MATH. 2443, HOMEWORK, EXAMS AND SOLUTIONS, SPRING 2023

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Contents

1. Review of calculus 1,2, 3

Given a rational number $\frac{m}{n}$ perform all cancellations between m and n. Then write m and n as products of primes

$$m = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}\,, \quad m = q_1^{b_1} q_2^{b_2} ... q_s^{b_s}\,.$$

Set

$$F(\frac{m}{n}) = p_1^{2a_1} p_2^{2a_2} ... p_r^{2a_r} \cdot q_1^{2b_1-1} q_2^{2b_2-1} ... q_s^{2b_s-1}\,.$$

This defines a function from the set $\mathbb{Q}^+$ of positive rational numbers to the set $\mathbb{Z}^+$ of positive integers.

Homework 1, due 8/30/2023.

1. Compute $\mathbb{F}(\frac{36}{30375})$. You may use any calculator or wolframalpha.com

Since 36=$2^2 3^2$ and $3375 = 3^3 5^3$ we see that

$$F(\frac{36}{3375}) = F(\frac{2^2 3^2}{3^3 5^3}) = 2^4 3^4 3^5 5^5 = 984150000\,.$$

2. Find the rational number x such that $\mathbb{F}(x) = 2023$. You may use any calculator or wolframalpha.com

Since 2023=$17^2 7 = 17^{2\cdot 1} 7^{2\cdot 1-1}$ we see that

$$2023 = F(\frac{17}{7})\,.$$

Date: September 13, 2023.

We checked in class that
$$F:\mathbb{Q}^+\to\mathbb{Z}^+$$
is a bijective function. This shows that there are as many positive integers as there are positive rational numbers.

3. Find a bijective function
$$f:\mathbb{Q}\to\mathbb{Z}$$
from ALL the rational numbers onto the set of ALL the integers.

Let F be as defined above. The
$$f(x)=\begin{cases} F(x) & \text{if} \quad x>0\,,\\ 0 & \text{if} \quad x=0\,,\\ -F(-x) & \text{if} \quad x<0\,. \end{cases}$$
does the job.

Recall the geometric series: for $0\le r<1$,
$$1+r+r^2+r^3+...=\frac{1}{1-r}\,.$$
This is used in the decimal expression of real numbers. For example
$$0.2340000=2\cdot\frac{1}{10}+3\cdot\left(\frac{1}{10}\right)^2+4\cdot\left(\frac{1}{10}\right)^3+0\cdot\left(\frac{1}{10}\right)^4+....$$

4. Is it true that $0.9999..=1$? Why?

Yes, because
$$\begin{aligned} 0.9999..=9\cdot\frac{1}{10}+9\cdot\left(\frac{1}{10}\right)^2+9\cdot\left(\frac{1}{10}\right)^3+...&=\frac{9}{10}\cdot\left(1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+..\right)\\ &=\frac{9}{10}\cdot\frac{1}{1-\frac{1}{10}}=1\,. \end{aligned}$$

5. Take a wild guess: are there more points on the plane than there are points on the real line, or not? (No grading - just enjoy it)

There is a bjiective function between the two sets, but the proof is not easy.

Homework 2, due 9/6/2023.

6. Get familiar with wolframalpha.com. Plot some graphs, do some computations, enjoy it. You'll find this program helpful.

7. Solve for x

$$e^{2x} - e^x - 6 = 0.$$

Let $u = e^x$. First we solve the quadratic equation

$$u^2 - u - 6 = 0.$$

The solutions are $u = 3$ and $u = -2$. There is no real number x such that $e^x = -2$, but the equation

$$e^x = 3$$

has a real solution $x = ln(3)$.

8. Solve for x

$$ln(ln(x)) = 1.$$

Since e is the unique positive real number such that

$$ln(e) = 1\,,$$

we see that

$$ln(x) = e\,.$$

Hence

$$x = \exp(ln(x)) = \exp(e) = e^e\,.$$

9. Use a program like wolframalpha.com to sketch the graphs of the functions

$$y = e^{|x|}, \quad y = ln(|x|), \quad y = x\,ln(x).$$

Type plot $e^{|x|}$ $, y = ln(|x|),\ y = x * ln(x)$.

10. On June 9, 2021, representative Louie Gohmert of Texas suggested that the US forest service and bureau of land management change the earth's orbit in order to prevent climate change.

https://www.nbcnews.com/video/rep-louie-gohmert-suggests-fighting-climate-change-by-altering-moon-s-orbit-114567237716

Compute how much energy would be required to move the earth one mile away from the sun. How far would the Earth move away from the sun if we used the total yield of the US nuclear arsenal as efficiently as possible.

Let G denote the gravitational constant, M the mass of sun, m the mass of earth and r

the average distance from earth to sun. The energy required to move the earth a distance d away from the sun is

$$GMm(\frac{1}{r} - \frac{1}{r+d}) = (GMmd)/(r(r+d))\,.$$

If d is one mile, then in terms of numbers that can be found on the internet the above is equal to

$$(6.67408{\cdot}10^{-11}{\cdot}1.9891{\cdot}10^{30}{\cdot}5.972{\cdot}10^{24}{\cdot}1600.9)/(93.693{\cdot}10^{6}{\cdot}(93.693{\cdot}10^{6}{+}1600.9){\cdot}4.184{*}10^{15})\,,$$

megatons. The factor $4.184{*}10^{15}$ translates Jules to megatons. According to wolframalpha the above number is equal to

$$3.45556 \cdot 10^{16} \quad \mathrm{Mt}\,.$$

This is the energy required to move the earth one mile away from the sun, neglecting all the technical problems which might increase this number.

The total energy in the US nuclear arsenal is less than

$$\begin{aligned}&\text{number of nuclear weapons} \times \text{energy in the most powerful nuclear bomb}\\ &\qquad = 3800 \times 1.2 \text{ megatons} = 4560 \text{ Mt} = 4.560 \cdot 10^3 \text{ Mt.}\end{aligned}$$

Hence we do not have enough power to move the Earth a mile away from the sun. Not even an inch.

Exercise X on page P below refers to the book "Calculus" by Gilbert Strang:

https://ocw.mit.edu/ans7870/resources/Strang/Edited/Calculus/Calculus.pdf

11. Exercise 8, page 324.

The length is equal to

$$\int_0^1 \sqrt{1 + \left(\frac{d}{dx}x^2\right)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx = \frac{1}{4}(2\sqrt{5} + \sinh^{-1}(2)) \approx 1.48\,.$$

12. Exercise 2, page 327.

Since $y = x^3$, we have

$$\left(\frac{dy}{dz}\right)^2 = \left(3x^2\right)^2 = 9x^4\,.$$

Hence the surface area is

$$\int_0^1 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi x^3\sqrt{1+9x^4}\,dx = \frac{\pi}{27}(9x^4+1)^{\frac{3}{2}}|_0^1 = \frac{\pi}{27}10^{\frac{3}{2}} - \frac{\pi}{27}\,.$$

13. Let S be the surface obtained by rotating the graph of the function $\cosh(x)$ for $-1 \leq x \leq 1$ around the x-axis. Compute the area or S.

The area of S is equal to

$$2\pi \int_{-1}^{1} \cosh(x)\sqrt{1 + (\cosh'(x))^2}\, dx\,.$$

This is equal to

$$2\pi \int_{-1}^{1} \cosh(x)\sqrt{1 + (\sinh(x))^2}\, dx = 2\pi \int_{-1}^{1} \cosh(x)\sqrt{(\cosh(x))^2 - (\sinh(x))^2 + (\sinh(x))^2}\, dx$$

$$= 2\pi \int_{-1}^{1} \cosh(x)\sqrt{(\cosh(x))^2}\, dx = 2\pi \int_{-1}^{1} (\cosh(x))^2\, dx$$

$$= 2\pi \int_{-1}^{1} \frac{e^{2x} + 2 + e^{-2x}}{4}\, dx = 2\pi\Big(\frac{\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}}{4}\Big|_{-1}^{1}\Big) = 2\pi\frac{e^2 + 4 - e^{-2}}{4}$$

14. Let S be the surface obtained by rotating the graph of the function $\cos(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ around the x-axis. Compute the volume enclosed by S.

The volume is equal to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\cos^2(x)\, dx = \pi\frac{1}{2}(x + \sin(x)\cos(x))\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi^2}{2}\,.$$

2. **Some Linear Algebra**

Here we follow sections 11.1, 11.2 in Gilbert's book.

Homework 3, due 9/13/2023.

15. Exercise 6, page 405 in the text.

There are many such examples. Here is one:

$$(0, 0, 1)\,, \quad (1, -1, 0)\,.$$

Both are perpendicular to $(1, 1, 0)$ and to each other.

16. Exercise 11, page 405 in the text.

Statement 1 is false. Vectors

$$u = (\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}))\,, \quad v = (\cos(\frac{\pi}{3}), -\sin(\frac{\pi}{3}))$$

make the angle 30 degrees with the vector $w = (1, 0)$, but the sum of them is equal to

$$u + v = (2\cos(\frac{\pi}{3}), 0)\,,$$

which makes angle zero with $w = (1, 0)$.

Statement 2 is true. Two nonzero vectors make an angle of ninety degrees means that their dot product is zero. Thus the assumption is that $u \cdot w = 0$ and $v \cdot w = 0$. Hence

$$(u + v) \cdot w = u \cdot w + v \cdot w = 0\,.$$

Statement 3 is false. Let $u = (1, 0, 0)$, $v = (1, 1, 0)$ and $w = (0, 0, 1)$. Then $u \cdot w = 0$, $v \cdot w = 0$ but $u \cdot v = 1 \neq 0$.

17. Exercise 43, page 406 in the text.

Statement (a) is false. Let $W = aV$ for some number a Then

$$|V + W|^2 = |V + aV|^2 = (1 + a)^2|V|^2$$

and

$$|V|^2 + |W|^2 = (1 + a^2)|V|^2\,.$$

Since, for $a > 0$,

$$(1 + a)^2 = 1 + 2a + a^2 > 1 + a^2\,,$$

we see that in our example for $V \neq 0$,

$$|V + W|^2 > |V|^2 + |W|^2\,.$$

Statement (b) is true. Since we are talking about a real triangle, we have $V \neq 0$ and $W \neq 0$. The equality

$$|V + W| = |V| + |W|$$

is equivalent to

$$|V + W|^2 = (|V| + |W|)^2\,,$$

which means that

$$V \cdot V + 2\mathsf{V} \cdot W + W \cdot W = |V|^2 + 2|\mathsf{V}| \cdot |W| + |W|^2\,.$$

This is equivalent to

$$\mathsf{V} \cdot W = |\mathsf{V}| \cdot |W|\,,$$

which means that the cosine of the angle between V and W is 1. Hence W is a multiple of V. Therefore they can't form a real triangle.

Statement (c) is obviously true.

Statement (d) is false. The vectors perpendicular to $(1, 1, 1)$ form a plane, hence they can't be on one line.

18. Exercise 10, page 414 in the text.

The equation is

$$0 = (1, 1, 1) \cdot (x - x_0, y - y_0, z - z_0)\,.$$

Equivalently

$$x + y + z = x_0 + y_0 + z_0 \,.$$

19. Exercise 36, page 415 in the text.

The vectors orthogonal to these two panes are $(1,1,5)$ and $(3,2,1)$. They are not parallel to each other. Hence the planes have a non-empty intersection. Therefore the distance between them is zero. In fact the point

$$(0, -\frac{2}{9}, \frac{13}{9})$$

is in the intersection of the two planes.

Homework 4, due 9/20/2023.

20. Exercise 10, page 423 in the text.

21. Exercise 24, page 424 in the text.

22. Compute the area of the triangle spanned by the three points $(1,2,3)$, $(3,4,5)$, $(6,7,10)$ in space

3. **Motion along a curve**

Here we follow sections 12.1, 12.2 and 12.4 in the text.

23. Exercise 20, page 452 in the text.

24. Exercise 24, page 452 in the text.

25. Exercise 46, page 453 in the text.

26. Use wolframalpha to plot the curve of problem 25.

27. Is there a velocity $v = (a,b,c)$ with $|v| = 1$ such that the line $R(t) = tv$ will ever meet the line $R(t) = (1,1,1) - t(1,2,3)$?