On Howe's Duality Theorem

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We adopt the Langlands classification to the context of real reductive dual pairs and prove that Howe's Duality Correspondence maps hermitian representations to hermitian representations. © 1988 Academic Press, Inc.

INTRODUCTION

The goal of this paper is to prove that the Duality Correspondence (5.3) maps hermitian representations to hermitian representations (Theorem 5.10). We achieve this by showing that the graph of this function is symmetric with respect to the operation of taking the hermitian dual of a representation (Theorem 5.5 and (5.11)).

In Sections 1 and 2 we prove some structural theorems about the reductive dual pairs (Definition 1.1) and in Section 3 we adopt the Langlands–Vogan classification [V1] to such pairs. These results are then used in the proof of the Theorem 5.5.

1. REDUCTIVE DUAL PAIRS IN THE SYMPLECTIC GROUP

Recall that if W is a finite dimensional vector space over a (commutative) field, then a bilinear form $\langle\, ,\, \rangle$ from $W \times W$ to that field is called symplectic (alternate) if $\langle w, w \rangle = 0$ for all $w \in W$. This form is non-degenerate if the zero vector is the only one perpendicular to all other vectors in $(W, \langle\, ,\, \rangle)$. Let then, in this paper, $(W, \langle\, ,\, \rangle)$ be a finite dimensional vector space with a nondegenerate symplectic form and let $Sp(W, \langle\, ,\, \rangle)$ denote the isometry group of $\langle\, ,\, \rangle$.

160

DEFINITION 1.1 [H3, H5].   A pair of subgroups G, G' of $Sp(W, \langle , \rangle)$ is called a reductive dual pair if

$$G' \text{is the centralizer of } G \text{ in } Sp(W, \langle , \rangle) \text{ and vice versa,} \qquad (1.2)$$

and

$$\text{both } G, G' \text{ act reductively on } W. \qquad (1.3)$$

These pairs have been classified [H4] over fields with characteristic $\neq 2$.

Suppose $W = W_1 \oplus W_2$ is an orthogonal direct sum decomposition of W and each W_j is invariant by G and G'. Let G_j be the restriction of G to W_j. Define G'_j similarly. Then $G \cong G_1 \times G_2$ and $G' \cong G'_1 \times G'_2$ and G_j, G'_j is a reductive dual pair in $Sp(W_j)$.

DEFINITION 1.4 [H3, H5].   We say that the reductive dual pair G, G' is irreducible if it has no nontrivial direct sum decomposition like that described above.

In this paper we consider the real reductive dual pairs only. Thus our symplectic vector space $(W, \langle , \rangle)$ will always be a real vector space.

Consider the field of real numbers $\mathbf{R}$ as a vector space over itself and define a symmetric, positive definite bilinear form on it by

$$(x, y)_1 = xy \qquad (x, y \in \mathbf{R}). \qquad (1.5)$$

Let $(\mathbf{R}^p, (,)_p)$ be the p-fold direct sum of $(\mathbf{R}, (,)_1)$ and

$$(\mathbf{R}^{p+q}, (,)_{p,q}) = (\mathbf{R}^p \oplus \mathbf{R}^q, (,)_p \oplus -(,)_q) \qquad (1.6)$$

for $p, q = 0, 1, 2, ...$ as usual.

We equip the vector space $\mathbf{R}^2$ with a symplectic form as

$$(x_1 \oplus y_1, x_2 \oplus y_2)'_2 = x_1 y_2 - x_2 y_1 \qquad (x_1, x_2, y_1, y_2 \in \mathbf{R}). \qquad (1.7)$$

Let

$$(\mathbf{R}^{2n}, (,)'_{2n}) \text{ denote the } n\text{-fold direct sum of } (\mathbf{R}^2, \subset (,)'_2). \qquad (1.8)$$

Fix a complex structure on $\mathbf{R}^2$:

$$J_0(x \oplus y) = y \oplus (-x) \qquad (x, y \in \mathbf{R}). \qquad (1.9)$$

Then $\mathbf{R}^2$ becomes a complex vector space and therefore admits a complex conjugation τ_0.
Explicitly

$$\tau_0(x \oplus y) = y \oplus x \qquad (x, y \in \mathbf{R}). \qquad (1.10)$$

Let $\mathbf{H}$ denote the division algebra of real quaternions [J1, Sect. 2.4]. Then

$$\mathbf{H} = \mathbf{R} \oplus \mathbf{R} i \oplus \mathbf{R} j \oplus \mathbf{R} k, \tag{1.11}$$

where

$$i^2 = j^2 = k^2 = -1, \, ij = -ji = k, \, jk = -kj = i, \, ki = -ik = j. \tag{1.12}$$

Let

$$\mathbf{C} = \mathbf{R} \oplus i\mathbf{R}. \tag{1.13}$$

Then $\mathbf{C}$ is the field of complex numbers and we have the following inclusions

$$\mathbf{R} \subseteq \mathbf{C} \subseteq \mathbf{H}. \tag{1.14}$$

We define an involution $\natural_1$ on $\mathbf{H}$ by

$$\natural_1(x_0 + x_1 i + x_2 j + x_3 k) = x_0 - x_1 i - x_2 j - x_3 k \qquad (x_0, x_1, x_2, x_3 \in \mathbf{R}). \tag{1.15}$$

This involution preserves $\mathbf{C}$ (1.13) and is trivial on $\mathbf{R}$. Define

$$\natural_0(x) = j \times j^{-1} \qquad (x \in \mathbf{H}). \tag{1.16}$$

Then

$\natural_0$ is an involutive automorphism of $\mathbf{H}$ which commutes with $\natural_1$ and is equal to $\natural_1$ when restricted to $\mathbf{C}$ (1.13). $\tag{1.17}$

In the rest of this paper $(D, \natural)$ will denote one of the pairs,

$$(\mathbf{R}, 1), (\mathbf{C}, 1), (\mathbf{C}, \natural_1), (\mathbf{H}, \natural_1), \tag{1.18}$$

where 1 stands for the trivial automorphism. On the right D-vector space

$$D^{p+q} = \mathbf{R}^{p+q} \otimes_{\mathbf{R}} D \tag{1.19}$$

define a $\natural$-hermitian form

$$(x \otimes a, y \otimes b)_{p,q} = \natural(a)(x, y)_{p,q} b \qquad (x, y \in \mathbf{R}; a, b \in D), \tag{1.20}$$

and if $\natural \neq 1$ a $\natural$-skew hermitian form

$$(u, v)'_{p,q} = i(u, v)_{p,q} \qquad (u, v \in D^{p+q}; i \text{ as in } (1.11). \tag{1.21}$$

If $\natural = 1$ we put

$$(x \otimes a, \, y \otimes b)'_{2n} = a(x, \, y)'_{2n} b \qquad (x, \, y \in \mathbf{R}^{2n}; \, a, \, b \in D), \qquad (1.22)$$

thus defining a symplectic form on D^{2n}. The notation (1.20), (1.21), (1.22) has been arranged so that (,) will always stand for a $\natural$-hermitian and (,)' for a $\natural$-skew hermitian form. We extend the maps (1.15), (1.16) to act on D^m via

$$\natural_j(x \otimes a) = x \otimes \natural_j(a) \qquad (x \in \mathbf{R}^m; \, a \in D; \, j = 0, \, 1). \qquad (1.23)$$

Let $(V, \natural, (\, , \,)'')$ denote the triple $(D^{p+q}, \natural, (\, , \,)_{p,q})$ or $(D^{p+q}, \natural, (\, , \,)'_{p,q})$ or $(D^{2n}, \natural, (\, , \,)'_{2n})$ as in (1.20), (1.21), (1.22), respectively. Then (,)'' is either a $\natural$-hermitian or $\natural$-skew hermitian form on V. Let us also allow (,)'' to be a zero form. Define

$$G(V, \natural, (\, , \,)'') = \text{the isometry group of the form } (\, , \,)''$$
$$\text{on the right } D\text{-vector space } V, \qquad (1.24)$$

and

$$K(V, \natural, (\, , \,)'') = G(V, \natural, (\, , \,)'') \cap G(V, \natural_1, (\, , \,)_{p+q,0}). \qquad (1.25)$$

Here $p + q = 2n$ if $(\, , \,)'' = (\, , \,)'_{2n}$.

Then (1.25) is a maximal compact subgroup of (1.24) and if (,)'' is zero then (1.24) is just the general linear group of V. Denote the Lie algebras of (1.24) and (1.25) by

$$\mathscr{g}(V, \natural, (\, , \,)'') \qquad \text{and} \qquad \mathscr{k}(V, \natural, (\, , \,)''), \qquad (1.26)$$

respectively. Fix the following (multiple of) the Killing form on $\mathscr{g} = g(V, \natural, (\, , \,)'')$,

$$B(X, \, Y) = \text{tr}(XY) \qquad (X, \, Y \in \mathscr{g}), \qquad (1.27)$$

where tr is the reduced trace over $\mathbf{R}$ on $\text{End}_D(V)$ (see, for example, [We, Chap. IX, Sect. 2]).

Now (1.26) and (1.27) imply the Cartan decomposition

$$\mathscr{g} = \mathscr{k} \oplus \mathscr{p}, \qquad (1.28)$$

where $\mathscr{p}$ is the orthogonal complement of $\mathscr{k}$ in $\mathscr{g}$ with respect to B, and the Cartan involution

$$\theta(X \oplus Y) = X \oplus (-Y) \qquad (X \in \mathscr{k}, \, Y \in \mathscr{p}). \qquad (1.29)$$

This map lifts to an automorphism of the group G and will be denoted by the same letter θ. We are going to use the notion of a θ-stable Cartan sub-

group H of G and the Weyl group of H in G, $W(G, H)$. The corresponding definitions may be found in [V1, Chap. 0].

LEMMA 1.30. *Let G denote the group* (1.24) *and K its maximal compact subgroup* (1.25). *Define an invertible endomorphism $\tau \in \mathrm{End}_{\mathbf{R}}(V)$ as follows:*

$$\tau = \natural_0 \qquad \text{if } G \text{ is a general, linear group } (\natural_0 \text{ as in (1.23))}, \qquad (1.31)$$

$$\tau = 1 \qquad \text{if } (D, \natural) = (\mathbf{C}, 1) \text{ or } (D, \natural) = (\mathbf{R}, 1) \text{ and } (\ ,\ )'' \text{ is symmetric,} \qquad (1.32)$$

$$\tau = \text{the } p\text{-fold direct sum of } \tau_0 \ (1.10) \text{ if } (D, \natural) = (\mathbf{R}, 1)$$
$$\text{and } (\ ,\ )'' \text{ is symplectic,} \qquad (1.33)$$

$$\tau = \natural_0 \ (\text{as in } (1.23)) \text{ if } \natural \neq 1. \qquad (1.34)$$

Then there is a maximal family $H_1, H_2, ..., H_m$ of θ-stable Cartan subgroups of G such that for each $j = 1, 2, ..., m$

$$\mathrm{Int}\ \tau(H_j) = H_j \qquad (1.35)$$

and

the Weyl group $W(G, H_j)$, when enlarged by adding the restriction of $\mathrm{Int}\ \tau|_{H_j}$, in the cases (1.32), (1.33), (1.34), *and* $(\mathrm{Int}\ \tau) \circ \theta|_{H_j}$ *in the case* (1.31), *contains the inverse mapping:* $g \to g^{-1}$. (1.36)

Proof. Our notation, modulo some small modifications, is compatible with that of Sugiura [Su, Sect. 3] so that for a fixed group G (1.24) one can take his list $\hbar_1, \hbar_2, ..., \hbar_m$ of Cartan subalgebras and define H_j to be the centralizer of $\hbar_j$ in G as usual $(j = 1, 2, ..., m)$. Since the Cartan involution θ is an inner automorphism in all cases (1.32), (1.33), (1.34), an inspection of Sugiura's list yields the result.   Q.E.D.

THEOREM 1.37. *Let $(W, \langle\ ,\ \rangle)$ be a real symplectic vector space and G, G' a reductive dual pair in $\mathrm{Sp}(W, \langle\ ,\ \rangle)$. Then there is an invertible endomorphism τ of W, maximal compact subgroups $K \subseteq G$, $K' \subseteq G'$ and maximal families of θ-stable Cartan subgroups $\{H_1, H_2, ..., H_m\}$ in G and $\{H'_1, H'_2, ..., H'_n\}$ in G' such that*

$$\tau\langle\ ,\ \rangle = -\langle\ ,\ \rangle, \qquad (1.38)$$

$\mathrm{Int}\ \tau$ preserves G, K, H_i and G', K', H'_j
$(i = 1, 2, ..., m; j = 1, 2, ..., n)$, (1.39)

$W(G, H_i)(W(G', H'_j))$ when extended by the restriction of $\mathrm{Int}\ \tau$ to H_i (H'_j) contains the inverse mapping $(i = 1, 2, ..., m; j = 1, 2, ..., n)$. (1.40)

Proof. It follows immediately from Definition 1.4 that G, G' can be assumed to be an irreducible dual pair. There are two types I and II of such pairs [H5, Sect. 4]. We will consider them separately and following [H4] construct G, G' explicitly.

Type I pairs. Let $(D, \natural)$ be as in (1.18). Construct $(D^{p+q}, (\ ,\ ))$ and $(D^{r+s}, (\ ,\ )')$ as in (1.20), and (1.21) or (1.22), respectively. Define $V = D^{p+q}, V' = D^{r+s}$ and

$$*: \operatorname{Hom}_D(V, V') \to \operatorname{Hom}_D(V', V),$$

$$*': \operatorname{Hom}_D(V', V) \to \operatorname{Hom}_D(V, V') \tag{1.41}$$

by

$$(Tv, v')' = (v, T^*v') \qquad (T \in \operatorname{Hom}_D(V, V')),$$

$$(Sv', v) = (v', S^*v')' \qquad (S \in \operatorname{Hom}_D(V', V)). \tag{1.42}$$

Let $W = \operatorname{Hom}_D(V, V')$ and

$$\langle T_1, T_2 \rangle = \operatorname{tr}(T_2^* T_1) \qquad (T_1, T_2 \in W). \tag{1.43}$$

Then $(W, \langle\ ,\ \rangle)$ is a real symplectic vector space. We let $g \in G(V, \natural, (\ ,\ ))$ and $g' \in G(V', \natural, (\ ,\ )')$ act on W by

$$g(T) = Tg^{-1}, \ g'(T) = g'T \qquad (T \in W). \tag{1.44}$$

This action preserves the symplectic form (1.43) and thus provides embeddings of the above two groups into $Sp(W, \langle\ ,\ \rangle)$. We call their images G and G', respectively. This is how all irreducible dual pairs of type I look like [H4, Chap. I, Sect. 6].

We consider two separate subcases: $\natural = 1$ and $\natural \neq 1$.

Assume that $\natural = 1$. Then $(D, \natural)$ is either $(\mathbf{R}, 1)$ or $(\mathbf{C}, 1)$. Define an endomorphism τ_1 of V' by

$$\tau_1 = \text{the } r\text{-fold direct sum of the map } \tau_0 \ (1.10), \qquad \text{if} \quad D = \mathbf{R}, \tag{1.45}$$

$$\tau_1(v) = iv \qquad \text{for} \quad v \in V', \text{if } D = \mathbf{C}. \tag{1.46}$$

Here i is as in (1.11). Let for $T \in W$

$$\tau(T) = \tau_1 T. \tag{1.47}$$

Then τ is an invertible endomorphism of W and it follows from (1.47), (1.44) that

Int τ preserves G, G', acts trivially on G, and is equal to Int τ_1 on G'.

$$\tag{1.48}$$

The formulas (1.42), (1.45), (1.46) imply that for any T_1, T_2 in W

$$(\tau_1 T_2)^*(\tau_1 T_1) = -T_2^* T_1 \qquad (1.49)$$

so that τ satisfies (1.38).

Assume that $\natural \neq 1$. Then $(D, \natural) = (\mathbf{C}, \natural_1)$ or $(H, \natural_1)$. It follows from (1.20), (1.21), and (1.16) that

$$(\natural_0(u), \natural_0(v)) = \natural_0(u, v) \qquad (u, v \in V) \qquad (1.50)$$

and

$$(\natural_0(u'), \natural_0(v'))' = -\natural_0(u', v')' \qquad (u', v' \in V'). \qquad (1.51)$$

Define τ in $\mathrm{End}_{\mathbf{R}}(W)$ by

$$\tau(T) = \natural_0 T \natural_0 \qquad (T \in W). \qquad (1.52)$$

Then (1.50), (1.51), and (1.42) imply that

$$(\tau(T))^* = -\tau(T) \qquad (T \in W). \qquad (1.53)$$

Therefore by the definition (1.43) and (1.17), τ satisfies (1.38). From the formulas (1.44) and the definition (1.52) we deduce that

$$\text{Int } \tau \text{ preserves } G, G' \text{ and acts on both of them by Int } \natural_0. \quad (1.54)$$

In both cases ($\natural = 1$ and $\natural \neq 1$) choose $K(K')$ as in (1.25). Then in view of (1.48), (1.54), (1.17) the statements (1.39) and (1.40) follow from Lemma 1.30.

Type II pairs. Define

$$W = \mathrm{Hom}_D(D^m, D^n) \oplus \mathrm{Hom}_D(D^n, D^m), \qquad (1.55)$$

and

$$\langle S_1 \oplus T_1, S_2 \oplus T_2 \rangle = \mathrm{tr}(S_1 T_2 - S_2 T_1), \text{ where}$$
$$S_1, S_2 \in \mathrm{Hom}_D(D^m, D^n) \text{ and } T_1, T_2 \in \mathrm{Hom}_D(D^n, D^m). \qquad (1.56)$$

Here tr is the reduced trace over $\mathbf{R}$ on $\mathrm{End}_D(D^n)$ as usual, and $D = \mathbf{H}, \mathbf{C}$, or $\mathbf{R}$. We let $g \in GL(D^m)$ and $g' \in GL(D^n)$ act on W as follows,

$$g(S \oplus T) = Sg^{-1} \oplus gT, \qquad (1.57)$$

$$g'(S \oplus T) = g'S \oplus Tg'^{-1}, \qquad (1.58)$$

where the notation is parallel to tat of (1.56). These actions preserve the symplectic form (1.56) and thus provide embeddings of $GL(D''')$ and $GL(D'')$ into $Sp(W, \langle\ ,\ \rangle)$. We call their images G and G', respectively.

All irreducible dual pairs of type II look like this [H4, Sect. 6]. Let $(\ ,\ )$ be the $\natural_1$-hermitian positive definite form on D^p as in (1.20). For $S \in \mathrm{Hom}_D(D''', D'')$ define $S^* \in \mathrm{Hom}_D(D'', D''')$ (not the same as (1.41)) by

$$(Sv, w) = (v, S^*w) \qquad (v \in D''', w \in D''). \tag{1.59}$$

Let

$$\tau_1(S \oplus T) = \natural_0 S \natural_0 \oplus \natural_0 T \natural_0 \qquad \text{(see (1.23))}, \tag{1.60}$$

$$\tau_2(S \oplus T) = T^* \oplus S^*. \tag{1.61}$$

Then τ_1, τ_2 are invertible endomorphisms of W and it follows from (1.56) that

$$\tau_1 \langle\ ,\ \rangle = \langle\ ,\ \rangle, \tag{1.62}$$

and

$$\tau_2 \langle\ ,\ \rangle = -\langle\ ,\ \rangle. \tag{1.63}$$

The formulas (1.57), (1.58), (1.60), (1.61) imply that both τ_1 and τ_2 preserve G and G'. Moreover

$$\mathrm{Int}\ \tau_2(g) = (g^{-1})^* \qquad (g \in G \text{ or } g \in G'), \text{ where } * \text{ is as in (1.59)}, \tag{1.64}$$

and

$$\mathrm{Int}\ \tau_1(g) = \natural_0\, g\, \natural_0 \qquad (g \in G \text{ or } g \in G'). \tag{1.65}$$

Define $\tau = \tau_1 \circ \tau_2$ and choose $K(K')$ as in (1.25). Then since $\mathrm{Int}\ \tau_2$ is the Cartan involution for G and G' (see (1.29)) and since $\tau \langle\ ,\ \rangle = -\langle\ ,\ \rangle$ by (1.62), (1.63), the theorem follows from Lemma 1.30.   Q.E.D.

2. REDUCTIVE DUAL PAIRS IN THE METAPLECTIC GROUP

By the metaplectic group $\widetilde{Sp}(W, \langle\ ,\ \rangle) = \widetilde{Sp}$ one understands the unique connected two-fold covering group of the symplectic group $Sp(W, \langle\ ,\ \rangle) = Sp$. Let $\widetilde{Sp}$ denote the universal covering group of Sp. Then for any automorphism ϕ of Sp we have the following commuting diagram

$$\Gamma \subseteq \tilde{\tilde{Sp}} \xrightarrow{\tilde{\tilde{\phi}}} \tilde{\tilde{Sp}} \supseteq \Gamma$$

$$\Big\downarrow \qquad \Big\downarrow \qquad (2.1)$$

$$1 \in Sp \xrightarrow{\phi} Sp \ni 1$$

where $\tilde{\tilde{\phi}}$ is an automorphism of $\tilde{\tilde{Sp}}$, the vertical arrows denote the covering map, and Γ is a discrete subgroup covering the identity element 1 of Sp and isomorphic to the additive group of integers $\mathbf{Z}$. The group Γ has only two automorphisms: the identity and the inverse mapping. Therefore, $\tilde{\tilde{\phi}}$ preserves the subgroup Γ_0 of index 2 in Γ. But the metaplectic group $\tilde{Sp} = \tilde{\tilde{Sp}}/\Gamma_0$. Therefore (cf. [D, 16.30.3])

for any automorphism ϕ of Sp there is a unique automorphism $\tilde{\phi}$ of $\tilde{Sp}$ such that the following diagram commutes:

$$\tilde{Sp} \xrightarrow{\tilde{\phi}} \tilde{Sp}$$

$$\Big\downarrow \qquad \Big\downarrow \qquad (2.2)$$

$$Sp \xrightarrow{\phi} Sp$$

For a reductive dual pair G, G' in the symplectic group Sp define

$\tilde{G}, \tilde{G}'$ to be the preimages in $\tilde{Sp}$ of G and G', respectively.   (2.3)

Recall that by a real reductive linear group Vogan [V1, 0.1.2] means a real Lie group G, a maximal compact subgroup K of G, and an involution θ of $\mathscr{g}$ satisfying the following conditions:

(a) $\mathscr{g}$ is a real reductive Lie algebra;

(b)   If $g \in G$, the automorphism $Ad(g)$ of $\mathscr{g}_{\mathbf{C}}$ is inner (for the corresponding complex connected group);

(c)   The fix point set of θ is $\mathscr{k}$;

(d)   Write $\mathscr{p}$ for the -1 eigenspace of θ, then the map $(g, X) \to g \exp(X)$ is a diffeomorphism from $K \times \mathscr{p}$ to G;

(e) G has a faithful finite dimensional representation;

(f)   Let $\mathscr{h} \subseteq \mathscr{g}$ be a Cartan subalgebra, and let H be the centralizer of $\mathscr{h}$ in G. Then H is abelian.

PROPOSITION 2.4.   *Let, G, G' be an irreducible dual pair. Then all G, G', $\tilde{G}$ and $\tilde{G}'$ satisfy* (a)–(f) *unles G, G' is a pair of type I and the associated division algebra with involution $(D, \natural) = (\mathbf{R}, 1)$. Then it may happen that $\tilde{G}'$*

is a metaplectic group (and thus fails to satisfy (e)) *and that* G, $\tilde{G}$ *violate* (b). *But then* G *(and* $\tilde{G}$) *contains a subgroup of index 2 which satisfies* (a)–(f).

The proof of this proposition is an exercise in the theory of Lie group coverings (see [D, 16.30]). It relies on the fact that one can embed the maximal compact subgroups, $K \subseteq G$, $K' \subseteq G'$ into a maximal compact subgroup U of $Sp(W, \langle\ ,\ \rangle)$, which is isomorphic to a compact complex unitary group, and the preimage $\tilde{U}$ of U in $\widetilde{Sp}(W, \langle\ ,\ \rangle)$ can be realized as

$$\tilde{U} = \{(g, z) \mid g \in U, z \in \mathbf{C}, \det g = z^2\}. \tag{2.5}$$

THEOREM 2.6.   *Let* G, G' *be a reductive dual pair in the symplectic group* $Sp(W, \langle\ ,\ \rangle)$. *Then there is an invertible endomorphism* τ *of* W, *maximal compact subgroups* $\tilde{K} \subset \tilde{G}$, $\tilde{K}' \subset \tilde{G}'$, *and maximal families of* θ-*stable Cartan subgroups* $\tilde{H}_1, \tilde{H}_2, ..., \tilde{H}_m$ *in* $\tilde{G}$ *and* $\tilde{H}'_1, \tilde{H}'_2, ..., \tilde{H}'_n$ *in* $\tilde{G}'$ *such that*

$$\tau \langle\ ,\ \rangle = -\langle\ ,\ \rangle, \tag{2.7}$$

the lifted automorphism $(\operatorname{Int} \tau)^{\sim}$ *(see* (2.2)) *preserves* $\tilde{G}$, $\tilde{K}$, $\tilde{H}_i$
and $\tilde{G}'$, $\tilde{K}'$, $\tilde{H}'_j$ $(i = 1, 2, ..., m; \ j = 1, 2, ..., n)$, $\tag{2.8}$

$W(\tilde{G}, \tilde{H}_i)(W(\tilde{G}', \tilde{H}'_j))$ *when extended by the restriction of* $(\operatorname{Int} \tau)^{\sim}$ *to* $\tilde{H}_i(\tilde{H}'_j)$ *contains the inverse mapping* $(i = 1, 2, ..., m;$ $j = 1, 2, ..., n)$. $\tag{2.9}$

Proof.   We take K, K', H_i, H'_j, and τ as in the Theorem 1.37 and define $\tilde{K}$, $\tilde{K}'$, $\tilde{H}_i$, $\tilde{H}'_j$ to be their preimages in the metaplectic group. The statements (2.7) and (2.8) are immediate. We shall prove (2.9). Recall [H4, Sect. 12] that by a compatible, positive complex structure on W one means an element J of the group $Sp(W, \langle\ ,\ \rangle)$ satisfying the following conditions:

$$J^2 = -1, \tag{2.10}$$

the symmetric bilinear form
$$B_J(w, w') = \langle Jw, w' \rangle \qquad (w, w' \in W) \tag{2.11}$$
is positive definite.

By inspecting the proof of Theorem 1.37, one checks that

there is a compatible, positive complex structure J on W,
which centralizes both K and K', $\tag{2.12}$

and

$$\operatorname{Int} \tau(J) = -J, \qquad (\operatorname{Int} \tau)^2 = 1. \tag{2.13}$$

Indeed, let

$J_1 \in \operatorname{End}_D(D^r)$ denote the r-fold direct sum of the map (1.9)
tensored with identity on D (1.19),   (2.14)

$I_{p,q} \in \operatorname{End}_D(D^{p+q})$ be the multiplication by $+1$ on D^p and by
-1 on D^q, where $D^{p+q} = D^p \oplus D^q$ via (1.6) and (1.19).   (2.15)

Define for $w \in W$

$$J(w) = \natural_1 \circ J_1 \circ w \quad \text{in the cases (1.45), (1.46)},   \qquad (2.16)$$

$$J(w) = -iI_{r,s} \circ w \circ I_{p,q} \quad \text{in the case (1.52)},   \qquad (2.17)$$

$$J(w) = T^* \oplus (-S^*) \quad \text{for } w = S \oplus T \text{ in the case (1.55)}.   \qquad (2.18)$$

A straightforward verification shows that J defined above satisfies (2.12)
and (2.13).

Denote by U_J the centralizer of J in Sp. Clearly (2.13) implies that

$$\text{Int } \tau \text{ preserves both } U_J \text{ and the center } Z_J \text{ of } U_J.   \qquad (2.19)$$

The group U_J is a compact unitary group [H4, Proposition 12.2], thus its
center Z_J is isomorphic to the multiplicative group of complex numbers
with absolute value equal to one. Therefore (2.13) implies that

$$\text{Int } \tau(g) = g^{-1} \qquad \text{for all } g \in Z_J.   \qquad (2.20)$$

The preimage $\tilde{U}_J$ of U_J in $\tilde{Sp}$ is a connected two-fold covering of U_J,
therefore (up to a Lie group isomorphism) we may realize it as

$$\tilde{U}_J = \{(g, z) \mid g \in U_J, z^2 = \det g\}.   \qquad (2.21)$$

In particular we see that the preimage $\tilde{Z}_J$ of Z_J in $\tilde{Sp}$ is connected and that
(2.20) holds with Z_J replaced by $\tilde{Z}_J$ and Int τ by (Int τ)$^\sim$.

Thus in the context of (2.21)

$$(\text{Int } \tau)^\sim(g, z) = (\text{Int } \tau(g), z^{-1}) \qquad (g \in U_J, z^2 = \det g).   \qquad (2.22)$$

Let $\mathfrak{u}_J$ be the Lie algebra of U_J. Then we have a Cartan decomposition

$$\mathfrak{sp} = \mathfrak{u}_J \oplus \mathfrak{p}_J,   \qquad (2.23)$$

which (by restriction) induces the Cartan decomposition of $\mathfrak{g}$ and of $\mathfrak{g}'$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'.   \qquad (2.24)$$

We decompose the Cartan subgroups H_j and H_l' (1.37) with respect to (2.24) as usual [V1, 0.2.1]:

$$H_j = T_j A_j, \quad H_l' = T_l' A_l' \qquad (j = 1, 2, ..., m; \ l = 1, 2, ..., n),$$

where $T_j = H_j \cap K$, $A_j = H_j \cap \exp(\not{p})$

and $T_l' = H_l' \cap K'$, $A_l' = H_l' \cap \exp(\not{p}')$. \hfill (2.25)

Clearly $(\text{Int } \tau)^{\sim}$ preserves $\tilde{T}_j(\tilde{T}_l')$ and since the covering (2.2) splits over the vector group $A_j(A_l')$, (2.9) follows from (1.37) and (2.22). \hfill Q.E.D.

DEFINITION 2.26. By a classical group we will mean any member of an irreducible dual pair (see Definition 1.4) or its preimage in the corresponding metaplectic group.

Notice that the Lie algebras of the classical groups defined above exhaust all the classical Lie algebras in the usual sense [He, Chap. IX, Sect. 4].

3. THE ADMISSIBLE DUAL OF A CLASSICAL GROUP

If Π is a continuous represnetation of a Lie group G then Π^{∞} will stand for the corresponding smooth representation. Π is called irreducible if $\Pi \neq 0$ and Π does not contain any proper, nonzero, closed G-invariant subspaces.

Let $\mathcal{g}$ be the Lie algebra of G and $K \subseteq G$ a compact subgroup.

DEFINITION 3.1 [V1, 0.3.9]. A $(\mathcal{g}, K)$-module is a complex vector space Ξ carrying representation of $\mathcal{g}$ and of K such that

Ξ is a (possibly infinite) direct sum of subspaces invariant and irreducible under K, \hfill (3.2)

the differential of the K-action is given by the inclusion of $\mathcal{k}$, the Lie algebra of K, in $\mathcal{g}$, \hfill (3.3)

the action of K on $\text{End}_\mathbb{C}(\Xi)$ by conjugation normalizes the image of $\mathcal{g}$ in $\text{End}_\mathbb{C}(\Xi)$ and induces the adjoint action of K on $\mathcal{g}$. \hfill (3.4)

A $(\mathcal{g}, K)$ module Ξ is admissible if all its K-isotypic components are finite dimensional [V1, 0.3.4]. Ξ is irreducible if $\Xi \neq 0$ and Ξ does not contain any proper nonzero $(\mathcal{g}, K)$ invariant subspaces.

Let G be a classical group (Eq. (2.10)) and K a maximal compact subgroup of G. For a continuous representation Π of G let Π_K be the associated Harish–Chandra module (consisting of K-finite vectors in Π)

[V1, 0.3.4]. Two representations of G are called infinitesimally equivalent if their Harish–Chandra modules are equivalent. The study of $(\mathcal{g}, K)$-modules instead of group representations is justified by several theorems of Harish–Chandra, Lepowsky, and Rader (see [V1, Theorems 0.3.5, 0.3.6, 0.3.10]). We will write

> $R(G)$ for the set of infinitesimal equivalence classes of
> irreducible admissible representations of G,   (3.5)

and

> $\hat{G}$ for the subset of $R(G)$ corresponding to the unitary
> representations of G.   (3.6)

For notational convenience we will identify the group representation Π with its Harish–Chandra module Π_K.

Let K be a maximal compact subgroup of G. From (Proposition 2.4) we know that K is not very far away from being connected. In fact one can check (using (2.21)) that $K = K^0 \times_s C$, where C is a finite (possibly trivial) group acting by automorphisms on the connected component K^0 of the identity of K and $\times_s$ denotes the semidirect product.

Let T^0 be a maximal torus (connected abelian subgroup) of K. The highest weight theory for a (disconnected) compact group defines a parametrization of irreducible representations of K on Hilbert spaces by dominant representations of $T^0 \times_s C$ (see [V3, remark after Definition 5.12]).

The point is that there is a Borel subalgebra $\mathcal{b}_C$ of $\mathcal{k}_C$ such that $T^0 \times_s C$ is equal to the normalizer of $\mathcal{b}_C$ in K. We are going to be more precise about that.

Any irreducible representation μ of T^0 is one dimensional and may be identified with its differential, also denoted by μ, which is an $\mathbf{R}$-linear mapping from the Lie algebra $\mathcal{t}$ of T^0 to $i\mathbf{R}$.

Let π be an irreducible (unitary) representation of K^0. For each irreducible unitary representation μ of T^0, the μ-weight space of π is

$$\pi_\mu = \{v \in \pi \mid \pi(g)v = \mu(g)v \text{ for all } g \in T^0\}. \qquad (3.7)$$

Obviously $\pi = \bigoplus_\mu \pi_\mu$; this is called the weight space decomposition of π. If $\pi_\mu \neq 0$, we call μ a weight of π. The multiplicity of μ in π is the dimension of π_μ.

The Borel subalgebra $\mathcal{b}_C$, mentioned above, has a d.irect sum decomposition,

$$\mathcal{b}_C = \mathcal{t}_C \oplus \mathcal{n}_C,$$

where $\mathcal{n}_C$ is the nilpotent radical in $\mathcal{b}_C$.

Naturally Lie algebra ℓ_C acts on π. There is a unique μ such that π_μ is annihilated by n_C. This space π_μ is one dimensional; a vector u_μ in π_μ is called a highest weight vector and μ the highest weight of π. It is well known that two irreducible, uniary representations of K^0 having the same highest weight are equivalent.

If π is an irreducible, unitary representation of K then $T^0 \times_s C$ acts irreducibly on the annihilator δ of n_C in π. Further restriction of δ to T^0 gives rise to a finite set of weights μ (3.7); we will call them the highest weights of π. The representation δ characterizes π.

Following Vogan [V1, Definition 5.4.18] let us write

$$\|\pi\|^2 = \min\{\mu + 2\rho_c, \mu + 2\rho_c)\,|\,\mu \in i\ell^* \text{ is a highest weight of } \pi\}, \quad (3.8)$$

where $2\rho_c = \Sigma\Delta(n_C, \ell_C)$ (see [V1, 5.3.2]). Note that all the μ's in the above set are conjugated by C, which preserves ρ_c, so the number $\|\mu + 2\rho_c\|^2$ is independent of μ.

An irreducible representation π of K is called a K-type of a (g, K)-module Ξ if the π-isotypic component of Ξ is nonzero. (3.9)

LEMMA 3.10. *Assume that K_V is a subgroup of K of index 2 (then K_V is normal). Let Π be an irreducible (g, K)-module. Then Π is a direct sum of at most two irreducible (g, K_V)-modules. If Π is a direct sum of two irreducible (g, K_V)-modules, then they are not isomorphic.*

Proof. Let $r \in K$ but $r \notin K_V$. Assume that there is a nonzero (g, K_V)-submodule $\Xi \subseteq \Pi$ such that

$$\Xi \cap r\Xi = 0.$$

Since $r^2 \in K_V$

$$\Xi + r\Xi$$

is a nonzero (g, K)-submodule of Π and therefore

$$\Xi \oplus r\Xi = \Pi.$$

Moreover Ξ is an irreducible (g, K_V)-module. Indeed if Y is a nonzero (g, K_V)-submodule of Ξ then

$$Y + rY$$

is a nonzero (g, K)-submodule of Π and therefore is equal to Π. Now if $v \in \Xi$ and $v \notin Y$ then

$$v = u + r w$$

for some $u, w \in \Upsilon$. Therefore

$$\mathbf{r} w = v - u \in (\mathbf{r} \Upsilon) \cap \Xi \subseteq (\mathbf{r} \Xi) \cap \Xi = 0$$

so that $v = u$. We see that $\Xi = \Upsilon$. Since $\mathbf{r}$ acts by conjugation on $(\mathcal{g}, K_V)$ it acts also on the set on irreducible $(\mathcal{g}, K_V)$-modules. It is clear that the action maps Ξ to $\mathbf{r}\Xi$.

Assume now that for every nonzero $(\mathcal{g}, K_V)$-submodule $\Xi \subseteq \Pi$ we have

$$\Xi \cap (\mathbf{r} \Xi) \neq 0.$$

Then $\Xi \cap (\mathbf{r} \Xi)$ is a nonzero $(\mathcal{g}, K)$-module and therefore

$$\Pi = \Xi \cap (\mathbf{r} \Xi) \subseteq \Xi \subseteq \Pi,$$

i.e.,

$$\Pi = \Xi$$

is irreducible as a $(\mathcal{g}, K_V)$-module. This proves the first statement of the lemma.

For the second one, assume that $\Xi = \Xi_1 \oplus \Xi_2$ as a $(\mathcal{g}, K_V)$ module and that there is a nonzero $(\mathcal{g}, K_V)$ intertwining operator T from Ξ_2 to Ξ_1. Take $\mathbf{r}$ from the complement of K_V in K. Then, since Ξ is irreducible as a $(\mathcal{g}, K)$ module, $\Xi_2 = \mathbf{r}\Xi_1$. Moreover the map

$$\Xi_1 v \to T(\mathbf{r}^{-1} T(\mathbf{r} v)) \in \Xi_1$$

is $(\mathcal{g}, K_V)$ intertwining and does not depend on $\mathbf{r}$. Since Ξ_1 is an irreducible $(\mathcal{g}, K_V)$ module we may assume that this is the identity map. In other words

$$T(\mathbf{r} v) = \mathbf{r} T^{-1}(v) \qquad \text{for} \quad v \in \Xi_1 \text{ and } \mathbf{r} \in K \text{ but } \mathbf{r} \notin K_V. \qquad (3.11)$$

Define

$$\Upsilon = \{ v \oplus T^{-1}(v) \mid v \in \Xi_1 \}.$$

One checks easily, using (3.11), that Υ is a proper $(\mathcal{g}, K)$ submodule of Ξ. This contradiction proves the second statement of the lemma.   Q.E.D.

DEFINITION 3.12 [V2, Definition 3.2].   An irreducible representation π of K is called a lowest K-type of a $(\mathcal{g}, K)$ module Π if

$$\|\pi\| \leqslant \|\sigma\| \qquad (\text{see } (3.8))$$

for any K-type σ of Π (see (3.9)).

THEOREM 3.13. *If Π is an irreducible admissible $(\mathscr{g}, K)$-module and $\pi \in \hat{K}$ a lowest K-type of Π, then π has multiplicity one in Π.*

Proof. Except for the case of the real orthogonal group or its preimage G this may be found in Vogan [V1, Sect. 6.5; V2, Theorem 3.3]. In this case the Proposition 2.4 implies that K contains a subgroup K_V of index two such that this theorem holds for $(\mathscr{g}, K_V)$-modules. The above lemma implies that the restriction of Π to $(\mathscr{g}, K_V)$ is an admissible $(\mathscr{g}, K_V)$-module of finite length. Let Y be an irreducible $(\mathscr{g}, K_V)$-quotient of that restriction.

By the Frobenius reciprocity law (see [V1, 0.3.25]) we have an injective morphism:

$$\Pi \to \operatorname{Ind}_{(\mathscr{g}, K_V)}^{(\mathscr{g}, K)}(Y).$$

Let $\sigma \in \hat{K}_V$ be the lowest K_V-type of Y. Then, by Eq. (3.11), the π-isotypic component of Π is mapped injectively via the above morphism into

$$\operatorname{Ind}_{K_V}^K(\sigma)$$

which decomposes into irreducibles each with multiplicity one (apply Lemma 3.10 to $(\mathscr{k}, K)$-modules). Therefore this isotypic component is equal to π.   Q.E.D.

For the definitions of a parabolic subgroup, cuspidal parabolic subgroup, Langlands decomposition, discrete series representation, parabolically induced representations, and character data we refer the reader to [V1].

THEOREM 3.14. *Let Π be an irreducible admissible representation of G, and let $\pi \in \hat{K}$ be a lowest K-type of Π. Then there is a set of cuspidal data (MA, η, ν) such that*

$$\pi \text{ is a lowest K-type in } \operatorname{Ind}_{M \cap K}^K(\eta|_{M \cap K}); \qquad (3.15)$$

for any parabolic subgroup P with Langlands decomposition MAN Π is infinitesimally equivalent to the unique irreducible subquotient of

$$\operatorname{Ind}_P^G(\eta \otimes \nu) \qquad (3.16)$$

containing the K-type π;

If $(M'A', \eta', \nu')$ is another set of cuspidal data for G satisfying (3.15) and such that Π is infinitesimally equivalent to the irreducible subquotient of

$$\operatorname{Ind}_{P'}^G(\eta' \otimes \nu') \qquad (P' = M'A'N'),$$

then $(M'A', \eta', \nu')$ is conjugated by K to (MA, η, ν).

Proof. Thanks to Vogan [V2, Proposition 9.11; V1, Theorems 6.6.15 and 6.5.10] and Proposition 2.4 we need only check the same case as in the proof of the previous Theorem 3.13. By Lemma 3.10 the restriction of Π to the subgroup G_V of index two in G contains an irreducible admissible quotient representation Υ of G_V with lowest K_V-type of $\sigma \in K_V$. Therefore Π is a subrepresentation of

$$\operatorname{Ind}_{G_V}^G(\Upsilon)$$

and π a subrepresentation of

$$\operatorname{Ind}_{K_V}^K(\sigma).$$

There exists a set of cuspidal data $(M_V A, \eta_V, \nu)$ for G_V such that (3.15) and (3.16) hold for Υ and G_V.

But then Π is a subquotient of

$$\operatorname{Ind}_{G_V}^G(\operatorname{Ind}_{M_V A N}^{G_V}(\eta_V \otimes \nu)) = \operatorname{Ind}_{MAN}^G(\operatorname{Ind}_{M_V}^M(\eta_V) \otimes \nu),$$

where MAN is a parabolic subgroup of G with the same Lie algebra as $M_V A N$ so that M_V is a subgroup of index two in M. The induced representation $\operatorname{Ind}_{M_V}^M(\eta_V)$ decomposes into a finite sum of irreducibles, each of which belongs to the discrete series of M. There is one of them, say η, such that Π is a subquotient of

$$\operatorname{Ind}_{MAN}^G(\eta \otimes \nu).$$

Since σ is a lowest K_V-type in

$$\operatorname{Ind}_{M_V \cap K_V}^{K_V}(\eta_V|_{M_V \cap K_V})$$

it follows that $\pi \subseteq \operatorname{Ind}_{K_V}^K(\sigma)$ is a lowest K-type in

$$\operatorname{Ind}_{M \cap K}^K(\eta|_{M \cap K}) \subseteq \operatorname{Ind}_{K_V}^K \operatorname{Ind}_{M_V \cap K_V}^{K_V}(\eta_V|_{M_V \cap K_V}).$$

Moreover since the multiplicity of σ in

$$\operatorname{Ind}_{M_V A N}^{G_V}(\eta_V \otimes \nu)$$

restricted to K_V is one and the multiplicity of π in

$$\operatorname{Ind}_{K_V}^K(\sigma)$$

is one, we see that the multiplicity of π in

$$\operatorname{Ind}_{MAN}^G(\eta \otimes \nu)$$

is one. Therefore this representation has only one subquotient containing the K-type π.

This shows (3.15) and (3.16). For (3.17) let (M', A', η', v') be another set of cuspidal data for G and such that Π is infinitesimally equivalent to an irreducible subquotient of

$$\text{Ind}_{M'A'N'}^{G}(\eta' \otimes v').$$

The group M' contains a subgroup M'_V of index two, which is also cotained in G_V. Moreover there is a discrete series representation η'_V of M'_V such that η' is a subrepresentation of

$$\text{Ind}_{M'_V}^{M'}(\eta'_V)$$

and Υ (as in the first part of the proof) is a subquotient of

$$\text{Ind}_{M'_V A'N'}^{G_V}(\eta'_V \otimes v').$$

Since the theorem holds for G_V, the cuspidal data $(M_V A, \eta_V, v)$ and $(M'_V A', \eta'_V, v')$ are conjugated by K_V. Therefore (MA, η, v) and $(M'A', \eta', v')$ are conjugated by K. Q.E.D.

The above theorem parametrizes the admissible dual of a classical group by K-conjugacy classes of cuspidal data and lowest K-types.

To each set of cuspidal data (MA, η, v) we can associate the set of character data (H, Γ, γ), and ice versa, as in [V1, Chap. VI, Sect. 6]. Let

$$A[H, \Gamma, \gamma] \text{ be the set of lowest } K\text{-types in } \text{Ind}_{M \cap K}^{K}(\eta|_{M \cap K}). \quad (3.18)$$

This set depends only on the K-conjugacy class of (H, Γ, γ). We conclude with the following.

COROLLARY 3.19. *Each (infinitesimal equivalence class of) irreducible admissible representation Π of a classical group G corresponds to a unique K-conjugacy class $[H, \Gamma, \gamma]$ of character data for G and a subset A of $A[H, \Gamma, \gamma]$, in such a way that $A = A(\Pi)$ is the set of lowest K-types of Π.* We will write

$$\Pi = \Pi_G[H, \Gamma, \gamma](A).$$

4. THE HERMITIAN REPRESENTATIONS

There are two natural involutions in the convolution ring [Wa, IA 2.4] $E'(G)$ of compactly supported distributions on a classical group G,

$$u^*(\phi) = \overline{u(\phi^*)} \qquad \text{and} \qquad u^\vee(\phi) = u(\phi^\vee), \quad (4.1)$$

where $u \in E'(G)$, $\phi \in C_c^\infty(G)$, $\phi^*(g) = \overline{\phi(g^{-1})}$, $\phi^\vee(g) = \phi(g^{-1})$, $g \in G$ (and "$-$" = $\natural_1$ (1.15)).

The first of them is antilinear

$$(au)^* = \bar{a}u^* \qquad (a \in \mathbf{C})$$

and the second is linear

$$(au)^\vee = au^\vee.$$

The composition of them will be denoted by "$-$":

$$\bar{u} := (u^*)^\vee = (u^\vee)^*, \qquad u \in E'(G). \tag{4.2}$$

Recall [K, Sect. 10.4; K–V, Sect. 1; F] that the Harish–Chandra–Hecke ring $U(g, K)$ is a subring of $E'(G)$ consisting of the left and right K-finite distributions with support in K. All of the above automorphisms (4.1), (4.2) preserve $U(g, K)$. This ring does not have the identity, only the approximate identity,

$$e_F = \sum_{\pi \in F} \dim \pi \cdot \chi_\pi, \tag{4.3}$$

where F varies over all finite sets in $\hat{K}$, and $\chi_\pi \in L^2(K)$ denotes the character of π.

Following [K–V] we call a $U(g, K)$ module Ξ unital if

for each $v \in \Xi$ there is a finite set $F_v \subseteq \hat{K}$ such that $e_F v = v$ for every finite subset F of $\hat{K}$ containing F_v. $\tag{4.4}$

Proposition 2.1 of [K–V] asserts that the category $C_{(g, K)}$ of (g, K)-modules is equivalent to the category of the unital $U(g, K)$-modules, denoted by the same symbol.

Let Π be a $U(g, K)$-module. Then

$$\text{Hom}_{\mathbf{C}}(\Pi, \mathbf{C})$$

becomes a $U(g, K)$ module if we define the action

$$U(g, K) \times \text{Hom}_{\mathbf{C}}(\Pi, \mathbf{C}), \qquad (u, f) \to u \cdot f \in \text{Hom}_{\mathbf{C}}(\Pi, \mathbf{C})$$

as

$$(u \cdot f)(v) = f(u^\vee \cdot v) \tag{4.5}$$

for $v \in \Pi$.

Assume that Π is an object in the category $C_{(\mathscr{g}, K)}$. Define the contragradient module Π^c, in this category, to be the submodule of $U(\mathscr{k}, K)$-finite vectors in $\operatorname{Hom}_C(\Pi, \mathbf{C})$. Explicitly

$$\Pi^c = \{ f \in \operatorname{Hom}_C(\Pi, \mathbf{C}) \,|\, \dim U(\mathscr{k}, K) f < \infty \}.$$

Naturally the association $\Pi \to \Pi^c$ extends to a contravariant, exact functor in $C_{(\mathscr{g}, K)}$.

Let's make $\operatorname{Hom}_C(\Pi, \mathbf{C})$ into a $U(\mathscr{g}, K)$-module in a different way. Namely

$$(u \cdot f)(v) = f(u^* \cdot v), \tag{4.6}$$

where $u \in U(\mathscr{g}, K)$, $f \in \operatorname{Hom}_C(\mathscr{g}, \mathbf{C})$, $v \in \Pi$. Then we can define

the hermitian dual Π^h of Π in $C_{(\mathscr{g}, K)}$ to be the $U(k, K)$-finite
part of $\operatorname{Hom}_C(\Pi, \mathbf{C})$. $\tag{4.7}$

Moreover the map $\Pi \to \Pi^h$ extends to a contravariant, exact functor in the category $C_{(\mathscr{g} K)}$.

Composing the two above functors we get an exact, covariant functor $\Pi \to \bar{\Pi}$ in the subcategory $CA_{(\mathscr{g}, K)}$ of the admissible modules.

Explicitly (in $C_{(\mathscr{g}, K)}$) $\bar{\Pi} = \Pi$ as an additive group, but the action of $U(\mathscr{g}, K)$ on it is different,

$$u \cdot v = \bar{u}(v),$$

for $u \in U(\mathscr{g}, K)$, $v \in \Pi$, where $\bar{u}(v)$ means the original action of $\bar{u} \in U(\mathscr{g}, K)$ or $v \in \Pi$.

Obviously (for Π in $CA_{(\mathscr{g}, K)}$) $\bar{\bar{\Pi}} = \Pi$, $\Pi^{cc} \cong \Pi$, $\Pi^{hh} \cong \Pi$. We will say that

$$\Pi \text{ is hermitian if } \Pi \cong \Pi^h. \tag{4.8}$$

Since we are interested only in the infinitesimal equivalence classes of representations of classical groups on locally convex spaces, we will say that

a representation Π_1 is a hermitian dual of a representation
Π_2 if $(\Pi_1)_K \cong (\Pi_2)_K^h$ in $C_{(\mathscr{g}, K)}$. Similarly Π_1 is a contra-
gradient representation of Π_2 if $(\Pi_1)_K = (\Pi_2)_K^c$. $\tag{4.9}$

In the context of Corollary 3.19 we have the following known theorem (cf. [V1, Proposition 8.5.6; K–Z] and Theorem 3.13).

THEOREM 4.10. *Let $[H, \Gamma, \gamma]$ be a set of character data, A a subset of $A[H, \Gamma, \gamma]$, and $\Pi[H, \Gamma, \gamma](A)$ the associated irreducible admissible representation of G. Then*

$$(\Pi[H, \Gamma, \gamma](A))^c = \Pi[H, \Gamma^{-1}, -\gamma](A^c),$$

where $A^c = \{\pi^c \mid \pi \in A\}$.

Moreover if $\kappa \in \operatorname{Aut}(G)$ is such that $\kappa(K) = K$, $\kappa(H) = H$, then

$$\kappa \cdot (\Pi[H, \Gamma, \gamma](A)) = \Pi[H, \kappa(\Gamma), \kappa(\gamma)](\kappa(A)),$$

where

$$\kappa(A) = \{\kappa(\pi) \mid \pi \in A\}.$$

5. HOWE'S DUALITY THEOREM

The metaplectic group $\tilde{Sp}(W, \langle \, , \, \rangle)$ has a unitary representation ω called the oscillator representation [A; H5; K–Ve]. It has the property that

if $\tilde{\phi}$ is an automorphism of $\tilde{Sp}(W, \langle \, , \, \rangle)$ as in (2.2) such that $\phi = \operatorname{Int} \tau$, $\tau \in GL(W)$, $\tau \langle \, , \, \rangle = -\langle \, , \, \rangle$, then $\phi(\omega) = \omega^c$—the contragradient oscillator representation.   (5.1)

For any reductive Lie subgroup $E \subseteq Sp(W, \langle \, , \, \rangle)$ let $\tilde{E}$ be the preimage of E in the metaplectic group $\tilde{Sp}(W, \langle \, , \, \rangle)$ and $R(\tilde{E}, \omega)$ the set of infinitesimal equivalence classes of continuous irreducible admissible representations of $\tilde{E}$ on locally convex topological vector spaces which can be realized as quotients by $\omega^\infty(\tilde{E})$ invariant closed subspaces of the space ω^∞. The following theorem of Roger Howe reveals the very special character of the oscillator representation.

THEOREM 5.2 [H2]. *The set $R(\tilde{G} \cdot \tilde{G}', \omega)$ is the graph of bijection between (all of) $R(\tilde{G}, \omega)$ and (all of) $R(\tilde{G}', \omega)$. In other words for each $\Pi \in R(\tilde{G}, \omega)$ there is a unique $\Pi' \in R(\tilde{G}', \omega)$ such that*

$$\Pi \,\check{\otimes}\, \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega),$$

and vice versa. (Here "$\check{\otimes}$" means the outer tensor product. The topology of $\Pi \,\check{\otimes}\, \Pi'$ is not uniquely defined but the infinitesimal equivalence class is.)

Moreover for Π and Π' as above

$$\dim(\operatorname{Hom}_{\tilde{G} \cdot \tilde{G}'}(\omega^\infty, \Pi \,\check{\otimes}\, \Pi')) = 1.$$

We will call the function (which is a bijection)

$$R(\tilde{G}, \omega) \Pi \to \Pi' \in R(G'\omega) \tag{5.3}$$

the "Duality Correspondence."

Let

$$R(\tilde{G} \cdot \tilde{G}', \omega)^c = \{(\Pi \,\check{\otimes}\, \Pi')^c \,|\, \Pi \,\check{\otimes}\, \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)\}. \tag{5.4}$$

THEOREM 5.5.   *For any reductive dual pair G, G' in $Sp(W, \langle\,,\,\rangle)$*

$$R(\tilde{G} \cdot \tilde{G}', \omega)^c = R(\tilde{G} \cdot \tilde{G}', \omega^c). \tag{5.6}$$

Here ω^c is the contragradient of ω as an $\tilde{S}p(\omega)$-module (not $\tilde{G} \cdot \tilde{G}'$-module).

Proof.   By (5.1) and Theorems 2.6, 4.10, there is an automorphism $\tilde{\kappa}$ of $\tilde{S}p(W, \langle\,,\,\rangle)$ such that

$$\tilde{\kappa}(\omega) = \omega^c, \tag{5.7}$$

$$\tilde{\kappa}(\tilde{G}) = \tilde{G}, \;\; \tilde{\kappa}(\tilde{G}') = \tilde{G}', \tag{5.8}$$

and

$$\tilde{\kappa}(\Pi) = \Pi^c, \;\; \tilde{\kappa}(\Pi') = (\Pi')^c \quad \text{for any} \quad \Pi \,\check{\otimes}\, \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega). \tag{5.9}$$

Therefore

$$R(\tilde{G} \cdot \tilde{G}', \omega)^c = \tilde{\kappa}(R(\tilde{G} \cdot \tilde{G}', \omega)) = R(\tilde{G} \cdot \tilde{G}', \kappa(\omega)) = R(\tilde{G} \cdot \tilde{G}', \omega^c). \qquad \text{Q.E.D.}$$

THEOREM 5.10.   *Let G, G' be a reductive dual pair and $\Pi \,\check{\otimes}\, \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$. Then Π is hermitian if and only if Π' is hermitian.*

In other words the Duality Correspondence (5.2) maps hermitian representations of $\tilde{G}$ to hermitian representations of $\tilde{G}'$.

Proof.   Theorem 5.5 implies that

$$R(\tilde{G} \cdot \tilde{G}', \omega)^h = R(\tilde{G} \cdot \tilde{G}', \omega)^{c-} = R(\tilde{G} \cdot \tilde{G}', \omega^c)^- = R(\tilde{G} \cdot \tilde{G}', \omega). \tag{5.11}$$

Since for any Π and Π', $(\Pi \,\check{\otimes}\, \Pi')^h = \Pi^h \,\check{\otimes}\, (\Pi')^h$ we see that this theorem follows from Theorem 5.2.   Q.E.D.

6. CONCLUDING REMARKS

The proof of Theorem 5.10 depends on the Langlands–Vogan classification of the irreducible admissible representations of classical groups (Corollary 3.19, Theorem 4.10). Howe's Duality Theorem (5.2) and its

proof are completely independent from this classification and are purely algebraic in their nature. Therefore, one should provide an independent proof of Theorem 5.10 in order to get a truly new construction of hermitian representations. One can do it for the so-called degree zero on degree one representations but I do not have any complete argument in general. Theorem 5.5, which actually asserts that

$$R(\widetilde{G} \cdot \widetilde{G}', \omega)^{\mathrm{h}} = R(\widetilde{G} \cdot G', \omega),$$

is interesting independently of Theorem 5.10. This theorem should also hold in the p-adic case [H3].

REFERENCES

[A]   J. D. ADAMS, Discrete spectrum of the reductive dual pair $(O(p, q), Sp(2m))$, *Invent. Math.* 74 (1983), 449–475.
[D]   J. DIEUDONNÉ, "Treatise on Analysis," Vol. III, Academic Press, New York/London, 1972.
[F]   D. FLATH, Decomposition of representations into tensor products, *in* "Proc. Symp. Pure Math.", Vol. XXXIII, AMS, Providence, RI, 1980.
[He]   S. HELGASON, "Differential Geometry and Symmetric Spaces," Academic Press, New York/London, 1962.
[H1]   R. HOWE, Remarks on classical invariant theory, preprint.
[H2]   R. HOWE, Transcending the classical invariant theory, preprint.
[H3]   R. HOWE, θ-series and invariant theory, *in* "Proc. Symp. Pure Math." Vol. XXXIII, pp. 275–285, AMS, Providence, RI, 1979.
[H4]   R. HOWE, manuscript in preparation on dual pairs.
[H5]   R. HOWE, Dual pairs in physics: Harmonic oscillators, photons, electrons, and singletons, *in* "Lectures in Applied Mathematics," Vol. 21, pp. 179–207, AMS, Providence, RI, 1985.
[H6]   R. HOWE, "On a Notion of Rank for Unitary Representations of the Classical Groups," CIME Summer School on Harmonic Analysis, Cortona, 1982.
[J1]   N. JACOSON, "Basic Algebra I," Freeman, San Francisco, 1974.
[J2]   N. JACOBSON, "Basic Algebra II," Freeman, San Francisco, 1980.
[Ki]   A. A. KIRYLLOV, "Elements of the Theory of Representation," Moscow, 1972. [in Russian]

[K]   A. KNAPP, "Representation Theory of Semisimple Groups—An Overview Based on Examples," Princeton Univ. Press, Princeton, NJ, 1986.

[K–V]   A. KNAPP AND D. A. VOGAN, Jr., Duality theorems in the relative Lie algebra cohomology, preprint.

[K–Ve] M. KASHIWARA AND M. VERGNE, On the Segal–Shale–Weil representation and harmonic polynomials, *Invent. Math.* **44** (1978), 1–97.

[K–Z]   A. KNAPP AND G. ZUCKERMAN, Classification theorems for representations of semisimple Lie groups, non-commutative harmonic analysis, *in* "Springer-Verlag Lecture Notes in Math. 587," pp. 138–159, 1977.

[Su]   M. SUGIURA, Conjugate classes of Cartan subalgebras in real semisimple Lie algebras, *J. Math. Soc. Japan* **11**, No. 4 (Oct. 1959).

[V1]   D. A. VOGAN, JR., "Representations of Real Reductive Lie Groups," Birkhäuser, Boston/Basel/Stuttgart, 1981.

[V2]   D. A. VOGAN, JR., Classifying representations by lowest K-types, *in* "Lectures in Applied Mathematics," Vol. 21, 1985.

[V3]   D. A. VOGAN, JR., The unitary dual of Gl_n over an Archimedian field, *Invent. Math.* **83**, Fasc. 3 (1986), 449–507.

[Wa]   G. WARNER, "Harmonic Analysis on Semisimple Lie Groups I," Springer-Verlag, Berlin, 1972.

[We]   A. WEIL, "Basic Number Theory," Springer-Verlag, Berlin, 1974.