STRICTLY POSITIVE DEFINITE FUNCTIONS
ON A COMPACT GROUP

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ABSTRACT. We recognize a result of Schreiner, concerning strictly positive definite functions on a sphere in an Euclidean space, as a generalization of Bochner's theorem for compact groups.

The purpose of this note is to provide a rapid proof of a relatively new result of Schreiner, [S], concerning strictly positive definite functions on a sphere in a Euclidean space. It is our pleasure to see that the theory of such functions has found applications in geosciences, and that it is well rooted in the existing mathematical literature.

Let G be a compact group, and let $H \subseteq G$ be a closed subgroup, such that the quotient G/H is infinite. Let $f : G \to \mathbb{C}$ be a continuous function, invariant under the left and right translations by elements of H. We denote the space of all such functions by $C(H\backslash G/H)$. Following Schreiner, [S], we say that the function f is *strictly positive definite* if and only if

$$(1) \qquad \sum_{i,j=1}^{n} c_i \overline{c}_j f(x_i^{-1} x_j) > 0$$

for any finite set $\{x_1, x_2, ..., x_n\} \subseteq G$ such that the cosets $x_1 H, x_2 H, ..., x_n H \subseteq G/H$ are distinct, and any complex numbers $c_1, c_2, ..., c_n$, not all equal to zero.

Let $\hat{G}$ denote the unitary dual of G, [K, 7.3]. This is the set of equivalence classes of irreducible unitary representations of G. For convenience, we choose an irreducible representation for each such class, and identify $\hat{G}$ with the set of these representations. Recall the Fourier transform

$$(2) \qquad \hat{f}(\pi) = \pi(f) = \int_G \pi(x) f(x) \, dx \qquad (\pi \in \hat{G}),$$

where f is any absolutely integrable function on G with respect to the Haar measure dx. Thus each $\pi(f)$ is a linear map on the finite dimensional Hilbert space $\mathcal{H}_\pi$, where the representation π is realized. Let $\mathcal{H}_\pi^H = \{v \in \mathcal{H}_\pi : \pi(h)v = v, \ h \in H\}$ be the subspace of H-fixed vectors. Let $(G/H)\hat{} = \{\pi \in \hat{G}; \ \mathcal{H}_\pi^H \neq 0\}$. By the Frobenius reciprocity theorem, $(G/H)\hat{}$ is the subset of $\hat{G}$, consisting of representations which occur in $L^2(G/H)$ (see [K, 8.4]).

Proposition 3. *A function $f \in C(H\backslash G/H)$ is strictly positive definite if $\pi(f) \geq 0$ for all $\pi \in (G/H)\hat{}$, and $\pi(f)|_{\mathcal{H}_\pi^H} > 0$ for all but finitely many $\pi \in (G/H)\hat{}$.*

Here "$\pi(f) \geq 0$" means that the operator $\pi(f)$ is positive semi-definite, i.e. $(\pi(f)v, v) \geq 0$, for any $v \in \mathcal{H}_\pi$, and the statement "$\pi(f)|_{\mathcal{H}_\pi^H} > 0$" means that the restriction of the operator $\pi(f)$ to $\mathcal{H}_\pi^H$ is positive definite, i.e. $(\pi(f)v, v) > 0$, for any $v \in \mathcal{H}_\pi^H \setminus \{0\}$.

Proof. With the notation (1), let μ be a bounded measure on G defined by

$$\int_G \phi(x)\, d\mu(x) = \sum_{i=1}^n c_i \int_H \phi(x_i h)\, dh,$$

where ϕ is a continuous function on G, and dh is the Haar measure on H, normalized so that the total measure of H is 1. Notice that the support of μ is contained in $\bigcup_{i=1}^n x_i H$, which (by our assumption) is a proper subset of G.

The Fourier transform of μ is defined as in (2):

$$\pi(\mu) = \int_G \pi(x) d\mu(x) \qquad (\pi \in \hat{G}).$$

Recall the Fourier inversion formula ([K, 12.2] or [H-R, 27.40]):

$$\phi(x) = \sum_{\pi \in \hat{G}} d(\pi) tr(\pi(\phi)\pi(x)^*) \quad (\phi \in C(G),\ x \in G),$$

where $d(\pi) = dim\, \mathcal{H}_\pi$. If each operator $\pi(\phi)$ is positive semi-definite ($\pi(\phi) \geq 0$), then the above Fourier series is absolutely, and hence uniformly, convergent to the continuous function ϕ (see [H-R, (34.9)]). Hence,

$$f(x_i^{-1} x_j) = \sum_{\pi \in \hat{G}} d(\pi) tr(\pi(f)\pi(x_i^{-1} x_j)^*)$$
$$= \sum_{\pi \in \hat{G}} d(\pi) tr(\pi(x_i)\pi(f)\pi(x_j)^*).$$

Notice that $\pi(h)\pi(f) = \pi(f) = \pi(f)\pi(h)$ for all $h \in H$. Hence $\pi(f) = P_{\mathcal{H}_\pi^H}\pi(f) = \pi(f)P_{\mathcal{H}_\pi^H}$, where $P_{\mathcal{H}_\pi^H} = \int_H \pi(h)\, dh$ is the orthogonal projection on $\mathcal{H}_\pi^H$. Therefore,

$$(4) \qquad f(x_i^{-1} x_j) = \sum_{\pi \in (G/H)\hat{}} d(\pi) tr(\pi(x_i) P_{\mathcal{H}_\pi^H} \pi(f)(\pi(x_j) P_{\mathcal{H}_\pi^H})^*).$$

After multiplying both sides of (4) by $c_i \overline{c}_j$ and summing over the indices i, j, we see that

$$\sum_{i,j=1}^n c_i \overline{c}_j f(x_i^{-1} x_j) = \sum_{\pi \in (G/H)\hat{}} d(\pi) tr(\pi(\mu) P_{\mathcal{H}_\pi^H} \pi(f)(\pi(\mu) P_{\mathcal{H}_\pi^H})^*).$$

Since $\pi(\mu) = \pi(\mu) P_{\mathcal{H}_\pi^H}$, the proposition shall follow as soon as we check that

$$(5) \qquad\qquad \pi(\mu) \neq 0 \text{ for infinitely many } \pi \in (G/H)\hat{}.$$

Suppose $\mu(\pi) = 0$ for all but finitely many $\pi \in (G/H)\hat{}$. Then, by Fourier inversion, [H-R, 27.40], μ coincides with a finite linear combination of matrix coefficients of irreducible unitary representations of G. Therefore, the space of all left translates of μ is finite dimensional. Hence, there are finitely many elements $y_1, y_2, ..., y_m \in G$, such that $\bigcup_{j=1}^m supp\, \mu = G$, where $supp\, \mu$ stands for the support of μ. Thus $G = \bigcup_{j=1}^m \bigcup_{i=1}^n y_j x_i H$, contrary to our assumption that the set G/H is infinite. $\qquad\square$

Suppose that (G, H) is a Gelfand pair, [D, 22.6.2]. Then for each $\pi \in \hat{G}$ and each $f \in C(H\backslash G/H)$, the operator $\pi(f)$ is a constant multiple of the projection $P_{\mathcal{H}_\pi^H}$: $\pi(f) = \lambda_\pi(f) P_{\mathcal{H}_\pi^H}$.

Corollary 6. *If (G, H) is a Gelfand pair, then a function $f \in C(H\backslash G/H)$ is strictly positive definite if $\lambda_\pi(f) > 0$ for all but finitely many $\pi \in (G/H)\hat{}$.*

As is well known, [D, 20.11.4, 22.6.3], $(SO(n), SO(n-1))$ is a Gelfand pair, and the quotient space $SO(n)/SO(n-1)$ coincides with the unit sphere in $\mathbb{R}^n$. Furthermore, in this case, the scalars $\lambda_\pi(f)$ can be expressed in terms of Legendre polynomials. Hence Corollary 6 generalizes a result of Schreiner, [S, Theorem 5.1]. We explain this in some detail.

Consider the Euclidean space $\mathbb{R}^n$ equipped with the usual dot product. Let $e_n = (0, 0, ..., 0, 1) \in \mathbb{R}^n$ be the "north pole" of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. Let $G = SO(n)$ be the group of isometries of the dot product, and let $H \subseteq G$ be the stabilizer of e_n. The restriction to the first $n-1$ coordinates identifies H with $SO(n-1)$.

It is customary, in representation theory, to normalize the Haar measure on any compact group to have the total mass 1. Then the total mass of our homogeneous space G/H is also 1. On the other hand the dot product in $\mathbb{R}^n$ forces a normalization of the rotation invariant measure ω on the sphere S^{n-1} so that the area of the sphere $|S^{n-1}| = \int_{S^{n-1}} d\omega \neq 1$. Hence,

$$\int_{S^{n-1}} \phi(\sigma)\, d\omega(\sigma) = |S^{n-1}| \int_G \phi(ge_n)\, dg \qquad (\phi \in C(S^{n-1})).$$

Therefore the map

$$L^2(G/H) \ni v \to \phi \in L^2(S^{n-1}), \quad \phi(ge_n) = |S^{n-1}|^{1/2} v(g) \qquad (g \in G)$$

is an isometry. This isometry is surjective and commutes with the action of the group G on both spaces. As is well known, [M], the space $L^2(S^{n-1})$ decomposes into a direct sum of irreducible subspaces:

$$L^2(S^{n-1}) = \sum_{m=0}^\infty L^2(S^{n-1})_m,$$

where $L^2(S^{n-1})_m$ stands for the space of spherical harmonics of degree m. Let π_m denote the representation of G on the space $\mathcal{H}_{\pi_m} = L^2(S^{n-1})_m$. Then the subspace $\mathcal{H}_{\pi_m}^H$ is one dimensional, and is spanned by the Legendre function:

$$L_m(\sigma) = P_m(e_n \cdot \sigma) \qquad (\sigma \in S^{n-1}),$$

where P_m is the Legendre polynomial of degree m (see [M, Lemma 2, page 16]). Notice that a function $f \in C(H\backslash G/H)$ is uniquely determined by a continuous function F on the interval $[-1, 1]$ by the formula

$$f(g) = F(e_n \cdot ge_n) \qquad (g \in G).$$

Notice that $f(g) = F(e_n \cdot ge_n) = F(g^{-1}e_n \cdot e_n) = F(e_n \cdot g^{-1}e_n) = f(g^{-1})$. In these terms, the constants which occur in Corollary 6 can be calculated as follows (see

[M, Lemma 1 on page 15, and formula (§4.6) on page 29]):

$$\lambda_{\pi_m}(f) = \lambda_{\pi_m}(f)L_m(e_n) = \pi_m(f)L_m(e_n)$$

$$= \int_G f(g)L_m(g^{-1}e_n)\, dg = \int_G F(e_n \cdot ge_n)P_m(e_n \cdot g^{-1}e_n)\, dg$$

$$= \int_G F(e_n \cdot ge_n)P_m(e_n \cdot ge_n)\, dg = \frac{|S^{n-2}|}{|S^{n-1}|}\int_{-1}^{1} F(t)P_m(t)(1-t^2)^{(n-3)/2}\, dt.$$

Thus, by Corollary 6, the integral kernel operator on $L^2(S^{n-1})$ corresponding to the integral kernel $F(\eta,\sigma) = F(\eta \cdot \sigma)$, $(\eta,\sigma \in S^{n-1})$ is strictly positive definite if $\lambda_{\pi_m} \geq 0$ for all $m \geq 0$, and the inequality is strict for all but finitely many m. This is the content of Theorem 5.1 of Schreiner, [S].

Acknowledgement

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