mu} - 1, -2), \pm\frac{1}{2\sqrt{5}}(\mu, \mu, 3\bar{\mu} - 1, \mu, -2),$$
$$\pm\frac{1}{2\sqrt{5}}(\mu, 3\bar{\mu} - 1, \mu, \mu, -2), \pm\frac{1}{2\sqrt{5}}(3\bar{\mu} - 1, \mu, \mu, \mu, -2),$$
$$\pm\frac{1}{2\sqrt{5}}(\bar{\mu}, \bar{\mu}, \bar{\mu}, 3\mu - 1, -2), \pm\frac{1}{2\sqrt{5}}(\bar{\mu}, \bar{\mu}, 3\mu - 1, \bar{\mu}, -2),$$
$$\pm\frac{1}{2\sqrt{5}}(\bar{\mu}, 3\mu - 1, \bar{\mu}, \bar{\mu}, -2), \pm\frac{1}{2\sqrt{5}}(3\mu - 1, \bar{\mu}, \bar{\mu}, \bar{\mu}, -2), \quad (5.4)$$

where $\mu = (1 + \sqrt{5})/2$, the golden mean, and $\bar{\mu} = (1 - \sqrt{5})/2$ are as in the previous section. Observe that all these wavelets (5.4) are obtained as the H-orbits of the vectors $\pm v$ and $\pm\bar{v}$, where v is the first vector in the list (5.4).

Now we come back to the icosahedron I. Recall that the group Alt(4) is contained in Alt(5) $= G$ as the stabilizer of the number 5. It has five orbits in F_I. Geometrically this means that one may color the faces of I using five different colors in such a way that the five faces around each vertex are all of different color. Then the midpoints of the four faces of a given color are the vertices of a tetrahedron.

Let us fix one vertex in I. Given a vector $(a, b, c, d, e) \in V$ we put the numbers a, b, c, d, e on the faces around this vertex in a cyclic order. We put these numbers also on the other faces respecting the coloring. Thus one number corresponds to each tetrahedron. After normalizing we get a G-intertwining isometry:

$$V \to L^2(F_I). \quad (5.5)$$

The image of (5.5) is irreducible but it does not fill up the whole isotypic component $L^2(F_I)_\rho$. Let

$$A_\pm\colon L^2(F_I) \to L^2(F_I) \quad (5.6)$$

be the orthogonal projection on the ±1-eigenspace of the antipodal map A. Then $L^2(F_I)_\rho$ is a direct sum of the projections of the image of (5.5) under A_+ and A_-.

There is another natural G-intertwining map

$$L^2(F_D) \to L^2(F_I) \quad (5.7)$$

which assigns to a function $f_D \in L^2(F_D)$ a function $f_I \in L^2(F_I)$ whose value on a face $\phi_I \in F_I$ is equal to the sum of the values of f_D on the faces of D adjacent to the vertex defining ϕ_I.

The adjoint to (5.7) is the map

$$L^2(F_I) \to L^2(F_D) \quad (5.8)$$

which assigns to a function $f_I \in L^2(F_I)$ a function $f_D \in L^2(F_D)$ whose value on a face

$\phi_D \in F_D$ is equal to the sum of the values of f_I on the faces of I adjacent to the vertex of I corresponding to ϕ_D in the $D \leftrightarrow I$ duality. It is clear that the kernel of (5.8) coincides with $L^2(F_I)_\rho$. Thus

$$L^2(F_I) = L^2(F_I)_\alpha \oplus L^2(F_I)_\beta \oplus L^2(F_I)_1 \oplus L^2(F_I)_\gamma \oplus L^2(F_I)_\rho, \tag{5.9}$$

where the first four summands are the image of (5.7). The last one is the kernel of (5.8). It consists of two copies of ρ. The discussion above verifies the following.

PROPOSITION 5.10.

(a) *The wavelets which occur in* $L^2(F_I)_\alpha$, $L^2(F_I)_\beta$, $L^2(F_I)_1$, $L^2(F_I)_\gamma$ *are obtained by applying the map* (5.7) *to the corresponding wavelets on the dodecahedron D, and normalizing.*

(b) *The wavelets in* $L^2(F_I)_\rho$ *are obtained by applying the map* (5.5) *to the vectors* (5.4), *then applying* A_+ *or* A_- *and normalizing.*

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