What is Howe correspondence?

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Thematic lectures

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Lecture 1:

The Weil representation of the metaplectic group

- The Schrödinger model
- The Robinson-Rawnsley model (on the Bargmann-Segal space)

O_2 commutes with the Fourier transform

Fourier transform: $\mathcal{F}v(x) := \int_{\mathbb{R}^2} e^{2\pi i x\cdot x'} v(x')\, dx' \qquad (v \in \mathcal{S}(\mathbb{R}^2))$

O_2-action on $\mathrm{L}^2(\mathbb{R}^2)$: $\omega(g)v(x) := v(g^{-1}x) \qquad (g \in \mathrm{O}_2)$

They commute: $\mathcal{F}\omega(g) = \omega(g)\mathcal{F} \qquad (g \in \mathrm{O}_2)$

Isotypic decomposition: $\mathrm{L}^2(\mathbb{R}^2) = \sum_{\rho \neq \det} \mathrm{L}^2(\mathbb{R}^2)_\rho$

Each $\mathcal{F}|_{\mathrm{L}^2(\mathbb{R}^2)_\rho}$ is described as an integral kernel operator in "Harmonic Analysis on Euclidean Spaces" by E. Stein and G. Weiss, 1971.

Hermite functions on $\mathbb{R}^2$

$$v_\beta(x) := P_{\beta_1}(x_1)P_{\beta_2}(x_2)e^{-\frac{\pi}{2}(x_1^2+x_2^2)}\,, \qquad \beta = (\beta_1, \beta_2) \in \mathbb{Z}^2_{\geq 0}$$

Eigenvectors for $\mathcal{F}$: $\quad \mathcal{F}v_\beta = \left(e^{i\frac{\pi}{2}}\right)^{|\beta|} v_\beta\,, \quad$ where $\quad |\beta| = \beta_1 + \beta_2$

Then $\quad \mathcal{S}(\mathbb{R}^2) = \sum_{d=0}^{\infty} \mathcal{S}(\mathbb{R}^2)_d\,, \quad$ where $\quad \mathcal{S}(\mathbb{R}^2)_d := \sum_{|\beta|=d} \mathbb{C}v_\beta.$

Hence, the diagonalization

$$\mathcal{F} = \sum_{d=0}^{\infty} \left(e^{i\frac{\pi}{2}}\right)^d I_{\mathcal{S}(\mathbb{R}^2)_d}$$

$\mathcal{F}$ is part of a one-parameter family of operators

$$\mathcal{F}_{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}} := \sum_{d=0}^{\infty} \left(e^{i\theta}\right)^d I_{\mathcal{S}(\mathbb{R}^2)_d}\,, \qquad \text{e.g.} \quad \mathcal{F}_{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}} = \mathcal{F}.$$

These operators commute with the SO_2-action.

O_2 also commutes with dilations and Gaussian multipliers

dilations: $\omega(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})v(x) := a^{-\frac{1}{2}} v(a^{-1}x)$

Gaussian multipliers: $\omega(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix})v(x) := e^{i\pi n(x_1^2+x_2^2)} v(x)$.

Altogether, O_2 commutes with the actions of the groups

$$\mathrm{K} = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\ \theta \in \mathbb{R}\},$$

$$\mathrm{A} = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix};\ a > 0\},$$

$$\mathrm{N} = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix};\ n \in \mathbb{R}\},$$

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Is there anything behind this?

Chomolungma

Gaussians and Weil factors on $\mathbb{R}$

Let dx denote the usual Lebesgue measure on $\mathbb{R}$.

Let $\chi(r) := e^{2\pi i r}$, $r \in \mathbb{R}$, and define

$$\gamma(a) := \lim_{b \to 0+} \int_{\mathbb{R}} \chi(\tfrac{1}{2}(a + ib)x^2)\, dx = |a|^{-\frac{1}{2}} \gamma_W(a)\,,$$

where

$$\gamma_W(a) := e^{\frac{\pi i}{4}\operatorname{sgn}(a)} \qquad (a \in \mathbb{R} \setminus \{0\})$$

is the Weil factor.

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Gaussians and Weil factors on a vector space

U finite dimensional vector space over $\mathbb{R}$ with Lebesgue measure μ_{U};
q a nondegenerate quadratic form on U.
Define

$$\gamma(q) := \lim_{p \to 0} \int_{\mathsf{U}} \chi(\tfrac{1}{2}(q + ip)(u))\, d\mu_{\mathsf{U}}(u)\,,$$

$$\gamma_W(q) := \frac{\gamma(q)}{|\gamma(q)|} = \chi(\tfrac{1}{4}\mathrm{sgn}(q))\,.$$

Back to Lie groups

$(\mathsf{W}, \langle\cdot,\cdot\rangle)$ a symplectic space;

Symplectic group:
$\mathrm{Sp} = \mathrm{Sp}(\mathsf{W}) = \{g \in \mathrm{End}(\mathsf{W});\ \langle gw, gw'\rangle = \langle w, w'\rangle, \forall w, w' \in \mathsf{W}\}$.

Symplectic Lie algebra:
$\mathfrak{sp} = \mathfrak{sp}(\mathsf{W}) = \{x \in \mathrm{End}(\mathsf{W});\ \langle xw, w'\rangle = -\langle w, xw'\rangle, \forall w, w' \in \mathsf{W}\}$.

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Determinants

Pick $J \in \mathfrak{sp}$ such that $J^2 = -I$ and let $B(\cdot,\cdot) := \langle J\cdot,\cdot\rangle > 0$.

Define

$$\det(g - 1 : \mathsf{W}/\,\mathrm{Ker}\,(g-1) \to (g-1)\mathsf{W}) := \det(\langle (g-1)w_i, w_j\rangle_{1\leq i,j\leq m}),$$

where $w_1, \ldots, w_m$ is any B-orthonormal basis of $\mathrm{Ker}\,(g-1)^{\perp_B} \subseteq \mathsf{W}$.

The Metaplectic Group

$$\gamma(a) := |a|^{-\frac{1}{2}} e^{\frac{\pi i}{4}\operatorname{sgn}(a)} \quad (a \in \mathbb{R}^\times)$$

For $g, g_1, g_2 \in \mathrm{Sp}$, let

$$\Theta^2(g) := \gamma(1)^{2\dim(g-1)\mathrm{W}-2}\left[\gamma\Big(\det(g-1 : \mathrm{W}/\operatorname{Ker}(g-1) \to (g-1)\mathrm{W})\Big)\right]^2$$

$$C(g_1, g_2) := \sqrt{\left|\frac{\Theta^2(g_1 g_2)}{\Theta^2(g_1)\Theta^2(g_2)}\right|}\,\gamma_W(q_{g_1,g_2}),$$

where

$$q_{g_1,g_2}(u', u'') := \frac{1}{2}\langle (g_1+1)(g_1-1)^{-1}u', u''\rangle + \frac{1}{2}\langle (g_2+1)(g_2-1)^{-1}u', u''\rangle \quad (u', u'' \in (g_1-1)\mathrm{W} \cap (g_2-1)\mathrm{W}).$$

The Metaplectic Group

$$\widetilde{\mathrm{Sp}} := \left\{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C},\ \ \xi^2 = \Theta^2(g)\right\}$$

$$(g_1, \xi_1)(g_2, \xi_2) := (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2)).$$

Normalization of Haar measures on vector spaces

Recall the positive definite form $B(\cdot,\cdot) = \langle J\cdot,\cdot\rangle$.

For any subspace $\mathsf{U} \subseteq \mathsf{W}$ we normalize the Haar measure μ_{U} on U so that the volume of the unit cube with respect to form B is 1.

If $\mathsf{V} \subseteq \mathsf{U}$ is a subspace, then B induces a positive definite form on the quotient U/V and hence a normalized Haar measure $\mu_{\mathsf{U}/\mathsf{V}}$ so that the volume of the unit cube is 1.

The Weil representation of $\widetilde{\mathrm{Sp}}$ (Schrödinger model)

$\mathsf{W} = \mathrm{X} \oplus \mathrm{Y}$ a complete polarization. We shall assume that $J\mathrm{X} = \mathrm{Y}$.

$\mathrm{Op} : \mathcal{S}'(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}'(\mathrm{X}))$

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x, x')v(x')\, d\mu_{\mathrm{X}}(x').$$

Weyl transform $\mathcal{K} : \mathcal{S}'(\mathsf{W}) \to \mathcal{S}'(\mathrm{X} \times \mathrm{X})$

$$\mathcal{K}(f)(x, x') = \int_{\mathrm{Y}} f(x - x' + y)\chi\big(\tfrac{1}{2}\langle y, x + x'\rangle\big)\, d\mu_{\mathrm{Y}}(y).$$

An imaginary Gaussian on the subspace $(g-1)\mathsf{W}$ of W:

$$\chi_{c(g)}(u) = \chi\big(\tfrac{1}{4}\langle \underbrace{(g+1)(g-1)^{-1}}_{c(g)} u, u\rangle\big) \qquad (u = (g-1)w,\ w \in \mathsf{W}).$$

For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}$ define

$$\Theta(\tilde{g}) = \xi, \qquad T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathsf{W}}, \qquad \omega(\tilde{g}) = \mathrm{Op} \circ \mathcal{K} \circ T(\tilde{g}).$$

Then $T : \widetilde{\mathrm{Sp}} \to \mathcal{S}'(\mathsf{W})$ is an injective homeomorphism.

$(\omega, \mathrm{L}^2(\mathrm{X}))$ is the Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ.

The Weil representation of $\mathrm{H}(\mathrm{W})$ (Schrödinger model)

The Heisenberg group:

$$\mathrm{H}(\mathrm{W}) = \mathrm{W} \times \mathbb{R}$$
$$(w, r)(w', r') := (w + w', r + r' + \tfrac{1}{2}\langle w, w' \rangle) .$$

Set

$$T(w, r) = \chi(r)\delta_w \qquad ((w, r) \in \mathrm{H}(\mathrm{W})).$$

Then

$$T : \mathrm{H}(\mathrm{W}) \to \mathcal{S}'(\mathrm{W})$$

is an injective homeomorphism.

Set $\omega := \mathrm{Op} \circ \mathcal{K} \circ T$.

$(\omega, \mathrm{L}^2(\mathrm{X}))$ is the Weil representation of $\mathrm{H}(\mathrm{W})$ with central character χ.

Explicitly, for $v \in \mathrm{L}^2(\mathrm{X})$ and $x \in \mathrm{X}$,

$$\omega(x_0, r)v(x) = \chi(r)v(x - x_0) \quad (x_0 \in \mathrm{X}, r \in \mathbb{R}) ,$$
$$\omega(y_0, r)v(x) = \chi(r)\chi(\langle y_0, x \rangle)v(x) \quad (y_0 \in \mathrm{Y}, r \in \mathbb{R}) .$$

Weil representation of $\widetilde{\mathrm{Sp}} \ltimes \mathrm{H}(\mathsf{W})$ (Schrödinger model)

Twisted convolution $\natural$:

$$\psi\natural\phi(w) = \int_{\mathsf{W}} \psi(u)\phi(w-u)\chi(\tfrac{1}{2}\langle u, w\rangle)\, d\mu_{\mathsf{W}}(u) \qquad (w \in \mathsf{W}).$$

Since the metaplectic group acts on the Heisenberg group via automorphisms

$$\tilde{g}(w, r) = (gw, r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W}), (w, r) \in \mathrm{H}(\mathsf{W})),$$

we have the semidirect product $\widetilde{\mathrm{Sp}}(\mathsf{W}) \ltimes \mathrm{H}(\mathsf{W})$, which we embed into the space of the tempered distributions by

$$T(\tilde{g}, (w, r)) = T(\tilde{g})\natural T(w, r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W}), (w, r) \in \mathrm{H}(\mathsf{W})).$$

Theorem

Let $\omega = \mathrm{Op} \circ \mathcal{K} \circ T$. *Then*

$$\omega\colon \widetilde{\mathrm{Sp}}(\mathsf{W}) \ltimes \mathrm{H}(\mathsf{W}) \to \mathrm{U}(\mathrm{L}^2(\mathrm{X}))$$

is an injective group homomorphism. For each $v \in \mathrm{L}^2(\mathrm{X})$, *the map*

$$\widetilde{\mathrm{Sp}}(\mathsf{W}) \ltimes \mathrm{H}(\mathsf{W}) \ni \tilde{g} \to \omega(\tilde{g})v \in \mathrm{L}^2(\mathrm{X})$$

is continuous. Hence $(\omega, \mathrm{L}^2(\mathrm{X}))$ *is a unitary representation of* $\widetilde{\mathrm{Sp}}(\mathsf{W}) \ltimes \mathrm{H}(\mathsf{W})$.

Moreover,

$$\omega(\tilde{g})\omega(w,r)\omega(\tilde{g}^{-1}) = \omega(gw,r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W}),\ (w,r) \in \mathrm{H}(\mathsf{W})).$$

The action of ω *extends to* $\mathcal{S}'(\mathrm{X})$ *and the above formula holds with* $\mathrm{L}^2(\mathrm{X})$ *replaced by* $\mathcal{S}'(\mathrm{X})$. *In particular,* $\omega(\widetilde{\mathrm{Sp}}(\mathsf{W}))$ *normalizes* $d\omega(\mathfrak{h}(\mathsf{W}))$.

The Robinson-Rawnsley model (on the Bargmann-Segal space)

The formula

$$\det{}^{-1/2}\left(\tfrac{1}{2i}(x+iy)\right) := \int_{\mathrm{W}} \chi\left(\tfrac{1}{4}\langle (x+iy)w, w\rangle\right) dw$$

$$(x, y \in \mathfrak{sp}(\mathrm{W}),\ \langle y\cdot, \cdot\rangle > 0)$$

defines the reciprocal of the unique holomorphic square root of the determinant of $\frac{1}{2i}(x+iy)$ which is positive for $x = 0$. In particular

$$\lim_{y \to 0} \det{}^{-1/2}\left(\tfrac{1}{2i}(x+iy)\right) = \gamma(q_x), \quad \text{where} \quad q_x(w) = \tfrac{1}{2}\langle xw, w\rangle\,.$$

For $g \in \mathrm{Sp}(\mathrm{W})$ set

$$C(g) := \frac{1}{2}(g + JgJ^{-1}), \quad A(g) := \frac{1}{2}(g - JgJ^{-1})\,.$$

$C(g)$ commutes with J and hence preserves the eigenspaces $\mathsf{W}_{\mathbb{C},J=\pm i} \subseteq \mathsf{W}_{\mathbb{C}}$.

Lemma

For any $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W})$, $C(g)$ is invertible and

$$\left(\Theta(\tilde{g})\det{}^{-1/2}(\tfrac{1}{2i}(x+iy))\right)^2 = \left(\det(C(g)|_{\mathsf{W}_{\mathbb{C},J=-i}}\right)^{-1}$$

Hence define

$$\det(C(\tilde{g})|_{\mathsf{W}_{\mathbb{C},J=-i}})^{-1/2} = \Theta(\tilde{g})\det{}^{-1/2}(\tfrac{1}{2i}(x+iy)).$$

View the real space W as a complex vector space where $-J$ plays the role of the multiplication by $\sqrt{-1}$.
Then $(\cdot,\cdot) : \mathsf{W} \times \mathsf{W} \to \mathbb{C}$ given by

$$(w, w') := \langle Jw, w'\rangle - i\langle w, w'\rangle$$

defines a positive definite hermitian form on W.

Let $\mathcal{H}$ denote the Bargmann-Segal space, i.e. the Hilbert space of holomorphic functions $h : \mathrm{W} \to \mathbb{C}$ such that

$$\int_{\mathrm{W}} |h(w)|^2 e^{-\pi(w,w)}\, dw < \infty.$$

For $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and $h \in \mathcal{H}$ set

$$\omega_{RR}(\tilde{g})h(w) = \det(C(\tilde{g})|_{\mathrm{W}_{\mathbb{C},J=-i}})^{-1/2} \int_{\mathrm{W}} h(u) e^{-\frac{\pi}{2}(w, C(g^{-1})^{-1} A(g^{-1}) w)} \times e^{-\frac{\pi}{2}(C(g)^{-1}A(g)u,u)} e^{\pi(C(g)^{-1}w,u)} e^{-\pi(u,u)}\, du\,.$$

In particular, if $g = JgJ^{-1}$ then

$$\omega_{RR}(\tilde{g})h(w) = \det(\tilde{g}|_{\mathrm{W}_{\mathbb{C},J=-i}})^{-1/2} h(g^{-1}w)\,.$$

Theorem

The two unitary representations $(\omega, \mathrm{L}^2(\mathrm{X}))$ *and* $(\omega_{RR}, \mathcal{H})$ *of* $\widetilde{\mathrm{Sp}}(\mathrm{W})$ *are unitarily equivalent.*

Notation: We shall write ω for ω_{RR} if there is no risk of confusion.

References

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Lecture 2:

- The Fock model of the Weil representation
- Basic invariants: matrix coefficients, character and wave front set
- Reducibility of the Weil representation
- Real reductive dual pairs

Some historical remarks

- John von Neumann (1926): two hermitian operators P and Q acting irreducibly on an infinite dimensional Hilbert space $\mathcal{H}$ and satisfying the canonical commutation relations
$$PQ - QP = \frac{1}{2\pi i}\,\mathrm{id}$$
are determined up to a "rotation in $\mathcal{H}$".

  In contemporary terms, up to a unitary equivalence, there is only one infinite dimensional irreducible unitary representation ω of the Heisenberg group $H(\mathrm{W})$ with a fixed central character.

  Therefore composing ω with an automorphism of $H(\mathrm{W})$ fixing the center gives an isomorphic representation. Sp acts on $H(\mathrm{W})$ by $g\cdot(w,r) = (gw,r)$. Hence there is a unitary **projective** representation ω_{pr} of Sp such that
$$\omega(gw,r) = \omega_{pr}(g)\omega(w,r)\omega_{pr}(g^{-1}) = \qquad (g\in\mathrm{Sp},\ (w,r)\in\mathrm{H(W)}).$$

- David Shale (a student of Irving Segal) (1962): this unitary operator may be chosen up to a sign $\pm$. Hence he obtained a unitary representation of the connected double cover of the symplectic group, which realizes the automorphism via conjugation.

- Ranga Rao (1993) gave an explicit formula for the action of every element of the metaplectic group.
- Teruji Thomas (a student of Vladimir Drinfeld) (2008) computed the Weyl symbols of the operators $\omega(\tilde{g})$.
- Anne-Marie Aubert and T.P. (2014): starting with Thomas' Weyl symbol, define the operator $\omega(\tilde{g})$ explicitly and prove in the Schrödinger model that $\omega(\tilde{g}_1)\omega(\tilde{g}_2) = \omega(\tilde{g}_1\tilde{g}_2)$, without the Stone von Neumann theorem. We eliminate the ± 1 ambiguity using the distribution character Θ of ω.
- What we refer to as the Robinson-Rawnsley model is a slight variation of the classical Bargmann-Segal (-Itzykson) model. In our Robinson-Rawnsley model, the sign ambiguity is eliminated, again using Θ.

In Lecture 1 we presented the two models of ω, as in the last two items.

Recap of a part of Lecture 1

- Symplectic space $(\mathsf{W}, \langle\cdot,\cdot\rangle)$ with the complexification $\mathsf{W}_{\mathbb{C}}$,
- $J \in \mathfrak{sp} \cap \mathrm{Sp}$, $J^2 = -1$, $\mathsf{W}_{\mathbb{C},J=-i}$ $-i$-eigenspace for J,
- $\mathcal{H}$ is the Bargmann-Segal space of holomorphic functions $h : \mathsf{W} \to \mathbb{C}$ such that

$$\int_{\mathsf{W}} |h(w)|^2 e^{-\pi(w,w)}\, dw < \infty.$$

- In our Robinson-Rawnsley model of the Weil representation ω the metaplectic group $\widetilde{\mathrm{Sp}}(\mathsf{W})$ acts on $\mathcal{H}$.

  In particular, if $g = JgJ^{-1}$ then

$$\omega(\tilde{g})h(w) = \det(\tilde{g}|_{\mathsf{W}_{\mathbb{C},J=-i}})^{-1/2} h(g^{-1}w)\,.$$

- An explicit $\widetilde{\mathrm{Sp}} \ltimes \mathrm{H}(\mathrm{W})$-intertwining isometry between our Robinson-Rawnsley and Schrödinger models is

$$\mathcal{H} \ni h \to \mathrm{Op} \circ \mathcal{K}(h\chi_{iJ})v_0 \in \mathrm{L}^2(\mathrm{X}),$$

where

$$v_0(x) = 2^{\frac{1}{4}\dim \mathrm{X}} e^{-\pi(x,x)} \qquad (x \in \mathrm{X}).$$

The derived representation $d\omega$ of our Robinson-Rawnsley results in the Fock model.

The Fock model

The space $\mathcal{P}(\mathrm{W}_{\mathbb{C},J=-i})$ of polynomial functions on $\mathrm{W}_{\mathbb{C},J=-i}$ is dense in $\mathcal{H}$. Pick a basis $e_1^+, e_2^+, ..., e_n^+$ of $\mathrm{W}_{\mathbb{C},J=i}$ and a basis $e_1^-, e_2^-, ..., e_n^-$ of $\mathrm{W}_{\mathbb{C},J=-i}$ such that

$$2\pi i\langle e_j^+, e_k^-\rangle = \delta_{j,k}\,.$$

Identify

$$\mathrm{W}_{\mathbb{C},J=-i} \ni z_1 e_1^- + ... + z_n e_n^- \to (z_1, ..., z_n)^t \in \mathbb{C}^n\,.$$

Then $\mathcal{P}(\mathrm{W}_{\mathbb{C},J=-i})$ is identified with $\mathbb{C}[z_1, ..., z_n]$.

For $g \in \mathrm{Sp}(\mathrm{W})^J$, denote by $[g] \in M_n(\mathbb{C})$ the matrix of $g|_{\mathrm{W}_{\mathbb{C},J=-i}}$ with respect to the ordered basis $e_1^-, e_2^-, ..., e_n^-$. Then

$$\left(\det(\tilde{g}|_{\mathrm{W}_{\mathbb{C},J=-i}})^{-1/2}\right)^2 = \det([g])^{-1}\,.$$

Therefore we set

$$\det{}^{-1/2}([g]) = \det(\tilde{g}|_{\mathrm{W}_{\mathbb{C},J=-i}})^{-1/2}\,.$$

For $1 \leq j, k \leq n$ define the following elements of $\mathfrak{sp}(\mathrm{W})_{\mathbb{C}}$:

$$E^{+}_{j,k} : e^{-}_{j} \to e^{+}_{k}\,,\ e^{-}_{k} \to e^{+}_{j}\,,\ e^{-}_{l} \to 0 \quad (l \notin \{j,k\})$$
$$E^{-}_{j,k} : e^{+}_{j} \to e^{-}_{k}\,,\ e^{+}_{k} \to e^{-}_{j}\,,\ e^{+}_{l} \to 0 \quad (l \notin \{j,k\})\,.$$

Then by taking derivatives of ω, we obtain the following formulas

$$\begin{aligned} d\omega(E^{+}_{j,j}) &= \frac{1}{2} z_j^2 \\ d\omega(E^{+}_{j,k}) &= z_j z_k \qquad j \neq k \\ d\omega(E^{-}_{j,j}) &= -\frac{1}{2}\partial_{z_j}^2 \\ d\omega(E^{-}_{j,k}) &= -\partial_{z_j}\partial_{z_k} \qquad j \neq k \end{aligned}$$

Furthermore, for $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})^J$,

$$\omega(\tilde{g})p(z) = \det{}^{-1/2}([g])p([g]^{-1}z) \qquad (p \in \mathbb{C}[z_1, ..., z_n], z \in \mathbb{C}^n)\,.$$

This is the Fock model.

The matrix coefficients of ω

For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}$
$\Theta(\tilde{g}) = \xi$,
$T(\tilde{g}) = \Theta(\tilde{g}) \chi_{c(g)} \mu_{(g-1)\mathrm{W}}$

Set

$$\chi_x(w) := \chi(\tfrac{1}{4}\langle xw, w\rangle) \qquad (x \in \mathfrak{sp}(\mathrm{W})_{\mathbb{C}},\ w \in \mathrm{W})\,.$$

(This function was used before for $x = c(g)$.)

The scalar function

$$\Omega(\tilde{g}) := T(\tilde{g})(\chi_{iJ}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}))$$

is $\det^{-1/2}$-spherical in the sense that

$$\Omega(\tilde{k}\tilde{g}\tilde{k}') = \det(\tilde{k}|_{\mathrm{W}_{\mathbb{C},J=-i}})^{-1/2}\Omega(\tilde{g}) \det(\tilde{k}'|_{\mathrm{W}_{\mathbb{C},J=-i}})^{-1/2}$$
$$(\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}),\ \tilde{k}, \tilde{k}' \in \widetilde{\mathrm{Sp}}(\mathrm{W})^J)\,.$$

There is a seminorm q on $\mathcal{S}(\mathrm{X})\hat{\otimes}\mathcal{S}(\mathrm{X})$ such that for any $v_1, v_2 \in \mathcal{S}(\mathrm{X})$,

$$|(\omega(\tilde{g})v_1, v_2)| \leq q(v_1 \otimes v_2)|\Omega(\tilde{g})| \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}))\,.$$

Let e_1, ..., e_n be a basis of X. Set $f_j = Je_j$. Assume that

$$\langle e_j, f_k\rangle = \delta_{j,k}\,.$$

For positive numbers a_1, ..., a_n define $a \in \mathrm{End}(\mathrm{W})$ by

$$ae_j = a_j e_j\,, \quad af_j = a_j^{-1} f_j\,.$$

Then $a \in \mathrm{Sp}(\mathrm{W})$ and the set A of all such elements forms the connected identity component of a maximally split Cartan subgroup of $\mathrm{Sp}(\mathrm{W})$. In these terms

$$\Omega(\tilde{a}) = \pm \prod_{j=1}^{n} \left(\frac{2}{a_j + a_j^{-1}} \right)^{1/2} \,.$$

Therefore by the "$\widetilde{K}\widetilde{A}\widetilde{K}$" decomposition

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} |\Omega(\tilde{g})|^p \, d\tilde{g} < \infty$$

if and only if $p > 4n$.

The distribution character of ω

Theorem

For any $f \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$, the operator

$$\omega(f) = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} f(\tilde{g})\omega(\tilde{g})\, d\tilde{g}$$

is of trace class and

$$\mathrm{tr}\,\omega(f) = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} f(\tilde{g})\Theta(\tilde{g})\, d\tilde{g}$$

where the integral is absolutely convergent.

Thus the function Θ introduced in the construction of the metaplectic group and the Weil representation is the distribution character of ω.

Let

$$\mathfrak{sp}^c(\mathrm{W}) := \{x \in \mathfrak{sp}(\mathrm{W});\ det(x-1) \neq 0\}\,.$$

This is the domain of the Cayley transform $c(x) = (x+1)(x-1)^{-1}$ in $\mathfrak{sp}(\mathrm{W})$.

Similarly we have $\mathrm{Sp}^c(\mathrm{W})$ and $\widetilde{\mathrm{Sp}^c}(\mathrm{W})$.

Fix a real analytic lift $\tilde{c} : \mathfrak{sp}^c(\mathrm{W}) \to \widetilde{\mathrm{Sp}^c}(\mathrm{W})$ of $c : \mathfrak{sp}^c(\mathrm{W}) \to \mathrm{Sp}^c(\mathrm{W})$ and let $\tilde{c}_-(x) = \tilde{c}(x)\tilde{c}(0)^{-1}$. Then $\tilde{c}_-(0)$ is the identity of the metaplectic group.

Theorem

$$\Theta(\tilde{c}_-(x)) = \Theta(\tilde{c}(0)^{-1})\Theta(\tilde{c}(x)) \int_{\mathrm{W}} \chi_x(w)\,dw \qquad (x \in \mathfrak{sp}^c(\mathrm{W}))\,.$$

The wave front set of a distribution

Let V be a finite dimensional $\mathbb{R}$-vector space. Recall the Fourier transform

$$\mathcal{F}(\phi)(v^*) = \int_{\mathrm{V}} \phi(v)\chi(-v^*(v))\, d\mu_{\mathrm{V}}(v) \qquad (\phi \in C_c^\infty(\mathrm{V}), v^* \in \mathrm{V}^*)\,.$$

The wave front set of a distribution u on V at a point $v \in V$, denoted $WF_v(u)$, is the complement of the set of all pairs (v, v^*), $v^* \in \mathrm{V}^*$, for which there is a $\phi \in C_c^\infty(V)$ with $\phi(v) \neq 0$ and an open cone $\Gamma \subseteq V^*$ containing v^* such that

$$|\mathcal{F}(\phi u)(v_1^*)| \leq C_N(1 + |v_1^*|)^{-N} \qquad (v_1^* \in \Gamma,\ N = 0, 1, 2, ...)\,.$$

This notion behaves well under diffeomorphisms. So for any distribution u on a manifold M, one defines $WF(u) \subseteq T^*M$ using charts.

For an admissible representation ρ of a real reductive Lie group with distribution character Θ_ρ, define the wave front set of ρ as $WF(\rho) = WF_1(\Theta_\rho)$.

The wave front set of ω

Define the unnormalized moment map

$$\tau_{\mathfrak{sp}} : \mathsf{W} \to \mathfrak{sp}^*(\mathsf{W}), \quad \tau_{\mathfrak{sp}}(w)(x) = \langle xw, w \rangle \qquad (x \in \mathfrak{sp}(\mathsf{W}), w \in \mathsf{W}).$$

Then the integral

$$\int_{\mathsf{W}} \psi(\tfrac{1}{4}\tau_{\mathfrak{sp}}(w))\, dw \qquad (\psi \in \mathcal{S}(\mathfrak{sp}^*(\mathsf{W})))$$

defines an invariant measure $\mu_{\mathcal{O}}$ on the minimal nilpotent coadjoint orbit $\mathcal{O} = \tau_{\mathfrak{sp}}(\mathsf{W} \setminus 0)$.

$$\int_{\mathsf{W}} \chi_x(w)\, dw = \int_{\mathsf{W}} \chi(\tfrac{1}{4}\tau_{\mathfrak{sp}}(w)(x))\, dw = \int_{\mathcal{O}} \chi(\xi(x))\, \mu_{\mathcal{O}}(\xi)$$

is a Fourier transform of $\mu_{\mathcal{O}}$.

Recall that

$$\Theta(\tilde{c}_-(x)) = \Theta(\tilde{c}(0)^{-1})\Theta(\tilde{c}(x)) \int_{\mathsf{W}} \chi_x(w)\, dw \qquad (x \in \mathfrak{sp}^c(\mathsf{W}))\,.$$

This shows that modulo the lift via Cayley transform and multiplication by a real analytic function, the character Θ is a Fourier transform of $\mu_{\mathcal{O}}$. In particular

$$WF_1(\Theta) = \tau_{\mathfrak{sp}}(\mathsf{W})\,.$$

One can show that as a subset of the cotangent bundle $\widetilde{\mathrm{Sp}}(\mathsf{W}) \times \mathfrak{sp}^*(\mathsf{W})$,

$$WF(\Theta) = \{(\tilde{g}, \xi);\ \xi \in WF_1(\Theta),\ Ad_g^*(\xi) = \xi,\ \tilde{g} \in \mathrm{supp}(\Theta)\}\,.$$

Question: does the above formula hold for the character of any admissible representation of any real reductive group?

Reducibility of ω

Let $\mathrm{Z} = \{\pm 1\}$ denote the center of Sp.
The preimage $\widetilde{\mathrm{Z}} \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W})$ is the center of $\widetilde{\mathrm{Sp}}$. It acts on $\mathrm{L}^2(\mathrm{X})$ as follows

$$\omega(\tilde{z})v(x) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} v(z^{-1}x)\,.$$

Set

$$\rho_+(\tilde{z}) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|}$$

and

$$\rho_-(\tilde{z}) = \begin{cases} \rho_+(\tilde{z}) \text{ if } z = 1\,, \\ -\rho_+(\tilde{z}) \text{ if } z = -1\,. \end{cases}$$

Then both ρ_+ and ρ_- are unitary characters of $\widetilde{\mathrm{Z}}$ and we have the $\widetilde{\mathrm{Z}}$ isotypic decomposition of ω

$$\mathrm{L}^2(\mathrm{X}) = \mathrm{L}^2(\mathrm{X})_{\rho_+} \oplus \mathrm{L}^2(\mathrm{X})_{\rho_-}\,,$$

where $\mathrm{L}^2(\mathrm{X})_{\rho_+}$ consists of even functions and $\mathrm{L}^2(\mathrm{X})_{\rho_-}$ of odd functions.

Since $\{0\}$ and $W \setminus \{0\}$ are the only Sp-orbits in W,

$$\mathcal{S}'(W)^{\mathrm{Sp}} = \mathbb{C}\delta \oplus \mathbb{C}\mu_W\,.$$

Hence, via the isomorphism $\mathrm{Op} \circ \mathcal{K}$,

$$\dim \mathrm{Hom}(\mathcal{S}(X), \mathcal{S}'(X))^{\omega(\widetilde{\mathrm{Sp}}(W))} = 2\,.$$

Therefore

$$\dim \mathrm{End}(\mathrm{L}^2(X))^{\omega(\widetilde{\mathrm{Sp}}(W))} \leq 2\,.$$

Thus the spaces $\mathrm{L}^2(X)_{\rho_\pm}$ are irreducible under the action of $\widetilde{\mathrm{Sp}}$.
Denote the resulting representations of $\widetilde{\mathrm{Sp}}$ by $\rho'_\pm$.
Hence as a representation of $\widetilde{\mathrm{Z}} \times \widetilde{\mathrm{Sp}}(W)$,

$$\mathrm{L}^2(X) = \mathrm{L}^2(X)_{\rho_+ \otimes \rho'_+} \oplus \mathrm{L}^2(X)_{\rho_- \otimes \rho'_-}\,.$$

This is the decomposition of ω into the sum of two irreducibles.

We just obtained the decomposition

$$\mathrm{L}^2(\mathrm{X}) = \mathrm{L}^2(\mathrm{X})_{\rho_+\otimes\rho'_+} \oplus \mathrm{L}^2(\mathrm{X})_{\rho_-\otimes\rho'_-} .$$

The relation

$$\begin{cases} \rho_+ \longleftrightarrow \rho'_+ \\ \rho_- \longleftrightarrow \rho'_- \end{cases}$$

is our first example of Howe correspondence $\rho \leftrightarrow \rho'$ between **some** irreducible representations of $\widetilde{\mathrm{Z}} = \widetilde{\mathrm{O}_1}$ and $\widetilde{\mathrm{Sp}} = \widetilde{\mathrm{Sp}_{2n}(\mathbb{R})}$.

The groups Z and Sp are mutual centralizers in Sp and they act reductively on W.

This makes them an example of a real reductive dual pair, as we are going to see next.

Dual Pairs

Two subgroups $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathrm{W})$ form a dual pair if they act reductively on W and they are mutual centralizers in $\mathrm{Sp}(\mathrm{W})$. The dual pair $(\mathrm{G}, \mathrm{G}')$ is called irreducible if there is no non-trivial direct sum orthogonal decomposition of W preserved by both G and G'.

Below we list the irreducible pairs, up to isomorphism.

G, G'
$\mathrm{GL}_n(\mathbb{D}),\ \mathrm{GL}_m(\mathbb{D})$
$\mathrm{O}_{p,q},\ \mathrm{Sp}_{2n}(\mathbb{R})$
$\mathrm{O}_p(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$
$\mathrm{U}_{p,q},\ \mathrm{U}_{r,s}$
$\mathrm{O}^*_{2n},\ \mathrm{Sp}_{p,q}$

Here $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$ or the quaternions $\mathbb{H}$.

The preimages $\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}' \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W})$ are also mutual centralizers in $\widetilde{\mathrm{Sp}}(\mathrm{W})$.

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Lecture 3:

- The First Fundamental Theorem of the Classical Invariant Theory
- Howe's Double Commutant Theorem
- Dual pairs with one member compact
- A dual pair as a supergroup
- The Capelli homomorphism

The unnormalized moment maps

Let $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp} = \mathrm{Sp}(\mathrm{W})$ be a dual pair with Lie algebras $\mathfrak{g}$, $\mathfrak{g}'$.

Example:

$$\mathrm{W} = \mathrm{M}_{m,2n}(\mathbb{R}), \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad \langle w', w\rangle = \mathrm{tr}(w'Jw^t),$$

$$g(w) = gw\,, \quad g'(w) = wg'^{-1}.$$

This way $\mathrm{G} = \mathrm{O}_m$, $\mathrm{G}' = \mathrm{Sp}_{2n}(\mathbb{R})$ are a dual pair inside $\mathrm{Sp}(\mathrm{W})$.

Unnormalized moment maps:

$$\tau_{\mathfrak{g}} : \mathrm{W} \to \mathfrak{g}^*\,,\ \tau_{\mathfrak{g}}(w)(x) = \langle xw, w\rangle\,,$$

$$\tau_{\mathfrak{g}'} : \mathrm{W} \to \mathfrak{g}'^*\,,\ \tau_{\mathfrak{g}'}(w)(x') = \langle x'w, w\rangle \qquad (x \in \mathfrak{g}\,,\ x' \in \mathfrak{g}'\,,\ w \in \mathrm{W})\,.$$

They intertwine the group action on the symplectic space with the coadjoint action on the dual of the Lie algebra,

$$\tau_{\mathfrak{g}}(gw)(x) = \tau_{\mathfrak{g}}(w)(g^{-1}xg)\,,$$

$$\tau_{\mathfrak{g}}(g'w)(x) = \tau_{\mathfrak{g}}(w)(g'^{-1}xg') \qquad (g \in \mathrm{G}\,,\ g' \in \mathrm{G}'\,,\ w \in \mathrm{W})\,.$$

The First Fundamental Theorem of the Classical Invariant Theory (FFTCIT)

For a finite dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$, let $\mathcal{P}(V)$ denote the space of the complex valued polynomial functions.

Theorem

Let $(\mathrm{G}, \mathrm{G}')$ *be a dual pair with* G *compact. Then*

$$
\begin{aligned}
&(a) \qquad \mathcal{P}(W)^{\mathrm{G}} = \mathcal{P}(\mathfrak{g}'^*) \circ \tau_{\mathfrak{g}'}\,, \\
&(b) \qquad C^\infty(W)^{\mathrm{G}} = C^\infty(\mathfrak{g}'^*) \circ \tau_{\mathfrak{g}'}\,, \\
&(c) \qquad \mathcal{S}(W)^{\mathrm{G}} = \mathcal{S}(\mathfrak{g}'^*) \circ \tau_{\mathfrak{g}'},.
\end{aligned}
$$

(a) Howe;
(b) Howe + Schwartz;
(c) Howe + Astengo, Di Blasio, Ricci.

Howe's Double Commutant Theorem

Let $\mathcal{U}(\mathfrak{e})$ denote the universal enveloping algebra of $\mathfrak{e}$.

Theorem

For any dual pair $(\mathrm{G}, \mathrm{G}')$,

$$d\omega(\mathcal{U}(\mathfrak{h}(\mathsf{W})))^{\omega(\widetilde{\mathrm{G}})} = d\omega(\mathcal{U}(\mathfrak{g}')) .$$

Since the action by conjugation factors to G the above formula may be rewritten as

$$d\omega(\mathcal{U}(\mathfrak{h}(\mathsf{W})))^{\mathrm{G}} = d\omega(\mathcal{U}(\mathfrak{g}')) .$$

In particular, applying this equation to the dual pair $(\mathrm{Z}, \mathrm{Sp})$, we see that

$$d\omega(\mathcal{U}(\mathfrak{h}(\mathsf{W})))^{\mathrm{Z}} = d\omega(\mathcal{U}(\mathfrak{sp}(\mathsf{W}))) .$$

Since $\mathrm{Z} \subseteq \mathrm{G}$, by taking G invariants on both sides, we get

$$d\omega(\mathcal{U}(\mathfrak{sp}(\mathsf{W})))^{\mathrm{G}} = d\omega(\mathcal{U}(\mathfrak{g}')) .$$

Howe Correspondence for dual pairs $(\mathrm{G}, \mathrm{G}')$ with G compact

We realize ω in the Fock model acting on the space $\mathcal{P} = \mathcal{P}(\mathsf{W}_{\mathbb{C}, J=-i})$. Assume that $\mathrm{G} \subseteq \mathrm{Sp}(\mathsf{W})^J$.

For $\rho \in \widehat{\widetilde{\mathrm{G}}}$, let $\mathcal{P}_\rho$ denote the ρ-isotypic component.

Denote by $\mathcal{R}(\mathrm{G}, \omega) \subseteq \widehat{\widetilde{\mathrm{G}}}$ the subset of the ρ such that $\mathcal{P}_\rho \neq 0$.

Theorem

For each $\rho \in \mathcal{R}(\mathrm{G}, \omega)$, the space $\mathcal{P}_\rho$ is irreducible under the joint action of $\widetilde{\mathrm{G}}$ and $\mathfrak{g}'$. Thus there is an irreducible representation $d\rho'$ of $\mathfrak{g}'$ such that

$$\mathcal{P}_\rho = \mathcal{P}_{\rho \otimes d\rho'}$$

as a $\widetilde{\mathrm{G}} \times \mathfrak{g}'$ module. If ρ_1 is not isomorphic to ρ_2 then $d\rho'_1$ is not isomorphic to $d\rho'_2$. Furthermore

$$\mathcal{P} = \bigoplus_{\rho \in \mathcal{R}(\mathrm{G}, \omega)} \mathcal{P}_{\rho \otimes d\rho'}$$

By taking closures we obtain irreducible unitary representations ρ' of $\widetilde{\mathrm{G}}'$ such that

$$\mathcal{H} = \sum_{\rho \in \mathcal{R}(\mathrm{G},\omega)} \mathcal{H}_{\rho \otimes \rho'} \,,$$

where the sum denotes direct orthogonal sum of Hilbert spaces.

In the next few slides we'll see how to determine $R(\mathrm{G}, \omega)$ and the correspondence $\rho \longleftrightarrow \rho'$.

The above decomposition in the Schrödinger model looks as follows,

$$\mathrm{L}^2(\mathrm{X}) = \sum_{\rho \in \mathcal{R}(\mathrm{G},\omega)} \mathrm{L}^2(\mathrm{X})_{\rho \otimes \rho'} \,.$$

Harmonic polynomials

Conjugation by J is a Cartan involution on $\mathfrak{g}'$. Let

$$\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$$

be the corresponding Cartan decomposition. Set

$$\mathfrak{p}'_{\mathbb{C}}{}^{\pm} = \{Z \in \mathfrak{p}'_{\mathbb{C}};\ [J, Z] = \pm 2Z\}$$

Then we have the Harish-Chandra decomposition

$$\mathfrak{g}'_{\mathbb{C}} = \mathfrak{p}'_{\mathbb{C}}{}^{+} \oplus \mathfrak{k}'_{\mathbb{C}} \oplus \mathfrak{p}'_{\mathbb{C}}{}^{-}.$$

Set

$$\mathit{Harm}(\mathrm{G}) = \{p \in \mathcal{P};\ d\omega(\mathfrak{p}'_{\mathbb{C}}{}^{-})p = 0\}.$$

This space is $\widetilde{\mathrm{G}}$ invariant. For $\rho \in \mathcal{R}(\mathrm{G}, \omega)$, let $\mathit{Harm}(\mathrm{G})_\rho$ be the ρ isotypic component.

Theorem

The space Harm$(\mathrm{G})_\rho$ *is irreducible under the joint action of* $\widetilde{\mathrm{G}}$ *and* $\widetilde{\mathrm{K}}'$. *As a representation of* $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{K}}'$ *it is of type* $\rho \otimes \sigma'$, *where* σ' *is an irreducible representation of* $\widetilde{\mathrm{K}}'$. *Thus*

$$\mathit{Harm}(\mathrm{G})_\rho = \mathit{Harm}(\mathrm{G})_{\rho\otimes\sigma'} \,.$$

The subspace Harm$(\mathrm{G})_\rho \subseteq \mathcal{P}_\rho$ *consists of the polynomials of lowest degree. The map*

$$\mathcal{R}(\mathrm{G},\omega) \ni \rho \to \sigma' \in \mathcal{R}(\mathrm{K}',\omega)$$

is injective. As a space of polynomials

$$\mathcal{P}_{\rho\otimes\rho'} = \mathcal{P}_\rho = \mathcal{P}^{\mathrm{G}} \cdot \mathit{Harm}(\mathrm{G})_\rho \,.$$

Denote by $\deg(\sigma')$ the degree of the polynomials where *Harm*$(\mathrm{G})_{\rho\otimes\sigma'}$ occurs.

Example: $\mathrm{G} = \mathrm{O}_2$, $\mathrm{G}' = \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$

$\mathcal{P} = \mathbb{C}[z_1, z_2]$

$$\omega(g)h(z_1, z_2) = h((z_1, z_2)g) \qquad (g \in \mathrm{G}, h \in \mathcal{P})$$

$$\omega(k_\theta)h(z_1, z_2) = e^{-i\theta}h(e^{i\theta}z_1, e^{i\theta}z_2) \quad \left(k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2 \subseteq \mathrm{G}'\right)$$

$$d\omega(\mathfrak{p}'_{\mathbb{C}}{}^{-}) = \mathbb{C}(\partial_{z_1}^2 + \partial_{z_2}^2), \qquad d\omega(\mathfrak{p}'_{\mathbb{C}}{}^{+}) = \mathbb{C}(z_1^2 + z_2^2)$$

For $k = 0, 1, 2, 3, \ldots$, let ρ_k be the irreducible representation of G acting on $\mathbb{C}(z_1 + iz_2)^k + \mathbb{C}(z_1 - iz_2)^k$ by the above formula. Then ρ_0 is the trivial representation of G.

$$Harm(\mathrm{G})_{\rho_0} = \mathbb{C}, \qquad Harm(\mathrm{G})_{\rho_k} = \mathbb{C}(z_1 + iz_2)^k + \mathbb{C}(z_1 - iz_2)^k$$

$$\mathcal{P}_{\rho_0} = \mathcal{P}^{\mathrm{G}} = \mathbb{C}[z_1^2 + z_2^2],$$
$$\mathcal{P}_{\rho_k} = \mathbb{C}[z_1^2 + z_2^2](\mathbb{C}(z_1 + iz_2)^k + \mathbb{C}(z_1 - iz_2)^k), \quad k = 1, 2, 3, \ldots$$

$$\sigma'_k(k_\theta) = e^{-ik\theta}$$

The harmonic correspondence is $\rho_k \longleftrightarrow \sigma'_{k+1}$ and $\deg(\sigma'_{k+1}) = k$

Decay of matrix coefficients

Let $\mathfrak{t}' \subseteq \mathfrak{k}'$ be a Cartan subalgebra. Fix a Borel subalgebras $\mathfrak{b}' \subseteq \mathfrak{k}'_{\mathbb{C}}$ containing $\mathfrak{t}'$. Then $\mathfrak{b}' \oplus \mathfrak{p}'_{\mathbb{C}}{}^{-}$ is Borel subalgebra of $\mathfrak{g}'_{\mathbb{C}}$. In these terms $d\rho'$ is a highest weight representation with highest weight $\lambda_{\rho'} \in \mathfrak{t}'_{\mathbb{C}}{}^{*}$.

There is a maximally split Cartan subalgebra of $\mathfrak{g}'$ with the split part $\mathfrak{a}'$ and a Cayley transform

$$C : \mathfrak{a}' \to i\mathfrak{t}' .$$

Example:
For the Lie algebra $\mathfrak{sp}_2(\mathbb{R})$

$$C : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

up to a sign.

Denote by $W(\mathfrak{a}')$ the Weyl group of $\mathfrak{a}'$.

The following theorem describes the decay of matrix coefficients of ρ', which are generally better than those coming from ω.

Theorem

There is a seminorm q on $\mathcal{S}(\mathrm{X}) \times \mathcal{S}(\mathrm{X})$ such that

$$|(\omega(\exp(x))u, v)| \leq q(u, v) \min_{s \in W(\mathfrak{a}')} e^{-|\lambda_{\rho'}(C(sx))|} \qquad (x \in \mathfrak{a}')\,.$$

The distribution character and the wave front set of ρ'

Denote by Θ_ρ the character of ρ and similarly for ρ'. Set

$$f_{\rho\otimes\rho'} = \int_{\widetilde{\mathrm{G}}} \Theta_\rho(\tilde{g}^{-1})\,T(\tilde{g})\,d\tilde{g}\,.$$

This is a tempered distribution on W and $\mathrm{Op}\circ\mathcal{K}(f_{\rho\otimes\rho'})$ is the orthogonal projection onto $\mathrm{L}^2(\mathrm{X})_{\rho\otimes\rho'}$, assuming the mass of $\widetilde{\mathrm{G}}$ is 1. Then, in terms of distributions

$$\Theta_{\rho'}(\tilde{c}_-(x)) = \Theta(\tilde{c}(0)^{-1})\Theta(\tilde{c}(x))\int_{\mathrm{W}} \chi_x(w) f_{\rho\otimes\rho'}(w)\,dw\,,\ \ x\in\mathfrak{sp}^c(\mathrm{W})\,.$$

Consequently

$$WF(\rho') = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(0))\,.$$

A dual pair as a supergroup

Fix two right vector spaces $V_{\bar{0}}$ and $V_{\bar{1}}$ over $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Set $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and define an element $S \in \mathrm{End}(V)$ by

$$S(v_0 + v_1) = v_0 - v_1 \qquad (v_0 \in V_{\bar{0}}, v_1 \in V_{\bar{1}}).$$

Let

$$\mathrm{End}(V)_{\bar{0}} = \{x \in \mathrm{End}(V);\ Sx = xS\}, \qquad \begin{array}{|c|c|}\hline \star & \\ \hline & * \\ \hline \end{array} \begin{array}{l} V_{\bar{0}} \\ V_{\bar{1}} \end{array}$$

$$\mathrm{End}(V)_{\bar{1}} = \{x \in \mathrm{End}(V);\ Sx = -xS\}, \qquad \begin{array}{|c|c|}\hline & \star \\ \hline * & \\ \hline \end{array} \begin{array}{l} V_{\bar{0}} \\ V_{\bar{1}} \end{array}$$

$$\mathrm{GL}(V)_{\bar{0}} = \mathrm{End}(V)_{\bar{0}} \cap \mathrm{GL}(V).$$

The anticommutant

$$\mathrm{End}(V)_{\bar{1}} \times \mathrm{End}(V)_{\bar{1}} \ni x, y \to \{x, y\} = xy + yx \in \mathrm{End}(V)_{\bar{0}}\,.$$

For $x, y \in \mathrm{End}(V)$. Set

$$\langle x, y \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(Sxy)\,.$$

The restriction of $\langle\cdot,\cdot\rangle$ to $\mathrm{End}(\mathsf{V})_{\overline{1}}$ is a nondegenerate symplectic form. The adjoint action $\mathrm{Ad} : \mathrm{GL}(\mathsf{V})_{\overline{0}} \to \mathrm{Sp}(\mathrm{End}(\mathsf{V})_{\overline{1}})$ maps the groups

$$\mathrm{G}_0 = \{g \in \mathrm{GL}(\mathsf{V})_{\overline{0}};\ g|_{\mathsf{V}_{\overline{1}}} = 1\}\,,\quad \mathrm{G}_1 = \{g \in \mathrm{GL}(\mathsf{V})_{\overline{0}};\ g|_{\mathsf{V}_{\overline{0}}} = 1\}$$

onto a dual pair $(\mathrm{G}_0, \mathrm{G}_1)$ with G_0 isomorphic to $\mathrm{GL}(\mathsf{V}_{\overline{0}})$ and G_1 isomorphic to $\mathrm{GL}(\mathsf{V}_{\overline{1}})$.

Suppose $(\cdot,\cdot)_0$ is a non-degenerate hermitian form on $\mathsf{V}_{\overline{0}}$ and $(\cdot,\cdot)_1$ is a non-degenerate skew-hermitian form on $\mathsf{V}_{\overline{1}}$. Denote by $(\cdot,\cdot)$ the direct sum of the two forms. Let

$$\mathfrak{s}_{\overline{0}} = \{x \in \mathrm{End}(\mathsf{V})_{\overline{0}};\ (xu, v) = -(u, xv),\ u, v \in \mathsf{V}\} \quad \subseteq \quad \begin{array}{|c|c|}\hline \star & \\ \hline & * \\ \hline\end{array}\ \begin{array}{c} \mathsf{V}_{\overline{0}} \\ \mathsf{V}_{\overline{1}} \end{array}$$

$$\mathfrak{s}_{\overline{1}} = \{x \in \mathrm{End}(\mathsf{V})_{\overline{1}};\ (xu, v) = (u, \mathsf{S}xv),\ u, v \in \mathsf{V}\} \quad \approx \quad \begin{array}{|c|c|}\hline & w \\ \hline w^* & \\ \hline\end{array}\ \begin{array}{c} \mathsf{V}_{\overline{0}} \\ \mathsf{V}_{\overline{1}} \end{array}$$

$$\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}},$$

$$\mathrm{S} = \{s \in \mathrm{GL}(\mathsf{V})_{\overline{0}};\ (su, sv) = (u, v),\ u, v \in \mathsf{V}\}\,.$$

The adjoint action $\mathrm{Ad} : \mathrm{S} \to \mathrm{Sp}(\mathfrak{s}_{\overline{1}})$ maps the groups

$$\mathrm{G}_0 = \{g \in \mathrm{S};\ g|_{\mathsf{V}_{\overline{1}}} = 1\}\,, \quad \mathrm{G}_1 = \{g \in \mathrm{S};\ g|_{\mathsf{V}_{\overline{0}}} = 1\}$$

onto a dual pair $(\mathrm{G}_0, \mathrm{G}_1)$ where G_0 is isomorphic to the isometry groups $\mathrm{G}((\cdot,\cdot)_0)$ and G_1 to the isometry group $\mathrm{G}((\cdot,\cdot)_1)$.

For the previous dual pair we shall also write $\mathrm{S} = \mathrm{GL}(\mathsf{V})_{\overline{0}}$ and $\mathfrak{s}_{\overline{1}} = \mathrm{End}(\mathsf{V})_{\overline{1}}$. Then for any dual pair we have the unnormalized moment maps

$$\mathfrak{s}_{\overline{1}} \ni w \to w^2|_{\mathsf{V}_{\overline{0}}} \in \mathfrak{g}_0, \qquad \mathfrak{s}_{\overline{1}} \ni w \to w^2|_{\mathsf{V}_{\overline{1}}} \in \mathfrak{g}_1\,.$$

In all case the restriction

$$\mathfrak{s}_{\overline{1}} \ni w \to w|_{\mathsf{V}_{\overline{1}}} \in \mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0}})$$

is a linear isomorphism.

Cartan subspaces in $\mathfrak{s}_{\overline{1}}$

An element $x \in \mathfrak{s}$ is called **semisimple** (resp., **nilpotent**) if x is semisimple (resp., nilpotent) as an endomorphism of V. We say that a semisimple element $x \in \mathfrak{s}_{\overline{1}}$ is **regular** if it is nonzero and $\dim(S.x) \geq \dim(S.y)$ for all semisimple $y \in \mathfrak{s}_{\overline{1}}$. The anticommutant and the double anticommutant of x in $\mathfrak{s}_{\overline{1}}$ are

$$
\begin{aligned}
{}^{x}\mathfrak{s}_{\overline{1}} &= \{y \in \mathfrak{s}_{\overline{1}} : \{x, y\} = 0\}\,, \\
{}^{{}^{x}\mathfrak{s}_{\overline{1}}}\mathfrak{s}_{\overline{1}} &= \bigcap_{y \in {}^{x}\mathfrak{s}_{\overline{1}}} {}^{y}\mathfrak{s}_{\overline{1}}\,,
\end{aligned}
$$

respectively. A Cartan subspace $\mathfrak{h}_{\overline{1}}$ of $\mathfrak{s}_{\overline{1}}$ is defined as the double anticommutant of a regular semisimple element $x \in \mathfrak{s}_{\overline{1}}$.
There are finitely many conjugacy classes of Cartan subspaces in $\mathfrak{s}_{\overline{1}}$.
Every semisimple element of $\mathfrak{s}_{\overline{1}}$ belongs to the G-orbit through an element of a Cartan subspace. The set of regular semisimple elements is dense in $\mathfrak{s}_{\overline{1}}$. Any two elements of a Cartan subspace $\mathfrak{h}_{\overline{1}} \subseteq \mathfrak{s}_{\overline{1}}$ commute as endomorphisms of V

Let $\mathfrak{h}_{\overline{1}}^2 \subseteq \mathfrak{s}_{\overline{0}}$ be the subspace spanned by all the squares w^2, $w \in \mathfrak{h}_{\overline{1}}$. If the rank of $\mathfrak{g}_0$ is smaller or equal to the rank of $\mathfrak{g}_1$ then the space $\mathfrak{h} = \mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{0}}}$ is a Cartan subalgebra of $\mathfrak{g}_0$. Similarly, If the rank of $\mathfrak{g}_1$ is smaller or equal to the rank of $\mathfrak{g}_0$ then the space $\mathfrak{h} = \mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{1}}}$ is a Cartan subalgebra of $\mathfrak{g}_1$. In general the relation

$$\{(w^2|_{\mathsf{V}_{\overline{0}}}, w^2|_{\mathsf{V}_{\overline{1}}}) : \ w \in \mathfrak{h}_{\overline{1}}\}$$

extends to a linear bijection

$$\mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{0}}} \longleftrightarrow \mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{1}}}\,.$$

We shall identify these two spaces, thus getting an embedding of a Cartan subalgebra of the Lie algebra of the smaller or equal rank ($\mathfrak{g}_0$ or $\mathfrak{g}_1$) into the Lie algebra of the greater or equal rank ($\mathfrak{g}_1$ or $\mathfrak{g}_0$).

The Capelli homomorphism

Howe's Double Commutant Theorem says that

$$d\omega(\mathcal{U}(\mathfrak{sp}(\mathsf{W})))^{\mathrm{G}} = d\omega(\mathcal{U}(\mathfrak{g}')) \quad \text{and} \quad d\omega(\mathcal{U}(\mathfrak{sp}(\mathsf{W})))^{\mathrm{G}'} = d\omega(\mathcal{U}(\mathfrak{g}))\,.$$

Hence we have the surjective algebra homomorphisms

$$\mathcal{U}(\mathfrak{g})^{\mathrm{G}} \longrightarrow d\omega(\mathcal{U}(\mathfrak{sp}(\mathsf{W})))^{\mathrm{G}\mathrm{G}'} \longleftarrow \mathcal{U}(\mathfrak{g}')^{\mathrm{G}'}\,.$$

Theorem

If the rank of $\mathfrak{g}'$ is smaller or equal to the rank of $\mathfrak{g}$, then the map

$$\mathcal{U}(\mathfrak{g}')^{\mathrm{G}'} \longrightarrow d\omega(\mathcal{U}(\mathfrak{sp}(\mathsf{W})))^{\mathrm{G}\mathrm{G}'}$$

is injective.

Hence the above defines a surjective algebra homomorphism

$$\mathcal{C} : \mathcal{U}(\mathfrak{g})^{\mathrm{G}} \longrightarrow \mathcal{U}(\mathfrak{g}')^{\mathrm{G}'}\,.$$

Given a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we have the Harish-Chandra isomorphism

$$\gamma_{\mathfrak{g}/\mathfrak{h}} : \mathcal{U}(\mathfrak{g})^{\mathrm{G}} \to \mathcal{U}(\mathfrak{h})^{W(\mathrm{G}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} .$$

where $W(\mathrm{G}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the corresponding Weyl group. Similarly there is the Harish-Chandra isomorphism

$$\gamma_{\mathfrak{g}'/\mathfrak{h}'} : \mathcal{U}(\mathfrak{g}')^{\mathrm{G}'} \to \mathcal{U}(\mathfrak{h}')^{W(\mathrm{G}'_{\mathbb{C}},\mathfrak{h}'_{\mathbb{C}})} .$$

Assume the rank of $\mathfrak{g}'$ is smaller or equal to the rank of $\mathfrak{g}$. then by viewing the dual pair as a supergroup we obtain an embedding

$$\mathfrak{h}' \subseteq \mathfrak{g} .$$

If the vector space V is the defining module for G, then

$$\mathrm{V} = \mathrm{V}_1 \oplus \mathrm{V}_0 ,$$

where V_0 is the intersection of the kernels of all the elements of $x \in \mathfrak{h}'$, $\mathfrak{h}' \subseteq \mathfrak{g} \subseteq \mathrm{End}(\mathrm{V})$.

Let $\mathfrak{z} \subseteq \mathfrak{g}$ denote the centralizer of $\mathfrak{h}'$. Then

$$\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{z}'',$$

where $\mathfrak{z}'' = \mathfrak{z}|_{\mathsf{V}_1}$. Denote by $\mathfrak{h}'' \subseteq \mathfrak{z}''$ a Cartan subalgebra. Then

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$$

is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{z}$. Let Z be the centralizer of $\mathfrak{h}'$ in G. Set $\mathrm{Z}'' = \mathrm{Z}|_{\mathsf{V}_1}$.

Theorem

Suppose $(\mathrm{G}, \mathrm{G}')$ *is not a complex dual pair, with rank of* G' *smaller or equal than the rank of* G*. If* G' *is isomorphic to* $\mathrm{O}_{p,q}$ *with* $p+q$ *odd then* Z'' *is isomorphic to a real symplectic group. Denote by*

$$\epsilon_{\mathfrak{z}''} : \mathcal{U}(\mathfrak{z}'')^{\mathrm{Z}''} \longrightarrow \mathbb{C}$$

the infinitesimal character of the Weil representation of $\widetilde{\mathrm{Z}}$*. for all other dual pairs, let* $\epsilon_{\mathfrak{z}''}$ *be the infinitesimal character of the trivial representation. Then the Capelli homomorphism* $\mathcal{C}$ *coincides with the composition of the following maps*

$$\begin{aligned} & \mathcal{U}(\mathfrak{g})^{\mathrm{G}} \underset{\gamma_{\mathfrak{z}/\mathfrak{h}}^{-1}\circ\gamma_{\mathfrak{g}/\mathfrak{h}}}{\longrightarrow} \mathcal{U}(\mathfrak{z})^{\mathrm{Z}} \\ = \ & \mathcal{U}(\mathfrak{h}')^{W(\mathrm{G}'_{\mathbb{C}},\mathfrak{h}'_{\mathbb{C}})} \otimes \mathcal{U}(\mathfrak{z}'')^{\mathrm{Z}''} \underset{\mathrm{id}\otimes\epsilon_{\mathfrak{z}''}}{\longrightarrow} \mathcal{U}(\mathfrak{h}')^{W(\mathrm{G}'_{\mathbb{C}},\mathfrak{h}'_{\mathbb{C}})} \otimes \mathbb{C} = \mathcal{U}(\mathfrak{h}')^{W(\mathrm{G}'_{\mathbb{C}},\mathfrak{h}'_{\mathbb{C}})} \\ & \underset{\gamma_{\mathfrak{g}'/\mathfrak{h}'}^{-1}}{\longrightarrow} \mathcal{U}(\mathfrak{g}')^{\mathrm{G}'} . \end{aligned}$$

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Lecture 4:

- Correspondence of simultaneous harmonics
- Howe correspondence for a general dual pair
- Open problems

Group representations and Harish-Chandra modules

Following Harish-Chandra's "Representations of a semisimple Lie group on a Banach space. I." 1951.

G - a real reductive group, $\mathrm{K} \subseteq \mathrm{G}$ - a maximal compact subgroup. A complex vector space V is called a $(\mathfrak{g}, \mathrm{K})$-module provided:

- both $\mathfrak{g}$ and K act on it so that

$$k \cdot x \cdot v = \mathrm{Ad}(k)x \cdot k \cdot v \qquad (k \in \mathrm{K}\,,\ x \in \mathfrak{g}\,,\ v \in \mathrm{V})\,;$$

- if $v \in \mathrm{V}$ then $\mathrm{K} \cdot v$ spans a finite dimensional subspace of V;
- the derivative of the action of K coincides with the action of $\mathfrak{k}$ given by the inclusion $\mathfrak{k} \subseteq \mathfrak{g}$.

U - a Banach space on which G acts. Let $\mathsf{U}^{an} \subseteq \mathsf{U}$ be the space of the analytic vectors.

For $\pi \in \hat{\mathrm{K}}$ let $\mathsf{U}(\pi) \subseteq \mathsf{U}$ be the subspace of vectors v with the following property:

there is a finite dimensional subspace $\mathsf{U}(v) \subseteq \mathsf{U}$ containing v which is semisimple under the action of K and so that each K-irreducible component of $\mathsf{U}(v)$ is isomorphic to π.

Set $\mathsf{U}_\pi = \mathsf{U}^{an} \cap \mathsf{U}(\pi)$ and let

$$\mathsf{U}_{\mathrm{K}} = \sum_{\pi \in \hat{\mathrm{K}}} \mathsf{U}_\pi$$

denote the subspace of the K-finite vectors.

Theorem

The space U_{K} is a $(\mathfrak{g}, \mathrm{K})$-module and is dense in U.

U (or U_K) is called admissible if $U_\pi < \infty$ for each $\pi \in \hat{K}$.

Theorem

Suppose U_K *is admissible and finitely generated. Then the map*

$$U \supseteq X \to X_K \subseteq U_K$$

is a bijection between closed G*-invariant subspaces and* $(\mathfrak{g}, K)$*-submodules.*

Theorem

U *is an irreducible unitary representation of* G *if and only if* U_K *is an irreducible unitarizable* $(\mathfrak{g}, K)$*-module.*

Two irreducible unitary representations of G *are equivalent if and only if the their* $(\mathfrak{g}, K)$*-modules are equivalent.*

Two group representations are called infinitesimally equivalent if and only if their $(\mathfrak{g}, K)$-modules are isomorphic. One calls U_K the Harish-Chandra module of U.

The correspondence of simultaneous harmonics

Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair with each member normalized by J.

Then $\mathrm{K} = \mathrm{G}^J$ and $\mathrm{K}' = \mathrm{G}'^J$ are maximal compact subgroups.

Let $\mathrm{M} \subseteq \mathrm{Sp}(\mathrm{W})$ be the centralizer of K' and let $\mathrm{M}' \subseteq \mathrm{Sp}(\mathrm{W})$ be the centralizer of K.

Then $\mathrm{M}^J \subseteq \mathrm{M}$ and $\mathrm{M}'^J \subseteq \mathrm{M}'$ are maximal compact subgroups and $(\mathrm{M}^J, \mathrm{M}'^J)$ happens to be a dual pair.

All together we obtain the following dual pairs

$(\mathrm{G}, \mathrm{G}')$,	$(\mathrm{K}, \mathrm{M}')$,	$(\mathrm{M}, \mathrm{K}')$,	$(\mathrm{M}^J, \mathrm{M}'^J)$.
(arbitrary, arbitrary)	(compact, arbitrary)	(arbitrary, compact)	(compact, compact)

Theorem

Let $\pi \in \mathcal{R}(\mathrm{M}^J, \omega)$ *correspond to* $\pi' \in \mathcal{R}(\mathrm{M}'^J, \omega)$. *Let d denote the degree of* $\pi \otimes \pi'$. *Assume that*

$$\mathcal{P}_{\pi\otimes\pi'} \cap \mathit{Harm}(\mathrm{K}) \neq 0 \neq \mathcal{P}_{\pi\otimes\pi'} \cap \mathit{Harm}(\mathrm{K}')\,.$$

Then there are unique representations $\sigma \in \mathcal{R}(\mathrm{K}, \omega)$ *and* $\sigma' \in \mathcal{R}(\mathrm{K}', \omega)$ *such that*

$$\mathcal{P}_{\pi\otimes\pi'} = \mathit{Harm}(\mathrm{K})_\sigma \cap \mathit{Harm}(\mathrm{K}')_{\sigma'} \oplus \sum \mathcal{R}$$

where $\mathcal{R}$ *is a direct sum of representations of* $\widetilde{\mathrm{K}} \times \widetilde{\mathrm{K}}'$ *of types* $\sigma_0 \otimes \sigma_0'$, *where* $\deg(\sigma_0) < d$ *or* $\deg(\sigma_0') < d$. *Furthermore the space*

$$\mathit{Harm}(\mathrm{K})_\sigma \cap \mathit{Harm}(\mathrm{K}')_{\sigma'}$$

is irreducible of type $\sigma \otimes \sigma'$. *The map* $\sigma \to \pi'$ *coincides with the lowest degree correspondence for the dual pair* $(\mathrm{K}, \mathrm{M}')$ *and* $\sigma' \to \pi$ *with the lowest degree correspondence for the dual pair* $(\mathrm{K}', \mathrm{M})$.

Howe correspondence for an arbitrary dual pair

Recall the metaplectic group $\widetilde{\mathrm{Sp}}$, with maximal compact subgroup $\widetilde{\mathrm{U}} = \widetilde{\mathrm{Sp}}^J$. Then $\mathcal{P}$ is the Harish-Chandra module (i.e. $(\mathfrak{sp}, \widetilde{\mathrm{U}})$-module) of ω. Consider:

- an irreducible dual pair $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}'})$ in $\widetilde{\mathrm{Sp}}$ with maximal compact subgroups $\widetilde{\mathrm{K}} \subseteq \widetilde{\mathrm{G}} \cap \widetilde{\mathrm{U}}$ and $\widetilde{\mathrm{K}'} \subseteq \widetilde{\mathrm{G}'} \cap \widetilde{\mathrm{U}}$;
- an irreducible $(\mathfrak{g}, \widetilde{\mathrm{K}})$-module ρ that occurs as a quotient $\rho = \mathcal{P}/\mathcal{N}$ of $\mathcal{P}$ by a $(\mathfrak{g}, \widetilde{\mathrm{K}})$ invariant subspace $\mathcal{N} \subseteq \mathcal{P}$;
- the intersection $\mathcal{N}_\rho$ of all subspaces $\mathcal{N}$ such that $\rho = \mathcal{P}/\mathcal{N}$.

Theorem (Howe 1989)

There is a quasisimple $(\mathfrak{g}', \widetilde{\mathrm{K}'})$-module ρ'_1 of finite length such that $\mathcal{P}/\mathcal{N}_\rho = \rho \otimes \rho'_1$ as a $(\mathfrak{g}, \widetilde{\mathrm{K}}) \times (\mathfrak{g}', \widetilde{\mathrm{K}'})$-module.
Moreover ρ'_1 has a unique irreducible quotient ρ'. By applying the above procedure to ρ' one recovers ρ.

We have just stated the main theorem, i.e.

Theorem (Howe 1989)

There is a quasisimple $(\mathfrak{g}', \widetilde{\mathrm{K}}')$-module ρ'_1 of finite length such that $\mathcal{P}/\mathcal{N}_\rho = \rho \otimes \rho'_1$ as a $(\mathfrak{g}, \widetilde{\mathrm{K}}) \times (\mathfrak{g}', \widetilde{\mathrm{K}}')$-module.
Moreover ρ'_1 has a unique irreducible quotient ρ'. By applying the above procedure to ρ' one recovers ρ.

- ρ'_1 is called the big Howe quotient, or $\Theta(\rho')$ or big Theta(ρ')
- ρ' is called the irreducible Howe quotient, or $\theta(\rho')$ or theta(ρ')
- The resulting bijection $\rho \longleftrightarrow \rho'$ is known as Howe correspondence or local θ correspondence.

General properties

Let $\mathrm{H}(\mathfrak{g}, \widetilde{\mathrm{K}})$ be the convolution algebra of left and right $\widetilde{\mathrm{K}}$-finite distributions on $\widetilde{\mathrm{G}}$ supported in $\widetilde{\mathrm{K}} \subseteq \widetilde{\mathrm{G}}$.

Theorem

$\rho_1' = \rho^\vee \otimes_{\mathrm{H}(\mathfrak{g},\widetilde{\mathrm{K}})} \mathcal{P}$, *where* $\rho^\vee$ *is the contragredient of* ρ.

Theorem

Suppose the rank of $\mathfrak{g}'$ *is smaller or equal to the rank of* $\mathfrak{g}$.

If ρ' *has infinitesimal character* $\gamma_{\rho'} : \mathcal{U}(\mathfrak{g}')^{\mathrm{G}} \to \mathbb{C}$, *then* ρ *(in fact* ρ_1*) has infinitesimal character* $\gamma_\rho = \gamma_{\rho'} \circ \mathcal{C} : \mathcal{U}(\mathfrak{g})^{\mathrm{G}} \to \mathbb{C}$.

Theorem

Suppose $\sigma \in \widetilde{\mathrm{K}}$ *is a lowest degree* K*-type of* ρ *and* $\sigma' \in \widetilde{\mathrm{K}}'$ *corresponds to* σ *via the correspondence of simultaneous harmonics. Then* $\sigma' \in \widetilde{\mathrm{K}}'$ *is a lowest degree* K'*-type of* ρ' *(in fact of* ρ_1'*).*

Theorem

Each irreducible $(\mathfrak{g}, \widetilde{\mathrm{K}})$*-module that occurs as a quotient of* $\mathcal{P}$ *is the Harish-Chandra module of a representation of* $\widetilde{\mathrm{G}}$ *that occurs as the quotient of the space of the smooth vectors of* ω *by a closed invariant subspace. The same holds for* ρ' *and* $\rho \otimes \rho'$*.*

This way the correspondence of the Harish-Chandra modules globalizes to a correspondence of group representations.

Theorem

If ρ *occurs as a quotient of* $\mathcal{P}$ *then* $WF(\rho) \subseteq \tau_{\mathfrak{g}}(\mathrm{W})$*.*

Theorem

If ρ *Hermitian then* ρ' *Hermitian.*

The Cauchy Harish-Chandra Integral

For a Cartan subgroup $\mathrm{H}' \subseteq \mathrm{G}'$. Define

- A' the split part of H';
- $\mathrm{A}'' \subseteq \mathrm{Sp}$ the centralizer of A';
- $\mathrm{A}''' \subseteq \mathrm{Sp}$ the centralizer of A''.

Then $(\mathrm{A}'', \mathrm{A}''')$ form a (reducible) dual pair in Sp.

There is an open dense subset $\mathsf{W}_{\mathrm{A}'''} \subseteq \mathsf{W}$ on which A''' acts freely.

Let $d\dot{w}$ be the measure on $\mathrm{A}'''\backslash\mathsf{W}_{\mathrm{A}'''}$ defined by

$$\int_{\mathsf{W}} \phi(w)\, d\mu_{\mathsf{W}}(w) = \int_{\mathrm{A}'''\backslash\mathsf{W}_{\mathrm{A}'''}} \int_{\mathrm{A}'''} \phi(aw)\, da\, d\dot{w}\,.$$

Theorem

For any $f \in C_c^\infty(\widetilde{\mathrm{A}''^c})$, *the distribution*

$$T(f) = \int_{\widetilde{\mathrm{A}''^c}} f(\tilde{g})T(\tilde{g})\,d\tilde{g} \in \mathcal{S}'(\mathrm{W})$$

is a function on W, *such that*

$$\int_{\mathrm{A}'''\setminus \mathrm{W}_{\mathrm{A}'''}} \left| \int_{\mathrm{A}''} f(g)T(g)(w)\,dx \right| d\dot{w} < \infty\,.$$

The formula

$$\mathrm{Chc}(f) = \int_{\mathrm{A}'''\setminus \mathrm{W}_{\mathrm{A}'''}} T(f)(w)\,d(\mathrm{A}'''w) \qquad (f \in C_c^\infty(\widetilde{\mathrm{A}''^c}))$$

defines a distribution on $\widetilde{\mathrm{A}''^c}$ *which coincides with a complex valued measure. This measure extends by zero to* $\widetilde{\mathrm{A}''}$ *and defines a distribution, which we denote by the same symbol.*

Moreover,

$$WF(\mathrm{Chc}) = \{(\tilde{g}, \tau_{\alpha''*}(w));\ \tilde{g} \in \widetilde{\mathrm{A}''}\,,\ \tau_{\alpha''*}(w) \neq 0\,, g(w) = -w\}\,.$$

The distribution Chc defined by

$$\mathrm{Chc}(f) = \int_{\mathrm{A}'''\backslash \mathsf{W}_{\mathrm{A}'''}} T(f)(w)\, d(\mathrm{A}'''w) \qquad (f \in C_c^\infty(\widetilde{\mathrm{A}''^c}))$$

is the Cauchy Harish-Chandra integral.

For any $h' \in \mathrm{H}'^{reg}$, the intersection of the wave front set of the distribution Chc with the conormal bundle of the embedding

$$\widetilde{\mathrm{G}} \ni \widetilde{g} \longrightarrow \widetilde{h'}\widetilde{g} \in \widetilde{\mathrm{A}''}$$

is empty. Hence there is a unique restriction of the distribution Chc to $\widetilde{\mathrm{G}}$, denoted $\mathrm{Chc}_{\widetilde{h'}}$.

The distribution $\Theta'_{\rho'}$

Recall the Weyl - Harish-Chandra integration formula

$$\int_{\widetilde{\mathrm{G}}'} \phi(g)\,dg = \sum_{\mathrm{H}'} c_{\mathrm{H}'} \int_{\widetilde{\mathrm{H}'^{reg}}} D(h) \int_{\widetilde{\mathrm{G}}'/\widetilde{\mathrm{H}}'} \phi(g\widetilde{h}g^{-1})\,d\dot{g}\,d\widetilde{h}\,.$$

Define

$$\Theta'_{\rho'}(f) = C_{\rho'} \sum c_{\mathrm{H}'} \int_{\widetilde{\mathrm{H}'^{reg}}} D(h)\,\Theta_{\rho'}(\widetilde{h}^{-1})\mathrm{Chc}_{\widetilde{h}}(f)\,d\widetilde{h}\,.$$

Theorem

$\Theta'_{\rho'}$ is an invariant eigendistribution on $\widetilde{\mathrm{G}}$ with infinitesimal character $\gamma_{\rho'} \circ \mathcal{C} : \mathcal{U}(\mathfrak{g})^{\mathrm{G}} \to \mathbb{C}$.

Let G'^0 be the Zariski identity component of G'.

(Then $\mathrm{G}'^0 = \mathrm{G}'$, unless G' is an even orthogonal group.)

Conjecture

If the character Θ_ρ is supported in G'^0, then, as distributions,

$$\Theta'_{\rho'} = \Theta_{\rho_1} \,,$$

where ρ_1 is the big Howe quotient of ρ'.

Pairs of type I in the stable range

The pair $(\mathrm{G}, \mathrm{G}')$ is of type I if it acts irreducibly on W and W is a single isotypic component under this action.
In this case, there is:

- a division algebra $\mathbb{D}$ with an involution over $\mathbb{F}$
- two vector spaces V and V′ with with non-degenerate Hermitian forms $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ of opposite type

such that

- $\mathsf{W} = \mathsf{V} \otimes_{\mathbb{F}} \mathsf{V}'$,
- G coincides with the isometry group of $(\mathsf{V}, (\cdot, \cdot))$,
- G' coincides with the isometry group of $(\mathsf{V}', (\cdot, \cdot)')$.

The pair $(\mathrm{G}, \mathrm{G}')$ is in the stable range with G' - the smaller member if the dimension of the maximal isotropic subspace of V is greater or equal to the dimension of V′.

The equality $\Theta'_{\rho'} = \Theta_\rho$

Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair of type I in the stable range with G' - the smaller member.

Assume that the representation ρ' of $\widetilde{\mathrm{G}}'$ is unitary.

Theorem

$\Theta'_{\rho'} = \Theta_\rho$.

Idea of the proof. We show that the two distributions are equal on a Zariski open subset $\widetilde{\mathrm{G}}'' \subseteq \widetilde{\mathrm{G}}$. Since both Θ_ρ and $\Theta'_{\rho'}$ are invariant eigendistributions, Harish-Chandra Regularity Theorem implies that they are equal everywhere.

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Some open problems

Preservation of unitarity: under what conditions, if ρ is unitary, then so is ρ'? (T.P., Jian-Shu Li, Hongy He, Sun Binyoung, Chengbo Zhu, Jajun Ma, Dan Barbasch,...)

Character correspondence: given Θ_ρ describe $\Theta_{\rho'}$. (T.P., Florent Bernon, Wee Teck Gan, Allan Merino,...)

Wave front set correspondence: given $WF(\rho)$ compute $WF(\rho')$. (T.P., Jajun Ma, Hung Yean Loke, Angela Pasquale, Mark McKee.)

Langlands parameters: given the Langlands parameters of ρ compute the Langlands parameters of ρ'. (T.P., Jeff Adams, Dan Barbasch, Annegret Paul, Colette Moeglin, Jean-Loup Waldspurger, Jian-Shu Li, Chengbo Zhu, Eng-Chye Tan, Xiang Fan.)

Thank You