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THE WAVE FRONT SET AND THE ASYMPTOTIC SUPPORT FOR p-ADIC GROUPS

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We prove that for p-adic groups the notion of the wave front set of a representation coincides with the notion of the asymptotic support.

1. The wave front sets of finite sums of homogeneous distributions. Let Ω be a p-adic field of characteristic zero, with valuation $|\cdot|$. Let $\mathbf{g}$ be a finite dimensional vector space over Ω. Fix a non-trivial character χ of the additive group Ω, and a non-degenerate symmetric bilinear form β on $\mathbf{g}$ with values in Ω.

For $f \in C_c^\infty(\mathbf{g})$ (compactly supported, locally constant functions on $\mathbf{g}$) define a Fourier Transform by

$$\hat{f}(Y) = \int_{\mathbf{g}} \chi(\beta(Y, X)) f(X)\, dX \qquad (Y \in \mathbf{g}). \tag{1.1}$$

Here dX is a Haar measure on the additive group of $\mathbf{g}$ (normalized so that the formula $(\hat{f})\hat{}\,(x) = f(-x)$ holds). Then $f \to \hat{f}$ is a bijective mapping of $C_c^\infty(\mathbf{g})$ onto itself (see **[Ha1]** or **[W**, p. 107]). If T is a distribution $\mathbf{g}$ then its Fourier transform $\hat{T}$ is given by

$$\hat{T}(f) = T(\hat{f}) \qquad (f \in C_c^\infty(\mathbf{g})). \tag{1.2}$$

Let $n = \dim_\Omega(\mathbf{g})$. For $f \in C_c^\infty(\mathbf{g})$ define

$$f_\lambda(X) = |\lambda|^{-n} f(\lambda^{-1}X) \qquad (X \in \mathbf{g},\ \lambda \in \Omega^\times). \tag{1.3}$$

Fix an open subgroup Λ of Ω^x with $[\Omega^x : \Lambda] < \infty$.

DEFINITION 1.4. A distribution T on $\mathbf{g}$ is Λ-homogeneous of degree $d \in \mathbf{C}$ if

$$T(f_\lambda) = |\lambda|^d T(f) \qquad (f \in C_c^\infty(\mathbf{g}),\ \lambda \in \Lambda).$$

Notice that

$$(f_\lambda)\hat{}\ = |\lambda|^{-n} (\hat{f})_{\lambda^{-1}} \qquad (f \in C_c^\infty(\mathbf{g}),\ \lambda \in \Omega^x), \tag{1.5}$$

so that if T is Λ-homogeneous of degree d then $\hat{T}$ is a Λ-homogeneous of degree $-n-d$. Clearly if T is a function:

$$T(f) = \int_{\mathbf{g}} T(X) f(X)\, dX,$$

then T is Λ-homogeneous of degree d iff for any $\lambda \in \Lambda$,

$$T(\lambda X)\,dX = |\lambda|^d T(X)\,dX.$$

The reader may safely focus on the case $\Lambda = \Omega^\times$. In order to justify the generality of Definition 1.4 we mention that a distribution homogeneous with respect to a quasicharacter of $\Omega^\times$ is Λ-homogeneous for a suitable Λ (see for example **[G-G-PS**, Ch. II]).

By fixing a base of $\mathbf{g}$ we can identify it with Ω^n and use the norm

$$|(\lambda, \lambda_2, \cdots, \lambda_n)| = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}. \tag{1.6}$$

The following simple fact will be used later.

LEMMA 1.7. *Let F and V be open-compact subsets of $\mathbf{g}$. Then there is $\delta > 0$ such that for any $\lambda \in \Omega$ with $|\lambda| < \delta$ the following inclusion holds*:

$$\lambda F + V \subseteq V.$$

It is known that any compactly supported distribution on $\mathbf{g}$ has a locally constant function as a Fourier Transform.

We are going to use (1.2) to analyze the singularities of T near zero.

DEFINITION 1.8 (**[He]** §2). A distribution T on $\mathbf{g}$ is Λ-smooth at $Y_0 \in \mathbf{g}\setminus\{0\}$ if there is an open neighborhood W of 0 and an open neighborhood V of Y_0 such that for any $f \in C_c^\infty(W)$ there is $N > 0$ for which $\lambda \in \Lambda$ and $|\lambda| > N$ imply

$$(fT)^\frown(\lambda Y) = 0 \quad \text{for any } Y \in V.$$

The complement of the set of Λ-smooth points of T in $\mathbf{g}\setminus\{0\}$ is called the Λ-*wave front set* of T at zero and is denoted $\mathrm{WF}^0_\Lambda(T)$.

The function $(fT)^\frown$, (1.9), is sometimes called a localized Fourier Transform of T (because $\mathrm{supp}(fT) \subseteq \mathrm{supp}(f)$). Of course this function can be expressed in terms of the convolution

$$(fT)^\frown = \hat{f} * \widehat{T}, \quad \text{where for } X, Y \in \mathbf{g}, \tag{1.10}$$
$$\hat{f} * \widehat{T}(X) = \widehat{T}(L_X \hat{f}), \quad L_X \hat{f}(Y) = \hat{f}(X - Y).$$

Using (1.10) and the notion of a lattice in $\mathbf{g}$ **[W**, p. 28] we rephrase the Definition 1.8. For a subset $U \subseteq \mathbf{g}$, let f_U denote the characteristic function of U.

LEMMA 1.11. *Let T be a distribution on $\mathbf{g}$ and let V be an open-compact subset of $\mathbf{g}\backslash\{0\}$. Then the following conditions on V are equivalent:*

(a) *$V \cap WF^0_\Lambda(T)$ is empty.*

(b) *There is a lattice U in $\mathbf{g}$ and a constant $c > 0$, such that*

$$f_U * \widehat{T}(\lambda Y) = 0 \quad \text{for } \lambda \in \Lambda,\ |\lambda| > c,\ Y \in V.$$

(c) *There is a lattice W in $\mathbf{g}$ and for any constant $1 > \varepsilon > 0$ a constant $c_\varepsilon > 0$ such that for any $f \in C_c^\infty(W)$,*

$$(*) \qquad (f_\gamma T)\widehat{\ }(\lambda Y) = 0 \quad \text{for } \lambda, \gamma \in \Lambda, |\lambda| > c_\varepsilon,\ \varepsilon < |\gamma| < 1, Y \in V.$$

Proof. Clearly $(*)$ implies (a). The equivalence of (a) and (b) was shown by Heifetz [**He**, Lemma 2.2] . We shall recall his proof to see that (b) implies $(*)$. Let W be the lattice dual to U, $f \in C_c^\infty(W)$, and let $F = -\operatorname{supp} \hat{f}$. Lemma 1.7 applied to the sets F and V provides a constant $\delta > 0$. Put $c_\varepsilon = \max\{\delta^{-1}\varepsilon^{-1}, c\}$. Since by (1.5) $\operatorname{supp}(f_\gamma)\widehat{\ } = \gamma^{-1} \operatorname{supp} \hat{f}$ we see that (under the assumptions of $(*)$)

$$\begin{aligned}(f_\gamma T)\widehat{\ }(\lambda Y) &= (f_\gamma f_W T)\widehat{\ }(\lambda Y) \\ &= \int_{\mathbf{g}} (f_\gamma)\widehat{\ }(Z) f_W T)\widehat{\ }(\lambda(-\lambda^{-1}Z + Y))\, dZ = 0. \qquad \square\end{aligned}$$

The reader may compare this proof with [**Hö**, 8.1.1] to see that the analogous argument in the classical situation is more complex.

Lemma 1.11 has the following immediate

COROLLARY 1.12. *The wave front set $\mathrm{WF}^0_\Lambda(T)$ contains the set A of those $Y \in \mathbf{g}\backslash\{0\}$ satisfying the condition that for any lattice $U \subseteq \mathbf{g}$ and any constant $c > 0$ there is $\lambda \in \lambda$ with $|\lambda| > c$ such that $f_U * \widehat{T}(\lambda Y) \neq 0$.*

Clearly Lemma 1.11 implies that

$$(1.13) \qquad \mathrm{WF}^0_\Lambda(T) \subseteq \Lambda \cdot \operatorname{supp} \widehat{T}.$$

Also, since for any lattice $U \subseteq \mathbf{g}$ the support of $f_U \widehat{T}$ is compact, the wave front set of T is the same as that associated to the truncation T_U of T at infinity, defined by $\widehat{T}_U = \widehat{T} - f_U \widehat{T}$. Therefore we have another

COROLLARY 1.14. *The wave front set $\mathrm{WF}^0_\Lambda(T)$ is contained in the set B, the intersection of all $\Lambda \cdot \operatorname{supp} \widehat{T}_U$, where U varies over all lattices in $\mathbf{g}$.*

Next we define a p-adic analog of the classical notion of an asymptotic cone (see [Hö, 8.1.7]). For any subset E of $\mathbf{g}\backslash\{0\}$ define its Λ-asymptotic cone to be the set

$$\text{(1.15)} \qquad \mathrm{AC}_\Lambda(E) = \left\{ \lim_{j+\infty} \lambda_j Z_j \mid \lambda_j \in \Lambda,\ \lim_{j\to\infty} \lambda_j = 0,\ Z_j \in E \right\}.$$

By a Λ-conical subset of $\mathbf{g}$ we will mean a subset closed under multiplication by elements of Λ. Then $\mathrm{AC}_\Lambda(E)$ is a closed Λ-conical subset of $\mathbf{g}$.

THEOREM 1.16. *For any distribution T on $\mathbf{g}$ define the sets A and B as in Corollaries* 1.12 *and* 1.14 *respectively. Then*

$$\text{(1.17)} \qquad A \subseteq \mathrm{WF}^0_\Lambda(T) \subseteq B \subseteq \mathrm{AC}_\Lambda(\operatorname{supp} \widehat{T}).$$

Moreover all these sets (1.17) *coincide if T is Λ-homogeneous.*

Proof. Only the last inclusion in (1.17) remains to be verified. It is obvious, however, if we realize that for any lattice U in $\mathbf{g}$ the support of $\widehat{T}_U$ is contained in the intersection of the support of $\widehat{T}$ with the complement of U in $\mathbf{g}$.

LEMMA 1.18. *For any finite sequence of real numbers $d_1 < d_2 < \cdots < d_r$ and a sequence $a_1, a_2, \ldots, a_r$ of complex numbers define the function*

$$F(x) = a_1 x^{d_1} + a_2 x^{d_2} + \cdots + a_r x^{d_r} \qquad (x > 0).$$

Then either F is identically equal to zero or F has at most $r-1$ zeros.

We omit the elementary proof.

THEOREM 1.19. *Let $T_1, T_2, \ldots, T_r$ be Λ-homogeneous distributions on $\mathbf{g}$ of degrees $d_1 < d_2 < \cdots < d_r$ respectively. Put $T = T_1 + T_2 + \cdots + T_r$. then*

$$\mathrm{WF}^0_\Lambda(T) = \bigcup_{j=1}^{r} \mathrm{WF}^0_\Lambda(T_j).$$

Proof. Since the wave front set of a finite sum of distributions is clearly contained in the union of the wave front sets of the summands, it will suffice to verify the inclusion

$$\text{(1.20)} \qquad \mathrm{WF}^0_\lambda(T) \supseteq \bigcup_{j=1}^{r} \mathrm{WF}^0_\Lambda(T_j).$$

Take V disjoint with $\mathrm{WF}^0_\Lambda(T)$ as in Lemma 1.11 (a). Then by (c)

$$0 = (f_\gamma T)^\wedge(\gamma^{-1}\lambda Y) = \sum_{j=1}^{r} |\gamma|^{d_j}(fT_j)^\wedge(\lambda Y) \tag{1.21}$$

for $f \in C_c^\infty(W)$, $\lambda \in \Lambda$, $\gamma \in \Lambda$, $|\gamma| > c_\varepsilon$, $\varepsilon < |\gamma| < 1$, $Y \in V$.

Choose $\varepsilon > 0$ so that there are at least r elements in the set $(\varepsilon, 1] \cap \{|\gamma| \,|\, \gamma \in \Lambda\}$. Then Lemma 1.18 implies that each summand in (1.21) is zero. □

2. ***P*-adic wave front sets of group representation.** Let $\mathbf{G}$ be a connected, reductive Ω-group and G the subgroup of all Ω-rational points in $\mathbf{G}$. Then G with its usual topology is a locally compact, totally disconnected, unimodular group. Let $\mathbf{g}$ be the Lie algebra of G. Then $\mathbf{g}$ is a vector space over Ω of finite dimension and G operates on $\mathbf{g}$ by means of the adjoint representation. Assume that the form β in (1.1) is G-invariant.

Let π be an irreducible admissible representation of G and

$$\Theta_\pi(f) = \operatorname{tr}\pi(f) \qquad (f \in C_c^\infty(G))$$

be its character.

Let N be the set of all elements of $\mathbf{g}$ which are nilpotent. Then N is the union of a finite number of G-orbits which are called the nilpotent orbits. For all this see **[Ha1]**, **[Ha2]**. Harish-Chandra **[He** 1, p. 180**]** has shown that one can choose an open neighborhood U of zero in $\mathbf{g}$ and, for each nilpotent orbit $\mathbf{O}$, a complex constant $c_{\mathbf{O}}$ such that

$$\Theta_\pi(\exp(X)) = \sum_{\mathbf{O}} c_{\mathbf{O}}\hat{\mu}_{\mathbf{O}}(X) \qquad (X \in U). \tag{2.1}$$

Here $\mu_{\mathbf{O}}$ is a Radon measure on $\mathbf{g}$ given by

$$\mu_{\mathbf{O}}(f) = \int_{G/G_0} f(\operatorname{Ad} g \cdot X_0)\, dg^* \qquad (f \in C_c^\infty(\mathbf{g}))$$

where $X_0 \in \mathbf{O}$ and G_0 is the stabilizer of X_0 in G (see **[R]**).

It follows from Theorem 1 in **[R]**, that $\mu_{\mathbf{O}}$ is a $\Omega^\times$-homogeneous distribution on $\mathbf{g}$ of degree $d = -n + \dim_\Omega(\mathbf{O})/2$. Therefore, via statement (1.5), $\hat{\mu}_0$ is a homogeneous distribution of degree $-\dim_\Omega(\mathbf{O})/2$.

Let π be an admissible representation of G of finite length. Put

$$T = \Theta_\pi \cdot \exp.$$

Then (2.1) implies that

$$T = \sum_{j=1}^{r} T_j$$

where the T_j's are homogeneous distributions on $\mathbf{g}$ of degrees d_j ($j = 1, 2, \ldots, r$). Explicitly

$$T_j = \sum_{\dim \mathbf{O}/2 = -d_j} c_{\mathbf{O}} \hat{\mu}_{\mathbf{O}}.$$

Retain the above notation. Then Theorem 1.19 implies the following

THEOREM 2.2. *Let π be an admissible representation of G of finite length. Then*

$$\mathrm{WF}^0_\Lambda(T) = \bigcup_{j=1}^{r} \operatorname{supp} \widehat{T}_j.$$

The left hand side of the first equation may be thought of as the wave front set of the representation π (see **[H]**, **[He]**) and the right hand side as the asymptotic support (see **[B-V]**) of π. Recall also **[He**, Theorem 3.4] that for π unitary $\mathrm{WF}^0_\Lambda(T)$ coincides with the wave front set of π defined by the trace class operators. A statement analogous to Theorem 2.2 for the real reductive Lie groups was conjectured in **[B-V]** (and should hold via the inverse of the Lefschetz principle). Theorem 1.19 is true in the real case and its proof is equally easy.

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REFERENCES

[B-V] D. Barbasch and P. Vogan, Jr., *The local structure of characters*, J. Funct. Anal., **37** (1980), 27–55.

[G-G-PS] I. M. Gelfand, M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, W. B. Saunders, Philadelphia (1969).

[H] Roger Howe, *Wave front sets of representation of Lie groups, automorphic forms, representation theory, and arithmetic*, Tata Inst., Bombay (1979).

[Ha1] Harish-Chandra, *The Characters of Reductive p-Adic Groups, Contributions to Algebra*, Academic Press, New York (1977).

[Ha2] ———, *Harmonic analysis on reductive p-adic groups*, in *Harmonic Analysis on Homogenous Spaces* Amer. Math. Soc., Providence, Rhode Island, 1973.

[He] D. B. Heifetz, *p-adic oscillatory integrals and wave front sets*, Pacific J. Math., **116**, No. 2, (1985), 285–305.

[Hö] Lars Hörmander, *The Analysis of Linear Partial Differential Operators* I, Springer Verlag, 1983.

[R] R. Ranga Rao, *Orbital integrals in reductive groups*, Ann. of Math., **96** (1972), 505–510.

[W] André Weil, *Basic Number Theory*, Springer Verlag (1974).

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