WEYL CALCULUS AND DUAL PAIRS

M. MCKEE, A. PASQUALE, AND T. PRZEBINDA

ABSTRACT. We consider a dual pair $(\mathrm{G}, \mathrm{G}')$, in the sense of Howe, with G compact acting on $\mathrm{L}^2(\mathbb{R}^n)$ for an appropriate n via the Weil Representation. Let $\widetilde{\mathrm{G}}$ be the preimage of G in the metaplectic group. Given a genuine irreducible unitary representation Π of $\widetilde{\mathrm{G}}$ we compute the Weyl symbol of orthogonal projection onto $\mathrm{L}^2(\mathbb{R}^n)_\Pi$, the Π-isotypic component. We apply the result to obtain an explicit formula for the character of the corresponding irreducible unitary representation Π' of $\widetilde{\mathrm{G}}'$ and to compute of the wave front set of Π' by elementary means.

CONTENTS

1. Introduction.

Let W be a vector space of finite dimension $2n$ over $\mathbb{R}$ with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. Denote by $\mathrm{Sp} \subseteq \mathrm{GL}(\mathrm{W})$ the symplectic group and let $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp} = \mathrm{Sp}(\mathrm{W})$ be an irreducible dual pair. Denote by $\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}'$ the preimages of G, G' in the metaplectic group

2010 *Mathematics Subject Classification.* Primary: 22E45; secondary: 22E46, 22E30.

Key words and phrases. Reductive dual pairs, Howe duality, Weyl calculus, Lie superalgebras.

The second author is grateful to the University of Oklahoma for hospitality and financial support. The third author gratefully acknowledges hospitality and financial support from the Université de Lorraine and partial support from the NSA grant H98230-13-1-0205.

$\widetilde{\mathrm{Sp}} = \widetilde{\mathrm{Sp}}(\mathrm{W})$. Consider irreducible admissible representations Π, Π' of $\widetilde{\mathrm{G}}$, $\widetilde{\mathrm{G}}'$ respectively which are in Howe's correspondence. By definition $\Pi \otimes \Pi'$ is realized as a quotient of the space of the smooth vectors of the Weil representation ω of $\widetilde{\mathrm{Sp}}(\mathrm{W})$. Hence, as explained in [Prz93], there is a unique, up to a non-zero constant multiple, GG'-invariant tempered distribution $f_{\Pi\otimes\Pi'}$ on W such that $\Pi \otimes \Pi'$ is realized on the range of the operator

$$\mathrm{Op} \circ \mathcal{K}(f_{\Pi\otimes\Pi'}). \tag{1}$$

(See (3) and (4) below for the precise definitions of Op and $\mathcal{K}$.) Thus, in principle, all the information about the representation $\Pi \otimes \Pi'$ is encoded in the distribution $f_{\Pi\otimes\Pi'}$. For example the group action is given by

$$(\Pi(\tilde{g}) \otimes \Pi'(\tilde{g}')) \circ (\mathrm{Op} \circ \mathcal{K}(f_{\Pi\otimes\Pi'})) = (\mathrm{Op} \circ \mathcal{K}(f_{\Pi\otimes\Pi'})) \circ \omega(\tilde{g}\tilde{g}') \qquad (\tilde{g} \in \widetilde{\mathrm{G}}, \tilde{g}' \in \widetilde{\mathrm{G}}').$$

This is why $f_{\Pi\otimes\Pi'}$ is called an intertwining distribution [Prz93]. In fact $f_{\Pi\otimes\Pi'}$ happens to be the Weyl symbol, [Hör83], of the operator (1), see (5) below.

Often $f_{\Pi\otimes\Pi'}$ may be computed in terms of the distribution character Θ_Π of Π, [Prz93]. If the group G is compact then the distribution character $\Theta_{\Pi'}$ may also be recovered from $f_{\Pi\otimes\Pi'}$ via an explicit formula, (19), [Prz91]. Thus we have a diagram

$$\Theta_\Pi \longrightarrow f_{\Pi\otimes\Pi'} \longrightarrow \Theta_{\Pi'}. \tag{2}$$

The asymptotic properties of $f_{\Pi\otimes\Pi'}$ determine the associated varieties of the primitive ideals of Π and Π' and, under some more assumptions, the wave front sets of these representations, see [Prz93] and [Prz91].

We believe that in general one should be able to have a diagram like (2) with the arrows in arbitrary direction. In particular deciding whether two representations are in Howe's correspondence should be done by comparing the intertwining distributions obtained from their characters.

The usual, often very successful, approach to Howe's correspondence avoids any work with the distributions on the symplectic space. Instead, one finds Langlands parameters (see [Moe89], [AB95], [Pau98], [Pau00], [Pau05], [LPTZ03]), character formulas (see [Ada98], [Ren98], [DP96]), or candidates for character formulas (as in [BP14]), or one establishes preservation of unitarity (as in [Li89], [He03], [Prz93] or [ABP+07]). However, in the background (explicit or not), there is the orbit correspondence induced by the moment maps

$$\mathfrak{g}^* \longleftarrow \mathrm{W} \longrightarrow \mathfrak{g}'^*,$$

see (16). This correspondence of orbits has been studied in [DKP97], [DKP05] and [Pan10]. Furthermore, in their recent work, [LM15], H. Y. Loke and J. J. Ma computed the associated variety of the representations for the dual pairs in the stable range in terms of the orbit correspondence. The p-adic case was studied in detail in [Moe98]. Moreover, still in the stable range, R. Gomez and C. Zhu computed their generalized Whittaker models, [GZ13].

Needless to say, working with the GG'-invariant distributions on W is a more direct approach than relying on the orbit correspondence. In this paper we consider the dual pairs with G compact. Then the representations Π and Π' are the irreducible unitary highest weight representations. They have been defined by Harish-Chandra in [Har55] and were classified in [EHW83]. They have been studied in terms of Zuckerman functors in [Wal84], [Ada83] and [Ada87].

As a complementary contribution to all this work, we compute the intertwining distributions $f_{\Pi\otimes\Pi'}$ explicitly. Our formula for the intertwining distribution $f_{\Pi\otimes\Pi'}$ is explicit enough to find its asymptotics, see Theorem 44. These allow us to compute the wave front set of the representation Π' within the Classical Invariant Theory, without using [Vog78]. See Corollary 46 below. Also, in the case when both groups are compact, we have the diagram (2) with the arrows in arbitrary direction. Therefore we see which representations Π and Π' occur in Howe's correspondence, which corresponds to which and we recover the fact that $\Pi\otimes\Pi'$ occurs with multiplicity one without using [How89a] or [Wey46]. This is a stepping stone for understanding the more general situation.

In order to describe more precisely the content of this paper we need to introduce some notation.

Denote by $\mathfrak{sp}$ the Lie algebra of Sp. Fix a compatible positive complex structure J on W. Hence $J\in\mathfrak{sp}$ is such that $J^2=-1$ (minus the identity) and the symmetric bilinear form $\langle J\cdot,\cdot\rangle$ is positive definite on W. Let dw be the Lebesgue measure on W such that the volume of the unit cube with respect to this form is 1. (Since all positive complex structures are conjugate by elements of Sp, this normalization does not depend on the particular choice of J.) Let $\mathrm{W}=\mathrm{X}\oplus\mathrm{Y}$ be a complete polarization. We normalize the Lebesgue measures on X and on Y similarly.

Each element $K\in\mathcal{S}^*(\mathrm{X}\times\mathrm{X})$ defines an operator $\mathrm{Op}(K)\in\mathrm{Hom}(\mathcal{S}(\mathrm{X}),\mathcal{S}^*(\mathrm{X}))$ by

$$\mathrm{Op}(K)v(x)=\int_{\mathrm{X}}K(x,x')v(x')\,dx'. \tag{3}$$

Here $\mathcal{S}(\mathrm{V})$ and $\mathcal{S}^*(\mathrm{V})$ denote the Schwartz space on the vector space V and the space of tempered distributions on V, respectively.

The map $\mathrm{Op}:\mathcal{S}^*(\mathrm{X}\times\mathrm{X})\to\mathrm{Hom}(\mathcal{S}(\mathrm{X}),\mathcal{S}^*(\mathrm{X}))$ is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [Hör83, Theorem 5.2.1].

Fix the unitary character $\chi(r)=e^{2\pi ir}$, $r\in\mathbb{R}$, and recall the Weyl transform $\mathcal{K}:\mathcal{S}^*(\mathrm{W})\to\mathcal{S}^*(\mathrm{X}\times\mathrm{X})$ given for $f\in\mathcal{S}(\mathrm{W})$ by

$$\mathcal{K}(f)(x,x')=\int_{\mathrm{Y}}f(x-x'+y)\chi\big(\frac{1}{2}\langle y,x+x'\rangle\big)\,dy\,, \tag{4}$$

see [How80, sec. 1.3]. The Weyl symbol of the operator $\mathrm{Op}\circ\mathcal{K}(f)$ is the symplectic Fourier transform $\widehat{f}$ of f, defined by

$$\widehat{f}(w')=2^{-n}\int_{\mathrm{W}}f(w)\chi(\tfrac{1}{2}\langle w,w'\rangle)\,dw \qquad (w'\in\mathrm{W}). \tag{5}$$

A theorem of Calderon and Vaillancourt asserts that the operator $\mathrm{Op}\circ\mathcal{K}(f)$ is bounded on $\mathrm{L}^2(\mathrm{X})$ if its Weyl symbol and all its derivatives are bounded functions on W, [How80, Theorem 3.1.3]. The intertwining distributions we compute are Weyl symbols of some bounded operators which naturally come from the representation theory of real reductive groups. Many of these symbols turn out to be singular distributions. In order to introduce them we recall the Weil representation.

For an element $g \in \mathrm{Sp}$, let $J_g = J^{-1}(g-1)$. Then its adjoint with respect to the form $\langle J\cdot, \cdot\rangle$ is $J_g^* = Jg^{-1}(1-g)$. In particular J_g and J_g^* have the same kernel. Hence the image of J_g is

$$J_g\mathrm{W} = (\mathrm{Ker} J_g^*)^\perp = (\mathrm{Ker} J_g)^\perp$$

where $\perp$ denotes the orthogonal with respect to $\langle J\cdot, \cdot\rangle$. Therefore, the restriction of J_g to $J_g\mathrm{W}$ defines an invertible element. Thus it makes sense to talk about $\det(J_g)^{-1}_{J_g\mathrm{W}}$, the reciprocal of the determinant of the restriction of J_g to $J_g\mathrm{W}$. Let

$$\widetilde{\mathrm{Sp}} = \{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C},\ \ \xi^2 = i^{\dim(g-1)\mathrm{W}} \det(J_g)^{-1}_{J_g\mathrm{W}}\}\,. \tag{6}$$

Then there exists a 2-cocycle $C : \mathrm{Sp} \times \mathrm{Sp} \to \mathbb{C}$, so that $\widetilde{\mathrm{Sp}}$ is a group, the metaplectic group, with respect to the multiplication

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2))\,. \tag{7}$$

In fact, by [AP14, Lemma 52],

$$|C(g_1, g_2)| = \sqrt{\left|\frac{\det(J_{g_1})_{J_{g_1}\mathrm{W}} \det(J_{g_2})_{J_{g_2}\mathrm{W}}}{\det(J_{g_1g_2})_{J_{g_1g_2}\mathrm{W}}}\right|} \tag{8}$$

and by [AP14, Proposition 46 and formula (109)],

$$\frac{C(g_1, g_2)}{|C(g_1, g_2)|} = \chi(\frac{1}{8}\,\mathrm{sgn}(q_{g_1,g_2})), \tag{9}$$

where $\mathrm{sgn}(q_{g_1,g_2})$ is the signature of the symmetric form

$$\begin{aligned} q_{g_1,g_2}(u', u'') \ &= \ \frac{1}{2}\langle (g_1+1)(g_1-1)^{-1}u', u''\rangle + \frac{1}{2}\langle (g_2+1)(g_2-1)^{-1}u', u''\rangle \\ &\quad (u', u'' \in (g_1-1)\mathrm{W} \cap (g_2-1)\mathrm{W}. \end{aligned} \tag{10}$$

By the signature of a (possibly degenerate) symmetric form we understand the difference between the maximal dimension of a subspace where the form is positive definite and the maximal dimension of a subspace where the form is negative definite.

Let

$$\chi_{c(g)}(u) = \chi\big(\frac{1}{4}\langle (g+1)(g-1)^{-1}u, u\rangle\big) \qquad (u = (g-1)w,\ w \in \mathrm{W}). \tag{11}$$

(In particular, if $g-1$ is invertible on W, then $\chi_{c(g)}(u) = \chi(\frac{1}{4}\langle c(g)u, u\rangle$ where $c(g) = (g+1)(g-1)^{-1}$ is the usual Cayley transform.) For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}$ define

$$\Theta(\tilde{g}) = \xi, \qquad T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathrm{W}}, \qquad \omega(\tilde{g}) = \mathrm{Op} \circ \mathcal{K} \circ T(\tilde{g}), \tag{12}$$

where $\mu_{(g-1)\mathrm{W}}$ is the Lebesgue measure on the subspace $(g-1)\mathrm{W}$ normalized so that the volume of the unit cube with respect to the form $\langle J\cdot, \cdot\rangle$ is 1. In these terms, $(\omega, \mathrm{L}^2(\mathrm{X}))$ is the Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ. A proof of this fact based on previous work of Ranga Rao [Rao93] may be found in [Tho09]. Conversely, one may take the above definition of ω and check directly that it is a representation with all the required properties. This was done in [AP14, Theorem 60].

We consider a dual pair $(\mathrm{G}, \mathrm{G}')$, in the symplectic group Sp, with G compact. Let $\widetilde{\mathrm{G}}$ be the preimage of G in the metaplectic group, equipped with the Haar measure of total mass 1. Fix an irreducible unitary representation Π of $\widetilde{\mathrm{G}}$ which occurs in the restriction of ω to $\widetilde{\mathrm{G}}$ and let Θ_Π be its (distribution) character. Then the operator

$$\omega(\check{\Theta}_\Pi) = \int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})\omega(\tilde{g})\, d\tilde{g}$$

is $(\dim \Pi)^{-1}$ times the orthogonal projection $\mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{X})_\Pi$ onto the Π-isotypic component of $\mathrm{L}^2(\mathrm{X})$. The Weyl symbol of this projection is equal to a constant multiple of

$$T(\check{\Theta}_\Pi) = \int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})T(\tilde{g})\, d\tilde{g}. \tag{13}$$

This is precisely the intertwining distribution we introduced before:

$$f_{\Pi\otimes\Pi'} = T(\check{\Theta}_\Pi).$$

Here we are using that the symplectic transform of $T(\check{\Theta}_\Pi)$ is $\pm T(\check{\Theta}_\Pi)$, [Prz91, (5.2) and (5.4.2)].

For example, if $\mathrm{G} = \mathrm{O}_1 = \{\pm 1\}$ and $\mathrm{G}' = \mathrm{Sp}$, then

$$\widetilde{\mathrm{G}} = \{(1,1),\ (1,-1),\ (-1, i^n 2^{-n}),\ (-1, -i^n 2^{-n})\}$$

with the multiplication given by $(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2))$, where

$$C(1, \pm 1) = C(\pm 1, 1) = 1 \ \text{ and } \ C(-1,-1) = 2^{2n}.$$

In these terms, the following two one-dimensional representations of $\widetilde{\mathrm{G}}$ occur in ω.

$$\Pi_+(g, \eta) = |\eta|^{-1}\eta,\ \ \Pi_-(g,\eta) = g|\eta|^{-1}\eta \tag{14}$$

A straightforward computation shows that

$$T(\check{\Theta}_{\Pi_\pm}) = \frac{1}{2}\left(\delta_0 \pm 2^{-n} dw\right), \tag{15}$$

where δ_0 is the Dirac delta at the origin in W.

In general, Classical Invariant Theory says that the space $\mathrm{L}^2(\mathrm{X})_\Pi$ is irreducible under the joint action of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$, [How89a]. Hence $\mathrm{L}^2(\mathrm{X})_\Pi = \mathrm{L}^2(\mathsf{X})_{\Pi\otimes\Pi'}$ for an irreducible unitary representation Π' of $\widetilde{\mathrm{G}}'$. We are interested in the character $\Theta_{\Pi'}$ of Π'.

The unnormalized moment map

$$\tau' : \mathrm{W} \to \mathfrak{g}'^*,\ \tau'(w)(y) = \langle yw, w\rangle \qquad (w \in \mathrm{W},\ y \in \mathfrak{g}') \tag{16}$$

is a quadratic polynomial map with compact fibers. Hence the pull-back

$$\mathcal{S}(\mathfrak{g}') \ni \psi \to \psi \circ \tau' \in \mathcal{S}(\mathrm{W})$$

is well defined and continuous, [Prz91, Lemma 6.1]. Therefore, by dualizing, we get a push-forward of distributions

$$\tau'_* : \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\mathfrak{g}'). \tag{17}$$

Recall the Cayley transform $c(x) = (x+1)(x-1)^{-1}$, which we view as a rational map from the Lie algebra $\mathfrak{sp}$ into the group Sp. In particular $c(0) = -1$. Let

$$\tilde{c} : \mathfrak{sp} \to \widetilde{\mathrm{Sp}} \tag{18}$$

be a real analytic lift of c. Set $\tilde{c}_-(x) = \tilde{c}(x)\tilde{c}(0)^{-1}$. Then $\tilde{c}_-(0)$ is the identity in the group $\widetilde{\mathrm{Sp}}$.

Let $\tilde{c}_-^*\Theta_{\Pi'}$ be the pullback of $\Theta_{\Pi'}$ by $\tilde{c}_-$, [Hör83, Theorem 6.1.2]. Then, as shown in [Prz91, Theorem 6.7], for an appropriately defined Fourier transform $\mathcal{F}$ on $\mathfrak{g}'$,

$$\frac{1}{\Theta \circ \tilde{c}}\, \tilde{c}_-^*\Theta_{\Pi'} = \frac{(\text{central character of }\Pi)(\tilde{c}(0))}{\dim \Pi}\, \mathcal{F}(\tau'_*(T(\check{\Theta}_\Pi)))\,. \tag{19}$$

Formula (19) allows us to determine $\Theta_{\Pi'}(\Psi)$ for every $\Psi \in C_c^\infty(\widetilde{G}')$ supported in the image of $\tilde{c}_-$. Indeed $\Theta_{\Pi'}(\Psi) = \big(\tilde{c}_-^*\Theta_{\Pi'}\big)(\psi)$ where $\psi(x) = \Psi(\tilde{c}_-(x))\,\mathrm{ch}^{-2r}(x)$ for $x \in \mathfrak{g}'$. Here ch^{-2r} is the Jacobian of $\tilde{c}_-$; see (153) and (156) in section 6.

For example, if $\mathrm{G} = \mathrm{O}_1$ then for $\psi \in \mathcal{S}(\mathfrak{g}')$ we have

$$\tau'_*(T(\check{\Theta}_\Pi))(\psi) = T(\check{\Theta}_\Pi)(\psi \circ \tau') = \frac{1}{2}\Big(\delta(\psi) + 2^{-n}\int_{\mathrm{W}} \psi(\tau'(w))\, dw\Big)\,.$$

Recall that $\tau'(\mathrm{W}\setminus\{0\}) \subseteq \mathfrak{g}'$ is one of the two non-zero minimal nilpotent orbits in $\mathfrak{g}' = \mathfrak{sp}$, which we denote $\mathcal{O}_{\min}$. Hence

$$\frac{1}{\Theta \circ \tilde{c}}\, \tilde{c}_-^*\Theta_{\Pi'_\pm} = \Pi_\pm(\tilde{c}(0))\mathcal{F}\Big(\frac{1}{2}(\delta \pm \mu_{\mathcal{O}_{\min}})\Big), \tag{20}$$

where $\mu_{\mathcal{O}_{\min}} = \tau'_*(2^{-n}dw)$ is an invariant measure on $\mathcal{O}_{\min}$ and $\Pi'_\pm$ are the corresponding two irreducible pieces of the Weil representation ω. Notice that, since $c(0) = -1$, the definition (14) gives $\Pi_-(\tilde{c}(0)) = -\Pi_+(\tilde{c}(0))$. It follows that

$$\frac{1}{\Theta \circ \tilde{c}}\, \tilde{c}_-^*\Theta = \Pi_+(\tilde{c}(0))\mathcal{F}(\mu_{\mathcal{O}_{\min}})\,. \tag{21}$$

This formula gives the character of ω as the Fourier transform of an invariant measure of a nilpotent orbits in $\mathfrak{sp}$. This is what Kirillov's orbit method would aim at (if it worked on semisimple Lie groups).

In any case, the formula (19) together with the precise description of the intertwinig distribution $f_{\Pi\otimes\Pi'} = T(\check{\Theta}_\Pi)$ given in Theorems 40 and 43, provides a description of the character $\Theta_{\Pi'}$ as a distribution on a Zariski open neighborhood of the identity. (Hence in principle, by Harish-Chandra's regularity theorem [Har63, Theorem 2], on the whole group.) A formula of Enright [Enr88, Corollary 2.3], describes the restriction of $\Theta_{\Pi'}$, as a function, to the regular set of a compact Cartan subgroup. However, despite the claim made there and in [Mar75], that formula does not give the character on other Cartan subgroups. This is easy to see in the case when G' is the metaplectic group and Π' is any of the two irreducible components of ω, see for example [AP14, Definition 56 and Theorem 60].

We compute explicitly the distribution $f_{\Pi\otimes\Pi'} = T(\check{\Theta}_\Pi)$ in terms of the GG'-orbital integrals on W, see Theorems 40 and 43. In particular we see that $T(\check{\Theta}_\Pi)$ is a smooth function if and only if $(\mathrm{G}, \mathrm{G}')$ is a pair of compact unitary groups, see Proposition 49. Also, modulo a few exceptions, $T(\check{\Theta}_\Pi)$ is a locally integrable function if and only if the rank of G is greater or equal to the rank of G′. Our results on the orbital integrals are based on the corresponding results of Harish-Chandra, which are transferred from the Lie algebra $\mathfrak{g}'$ to the symplectic space W via a theorem of G. Schwarz, [Sch74]. We hope to circumvent it in the future in order to treat the case of a non-compact group G.

Let $\tau : \mathrm{W} \to \mathfrak{g}^*$ be the unnormalized moment map for $\mathfrak{g}$ given, as in (16), by $\tau(w)(x) = \langle xw, w\rangle$. The variety $\tau^{-1}(0) \subseteq \mathrm{W}$ is the closure of a single orbit $\mathcal{O}$; see e.g. [Prz91, Lemma 2.16]. There is a positive GG′-invariant measure $\mu_\mathcal{O}$ on this orbit which defines a homogeneous tempered distribution. We denote its degree by $\deg \mu_\mathcal{O}$. For $t > 0$ let $M_t(w) = tw$, $w \in \mathrm{W}$. Denote by M_t^* the corresponding pullback of distributions. In particular $M_t^* \mu_\mathcal{O} = t^{\deg \mu_\mathcal{O}} \mu_\mathcal{O}$. We show that

$$t^{\deg \mu_\mathcal{O}} M_{t^{-1}}^* T(\check{\Theta}_\Pi) \underset{t \to 0}{\to} const\, \mu_\mathcal{O},$$

as tempered distributions, where *const* is a non-zero constant, see Proposition 44. This last statement leads to an elementary proof of the equality

$$WF_1(\Theta_{\Pi'}) = \tau' \tau^{-1}(0)\,, \tag{22}$$

see Corollary 46. Here $WF_1(\Theta_{\Pi'})$ denotes the fiber of the wave front set $WF(\Theta_{\Pi'})$ over the identity $1 \in \widetilde{G'}$. As proven by Rossmann in [Ros95], $WF_1(\Theta_{\Pi'})$ agrees with $WF(\Pi')$, the wave front set of the representation Π' in the sense of Howe [How81].

The equality (22) was already verified in [Prz91, Theorem 6.11], but the proof used a theorem of Vogan concerning the restriction of a representation to a maximal compact subgroup, [Vog78], which is not needed in our present approach. In order to stay within the theory of the almost semisimple orbital integrals on the symplectic space, see section 4, we consider only representations Π such that the distribution character Θ_Π is supported in the preimage $\widetilde{\mathrm{G}_1}$ of the Zariski identity component G_1 of G. This eliminates some representations of the groups O_{2l}. For reader's convenience one should mention here that there is a notion of an associated variety of a presentation introduced by Vogan [Vog89] and that associated variety determines the wave front set of a representation [SV00]. In this context a recent work of Loke and Ma [LM15] provides a vast generalization of our formula for the wave front set of Π', for dual pairs in the stable range with G-the smaller member. Needless to say this approach is much more sophisticated and much less direct than ours. We point out that the Gelfand-Kirillov dimension of Π' $(=\frac{1}{2}\dim WF(\Pi'))$ was determined in [Prz91, Theorem C.1] and rediscovered later independently in [NOT+01] and in [EW04, Theorem 6]. Also, [NOT+01] gives a formula for the associated variety of Π' in the case G-compact and without the stable range assumption.

In section 7 we demonstrate that our computation are precise enough to recover Weyl's theorem, saying that if both G and G′ are compact then the restriction of the Weil representation to $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}'}$ decomposes with multiplicity one.

In section 8 we compute limits of the almost semisimple orbital integrals using van der Corput lemma rather than techniques based on unpublished work of Ranga Rao and obtain in that case a stronger version of Rossmann's limit formula, [Ros90].

2. The group $\widetilde{\mathrm{G}}$ and the Weyl denominator.

We keep the notation from the introduction. In particular, $(\mathrm{G}, \mathrm{G}')$ is a dual pair with G compact and J is a fixed compatible positive complex structure on W.

In this section we describe the restriction of the covering

$$\widetilde{\mathrm{Sp}} \ni (g, \xi) \to g \in \mathrm{Sp} \tag{23}$$

to the group G. This is then applied to study the analytic lift of the Weyl denominator.

Let $\mathrm{Sp}^J \subseteq \mathrm{Sp}$ denote the centralizer of J. Since Sp^J is a maximal compact subgroup of Sp, we may assume that $\mathrm{G} \subseteq \mathrm{Sp}^J$ and begin by studying the restriction of the map (23) to $\widetilde{\mathrm{Sp}^J}$.

Let $\mathrm{W}_{\mathbb{C}}$ be the complexification of W. Denote by the same symbol $\langle \cdot, \cdot \rangle$ the complex bilinear extension of the symplectic form from W to $\mathrm{W}_{\mathbb{C}}$. Let $\mathrm{W}_{\mathbb{C}}^+ \subseteq \mathrm{W}_{\mathbb{C}}$ be the i-eigenspace for J. Denote by $\mathrm{W}_{\mathbb{C}} \ni w \to \overline{w} \in \mathrm{W}_{\mathbb{C}}$ the conjugation with respect to the real form $\mathrm{W} \subseteq \mathrm{W}_{\mathbb{C}}$. Then $\mathrm{W}_{\mathbb{C}}^- = \overline{\mathrm{W}_{\mathbb{C}}^+}$ is the $(-i)$-eigenspace for J and

$$\mathrm{W}_{\mathbb{C}} = \mathrm{W}_{\mathbb{C}}^+ \oplus \mathrm{W}_{\mathbb{C}}^- \tag{24}$$

is a complete polarization. The formula

$$H(w, w') = i\langle w, \overline{w'} \rangle \qquad (w, w' \in \mathrm{W}_{\mathbb{C}}) \tag{25}$$

defines a non-degenerate hermitian form on $\mathrm{W}_{\mathbb{C}}$ and the map

$$\frac{1}{2}(1 - iJ) : \mathrm{W} \to \mathrm{W}_{\mathbb{C}}^+ \tag{26}$$

is an $\mathbb{R}$-linear isomorphism. Moreover, if $w = \frac{1}{2}(1 - iJ)w_0$ and $w' = \frac{1}{2}(1 - iJ)w_0'$ for some $w_0, w_0' \in \mathrm{W}$, then

$$H(w, w') = \frac{1}{2}(\langle Jw_0, w_0' \rangle + i\langle w_0, w_0' \rangle). \tag{27}$$

In particular, the restriction $H|_{\mathrm{W}_{\mathbb{C}}^+}$ of H to $\mathrm{W}_{\mathbb{C}}^+$ is positive definite. Let $\mathrm{U} \subseteq \mathrm{End}(\mathrm{W}_{\mathbb{C}}^+)$ denote the isometry group of the form $H|_{\mathrm{W}_{\mathbb{C}}^+}$.

Let $g \in \mathrm{Sp}^J$. Then g can be extended to a complex linear endomorphism, still denoted by g, which belongs to $\mathrm{Sp}(\mathrm{W}_{\mathbb{C}})^J$. Clearly $g(\mathrm{W}_{\mathbb{C}}^+) = \mathrm{W}_{\mathbb{C}}^+$. Set $u = g|_{\mathrm{W}_{\mathbb{C}}^+}$. Then for every $w \in \mathrm{W}_{\mathbb{C}}^+$ with $w = \frac{1}{2}(1 - iJ)w_0$ for $w_0 \in \mathrm{W}$, we have $uw = g\left[\frac{1}{2}(1 - iJ)w_0\right] = \frac{1}{2}(1 - iJ)gw_0$, i.e.

$$u = \frac{1}{2}(1 - iJ) \circ g \circ [\frac{1}{2}(1 - iJ)]^{-1}.$$

It follows that $u \in U$.

Let

$$\widetilde{\mathrm{U}} = \{(u, \zeta);\ \det u = \zeta^2,\ u \in \mathrm{U}\} \subseteq \mathrm{GL}(\mathrm{W}_{\mathbb{C}}^+) \times \mathbb{C}^\times. \tag{28}$$

endowed with the coordinate-wise multiplication. This is a connected two-fold covering group of U.

Proposition 1. *The group isomorphism*

$$\mathrm{Sp}^J \in g \to u = g|_{\mathrm{W}_{\mathbb{C}}^+} = \frac{1}{2}(1-iJ)\circ g\circ [\frac{1}{2}(1-iJ)]^{-1} \in U \tag{29}$$

lifts to a group isomorphism

$$\widetilde{\mathrm{Sp}}^J \ni (g,\xi) \to (u, \xi \det(g-1)_{(g-1)\mathrm{W}_{\mathbb{C}}^+}) \in \tilde{\mathrm{U}}\,. \tag{30}$$

Therefore the restriction of the covering (23) to $\widetilde{\mathrm{Sp}}^J$ *is isomorphic to the covering*

$$\widetilde{\mathrm{U}} \ni (u,\zeta) \to u \in \mathrm{U}. \tag{31}$$

Proof. Notice that any element $g \in \mathrm{Sp}(\mathrm{W}_{\mathbb{C}})^J$ preserves the decomposition (24) and satisfies the following formula

$$\langle g^{-1}w, w'\rangle = \langle w, gw'\rangle \qquad (w \in \mathrm{W}_{\mathbb{C}}^+,\ w' \in \mathrm{W}_{\mathbb{C}}^-). \tag{32}$$

Since the symplectic form identifies $\mathrm{W}_{\mathbb{C}}^+$ with the dual of $\mathrm{W}_{\mathbb{C}}^-$ and since the determinant of the adjoint linear map is equal to the determinant of the original linear map, (32) shows that

$$\det(g-1)_{\mathrm{W}_{\mathbb{C}}^-} = \det(g^{-1}-1)_{\mathrm{W}_{\mathbb{C}}^+},$$

where we take the determinant of the linear map restricted to the indicated subspace. Hence,

$$\det(g-1) = (-1)^n \det(g-1)^2_{\mathrm{W}_{\mathbb{C}}^+} \det(g)^{-1}_{\mathrm{W}_{\mathbb{C}}^+}. \tag{33}$$

Let $g \in \mathrm{Sp}^J$. The restriction $\langle\cdot,\cdot\rangle_0$ of $\langle\cdot,\cdot\rangle$ to $(g-1)\mathrm{W}$ is nondegenerate. Indeed, since g and J commute, the orthogonal complement $((g-1)\mathrm{W})^\perp$ of $(g-1)\mathrm{W}$ with respect to $\langle\cdot,\cdot\rangle$ coincides with the orthogonal complement of $(g-1)\mathrm{W}$ with respect to the positive definite form $\langle J\cdot,\cdot\rangle$. Hence $((g-1)\mathrm{W})^\perp \cap (g-1)\mathrm{W} = 0$.

Consider $(g-1)\mathrm{W}$ as a symplectic space with $\langle\cdot,\cdot\rangle_0$ as a symplectic form, and let Sp_0 be the corresponding symplectic group. Since J preserves $(g-1)\mathrm{W}$, we can consider its restriction J_0 to $(g-1)\mathrm{W}$. It satisfies $J_0^2 = -1$ and the bilinear form $\langle J_0\cdot,\cdot\rangle_0$ is symmetric and positive definite. So J_0 is a positive compatible complex structure on $(g-1)\mathrm{W}$. In particular, $J_0 \in \mathrm{Sp}_0$. So $\det(J)_{(g-1)\mathrm{W}} = \det(J_0) = 1$.

Formula (33) applied to $g \in \mathrm{Sp}((g-1)\mathrm{W}_{\mathbb{C}})^{J_0}$ shows that

$$\det(J_g)_{(g-1)\mathrm{W}} = \det(J^{-1}(g-1))_{(g-1)\mathrm{W}} = i^{\dim(g-1)\mathrm{W}} \det(g-1)^2_{(g-1)\mathrm{W}_{\mathbb{C}}^+} \det(g)^{-1}_{(g-1)\mathrm{W}_{\mathbb{C}}^+}.$$

Recall the notation $u = g|_{\mathrm{W}_{\mathbb{C}}^+}$. Since $\mathrm{W}_{\mathbb{C}}^+$ is the direct sum of $(g-1)\mathrm{W}_{\mathbb{C}}^+$ and the subspace where $g = 1$,

$$\det(g)_{(g-1)\mathrm{W}_{\mathbb{C}}^+} = \det(g)_{\mathrm{W}_{\mathbb{C}}^+} = \det(u). \tag{34}$$

Thus, in terms of (6)

$$\det(J_g)^{-1}_{(g-1)\mathrm{W}} = i^{\dim(g-1)\mathrm{W}} \det(g-1)^{-2}_{(g-1)\mathrm{W}_{\mathbb{C}}^+} \det(u). \tag{35}$$

Hence, the map (30) is a well defined bijection. We now prove that it is a group homomorphism.

We see from (7) that the map (30) is a group homomorphism if and only if

$$C(g_1, g_2) = \frac{\det(g_1 - 1)_{(g_1-1)\mathrm{W}_{\mathbb{C}}^+} \det(g_2 - 1)_{(g_2-1)\mathrm{W}_{\mathbb{C}}^+}}{\det(g_1 g_2 - 1)_{(g_1 g_2-1)\mathrm{W}_{\mathbb{C}}^+}}. \tag{36}$$

The equations (35) and (8) show that the absolute values of the two sides of (36) are equal.

In order to shorten the notation, let us write $c(g)u = (g+1)(g-1)^{-1}u$ for u in the image of $g-1$. Set

$$h_{g_1,g_2}(w', w'') = H(-ic(g_1)w', w'') \quad + \quad H(-ic(g_2)w', w'') \qquad (w', w'' \in (g_1-1)\mathrm{W}_{\mathbb{C}}^+ \cap (g_2-1)\mathrm{W}_{\mathbb{C}}^+.$$

Then, since g_1 and g_2 commute with J,

$$\begin{aligned} h_{g_1,g_2}(w', w'') &= \frac{1}{2}(\langle c(g_1)w_0', w_0''\rangle - i\langle Jc(g_1)w_0', w_0''\rangle) \\ &+ \frac{1}{2}(\langle c(g_2)w_0', w_0''\rangle - i\langle Jc(g_2)w_0', w_0''\rangle), \end{aligned}$$

where w' and w'' are the images of w_0' and w_0'' under the map (26) respectively. Moreover, $w_0', w_0'' \in (g_1-1)\mathrm{W} \cap (g_2-1)\mathrm{W}$. In particular we see that the form h_{g_1,g_2} is hermitian and

$$\mathrm{Re}(h_{g_1,g_2}(w', w'')) = q_{g_1,g_2}(w_0', w_0'').$$

Let w^1, w^2, ..., w^n be an h_{g_1,g_2}-orthogonal basis of the complex vector space $\mathrm{W}_{\mathbb{C}}^+$ with $h_{g_1,g_2}(w^k, w^k) = \pm 1$ or 0. Then w_0^1, Jw_0^1, w_0^2, Jw_0^2, ..., w_0^n, Jw_0^n is an q_{g_1,g_2}-orthogonal basis of the real vector space W with $h_{g_1,g_2}(w^k, w^k) = q_{g_1,g_2}(w_0^k, w_0^k) = q_{g_1,g_2}(Jw_0^k, Jw_0^k)$. The signature of h_{g_1,g_2} is the difference between the number of the positive $h_{g_1,g_2}(w^k, w^k)$ and the number of the negative $h_{g_1,g_2}(w^k, w^k)$, and similarly for the symmetric form q_{g_1,g_2}. Hence

$$\mathrm{sgn}\, h_{g_1,g_2} = \frac{1}{2}\,\mathrm{sgn}\, q_{g_1,g_2}.$$

Therefore

$$i^{\mathrm{sgn}\, h_{g_1,g_2}} = e^{\frac{\pi i}{2}\mathrm{sgn}\, h_{g_1,g_2}} = e^{\frac{\pi i}{4}\mathrm{sgn}\, q_{g_1,g_2}} = \chi(\frac{1}{8}\,\mathrm{sgn}\, q_{g_1,g_2}). \tag{37}$$

We see from (9) and (37) that it will suffice to prove that

$$\begin{aligned} &\frac{\det(g_1 - 1)_{(g_1-1)\mathrm{W}_{\mathbb{C}}^+} \det(g_2 - 1)_{(g_2-1)\mathrm{W}_{\mathbb{C}}^+}}{\det(g_1 g_2 - 1)_{(g_1 g_2-1)\mathrm{W}_{\mathbb{C}}^+}} \left| \frac{\det(g_1 g_2 - 1)_{(g_1 g_2-1)\mathrm{W}_{\mathbb{C}}^+}}{\det(g_1 - 1)_{(g_1-1)\mathrm{W}_{\mathbb{C}}^+} \det(g_2 - 1)_{(g_2-1)\mathrm{W}_{\mathbb{C}}^+}} \right| \\ = \;& i^{\mathrm{sgn}\, h_{g_1,g_2}}. \end{aligned} \tag{38}$$

This requires a significant amount of additional notation and therefore will be done in Appendix E. In fact, since both sides of (36) are cocycles we may (and shall) assume that $\mathrm{Ker}(g_1 - 1) = \{0\}$, see [AP14, proof of Theorem 31]. □

Proposition 1 allows us to study the covering $\widetilde{\mathrm{G}} \to \mathrm{G}$ by means of the (explicitly given) covering $\widetilde{\mathrm{U}} \to \mathrm{U}$.

In the following we restrict ourselves to dual pairs $(\mathrm{G}, \mathrm{G}')$ which are irreducible, that is no nontrivial direct sum decomposition of the symplectic space W is simultaneously preserved by G and G′. Irreducible dual pairs have been classified by Howe [How79]. Those for which G is compact are all of type I. They are

$$(\mathrm{O}_d, \mathrm{Sp}_{2m}(\mathbb{R}))\,, \qquad (\mathrm{U}_d, \mathrm{U}_{p,q})\,, \qquad (\mathrm{Sp}_d, \mathrm{O}^*_{2m}). \tag{39}$$

More precisely, given the dual pair $(\mathrm{G}, \mathrm{G}')$, there is a division algebra $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, an involution $\mathbb{D} \ni a \to \overline{a} \in \mathbb{D}$ over $\mathbb{R}$, a right $\mathbb{D}$-vector space V with a positive definite hermitian form $(\cdot,\cdot)$ and a left $\mathbb{D}$-vector space V′ with a non-degenerate skew-hermitian form $(\cdot,\cdot)'$ so that G is the isometry group of the form $(\cdot,\cdot)$, G′ is the isometry group of the form $(\cdot,\cdot)'$ and the symplectic space $\mathrm{W} = \mathrm{V} \otimes_{\mathbb{D}} \mathrm{V}'$ with $\langle\cdot,\cdot\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}((\cdot,\cdot) \otimes (\cdot,\cdot)')$, see [How79]. The group G is viewed as a subgroup of the symplectic group via the identification $\mathrm{G} \ni g = g \otimes 1 \in \mathrm{Sp}$. Similarly, the group G′ is viewed as a subgroup of the symplectic group via the identification $\mathrm{G}' \ni g' = 1 \otimes g' \in \mathrm{Sp}$.

Since G commutes with J, there is $J' \in \mathrm{G}'$ such that $J = 1 \otimes J'$. Let $\mathrm{V}'_{\mathbb{C}} = \mathrm{V}' \otimes_{\mathbb{R}} \mathbb{C}$ and let ${\mathrm{V}'_{\mathbb{C}}}^{+} \subseteq \mathrm{V}'_{\mathbb{C}}$ be the i-eigenspace for J'. Then $\mathrm{W}_{\mathbb{C}}^{+} = \mathrm{V} \otimes_{\mathbb{D}} {\mathrm{V}'_{\mathbb{C}}}^{+}$ and

$$\det(g)_{\mathrm{W}_{\mathbb{C}}^{+}} = \det(g \otimes 1)_{\mathsf{V}\otimes_{\mathbb{D}}{\mathsf{V}'_{\mathbb{C}}}^{+}} \qquad (g \in \mathrm{G}).$$

The description of G as isometry group of the form $(\cdot,\cdot)$ on V provides an irreducible complex representation of G on $\mathrm{V}_{\mathbb{C}} = \mathrm{V} \otimes_{\mathbb{R}} \mathbb{C}$, of dimension $d = \dim \mathrm{V}$.

Recall that the group O_d has, up to an equivalence, one irreducible complex representation of dimension d. Call it δ. Then $\det(\delta(g)) = \pm 1$, $g \in \mathrm{G}$.

The group U_d has two irreducible complex representations of dimension d, δ and the contragredient δ^c. Then $\det(\delta^c(g)) = \det(\delta(g))^{-1}$, $g \in \mathrm{G}$.

The group Sp_d has, up to an equivalence, one irreducible complex representation δ of dimension $2d$. In this case $\det(\delta(g)) = 1$, $g \in \mathrm{G}$.

In these terms

$$\det(g \otimes 1)_{\mathsf{V}\otimes_{\mathbb{D}}{\mathsf{V}'_{\mathbb{C}}}^{+}} = \begin{cases} \det(\delta(g))^m & \text{if } \mathrm{G}' \text{ is isomorphic to } \mathrm{Sp}_{2m}(\mathbb{R}), \\ \det(\delta(g))^{p-q} & \text{if } \mathrm{G}' \text{ is isomorphic to } \mathrm{U}_{p,q}, \\ 1 & \text{if } \mathrm{G}' \text{ is isomorphic to } \mathrm{O}^*_{2m}. \end{cases} \tag{40}$$

We set

$$\sqrt{\mathrm{G}} = \{(g,\zeta);\ g \in \mathrm{G}, \zeta^2 = \det(\delta(g))\}\,. \tag{41}$$

Proposition 2. *The covering*

$$\widetilde{\mathrm{G}} \to \mathrm{G} \tag{42}$$

splits if and only if $\det(g \otimes 1)_{\mathsf{V}\otimes_{\mathbb{D}}{\mathsf{V}'_{\mathbb{C}}}^{+}}$ *is a square. This does NOT happen if and only if either* G′ *is isomorphic to* $\mathrm{Sp}_{2m}(\mathbb{R})$ *with* m *odd or* G′ *is isomorphic to* $\mathrm{U}_{p,q}$ *with* $p+q$ *odd. In these cases* $\widetilde{\mathrm{G}}$ *is isomorphic to* $\sqrt{\mathrm{G}}$.

Proof. Because of Proposition 1 we can identify the cover $\widetilde{\mathrm{Sp}^J} \to \mathrm{Sp}^J$ with $\widetilde{\mathrm{U}} \to \mathrm{U}$. Hence the cover $\widetilde{\mathrm{G}} \to \mathrm{G}$ splits if and only if there is a group homomorphism $\mathrm{G} \in g \to \zeta(g) \in \mathrm{U}_1 \subseteq \mathbb{C}^\times$ so that $\zeta(g)^2 = \det(g)_{\mathrm{W}_\mathbb{C}^+}$. Here we are using (34) and that G is compact. By (40), this happens except at most in the two cases listed in the statement of the Proposition.

Suppose that $\mathrm{G}' = \mathrm{Sp}_{2m}(\mathbb{R})$, and let $\zeta : \mathrm{O}_d \to \mathrm{U}_1$ be a continuous group homomorphism so that $\zeta(g)^2 = \det(\delta(g))^m = (\pm 1)^m$. Then $\zeta(\mathrm{O}_d) \subseteq \{\pm 1, \pm i\}$ and it is a subgroup with at most two elements. So $\zeta(\mathrm{O}_d) \subseteq \{\pm 1\}$. On the other hand, if $g \in \mathrm{O}_d \setminus \mathrm{SO}_d$, then $\det(\delta(g)) = -1$. Thus $\zeta(g)^2 \neq \det(\delta(g))^m$ if m is odd.

Suppose now that $\mathrm{G}' = \mathrm{U}_{p,q}$, and let $\zeta : \mathrm{U}_d \to \mathrm{U}_1$ be a continuous group homomorphism so that $\zeta(g)^2 = \det(\delta(g))^{p-q}$. Restriction to $\mathrm{U}_1 \equiv \{\mathrm{diag}(h, 1 \ldots, 1) : h \in \mathrm{U}_1\} \subseteq \mathrm{U}_d$ yields a continuous group homomorphism $h \in \mathrm{U}_1 \to \zeta(h) \in \mathrm{U}_1$. Thus, there is $k \in \mathbb{Z}$ so that $\zeta(h) = h^k$ for all $h \in \mathrm{U}_1$. So $h^{2k} = \zeta(h)^2 = \det(\mathrm{diag}(h, 1 \ldots, 1))^{p-q}$ implies that $p + q$ must be even.

For the last statement, consider for $k \in \mathbb{Z}$ the covering $\mathrm{M}_k = \{(g, \zeta) : \zeta^2 = \det(\delta(g))^{2k+1}\}$ of G. Then $(g, \zeta) \to (g, \zeta(\det(\delta(g)))^{-k})$ is a covering isomorphism between M_k and M_0. □

Let $\mathrm{G}^\sharp = \sqrt{\mathrm{G}}$ if the covering (42) does not split and let $\mathrm{G}^\sharp = (\mathbb{Z}/2\mathbb{Z}) \times \sqrt{\mathrm{G}}$ if it does. Then we have the obvious (possibly trivial) covering

$$\mathrm{G}^\sharp \to \widetilde{\mathrm{G}}. \tag{43}$$

Let $\mathrm{H} \subseteq \mathrm{G}$ be the diagonal Cartan subgroup. Denote by $\mathrm{H}^\sharp \subseteq \mathrm{G}^\sharp$ the preimage of $\widetilde{\mathrm{H}}$ and let $\mathrm{H}^\sharp_o \subseteq \mathrm{H}^\sharp$ be the connected identity component. Fix a system of positive roots of $\mathfrak{h}$ in $\mathfrak{g}_\mathbb{C}$ and let $\rho \in i\mathfrak{h}^*$ denote one half times the sum of all the positive roots. Then there is a group homomorphism $\xi_\rho : \mathrm{H}^\sharp_o \to \mathbb{C}^\times$ whose derivative at the identity coincides with ρ. More generally, for any $\mu \in i\mathfrak{h}^*$, let $\xi_\mu : \mathrm{H}^\sharp_o \to \mathbb{C}^\times$ denote the unique group homomorphism which has derivative at the identity equal to μ, if it exists. In particular the Weyl denominator

$$\Delta(h) = \xi_\rho(h) \prod_{\alpha>0} (1 - \xi_{-\alpha}(h)) \qquad (h \in \mathrm{H}^\sharp_o), \tag{44}$$

where the product is taken over all the positive roots α, is well defined and analytic.

3. **A Theorem of G. W. Schwartz.**

Since the group G is compact, the involution $\mathbb{D} \ni a \to \overline{a} \in \mathbb{D}$ is trivial if and only if $\mathbb{D} = \mathbb{R}$. Let $M_{d,d'} = M_{d,d'}(\mathbb{D})$ denote the real vector space of the d by d' matrices with the entries in $\mathbb{D}$ and let $\mathcal{H}_{d'} = \mathcal{H}_{d'}(\mathbb{D}) = \{X \in M_{d',d'};\ \overline{X}^t = X\}$ denote the real vector space of the $\mathbb{D}$-hermitian matrices of size d'. In this section we shall be concerned with the map

$$\beta : M_{d,d'} \ni w \to \overline{w}^t w \in \mathcal{H}_{d'}. \tag{45}$$

The group $\mathrm{U}_d = \mathrm{U}_d(\mathbb{D}) = \{g \in \mathrm{GL}_d(\mathbb{D});\ \overline{g}^t = g^{-1}\}$ acts on $M_{d,d'}$ via the left multiplication and preserves the fibers of β. (In the standard notation $\mathrm{U}_d(\mathbb{D})$ is equal to O_d if $\mathbb{D} = \mathbb{R}$, U_d

if $\mathbb{D}=\mathbb{C}$ and $\mathrm{Sp}_m d$ if $\mathbb{D}=\mathbb{H}$.) The title of this section refers to the following proposition, which is based on a theorem by G. Schwartz [Sch74] .

Proposition 3. *With the above notation,*

$$C_c^\infty(\mathcal{H}_{d'})\circ\beta = C_c^\infty(M_{d,d'})^{\mathrm{U}_d}, \tag{46}$$

where X^Y means the Y-invariants in X.

Proof. Since each element of $\mathbb{D}$ may be viewed as a matrix with real entries of size equal to the dimension of $\mathbb{D}$ over $\mathbb{R}$, we have an inclusion $M_{d,d'}(\mathbb{D})\subseteq M_{d\,\dim\mathbb{D},d'\,\dim\mathbb{D}}(\mathbb{R})$. Cauchy-Schwarz inequality shows that

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}\,\beta(w)=\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(I\beta(w))\leq\sqrt{\mathrm{tr}_{\mathbb{D}/\mathbb{R}}\,I}\,\sqrt{\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\beta(w)^2)}\qquad (w\in M_{d,d'}). \tag{47}$$

In particular we see that β is a proper map. Since the composition with β maps smooth functions to smooth functions, the left hand side of (46) is contained in the right hand side.

Consider a function $\phi\in C_c^\infty(M_{d,d'})^{\mathrm{U}_d}$. Then $\beta(\mathrm{supp}(\phi))$ is a compact subset of $\mathcal{H}_{d'}$. Hence there exists a function $f\in C_c^\infty(\mathcal{H}_{d'})$ equal to 1 in a neighborhood of $\beta(\mathrm{supp}(\phi))$. Also, we know from [Sch74] that there is a function $\Psi\in C^\infty(\mathcal{H}_{d'})$ such that $\Psi\circ\beta=\phi$. Let $\psi=\Psi f$. Then $\psi\in C_c^\infty(\mathcal{H}_{d'})$ and $\psi\circ\beta=\phi$. Thus the right hand side of (46) is contained in the left hand side. □

Corollary 4. *The following equality holds, $C_c^\infty(\mathfrak{g}')\circ\tau'=C_c^\infty(\mathrm{W})^{\mathrm{G}}$.*

Proof. Fix a matrix $F=-\overline{F}^t\in\mathrm{GL}_{d'}(\mathbb{D})$. The classification of the dual pairs, [How89b], implies that we may realize the symplectic space W as $M_{d,d'}$ with the symplectic form

$$\langle w',w\rangle=\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(-F\overline{w}^t w')\qquad (w\in\mathrm{W}).$$

Then $\mathrm{G}=\mathrm{U}_d$ acts on W by the left multiplication and $\mathrm{G}'=\{g\in\mathrm{GL}_{d'}(\mathbb{D});\ \overline{g}^t F g=F\}$ via the right multiplication by the inverse. In particular,

$$\langle y(w),w\rangle=\langle -wy,w\rangle=\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(F\overline{w}^t wy)\qquad (y\in\mathfrak{g}',\ w\in\mathrm{W}).$$

Notice that the Lie algebra $\mathfrak{g}'=\{FX;\ X\in\mathcal{H}_{d'}\}$. Also, if we identify $\mathfrak{g}'=\mathfrak{g}'^*$ via the trace, then

$$\tau'(w)=F\overline{w}^t w=F\beta(w)\qquad (w\in\mathrm{W}).$$

Therefore

$$C_c^\infty(\mathfrak{g}')\circ\tau'=C_c^\infty(\mathcal{H}_{d'})\circ\beta.$$

Hence the corollary follows from Proposition 3. □

Corollary 5. *The map $\tau'_*:\mathcal{S}^*(\mathrm{W})^{\mathrm{G}}\to\mathcal{S}^*(\mathfrak{g}')$, (17), is injective.*

4. An almost semisimple orbital integral on the symplectic space.

In this section we describe the orbital integrals needed to express the distribution (13). For that purpose it is convenient to view our dual pair as a supergroup as follows.

Let $\mathsf{V}_{\overline{0}} = \mathrm{V}$ and let $\mathsf{V}_{\overline{1}} = \mathrm{V}'$. From now on we assume that both are left vector spaces over $\mathbb{D}$. Set $\mathsf{V} = \mathsf{V}_{\overline{0}} \oplus \mathsf{V}_{\overline{1}}$ and define an element $\mathsf{S} \in \mathrm{End}(\mathsf{V})$ by

$$\mathsf{S}(v_0 + v_1) = v_0 - v_1 \qquad (v_0 \in \mathsf{V}_{\overline{0}}, v_1 \in \mathsf{V}_{\overline{1}}).$$

Let

$$\begin{aligned} \mathrm{End}(\mathsf{V})_{\overline{0}} &= \{x \in \mathrm{End}(\mathsf{V});\ \mathsf{S}x = x\mathsf{S}\}, \\ \mathrm{End}(\mathsf{V})_{\overline{1}} &= \{x \in \mathrm{End}(\mathsf{V});\ \mathsf{S}x = -x\mathsf{S}\}, \\ \mathrm{GL}(\mathsf{V})_{\overline{0}} &= \mathrm{End}(\mathsf{V})_{\overline{0}} \cap \mathrm{GL}(\mathsf{V}). \end{aligned}$$

Denote by $(\cdot,\cdot)''$ the direct sum of the two forms $(\cdot,\cdot)$ and $(\cdot,\cdot)'$. Let

$$\begin{aligned} \mathfrak{s}_{\overline{0}} &= \{x \in \mathrm{End}(\mathsf{V})_{\overline{0}};\ (xu, v)'' = -(u, xv)'',\ u, v \in \mathsf{V}\}, \\ \mathfrak{s}_{\overline{1}} &= \{x \in \mathrm{End}(\mathsf{V})_{\overline{1}};\ (xu, v)'' = (u, \mathsf{S}xv)'',\ u, v \in \mathsf{V}\}, \\ \mathfrak{s} &= \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}, \\ \mathrm{S} &= \{s \in \mathrm{GL}(\mathsf{V})_{\overline{0}};\ (su, sv)'' = (u, v)'',\ u, v \in \mathsf{V}\}, \\ \langle x, y\rangle &= \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xy). \end{aligned} \tag{48}$$

Then $(\mathrm{S}, \mathfrak{s})$ is a real Lie supergroup, i.e. a real Lie group S together with a real Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$, whose even component $\mathfrak{s}_{\overline{0}}$ is the Lie algebra of S. We shall write $\mathfrak{s}(\mathsf{V})$ instead of $\mathfrak{s}$ whenever we shall want to specify the Lie superalgebra $\mathfrak{s}$ constructed as above from V and $(\cdot,\cdot)''$. By restriction, we have the identification

$$\mathfrak{s}_{\overline{1}} = \mathrm{Hom}_{\mathbb{D}}(\mathsf{V}_{\overline{0}}, \mathsf{V}_{\overline{1}})\,. \tag{49}$$

The group S acts on $\mathfrak{s}$ by conjugation and $\langle\cdot,\cdot\rangle$ is a non-degenerate S-invariant form on the real vector space $\mathfrak{s}$, whose restriction to $\mathfrak{s}_{\overline{0}}$ is symmetric and restriction to $\mathfrak{s}_{\overline{1}}$ is skew-symmetric. We shall employ the notation $s.x = sxs^{-1}$ for the action of $s \in \mathrm{S}$ on $x \in \mathfrak{s}$. In terms of our previous notation,

$$\mathfrak{g} = \mathfrak{s}_{\overline{0}}|_{\mathsf{V}_{\overline{0}}}, \quad \mathfrak{g}' = \mathfrak{s}_{\overline{0}}|_{\mathsf{V}_{\overline{1}}}, \quad \mathrm{W} = \mathfrak{s}_{\overline{1}}, \quad \mathrm{G} = \mathrm{S}|_{\mathsf{V}_{\overline{0}}}, \quad \mathrm{G}' = \mathrm{S}|_{\mathsf{V}_{\overline{1}}},$$

so that

$$\mathfrak{s}_{\overline{0}} = \mathfrak{g} \oplus \mathfrak{g}' \quad \text{and} \quad \mathrm{S} = \mathrm{G} \times \mathrm{G}'.$$

Notice that the action of $\mathrm{S} = \mathrm{G} \times \mathrm{G}'$ on $\mathfrak{s}_{\overline{1}} = \mathrm{W}$ by conjugation corresponds to the action of G on W by the left multiplication and of G′ on W via the right multiplication by the inverse. Also, we have the unnormalized moment maps

$$\tau : \mathrm{W} \ni w \to w^2|_{\mathsf{V}_{\overline{0}}} \in \mathfrak{g}, \qquad \tau' : \mathrm{W} \ni w \to w^2|_{\mathsf{V}_{\overline{1}}} \in \mathfrak{g}'. \tag{50}$$

An element $x \in \mathfrak{s}$ is called semisimple (resp., nilpotent) if x is semisimple (resp., nilpotent) as an endomorphism of V. We say that a semisimple element $x \in \mathfrak{s}_{\overline{1}}$ is regular if it is nonzero and $\dim(S.x) \geq \dim(S.y)$ for all semisimple $y \in \mathfrak{s}_{\overline{1}}$. Let $x \in \mathfrak{s}_{\overline{1}}$ be fixed. The anticommutant and the double anticommutant of x in $\mathfrak{s}_{\overline{1}}$ are

$$\begin{aligned} {}^{x}\mathfrak{s}_{\overline{1}} &= \{y \in \mathfrak{s}_{\overline{1}} : \{x, y\} = 0\}\,, \\ {}^{{}^{x}\mathfrak{s}_{\overline{1}}}\mathfrak{s}_{\overline{1}} &= \bigcap_{y \in {}^{x}\mathfrak{s}_{\overline{1}}} {}^{y}\mathfrak{s}_{\overline{1}}\,, \end{aligned}$$

respectively. A Cartan subspace $\mathfrak{h}_{\overline{1}}$ of $\mathfrak{s}_{\overline{1}}$ as the double anticommutant of a regular semisimple element $x \in \mathfrak{s}_{\overline{1}}$. We denote by ${\mathfrak{h}_{\overline{1}}}^{reg}$ the set of regular elements in $\mathfrak{h}_{\overline{1}}$.

Next we describe the Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathfrak{s}_{\overline{1}}$ for the supergroups associated with the irreducible dual pairs $(\mathrm{G}, \mathrm{G}')$ with G compact. We refer to [Prz06, §6] and [MPP15, §4] for the proofs omitted here. Given a Cartan subspace $\mathfrak{h}_{\overline{1}}$, there are $\mathbb{Z}/2\mathbb{Z}$-graded subspaces $\mathsf{V}^j \subseteq \mathsf{V}$ such that the restriction of the form $(\cdot,\cdot)''$ to each V^j is non-degenerate, V^j is orthogonal to V^k for $j \neq k$ and

$$\mathsf{V} = \mathsf{V}^0 \oplus \mathsf{V}^1 \oplus \mathsf{V}^2 \oplus \cdots \oplus \mathsf{V}^{l''}. \tag{51}$$

The subspace V^0 coincides with the intersection of the kernels of the elements of $\mathfrak{h}_{\overline{1}}$ (equivalently, $\mathsf{V}^0 = \mathrm{Ker}(x)$ if $\mathfrak{h}_{\overline{1}} = {}^{{}^{x}\mathfrak{s}_{\overline{1}}}\mathfrak{s}_{\overline{1}}$). For $1 \leq j \leq l''$, the subspaces V^j are described as follows. Suppose $\mathbb{D} = \mathbb{R}$. Then there is a basis v_0, v_0' of $\mathsf{V}^j_{\overline{0}}$ and basis v_1, v_1' of $\mathsf{V}^j_{\overline{1}}$ such that

$$\begin{aligned} (v_0, v_0)'' = (v_0', v_0')'' = 1, \qquad (v_0, v_0')'' = 0, \\ (v_1, v_1)'' = (v_1', v_1')'' = 0, \qquad (v_1, v_1')'' = 1. \end{aligned}$$

The following formulas define an element $u_j \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$,

$$\begin{aligned} u_j(v_0) &= \frac{1}{\sqrt{2}}(v_1 - v_1'), \qquad & u_j(v_1) &= \frac{1}{\sqrt{2}}(v_0 - v_0'), \\ u_j(v_0') &= \frac{1}{\sqrt{2}}(v_1 + v_1'), \qquad & u_j(v_1') &= \frac{1}{\sqrt{2}}(v_0 + v_0'). \end{aligned}$$

Suppose $\mathbb{D} = \mathbb{C}$. Then $\mathsf{V}^j_{\overline{0}} = \mathbb{C}v_0$, $\mathsf{V}^j_{\overline{1}} = \mathbb{C}v_1$, where $(v_0, v_0)'' = 1$ and $(v_1, v_1)'' = \delta_j i$, with $\delta_j = \pm 1$. The following formulas define an element $u_j \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$,

$$u_j(v_0) = e^{-i\delta_j \frac{\pi}{4}} v_1, \qquad u_j(v_1) = e^{-i\delta_j \frac{\pi}{4}} v_0. \tag{52}$$

Suppose $\mathbb{D} = \mathbb{H}$. Then $\mathsf{V}^j_{\overline{0}} = \mathbb{H}v_0$, $\mathsf{V}^j_{\overline{1}} = \mathbb{H}v_1$, where $(v_0, v_0)'' = 1$ and $(v_1, v_1)'' = i$. The following formulas define an element $u_j \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$,

$$u_j(v_0) = e^{-i\frac{\pi}{4}} v_1, \qquad u_j(v_1) = e^{-i\frac{\pi}{4}} v_0.$$

In any case, by extending each u_j by zero outside V^j, we have

$$\mathfrak{h}_{\overline{1}} = \sum_{j=1}^{l''} \mathbb{R}u_j\,. \tag{53}$$

The formula (53) describes all Cartan subspaces in $\mathfrak{s}_{\overline{1}}$, up to conjugation by S. In other words it describes a maximal family of mutually non-conjugate Cartan subspaces. Notice that there is only one such subspace unless the dual pair (G,G') is isomorphic to $(\mathrm{U}_l,\mathrm{U}_{p,q})$ with $l''=l<p+q$. In the last case there are $\min(l,p)-\max(l-q,0)+1$ such subspaces, assuming $p\leq q$. For each m such that $\max(l-q,0)\leq m\leq \min(p,l)$ there is a Cartan subspace $\mathfrak{h}_{\bar 1,m}$ determined by the condition that m is the number of the positive δ_j in (52). We may assume that $\delta_1=\cdots=\delta_m=1$ and $\delta_{m+1}=\cdots=\delta_l=-1$. The choice of the spaces $\mathsf{V}_{\bar 0}^j$ may be done independently of m. The spaces $\mathsf{V}_{\bar 1}^j$ depend on m.

The Weyl group $W(\mathrm{S},\mathfrak{h}_{\bar 1})$ is the quotient of the stabilizer of $\mathfrak{h}_{\bar 1}$ in S by the subgroup $\mathrm{S}^{\mathfrak{h}_{\bar 1}}$ fixing each element of $\mathfrak{h}_{\bar 1}$. If $\mathbb{D}\neq\mathbb{C}$, then the group $W(\mathrm{S},\mathfrak{h}_{\bar 1})$ acts by all the sign changes and all permutations of the u_j's. If $\mathbb{D}=\mathbb{C}$, then the group $W(\mathrm{S},\mathfrak{h}_{\bar 1})$ acts by all the sign changes of the u_j's and all permutations which preserve $(\delta_1,\ldots,\delta_{l''})$, see [Prz06, (6.3)].

Set $\delta_j=1$ for all $1\leq j\leq l''$, if $\mathbb{D}\neq\mathbb{C}$. In general, let

$$J_j=\delta_j\tau(u_j),\qquad J_j'=\delta_j\tau'(u_j)\qquad (1\leq j\leq l''). \tag{54}$$

Then J_j, J_j' are complex structures on $\mathsf{V}_{\bar 0}^j$ and $\mathsf{V}_{\bar 1}^j$ respectively. Explicitly,

$$\begin{aligned}&J_j(v_0)=-v_0',\quad J_j(v_0')=v_0,\quad J_j'(v_1)=-v_1',\quad J_j'(v_1')=v_1,\quad \text{if } \mathbb{D}=\mathbb{R},\\ &J_j(v_0)=-iv_0,\quad J_j'(v_1)=-iv_1,\quad \text{if } \mathbb{D}=\mathbb{C} \text{ or } \mathbb{D}=\mathbb{H}.\end{aligned} \tag{55}$$

(The point of the multiplication by the δ_j in (54) is that the complex structures J_j, J_j' do not depend on the Cartan subspace $\mathfrak{h}_{\bar 1}$.) In particular, if $w=\sum_{j=1}^{l''}w_ju_j\in\mathfrak{h}_{\bar 1}$, then

$$\tau(w)=\sum_{j=1}^{l''}w_j^2\delta_jJ_j \quad\text{and}\quad \tau'(w)=\sum_{j=1}^{l''}w_j^2\delta_jJ_j'. \tag{56}$$

Let $\mathfrak{h}_{\bar 1}^2\subseteq\mathfrak{s}_{\bar 0}$ be the subspace spanned by all the squares w^2, $w\in\mathfrak{h}_{\bar 1}$. Then

$$\mathfrak{h}_{\bar 1}^2=\sum_{j=1}^{l''}\mathbb{R}(J_j+J_j').$$

We shall use the following identification

$$\mathfrak{h}_{\bar 1}^2|_{\mathsf{V}_{\bar 0}}\ni\sum_{j=1}^{l''}y_jJ_j=\sum_{j=1}^{l''}y_jJ_j'\in\mathfrak{h}_{\bar 1}^2|_{\mathsf{V}_{\bar 1}} \tag{57}$$

and denote both spaces by $\mathfrak{h}$. Denote by l the rank of $\mathfrak{g}$ and by l' the rank of $\mathfrak{g}'$. Then $\mathfrak{h}$ is an elliptic Cartan subalgebra of $\mathfrak{g}$, if $l''=l$, and an elliptic Cartan subalgebra of $\mathfrak{g}'$, if $l''=l'$. Let $d=\dim_{\mathbb{D}}\mathsf{V}_{\bar 0}$ and let $d'=\dim_{\mathbb{D}}\mathsf{V}_{\bar 1}$. The proofs of the following two lemmas is straightforward and left to the reader.

Lemma 6. *Suppose $l \leq l'$. Then $l'' = l$ and one may choose the system of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ so that the product of all of them is given by the formula*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(\sum_{j=1}^{l} y_j J_j) = \begin{cases} \prod_{1\leq j<k\leq l} i(-y_j+y_k) & \text{if } \mathbb{D}=\mathbb{C}, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l} 2iy_j & \text{if } \mathbb{D}=\mathbb{H}, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2) & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l}, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l} iy_j & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}. \end{cases} \tag{58}$$

Let $\mathfrak{z}' \subseteq \mathfrak{g}'$ be the centralizer of $\mathfrak{h}$. We may choose the order of roots of $\mathfrak{h}$ in $\mathfrak{g}'_{\mathbb{C}}/\mathfrak{z}'_{\mathbb{C}}$ so that the product of all of them is equal to

$$\pi_{\mathfrak{g}'/\mathfrak{z}'}(y) = \tag{59}$$
$$\begin{cases} \prod_{1\leq j<k\leq l} i(-y_j+y_k)\cdot\prod_{j=1}^{l}(-iy_j)^{d'-d} & \text{if } \mathbb{D}=\mathbb{C}, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l}(-y_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{H}, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l} 2iy_j\cdot\prod_{j=1}^{l}(iy_j)^{d'-d} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l}, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l} 2iy_j\cdot\prod_{j=1}^{l}(iy_j)^{d'-d+1} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}. \end{cases}$$

The product of the positive roots of $\mathfrak{h}_{\overline{1}}^2$ in the complexification of $\mathfrak{s}_{\overline{0}}$ evaluated at w^2, where $w=\sum_{j=1}^{l''} w_j u_j \in \mathfrak{h}_{\overline{1}}$, is equal to

$$\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) = \pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w)) =$$
$$\begin{cases} \left(\prod_{1\leq j<k\leq l} i(-\delta_j w_j^2+\delta_k w_k^2)\right)^2\cdot\prod_{j=1}^{l}(-i\delta_j w_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{C}, \\ \left(\prod_{1\leq j<k\leq l}(-w_j^4+w_k^4)\right)^2\cdot\prod_{j=1}^{l} 2iw_j^2\cdot\prod_{j=1}^{l}(-w_j^4)^{d'-d} & \text{if } \mathbb{D}=\mathbb{H}, \\ \left(\prod_{1\leq j<k\leq l}(-w_j^4+w_k^4)\right)^2\cdot\prod_{j=1}^{l} 2iw_j^2\cdot\prod_{j=1}^{l}(iw_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l}, \\ \left(\prod_{1\leq j<k\leq l}(-w_j^4+w_k^4)\right)^2\cdot\prod_{j=1}^{l} iw_j^2\cdot\prod_{j=1}^{l} 2iw_j^2\cdot\prod_{j=1}^{l}(iw_j^2)^{d'-d+1} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}. \end{cases}$$

Lemma 7. *Suppose $l \geq l'$. Then $l' = l''$. Set $\mathfrak{h}' = \mathfrak{h}$. Then one may choose the system of the positive roots of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ so that the product of all of them is given by the formula*

$$\pi_{\mathfrak{g}'/\mathfrak{h}'}(\sum_{j=1}^{l'} y_j J_j) = \begin{cases} \prod_{1\leq j<k\leq l'} i(-y_j+y_k) & \text{if } \mathbb{D}=\mathbb{C}, \\ \prod_{1\leq j<k\leq l'}(-y_j^2+y_k^2) & \text{if } \mathbb{D}=\mathbb{H}, \\ \prod_{1\leq j<k\leq l'}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'} 2iy_j & \text{if } \mathbb{D}=\mathbb{R}. \end{cases} \tag{60}$$

Let $\mathfrak{z} \subseteq \mathfrak{g}$ be the centralizer of $\mathfrak{h}$. We may choose the order of roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}/\mathfrak{z}_{\mathbb{C}}$ so that the product of all of them is equal to

$$\pi_{\mathfrak{g}/\mathfrak{z}}(y) = \tag{61}$$
$$\begin{cases} \prod_{1\leq j<k\leq l'} i(-y_j+y_k)\cdot\prod_{j=1}^{l'}(-iy_j)^{d-d'} & \text{if } \mathbb{D}=\mathbb{C}, \\ \prod_{1\leq j<k\leq l'}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'} 2iy_j\cdot\prod_{j=1}^{l'}(-y_j^2)^{d-d'} & \text{if } \mathbb{D}=\mathbb{H}, \\ \prod_{1\leq j<k\leq l'}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'}(iy_j)^{d-d'} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l}, \\ \prod_{1\leq j<k\leq l}(-y_j^2+y_k^2)\cdot\prod_{j=1}^{l'} iy_j\cdot\prod_{j=1}^{l'}(iy_j)^{d-d'} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}. \end{cases}$$

The product of the positive roots of $\mathfrak{h}_{\overline{1}}^2$ *in the complexification of* $\mathfrak{s}_{\overline{0}}$ *evaluated at* w^2, *where* $w = \sum_{j=1}^{l''} w_j u_j \in \mathfrak{h}_{\overline{1}}$, *is equal to*

$$\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) = \pi_{\mathfrak{g}'/\mathfrak{h}'}(\tau'(w))\pi_{\mathfrak{g}/\mathfrak{z}}(\tau(w)) =$$
$$\begin{cases} \left(\prod_{1\le j<k\le l'} i(-\delta_j w_j^2 + \delta_k w_k^2)\right)^2 \cdot \prod_{j=1}^{l'}(-i\delta_j w_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{C},\\ \left(\prod_{1\le j<k\le l'}(-w_j^4 + w_k^4)\right)^2 \cdot \prod_{j=1}^{l'} 2iw_j^2 \cdot \prod_{j=1}^{l'}(-w_j^4)^{d'-d} & \text{if } \mathbb{D}=\mathbb{H},\\ \left(\prod_{1\le j<k\le l'}(-w_j^4 + w_k^4)\right)^2 \cdot \prod_{j=1}^{l'} 2iw_j^2 \cdot \prod_{j=1}^{l'}(iw_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l},\\ \left(\prod_{1\le j<k\le l'}(-w_j^4 + w_k^4)\right)^2 \cdot \prod_{j=1}^{l'} iw_j^2 \cdot \prod_{j=1}^{l'} 2iw_j^2 \cdot \prod_{j=1}^{l'}(iw_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}. \end{cases}$$

The following lemma is an immediate consequence of Lemmas 6 and 7.

Lemma 8. *There is a constant* $C(\mathfrak{h}_{\overline{1}})$, *which depends on* $\mathfrak{h}_{\overline{1}}$, *such that* $|C(\mathfrak{h}_{\overline{1}})| = 1$ *and*

$$|\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| = C(\mathfrak{h}_{\overline{1}})\, \pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) \qquad (w \in \mathfrak{h}_{\overline{1}}).$$

If $\mathfrak{h}_{\overline{1}}$ is a Cartan subspace of W, then

$$\mathfrak{h}_{\overline{1}}^{reg} = \{w \in \mathfrak{h}_{\overline{1}} : \pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) \neq 0\}\,. \tag{62}$$

Fix a Cartan subspace $\mathfrak{h}_{\overline{1}} \subseteq$ W, an element $w \in \mathfrak{h}_{\overline{1}}^{reg}$ and a function $\phi \in \mathcal{S}(\mathrm{W})$. Suppose $\mathrm{G} = \mathrm{O}_{2l+1}$ with $l < l'$. Let $w_0 \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^0)$ be a non-zero element. Then $w + w_0$ is a regular almost semisimple element whose centralizer in S is denoted by $\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}$. Set $\mathcal{O}(w) = \mathrm{S}.(w + w_0)$ and define

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}} \phi(s.(w + w_0))\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}). \tag{63}$$

Then, up to a constant multiple,

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \int_{\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)} \phi(s.(w + w^0))\, dw^0\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}). \tag{64}$$

In all the remaining cases let $\mathcal{O}(w) = \mathrm{S}.w$ and let

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \phi(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}). \tag{65}$$

Let $\mathrm{H} \subseteq \mathrm{G}$ be the Cartan subgroup with the Lie algebra $\mathfrak{h}$. Denote by $\Delta(\mathrm{H}) \subseteq \mathrm{G} \times \mathrm{G}'$ be the diagonal embedding. Then, explicitly,

$$\mathrm{S}^{\mathfrak{h}_{\overline{1}}} = \Delta(\mathrm{H})(\{1\} \times \mathrm{Z}'), \tag{66}$$

where $\mathrm{Z}' \subseteq \mathrm{G}'$ is the centralizer of $\mathfrak{h} \subseteq \mathfrak{g}'$.

To simplify the notation, when the Cartan subspace $\mathfrak{h}_{\overline{1}}$ is fixed, we shall simply write $\mu_{\mathcal{O}(w)}$ instead of $\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}$. These are well defined, tempered distribution on W, see [MPP15], which depend only on $\tau(w)$, or equivalently $\tau'(w)$ via the identification (57). Let μ_{W} be the Lebesgue measure on W normalized as in the Introduction. Choose a positive Weyl chamber $\mathfrak{h}_{\overline{1}}^{+} \subseteq \mathfrak{h}_{\overline{1}}{}^{reg}$. We shall normalize the above orbital integrals so that the Weyl integration formula reads

$$\mu_{\mathrm{W}} = \sum_{\mathfrak{h}_{\overline{1}}} \int_{\tau(\mathfrak{h}_{\overline{1}}^{+})} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \mu_{\mathcal{O}(w)}(\phi)\, d\tau(w) \tag{67}$$

if $l \leq l'$, and

$$\mu_{\mathrm{W}} = \int_{\tau'(\mathfrak{h}_{\overline{1}}^{+})} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \mu_{\mathcal{O}(w)}(\phi)\, d\tau'(w) \tag{68}$$

if $l \geq l'$.

Lemma 9. *Suppose $l \leq l'$ and $\mathbb{D} = \mathbb{C}$. Then for* $\max(l-q, 0) \leq m < m' \leq \min(p, l)$,

$$\tau(\mathfrak{h}_{\overline{1},m}^{reg}) \cap \tau(\mathfrak{h}_{\overline{1},m'}^{reg}) = \emptyset.$$

Proof. We see from (56) and (62) that

$$\tau(\mathfrak{h}_{\overline{1},m}^{reg}) \quad = \quad \{\sum_{j=1}^{l} y_j J_j;\ y_1, \dots, y_m > 0 > y_{m+1}, \dots, y_l,\ \ y_j \neq y_k \text{ for } j \neq k\}. \tag{69}$$

Hence

$$\begin{aligned} & \tau(\mathfrak{h}_{\overline{1},m}^{reg}) \cap \tau(\mathfrak{h}_{\overline{1},m'}^{reg}) \\ = \ & \{\sum_{j=1}^{l} y_j J_j;\ y_1, \dots, y_m > 0 = y_{m+1} = \cdots = y_{m'} > y_{m'+1}, \dots, y_l,\ \ y_j \neq y_k \text{ for } j \neq k\} \\ = \ & \emptyset. \end{aligned}$$

□

Definition 10. *Let $C_{\mathfrak{h}_{\overline{1}}} = C(\mathfrak{h}_{\overline{1}}) \cdot i^{\dim \mathfrak{g}/\mathfrak{h}}$, where $C(\mathfrak{h}_{\overline{1}})$ is as in Lemma 8. Define the Harish-Chandra regular almost semisimple orbital integral on* W *by the following formula*

$$f(y) = \sum_{\mathfrak{h}_{\overline{1}}} C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \mu_{\mathcal{O}(w)} \qquad (y \in \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg}),\ y = \tau(w) = \tau'(w)).$$

(Since, by Lemma 9, the union is disjoint, the definition makes sense. If $l > l'$, then there is only one Cartan subspace $\mathfrak{h}_{\overline{1}}$ and $\mathfrak{z}' = \mathfrak{h}'$.)

In the remainder of this section we shall extend f and its partial derivatives continuously to a larger domain inside $\mathfrak{h}$. This domain will depend on l and l'. If $l \leq l'$, then we will provide a continuous extension of f (and its partial derivatives up to a specific order) to a distribution valued function $f : \mathfrak{h} \cap \tau(\mathrm{W}) \to S^*(\mathrm{W})^S$. If $l > l'$, then f and all of its partial derivatives extend continuously to the closure of every connected component of $\mathfrak{h}'^{In-reg} \cap \tau'(\mathfrak{h}_{\overline{1}})$. See Theorem 17 for the precise statement.

Let $\mu_{\mathfrak{g}}$ be the Lebesgue measure on $\mathfrak{g}$. Let us normalize the orbital integrals $\mu_{\mathrm{G}.y} \in \mathcal{S}^*(\mathfrak{g})$, $y \in \mathfrak{h}^{reg}$, so that

$$\mu_{\mathfrak{g}} = \int_{\mathfrak{h}^{+}} |\pi_{\mathfrak{g}/\mathfrak{h}}(y)|^2 \mu_{\mathrm{G}.y}\, dy, \tag{70}$$

where $\mathfrak{h}^+ \subseteq \mathfrak{h}^{reg}$ is a Weyl chamber.

Let $\mathrm{W}_\mathfrak{g} \subseteq \mathrm{W}$ be the maximal subset such that $\tau|_{\mathrm{W}_\mathfrak{g}} : \mathrm{W}_\mathfrak{g} \to \mathfrak{g}$, the restriction of τ to $\mathrm{W}_\mathfrak{g}$, is a submersion. Then $\mathrm{W}_\mathfrak{g} \neq \emptyset$ if and only if $l \leq l'$, see Appendix A. In this case we shall assume that

$$\tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_\mathfrak{g}) = \mu_\mathrm{W}|_{\mathrm{W}_\mathfrak{g}}. \tag{71}$$

Lemma 11. *Suppose $l \leq l'$. Then*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.\tau(w)}) = f(y)|_{\mathrm{W}_\mathfrak{g}} \qquad (w \in \mathfrak{h}_{\bar{1}}{}^{reg}).$$

Proof. We see from (71) that

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|^2\tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.\tau(w)}) = |\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))|\mu_{\mathcal{O}(w)}|_{\mathrm{W}_\mathfrak{g}} \qquad (w \in \mathfrak{h}_{\bar{1}}^+).$$

Hence,

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|\tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.\tau(w)}) = |\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))|\mu_{\mathcal{O}}(w)|_{\mathrm{W}_\mathfrak{g}} \qquad (w \in \mathfrak{h}_{\bar{1}}{}^{reg}),$$

because both sides are $W(\mathrm{S}, \mathfrak{h}_{\bar{1}})$-invariant. Thus,

$$\begin{aligned} &\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.\tau(w)}) \\ = &\left(\frac{|\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))|}{\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))}\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))}{|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|}\right)\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))\mu_{\mathcal{O}}(w)|_{\mathrm{W}_\mathfrak{g}} \qquad (w \in \mathfrak{h}_{\bar{1}}{}^{reg}). \end{aligned}$$

Let $C(\mathfrak{h}_{\bar{1}})$ be the constant in Lemma 8. Then

$$\frac{|\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))|}{\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))}\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))}{|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|} = C(\mathfrak{h}_{\bar{1}})\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))^2}{|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|^2} = C(\mathfrak{h}_{\bar{1}})i^{\dim \mathfrak{g}/\mathfrak{h}}.$$

Hence, the lemma follows. ☐

Let

$$W(\mathrm{G}, \mathfrak{h}) = \begin{cases} \Sigma_l & \text{if } \mathbb{D} = \mathbb{C}, \\ \Sigma_l \ltimes \{\pm 1\}^l & \text{otherwise.} \end{cases} \tag{72}$$

Denote the elements of Σ_l by σ and the elements of $\{\pm 1\}^l$ by $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_l)$, so that an arbitrary element of the group (72) looks like $\epsilon\sigma$, with $\epsilon = (1, 1, \dots, 1)$, if $\mathbb{D} = \mathbb{C}$. This group acts on $\mathfrak{h}$ as follows:

$$(\epsilon\sigma)\sum_{j=1}^{l} y_j J_j = \sum_{j=1}^{l} \epsilon_j y_{\sigma^{-1}(j)} J_j \tag{73}$$

and coincides with the Weyl group, equal to the normalizer of $\mathfrak{h}$ in G divided by the centralizer of $\mathfrak{h}$ in G, as the indicated by the notation.

Since the moment map τ intertwines the action of the Weyl group $W(\mathrm{S}, \mathfrak{h}_{\bar{1}})$ with the subgroup $W(\mathrm{S}, \mathfrak{h}_{\bar{1}}, \mathfrak{h}) \subseteq \Sigma_l \subseteq W(\mathrm{G}, \mathfrak{h})$ leaving the sequence $\delta_1,\ \delta_2,\ \dots,\ \delta_l$ fixed. The function $f(y)$ is invariant under that subgroup. Furthermore,

$$W(\mathrm{G}, \mathfrak{h})\bigcup_{\mathfrak{h}_{\bar{1}}}\tau(\mathfrak{h}_{\bar{1}}{}^{reg}) = \bigcup_{\mathfrak{h}_{\bar{1}}}(W(\mathrm{G}, \mathfrak{h})/W(\mathrm{S}, \mathfrak{h}_{\bar{1}}, \mathfrak{h}))\tau(\mathfrak{h}_{\bar{1}}{}^{reg}), \tag{74}$$

where the union on the right hand side is disjoint. Hence, in any case ($l \le l'$ or $l > l'$) we may extend the function f uniquely to $W(\mathrm{G},\mathfrak{h})\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}{}^{reg})$ so that the extension satisfies the following symmetry condition

$$f(sy)=\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)f(y) \qquad (s\in W(\mathrm{G},\mathfrak{h}),\ y\in W(\mathrm{G},\mathfrak{h})\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}{}^{reg})), \tag{75}$$

where $\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}$ is defined by

$$\pi_{\mathfrak{g}/\mathfrak{h}}(sy)=\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\,\pi_{\mathfrak{g}/\mathfrak{h}}(y) \qquad (y\in\mathfrak{h})\,. \tag{76}$$

One motivation for such a definition of the extension is that left hand side of the equality in Lemma 11 extends to all $y\in\mathfrak{h}$ and satisfies the symmetry condition (75). We would like to extend the function f from the set $W(\mathrm{G},\mathfrak{h})\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}{}^{reg})$ to $\mathfrak{h}\cap\tau(\mathrm{W})$. This will require some more work.

Suppose $l\le l'$. Fix an elliptic Cartan subalgebra $\mathfrak{h}'\subseteq\mathfrak{g}'$ containing $\tau'(\mathfrak{h}_{\overline{1}})$. We may assume that $\mathfrak{h}'$ does not depend on $\mathfrak{h}_{\overline{1}}$. Let $\mathfrak{h}'^{In-reg}\subseteq\mathfrak{h}'$ be the subset where no non-compact roots vanish. Set $\mathfrak{h}_{\overline{1}}^{In-reg}=\tau'^{-1}(\mathfrak{h}'^{In-reg})\cap\mathfrak{h}_{\overline{1}}$. Then $\tau(\mathfrak{h}_{\overline{1}}^{In-reg})$ is the set of the elements $y\in\tau(\mathfrak{h}_{\overline{1}})$ such that, under the identification (57), no non-compact imaginary root of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ vanishes on y.

Lemma 12. *Suppose $l\le l'$. For our specific Cartan subspace (53), the set $\tau(\mathfrak{h}_{\overline{1}}^{In-reg})$ consists of elements $y=\sum_{j=1}^{l}y_jJ_j$, such that*

$$\begin{aligned}&y_j>0 \text{ for all } j,\ \text{if } \mathrm{G}=\mathrm{O}_{2l} \text{ or } \mathrm{G}=\mathrm{O}_{2l+1} \text{ or } \mathrm{G}=\mathrm{Sp}_l \text{ with } l<l' \text{ or } 1=l=l',\\ &y_j\ge 0 \text{ for all } j\,,\ y_j+y_k>0 \text{ for all } j\ne k,\ \text{if } \mathrm{G}=\mathrm{Sp}_l \text{ and } 1<l=l',\end{aligned} \tag{77}$$

and

$$\begin{aligned}&y_j>0 \text{ if } j\le m \text{ and } l-m<q;\quad y_j\ge 0 \text{ if } j\le m \text{ and } l-m=q \text{ when } l\ge q\,;\\ &y_j<0 \text{ if } m<j \text{ and } m<p;\quad y_j\le 0 \text{ if } m<j \text{ and } m=p \text{ when } l\ge p\,;\\ &y_j-y_k>0 \text{ if }\ j\le m<k\,,\end{aligned} \tag{78}$$

if $\mathrm{G}'=\mathrm{U}_{p,q}$ and $\mathfrak{h}_{\overline{1}}=\mathfrak{h}_{\overline{1},m}$. In particular, in the last case,

$$\tau(\mathfrak{h}_{\overline{1},m}^{In-reg})\cap\tau(\mathfrak{h}_{\overline{1},m'}^{In-reg})\ne\emptyset \quad \text{implies} \quad |m-m'|\le 1, \tag{79}$$

$$\tau(\mathfrak{h}_{\overline{1},m}^{In-reg})\cap\tau(\mathfrak{h}_{\overline{1},m+1}^{In-reg})\subseteq\{\sum_{j=1}^{l}y_jJ_j;\ y_1,\dots,y_m\ge 0=y_{m+1}\ge y_{m+2},\dots,y_l\}.$$

Proof. We see from (56) that the set $\tau(\mathfrak{h}_{\overline{1}})$ consists of elements $y=\sum_{j=1}^{l}y_jJ_j$, such that $\delta_jy_j\ge 0$ for all $1\le j\le l$. Hence $\sum_{j=1}^{l}y_jJ'_j\in\mathfrak{h}'$ not annihilated by any imaginary non-compact root of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ implies (77) when $\mathbb{D}\ne\mathbb{C}$.

If $G' = U_{p,q}$, then the non-compact roots of of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ acting on elements of $\mathfrak{h} \subseteq \mathfrak{h}'$ are given by

$$\mathfrak{h} \ni \sum_{j=1}^{l} y_j J'_j \to \pm i(y_j - y_k) \in i\mathbb{R}, \quad \text{if } \ j \leq m < k \text{ or } k \leq m < j,$$

$$\mathfrak{h} \ni \sum_{j=1}^{l} y_j J'_j \to \pm i y_j \in i\mathbb{R}, \ \text{if } j \leq m \text{ and } l-m < q \ \text{ or } \ m < j \text{ and } m < p.$$

Hence, (78) follows. The last statement follows from the equality

$$\begin{aligned} & \tau(\mathfrak{h}_{\overline{1},m}) \cap \tau(\mathfrak{h}_{\overline{1},m+k}) \\ = \ & \{\sum_{j=1}^{l} y_j J_j;\ y_1, \dots, y_m \geq 0 = y_{m+1} = \cdots = y_{m+k} \geq y_{m+k+1}, \dots, y_l\}, \end{aligned} \tag{80}$$

which is a consequence of (69). ☐

Lemma 13. *Suppose $l \leq l'$. For a fixed Cartan subspace $\mathfrak{h}_{\overline{1}}$, the function*

$$\tau'_* \circ f : \tau(\mathfrak{h}_{\overline{1}}^{reg}) \to \mathcal{S}(\mathfrak{g}')^{G'} \tag{81}$$

extends to a smooth function

$$\tau'_* \circ f : \tau(\mathfrak{h}_{\overline{1}}^{In-reg}) \to \mathcal{S}(\mathfrak{g}')^{G'} \tag{82}$$

whose all derivatives are bounded. Further, any derivative of (82) extends to a continuous function on the closure of any connected component of $\tau(\mathfrak{h}_{\overline{1}}^{In-reg})$.

Proof. For a moment let us exclude the case $G = O_{2l+1}$ with $l < l'$. Let $\psi \in C_c^\infty(\mathfrak{g}')$. Then

$$\tau'_*(\mu_{\mathcal{O}(w)})(\psi) = \int_{S/S^{\mathfrak{h}_{\overline{1}}}} \psi(\tau'(s.w))\, d(sS^{\mathfrak{h}_{\overline{1}}}). \tag{83}$$

Let $Z' \subseteq G'$ is the centralizer of $\mathfrak{h} \subseteq \mathfrak{g}'$ and recall formula (66) for $S^{\mathfrak{h}_{\overline{1}}}$. Since G is compact, (83) is a constant multiple of

$$\int_{G'/Z'} \psi(g'.y)\, d(g'Z'). \tag{84}$$

As checked in [MPP15, (23)], there is a positive constant C such that

$$\begin{aligned} & \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{G'/Z'} \psi(g'.y)\, d(g'Z') \\ = \ & C\partial(\pi_{\mathfrak{z}'/\mathfrak{h}'}) \left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'') \int_{G'} \psi(g'.(y+y'')\, dg' \right)|_{y''=0}, \end{aligned} \tag{85}$$

where $y \in \mathfrak{h}$ and $y'' \in \mathfrak{h}' \cap [\mathfrak{z}', \mathfrak{z}']$. Hence, the lemma follows from [Har57b, Theorem 2, page 207 and Lemma 25, page 232] and the fact the space of the distributions is weakly complete, [Hör83, Theorem 2.1.8].

Suppose $\mathrm{G} = \mathrm{O}_{2l+1}$ with $l < l'$. Let $w_0 \in \mathfrak{s}_{\overline{1}}(\mathsf{V}^0)$ be as in (63). Then $(w+w_0)^2 = w^2+w_0^2$. Hence,

$$\begin{aligned}\tau'_*(\mu_{\mathcal{O}(w)})(\psi) &= \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}} \psi(\tau'(s.(w+w_0)))\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}) \qquad (86)\\ &= \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}} \psi(s.(\tau'(w)+\tau'(w_0)))\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0})\\ &= C_1 \int_{\mathrm{G}'/\mathrm{Z}'^n} \psi(g.(y+n))\, d(g\mathrm{Z}'^n),\end{aligned}$$

where C_1 is a positive constants, $y = \tau'(w)$, $n = \tau'(w_0)$ and Z'^n is the centralizer of n in Z'.

Let $\pi^{short}_{\mathfrak{z}'/\mathfrak{h}'}$ denote the product of the positive short roots of $\mathfrak{h}'$ in $\mathfrak{z}'_{\mathbb{C}}$. As checked in [MPP15, (35)], there is a positive constant C such that

$$\begin{aligned}\partial(\pi^{short}_{\mathfrak{z}'/\mathfrak{h}'}) \left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+x) \int_{\mathrm{G}'/\mathrm{H}'} \psi(g.(y+y''))\, d(g\mathrm{H})\right)\Bigg|_{y''=0} & \qquad (87)\\ = C\pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{\mathrm{G}'/\mathrm{Z}'^n} \psi(g.(y+n))\, d(g\mathrm{Z}'^n). &\end{aligned}$$

Hence the lemma follows from theorems of Harish-Chandra, as before. □

For a test function ϕ on a vector space U set

$$\phi_t(u) = t^{-\dim \mathrm{U}}\phi(t^{-1}u) \qquad (t>0,\ u \in \mathrm{U}). \qquad (88)$$

Then a distribution Φ on U is homogeneous of degree $a \in \mathbb{C}$ if and only if

$$\Phi(\phi_t) = t^a\Phi(\phi) \qquad (t>0,\ \phi \in C_c^\infty(\mathrm{U})).$$

Lemma 14. *Suppose $l = 1$. Set*

$$f^{(k)} = \lim_{y\to 0} \partial(J_1^k)f(y) \qquad (k = 0,\ 1,\ ,\dots).$$

Then $\tau'_*(f^{(k)})$ is homogeneous of degree

$$\begin{aligned}&-\dim(\mathfrak{g}') + \deg(\pi_{\mathfrak{g}'/\mathfrak{z}'}) + l' - 1 - k, \text{ if } \mathrm{G} = \mathrm{O}_{2l+1} \text{ and } l < l', \qquad (89)\\ &-\dim(\mathfrak{g}') + \deg(\pi_{\mathfrak{g}'/\mathfrak{z}'}) - k, \text{ otherwise.}\end{aligned}$$

(Here $\deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})$ denotes the degree of the polynomial $\pi_{\mathfrak{g}'/\mathfrak{z}'}$.) Furthermore,

$$\mathrm{supp}(\tau'_*(f^{(k)})) \subseteq \tau'(\tau^{-1}(0)). \qquad (90)$$

Proof. It suffices to consider the restriction of f to $\tau(\mathfrak{h}_{\overline{1}}^{reg})$ for one of the Cartan subspaces $\mathfrak{h}_{\overline{1}}$. Let $\psi \in C_c^\infty(\mathfrak{g}')$. For a moment let us exclude the case $\mathrm{G} = \mathrm{O}_{2l+1}$, $l < l'$. As we have seen in the proof of Lemma 13, there is a non-zero constant C, such that for $t>0$

$$\begin{aligned} \tau'_*(f^{(0)})(\psi_t) &= C\lim_{y\to 0}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}'/\mathrm{Z}'}\psi_t(g.y)\,d(g\mathrm{Z}') \\ &= C\lim_{y\to 0}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}'/\mathrm{Z}'}t^{-\dim(\mathfrak{g}')}\psi(g.t^{-1}y)\,d(g\mathrm{Z}') \\ &= t^{-\dim(\mathfrak{g}')+\deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})}\tau'_*(f^{(0)})(\psi). \end{aligned}$$

Thus, by taking the derivative, (89) follows.

Let $U\subseteq \mathrm{W}$ be an open subset with the compact closure $\overline{U}$ such that $\overline{U}\cap\tau^{-1}(0)=\emptyset$. For $w'\in\overline{U}$ let $w'=w'_s+w'_n$ be the Jordan decomposition and let ϵ be the minimum of all the $|w'_s|$ (for some fixed norm $|\cdot|$ on W) such that $w'\in\overline{U}$. Then $\epsilon>0$ because otherwise there would be a non-zero nilpotent element of W outside of $\tau^{-1}(0)$, which is impossible. Hence

$$\mathrm{S}.w\cap U=\emptyset\qquad (|w|<\epsilon, w\in\mathfrak{h}_{\overline{1}}). \tag{91}$$

Since $\operatorname{supp} f(\tau(w))=\mathrm{S}w$ this implies (90).

Suppose $\mathrm{G}=\mathrm{O}_{2l+1}$ and $l<l'$. Then for $t>0$,

$$\tau'_*(f^{(0)})(\psi_t)=C\lim_{y\to 0}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}}\int_{\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)}\psi_t(\tau'(s.(w+w^0)))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}}_{\overline{1}})$$

But,

$$\begin{aligned} &\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}}\int_{\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)}\psi_t(\tau'(s.(w+w^0)))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}}_{\overline{1}}) \\ &= t^{-\dim(\mathfrak{g}')}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}}\int_{\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)}\psi(\tau'(s.(t^{-1/2}w+t^{-1/2}w^0)))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}}_{\overline{1}}) \\ &= t^{-\dim(\mathfrak{g}')+\frac{1}{2}\dim(\mathfrak{s}_{\overline{1}}(\mathsf{V}^0))}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}}\int_{\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)}\psi(\tau'(s.(t^{-1/2}w+w^0)))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}}_{\overline{1}}) \\ &= t^{-\dim(\mathfrak{g}')+\deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})+\frac{1}{2}\dim(\mathfrak{s}_{\overline{1}}(\mathsf{V}^0))}\pi_{\mathfrak{g}'/\mathfrak{z}'}(t^{-1}y)\int_{\mathrm{G}'/\mathrm{Z}'^n}\psi(g.(t^{-1}y+n))\,d(g\mathrm{Z}'^n)). \end{aligned}$$

Hence, by taking the limit if $y\to 0$ we conclude that

$$\tau'_*(f^{(0)})(\psi_t)=t^{-\dim(\mathfrak{g}')+\deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})+\frac{1}{2}\dim(\mathfrak{s}_{\overline{1}}(\mathsf{V}^0))}\tau'_*(f^{(0)})(\psi).$$

Since $\dim(\mathfrak{s}_{\overline{1}}(\mathsf{V}^0))=2l'-2$, (89) follows.

Also, with the above notation, $w+w_0$ is a Jordan sum with w, the semisimple part, and w_0, the nilpotent part. Hence, as in (91), we have

$$\mathrm{S}.(w+w_0)\cap U=\emptyset\qquad (|w|<\epsilon, w\in\mathfrak{h}_{\overline{1}}).$$

Since $\operatorname{supp}(f(\tau(w)))=\mathrm{S}.(w+w_0)$, (90) follows. □

Lemma 15. *Let $l=1$. Then $\mathfrak{h}=\mathbb{R}J_1$ and*

$$W(\mathrm{G},\mathfrak{h})\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}{}^{reg})=\begin{cases}\mathbb{R}^+J_1 & \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{U}_1,\mathrm{U}_{l'}=\mathrm{U}_{l',0}),\\ \mathbb{R}^-J_1 & \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{U}_1,\mathrm{U}_{l'}=\mathrm{U}_{0,l'}),\\ \mathbb{R}^\times J_1 & \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{O}_3,\mathrm{Sp}_{2l'}),\ (\mathrm{O}_2,\mathrm{Sp}_{2l'}),\ (\mathrm{Sp}_1,\mathrm{O}^*_{2l'})\\ & \text{or } (\mathrm{U}_1,\mathrm{U}_{p,q}) \text{ with } 1\le p\le q.\end{cases}$$

Let $f(y)$ denote the function (75). For an integer $k=0,1,2,\dots$ define

$$\langle f^{(k)}\rangle=\lim_{y\to 0\pm}\partial(J_1^k)f(yJ_1)$$

if $(\mathrm{G},\mathrm{G}')=(\mathrm{U}_1,\mathrm{U}_{l'})$ and

$$\langle f^{(k)}\rangle=\lim_{y\to 0+}(\partial(J_1^k)f(yJ_1)-\lim_{y\to 0-}(\partial(J_1^k)f(yJ_1)$$

in the remaining cases. Assume that $1<l'$. Then

$$\langle f^{(k)}\rangle=0 \ \text{if } 0\le k<\begin{cases}2l'-2 & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathrm{G}=\mathrm{O}_3,\\ 2l'-1 & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathrm{G}=\mathrm{O}_2,\\ l'-1 & \text{if } \mathbb{D}=\mathbb{C},\\ 2(l'-1) & \text{if } \mathbb{D}=\mathbb{H}.\end{cases}$$

Proof. Suppose $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_3,\mathrm{Sp}_{2l'})$. We know from Lemma 14 that the distribution $\tau'_*(\langle f^{(k)}\rangle)$ is supported in $\tau'(\tau^{-1}(0))$. However Lemma 11 shows that for any $\phi\in C_c^\infty(\mathrm{W}_\mathfrak{g})$, $f(y)(\phi)$ is a smooth function of $y\in\mathfrak{h}$. Therefore, $\langle f^{(k)}\rangle|_{\mathrm{W}_\mathfrak{g}}=0$. Hence,

$$\mathrm{supp}(\tau'_*(\langle f^{(k)}\rangle))\subseteq\tau'(\tau^{-1}(0)\setminus\mathrm{W}_\mathfrak{g}).$$

A straightforward argument shows that $\tau'(\tau^{-1}(0)\setminus\mathrm{W}_\mathfrak{g})$ is the union of one of the two minimal nilpotent orbits in $\mathfrak{g}'$, call it $\mathcal{O}_{min}$, and the zero orbit. Furthermore, $\dim(\mathcal{O}_{min})=2l'$. (See Appendix A.) Lemma 14 and (58) show that $\tau'_*(\langle f^{(k)}\rangle)$ is a homogeneous distribution of degree

$$-\dim\mathfrak{g}'+\deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})+l'-1-k=-\dim\mathfrak{g}'+3l'-2-k$$

However, as shown in [Wal93, Lemma 6.2], $\tau'_*(\langle f^{(k)}\rangle)=0$ if the homogeneity degree is greater than $-\dim\mathfrak{g}'+\frac{1}{2}\dim\mathcal{O}_{min}$. Hence the claim follows.

Exactly the same argument works if $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_2,\mathrm{Sp}_{2l'})$, or $(\mathrm{Sp}_1,\mathrm{O}^*_{2l'})$, or $(\mathrm{U}_1,\mathrm{U}_{p,q})$ with $1\le p\le q$, except that $\tau'(\tau^{-1}(0)\setminus\mathrm{W}_\mathfrak{g})=\{0\}$, see Appendix A. So, instead of relying on [Wal93, Lemma 6.2], we may use the classical description of distributions supported at $\{0\}$, [Hör83, Theorem 2.3.4.].

Suppose $(\mathrm{G},\mathrm{G}')=(\mathrm{U}_1,\mathrm{U}_{l'})$. Then (83) and (84) show that for $\psi\in C_c^\infty(\mathfrak{g}')$ and $0\neq y=\tau(w)=\tau'(w)$,

$$\tau'_*(f(y))(\psi)=const\,\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}'}\psi(g'.y)\,d(g').$$

Since the group G' is compact, the last integral defines a smooth function of $y=y'J_1$. Also, in this case, $\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)=(iy')^{l'-l}$. Hence, the claim follows. □

Lemma 16. *Suppose $l \leq l'$. Let $f(y)$ denote the function (75), with $y = \sum_{j=1}^{l} y_j J_j \in W(\mathrm{G},\mathfrak{h}) \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_l)$ set $\partial(J)^\alpha = \partial(J_1)^{\alpha_1} \dots \partial(J_l)^{\alpha_l}$. For $1 \leq j \leq l$ define*

$$\langle \partial(J)^\alpha f \rangle_{y_j=0} = \lim_{y_j \to 0\pm} \partial(J)^\alpha f(y)$$

if $\{y_j \neq 0;\ y \in W(\mathrm{G},\mathfrak{h}) \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})\} = \mathbb{R}^\pm$, and

$$\langle \partial(J)^\alpha f \rangle_{y_j=0} = \lim_{y_j \to 0+} \partial(J)^\alpha f(y) - \lim_{y_j \to 0-} \partial(J)^\alpha f(y)$$

if $\{y_j \neq 0;\ y \in W(\mathrm{G},\mathfrak{h}) \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})\} = \mathbb{R}^\times$. Then for $1 \leq j \leq l$

$$\langle \partial(J)^\alpha f \rangle_{y_j=0} = 0 \text{ if } 0 \leq \alpha_j < \begin{cases} 2(l'-l)+1 & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathrm{G} = \mathrm{O}_{2l}, \\ 2l'-2l & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathrm{G} = \mathrm{O}_{2l+1}, \\ l'-l & \text{if } \mathbb{D} = \mathbb{C}, \\ 2(l'-l) & \text{if } \mathbb{D} = \mathbb{H}. \end{cases}$$

(Here $\langle \partial(J)^\alpha f \rangle_{y_j=0}$ is a function of the y with $y_j = 0$.)

Proof. Without any loss of generality we may assume that $j = l$. Let $w = \sum_{j=1}^{l-1} w_j u_j$, where $\delta_j w_j^2 = y_j$, $1 \leq j \leq l-1$. Recall the decomposition (51). The centralizer of w in $\mathrm{W} = \mathfrak{s}_{\overline{1}}$ is equal to

$$\mathfrak{s}_{\overline{1}}^w = \mathfrak{s}_{\overline{1}}(\mathsf{V})^w = \mathfrak{s}_{\overline{1}}(\mathsf{V}^1)^w \oplus \cdots \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^{l-1})^w \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l). \tag{92}$$

As checked in the proof of [Prz06, Theorem 4.5] [1], there is a slice through w equal to

$$U_w = (w_1 - \epsilon, w_1 + \epsilon)u_1 + \cdots + (w_{l-1} - \epsilon, w_{l-1} + \epsilon)u_{l-1} + \mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l),$$

where $\epsilon > 0$ is sufficiently small. In order to indicate its dependence on the graded space V, let us denote by $f_{\mathsf{V}}(y)$ the function (75). Recall that $y = \sum_{j=1}^{l} y_j J_j \in W(\mathrm{G},\mathfrak{h}) \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})$ and let w_y be such that $\tau'(w_y) = y$. The Lebesgue measure on $\mathfrak{s}_{\overline{1}}(\mathsf{V})$ is fixed and the orbital integral $\mu_{\mathcal{O}(w_y)}$ is normalized as in (67). We normalize the Lebesgue measure on each $\mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$ and on $\mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l)$ so that via the direct sum decomposition

$$\mathfrak{s}_{\overline{1}}(\mathsf{V}) = \mathfrak{s}_{\overline{1}}(\mathsf{V}^1) \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^2) \oplus \cdots \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^{l-1}) \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l)$$

we get the same measure on $\mathrm{W} = \mathfrak{s}_{\overline{1}}(\mathsf{V})$. Then the $\mathrm{S}(\mathsf{V})$-orbital integral $\mu_{\mathcal{O}(w_y)}$ restricts to U_w and the result is the tensor product of $\mathrm{S}(\mathsf{V}^j)^w$-orbital integrals and the $\mathrm{S}(\mathsf{V}^0 \oplus \mathsf{V}^l)$-orbital integral, because U_w is a slice. Therefore,

$$\begin{aligned} & f_{\mathsf{V}}(y)|_{\mathrm{U}_w} \\ = \ & P(y)(f_{\mathsf{V}^1}(y_1 J_1)|_{(w_1-\epsilon,w_1+\epsilon)u_1} \otimes \cdots \otimes f_{\mathsf{V}^{l-1}}(y_{l-1}J_{l-1})|_{(w_{l-1}-\epsilon,w_{l-1}+\epsilon)u_{l-1}} \otimes f_{\mathsf{V}^0 \oplus \mathsf{V}^l}(y_l J_l)), \end{aligned} \tag{93}$$

[1] The statement of that theorem needs to be modified as follows. "Let $x \in \mathfrak{g}_1$ be semisimple. Then $\mathfrak{g}_{\overline{1}}^x$ has a basis of G^x-invariant neighborhoods of x consisting of admissible slices U_x through x. If $\ker(x) = 0$ then one may choose the U_x so that, for $i = \overline{0}, \overline{1}$,

$$\mathrm{U}_x \ni y \to y^2|_{\mathsf{V}_i} \in \mathfrak{g}_0(\mathsf{V}_i)^{x^2}$$

is an (injective) immersion."

where $P(y)$ is a polynomial, whose precise expression may be found from (58). In (93)

$$f_{\mathsf{V}^j}(y_jJ_j)|_{(w_j-\epsilon,w_j+\epsilon)u_j} \in \mathcal{D}'((w_j-\epsilon,w_j+\epsilon)u_j) \qquad (1\leq j\leq l-1)$$

and

$$f_{\mathsf{V}^0\oplus\mathsf{V}^l}(y_lJ_l) \in \mathcal{D}'(\mathfrak{s}_{\overline{1}}(\mathsf{V}^0\oplus\mathsf{V}^l)). \tag{94}$$

Here $\mathcal{D}'(\mathrm{X})$ denotes the space of distributions on X. Since the dimension of a Cartan subalgebra of $\mathrm{S}(\mathsf{V}^0\oplus\mathsf{V}^l)|_{\mathsf{V}_{\overline{1}}}$ is equal to $l'-l+1$, Lemma 16 follows from (93), (94) and Lemma 15. This verifies the claim with $\alpha=(0,\dots,0,k,0,\dots,0)$ with the k on the place j. In order to complete the proof we repeat the same argument with the f replaced by $\partial(J)^\beta f$, where $\beta_j=0$. □

In the case $l\leq l'$, Lemmas 13 and 16 provide a further extension of the function f to a continuous function

$$f: W(\mathrm{G},\mathfrak{h})\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}^{In-reg}) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{95}$$

which satisfies the symmetry condition (75).

Let

$$r=\frac{2\dim(\mathfrak{g})}{\dim(\mathsf{V}_{\overline{0}})}, \tag{96}$$

where we view both $\mathfrak{g}$ and $\mathsf{V}_{\overline{0}}$ as vector spaces over $\mathbb{R}$. Explicitly,

$$r=\begin{cases} 2l-1 & \text{if } \mathrm{G}=\mathrm{O}_{2l},\\ 2l & \text{if } \mathrm{G}=\mathrm{O}_{2l+1},\\ l & \text{if } \mathrm{G}=\mathrm{U}_l,\\ l+\frac{1}{2} & \text{if } \mathrm{G}=\mathrm{Sp}_l.\end{cases} \tag{97}$$

Let

$$\iota=\begin{cases} 1 & \text{if } \mathbb{D}=\mathbb{R} \text{ or } \mathbb{C}\\ \frac{1}{2} & \text{if } \mathbb{D}=\mathbb{H}.\end{cases} \tag{98}$$

The formula (97) together with (58) show that

$$\max\{\deg_{y_j}\pi_{\mathfrak{g}/\mathfrak{h}};\ 1\leq j\leq l\}=\frac{1}{\iota}(r-1), \tag{99}$$

where $\deg_{y_j}\pi_{\mathfrak{g}/\mathfrak{h}}$ denote the degree of $\pi_{\mathfrak{g}/\mathfrak{h}}(\sum_{j=1}^l y_jJ_j)$ with respect to the variable y_j.

The following theorem collects the required properties of the Harish-Chandra almost semisimple orbital integral on W.

Theorem 17. *Suppose $l\leq l'$. Then the closure of the subset $W(\mathrm{G},\mathfrak{h})\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}^{reg})\subseteq\mathfrak{h}$ is equal to*

$$\mathfrak{h}\cap\tau(\mathrm{W})=\begin{cases}\mathfrak{h} & \text{if } \mathbb{D}\neq\mathbb{C},\\ W(\mathrm{G},\mathfrak{h})\{\sum_{j=1}^l y_jJ_j;\ y_1,\dots,y_{\max(l-q,0)}\geq 0\geq y_{\min(p,l)+1},\dots,y_l\} & \text{if } \mathbb{D}=\mathbb{C}.\end{cases}$$

The function (95) is smooth on the subset where each $y_j \neq 0$ *and, for any multi-index* $\alpha = (\alpha_1, \dots, \alpha_l)$ *with*

$$\max(\alpha_1, \dots, \alpha_l) \leq \begin{cases} d' - r - 1 & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C}, \\ 2(d' - r) & \text{if } \mathbb{D} = \mathbb{H}, \end{cases}$$

the function $\partial(J^\alpha)f(y)$ *extends to a continuous function on* $\mathfrak{h} \cap \tau(\mathrm{W})$. *This extension is equal to zero on the boundary of this set. We shall therefore extend it from* $W(\mathrm{G}, \mathfrak{h}) \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{In-reg})$ *to* $\mathfrak{h} \cap \tau(\mathrm{W})$ *by zero.*

Suppose now $l > l'$. *Then* f *extends to a smooth function*

$$f : \mathfrak{h}'^{In-reg} \cap \tau'(\mathfrak{h}_{\overline{1}}) \to \mathbb{C} \tag{100}$$

and any derivative of f *extends to a continuous function on the closure of any connected component of* $\mathfrak{h}'^{In-reg} \cap \tau'(\mathfrak{h}_{\overline{1}})$.

Proof. The formula for $\mathfrak{h} \cap \tau(\mathrm{W})$ follows from (56), (69) and (73), via a case by case verification. The extension of $\partial(J^\alpha)f(y)$ is a consequence of Lemma 16. Finally, the extension in the case $l \leq l'$ is done as in Lemma 13. □

In general we shall write $f_\phi(y)$ for $f(y)(\phi)$.

5. **Limits of orbital integrals.**

In this section we consider weighted dilations of the almost semisimple orbital using a positive variable t. It turns out that the limit as t tends to 0 is a constant multiple of the invariant measure on a nilpotent orbit. So we begin by describing the nilpotent orbits. It turns out that there is one maximal orbit.

Let m denote the minimum of $d = \dim_{\mathbb{D}} \mathsf{V}_{\overline{0}}$ and the dimension, over $\mathbb{D}$, of a maximal isotropic subspace of $\mathsf{V}_{\overline{1}}$ with respect to the form $(\cdot, \cdot)'$ in (48). Recall that the pair $(\mathrm{G}, \mathrm{G}')$ is said to be in the stable range with G the smaller member if $m = d$.

Lemma 18. *Let* m *be as above and let* $d' = \dim_{\mathbb{D}} \mathsf{V}_{\overline{1}}$. *Let* $\mathcal{SH}_k(\mathbb{D})$ *denote the space of the skew-hermitian matrices of size* k *with coefficients on* $\mathbb{D}$, $0 \leq k \leq m$. *There are nilpotent orbits* $\mathcal{O}'_k \subseteq \mathfrak{g}$ *such that*

$$\begin{aligned} &\tau'\tau^{-1}(0) = \mathcal{O}'_m \cup \mathcal{O}'_{m-1} \cup \dots \cup \mathcal{O}'_0, \\ &\mathcal{O}'_k \cup \mathcal{O}'_{k-1} \cup \dots \cup \mathcal{O}'_0 \;\; \text{is the closure of } \; \mathcal{O}'_k \;\; \text{for } \; 0 \leq k \leq m, \\ &\dim \mathcal{O}'_k = d'k \dim_{\mathbb{R}}(\mathbb{D}) - 2 \dim_{\mathbb{R}} \mathcal{SH}_k(\mathbb{D}). \end{aligned} \tag{101}$$

Explicitly

$$\dim \mathcal{O}'_k = \begin{cases} kd' - k(k-1) & \text{if } \; \mathbb{D} = \mathbb{R}, \\ 2kd' - 2k^2 & \text{if } \; \mathbb{D} = \mathbb{C}, \\ 4kd' - 2k(2k+1) & \text{if } \; \mathbb{D} = \mathbb{H}. \end{cases} \tag{102}$$

Suppose $\mathbb{D} = \mathbb{R}$ *or* $\mathbb{C}$. *Then the partition of* d' *corresponding to the complexification* $\mathcal{O}'_{k\mathbb{C}} = \mathrm{G}'_{\mathbb{C}}\mathcal{O}'_k$ *of the orbit* $\mathcal{O}'_k$ *is* $\lambda' = (2^k, 1^{d'-2k})$. *In other words, the Young diagram corresponding to the orbit* $\mathcal{O}'_{k\mathbb{C}}$ *has* k *rows of length* 2 *and* $d' - 2k$ *rows of length* 1. *If* $\mathbb{D} = \mathbb{H}$, *then* $\mathcal{O}'_{k\mathbb{C}}$ *corresponds to the partition* $\lambda' = (2^{2k}, 1^{2d'-4k})$ *of* $2d'$.

The equality

$$\dim \mathcal{O}'_m = \dim \mathrm{W} - 2 \dim \mathfrak{g}, \tag{103}$$

holds if and only if either the dual pair $(\mathrm{G}, \mathrm{G}')$ *is in the stable range with* G *- the smaller member or if* $(\mathrm{G}, \mathrm{G}')$ *is one of the following pairs*

$$(\mathrm{O}_{m+1}, \mathrm{Sp}_{2m}(\mathbb{R})), \quad (\mathrm{U}_{d'-m}, \mathrm{U}_{m,d'-m}) \quad \textit{with} \quad 2m < d'.$$

Proof. As is well known, the variety $\tau'(\tau^{-1}(0))$ is the closure of a single G' orbit $\mathcal{O}'_m$, see for example [Prz91, Lemma (2.16)].

Let $w \in \mathfrak{s}_{\bar{1}}$ be a nilpotent element. Then

$$\mathsf{V} = \mathsf{V}^{(0)} \oplus \mathsf{V}^{(1)} \oplus \cdots \oplus \mathsf{V}^{(k)},$$

where $\mathsf{V}^{(0)}$ is the kernel of w and each $(w, \mathsf{V}^{(j)})$, $1 \leq j \leq k$ is indecomposable (see [DKP05, Def. 3.14]). (If $w = 0$ then $\mathsf{V} = \mathsf{V}^{(0)}$.) The orbit of w, call it $\mathcal{O}_k$, is of maximal dimension if the kernel of w is minimal, which happens if and only if $k = m$. Since the only nilpotent element of $\mathfrak{g}$ is zero, we have

$$0 = \tau(w) = w^2|_{\mathsf{V}_{\bar{0}}}.$$

Fix $j \geq 1$. Then $(w, \mathsf{V}^{(j)})$ is non-zero and indecomposable. The structure of such elements is well known. In particular we see from [DKP05, Prop.5.2(e)] that there are vectors $v_1, v_3 \in \mathsf{V}_{\bar{1}}$ and $v_2 \in \mathsf{V}_{\bar{0}}$ such that

$$\mathsf{V}^{(j)} = \mathbb{D}v_2 \oplus (\mathbb{D}v_1 \oplus \mathbb{D}v_3), \quad wv_1 = v_2, \; wv_2 = v_3, \; wv_3 = 0.$$

Hence,

$$\mathsf{V}^{(j)}_{\bar{1}} = \mathbb{D}v_1 \oplus \mathbb{D}v_3, \quad w^2 v_1 = v_3, \; w^2 v_3 = 0$$

and the decomposition

$$\mathsf{V}_{\bar{1}} = \mathsf{V}^{(0)}_{\bar{1}} \oplus \mathsf{V}^{(1)}_{\bar{1}} \oplus \cdots \oplus \mathsf{V}^{(k)}_{\bar{1}},$$

determines the G'-orbit $\mathcal{O}'_k$ of $\tau'(w) = w^2|_{\mathsf{V}_{\bar{1}}}$.

The dual pair corresponding to $\mathrm{S}|_{\mathsf{V}^{(j)}}$ is $(\mathrm{O}_1, \mathrm{Sp}_2(\mathbb{R}))$, if $\mathbb{D} = \mathbb{R}$, $(\mathrm{U}_1, \mathrm{U}_{1,1})$, if $\mathbb{D} = \mathbb{C}$, $(\mathrm{Sp}_1, \mathrm{O}^*_4)$, if $\mathbb{D} = \mathbb{H}$. The complexifications of these pairs are $(\mathrm{O}_1, \mathrm{Sp}_2(\mathbb{C}))$, $(\mathrm{GL}_1(\mathbb{C}), \mathrm{GL}_2(\mathbb{C}))$ and $(\mathrm{Sp}_2(\mathbb{C}), \mathrm{O}_4(\mathbb{C}))$ respectively. In particular, this leads to the description of the complexification of the orbit in terms of the Young diagrams, as in [CM93].

The closure relations between the orbits $\mathcal{O}'_k$ are well known and their dimension may be computed using [CM93, Corollary 6.1.4] leading to (102). The dimension formula in (101) follows from (102).

The fact that (103) holds if the pair is in the stable range was checked in [Prz91, Lemma (2.19)]. The last statement follows from (102) via a direct computation. □

Now we construct a slice through an element of the maximal nilpotent orbit.

Recall the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ and the automorphism θ on $\mathfrak{s}$, [Prz06, sec. 2.1]. (The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{s}_{\bar{0}}$ is a Killing form and the restriction to $\mathfrak{s}_{\bar{1}}$ is a symplectic form. Also, the restriction of θ to $\mathfrak{s}_{\bar{0}}$ is a Cartan involution and the restriction of $-\theta$ to $\mathfrak{s}_{\bar{1}}$ is a positive definite compatible complex structure.) In particular the bilinear form $B(\cdot, \cdot) = -\langle \theta \cdot, \cdot \rangle$ is symmetric and positive definite.

Fix an element $N \in \mathfrak{s}_{\overline{1}}$. Then $N + [\mathfrak{s}_{\overline{0}}, N] \subseteq \mathfrak{s}_{\overline{1}}$ may be thought of as the tangent space at N to the S-orbit through N. Denote by $[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \subseteq \mathfrak{s}_{\overline{1}}$ the B-orthogonal complement of $[\mathfrak{s}_{\overline{0}}, N]$. Since the form B is positive definite,

$$\mathfrak{s}_{\overline{1}} = [\mathfrak{s}_{\overline{0}}, N] \oplus [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}. \tag{104}$$

Consider the map

$$\sigma : \mathrm{S} \times \left(N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}\right) \ni (s, u) \to su \in \mathfrak{s}_{\overline{1}}. \tag{105}$$

The range of the derivative of the map σ at (s, u) is equal to

$$[\mathfrak{s}_{\overline{0}}, su] + s[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = s\left([\mathfrak{s}_{\overline{0}}, u] + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}\right). \tag{106}$$

Let

$$U = \{u \in N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B};\ [\mathfrak{s}_{\overline{0}}, u] + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = \mathfrak{s}_{\overline{1}}\}. \tag{107}$$

The equality (104) implies that $N \in U$ and U is the slice we were looking for. Next, we consider the orbits passing through U. The maximal nilpotent orbit corresponds to a point N and the almost semisimple ones to others points in U which will approach N in a suitable sense, as explained below.

Notice that U is the maximal open neighborhood of N in $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ such that the map

$$\sigma : \mathrm{S} \times U \ni (s, u) \to su \in \mathfrak{s}_{\overline{1}} \tag{108}$$

is a submersion. Then $\sigma(\mathrm{S} \times U) \subseteq \mathfrak{s}_{\overline{1}}$ is an open S-invariant subset and

$$\sigma : \mathrm{S} \times U \ni (s, u) \to su \in \sigma(\mathrm{S} \times U) \tag{109}$$

is a surjective submersion.

We shall use the map (109) to study the S-orbital integrals in $\mathfrak{s}_{\overline{1}}$. This is parallel to Ranga Rao's unpublished study of the orbital integrals in a semisimple Lie algebra, [BV80].

From now on we assume that N is nilpotent.

Lemma 19. *The map*

$$N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \ni u \to u^2 \in \mathfrak{s}_{\overline{0}} \tag{110}$$

is proper.

Proof. We proceed in terms of matrices. Thus $\mathsf{V}_{\overline{0}} = \mathbb{D}^d$ is a left vector space over $\mathbb{D}$ via

$$av := v\overline{a} \qquad (v \in \mathsf{V}_{\overline{0}},\ a \in \mathbb{D}).$$

Then $\mathrm{End}_{\mathbb{D}}(\mathsf{V}_{\overline{0}})$ may be identified with the space of matrices $M_d(\mathbb{D})$ acting on $\mathbb{D}^d$ via left multiplication. Let

$$(v, v') = \overline{v}^t v' \qquad (v, v' \in \mathbb{D}^d).$$

This is a positive definite hermitian form on $\mathbb{D}^d$. The isometry group of this form is

$$\mathrm{G} = \{g \in M_d(\mathbb{D});\ \overline{g}^t g = I_d\}.$$

Similarly, $\mathsf{V}_{\overline{1}} = \mathbb{D}^{d'}$ is a left vector space over $\mathbb{D}$ and

$$\mathrm{G}' = \{g \in M_{d'}(\mathbb{D});\ \overline{g}^t F g = F\},$$

where $F = -\overline{F}^t \in \mathrm{GL}_{d'}(\mathbb{D})$. This is the isometry group of the form

$$(v, v')' = \overline{v}^t F v' \qquad (v, v' \in \mathbb{D}^{d'}).$$

Furthermore we have the identifications

$$\mathfrak{s}_{\overline{1}} = \mathrm{Hom}_{\mathbb{D}}(\mathsf{V}_{\overline{0}}, \mathsf{V}_{\overline{1}}) = M_{d',d}(\mathbb{D}),$$

with the symplectic form

$$\langle w', w\rangle = tr_{\mathbb{D}/\mathbb{R}}(w^* w') \qquad (w, w' \in M_{d',d}(\mathbb{D})),$$

where $w^* = \overline{w}^t F$. Also, $-\theta(w) = F^{-1}w$ so that

$$B(w', w) = tr_{\mathbb{D}/\mathbb{R}}(\overline{w}^t w').$$

Two elements $w, w' \in M_{d',d}(\mathbb{D})$ anticommute if and only if

$$ww'^* + w'w^* = 0 \ \text{ and } \ w^*w' + w'^*w = 0. \tag{111}$$

From now on we choose the matrix F as follows

$$F = \begin{pmatrix} 0 & 0 & I_k \\ 0 & F' & 0 \\ -I_k & 0 & 0 \end{pmatrix} \tag{112}$$

where $0 \leq k \leq m$, where m is the minimum of d and the Witt index of the form $(\ ,\ )'$, as in Lemma 18. Then, with the block decomposition of an element of $M_{d',d}(\mathbb{D}) = M_{d',k}(\mathbb{D}) \oplus M_{d',d-k}(\mathbb{D})$ dictated by (112),

$$\begin{pmatrix} w_1 & w_4 \\ w_2 & w_5 \\ w_3 & w_6 \end{pmatrix}^* = \begin{pmatrix} -\overline{w}_3^t & \overline{w}_2^t F' & \overline{w}_1^t \\ -\overline{w}_6^t & \overline{w}_5^t F' & \overline{w}_4^t \end{pmatrix}.$$

We may choose

$$N = N_k = \begin{pmatrix} I_k & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{113}$$

Notice that

$$[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = \theta\left([\mathfrak{s}_{\overline{0}}, N]^{\perp}\right) = \theta\left({}^N\mathfrak{s}_{\overline{1}}\right) = {}^{\theta N}\mathfrak{s}_{\overline{1}},$$

where the second equality is taken from [Prz06, Lemma 3.5]. Hence, a straightforward computation using (111) shows that

$$[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix};\ w_3 = -\overline{w}_3^t \right\}.$$

The image of w under the map (110) consists of pairs of matrices

$$\begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix}\begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix}^* = \begin{pmatrix} w_3 & 0 & I_k \\ -w_5\overline{w}_6^t & w_5\overline{w}_5^t F' & 0 \\ -w_3\overline{w}_3^t - w_6\overline{w}_6^t & w_6\overline{w}_5^t F' & w_3 \end{pmatrix} \tag{114}$$

and

$$\begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix}^* \begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} = \begin{pmatrix} 2w_3 & w_6 \\ -\overline{w}_6^t & \overline{w}_5^t F' w_5 \end{pmatrix}. \tag{115}$$

Hence the claim follows. □

Suppose $k = m$. Then it is easy to see from (113) and (A.1) that $N = N_m \in \mathrm{W}_{\mathfrak{g}}$, or equivalently $U \subseteq \mathrm{W}_{\mathfrak{g}}$, if and only if either the pair $(\mathrm{G}, \mathrm{G}')$ is in the stable range with G the smaller member or $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{l'+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$.

Corollary 20. *If $k = m$, then the map*

$$\tau : N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \ni w \to w^* w \in \mathfrak{g} \tag{116}$$

is proper.

Proof. This follows from the formula (115). Indeed, it is enough to see that the map

$$w_5 \to \overline{w}_5^t F' w_5$$

is proper. The variable w_5 does not exist unless $\mathbb{D} = \mathbb{C}$ and $d > m$. This means that m is the Witt index of the form $(\ ,\ )'$. Hence iF' is a definite hermitian matrix. Therefore the above map is proper. □

Corollary 21. *Suppose $k = m$. If $E \subseteq \mathfrak{s}_{\overline{1}}$ is a subset such that $\tau(E) \subseteq \mathfrak{g}$ is bounded, then*

$$E \cap \left(N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}\right)$$

is bounded.

Proof. This is immediate from Corollary 20. □

Lemma 22. *For each $k = 0, 1, 2, \ldots, m$, the orbital integral $\mu_{\mathcal{O}_k}$ is* S *- invariant and defines a tempered distribution on* W*, homogeneous of degree* $\deg \mu_{\mathcal{O}_k} = \dim \mathcal{O}_k' - \dim \mathrm{W}$.

Proof. The stabilizer of the image of N_k in $\mathsf{V}_{\overline{1}}$ is a parabolic subgroup $\mathrm{P}' \subseteq \mathrm{G}'$ with the Langlands decomposition $P' = \mathrm{GL}_k(\mathbb{D})\mathrm{G}''\mathrm{N}'$, where G'' is an isometry group of the same type as G' and N' is the unipotent radical. As a $\mathrm{GL}_k(\mathbb{D})$ - module, $\mathfrak{n}'$, the Lie algebra of N', is isomorphic to $M_{k,d'-2k}(\mathbb{D}) \oplus \mathcal{H}_k(\mathbb{D})$, where $\mathcal{H}_k(\mathbb{D}) \subseteq M_{k,k}(\mathbb{D})$ stands for the space of the hermitian matrices. Hence the absolute value of the determinant of the adjoint action of an element $a \in \mathrm{GL}_k(\mathbb{D})$ on the real vector space $\mathfrak{n}'$ is equal to

$$|\det \mathrm{Ad}(a)_{\mathfrak{n}'}| = |\det_{\mathbb{R}}(a)|^{d'-2k+\frac{2\dim \mathcal{H}_k(\mathbb{D})}{k \dim_{\mathbb{R}} \mathbb{D}}}$$

Since $\mathrm{G}' = \mathrm{K}'\mathrm{P}'$, where K' is a maximal compact subgroup, the Haar measure on G' may be written as

$$dg' = |\det \mathrm{Ad}(a)_{\mathfrak{n}'}|\, dk\, da\, dg''\, dn'.$$

Recall that $da = |\det_{\mathbb{R}}(a)|^{-k}\, d^+a$, where d^+a stands for the Lebesgue measure on the real vector space $M_{k,k}(\mathbb{D})$. Also,

$$\frac{2\dim\mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}} - k = \begin{cases} 1 & \text{if } \mathbb{D}=\mathbb{R},\\ 0 & \text{if } \mathbb{D}=\mathbb{C},\\ -\frac{1}{2} & \text{if } \mathbb{D}=\mathbb{H}. \end{cases}$$

Hence,

$$|\det \mathrm{Ad}(a)_{\mathfrak{n}'}||\det_{\mathbb{R}}(a)|^{-k} = |\det_{\mathbb{R}}(a)|^{d'-2k+\frac{2\dim\mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}}-k}$$

is locally integrable on the real vector space $M_{k,k}(\mathbb{D})$.

Since the stabilizer of N_k in G' is equal to $\mathrm{G}''\mathrm{N}' \subseteq \mathrm{P}'$, the G' orbit of N_k defines a tempered distribution on W by

$$\int_{\mathrm{W}} \phi(w)\, d\mu_{\mathrm{G}'N_k}(w) = \int_{\mathrm{GL}_k(\mathbb{D})}\int_{\mathrm{K}'} \phi(kaN_k)\det\mathrm{Ad}(a)_{\mathfrak{n}'}\, dk\, da \qquad (\phi\in\mathcal{S}(\mathrm{W}).$$

This distribution is homogeneous of degree

$$(d'-2k+\frac{2\dim\mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}})dim_{\mathbb{R}}\mathbb{D} - \dim\mathrm{W}$$

Thus it remains to check that

$$(d'-2k+\frac{2\dim\mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}})\, k\dim_{\mathbb{R}}\mathbb{D} = d'k\dim_{\mathbb{R}}(\mathbb{D}) - 2\dim_{\mathbb{R}}\mathcal{SH}_k(\mathbb{D}),$$

which is easy, because $M_{k,k}(\mathbb{D}) = \mathcal{H}_k(\mathbb{D})\oplus\mathcal{SH}_k(\mathbb{D})$. In order to conclude the proof we notice that the orbital integral on the orbit $\mathrm{S}N_k$ is the G - average of the orbital integral we just considered:

$$\int_{\mathrm{W}} \phi(w)\, d\mu_{\mathrm{S}N_k}(w) = \int_{\mathrm{G}}\int_{\mathrm{GL}_k(\mathbb{D})}\int_{\mathrm{K}'} \phi(kaN_kg)\det\mathrm{Ad}(a)_{\mathfrak{n}'}\, dk\, da\, dg.$$

□

Now we want to see dilations by $t>0$ in $\mathfrak{s}_{\bar{1}}$ as transformations in the slice U modulo the action of the group S, which is permissible as we consider S-orbits.

For $t>0$ let

$$s_t = \begin{pmatrix} t^{-1} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & t \end{pmatrix}$$

where the blocks are as in (112). Then $s_t\in\mathrm{G}'$. We view s_t as an element $s_t\in\mathrm{GL}(\mathfrak{s}_{\bar{1}})$ by

$$s_t(w) = s_tw \qquad (w\in\mathfrak{s}_{\bar{1}}).$$

Also, define an element $M_t\in\mathrm{GL}(\mathfrak{s}_{\bar{1}})$ by

$$M_t(w) = tw \qquad (w\in\mathfrak{s}_{\bar{1}}).$$

Set $g_t = M_t\circ s_t\in\mathrm{GL}(\mathfrak{s}_{\bar{1}})$. Thus

$$g_t(w) = ts_tw \qquad (w\in\mathfrak{s}_{\bar{1}}).$$

Lemma 23. *The linear map $g_t \in \mathrm{GL}(\mathfrak{s}_{\overline{1}})$ preserves the set $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ and the subset $U \subseteq N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$. In fact*

$$g_t \begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & tw_5 \\ t^2 w_3 & t^2 w_6 \end{pmatrix}. \tag{117}$$

Hence,

$$\tau|_U \circ g_t|_U = M_{t^2} \circ \tau|_U. \tag{118}$$

Furthermore

$$g_t \circ \sigma = \sigma \circ (\mathrm{Ad}\ s_t \times g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}), \tag{119}$$

where the $g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}$ on the right hand side stands for the restriction of g_t to $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$. In particular, the subset $\sigma(\mathrm{S} \times U) \subseteq \mathfrak{s}_{\overline{1}}$ is closed under the multiplication by the positive reals. Also,

$$\det((g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}})') = t^{\dim \mathfrak{s}_{\overline{1}} - \dim \mathcal{O}'_k}. \tag{120}$$

and

$$\det(g'_t) = t^{\dim \mathfrak{s}_{\overline{1}}} \tag{121}$$

Proof. The formula (117) is clear from the definition of g_t and (118) follows from (117) and (115).

In order to verify (119) we notice that for $s \in \mathrm{S}$ and $u \in N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ we have

$$\begin{aligned} g_t \circ \sigma(s,u) &= g_t(su) = t(s_t s)u = (s_t s s_t^{-1})(t s_t u) \\ &= \sigma(s_t s s_t^{-1}, g_t u) = \sigma \circ (\mathrm{Ad}\ s_t \times g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}})(s,u). \end{aligned}$$

By the Chain Rule, the derivative of σ at (s,u) is surjective if and only if the derivative of $g_t \circ \sigma$ at (s,u) is surjective. Then (117) shows that this happens if and only if the derivative of σ at $(s_t s s_t^{-1}, g_t u)$ is surjective. By (106), the last statement is equivalent to the derivative of σ being surjective at $(s, g_t u)$. In other words, g_t preserves U.

Since $g'_t = M'_t = M_t$ and since $\det s_t = 1$, (121) is obvious. In order to verify (120) we proceed as follows. The derivative of the map $g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}$ coincides with the following linear map

$$\begin{pmatrix} 0 & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & tw_5 \\ t^2 w_3 & t^2 w_6 \end{pmatrix}.$$

The determinant of this map is equal to the determinant of the following map

$$\begin{pmatrix} 0 & w_4 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} \to \begin{pmatrix} 0 & tw_4 \\ 0 & tw_5 \\ t^2 w_3 & tw_6 \end{pmatrix},$$

which equals

$$t^{2 \dim_{\mathbb{R}} \mathcal{SH}_k(\mathbb{D})} t^{d'(d-k) \dim_{\mathbb{R}} \mathbb{D}}.$$

Since, by (101),

$$2\dim_{\mathbb{R}}\mathcal{SH}_k(\mathbb{D})+d'(d-k)\dim_{\mathbb{R}}\mathbb{D}=\dim\mathfrak{s}_{\overline{1}}-\dim\mathcal{O}'_k,$$

(120) follows. □

Next we consider an S-invariant distribution F on $\sigma(\mathrm{S}\times U)$ and its restriction $F|_U$ to the slice U. The following lemma proves that the restriction to U of the t-dilation of F is equal to $g_t|_U$ applied to $F|_U$.

Lemma 24. *Suppose $F\in\mathcal{D}'(\sigma(\mathrm{S}\times U))^{\mathrm{S}}$. Then the intersection of the wave front set of F with the conormal bundle to U is zero, so that the restriction $F|_U$ is well defined. Furthermore, $\sigma^*F=\mu_{\mathrm{S}}\otimes F|_U$. Moreover, for $t>0$,*

$$M_t^*F=g_t^*F, \tag{122}$$

and

$$(M_t^*F)|_U=(g_t|_U)^*F|_U. \tag{123}$$

Proof. The wave front set of F is contained in the union of the conormal bundles to the S-orbits through elements of $\mathfrak{s}_{\overline{1}}$. This is because the characteristic variety of the system of differential equations expressing the condition that this distribution is annihilated by the action of the Lie algebra $\mathfrak{s}_{\overline{0}}$ coincides with that set. The intersection of this set with the conormal bundle to U is zero. Indeed, this intersection is equal to the orthogonal complement to the sum of the union of the tangent bundles to the orbits and the tangent bundle to U, which, by the submersivity of the map σ, is equal to the whole tangent bundle. Hence, F restricts uniquely to U. The formula $\sigma^*F=\mu_{\mathrm{S}}\otimes F|_U$ follows from the diagram

$$U\to\mathrm{S}\times U\to\sigma(\mathrm{S}\times U),\quad u\to(1,u)\to u,$$

which shows that the restriction to U equals the composition of σ^* and the pullback via the embedding of U into $\mathrm{S}\times U$.

Since $s_t^*F=F$ we see that $g_t^*F=M_t^*s_t^*F=M_t^*F$. Hence,

$$(g_t|_U)^*F|_U=(g_t^*F)|_U=(M_t^*F)|_U.$$

Thus we are done with (122) and (123). □

Lemma 25. *Suppose $F,F_0\in\mathcal{D}'(\sigma(\mathrm{S}\times U))^{\mathrm{S}}$ and $a\in\mathbb{C}$ are such that*

$$t^a(g_{t^{-1}}|_U)^*F|_U\underset{t\to0+}{\to}F_0|_U.$$

Then

$$t^aM_{t^{-1}}^*F\underset{t\to0+}{\to}F_0$$

in $\mathcal{D}'(\sigma(\mathrm{S}\times U))$.

Proof. This is immediate from (123) and Proposition B.1. □

Now we are ready to compute the limit of the weighted dilatation of the unnormalized almost semisimple orbital integral $\mu_{\mathcal{O}}$.

Proposition 26. *Let $k = m$. Let $\mathcal{O} \subseteq \sigma(\mathrm{S} \times U)$ be an* S*-orbit and let $\mu_{\mathcal{O}} \in \mathcal{D}'(\mathfrak{s}_{\overline{1}})$ be the corresponding orbital integral. Assume $\mu_{\mathcal{O}}$ is* S *- invariant. Then*

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_m}} \mathrm{M}^*_{t^{-1}} \mu_{\mathcal{O}}|_{\sigma(\mathrm{S}\times U)} = \mu_{\mathcal{O}}|_U(U)\, \mu_{\mathcal{O}_m}|_{\sigma(\mathrm{S}\times U)}, \tag{124}$$

where $\mu_{\mathcal{O}_m} \in \mathcal{D}'(\sigma(\mathrm{S} \times U))$ is the orbital integral on the orbit $\mathcal{O}_m = \mathrm{S}N_m$ normalized so that $\mu_{\mathcal{O}_m}|_U$ is the Dirac delta at N_m and the convergence is in $\mathcal{D}'(\sigma(\mathrm{S} \times U))$. This orbital integral is a homogeneous distribution of degree $\deg \mu_{\mathcal{O}_m} = \dim \mathcal{O}'_m - \dim \mathrm{W}$.

A few remarks before the proof. The scalar $\mu_{\mathcal{O}}|_U(U)$ may be thought of as the volume of the intersection $\mathcal{O} \cap U$. This volume is finite because the restriction $\mu_{\mathcal{O}}|_U$ is a distribution on U with the support equal to the closure of $\mathcal{O} \cap U$, which is compact by Corollary 21. Hence $\mu_{\mathcal{O}}|_U$ applies to any smooth function on U, which may be chosen to be constant on $\mathcal{O} \cap U$.

A straightforward argument shows that every regular orbit $\mathcal{O}$ passes through U, i.e. is contained in $\sigma(\mathrm{S} \times U)$, if the pair $(\mathrm{G}, \mathrm{G}')$ is in the stable range.

Our normalization of $\mu_{\mathrm{S}N_m}$ does not depend on the normalization of $\mu_{\mathcal{O}}$, which is absorbed by the factor $\mu_{\mathcal{O}}|_U(U)$.

Proof. By (120)

$$t^{\dim \mathcal{O}'_m - \dim \mathfrak{s}_{\overline{1}}} (g_{t^{-1}}|_U)^* \mu_{\mathcal{O}}|_U(\psi) = \mu_{\mathcal{O}}|_U(\psi \circ g_t|_U).$$

We see from (117) that

$$\lim_{t\to 0} g_t u = N_m \qquad (u \in U).$$

Hence, for any $\psi \in C_c^\infty(U)$,

$$\lim_{t\to 0} \mu_{\mathcal{O}}|_U(\psi \circ g_t) = \mu_{\mathcal{O}}|_U(\psi(N)\mathbb{I}_U) = \mu_{\mathcal{O}}|_U(\mathbb{I}_U)\psi(N) = \mu_{\mathcal{O}}|_U(U)\psi(N_m),$$

where $\mathbb{I}_U$ is the indicator function of U. Thus (124) follows from Lemma 25. □

In order to find the limit of the weighted dilations of the normalized almost semisimple orbital integral $f(y)$, see Corollary 29, we still need to compute the weight.

A direct computation involving the formulas (58) and (61) verifies the following lemma.

Lemma 27. *Suppose $d \leq d'$. Then*

$$\dim \mathrm{W} = \dim \mathfrak{g} + \dim \mathfrak{g}'/\mathfrak{z}' + \dim \mathfrak{h} + \dim \mathfrak{s}_{\overline{1}}(\mathsf{V}^0), \tag{125}$$

or equivalently

$$\dim \mathrm{W} = 2 \deg \pi_{\mathfrak{g}/\mathfrak{h}} + 2 \deg \pi_{\mathfrak{g}'/\mathfrak{z}'} + 2 \dim \mathfrak{h} + \dim \mathfrak{s}_{\overline{1}}(\mathsf{V}^0). \tag{126}$$

Here $\dim \mathfrak{s}_{\overline{1}}(\mathsf{V}^0) = 0$ *unless* $\mathrm{G} = \mathrm{O}_{2l+1}$, $\mathrm{G}' = \mathrm{Sp}_{2l'}(\mathbb{R})$ *and* $d = 2l + 1 < 2l' = d'$.

If $d < d'$, then $\dim \mathfrak{s}_{\overline{1}}(\mathsf{V}^0) = 0$ *and*

$$\dim \mathrm{W} = \dim \mathfrak{g}' + \dim \mathfrak{g}/\mathfrak{z} + \dim \mathfrak{h}, \tag{127}$$

or equivalently

$$\dim \mathrm{W} = 2 \deg \pi_{\mathfrak{g}'/\mathfrak{h}} + 2 \deg \pi_{\mathfrak{g}/\mathfrak{z}} + 2 \dim \mathfrak{h}. \tag{128}$$

Recall Harish-Chandra's semisimple orbital integral on $f(y) \in \mathcal{S}^*(\mathrm{W})^{\mathrm{S}}$, (95) and (100). Lemma 27 plus a direct computation implies the following lemma.

Lemma 28. *For $\phi \in C_c^\infty(\mathrm{W})$ and $t > 0$*

$$\mu_{\mathcal{O}(w)}(t^{\dim \mathrm{W}}\phi_t) = t^{\dim \mathfrak{s}_{\overline{1}}(\mathrm{V}^0)}\mu_{\mathcal{O}(t^{-1}w)}(\phi) \qquad (w \in \mathfrak{h}_{\overline{1}}^{reg}). \tag{129}$$

Equivalently,

$$M_{t^{-1}}^* f(y) = t^{2\,\deg\,\pi_{\mathfrak{g}/\mathfrak{z}}+2\dim\mathfrak{h}} f(t^2 y). \tag{130}$$

Also, without any assumptions, we have the following equivalent formulas

$$\begin{aligned}(\psi\circ\tau')_t &= t^{2\,\dim\,\mathfrak{g}'-\dim\,\mathrm{W}}\psi_{t^2}\circ\tau' \qquad (\psi\in\mathcal{S}(\mathfrak{g}')),\\ \tau'_*(M_{t^{-1}}^* u) &= t^{\dim \mathrm{W}-2\dim\mathfrak{g}'}M_{t^{-2}}^*\tau'_*(u) \qquad (u \in \mathcal{S}^*(\mathrm{W})).\end{aligned} \tag{131}$$

Corollary 29. *Assume that $k = m$. Then,*

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* f(y)|_{\sigma(\mathrm{S}\times U)} = f(y)|_U(U)\,\mu_{\mathcal{O}_m}(\phi)|_{\sigma(\mathrm{S}\times U)}. \tag{132}$$

The distribution $f(0)$ is homogeneous of degree $\deg f(0) = -(2\deg\pi_{\mathfrak{g}/\mathfrak{z}} + 2\dim\mathfrak{h})$. *(If $l \le l'$, then $2\deg\pi_{\mathfrak{g}/\mathfrak{z}} + 2\dim\mathfrak{h} = \dim\mathfrak{g} + \dim\mathfrak{h}$.) Moreover (132) is equivalent to*

$$\lim_{t\to 0+} t^{\frac{1}{2}(\deg\mu_{\mathcal{O}_m}-\deg f(0))} f(ty)|_{\sigma(\mathrm{S}\times U)} = f(y)|_U(U)\,\mu_{\mathcal{O}_m}|_{\sigma(\mathrm{S}\times U)}. \tag{133}$$

Proof. The statement (132) is immediate from Proposition 26. In order to see that (132) is equivalent to (133) we recall (130), which also shows that $\deg f(0) = -2\deg\pi_{\mathfrak{g}/\mathfrak{z}} - 2\dim\mathfrak{h}$. □

The set of almost semisimple elements in $\mathfrak{s}_{\overline{1}}$ coincides with the union of the S-orbits through the generalized Cartan subspaces $\bigcup_{\tilde{\mathfrak{h}}_{\overline{1}}} \mathrm{S}\tilde{\mathfrak{h}}_{\overline{1}}$, [MPP15, (47)]. Since the set of the regular almost semisimple elements is dense in $\bigcup_{\tilde{\mathfrak{h}}_{\overline{1}}} \mathrm{S}\tilde{\mathfrak{h}}_{\overline{1}}$, [MPP15, Theorem 19] implies that the set of the regular almost semisimple elements is dense in $\mathfrak{s}_{\overline{1}}$. Hence there is a regular y such that the corresponding orbit in W is contained in $\sigma(\mathrm{S}\times U)$. Then all the orbits corresponding to ty, with $t \ge 0$, are contained in $\sigma(\mathrm{S}\times U)$.

Next we shall try to shed some light at the limits of the derivatives of the orbital integrals. We assume that $l \le l'$. As in [Har57a] we identify the symmetric algebra on $\mathfrak{g}$ with $\mathbb{C}[\mathfrak{g}]$, the algebra of the polynomials on $\mathfrak{g}$ using the invariant symmetric bilinear form B on $\mathfrak{g}$.

Lemma 30. *Let $y \in \mathfrak{h}\cap\tau(\mathrm{W})$ and let $Q \in \mathbb{C}[\mathfrak{h}]$ be such that* $\deg(Q)$ *is small enough so that, by Corollary 17, $\partial(Q)f(y)$ exists. Then*

$$t^{\deg\mu_{\mathcal{O}_m}} M_{t^{-1}}^*\partial(Q)f(y)|_{\sigma(\mathrm{S}\times U)} \underset{t\to 0+}{\to} C\mu_{\mathcal{O}_m}, \tag{134}$$

in $\mathcal{D}'(\sigma(\mathrm{S}\times U))$, where $C = \partial(Q)f(y)|_U(\mathbb{I}_U)$ is the value of the compactly supported distribution $\partial(Q)f(y)|_U$ on U applied to the constant function $\mathbb{I}_U$.

Proof. We see from Lemma 24 that it suffices to prove the lemma with (134) replaced by

$$t^{\deg \mu_{\mathcal{O}_m}} g_{t^{-1}}|_U^* \partial(Q) f(y)|_U \underset{t\to 0+}{\to} C\delta_{N_m}, \tag{135}$$

Let $\psi \in C_c^\infty(U)$. Since $\partial(Q)f(y)|_U$ is a compactly supported distribution on U,

$$\begin{aligned} t^{\deg \mu_{\mathcal{O}_m}} g_{t^{-1}}|_U^* \partial(Q) f(y)|_U(\psi) &= \partial(Q)f(y)|_U(\psi\circ g_t) \\ &\underset{t\to 0+}{\to} \partial(Q)f(y)|_U(\psi(N_m)\mathbb{I}_U) \\ &= \partial(Q)f(y)|_U(\mathbb{I}_U)\delta_{N_m}(\psi). \end{aligned}$$

□

Proposition 31. *Let $y \in \mathfrak{h}\cap\tau(\mathrm{W})$ and let $Q\in\mathbb{C}[\mathfrak{h}]$ be such that $\deg(Q)$ is small enough so that, by Corollary 17, $\partial(Q)f(y)$ exists. Then*

$$t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* \partial(Q) f(y) \underset{t\to 0+}{\to} C\mu_{\mathcal{O}_m} \tag{136}$$

in the topology of $\mathcal{S}^*(\mathrm{W})$, where $C = \partial(Q)f(y)|_U(\mathbb{I}_U)$. Moreover, there is a seminorm q on $\mathcal{S}(\mathrm{W})$ and $N\geq 0$ such that

$$\left|t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* \partial(Q) f(y)(\phi)\right| \leq (1+|y|)^N q(\phi)$$
$$(0<t\leq 1,\ y\in\mathfrak{h}\cap\tau(\mathrm{W}),\ \phi\in\mathcal{S}(\mathrm{W})). \tag{137}$$

Proof. As we have seen in (87) and (85), there is a positive constant *const* such that for $\psi\in\mathcal{S}(\mathfrak{g}')$

$$\begin{aligned} &\tau'_*(\partial(Q)f(y))(\psi) = \partial(Q)\tau'_*(f(y))(\psi) \\ &= \ const\ \partial(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\psi(g.(y+y''))\,dg\right)|_{y''=0}, \end{aligned} \tag{138}$$

where $\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'} = \pi^{short}_{\mathfrak{z}'/\mathfrak{h}'}$ if $\mathrm{G}=\mathrm{O}_{2l+1}$ with $l<l'$, and $\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'} = \pi_{\mathfrak{z}'/\mathfrak{h}'}$ otherwise. Let $P\in\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'}$. Then

$$\begin{aligned} &\partial(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}(P\psi)(g.(y+y''))\,dg\right)|_{y''=0} \\ &= \ \partial(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})\left(P(y+y'')\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\psi(g.(y+y''))\,dg\right)|_{y''=0}. \end{aligned} \tag{139}$$

By commuting the operators of the multiplication by a polynomial with differentiation we may write

$$\partial(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})P(y+y'') = \sum_{|\alpha|\leq\deg(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})} P_\alpha(y+y'')\partial^\alpha,$$

where $\partial^\alpha = \prod_{j=1}^{l'}\partial(J'_j)^{\alpha_j}$. Hence, (139) is equal to

$$\sum_{|\alpha|\leq\deg(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})} P_\alpha(y)\partial^\alpha\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\psi(g.(y+y''))\,dg\right)|_{y''=0}. \tag{140}$$

We see from (138) - (140) that the range of the map

$$\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'} \ni P \to \tau'_*(\partial(Q)f(y)) \circ P \in \mathcal{S}^*(\mathfrak{g}') \tag{141}$$

is contained in the space spanned by the distributions

$$\partial^\alpha \left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'') \int_{\mathrm{G}'} \psi(g.(y+y''))\, dg\right)|_{y''=0} \qquad (|\alpha| \leq \deg(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'}).$$

In particular this range is finite dimensional. Therefore the distribution (138) is annihilated by an ideal of finite co-dimension in $\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'}$. Hence the Fourier transform

$$(\tau'_*(\partial(Q)f(y)))\hat{}\in \mathcal{S}^*(\mathfrak{g}') \tag{142}$$

is annihilated by an ideal of finite co-dimension in $\partial(\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'})$. Now Harish-Chandra Regularity Theorem [Har65, Theorem 1, page 11] implies that the distribution (142) is a locally integrable function whose restriction to the set of the regular semisimple elements has a known structure. Specifically, let $\mathfrak{h}'_1 \subseteq \mathfrak{g}'$ be a Cartan subalgebra and let π_1 be the product of the positive roots (with respect to some order of the roots). Then Harish-Chandra's formula for the radial component of a G'-invariant differential operator with constant coefficients on $\mathfrak{g}'$ together with [HC64, Lemma 19] show that the restriction

$$\pi_1\, (\tau'_*(\partial(Q)f(y)))\hat{}|_{\mathfrak{h}'^{reg}_1}$$

is annihilated by an ideal of finite co-dimension in $\partial(\mathbb{C}[\mathfrak{h}'_1])$. Hence, for any connected component $C(\mathfrak{h}'^{reg}_1) \subseteq \mathfrak{h}'^{reg}_1$ there is an exponential polynomial $\sum_j p_j e^{\lambda_j}$. such that

$$(\tau'_*(\partial(Q)f(y)))\hat{}|_{C(\mathfrak{h}'^{reg}_1)} = \frac{1}{\pi_1}\sum_j p_j e^{\lambda_j}. \tag{143}$$

Let

$$F(x) = \sum_j p_j(x) e^{\lambda_j(x)} \qquad (x \in C(\mathfrak{h}'^{reg}_1)).$$

This function extends analytically beyond the connected component and for any $k = 1, 2, 3, ...$ we have Taylor's formula, as in [Hör83],

$$F(x) = \sum_{|\alpha|<k} \partial^\alpha F(0)\frac{x^\alpha}{\alpha!} + k\int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \partial^\alpha F(tx)\, dt \frac{x^\alpha}{\alpha!}. \tag{144}$$

Since the distribution (142) is tempered, we see from Harish-Chandra's theory of the orbital integrals

$$\psi(x) = \pi_1(x) \int_{\mathrm{G}'/\mathrm{H}'} \psi(g.x)\, dg\mathrm{H}' \qquad (\psi \in \mathcal{S}(\mathfrak{g}')) \tag{145}$$

that the real parts of the λ_j are non-positive on the $C(\mathfrak{h}'^{reg}_1)$. Furthermore, they depend linearly on y and the p_j depend polynomially on the y. Therefore a straightforward argument shows that there is $N > 0$ such that

$$|\partial^\alpha F(tx)| \leq constant\,(1+|y|)^N (1+|x|)^N \sum_{|\alpha|=k} \left|\frac{x^\alpha}{\alpha!}\right|. \tag{146}$$

Therefore (142) is a finite sum of homogeneous distributions, of possibly negative degrees, plus the error term which is bounded by (146). Therefore there is an integer a such that the following limit exists in $\mathcal{S}^*(\mathfrak{g}')$:

$$\lim_{t\to 0+} t^a M_t^* \left(\tau'_*(\partial(Q)f(y))\right)\hat{}. \tag{147}$$

Moreover, there is a seminorm q on $\mathcal{S}(\mathfrak{g}')$ and $N \geq 0$ such that

$$|t^a M_t^* \left(\tau'_*(\partial(Q)f(y))\right)\hat{}(\psi)| \leq (1+|y|)^N q(\psi)$$
$$(0 < t \leq 1,\ y \in \mathfrak{h} \cap \tau(\mathrm{W}),\ \psi \in \mathcal{S}(\mathfrak{g}')). \tag{148}$$

(All we did here was an elaboration of the argument used in the proof of [BV80, Theorem 3.2].) By taking the inverse Fourier transform we see that there is an integer b such that the following limit exists in $\mathcal{S}^*(\mathfrak{g}')$:

$$\lim_{t\to 0+} t^b M_{t^{-1}}^* \tau'_*(\partial(Q)f(y)). \tag{149}$$

Moreover, there is a seminorm q on $\mathcal{S}(\mathfrak{g}')$ and $N \geq 0$ such that

$$\left|\lim_{t\to 0+} t^b M_{t^{-1}}^* \tau'_*(\partial(Q)f(y))(\psi)\right| \leq (1+|y|)^N q(\psi)$$
$$(0 < t \leq 1,\ y \in \mathfrak{h} \cap \tau(\mathrm{W}),\ \psi \in \mathcal{S}(\mathfrak{g}')). \tag{150}$$

But then the injectivity of the map τ'_*, see Corollary 5, and (131) imply that there is an integer d such that the following limit exists in $\mathcal{S}^*(\mathrm{W})$.

$$\lim_{t\to 0+} t^d M_{t^{-1}}^* \partial(Q) f(y). \tag{151}$$

Now Lemma 30 shows that $d = \deg \mu_{\mathcal{O}_m}$ and the proposition follows. □

6. **Intertwining distributions.**

In the following we consider an irreducible unitary representation Π of $\widetilde{\mathrm{G}}$. We suppose that Π is genuine in the sense that it is non-trivial on the kernel of the covering map $\widetilde{\mathrm{G}} \to \mathrm{G}$. Let $\mu \in i\mathfrak{h}^*$ represent the infinitesimal character of Π. In particular, when μ is dominant, then we will refer to it as the Harish-Chandra parameter of Π. This is consistent with the usual terminology; see e.g. [Kna86, Theorem 9.20].

Assume that the distribution character Θ_Π is supported in the preimage $\widetilde{\mathrm{G}_1}$ of the Zariski identity component G_1 of G. (Recall that $\mathrm{G}_1 = \mathrm{G}$ unless G is an even orthogonal group.) Then

$$\Theta_\Pi(h)\Delta(h) = \sum_{s\in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\xi_{s\mu}(h) \qquad (h \in \mathrm{H}_o^\sharp), \tag{152}$$

where we lift Π from $\widetilde{\mathrm{G}}$ to $\mathrm{G}^\sharp$ via the covering (43) if necessary, Δ is the Weyl denominator (44) and $\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}$ is as in (76). Since $\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}} = \mathrm{sgn}$, the sign character of the Weyl group $W(\mathfrak{g},\mathfrak{h})$, unless $\mathrm{G} = \mathrm{O}_{2l}$, we need to justify the formula (152) only in this case. Our assumption that the distribution character Θ_Π is supported in the preimage $\widetilde{\mathrm{G}_1}$ implies that

$$\Theta_\Pi(h) = \frac{\sum_{s\in W(\mathfrak{g},\mathfrak{h})}\mathrm{sgn}(s)\xi_{s\mu}(h)}{\Delta(h)} + \frac{\sum_{s\in W(\mathfrak{g},\mathfrak{h})}\mathrm{sgn}(s)\xi_{s\mu}(th)}{\Delta(th)} \qquad (h\in \mathrm{H}_o^\sharp),$$

where $W(\mathfrak{g},\mathfrak{h})$ is the Weyl group of $\mathrm{G}_1 = \mathrm{SO}_{2l}$ and t is any element of $W(\mathrm{G},\mathfrak{h})$ which does not belong to $W(\mathfrak{g},\mathfrak{h})$. Since $\mathrm{sgn}(s) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)$ for $s\in W(\mathfrak{g},\mathfrak{h})$ and

$$\Delta(th) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\Delta(h),$$

(152) follows.

In this section we study the analytic properties of the distribution $f_{\Pi\otimes\Pi'} = T(\check{\Theta}_\Pi) \in \mathcal{S}(\mathrm{W})$ introduced in (13). For $x\in\mathfrak{g}$ define

$$\mathrm{ch}(x) = |\det_{\mathbb{R}}(x-1)|^{1/2}\,, \tag{153}$$

where the subscript $\mathbb{R}$ indicates that the element $x\in\mathfrak{g}\subseteq \mathrm{End}(\mathsf{V}_{\overline{0}})$ is viewed as an endomorphism of $\mathsf{V}_{\overline{0}}$ over $\mathbb{R}$. Since all eigenvalues of $x\in\mathfrak{g}$ are purely imaginary, $x-1$ is invertible and the function $\mathrm{ch}(x)$ is non-zero on $\mathfrak{g}$ and we can raise it to any real power.

For an endomorphism x of W we set

$$\chi_x(w) = \chi\big(\frac{1}{4}\langle xw, w\rangle\big) \qquad w\in \mathrm{W}\,. \tag{154}$$

This definition coincides with (11) when $x = c(g)$ for $g\in \mathrm{Sp}$ and $g-1$ is invertible.

Recall the functions $\tilde{c}$ and $\tilde{c}_-$ on $\mathfrak{sp}$ to $\widetilde{\mathrm{Sp}}$ introduced in (18) and the constants r and ι defined by (96) and (98), respectively. Recall also that $d' = \dim_{\mathbb{D}}\mathsf{V}_{\overline{1}}$ and that $\widetilde{\mathrm{G}}$ is equipped with the Haar measure $d\widetilde{g}$ of total mass 1.

Lemma 32. *Let $c_-^\sharp : \mathfrak{h}\to \mathrm{H}_o^\sharp$ be a real analytic lift of $\tilde{c}_- : \mathfrak{h}\to\widetilde{\mathrm{H}}$, via the covering (43). For any $\phi\in\mathcal{S}(\mathrm{W})$*

$$\begin{aligned} T(\check{\Theta}_\Pi)(\phi) &= C\int_{\mathfrak{h}}\Big((\check{\Theta}_\Pi\Delta)(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\Big)\Big(\pi_{\mathfrak{g}/\mathfrak{h}}(x)\int_{\mathrm{W}}\chi_x(w)\phi^{\mathrm{G}}(w)\,dw\Big)\,dx \\ &= C'\int_{\mathfrak{h}}\Big(\xi_{-\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\Big)\Big(\pi_{\mathfrak{g}/\mathfrak{h}}(x)\int_{\mathrm{W}}\chi_x(w)\phi^{\mathrm{G}}(w)\,dw\Big)\,dx, \end{aligned}$$

where $\phi^{\mathrm{G}}(w) = \int_{\mathrm{G}}\phi(gw)\,dg$, C is a non-zero constant, $C' = C|W(\mathrm{G},\mathfrak{h})|$ and each consecutive integral is absolutely convergent.

Proof. By definition

$$T(\check{\Theta}_\Pi)(\phi) = \int_{\widetilde{\mathrm{G}}}\check{\Theta}_\Pi(\widetilde{g})T(\widetilde{g})(\phi)\,d\widetilde{g}, \tag{155}$$

where the integral is absolutely convergent because both, the character and the function $T(\widetilde{g})(\phi)$ are bounded (see for example [Prz93, Proposition 1.13]) and the group $\widetilde{\mathrm{G}}$ is compact.

Since the support of the character is contained in $\widetilde{\mathrm{G}}_1$ and since the image of the Cayley transform $c_- : \mathfrak{g}\to\mathrm{G}$ is contained and dense in G_1, we may integrate over $\mathfrak{g}$ rather than $\widetilde{\mathrm{G}}$

in (155). As checked in [Prz91, (3.11)], the Jacobian of $\tilde{c}_-(x) = \tilde{c}(x)\tilde{c}(0)^{-1}$ is a constant multiple of $\mathrm{ch}^{-2r}(x)$. Also, the element $\tilde{c}(0)$ is in the center of the metaplectic group. In particular, $\check{\Theta}_\Pi(\tilde{g}\tilde{c}(0)^{-1})$ is a constant multiple of $\check{\Theta}_\Pi(\tilde{g})$, where the constant has the absolute value 1. Thus

$$T(\check{\Theta}_\Pi)(\phi) = C_1 \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}_-(x)) T(\tilde{c}(x))(\phi)\, \mathrm{ch}^{-2r}(x)\, dx, \tag{156}$$

where C_1 is a non-zero constant and, by (12), $T(\tilde{c}(x)) = \Theta(\tilde{c}(x))\chi_x\mu_{\mathrm{W}}$. Since $c(g.x) = g.c(x)$, there is $s \in \widetilde{\mathrm{Sp}}$ in the preimage of $1 \in \mathrm{Sp}$ such that $s\tilde{g}\tilde{c}(x)\tilde{g}^{-1} = \tilde{c}(g.x)$. Since Π occurs in Weil representation, for every $\tilde{h} \in \widetilde{\mathrm{Sp}}$ we have $\check{\Theta}_\Pi(s\tilde{h})\Theta(s\tilde{h}) = \check{\Theta}_\Pi(\tilde{h})\Theta(\tilde{h})$. As $\check{\Theta}_\Pi$ and Θ are characters of $\widetilde{G}$, it follows that $\check{\Theta}_\Pi(\tilde{c}(g.x))\Theta(\tilde{c}(g.x)) = \check{\Theta}_\Pi(\tilde{c}(x))\Theta(\tilde{c}(x))$. Weyl integration formula on $\mathfrak{g}$ shows that

$$T(\check{\Theta}_\Pi)(\phi) = C_2 \int_{\mathfrak{h}} |\pi_{\mathfrak{g}/\mathfrak{h}}(x)|^2 \check{\Theta}_\Pi(\tilde{c}_-(x)) T(\tilde{c}(x))(\phi^{\mathrm{G}})\, \mathrm{ch}^{-2r}(x)\, dx, \tag{157}$$

where C_2 is a non-zero constant and ϕ^{G} is as in the statement of the Lemma. Recall [Prz93, Lemma 5.7] that $\pi_{\mathfrak{g}/\mathfrak{h}}(x)$ is a constant multiple of $\Delta(c_-^\sharp(x))\,\mathrm{ch}^{r-\iota}(x)$. Also $\overline{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}$ is a constant multiple of $\pi_{\mathfrak{g}/\mathfrak{h}}(x)$. Hence, if we set $\check{\Theta}_\Pi(c_-^\sharp(x)) = \check{\Theta}_\Pi(\tilde{c}_-(x))$, then

$$T(\check{\Theta}_\Pi)(\phi) = C_3 \int_{\mathfrak{h}} (\check{\Theta}_\Pi\Delta)(c_-^\sharp(x))\, \mathrm{ch}^{r-\iota}(x) \pi_{\mathfrak{g}/\mathfrak{h}}(x) T(\tilde{c}(x))(\phi^{\mathrm{G}})\, \mathrm{ch}^{-2r}(x)\, dx, \tag{158}$$

where C_3 is a non-zero constant. Observe that, by (6) and (12),

$$\Theta(\tilde{c}(x))^2 = i^{\dim \mathrm{W}} \det(J_{c(x)})_{\mathrm{W}}^{-1} = i^{\dim \mathrm{W}} \det(c(x)-1)_{\mathrm{W}}^{-1} = i^{\dim \mathrm{W}} \det\left(2^{-1}(x-1)\right)_{\mathrm{W}}.$$

Since the determinant is taken on W, (153) implies that $\Theta(\tilde{c}(x))$ is a constant multiple of $\mathrm{ch}^{d'}(x)$, where $d' = \dim_{\mathbb{D}} \mathsf{V}_{\bar{1}}$ as before. Hence, by (152),

$$\begin{aligned} T(\check{\Theta}_\Pi)(\phi) &= C_4 \int_{\mathfrak{h}} (\check{\Theta}_\Pi\Delta)(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw\, dx \qquad (159)\\ &= C_4 \sum_{s \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \int_{\mathfrak{h}} \left(\xi_{-s\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x)\right) \left(\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw\right) dx, \end{aligned}$$

where C_4 is a non-zero constant and each consecutive integral is absolutely convergent. Notice that for $s \in W(\mathrm{G}, \mathfrak{h})$

$$\begin{aligned} \int_{\mathrm{W}} \chi_{sx}(w)\phi^{\mathrm{G}}(w)\, dw &= \int_{\mathrm{W}} \chi_x(s^{-1}w)\phi^{\mathrm{G}}(w)\, dw = \int_{\mathrm{W}} \chi_x(s^{-1}w)\phi^{\mathrm{G}}(s^{-1}w)\, dw \\ &= \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw \end{aligned}$$

and that $\pi_{\mathfrak{g}/\mathfrak{h}}(sx) = \mathrm{sgn}(s)\,\pi_{\mathfrak{g}/\mathfrak{h}}(x)$. Therefore

$$\begin{aligned}
&\sum_{s\in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \int_{\mathfrak{h}} \left(\xi_{-s\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\right) \left(\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw\right) dx\\
=& \sum_{s\in W(\mathrm{G},\mathfrak{h})} \int_{\mathfrak{h}} \left(\xi_{-s\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\right) \left(\pi_{\mathfrak{g}/\mathfrak{h}}(sx) \int_{\mathrm{W}} \chi_{sx}(w)\phi^{\mathrm{G}}(w)\,dw\right) dx\\
=& \sum_{s\in W(\mathrm{G},\mathfrak{h})} \int_{\mathfrak{h}} \left(\xi_{-s\mu}(c_-^\sharp(s^{-1}x))\,\mathrm{ch}^{d'-r-\iota}(s^{-1}x)\right) \left(\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw\right) dx\\
=& \sum_{s\in W(\mathrm{G},\mathfrak{h})} \int_{\mathfrak{h}} \left(\xi_{-\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\right) \left(\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw\right) dx\\
=& |W(\mathrm{G},\mathfrak{h})| \int_{\mathfrak{h}} \left(\xi_{-\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\right) \left(\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw\right) dx.
\end{aligned}$$

We can verify the absolute convergence as follows. The boundedness of the function $T(\widetilde{g})(\phi)$, $\tilde{g} \in \widetilde{\mathrm{G}}$, means that there is a seminorm $q(\phi)$ on $\mathcal{S}(\mathfrak{g})$ such that

$$\left|\Theta(\tilde{c}(x)) \int_{\mathrm{W}} \chi_x(w)\phi(w)\,dw\right| \leq q(\phi) \qquad (x \in \mathfrak{g}). \tag{160}$$

Equivalently, replacing $q(\phi)$ by a constant multiple,

$$\left|\int_{\mathrm{W}} \chi_x(w)\phi(w)\,dw\right| \leq q(\phi)\,\mathrm{ch}^{-d'}(x) \qquad (x \in \mathfrak{g}). \tag{161}$$

(This is the van der Corput estimate, [Ste93, formula (23) on page 345].) Also, (99) and (164) below imply that

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(x)| \leq C_5\,\mathrm{ch}^{r-1}(x) \leq C_5\,\mathrm{ch}^{r-\iota}(x) \qquad (x \in \mathfrak{h}),$$

where C_5 is a constant. Hence

$$\left|\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi(w)\,dw\right| \leq q(\phi)\,\mathrm{ch}^{-d'+r-\iota}(x) \qquad (x \in \mathfrak{h}). \tag{162}$$

Therefore the integral over $\mathfrak{h}$ in (159) may be dominated by (i.e. is less or equal a constant times the following expression)

$$\int_{\mathfrak{h}} \mathrm{ch}^{d'-r-\iota}(x)\,\mathrm{ch}^{-d'+r-\iota}(x)\,dx = \int_{\mathfrak{h}} \mathrm{ch}^{-2\iota}(x)\,dx < \infty.$$

□

Let us fix the branch of the square root:

$$\mathbb{C} \setminus \mathbb{R}^- \ni z \to z^{\frac{1}{2}} \in \mathbb{C} \tag{163}$$

so that $z^{\frac{1}{2}} > 0$, if $z > 0$. Then for $y = \sum_{j=1}^{l} y_j J_j \in \mathfrak{h}$,

$$\mathrm{ch}(y) = \prod_{j=1}^{l} (1 + y_j^2)^{\frac{1}{2\iota}} = \prod_{j=1}^{l} (1 + iy_j)^{\frac{1}{2\iota}} (1 - iy_j)^{\frac{1}{2\iota}}. \tag{164}$$

The elements J_j, $1 \le j \le l$, form a basis of the real vector space $\mathfrak{h}$. Let J_j^*, $1 \le j \le l$, be the dual basis of the space $\mathfrak{h}^*$ and set

$$e_j = -iJ_j^*, \qquad 1 \le j \le l\,. \tag{165}$$

If $\mu \in i\mathfrak{h}^*$, then $\mu = \sum_{j=1}^{l} \mu_j e_j$ with $\mu_j \in \mathbb{R}$. We say that μ is strictly dominant if $\mu_1 > \mu_2 > \cdots > \mu_l$.

The action of $\mathrm{W}(\mathrm{G}, \mathfrak{h})$ on $\mathfrak{h}$ extends by duality to $i\mathfrak{h}^*$: if $\mu = \sum_{j=1}^{l} \mu_j e_j \in i\mathfrak{h}^*$ and $t = \epsilon\sigma \in \mathrm{W}(\mathrm{G}, \mathfrak{h})$ is as in (73), then

$$t\Big(\sum_{j=1}^{l} \mu_j e_j\Big) = \sum_{j=1}^{l} \epsilon_{\sigma^{-1}(j)} \mu_{\sigma^{-1}(j)} e_j\,. \tag{166}$$

Lemma 33. *Let $c_-^\sharp$ be as in Lemma 32. Then*

$$\xi_{-\mu}(c_-^\sharp(ty)) = \xi_{-t^{-1}\mu}(c_-^\sharp(y)) \qquad (t \in \mathrm{W}(\mathrm{G}, \mathfrak{h}\,, \mu \in i\mathfrak{h}^*\,, y \in \mathfrak{h})\,. \tag{167}$$

Moreover, let

$$\delta = \frac{1}{2\iota}(d' - r + \iota)\,. \tag{168}$$

Then, with the notation of Lemma 32 and (164),

$$\xi_{-\mu}(c_-^\sharp(y))\, \mathrm{ch}^{d'-r-\iota}(y) = \prod_{j=1}^{l} (1 + iy_j)^{-\mu_j+\delta-1} (1 - iy_j)^{\mu_j+\delta-1}, \tag{169}$$

where all the exponents are integers:

$$\pm\mu_j + \delta \in \mathbb{Z} \qquad (1 \le j \le l). \tag{170}$$

In particular, (169) is a rational function in the variables y_1, y_2, ..., y_l.

Proof. Since

$$\xi_{-\mu}(c_-^\sharp(y)) = \prod_{j=1}^{l} \left(\frac{1 + iy_j}{1 - iy_j}\right)^{-\mu_j} = \prod_{j=1}^{l} (1 + iy_j)^{-\mu_j} (1 - iy_j)^{\mu_j}\,,$$

(167) and (169) follow from (164).

Let $\lambda = \sum_{j=1}^{l} \lambda_j e_j$ be the highest weight of the representation Π and let $\rho = \sum_{j=1}^{l} \rho_j e_j$ be one half times the sum of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}_\mathbb{C}$. If μ is the Harish-Chandra parameter of Π, then $\mu = \lambda + \rho = \sum_{j=1}^{l} \mu_j e_j$. Hence, the statement (170) is equivalent to

$$\lambda_j + \rho_j + \frac{1}{2\iota}(d' - r + \iota) \in \mathbb{Z}\,, \tag{171}$$

which holds because of the assumption that Π is a genuine representation of $\widetilde{\mathrm{G}}$. Indeed, if $\mathrm{G}=\mathrm{O}_d$, then with the standard choice of the positive root system, $\rho_j=\frac{d}{2}-j$. Also, $\lambda_j\in\mathbb{Z}$, $\iota=1$, $r=d-1$. Hence, (171) follows. Similarly, if $\mathrm{G}=\mathrm{U}_d$, then $\rho_j=\frac{d+1}{2}-j$, $\lambda_j+\frac{d'}{2}\in\mathbb{Z}$, $\iota=1$, $r=d$, which implies (171). If $\mathrm{G}=\mathrm{Sp}_d$, then $\rho_j=d+1-j$, $\lambda_j\in\mathbb{Z}$, $\iota=\frac{1}{2}$, $r=d+\frac{1}{2}$, and (171) follows. □

In order to study the inner integral occurring in the formula for $T(\check{\Theta}_\Pi)$ in Lemma 32, we shall need the following lemma.

Lemma 34. *Fix an element $z\in\mathfrak{h}$. Let $\mathfrak{z}\subseteq\mathfrak{g}$ and $\mathrm{Z}\subseteq\mathrm{G}$ denote the centralizer of z. (Then Z is a real reductive group with the Lie algebra $\mathfrak{z}$.) Denote by $\mathfrak{c}$ the center of $\mathfrak{z}$ and by $\pi_{\mathfrak{g}/\mathfrak{z}}$ be the product of the positive roots for $(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ which do not vanish on z. Let $B(\cdot,\cdot)$ be any non-degenerate symmetric G-invariant real bilinear form on $\mathfrak{g}$. Then there is a constant $C_\mathfrak{z}$ such that for and $x\in\mathfrak{h}$ and $x'\in\mathfrak{c}$,*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x)\pi_{\mathfrak{g}/\mathfrak{z}}(x')\int_{\mathrm{G}} e^{iB(g.x,x')}\,dg=C_\mathfrak{z}\sum_{tW(\mathrm{Z},\mathfrak{h})\in W(\mathrm{G},\mathfrak{h})/W(\mathrm{Z},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\pi_{\mathfrak{z}/\mathfrak{h}}(t^{-1}x)e^{iB(x,tx')}.$$

(Here $\pi_{\mathfrak{z}/\mathfrak{h}}=1$ if $\mathfrak{z}=\mathfrak{h}$.)

Proof. The proof is a straightforward modification of the argument proving Harish-Chandra's formula for the Fourier transform of a regular semisimple orbit, [Har57a, Theorem 2, page 104]. A more general result was obtained in [DV90, Proposition 34, p. 49]. □

We shall fix the symplectic form $\langle\cdot,\cdot\rangle$ on W according to the Lie superalgebra structure introduced at the beginning of section 4 as follows

$$\langle w,w'\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}ww') \qquad (w,w'\in\mathrm{W}). \tag{172}$$

Then

$$\langle xw,w\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xw^2) \qquad (x\in\mathfrak{g}\oplus\mathfrak{g}', w\in\mathrm{W}). \tag{173}$$

(See [Prz06, (2.4')].) Let

$$B(x,y)=\frac{\pi}{2}\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xy)=\frac{\pi}{2}\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) \qquad (x,y\in\mathfrak{g}). \tag{174}$$

Then, using the expression (50) for the unnormalized moment map τ, we have

$$\chi_x(w)=e^{\frac{\pi i}{2}\langle xw,w\rangle}=e^{iB(x,\tau(w))} \qquad (x\in\mathfrak{g}, w\in\mathrm{W}). \tag{175}$$

Lemma 35. *Suppose $l\le l'$. Then, with the notation of Lemma 32,*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x)\int_{\mathrm{W}}\chi_x(w)\phi^{\mathrm{G}}(w)\,dw=C\int_{\mathfrak{h}\cap\tau(\mathrm{W})} e^{iB(x,y)}f_\phi(y)\,dy\,.$$

where C is a non-zero constant and $f_\phi(y)=f(y)(\phi)$ for the Harish-Chandra regular almost semisimple orbital integral $f(y)$ of Definition 10.

Proof. By Lemmas 6 and 8 and the Weyl integration formula (67) on W,

$$\int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\,dw = \sum_{\mathfrak{h}_{\overline{1}}} \int_{\tau(\mathfrak{h}_{\overline{1}}^+)} \pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau(w))C(\mathfrak{h}_{\overline{1}})\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}})\,d\tau(w).$$

Let us consider first the case (65)

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}}) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} (\chi_x\phi^{\mathrm{G}})(s.w)\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}).$$

Recall from (57) the identification $y = \tau(w) = \tau'(w)$ and let us write $s = gg'$, where $g \in \mathrm{G}$ and $g' \in \mathrm{G}'$. Then

$$\chi_x(s.w) = e^{i\frac{\pi}{2}\langle x(s.w), s.w\rangle} = e^{iB(x,\tau(s.w))} = e^{iB(x,g.\tau(w))} = e^{iB(x,g.y)}$$

and

$$\phi^{\mathrm{G}}(s.w) = \phi^{\mathrm{G}}(g'.w).$$

Since

$$(\{1\}\times\mathrm{G}')\cap\mathrm{S}^{\mathfrak{h}_{\overline{1}}} = \{1\}\times\mathrm{Z}',$$

we see that for a positive constant C_1

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}}) = C_1\int_{\mathrm{G}} e^{iB(x,g.y)}\,dg \int_{\mathrm{G}'/\mathrm{Z}'} \phi^{\mathrm{G}}(g'.w)\,d(g'\mathrm{Z}').$$

However we know from Harish-Chandra (Lemma 34) that

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x)\left(\int_{\mathrm{G}} e^{iB(x,g.y)}\,dg\right)\pi_{\mathfrak{g}/\mathfrak{h}}(y) = C_2\sum_{t\in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)e^{iB(x,ty)}.$$

Hence, by Definition 10 and (75) and for some suitable positive constants C_k,

$$\begin{aligned}
&\pi_{\mathfrak{g}/\mathfrak{h}}(x)\int_{\mathrm{W}}\chi_x(w)\phi^{\mathrm{G}}(w)\,dw \qquad (176)\\
=&\ C_3\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\sum_{\mathfrak{h}_{\overline{1}}}\int_{\tau(\mathfrak{h}_{\overline{1}}^{+})}e^{iB(x,ty)}C(\mathfrak{h}_{\overline{1}})\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}'/\mathrm{Z}'}\phi^{\mathrm{G}}(g'.w)\,d(g'\mathrm{Z}')\,dy\\
=&\ C_3\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\sum_{\mathfrak{h}_{\overline{1}}}\int_{\tau(\mathfrak{h}_{\overline{1}}^{+})}e^{iB(x,ty)}i^{-\dim\mathfrak{g}/\mathfrak{h}}C_{\mathfrak{h}_{\overline{1}}}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}'/\mathrm{Z}'}\phi^{\mathrm{G}}(g'.w)\,d(g'\mathrm{Z}')\,dy\\
=&\ C_4\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\int_{\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}^{+})}e^{iB(x,ty)}f_{\phi^{\mathrm{G}}}(y)\,dy\\
=&\ C_4\sum_{t\in W(\mathrm{G},\mathfrak{h})}\int_{\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}^{+})}e^{iB(x,ty)}f_{\phi^{\mathrm{G}}}(t.y)\,dy\\
=&\ C_4\int_{W(\mathrm{G},\mathfrak{h})(\bigcup_{\mathfrak{h}_{\overline{1}}}\tau(\mathfrak{h}_{\overline{1}}^{+}))}e^{iB(x,y)}f_{\phi^{\mathrm{G}}}(y)\,dy\\
=&\ C_4\int_{\mathfrak{h}\cap\tau(\mathrm{W})}e^{iB(x,y)}f_{\phi^{\mathrm{G}}}(y)\,dy.
\end{aligned}$$

Since $f_{\phi^{\mathrm{G}}}=\mathrm{vol}(\mathrm{G})f_\phi$, the formula follows.

Consider now the case

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}})=\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}}\int_{\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)}(\chi_x\phi^{\mathrm{G}})(s.(w+w^0))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}).$$

Then, as in (64),

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}})=\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}}(\chi_x\phi^{\mathrm{G}})(s.(w+w_0))\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}).$$

Furthermore,

$$\begin{aligned}
\chi_x(s.(w+w_0)) &= e^{i\frac{\pi}{2}\langle x(s.(w+w_0)),s.(w+w_0)\rangle}=e^{i\frac{\pi}{2}\langle x(s.w),sw\rangle}\\
&= e^{iB(x,\tau(sw))}=e^{iB(x,g.\tau(w))}=e^{iB(x,g.y)}
\end{aligned}$$

and

$$\phi^{\mathrm{G}}(s.(w+w_0))=\phi^{\mathrm{G}}(g'.(w+w_0)).$$

Hence, with $n=\tau'(w_0)$,

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}})=C_1\int_{\mathrm{G}}e^{iB(x,g.y)}\,dg\int_{\mathrm{G}'/\mathrm{Z}'^n}\phi^{\mathrm{G}}(g'.w)\,d(g'\mathrm{Z}'^n).$$

Therefore, the computation (176) holds again, and we are done. □

Lemma 36. *Suppose $l > l'$. Let $\mathfrak{z} \subseteq \mathfrak{g}$ and $\mathrm{Z} \subseteq \mathrm{G}$ be the centralizers of $\tau(\mathfrak{h}_{\overline{1}})$. Then for $\phi \in \mathcal{S}(\mathrm{W})$*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw$$
$$= C \sum_{tW(\mathrm{Z},\mathfrak{h}) \in W(\mathrm{G},\mathfrak{h})/W(\mathrm{Z},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\pi_{\mathfrak{z}/\mathfrak{h}}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,t.y)} f_\phi(y)\, dy,$$

where C is a non-zero constant.

Proof. By the Weyl integration formula (67) with the roles of G and G′ reversed and Lemmas 7 and 8,

$$\int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw = C_1 \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \pi_{\mathfrak{g}/\mathfrak{z}}(y)\pi_{\mathfrak{g}'/\mathfrak{h}'}(y)\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}})\, d\tau'(w),$$

where

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}}) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} (\chi_x\phi^{\mathrm{G}})(s.w)\, d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}}).$$

Recall the identification $y = \tau(w) = \tau'(w)$ and let us write $s = gg'$, where $g \in \mathrm{G}$ and $g' \in \mathrm{G}'$. Then

$$\chi_x(s.w) = e^{i\frac{\pi}{2}\langle x(s.w), s.w\rangle} = e^{iB(x,\tau(s.w))} = e^{iB(x,g.\tau(w))} = e^{iB(x,g.y)}$$

and

$$\phi^{\mathrm{G}}(s.w) = \phi^{\mathrm{G}}(g'.w).$$

Since

$$(\{1\} \times \mathrm{G}') \cap \mathrm{S}^{\mathfrak{h}_{\overline{1}}} = \{1\} \times \mathrm{H}',$$

we see that

$$\mu_{\mathcal{O}(w)}(\chi_x\phi^{\mathrm{G}}) = C_2 \int_{\mathrm{G}} e^{iB(x,g.y)}\, dg \int_{\mathrm{G}'/\mathrm{H}'} \phi^{\mathrm{G}}(g'.w)\, d(g'\mathrm{H}').$$

However we know from Harish-Chandra (Lemma 34) that

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{G}} e^{iB(g.x,y)}\, dg\, \pi_{\mathfrak{g}/\mathfrak{z}}(y) = C_3 \sum_{tW(\mathrm{Z},\mathfrak{h}) \in W(\mathrm{G},\mathfrak{h})/W(\mathrm{Z},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\pi_{\mathfrak{z}/\mathfrak{h}}(t^{-1}x)e^{iB(x,ty)}.$$

Hence,

$$\begin{aligned}
&\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw \qquad (177)\\
= &\; C_4 \sum_{tW(\mathrm{Z},\mathfrak{h}) \in W(\mathrm{G},\mathfrak{h})/W(\mathrm{Z},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\pi_{\mathfrak{z}/\mathfrak{h}}(t^{-1}x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,ty)}\pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \int_{\mathrm{G}'/\mathrm{H}'} \phi^{\mathrm{G}}(g'.w)\, d(g'\mathrm{H}')\, dy\\
= &\; C_5 \sum_{tW(\mathrm{Z},\mathfrak{h}) \in W(\mathrm{G},\mathfrak{h})/W(\mathrm{Z},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\pi_{\mathfrak{z}/\mathfrak{h}}(t^{-1}x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,ty)} f_{\phi^{\mathrm{G}}}(y)\, dy.
\end{aligned}$$

Since $f_{\phi^{\mathrm{G}}} = \mathrm{vol}(\mathrm{G}) f_\phi$, the formula follows. □

Lemma 37. *Suppose $l \le l'$. Let $f(y)$ denote the function (95). Then there is a seminorm q on $\mathcal{S}(\mathrm{W})$ such that*

$$\Big|\int_{\mathfrak{h}\cap\tau(\mathrm{W})} f(y)(\phi)\, e^{iB(x,y)}\, dy\Big| \le q(\phi)\ \mathrm{ch}(x)^{-d'+r-1} \le q(\phi)\ \mathrm{ch}(x)^{-d'+r-\iota}$$

$$(x \in \mathfrak{h},\ \phi \in \mathcal{S}(\mathrm{W})).$$

Proof. Since $l \le l'$, Lemma 35 and van der Corput estimate (162) prove that there is a seminorm q on $\mathcal{S}(\mathrm{W})$ such that for all $y \in \mathfrak{h}$ and $\phi \in \mathcal{S}(\mathrm{W})$ we have

$$\Big|\int_{\mathfrak{h}\cap\tau(\mathrm{W})} f(y)(\phi)\, e^{iB(x,y)}\, dy\Big| \le q(\phi)\, |\pi_{\mathfrak{g}/\mathfrak{h}}(x)|\, \mathrm{ch}(x)^{-d'}.$$

The result therefore follows from (99) and (164). □

Corollary 38. *Suppose $l \le l'$. Then for any $\phi \in \mathcal{S}(\mathrm{W})$*

$$T(\check{\Theta}_\Pi)(\phi) = C\int_{\mathfrak{h}} \xi_{-\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\mathfrak{h}\cap\tau(\mathrm{W})} e^{iB(x,y)} f_\phi(y)\, dy\, dx,$$

where C is a non-zero constant and each consecutive integral is absolutely convergent.

Proof. The equality is immediate from Lemmas 32 and 35. The absolute convergence of the outer integral over $\mathfrak{h}$ follows from Lemma 37. □

Suppose $l > l'$. Recall from (57) with $l'' = l'$ that $\mathfrak{h}' = \sum_{j=1}^{l'} \mathbb{R}J'_j$ is identified with $\sum_{j=1}^{l'} \mathbb{R}J_j \subseteq \mathfrak{h}$. Let $\mathfrak{h}'' = \sum_{j=l'+1}^{l} \mathbb{R}J_j$, so that

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''. \tag{178}$$

Then $\mathfrak{z}$, the centralizer of $\tau(\mathfrak{h}_{\overline{1}})$, is the centralizer of $\mathfrak{h}'$ in $\mathfrak{g}$ and $\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{g}''$, where $\mathfrak{g}''$ is the Lie algebra of the group G'' of the isometries of the restriction of the form $(\cdot,\cdot)$ to

$$\sum_{j=l'+1}^{l} \mathsf{V}_{\overline{0}}^{j}. \tag{179}$$

Furthermore, the derived Lie algebras of $\mathfrak{z}$ and $\mathfrak{g}''$ coincide (i.e. $[\mathfrak{z},\mathfrak{z}] = [\mathfrak{g}'',\mathfrak{g}'']$) and $\mathfrak{h}''$ is a Cartan subalgebra of $\mathfrak{g}''$.

Corollary 39. *Suppose $l > l'$. Then for any $\phi \in \mathcal{S}(\mathrm{W})$*

$$\begin{aligned} & T(\check{\Theta}_\Pi)(\phi) \\ = \;& C \sum_{s\in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \int_{\mathfrak{h}} \xi_{-s\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x)\pi_{\mathfrak{z}/\mathfrak{h}}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} f_\phi(y)\, dy\, dx \end{aligned}$$

where C is a non-zero constant and each consecutive integral is absolutely convergent.

Proof. The formula is immediate from Lemmas 32 and 36 together with formula (167):

$$
\begin{aligned}
& T(\check\Theta_\Pi)(\phi)\\
= \ & C_1\int_{\mathfrak{h}}\Big(\xi_{-\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\Big)\Big(\pi_{\mathfrak{g}/\mathfrak{h}}(x)\int_{\mathrm{W}}\chi_x(w)\phi^{\mathrm{G}}(w)\,dw\Big)\,dx\\
= \ & C_2\int_{\mathfrak{h}}\Big(\xi_{-\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\Big)\\
& \left(\sum_{tW(\mathrm{Z},\mathfrak{h})\in W(\mathrm{G},\mathfrak{h})/W(\mathrm{Z},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\pi_{\mathfrak{z}/\mathfrak{h}}(t^{-1}x)\int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})}e^{iB(x,ty)}f_\phi(y)\,dy\right)\,dx\\
= \ & \frac{C_2}{|W(\mathrm{Z},\mathfrak{h})|}\int_{\mathfrak{h}}\Big(\xi_{-\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\Big)\\
& \left(\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\pi_{\mathfrak{z}/\mathfrak{h}}(t^{-1}x)\int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})}e^{iB(x,ty)}f_\phi(y)\,dy\right)\,dx\\
= \ & C_3\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\int_{\mathfrak{h}}\Big(\xi_{-\mu}(c_-^\sharp(tx))\,\mathrm{ch}^{d'-r-\iota}(tx)\Big)\\
& \left(\pi_{\mathfrak{z}/\mathfrak{h}}(x)\int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})}e^{iB(tx,ty)}f_\phi(y)\,dy\right)\,dx\\
= \ & C_3\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(t)\int_{\mathfrak{h}}\Big(\xi_{-t^{-1}\mu}(c_-^\sharp(x))\,\mathrm{ch}^{d'-r-\iota}(x)\Big)\\
& \left(\pi_{\mathfrak{z}/\mathfrak{h}}(x)\int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})}e^{iB(x,y)}f_\phi(y)\,dy\right)\,dx.
\end{aligned}
$$

As in (99) we check that

$$\max\{\deg_{x_j}\pi_{\mathfrak{z}/\mathfrak{h}};\ 1\le j\le l\}=\max\{\deg_{x_j}\pi_{\mathfrak{z}''/\mathfrak{h}''};\ 1\le j\le l''\}=\frac{1}{\iota}(r''-1).$$

Also, $r-r''=d'$. Hence,

$$\mathrm{ch}^{d'-r-1}(x)|\pi_{\mathfrak{z}/\mathfrak{h}}(x)|\le const\ \mathrm{ch}^{d'-r-1+r''-1}(x)=const\ \mathrm{ch}^{-2}(x).$$

Furthermore, f_ϕ is absolutely integrable. Therefore, the absolute convergence of the last integral over $\mathfrak{h}$ follows from the fact that $\mathrm{ch}^{-2\iota}$ is absolutely integrable. ☐

Let $\beta=\frac{\pi}{\iota}$, where ι is as in (98). Then

$$B(x,y)=-\sum_{j=1}^{l}x_j\beta y_j\qquad\Big(x=\sum_{j=1}^{l}x_jJ_j\,,\,y=\sum_{j=1}^{l}y_jJ_j\in\mathfrak{h}\Big).\tag{180}$$

Indeed, the definition of the form B, (174), shows that

$$\begin{aligned} B(x,y) &= \frac{\pi}{2}\operatorname{tr}_{\mathbb{D}/\mathbb{R}}(xy) = \frac{\pi}{2}\sum_{j,k}\operatorname{tr}_{\mathbb{D}/\mathbb{R}}(J_jJ_k)x_jy_k \\ &= \frac{\pi}{2}\sum_j \operatorname{tr}_{\mathbb{D}/\mathbb{R}}(-1)x_jy_j = -\frac{\pi}{\iota}\sum_j x_jy_j. \end{aligned} \tag{181}$$

For a subset $\gamma \subseteq \{1,2,\dots,l\}$ let

$$\mathfrak{h}\cap\gamma^\perp = \{y \in \mathfrak{h};\ y_j = 0 \text{ for all } j\in\gamma\}.$$

Theorem 40. *Let $l \le l'$. Fix a genuine representation Π of $\widetilde{\mathrm{G}}$ with the Harish-Chandra parameter $\mu \in i\mathfrak{h}^*$. Let $P_{a,b}$ and $Q_{a,b}$ be the polynomials defined in (C.4) and (C.5). Let δ be as in (168) and set*

$$a_j = -\mu_j - \delta + 1\,, \qquad b_j = \mu_j - \delta + 1 \tag{182}$$
$$p_j(y_j) = P_{a_j,b_j}(\beta y_j)e^{-\beta|y_j|}\,, \qquad q_j(y_j) = \beta^{-1}Q_{a_j,b_j}(\beta^{-1}y_j) \qquad (1 \le j \le l).$$

There is a non-zero constant C such that for $\phi \in \mathcal{S}(\mathrm{W})$

$$\begin{aligned} T(\check\Theta_\Pi)(\phi) &= C\int_{\mathfrak{h}\cap\tau(\mathrm{W})}\left(\prod_{j=1}^{l}\left(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)\right)\right)\cdot f_\phi(y)\,dy \\ &= C\sum_{\gamma\subseteq\{1,2,\dots,l\}}\int_{\mathfrak{h}\cap\tau(\mathrm{W})\cap\gamma^\perp}\prod_{j\notin\gamma}p_j(y_j)\cdot\prod_{j\in\gamma}q_j(\partial(J_j))\cdot f_\phi(y)\,d_\gamma y \end{aligned} \tag{183}$$

where $d_\gamma y = \prod_{j\notin\gamma} dy_j$ and δ_0 is the Dirac delta at 0. Equivalently, if we define the following generalized function

$$u(y) = C\prod_{j=1}^{l}(p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)) \qquad (y \in \mathfrak{h}\cap\tau(\mathrm{W})),$$

then

$$T(\check\Theta_\Pi)(\phi) = \int_{\mathfrak{h}\cap\tau(\mathrm{W})} u(y)\cdot f_\phi(y)\,dy\,. \tag{184}$$

Theorem 17 implies that the function f_ϕ has the required number of continuous derivatives for the formula (183) to make sense. Notice that the operators appearing in the integrand of (183) act on different variables and therefore commute.

Proof. Notice that the a_j, b_j are integers by (170). Equation (183) follows from Corollary 38, Lemma 33, Theorem 17 and from Proposition C.4. □

In the case $\mathbb{D} = \mathbb{C}$ the boundary $\partial(\mathfrak{h}\cap\tau(\mathrm{W}))$ may be non-empty. Then the integrals in (183) with $(\mathfrak{h}\setminus\gamma^\perp)\cap\partial(\mathfrak{h}\cap\tau(\mathrm{W})) \ne \emptyset$ vanish. In any case we sum only over the γ such that $\partial(\mathfrak{h}\cap\tau(\mathrm{W})) \subseteq \gamma^\perp$. In particular, all terms of the sum corresponding to $\gamma \ne \emptyset$ vanish provided all hyperplanes $y_j = 0$ are boundaries of $\mathfrak{h}\cap\tau(\mathrm{W})$. From Theorem 17 we see that this is the case if and only if $\max(l-q,0) = \min(p,l)$. In turn, this is equivalent to either $l' = q \geq l$ and $p = 0$ or $l = p + q = l'$. In the first case $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{l'})$; in the second $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_{p+q}, \mathrm{U}_{p,q})$.

Notice also that the degree of the polynomial Q_{a_j,b_j} is $2\delta - 2$. Hence, there are no derivatives in the formula (183) if $2\delta - 2 \leq 0$. However,

$$2\delta - 2 = \frac{1}{\iota}(d' - r - \iota) = \begin{cases} d' - r - 1 & \text{if } \mathbb{D} \neq \mathbb{H}, \\ 2(d' - r) - 1 & \text{if } \mathbb{D} = \mathbb{H}. \end{cases}$$

Also,

$$d' - r - \iota = \begin{cases} 2l' - 2l & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}), \\ 2l' - 2l - 1 & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}), \\ l' - l - 1 & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,q}), p + q = l', \\ l' - l & \text{if } (\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_l, \mathrm{O}^*_{2l'}). \end{cases}$$

Thus, since we assume $l \leq l'$, there are no derivatives in the formula (183) if

$$d' - r - \iota \leq 0,$$

which means that $l = l'$ if $\mathbb{D} \neq \mathbb{C}$ and $l \in \{l' - 1, l'\}$ if $\mathbb{D} = \mathbb{C}$.

Lemma 41. *Suppose $l > l'$. In terms of Corollary 39 and the decomposition (178)*

$$\begin{aligned} &\xi_{-s\mu}(c_-^\sharp(x)) \operatorname{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{z}/\mathfrak{h}}(x) \qquad (185)\\ = \;& \left(\xi_{-s\mu}(c_-^\sharp(x')) \operatorname{ch}^{d'-r-\iota}(x')\right) \left(\xi_{-s\mu}(c_-^\sharp(x'')) \operatorname{ch}^{d'-r-\iota}(x'') \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'')\right), \end{aligned}$$

where $x' \in \mathfrak{h}'$ and $x'' \in \mathfrak{h}''$. Moreover,

$$\begin{aligned} &\int_{\mathfrak{h}''} \xi_{-s\mu}(c_-^\sharp(x'')) \operatorname{ch}^{d'-r-\iota}(x'') \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') \, dx'' \qquad (186)\\ = \;& C \sum_{s'' \in W(\mathfrak{g}'', \mathfrak{h}'')} \operatorname{sgn}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho''), \end{aligned}$$

where C is a constant, ρ'' is one half times the sum of positive roots for $(\mathfrak{g}''_{\mathbb{C}}, \mathfrak{h}''_{\mathbb{C}})$ and $\mathbb{I}_{\{0\}}$ is the indicator function of zero.

Proof. Part (185) is obvious, because $\pi_{\mathfrak{z}/\mathfrak{h}}(x' + x'') = \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'')$. We shall verify (186). Let r'' denotes the number defined in (96) for the Lie algebra $\mathfrak{g}''$. A straightforward computation verifies the following table.

$\mathfrak{g}$	r	r''	$d' - r + r''$
$\mathfrak{u}_d$	d	$d - d'$	0
$\mathfrak{o}_d$	$d - 1$	$d - d' - 1$	0
$\mathfrak{sp}_d$	$d + \frac{1}{2}$	$d - d' + \frac{1}{2}$	0

By [Prz93, Lemma 5.7] applied to $\mathrm{G}'' \supseteq \mathrm{H}''$ and $\mathfrak{g}'' \supseteq \mathfrak{h}''$,

$$\pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') = C_1'' \Delta''(c_-^\sharp(x'')) \operatorname{ch}^{r''-\iota}(x'') \qquad (x = x' + x'',\ x' \in \mathfrak{h}',\ x'' \in \mathfrak{h}''),$$

where Δ'' is the Weyl denominator for G'',

$$\Delta'' = \sum_{s''\in W(\mathfrak{g}'',\mathfrak{h}'')} \mathrm{sgn}(s'')\,\xi_{s''\rho''}. \tag{187}$$

Hence, the integral (186) is a constant multiple of

$$\begin{aligned}
&\int_{\mathfrak{h}''} \xi_{-s\mu}(c_-^\sharp(x''))\Delta''(c_-^\sharp(x''))\,\mathrm{ch}^{d'-r+r''}(x'')\left(\mathrm{ch}^{-2\iota}(x'')\,dx''\right)\\
=\ &\int_{c_-^\sharp(\mathfrak{h}'')} \xi_{-s\mu}(h)\Delta''(h)\,dh,
\end{aligned}$$

where $c_-^\sharp(\mathfrak{h}'') \subseteq \mathrm{H}_o''^\sharp$.

Notice that the function

$$\mathrm{H}_o''^\sharp \ni h \to \xi_{-s\mu}(h)\Delta''(h) \in \mathbb{C} \tag{188}$$

is constant on the fibers of the covering map $\mathrm{H}_o''^\sharp \to \mathrm{H}_o''$. Indeed, (187) shows that we'll be done as soon as we check that the weight $-s\mu + s''\rho''$ is integral for the Cartan subgroup H'' (i.e. it is equal to the derivative of a character of H''). Only in the two cases, $\mathrm{G} = \mathrm{U}_d$ and $\mathrm{G} = \mathrm{O}_{2n+1}$, the covering (188) is non-trivial.

Recall the notation used in the proof of (171), and let d'' denote the dimension of the vector space (179) over $\mathbb{D}$. Suppose $\mathrm{G} = \mathrm{U}_d$. Then $\mathrm{G}'' = \mathrm{U}_{d''}$, $\lambda_j + \frac{d'}{2} \in \mathbb{Z}$ and $\rho_j + \frac{d+1}{2} \in \mathbb{Z}$. Hence, $(-s\mu)_j + \frac{d'+d+1}{2} \in \mathbb{Z}$. Since, $\rho_j'' + \frac{d''+1}{2} \in \mathbb{Z}$, we see that

$$\mathbb{Z} \ni (-s\mu)_j + \frac{d'+d+1}{2} + \rho_j'' + \frac{d''+1}{2} = (-s\mu)_j + \rho_j'' + d + 1.$$

Therefore $(-s\mu)_j + \rho_j'' \in \mathbb{Z}$.

Suppose $\mathrm{G} = \mathrm{SO}_{2n+1}$. Then $\mathrm{G}'' = \mathrm{SO}_{2n''+1}$, $\lambda_j \in \mathbb{Z}$ and $\rho_j + \frac{1}{2} \in \mathbb{Z}$. Hence, $(-s\mu)_j + \frac{1}{2} \in \mathbb{Z}$. Since, $\rho_j'' + \frac{1}{2} \in \mathbb{Z}$, we see that $(-s\mu)_j + \rho_j'' \in \mathbb{Z}$.

Therefore, (188) is a constant multiple of

$$\begin{aligned}
&\sum_{s''\in W(\mathfrak{g}'',\mathfrak{h}'')} \mathrm{sgn}(s'') \int_{\mathrm{H}_o''} \xi_{-s\mu}(h)\xi_{s''\rho''}(h)\,dh \qquad (189)\\
=\ &\begin{cases} \mathrm{vol}(\mathrm{H}_o'')\,\mathrm{sgn}(s'') & \text{if } (s\mu)|_{\mathfrak{h}''} = s''\rho'',\\ 0 & \text{if } (s\mu)|_{\mathfrak{h}''} \notin W(\mathfrak{g}'',\mathfrak{h}'')\rho'', \end{cases}\\
=\ &\mathrm{vol}(\mathrm{H}_o'') \sum_{s''\in W(\mathfrak{g}'',\mathfrak{h}'')} \mathrm{sgn}(s'')\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'').
\end{aligned}$$

□

Corollary 42. *Suppose $l > l'$. Then for any $\phi \in \mathcal{S}(\mathrm{W})$*

$$\begin{aligned} & T(\check\Theta_\Pi)(\phi) & (190)\\ = \ & C \sum_{s\in W(\mathrm{G}',\mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \int_{\mathfrak{h}'} \xi_{-s\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} f_\phi(y)\, dy\, dx \\ = \ & C' \int_{\mathfrak{h}'} \xi_{-\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} f_\phi(y)\, dy\, dx \end{aligned}$$

where C is a non-zero constant, $C' = C|W(\mathrm{G}',\mathfrak{h}')|$ and each consecutive integral is absolutely convergent. The expression (190) is zero unless one can choose the Harish-Chandra parameter μ of Π so that

$$\mu|_{\mathfrak{h}''} = \rho''. \tag{191}$$

Proof. The second equality in (190) follows from the fact that $f_\phi(sy) = \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) f_\phi(y)$, $s \in W(\mathrm{G}',\mathfrak{h}')$.

Observe that $B(x'+x'',y) = B(x',y)$ for $x' \in \mathfrak{h}'$, $x'' \in \mathfrak{h}''$ and $y \in \tau'(\mathfrak{h}_{\overline{1}}{}^{reg}) \subseteq \mathfrak{h}'$. We see therefore from Corollary 39 and Lemma 41 that

$$\begin{aligned} T(\check\Theta_\Pi)(\phi) = C \sum_{s\in W(\mathrm{G},\mathfrak{h})} \sum_{s''\in W(\mathrm{G}'',\mathfrak{h}'')} & \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\, \mathrm{sgn}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') \\ & \int_{\mathfrak{h}'} \xi_{-s\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} f_\phi(y)\, dy\, dx. \end{aligned} \tag{192}$$

Notice that for $x \in \mathfrak{h}'$ and $s'' \in W(\mathrm{G}'',\mathfrak{h}'')$, we have $s''x = x$. Thus $\xi_{-s\mu}(c_-^\sharp(x)) = \xi_{-s''s\mu}(c_-^\sharp(x))$ by (167). Notice also that $W(\mathrm{G}'',\mathfrak{h}'') \subseteq W(\mathrm{G},\mathfrak{h})$ and $\mathrm{sgn}(s'') = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s'')$. Moreover, $\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') = \mathbb{I}_{\{0\}}(-(s''^{-1}s\mu)|_{\mathfrak{h}''} + \rho'')$. Hence, replacing s by $s''s$ in (192), we see that this expression is equal to

$$\begin{aligned} = \ & C \sum_{s\in W(\mathrm{G},\mathfrak{h})} \sum_{s''\in W(\mathrm{G}'',\mathfrak{h}'')} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + \rho'') & (193)\\ & \int_{\mathfrak{h}'} \xi_{-s\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} f_\phi(y)\, dy\, dx. \\ = \ & C|W(\mathrm{G}'',\mathfrak{h}'')| \sum_{s\in W(\mathrm{G},\mathfrak{h}),\ (s\mu)|_{\mathfrak{h}''}=\rho''} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \\ & \int_{\mathfrak{h}'} \xi_{-s\mu}(c_-^\sharp(x))\, \mathrm{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} e^{iB(x,y)} f_\phi(y)\, dy\, dx. \end{aligned}$$

Clearly (193) is zero if there is no s such that $(s\mu)|_{\mathfrak{h}''} = \rho''$. Since μ is determined only up to the conjugation by the Weyl group, we may thus assume that $\mu|_{\mathfrak{h}''} = \rho''$ and (191) follows. Under this assumption the summation in (193) is over the s such that $(s\mu)|_{\mathfrak{h}''} = \rho''$. For such s we have

$$(s\mu)|_{\mathfrak{h}'} + \rho'' = s\mu = s(\mu|_{\mathfrak{h}'} + \rho''). \tag{194}$$

Since μ is regular, (194) shows that $s \in W(\mathrm{G}', \mathfrak{h}')$. Hence, (190) follows. The absolute convergence of the integrals was checked in the proof of Corollary 39. $\square$

Theorem 43. *Let $l > l'$. Fix a genuine representation Π of $\widetilde{\mathrm{G}}$ with the Harish-Chandra parameter $\mu \in i\mathfrak{h}^*$. The distribution $T(\check{\Theta}_\Pi)$ is zero unless one can choose μ so that*

$$\mu|_{\mathfrak{h}''} = \rho''. \tag{195}$$

Let us assume (195) and let

$$a_{s,j} = (s\mu)_j - \delta + 1, \qquad b_{s,j} = -(s\mu)_j - \delta + 1 \qquad (s \in W(\mathrm{G}', \mathfrak{h}'),\ 1 \le j \le l').$$

There is a constant C such that for $\phi \in \mathcal{S}(\mathrm{W})$

$$\begin{aligned} T(\check{\Theta}_\Pi)(\phi) &= C \sum_{s \in W(\mathrm{G}', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}(\beta y_j) e^{-\beta|y_j|} \cdot f_\phi(y)\, dy \quad (196) \\ &= C' \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \prod_{j=1}^{l'} p_j(y_j) \cdot f_\phi(y)\, dy, \end{aligned}$$

where $C' = C|W(\mathrm{G}', \mathfrak{h}')|$, the constant β is as in (180) and $p_j(y_j) = P_{a_{1,j}, b_{1,j}}(\beta y_j) e^{-\beta|y_j|}$. In particular $T(\check{\Theta}_\Pi)$ is a locally integrable function whose restriction to $\mathfrak{h}_{\overline{1}}{}^{reg}$ is equal to a non-zero constant multiple of

$$\frac{1}{\pi_{\mathfrak{g}/\mathfrak{z}}(\tau'(w))} \sum_{s \in W(\mathrm{G}', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}(\beta \delta_j \tau'(w)_j) e^{-\beta \tau'(w)_j} \qquad (w \in \mathfrak{h}_{\overline{1}}{}^{reg}). \tag{197}$$

Proof. The formula (196) follows from Corollary 42, Lemma 33 and from Proposition C.5, because under our assumption, $a_{s,j} + b_{s,j} = -2\delta + 2 = -\frac{1}{\iota}(d' - r) + 1 \ge 1$. Weyl integration on W, (68), together with (196) implies (197). $\square$

Our formula for the intertwining distribution $T(\check{\Theta}_\Pi)$ is explicit enough to find its asymptotics, see Theorem 44. These allow us to compute the wave front set of the representation Π' within the Classical Invariant Theory, without using [Vog78]. See Corollary 46 below. We keep the notation of section 5.

Theorem 44. *In the topology of $\mathcal{S}^*(\mathrm{W})$,*

$$t^{\deg \mu_{\mathcal{O}_m}} M^*_{t^{-1}} T(\check{\Theta}_\Pi) \underset{t \to 0+}{\to} C \mu_{\mathcal{O}_m},$$

where $C \neq 0$, if $T(\check{\Theta}_\Pi) \neq 0$.

Proof. Proposition 31, the Lebesgue dominated convergence theorem and Theorem 40 imply that for $l \le l'$

$$\begin{aligned} &\lim_{t \to 0+} t^{\deg \mu_{\mathcal{O}_m}} M^*_{t^{-1}} T(\check{\Theta}_\Pi) \qquad (198) \\ &= \left(C \sum_{\gamma \subseteq \{1,2,\ldots,l\}} \int_{\mathfrak{h} \cap \tau(\mathrm{W}) \cap \gamma^\perp} \prod_{j \notin \gamma} p_j(y_j) \cdot \prod_{j \in \gamma} q_j(\partial(J_j)) \cdot f(y)|_U(\mathbb{I}_U)\, d_\gamma y \right) \mu_{\mathcal{O}_m}. \end{aligned}$$

Similarly, Proposition 31, the Lebesgue dominated convergence theorem and Theorem 43 imply that for $l \geq l'$

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi) = \left(C' \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \prod_{j=1}^{l'} p_j(y_j) \cdot f(y)|_U(\mathbb{I}_U)\, dy \right) \mu_{\mathcal{O}_m}. \tag{199}$$

Thus in each case the limit is a constant multiple of the measure $\mu_{\mathcal{O}_m}$. (Notice that it is not easy to see directly that the constants (198) and (199) are finite, but as the byproduct of the convergence, we see that they are.) The constant is equal to the integral, over U, of the restriction the distribution $T(\check{\Theta}_\Pi)$ to U and can be computed, without the use of the orbital integrals as follows.

Recall from (175) that

$$\langle xw, w\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x\tau(w)) \qquad (x \in \mathfrak{g}, w \in \mathrm{W}).$$

Hence

$$\begin{aligned} T(\check{\Theta}_\Pi)|_U(\mathbb{I}_U) &= \int_U T(\check{\Theta}_\Pi)|_U(u)\, du = \int_U \int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})\Theta(\tilde{g})\chi_{c(g)}(u)\, d\tilde{g}\, du \\ &= \int_U \int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})\Theta(\tilde{g})\chi\big(\frac{1}{4}\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(c(g)\tau(u))\big)\, d\tilde{g}\, du. \end{aligned} \tag{200}$$

We shall use the notation introduced in the proof of Lemma 19 with $k = m$.

In the stable range the elements of U are of the form

$$u = \begin{pmatrix} I_m \\ 0 \\ w_3 \end{pmatrix}$$

with $w_3 = -\overline{w}_3^t$. So, by (115), $\tau(u) = 2w_3$ and the last integral in (200) is a non-zero constant multiple of

$$\begin{aligned} &\int_{\mathfrak{g}} \int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})\Theta(\tilde{g})\chi\big(\frac{1}{4}\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(c(g)x)\big)\, d\tilde{g}\, dx \\ =\ &\int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})\Theta(\tilde{g})\Big(\int_{\mathfrak{g}} \chi\big(\frac{1}{4}\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(c(g)x)\big)\, dx\Big)\, d\tilde{g} \\ =\ &\mathit{const} \int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})\Theta(\tilde{g})\delta_0(c(g))\, d\tilde{g} \\ =\ &\mathit{const}\ \Theta_\Pi(-\tilde{I})\Theta(-\tilde{I}), \end{aligned}$$

where $\Theta_\Pi(\tilde{g}^{-1})\Theta(\tilde{g})$ does not depend on the preimage $\tilde{g}$ of $g \in \mathrm{Sp}$ and *const* denotes some non-zero constant.

Suppose now that the dual pair is not in the stable range, so $m < d$. Suppose moreover that U consists of the matrices of the form

$$u = \begin{pmatrix} I_m & 0 \\ w_3 & w_6 \end{pmatrix}$$

with $w_3 = -\overline{w}_3^t$. This means that $F' = 0$ in (112), i.e. that G' is not equal to $\mathrm{U}_{p,q}$ with $p \neq q$. By (115),

$$\tau(u) = \begin{pmatrix} 2w_3 & w_6 \\ -\overline{w}_6^t & 0 \end{pmatrix}.$$

For $g \in \mathrm{G}$, write $c(g) = \begin{pmatrix} x_{11} & -\overline{x_{12}}^t \\ x_{12} & x_{22} \end{pmatrix} \in \mathfrak{g}$ with $x_{11} = -\overline{x_{11}}^t \in M_m(\mathbb{D})$ and $x_{22} = -\overline{x_{22}}^t \in M_{d-m}(\mathbb{D})$. Then

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}\Big(c(g)\begin{pmatrix} 2w_3 & w_6 \\ -\overline{w}_6^t & 0 \end{pmatrix}\Big) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(2x_{11}w_3 + \overline{x_{12}}^t\overline{w_6}^t + x_{12}w_6) = 2\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x_{11}w_3) + 2\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x_{12}w_6)\,.$$

Hence, the last integral in (200) is a non-zero constant multiple of

$$\begin{aligned}
&\int\int\int_{\widetilde{\mathrm{G}}} \Theta_\Pi(\tilde{g}^{-1})\Theta(\tilde{g})\chi\Big(\frac{1}{4}\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}\Big(c(g)\begin{pmatrix} 2w_3 & w_6 \\ -\overline{w}_6^t & 0 \end{pmatrix}\Big)\Big)\,d\tilde{g}\,dw_3\,dw_6 \qquad (201)\\
&= C_1\int\int\int_{\mathfrak{g}} \Theta_\Pi(\tilde{c}(-x))\Theta(\tilde{c}(x))\chi\Big(\frac{1}{4}\,\mathrm{tr}_{\mathbb{D}/\mathbb{R}}\Big(x\begin{pmatrix} 2w_3 & w_6 \\ -\overline{w}_6^t & 0 \end{pmatrix}\Big)\Big)|\det(x-1)|^{-r}\,dx\,dw_3\,dw_6\\
&= C_2\int_{\mathfrak{g}} \Theta_\Pi(\tilde{c}(-\begin{pmatrix} x_{11} & -\overline{x_{12}}^t \\ x_{12} & x_{22} \end{pmatrix}))\Theta(\tilde{c}(\begin{pmatrix} x_{11} & -\overline{x_{12}}^t \\ x_{12} & x_{22} \end{pmatrix}))\Big|\det(\begin{pmatrix} x_{11} & -\overline{x_{12}}^t \\ x_{12} & x_{22} \end{pmatrix}-1)\Big|^{-r}\\
&\quad \delta_0(x_{11})\delta_0(x_{12})\,dx_{11}\,dx_{12}\,dx_{22}\\
&= C_3\int_{M_{d-m}(\mathbb{D})} \Theta_\Pi(\tilde{c}(-\begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix}))\Theta(\tilde{c}(\begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix}))\Big|\det(\begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix}-1)\Big|^{-r}\,dx_{22}\\
&= C_4\int_{M_{d-m}(\mathbb{D})} \Theta_\Pi(\tilde{c}(-\begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix}))\Big|\det(\begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix}-1)\Big|^{d'/2-r}\,dx_{22}\,,
\end{aligned}$$

where the C_j are non-zero constants, and we used the fact that the Jacobian of the Cayley transform is equal to $|\det(x-1)|^{-r}$ and $\Theta(\tilde{c}(x))$ is a constant multiple of $|\det(x-1)|^{d'/2}$. (See the proof of Lemma 32.) Here r depends on G as in (97). Also, we see from (97) that $d'/2 - r$ is equal to minus "the r" for the subgroup $\mathrm{G}_2 \subseteq \mathrm{G}$ consisting of the matrices of the form $\begin{pmatrix} I_m & 0 \\ 0 & g_{22} \end{pmatrix}$. In the case considered the covering, $\widetilde{\mathrm{G}} \to \mathrm{G}$ splits and the representation Π is the tensor product of the unique non-trivial character χ_Π of the two element group and the trivial lift of an irreducible representation Π_0 of the group G. Hence, (201) is a non-zero constant multiple of

$$\int_{\mathrm{G}_2} \Theta_{\Pi_0}\Big(\begin{pmatrix} -I_m & 0 \\ 0 & g_{22} \end{pmatrix}^{-1}\Big)\chi_\Pi\Big(\widetilde{\begin{pmatrix} -I_m & 0 \\ 0 & g_{22} \end{pmatrix}}\Big)\,dg_{22}. \qquad (202)$$

Since the Cayley transform c is defined on the whole Lie algebra and is continuous, the term $\chi_\Pi\Big(\begin{pmatrix} -I_m & 0\\ 0 & g_{22}\end{pmatrix}\Big)$ is constant. Thus (201) is a non-zero constant multiple of

$$\int_{\mathrm{G}_2}\Theta_{\Pi_0}\Big(\begin{pmatrix} -I_m & 0\\ 0 & g_{22}\end{pmatrix}^{-1}\Big)\,dg_{22}=\chi_{\Pi_0}(-I_d)\int_{\mathrm{G}_2}\Theta_{\Pi_0}\Big(\begin{pmatrix} I_m & 0\\ 0 & g_{22}\end{pmatrix}^{-1}\Big)\,dg_{22},$$

where χ_{Π_0} is the central character of Π_0. Therefore (201) is a non-zero constant multiple of the multiplicity of the trivial representation of G_2 in Π_0. However, we know the highest weight of Π_0, see Theorem 43, and see from there that this multiplicity is non-zero. Therefore (201) is non-zero.

Suppose now that U consists of the matrices of the form

$$u=\begin{pmatrix} I_m & 0\\ 0 & w_5\\ w_3 & w_6\end{pmatrix}.$$

Then $\mathbb{D}=\mathbb{C}$, the dual pair is $(\mathrm{U}_d,\mathrm{U}_{m+p,m})$ and we may assume that

$$\tau(u)=\begin{pmatrix} 2w_3 & w_6\\ -\overline{w}_6^t & \overline{w_5}^t i w_5\end{pmatrix}.$$

Computations similar to those of the previous case show that the last integral in (200) is a non-zero constant multiple of

$$\begin{aligned}
&\iiiint_{\widetilde{\mathrm{G}}}\Theta_\Pi(\tilde g^{-1})\Theta(\tilde g)\chi\Big(\frac14\operatorname{tr}_{\mathbb{C}/\mathbb{R}}\big(c(g)\begin{pmatrix} 2w_3 & w_6\\ -\overline{w}_6^t & \overline{w_5}^t i w_5\end{pmatrix}\big)\Big)\,d\tilde g\,dw_3\,dw_6\,dw_5\\
=\;& C_1\int\int\Theta_\Pi(\tilde c(-\begin{pmatrix} 0 & 0\\ 0 & x_{22}\end{pmatrix}))\Theta(\tilde c(\begin{pmatrix} 0 & 0\\ 0 & x_{22}\end{pmatrix}))\chi\big(\frac14\operatorname{tr}_{\mathbb{C}/\mathbb{R}}(x_{22}\overline{w_5}^t i w_5)\big)\\
&\Big|\det(\begin{pmatrix} 0 & 0\\ 0 & x_{22}\end{pmatrix}-1)\Big|^{-r}\,dx_{22}\,dw_5\\
=\;& C_1\int\int\Theta_\Pi(\tilde c(-\begin{pmatrix} 0 & 0\\ 0 & x_{22}\end{pmatrix}))\chi\big(\frac14\operatorname{tr}_{\mathbb{C}/\mathbb{R}}(x_{22}\overline{w_5}^t i w_5)\big)\\
&\Big|\det(\begin{pmatrix} 0 & 0\\ 0 & x_{22}\end{pmatrix}-1)\Big|^{d'/2-r}\,dx_{22}\,dw_5\\
=\;& C_1\int\int\Theta_\Pi(\tilde c(-\begin{pmatrix} 0 & 0\\ 0 & x_{22}\end{pmatrix}))\chi\big(\frac14\operatorname{tr}_{\mathbb{C}/\mathbb{R}}(x_{22}\overline{w_5}^t i w_5)\big)\\
&|\det(x_{22}-1)|^{d'/2-r}\,dx_{22}\,dw_5.
\end{aligned}\tag{203}$$

Let $\mathrm{W}_5 = M_{p,d-m}(\mathbb{C})$ endowed with the structure of real symplectic space induced by W under the identification of w_5 with $\begin{pmatrix} 0 & 0 \\ 0 & w_5 \\ 0 & 0 \end{pmatrix}$. If $g_{22} - 1$ is invertible, then [2]

$$\int_{\mathrm{W}_5} \chi\big(\frac{1}{4}\,\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(c(g_{22})\overline{w_5}^t i w_5)\big)\, dw_5 = \int_{\mathrm{W}_5} \chi_{c(g_{22})}(w_5)\, dw_5 = \frac{\Theta_5(\tilde{c}(0)\tilde{g}_{22})}{\Theta_5(\tilde{c}(0))\Theta_5(\tilde{g}_{22})}\,, \tag{204}$$

where Θ_5 is the character of the Weil representation for the metaplectic group $\widetilde{\mathrm{Sp}}(\mathrm{W}_5)$, as defined in (12).

Notice that $d'/2 - r = p/2 - (d-m)$ and $d-m$ is "the r" for the group U_{d-m}. Hence (203) is a non-zero constant multiple of

$$\begin{aligned} &\int_{\widetilde{\mathrm{G}}_2} \Theta_\Pi(\begin{pmatrix} \widetilde{-I_m} & 0 \\ 0 & \tilde{g}_{22} \end{pmatrix}^{-1})\Theta_5(\tilde{g}_{22})\frac{\Theta_5(\tilde{c}(0)\tilde{g}_{22})}{\Theta_5(\tilde{c}(0))\Theta_5(\tilde{g}_{22})}\, d\tilde{g}_{22} \\ = \;& C_1 \int_{\widetilde{\mathrm{G}}_2} \Theta_\Pi(\begin{pmatrix} \widetilde{-I_m} & 0 \\ 0 & \tilde{c}(0)^{-1}\tilde{g}_{22} \end{pmatrix}^{-1})\frac{\Theta_5(\tilde{g}_{22})}{\Theta_5(\tilde{c}(0))}\, d\tilde{g}_{22} \\ = \;& C_2 \int_{\mathrm{G}_2} \Theta_{\Pi|_{\widetilde{\mathrm{G}}_2}}(\tilde{g}_{22}^{-1})\Theta_5(\tilde{g}_{22})\, d\tilde{g}_{22}. \end{aligned} \tag{205}$$

This is a non-zero constant multiple of the sum of the multiplicities of the irreducible components of $\Pi|_{\widetilde{\mathrm{G}}_5}$ in ω_5. This sum is positive because ω_5 is the restriction of ω to $\widetilde{\mathrm{Sp}}(\mathrm{W}_5)$, $\widetilde{\mathrm{G}}_2 = \widetilde{\mathrm{G}} \cap \mathrm{Sp}(\mathrm{W}_5)$ and Π occurs in ω. Also, this sum is finite because the centralizer of G_2 in $\mathrm{Sp}(\mathrm{W}_5)$ is compact, i.e. G_2 belongs to a dual pair with both members compact. □

Corollary 45. *In the topology of $\mathcal{S}^*(\mathfrak{g}')$,*

$$t^{\dim \mathcal{O}'_m} M^*_{t^2}\mathcal{F}(\tau'_*(T(\check{\Theta}_\Pi))) \underset{t\to 0+}{\to} C\mathcal{F}\mu_{\mathcal{O}'_m}$$

or equivalently

$$t^{\dim \mathcal{O}'_m - 2\dim \mathfrak{g}'} M^*_{t^{-2}}\tau'_*(T(\check{\Theta}_\Pi)) \underset{t\to 0+}{\to} C\mu_{\mathcal{O}'_m},$$

where $C \neq 0$, if $T(\check{\Theta}_\Pi) \neq 0$.

Proof. Observe that, by the uniqueness of the GG'-invariant measure, $\tau'_*(\mu_{\mathcal{O}_m})$ is a positive constant multiple of $\mu_{\mathcal{O}'_m}$. Hence Proposition 44 together with (131) show that

$$t^{\deg \mu_{\mathcal{O}_m} + \dim \mathrm{W} - 2\dim \mathfrak{g}'} M^*_{t^{-2}}\tau'_*(T(\check{\Theta}_\Pi)) \underset{t\to 0+}{\to} C\mu_{\mathcal{O}'_m}.$$

---

[2]Formula (204) can be easily verified using twisted convolution $\natural$. Indeed

$$\Theta(\tilde{c}(0)) \int_{\mathrm{W}} T(\tilde{g})(w)\, dw = T(\tilde{c}(0))\natural T(g)(0) = \Theta(\tilde{c}(0)\tilde{g}),$$

so that

$$\int_{\mathrm{W}} \chi_{c(g)}(w)\, dw = \frac{\Theta(\tilde{c}(0)\tilde{g})}{\Theta(\tilde{c}(0))\Theta(\tilde{g})}.$$

See e.g. [AP14, Lemma 57].

Now we apply the Fourier transform $\mathcal{F}$ and use the fact that

$$\mathcal{F}\circ M_{t^{-2}}^* = t^{2\dim\mathfrak{g}'} M_{t^2}^*\circ\mathcal{F}$$

to see that

$$t^{\deg\mu_{\mathcal{O}_m}+\dim \mathrm{W}} M_{t^2}^*\mathcal{F}(\tau'_* T(\check{\Theta}_\Pi)) \underset{t\to 0+}{\to} C\mathcal{F}\mu_{\mathcal{O}'_m}.$$

But Lemma 22 implies that $\deg\mu_{\mathcal{O}_m}+\dim\mathrm{W}=\dim\mathcal{O}'_m$. Hence the corollary follows. □

Corollary 46. *Suppose the representation* Π *occurs in Howe's correspondence and the distribution character* Θ_Π *is supported in the preimage* $\widetilde{\mathrm{G}_1}$ *of the Zariski identity component* G_1 *of* G. *Then*

$$WF(\Pi') = \overline{\mathcal{O}'_m}.$$

Proof. This is immediate from Corollary 45, Lemma D.1 and the easy to verify inclusion $WF(\Pi')\subseteq\overline{\mathcal{O}'_m}$, [Prz91, (6.14)]. □

7. The pair $\mathrm{G}=\mathrm{U}_l$, $\mathrm{G}'=\mathrm{U}_{l'}$, $l\leq l'$.

In this section we consider a dual pair (G,G') with both members compact. By the classification of the dual pairs, both G and G' are compact unitary groups and we may assume that $\mathrm{G}=\mathrm{U}_l$ and $\mathrm{G}'=\mathrm{U}_{l'}$ with $l\leq l'$. Furthermore, since we want to use the results of section 6, we view $\mathrm{U}_{l'}$ as $\mathrm{U}_{p,q}$ with $p=0$ and $q=l'$. In particular, by Theorem (17),

$$\mathfrak{h}\cap\tau(\mathrm{W})=\Big\{\sum_{j=1}^{l} y_jJ_j;\ y_j\leq 0 \text{ for all } 1\leq j\leq l\Big\},$$

which is a $W(\mathrm{G},\mathfrak{h})$-invariant domain, where $W(\mathrm{G},\mathfrak{h})$ acts on $\mathfrak{h}$ by permutation of the coordinates, as indicated in (73). The constant δ introduced in (168) has value

$$\delta=\frac{1}{2}(l'-l+1)\,, \tag{206}$$

and $\beta=\pi$ in (180).

Fix a genuine representation Π of $\widetilde{\mathrm{G}}$ with the Harish-Chandra parameter $\mu\in i\mathfrak{h}^*$ and define

$$a_j=\mu_j-\delta+1,\qquad b_j=-\mu_j-\delta+1\quad (1\leq j\leq l)\,, \tag{207}$$

$$a_{s,j}=(s\mu)_j-\delta+1,\quad b_{s,j}=-(s\mu)_j-\delta+1\quad (s\in W(\mathrm{G},\mathfrak{h}), 1\leq j\leq l)\,. \tag{208}$$

(Hence $a_j=a_{1,j}$ and $b_j=b_{1,j}$.)

Lemma 47. *There is a constant* C *such that for* $\phi\in\mathcal{S}(\mathrm{W})$

$$\begin{aligned} T(\check{\Theta}_\Pi)(\phi) &= C\int_{\mathfrak{h}\cap\tau(\mathrm{W})}\prod_{j=1}^{l} P_{a_j,b_j}(\pi y_j)e^{\pi y_j}\cdot f_\phi(y)\,dy \\ &= \frac{C}{|W(\mathrm{G},\mathfrak{h})|}\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\int_{\mathfrak{h}\cap\tau(\mathrm{W})}\prod_{j=1}^{l} P_{a_{s,j},b_{s,j}}(\pi y_j)e^{\pi y_j}\cdot f_\phi(y)\,dy. \end{aligned} \tag{209}$$

Equivalently, with a possibly different constant C,

$$T(\check{\Theta}_\Pi)(\phi) = C \int_{\mathfrak{h}\cap\tau(\mathrm{W})} \prod_{j=1}^{l} P_{a_j,b_j,-2}(\pi y_j) e^{\pi y_j} \cdot f_\phi(y)\, dy. \tag{210}$$

Proof. We are going to use Theorem 40. Since we view $\mathrm{U}_{l'}$ as $\mathrm{U}_{p,q}$ with $p = 0$ and $q = l'$, we have $\delta_j = -1$ for all j. Since the degree of the polynomial $Q_{a_{s,j},b_{s,j}}$ is equal to $-a_{s,j} - b_{s,j} = 2\delta - 2 = l' - l - 1$, Corollary 17 implies that $Q_{a_{s,j},b_{s,j}}(\pi^{-1}\partial(J_j)) f_\phi(y)_{y_j=0} = 0$. Hence all the terms with $\gamma \neq \emptyset$ are zero. Thus the first equality in (209) follows. For the second, notice that, since $(sy)_j = y_{s^{-1}(j)}$, we have

$$\begin{aligned} \prod_{j=1}^{l} P_{a_{s,j},b_{s,j}}(\pi y_j) e^{\pi y_j} &= e^{\sum_{j=1}^{l} \pi y_j} \prod_{j=1}^{l} P_{a_{s,j},b_{s,j}}(\pi y_j) \\ = \; e^{\sum_{j=1}^{l} \pi y_{s^{-1}(j)}} \prod_{j=1}^{l} P_{a_{1,j},b_{1,j}}(\pi y_{s^{-1}(j)}) &= \prod_{j=1}^{l} P_{a_j,b_j}(\pi (s^{-1}y)_j) e^{\pi (s^{-1}y)_j} \end{aligned}$$

and recall that $f_\phi(sy) = \mathrm{sgn}(s) f_\phi(y)$. Finally, since $\pi y_j \leq 0$, (C.4) implies that we may replace $P_{a_j,b_j}(\pi y_j)$ by $2\pi P_{a_j,b_j,-2}(\pi y_j)$. ☐

Lemma 48. *The distribution $T(\check{\Theta}_\Pi)$ is non-zero if and only if the highest weight $\lambda = \sum_{j=1} \lambda_j e_j \in i\mathfrak{h}^*$ of Π satisfies the following condition:*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq \frac{l'}{2}. \tag{211}$$

Equivalently, if and only if the Harish-Chandra parameter μ of Π satisfies

$$\mu_j \in \delta + \mathbb{Z}^+ \qquad (j = 1, 2, \ldots, l) \tag{212}$$

where $\mathbb{Z}^+$ denotes the set of non-negative integers.

The condition (211) means exactly that Π occurs in Howe's correspondence, see for example [Prz96, (A.5.2)]. Recall that we have chosen the Harish-Chandra parameter μ to be strictly dominant, i.e. so that $\mu_1 > \mu_2 > \cdots > \mu_l$, but, in fact, the condition (212) does not depend on the choice of the order of roots.

Proof. Let $\rho = \sum_{j=1} \rho_j e_j$ be one half times the sum of the positive roots. Then $\mu_j = \lambda_j + \rho_j$ and $\rho_j = \frac{l+1}{2} - j$. Hence

$$\begin{aligned} \mu_j - \mu_{j+1} &= \lambda_j - \lambda_{j+1} + 1 \qquad (1 \leq j \leq l-1)\,, \\ \mu_j - \delta &= \lambda_j - \frac{l'}{2} + l - j \qquad (1 \leq j \leq l)\,. \end{aligned}$$

This proves the equivalence of the conditions (211) and (212) since μ is strictly dominant.

Notice that, since $a_j = \mu_j - \delta + 1$, the condition (212) is also equivalent to $a_j \geq 1$ for all $1 \leq j \leq l$. If the distribution $T(\check{\Theta}_\Pi)$ is non-zero, then none of the $P_{a_j,b_j,-2}$ can be identically 0. So $a_j \geq 1$ for all $1 \leq j \leq l$ by (C.2).

It remains to see that the condition $a_j \geq 1$ for all $1 \leq j \leq l$ suffices for the non-vanishing of the expression (210). We are going to use a non-direct argument, though an alternative one is going to be evident from Proposition 49 below.

In the case when both members of the dual pair are compact

$$f_\phi(y) = C_1 \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{\mathrm{S}} \phi(s.w)\, ds \tag{213}$$

where $w \in \mathfrak{h}_{\overline{1}}$ and $\tau(w) = y$ is identified with $\tau'(w)$ and C_1 is the appropriate constant. Clearly the integral in (213) converges if $\phi \in C_c^\infty(\mathrm{W})$ and gives an element of $C_c^\infty(\mathrm{W})^{\mathrm{G}}$. By Corollary 4, it is of the form $\psi \circ \tau'$ where $\psi \in C_c^\infty(\mathfrak{g}')$. Since G' is compact, a theorem of Dadok, [Dad82, Corollary 1.5], applied to $i\mathfrak{g}'$ shows that the function

$$\mathfrak{h} \cap \tau(\mathrm{W}) \ni y \to \int_{\mathrm{S}} \phi(s.w)\, ds \in \mathbb{C}, \qquad (y = \tau(w),\ w \in \mathfrak{h}_{\overline{1}})$$

may be an arbitrary $W(\mathrm{G}, \mathfrak{h})$-invariant compactly supported C^∞ function on $\mathfrak{h} \cap \tau(\mathrm{W})$, as ϕ varies through $C_c^\infty(\mathrm{W})$. Therefore

$$\mathfrak{h} \ni y \to \pi_{\mathfrak{g}/\mathfrak{h}}(y) \int_{\mathrm{S}} \phi(s.w)\, ds, \qquad (y = \tau(w),\ w \in \mathfrak{h}_{\overline{1}})$$

may be an arbitrary $W(\mathrm{G}, \mathfrak{h})$-skew-invariant compactly supported C^∞ function on $\mathfrak{h} \cap \tau(\mathrm{W})$. Hence, if (209) were zero, then the function

$$\prod_{j=1}^{l} P_{a_j, b_j, -2}(\pi y_j) e^{-\pi y_j} \cdot \frac{\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)}$$

would have to be $W(\mathrm{G}, \mathfrak{h})$-invariant. Equivalently,

$$\prod_{j=1}^{l} P_{a_j, b_j, -2}(\pi y_j)$$

would have to be $W(\mathrm{G}, \mathfrak{h})$-invariant. This is not possible if $a_j \geq 1$ for all j. Indeed, μ is strictly dominant and hence, by (C.2), the $P_{a_j, b_j, -2}$ are non-zero polynomials of different degrees. Thus, the distribution (209) is not zero. □

Proposition 49. *With the notation of Lemma 48, let*

$$P_\mu(y) = \prod_{j=1}^{l} P_{a_j, b_j, -2}(\pi y_j) \qquad (y \in \mathfrak{h}).$$

The distribution $\mathrm{T}(\check\Theta_\Pi)$ *is a smooth* GG'*-invariant function on* W*. For* $w \in \mathfrak{h}_{\overline{1}}$ *it is given by the following formula:*

$$\begin{aligned}\mathrm{T}(\check\Theta_\Pi)(w) &= c_\Pi\, e^{-\frac{\pi}{2}\langle Jw,w\rangle}\Big(\frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)}\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)P_\mu(sy)\Big)\\ &= c_\Pi\, e^{-\frac{\pi}{2}\langle Jw,w\rangle}\Big(\frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)}\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)P_{s\mu}(y)\Big), \qquad (214)\end{aligned}$$

where c_Π *is a constant,* J *is the unique positive compatible complex structure on* W *centralized by* G *and* G'*,* β *is as in (180) and* $y = \tau(w) \in \mathfrak{h}$*. The sum in (214) is a* $W(\mathrm{G},\mathfrak{h})$*-skew symmetric polynomial. Hence the quotient by* $\pi_{\mathfrak{g}/\mathfrak{h}}$ *is a* $W(\mathrm{G},\mathfrak{h})$ *invariant polynomial on* $\mathfrak{h}$*. It extends uniquely to a* G*-invariant polynomial* $\tilde P_\mu$ *on* $\mathfrak{g}$*. Thus*

$$\mathrm{T}(\check\Theta_\Pi)(w) = c_\Pi\, e^{-\frac{\pi}{2}\langle Jw,w\rangle}\tilde P_\mu(\tau(w)) \qquad (w\in \mathrm{W}). \qquad (215)$$

Proof. We see from Lemma 47 and the formula (213) that for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$T(\check\Theta_\Pi)(\phi) = C_1\int_{\mathfrak{h}\cap\tau(\mathrm{W})} e^{\pi\sum_{j=1}^l y_j}P_\mu(y)\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}\times\mathrm{G}'}\phi(gg'.w)\,dg\,dg'\,dy. \qquad (216)$$

However, by (181) and (175) with $\beta = \pi$,

$$-\pi\sum_{j=1}^l y_j = B(\sum_{j=1}^l J_j, y) = \frac{\pi}{2}\langle\sum_{j=1}^l J_j w, w\rangle = \frac{\pi}{2}\langle Jw,w\rangle,$$

where $J = \sum_{j=1}^l J_j$ has the required properties. Furthermore,

$$\pi_{\mathfrak{g}'/\mathfrak{z}'}(y) = \frac{1}{|W(\mathrm{G},\mathfrak{h})|}\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\pi_{\mathfrak{g}'/\mathfrak{z}'}(sy).$$

Hence, (216) implies

$$\begin{aligned}&T(\check\Theta_\Pi)(\phi) \qquad (217)\\ = &\ C_2\int_{\mathfrak{h}\cap\tau(\mathrm{W})} e^{-\frac{\pi}{2}\langle Jw,w\rangle}\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)P_\mu(sy)\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}\times\mathrm{G}'}\phi(gg'.w)\,dg\,dg'\,dy.\end{aligned}$$

Weyl integration formula on W, (67), together with (217) implies (214). □

We now reverse the role of G and G′ to compute the intertwining distribution $\mathrm{T}(\check\Theta_{\Pi'})$ for a genuine irreducible unitary representation Π' of $\widetilde{\mathrm{G}}'$. Since we assume that $l \le l'$, the decomposition (178) becomes

$$\mathfrak{h}' = \mathfrak{h}\oplus\mathfrak{h}''. \qquad (218)$$

Proposition 50. *Let* Π' *be a genuine representation of* $\widetilde{\mathrm{G}'}$ *with the Harish-Chandra parameter* $\mu' \in i\mathfrak{h}'^*$*. Then* $\mathrm{T}(\check\Theta_{\Pi'}) \neq 0$ *if and only if*

$$\begin{aligned}&-\mu'_j \in \delta + \mathbb{Z}^+, && (1\le j\le l') \qquad (219)\\ &\text{and }\ \mu'|_{\mathfrak{h}''} = \rho'' && \text{(up to permutation of the coordinates).}\end{aligned}$$

$\mathrm{T}(\check{\Theta}_{\Pi'})$ *is a non-zero constant multiple of* $\mathrm{T}(\check{\Theta}_{\Pi})$ *if and only if* μ *and* μ' *can be chosen in their Weyl group orbits so that*

$$\mu'|_{\mathfrak{h}} = -\mu \quad \text{and} \quad \mu'|_{\mathfrak{h}''} = \rho''. \tag{220}$$

Proof. As one may see from (174), reversing the roles of the members of the dual pair in Theorem 43 results in replacing the form B by $-B$. The constant $\beta = \pi$ get therefore replaced by $-\beta = -\pi$. Also, (213) gets replaced by

$$f'_\phi(y) = C_1\, \pi_{\mathfrak{g}/\mathfrak{h}}(y) \int_{\mathrm{S}} \phi(s.w)\, ds$$

with an appropriate constant C_1. Since this plays no role in Lemma 41 and Corollary 42, $T(\check{\Theta}_{\Pi'})$ is zero unless one can choose μ' so that

$$\mu'|_{\mathfrak{h}''} = \rho''. \tag{221}$$

Since the roles of l and l' are reversed, $\delta = \frac{1}{2}(l' - l + 1)$ is replaced by $\delta' = \frac{1}{2}(l - l' + 1)$. As before, $\delta_j = -1$ for all j. Let

$$a'_{s,j} = -(s\mu')_j - \delta' + 1,\ b'_{s,j} = (s\mu')_j - \delta' + 1 \qquad (s \in W(\mathrm{G}, \mathfrak{h}),\ 1 \le j \le l).$$

Then Theorem 43 says that

$$T(\check{\Theta}_{\Pi'})(\phi) \;=\; C \sum_{s \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}(s) \int_{\mathfrak{h}\cap\tau(\mathrm{W})} \prod_{j=1}^{l} P_{a'_{s,j}, b'_{s,j}}(-\pi y_j) e^{-\pi y_j} \cdot f'_\phi(y)\, dy, \tag{222}$$

where there are no derivatives because the degree of the polynomial $Q_{a'_{s,j}, b'_{s,j}}$ is equal to $-a'_{s,j} - b'_{s,j} = 2\delta' - 2 = l - l' - 1 < 0$.

Since $-\pi y_j \ge 0$, we have $P_{a'_{s,j}, b'_{s,j}}(-\pi y_j) = 2\pi P_{a'_{s,j}, b'_{s,j}, 2}(-\pi y_j)$. Recall, (C.3), that $P_{a'_{s,j}, b'_{s,j}, 2}(-\pi y_j) = P_{b'_{s,j}, a'_{s,j}, -2}(\pi y_j)$. Hence, (222) may be rewritten as

$$T(\check{\Theta}_{\Pi'})(\phi) \;=\; C \sum_{s \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}(s) \int_{\mathfrak{h}\cap\tau(\mathrm{W})} \prod_{j=1}^{l} P_{b'_{s,j}, a'_{s,j}, -2}(\pi y_j) e^{-\pi y_j} \cdot f'_\phi(y)\, dy, \tag{223}$$

with a different constant C. Recall also that

$$\pi_{\mathfrak{g}/\mathfrak{h}}(y) \cdot \prod_{j=1}^{l} y_j^{l'-l} = C_2 \pi_{\mathfrak{g}'/\mathfrak{z}'}(y).$$

Hence, Proposition C.6 shows that

$$\prod_{j=1}^{l} P_{b'_{s,j}, a'_{s,j}, -2}(\pi y_j) \cdot \pi_{\mathfrak{g}/\mathfrak{h}}(y) = C_3 \prod_{j=1}^{l} P_{b'_{s,j} - (l'-l), a'_{s,j} - (l'-l), -2}(\pi y_j) \cdot \pi_{\mathfrak{g}'/\mathfrak{z}'}(y).$$

Therefore (223) coincides with

$$\begin{aligned} & T(\check\Theta_{\Pi'})(\phi) \qquad (224)\\ = \ & C\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\int_{\mathfrak{h}\cap\tau(\mathrm{W})}\prod_{j=1}^{l}P_{(s\mu')_j-\delta+1,-(s\mu')_j-\delta+1,-2}(\pi y_j)e^{-\pi y_j}\cdot f_\phi(y)\,dy, \end{aligned}$$

with a possibly different constant C. By comparing (224) with (209) we see that (220) holds. □

Since, by the definition (12),

$$\mathrm{OP}(\mathcal{K}(T(\check\Theta_\Pi)))=\omega(\check\Theta_\Pi) \text{ and } \mathrm{OP}(\mathcal{K}(T(\check\Theta_{\Pi'})))=\omega(\check\Theta_{\Pi'}),$$

Proposition 50 implies the following corollary.

Corollary 51. *When restricted to the group $\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'$, the oscillator representation decomposes into the Hilbert direct sum of irreducible components of the form $C_\Pi\cdot\Pi\otimes\Pi'$, where Π is determined by Π' via the condition (220) and C_Π are some positive integral constants.*

The final part of this section is devoted to the proof that $C_\Pi=1$, that is that each irreducible representation $\Pi\otimes\Pi'$ of $\widetilde{\mathrm{G}}\times\widetilde{\mathrm{G}}'$ contained in the oscillatory representation occurs with multiplicity one, see Proposition 60 below. This is a well known and fundamental fact due to Hermann Weyl, [Wey46]. We include a proof using the interwining distributions.

For $\beta\in\mathfrak{h}^*$ we denote by H_β the unique element in $\mathfrak{h}$ such that $\beta(H)=\mathrm{tr}(HH_\beta)$ for all $H\in\mathfrak{h}$. We define the operator $\partial(\beta)$ as the directional derivative in the direction of H_β. This defines in particular $\partial(\pi_{\mathfrak{g}/\mathfrak{h}})=\prod_{\alpha>0}\partial(\alpha)$.

Lemma 52. *The constant $C_\mathfrak{z}$ in Lemma 34 is equal in absolute value to*

$$\frac{\pi^{-ll'+\frac{l'(l'+1)}{2}}}{\mathrm{vol}(\mathrm{G})}\frac{|W(\mathfrak{z},\mathfrak{h})|}{|W(\mathfrak{g},\mathfrak{h})|}\frac{\partial(\pi_{\mathfrak{g}/\mathfrak{h}})(\pi_{\mathfrak{g}/\mathfrak{h}})}{\partial(\pi_{\mathfrak{z}/\mathfrak{h}})(\pi_{\mathfrak{z}/\mathfrak{h}})}=\frac{(l-l')!}{l!}\frac{\pi^{-ll'+\frac{l'(l'+1)}{2}}}{\mathrm{vol}(\mathrm{G})}\frac{\partial(\pi_{\mathfrak{g}/\mathfrak{h}})(\pi_{\mathfrak{g}/\mathfrak{h}})}{\partial(\pi_{\mathfrak{z}/\mathfrak{h}})(\pi_{\mathfrak{z}/\mathfrak{h}})}\,. \qquad (225)$$

Proof. The proof is a straightforward modification of the argument proving Harish-Chandra's formula for the Fourier transform of a regular semisimple orbit, [Har57a, Theorem 2, page 104]. Notice that the constant $\pi^{-ll'+\frac{l'(l'+1)}{2}}=\pi^{-\frac{l(l-1)}{2}}\pi^{\frac{(l-l')(l-l'-1)}{2}}$ is due to the normalization of B in (181). □

Recall that we denote by Σ_m the group of permutations of $\{1,2,\dots,m\}$.

Lemma 53. *Let $z_j\in\mathbb{C}$ for $1\le j\le m$. Then, with the convention that empty products are equal to 1,*

$$\sum_{s\in\Sigma_m}\mathrm{sgn}(s)\prod_{j=1}^{m}\prod_{k=1}^{s(j)-1}(z_j-k)=\prod_{1\le j<k\le m}(z_j-z_k)\,. \qquad (226)$$

Proof. The left-hand side is a Vandermonde determinant. Indeed

$$\begin{aligned}
&\sum_{s\in\Sigma_m} \mathrm{sgn}(s) \prod_{j=1}^{m} \prod_{k=1}^{s(j)-1} (z_j - k)\\
&= \det \begin{bmatrix} 1 & (z_1-1) & (z_1-1)(z_1-2) & \dots & (z_1-1)(z_1-2)\dots(z_1-m+1) \\ 1 & (z_2-1) & (z_2-1)(z_2-2) & \dots & (z_2-1)(z_2-2)\dots(z_2-m+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (z_m-1) & (z_m-1)(z_m-2) & \dots & (z_m-1)(z_m-2)\dots(z_m-m+1) \end{bmatrix}\\
&= \det \begin{bmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{m-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_m & z_m^2 & \dots & z_m^{m-1} \end{bmatrix}.
\end{aligned}$$

This proves the result. ☐

Lemma 54. *Let $n \in \mathbb{Z}$ with $n \geq 2$ and let $a \in \mathbb{C}$. Set*

$$F(a,n) = \det \begin{bmatrix} 1 & a & a(a+1) & \dots & a(a+1)\cdots(a+n-2) \\ 1 & a+1 & (a+1)(a+2) & \dots & (a+1)(a+2)\cdots(a+n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a+n-1 & (a+n-1)(a+n) & \dots & (a+n-1)(a+n)\cdots(a+2n-3) \\ 1 & a+n & (a+n)(a+n+1) & \dots & (a+n)(a+n+1)\cdots(a+2n-2) \end{bmatrix}.$$

Then

$$F(a,n) = \prod_{k=1}^{n-1} k!\,.$$

In particular, $F(a,n)$ is independent of a.

Proof. For $2 \le j \le n$ we replace the j-th row by the difference between the j-th and the $(j-1)$-th row. We obtain:

$$\begin{aligned}
F(a,n) &= \det \begin{bmatrix} 1 & a & a(a+1) & \dots & a(a+1)\cdots(a+n-2) \\ 0 & 1 & 2(a+1) & \dots & (n-1)(a+1)\cdots(a+n-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2(a+n-1) & \dots & (n-1)(a+n-1)\cdots(a+2n-4) \\ 0 & 1 & 2(a+n) & \dots & (n-1)(a+n)\cdots(a+2n-3) \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 2(a+1) & \dots & (n-1)(a+1)\cdots(a+n-2) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2(a+n-1) & \dots & (n-1)(a+n-1)\cdots(a+2n-4) \\ 1 & 2(a+n) & \dots & (n-1)(a+n)\cdots(a+2n-3) \end{bmatrix} \\
&= (n-1)! \det \begin{bmatrix} 1 & (a+1) & \dots & (a+1)\cdots(a+n-2) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (a+n-1) & \dots & (a+n-1)\cdots(a+2n-4) \\ 1 & (a+n) & \dots & (a+n)\cdots(a+2n-3) \end{bmatrix} \\
&= (n-1)! F(a+1, n-1)\,.
\end{aligned}$$

Iterating, we conclude

$$F(n,a) = (n-1)!F(a+1,n-1) = \cdots = (n-1)!\cdots 2!\, F(a+n-2,2) = \prod_{k=1}^{n-1} k!$$

since

$$F(a+n-2,2) = \det \begin{bmatrix} 1 & a+n-2 \\ 1 & a+n-1 \end{bmatrix} = 1\,.$$

□

We may identify

$$\mathfrak{s}_{\overline{1}} = \mathrm{W} = \mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0}}) = M_{l,l'}(\mathbb{C}),$$

so that $\tau(w) = wi\overline{w}^t$ and $J(w) = -iw$, $w \in M_{l,l'}(\mathbb{C})$, is a compatible positive complex structure on W contained in $\mathfrak{g}$. Then (173) implies that

$$\langle J(w), w\rangle = \mathrm{Re}\ \mathrm{tr}(J\tau(w)) = \mathrm{tr}(w\overline{w}^t).$$

Hence our normalization of the Lebesgue measure dw on W is such that for each entry, $dz = dx\,dy$ if $z = x+iy \in \mathbb{C}$. In particular, if we let $\Phi(w) = e^{-\mathrm{tr}(w\overline{w}^t)}$ then

$$\int_{\mathrm{W}} \Phi(w)\, dw = \pi^{ll'}\,.$$

Lemma 55. *With the above notation,*

$$\int_{\mathfrak{h}_{\overline{1}}^2} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \int_{\mathrm{G}\times\mathrm{G}'} \Phi(s.w)\, ds\, dw^2 = \mathrm{vol}(G)\,\mathrm{vol}(G') \prod_{k=0}^{l} k! \prod_{k=0}^{l-1} (k+l'-l)!\,. \qquad (227)$$

Consequently,

$$\int_{\mathrm{W}} f(w)\, dw = C \int_{\mathfrak{h}_{\overline{1}}^2} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \int_{\mathrm{GG}'} f(s.w)\; ds\, dw^2 \qquad (f \in C_c(\mathrm{W}))\,,$$

where

$$C = \frac{\pi^{ll'}}{\mathrm{vol}(\mathrm{G})\,\mathrm{vol}(\mathrm{G}')\prod_{k=0}^{l} k! \prod_{k=0}^{l-1}(k+l'-l')!}\,. \tag{228}$$

Proof. By $\mathrm{G}\times\mathrm{G}'$-invariance,

$$\int_{\mathrm{G}\times\mathrm{G}'} \Phi(s.w)\, ds = \mathrm{vol}(G)\,\mathrm{vol}(G')\Phi(w)\,.$$

Moreover,

$$|\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| = |\pi_{\mathfrak{g}'/\mathfrak{h}'}(\tau'(w))||\pi_{\mathfrak{g}/\mathfrak{z}}(\tau(w))|\,.$$

Using (58) and (61), up to a constant of absolute value one,

$$\begin{aligned}
&\int_{\mathfrak{h}_{\overline{1}}^2} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)| \int_{\mathrm{GG}'} \Phi(s.w)\; ds\, dw^2 \\
&\quad = \int_{(\mathbb{R}^+)^l} \prod_{1\le j<k\le l} (x_j-x_k)^2 \Big(\prod_{j=1}^{l} x_j^{l'-l}\Big) e^{-x_1-\cdots-x_l}\; dx_1\cdots dx_l\,.
\end{aligned} \tag{229}$$

Recall that

$$\int_0^\infty x^\alpha e^{-x}\; dx = \alpha!\,.$$

Since

$$\begin{aligned}
\prod_{1\le j<k\le l}(x_j-x_k)^2 &= \Big(\sum_{s\in\Sigma_l}\mathrm{sgn}(s)\; x_1^{s(1)-1}\cdots x_l^{s(l)-1}\Big)^2 \\
&= \sum_{s,t\in\Sigma_l}\mathrm{sgn}(st)\; x_1^{s(1)+t(1)-2}\cdots x_{d'}^{s(l)+t(l)-2}\,,
\end{aligned}$$

formula (229) is equal to

$$\begin{aligned}
&\sum_{s,t\in\Sigma_l}\mathrm{sgn}(st)\int_{(\mathbb{R}^+)^l} x_1^{s(1)+t(1)+l'-l-2}\cdots x_l^{s(l)+t(l)+l'-l-2} e^{-x_1-\cdots-x_l}\; dx_1\cdots dx_l \\
&\quad = \sum_{s,t\in\Sigma_l}\mathrm{sgn}(st)\big(s(1)+t(1)+l'-l-2\big)!\cdots\big(s(l)+t(l)+l'-l-2\big)! \\
&\quad = |\Sigma_l|\sum_{s\in\Sigma_l}\mathrm{sgn}(s)\prod_{j=1}^{l}\big(s(j)+j+l'-l-2\big)! \\
&\quad = l!\,\det\big[\big(k+j+l'-l-2\big)!\big]_{j,k=1}^{l}\,.
\end{aligned} \tag{230}$$

Applying Lemma 54, we obtain

$$\begin{aligned} & l!\det\left[(k+j+l'-l-2)!\right]_{j,k=1}^{l} \\ & \qquad = l!\prod_{k=0}^{l-1}(k+l'-l))!F(3+(l'-l-2),l) = \prod_{k=0}^{l-1}(k+l'-l)!\prod_{k=0}^{l}k!\,, \end{aligned}$$

which proves the lemma. □

One can relate the Haar measure on the group to the Lebesgue measure on the Lie algebra via the following formulas

$$\begin{aligned} dc_-^\sharp(x) &= 2^{l^2}\ \mathrm{ch}^{-2l}(x)\,dx \qquad (x\in\mathfrak{g}), \\ dc_-^\sharp(x) &= 2^{l}\ \mathrm{ch}^{-2}(x)\,dx \qquad (x\in\mathfrak{h})\,. \end{aligned} \tag{231}$$

Also, we have the following, easy to verify, equation

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) = 2^{-\frac{l(l-1)}{2}}\,\Delta(c_-^\sharp(x))\ \mathrm{ch}^{l-1}(x) \qquad (x\in\mathfrak{h})\,. \tag{232}$$

Lemma 56. *The constant C in (186), with the roles of* G *and* G′ *reversed, is equal to*

$$2^{-(l'-l)(l'-l+1)/2}\,\mathrm{vol}(\mathrm{H}''^{\sharp}_{o})\,.$$

Proof. This follows from (232) and (231) with $l'-l$ in place of l. □

Lemma 57. *The constant c_Π in Proposition 49 is equal in absolute value to*

$$(2\pi)^l 2^{1-l'l+\frac{l(l+1)}{2}}\,\frac{\partial(\pi_{\mathfrak{g}/\mathfrak{h}})\big(\pi_{\mathfrak{g}/\mathfrak{h}}\big)}{|W(\mathrm{G},\mathfrak{h})|}\,\frac{\mathrm{vol}(\mathrm{G})}{\mathrm{vol}(\mathrm{H})}\,\pi^{-\frac{l(l-1)}{2}} = (2\pi)^l 2^{1+l(l-l')}\,. \tag{233}$$

Proof. The value of the constant c_Π is obtained by repeating the computation in the proof of Proposition 49 keeping track of the constants, and the following formula, due to Macdonald [Mac80, p. 95]:

$$\frac{\partial(\pi_{\mathfrak{g}/\mathfrak{h}})\big(\pi_{\mathfrak{g}/\mathfrak{h}}\big)}{|W(\mathrm{G},\mathfrak{h})|}\,\frac{\mathrm{vol}(\mathrm{G})}{\mathrm{vol}(\mathrm{H})} = (2\pi)^N\,, \tag{234}$$

where $N=\frac{l(l-1)}{2}$ is the number of positive roots. □

Lemma 58. *For every smooth function $F:\mathfrak{h}\to\mathbb{C}$*

$$\big(\partial(\pi_{\mathfrak{g}/\mathfrak{h}})(\pi_{\mathfrak{g}/\mathfrak{h}}F)\big)(0) = \Big(\prod_{k=0}^{l}k!\Big)\,F(0)\,. \tag{235}$$

Proof. In terms of the coordinates $x_j = iy_j$ $(1\le j\le l)$,

$$\partial(\pi_{\mathfrak{g}/\mathfrak{h}}) \;=\; \prod_{1\le j<k\le l}(\partial_{x_j}-\partial_{x_k}) = \sum_{s\in\Sigma_l}\mathrm{sgn}(s)\partial_{x_1}^{s(1)-1}\cdots\partial_{x_l}^{s(l)-1}\,. \tag{236}$$

By the product rule,

$$\big(\partial(\pi_{\mathfrak{g}/\mathfrak{h}})(\pi_{\mathfrak{g}/\mathfrak{h}}F)\big)(0) = \partial(\pi_{\mathfrak{g}/\mathfrak{h}})(\pi_{\mathfrak{g}/\mathfrak{h}})F(0)\,.$$

Moreover, if $\delta_{s,t}$ denotes Kronecker's delta, then

$$\begin{aligned}
\partial(\pi_{\mathfrak{g}/\mathfrak{h}})(\pi_{\mathfrak{g}/\mathfrak{h}}) &= \sum_{s\in\Sigma_l} \mathrm{sgn}(s)\partial_{x_1}^{s(1)-1}\cdots\partial_{x_l}^{s(l)-1}\Big(\sum_{t\in\Sigma_l}\mathrm{sgn}(t){x_1}^{t(1)-1}\cdots {x_l}^{t(l)-1}\Big)\\
&= \sum_{s\in\Sigma_l}\mathrm{sgn}(s)\Big(\sum_{t\in\Sigma_l}\mathrm{sgn}(t)\,\delta_{s,t}\,\prod_{k=1}^{l}(t(k)-1)!\Big)\\
&= |\Sigma_l|\,\prod_{k=1}^{l}(k-1)!\\
&= \prod_{k=0}^{l} k! \qquad (237)
\end{aligned}$$

□

Lemma 59. *The following equality holds*

$$\dim\Pi' = \frac{1}{\prod_{j=1}^{l}(l'-j)!}\prod_{j=1}^{l}\frac{(\delta+\mu_j-1)!}{(\mu_j-\delta)!}\prod_{1\le j<k\le l}(\mu_j-\mu_k)\,. \qquad (238)$$

Proof. By Weyl's dimension formula,

$$\dim\Pi' = \prod_{\alpha>0}\frac{\langle\mu',\alpha\rangle}{\langle\rho,\alpha\rangle} = \frac{\prod_{k=2}^{l'}(\mu_1'-\mu_k')\cdots\prod_{k=l+1}^{l'}(\mu_l'-\mu_k')}{\prod_{k=2}^{l'}(k-1)\cdots\prod_{k=l+1}^{l'}(k-l)}\,, \qquad (239)$$

where $\langle\cdot,\cdot\rangle$ is the form on $\mathfrak{h}'^*$ induced by (any nonzero multiple) of the Killing form on $\mathfrak{g}'$. Indeed, recall that, in our conventions, the positive roots of $(\mathfrak{g}'_{\mathbb{C}},\mathfrak{h}'_{\mathbb{C}})$ are of the form

$$\alpha_{j,k}(x) = x_j - x_k \qquad (1\le j<k\le l'\,, x\in\mathfrak{h}')\,,$$

and

$$\rho' = (\rho_1',\dots,\rho_{l'}') \qquad \text{with} \qquad \rho_j' = \tfrac{1}{2}(l'-2j+1) \text{ for } 1\le j\le l'\,.$$

Let ρ'' be the ρ-function for $\mathfrak{u}_{d-d'}$. Let us fix the form on $\mathfrak{h}'^*$ associated with the trace form $\langle x,y\rangle = -\operatorname{tr}(xy)$ on $\mathfrak{h}'$. Then

$$\langle\rho,\alpha_{j,k}\rangle = k-j\,.$$

Recall that

$$\mu_j' = \delta + n_j \qquad (1\le j\le l\,, n_j\in\mathbb{Z}^+) \qquad (240)$$

$$\mu_j' = \rho''_{j-l} = \tfrac{1}{2}(l'-l+1-2(j-l)) = \delta - j + l \qquad (l+1\le j\le l')\,. \qquad (241)$$

Hence

$$\langle\mu',\alpha_{j,k}\rangle = k-j \qquad (l+1<j<k\le l')\,.$$

This proves the second equality in (239). Observe now that

$$\prod_{k=j}^{l'}(k-j) = \prod_{k=1}^{l'-j}k = (l'-j)!\,.$$

The denominator of (239) is therefore equal to

$$\prod_{j=1}^{l}(l'-j)!$$

For the numerator of (239), using (241), we have

$$\begin{aligned}
&\prod_{k=2}^{l'}(\mu_1'-\mu_k')\cdots\prod_{k=l+1}^{l'}(\mu_l'-\mu_k')\\
&= (\mu_1'-\mu_2')\ (\mu_1'-\mu_3')\ \cdots\ (\mu_1'-\mu_l')\ (\mu_1'-\mu_{l+1}')\ \cdots\ (\mu_1'-\mu_{l'}')\\
&\qquad \times\ (\mu_2'-\mu_3')\ \cdots\ (\mu_2'-\mu_l')\ (\mu_2'-\mu_{l+1}')\ \cdots\ (\mu_2'-\mu_{l'}')\\
&\qquad\qquad \times\ \ddots \qquad\qquad\qquad \vdots\\
&\qquad\qquad\qquad \times\ (\mu_{l-1}'-\mu_l')(\mu_{l-1}'-\mu_{l+1}')\cdots(\mu_{l-1}'-\mu_{l'}')\\
&\qquad\qquad\qquad\qquad \times\ (\mu_l'-\mu_{l+1}')\ \cdots\ (\mu_l'-\mu_{l'}')\\
&= \prod_{1\le j<k\le l}(\mu_j'-\mu_k')\prod_{\substack{1\le j\le l\\ l+1\le k\le l'}}(\mu_j'-\mu_k')\\
&= \prod_{1\le j<k\le l}(\mu_j'-\mu_k')\prod_{\substack{1\le j\le l\\ l+1\le k\le l'}}(n_j+k-l)\\
&= \prod_{1\le j<k\le l}(\mu_j'-\mu_k')\prod_{1\le j\le l}\ \prod_{1\le k\le l'-l}(n_j+k)\\
&= \prod_{1\le j<k\le l}(\mu_j'-\mu_k')\prod_{j=1}^{l}\frac{(2\delta+n_j-1)!}{n_j!}
\end{aligned}$$

□

Proposition 60. *Up to a constant of absolute value one,*

$$T(\check{\Theta}_\Pi)(0)=\mathrm{vol}(\widetilde{G})\cdot\dim\Pi'\,. \tag{242}$$

Equivalently, $\Pi\otimes\Pi'$ *is contained in* ω *exactly once.*

Proof. The projection of ω onto its isotypic component of type Π is given by

$$\begin{aligned}
\mathrm{vol}(\widetilde{\mathrm{G}})\cdot P_\Pi &= \dim\Pi\cdot\int_{\widetilde{G}}\check{\Theta}_\Pi(\widetilde{g})\omega(\widetilde{g})\,d\widetilde{g}\\
&= \dim\Pi\cdot\int_{\widetilde{G}}\check{\Theta}_\Pi(\widetilde{g})\,\mathrm{OP}(\mathcal{K}(T(\widetilde{g})))\,d\widetilde{g}\\
&= \dim\Pi\cdot\mathrm{OP}\circ\mathcal{K}\Big(\int_{\widetilde{G}}\check{\Theta}_\Pi(\widetilde{g})T(\widetilde{g})\,d\widetilde{g}\Big)\\
&= \dim\Pi\cdot\mathrm{OP}\circ\mathcal{K}\big(T(\check{\Theta}_\Pi)\big)\,.
\end{aligned}$$

Also, with $K = \mathcal{K}\big(T(\check\Theta_\Pi)\big)$, we have

$$\mathrm{tr}(\mathrm{OP}(K)) = \int_{\mathsf{X}} K(x,x)\, dx = T(\check\Theta_\Pi)(0)\,.$$

It follows that the dimension of the isotypic component of type Π is

$$\begin{aligned}\mathrm{tr}(P_\Pi) &= \dim\Pi\cdot \mathrm{tr}\left(\mathrm{OP}\left(\mathcal{K}(T(\check\Theta_\Pi))\right)\right)\frac{1}{\mathrm{vol}(\widetilde{\mathrm{G}})}\\ &= \dim\Pi\cdot\frac{T(\check\Theta_\Pi)(0)}{\mathrm{vol}(\widetilde{\mathrm{G}})}\,. \qquad (243)\end{aligned}$$

Hence $\Pi\otimes\Pi'$ is contained in ω exactly once if and only if (242) holds.

Notice first that, by definition, $(b-1)!P_{a,b,2}(\xi)$ is a polynomial function of a and b. Hence, using (214), we have that

$$\prod_{j=1}^{l}(-\mu_j-\delta)!\; T(\check\Theta_\Pi)(0)$$

is polynomial function of $(\mu_1,\mu_2,\dots,\mu_l)$. It therefore suffices to prove (242) for $-\mu_j$ satisfying (212).

By Lemma 58, $T(\check\Theta_\Pi)(0)$ can be computed from (235) by evaluating at 0 the function

$$e^{\frac{\pi}{4}\langle Jw,w\rangle}T(\check\Theta_{\Pi'})(w)$$

determined in Proposition 49. Set $x_j=\pi y_j$ and $\partial_j=\partial_{x_j}$. Then, by (214), (233) and (235),

$$\begin{aligned}&\Big(\prod_{k=0} k!\Big)T(\check\Theta_\Pi)(0)\\ &= c_\Pi\partial(\pi_{\mathfrak{g}/\mathfrak{h}})\Big(\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(t)\prod_{j=1}^{l}P_{(t\mu)_j-\delta+1,-(t\mu)_j-\delta+1,-2}(\pi y_j)\Big)(0)\\ &= c_\Pi(-i\pi)^{\frac{l(l-1)}{2}}\prod_{1\le j<k\le l}(\partial_j-\partial_k)\Big(\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(t)\prod_{j=1}^{l}P_{(t\mu)_j-\delta+1,-(t\mu)_j-\delta+1,-2}(x_j)\Big)(0)\\ &= c_\Pi(-i\pi)^{\frac{l(l-1)}{2}}\sum_{t\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(t)\sum_{s\in W(\mathrm{G}/\mathfrak{h})}\mathrm{sgn}(s)\prod_{j=1}^{l}\big(\partial_j^{s(j)-1}P_{(t\mu)_j-\delta+1,-(t\mu)_j-\delta+1,-2}\big)(0)\\ &= c_\Pi(-i\pi)^{\frac{l(l-1)}{2}}\sum_{t,s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(ts)\prod_{j=1}^{l}\big(\partial_j^{s(j)-1}P_{(t\mu)_j-\delta+1,-(t\mu)_j-\delta+1,-2}\big)(0)\\ &= c_\Pi(-i\pi)^{\frac{l(l-1)}{2}}|W(\mathrm{G},\mathfrak{h})|\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\prod_{j=1}^{l}\big(\partial_j^{s(j)-1}P_{\mu_j-\delta+1,-\mu_j-\delta+1,-2}\big)(0)\,. \qquad (244)\end{aligned}$$

According to Lemma C.8, we have

$$\begin{aligned}
\partial_j^{s(j)-1} & P_{\mu_j-\delta+1,-\mu_j-\delta+1,-2}(0) \\
&= P_{\mu_j-s(j)-\delta+2,-\mu_j-\delta+1,-2}(0) \\
&= (-1)^{\mu_j-\delta-s(j)+2}\ 2^{s(j)+2(\delta-1)}\binom{\mu_j+\delta-1}{s(j)+2(\delta-1)},
\end{aligned} \tag{245}$$

where last equality holds under the assumption (212). Notice that

$$\begin{aligned}
\prod_{j=1}^{l}\binom{\mu_j+\delta-1}{s(j)+2(\delta-1)} &= \prod_{j=1}^{l}\frac{(\mu_j+\delta-1)!}{(s(j)+2(\delta-1))!(\mu_j-s(j)-\delta+1)!} \\
&= \prod_{j=1}^{l}\frac{1}{(j+2(\delta-1))!}\prod_{j=1}^{l}\frac{(\mu_j+\delta-1)!}{(\mu_j-s(j)-\delta+1)!} \\
&= \prod_{j=1}^{l}\frac{1}{(l'-j)!}\prod_{j=1}^{l}\frac{(\mu_j+\delta-1)!}{(\mu_j-s(j)-\delta+1)!}\,.
\end{aligned} \tag{246}$$

Hence, omitting constants of absolute value one, we have

$$\begin{aligned}
&\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\prod_{j=1}^{l}\partial_j^{s(j)-1}P_{\mu_j-\delta+1,-\mu_j-\delta+1,-2}(0) \qquad (247)\\
&= 2^{ll'-\frac{l(l+1)}{2}}\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\binom{\mu_j+\delta-1}{s(j)+2(\delta-1)} \\
&= 2^{ll'-\frac{l(l+1)}{2}}\prod_{j=1}^{l}\frac{(\mu_j+\delta-1)!}{(l'-j)!}\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\prod_{j=1}^{l}\frac{1}{(\mu_j-s(j)-\delta+1)!}
\end{aligned}$$

By (226),

$$\sum_{s\in W(\mathrm{G},\mathfrak{h})}\mathrm{sgn}(s)\prod_{j=1}^{l}\frac{(\mu_j-\delta)!}{(\mu_j-s(j)-\delta+1)!} = \prod_{1\le j<k\le l}(\mu_j-\mu_k)\,. \tag{248}$$

Comparing Lemma 59, (244), (247) and (248), we deduce the following equality

$$\Big(\prod_{k=0}^{l}k!\Big)T(\check{\Theta}_\Pi)(0) = C_\Pi(i\pi)^{\frac{l(l-1)}{2}}2^{ll'-\frac{l(l+1)}{2}}|W(\mathrm{G},\mathfrak{h})|\dim\Pi'\,. \tag{249}$$

We therefore conclude that $T(\check{\Theta}_\Pi)(0) = K_\Pi \dim \Pi'$, where

$$\begin{aligned} |K_\Pi| &= C_\Pi \pi^{\frac{l(l-1)}{2}} 2^{ll' - \frac{l(l+1)}{2}} \frac{|W(\mathrm{G}, \mathfrak{h})|}{\prod_{k=0}^{l} k!} \\ &= 2\, \frac{(2\pi)^{\frac{l(l-1)}{2}}}{\prod_{k=0}^{l-1} k!}\, (2\pi)^l \\ &= 2 \,\mathrm{vol}(\mathrm{G}) \qquad (250) \\ &= \mathrm{vol}(\widetilde{\mathrm{G}})\,. \end{aligned}$$

In (250) we have used (234), (237) and that $\mathrm{vol}(\mathrm{H}) = (2\pi)^l$. □

8. Limits of orbital integrals in the stable range.

The results on the limits of the orbital integrals obtained in the section 6 did not make any assumption on the relative sizes of the groups G and G′. They are based on the adaptation to orbital integrals on the symplectic space W of Harish-Chandra's study of orbital integrals on a Lie algebra. However, if we restrict ourselves to the stable range, we may obtain the same results without any reference to Harish-Chandra's work. This is another indication on how natural is the stable range assumption in the theory of reductive dual pairs. For instance, recently Lock and Ma, [LM15], computed the correspondence of the associated varieties under this assumption. This is equivalent to computing the wave front set correspondence by the work of Schmid and Vilonen, [SV00].

Lemma 61. *The following inequality holds*

$$\dim \mathrm{W} - \dim \mathcal{O}'_m - \dim \mathfrak{g} - \dim \mathfrak{h} \geq 0. \qquad (251)$$

The two sides of (251) are equal if and only if

$$(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R})); \quad (\mathrm{O}_3, \mathrm{Sp}_{2l'}(\mathbb{R})); \quad (\mathrm{U}_1, \mathrm{U}_{p,q}),\ 1 \leq p \leq q. \qquad (252)$$

Proof. Let $F(d)$ denote The quantity (251) as a function of $d = \dim \mathsf{V}_{\overline{0}} \geq m$. This is a concave down quadratic function. We know from (103) that $F(m) = \dim \mathfrak{g} - \dim \mathfrak{h} \geq 0$. Also, (102) gives the following explicit formula

$$F(d) = \begin{cases} 2dm - m^2 - m - \frac{d^2}{2} & \text{if } \mathrm{G} = \mathrm{O}_{2l},\ \ d' = 2m, \\ 2dm - m^2 - m - \frac{d^2-1}{2} & \text{if } \mathrm{G} = \mathrm{O}_{2l+1},\ \ d' = 2m, \\ 2dd' - 2md' + 2m^2 - d^2 - d & \text{if } \mathrm{G} = \mathrm{U}_d, \\ 8dm - 4m^2 + 2m - 2d^2 - 2d & \text{if } \mathrm{G} = \mathrm{Sp}_d,\ \ d' = 2m. \end{cases}$$

Hence, $F(d') \geq 0$ if $\mathrm{G} \neq \mathrm{O}_{2l+1}$ and $F(d'+1) \geq 0$ if $\mathrm{G} = \mathrm{O}_{2l+1}$. This verifies (251).

Since $m \leq d \leq d'$ if $\mathrm{G} \neq \mathrm{O}_{2l+1}$, and $m \leq d \leq d'+1$ if $\mathrm{G} = \mathrm{O}_{2l+1}$, the equality in (251) implies that $F(m) = 0$. But $F(m) = \dim \mathfrak{g} - \dim \mathfrak{h}$. Hence, (252) follows. □

Lemma 62. *Assume $d' \geq 2r$. If the pair $(\mathrm{G}, \mathrm{G}')$ is in the stable range with* G*-the smaller member, then one may normalize the positive orbital integral $\mu_{\mathcal{O}_m}$ so that*

$$\lim_{y \to 0} \frac{1}{C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}/\mathfrak{h}}(y)} f_\phi(y) = \mu_{\mathcal{O}_m}(\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W})),$$

where the limit is taken over $y \in \mathfrak{h}^{reg}$*. (The stable range assumption implies* $d' \geq 2r$*.) If the pair* $(\mathrm{G}, \mathrm{G}')$ *is not in the stable range with* G*-the smaller member then the limit is zero.*

Proof. Since G is compact, the orbit $\mathrm{G} \cdot y \subseteq \mathfrak{g}$ is compact and hence the Fourier transform

$$\hat{\mu}_{\mathrm{G}.y}(x) = \int_{\mathfrak{g}} e^{-iB(z,x)}\, d\mu_{\mathrm{G}.y}(z) \qquad (x \in \mathfrak{g})$$

is a uniformly bounded function as y varies through $\mathfrak{h}$. Furthermore, by (99)

$$\int_{\mathfrak{g}} \mathrm{ch}^{-d'}(x)\, dx = \int_{\mathfrak{h}} |\pi_{\mathfrak{g}/\mathfrak{h}}(x)|^2\, \mathrm{ch}^{-d'}(x)\, dx \leq \int_{\mathfrak{h}} \mathrm{ch}^{-d'+2r-2\iota}(x)\, dx \tag{253}$$
$$\leq \int_{\mathfrak{h}} \mathrm{ch}^{-2\iota}(x)\, dx < \infty.$$

Hence, by van der Corput estimate (162) for $\phi \in \mathcal{S}(\mathrm{W})$, the consecutive integrals in the following computation are absolutely convergent:

$$\begin{aligned}
&\int_{\mathfrak{g}} \hat{\mu}_{\mathrm{G}.y}(x) \int_{\mathrm{W}} \chi_x(w)\phi(w)\, dw\, dx \qquad (254)\\
=\ & C_1 \int_{\mathfrak{h}} \hat{\mu}_{\mathrm{G}.y}(x)\overline{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathrm{W}} \chi_x(w)\phi^{\mathrm{G}}(w)\, dw\, dx\\
=\ & C_2 \int_{\mathfrak{h}} \frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)} \sum_{s \in W(\mathrm{G},\mathfrak{h})} \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) e^{-iB(y,sx)} \int_{\mathfrak{h}} e^{iB(x,z)} f_\phi(z)\, dz\, dx\\
=\ & C_2 \int_{\mathfrak{h}} \frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)} \sum_{s \in W(\mathrm{G},\mathfrak{h})} e^{-iB(y,sx)} \int_{\mathfrak{h}} e^{iB(x,z)} f_\phi(sz)\, dz\, dx\\
=\ & C_2 \int_{\mathfrak{h}} \frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)} \sum_{s \in W(\mathrm{G},\mathfrak{h})} e^{-iB(y,sx)} \int_{\mathfrak{h}} e^{iB(sx,z)} f_\phi(z)\, dz\, dx\\
=\ & C_2 |W(\mathrm{G},\mathfrak{h})| \int_{\mathfrak{h}} \frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)} e^{-iB(y,x)} \int_{\mathfrak{h}} e^{iB(x,z)} f_\phi(z)\, dz\, dx\\
=\ & C_3 \frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)} f_\phi(y),
\end{aligned}$$

where C_1, C_2 and C_3 are some constants and in the second equality we used Lemmas 34 and 35. Hence,

$$\lim_{y \to 0} C_3 \frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)} f_\phi(y) = \lim_{y \to 0} \int_{\mathfrak{g}} \hat{\mu}_{\mathrm{G}.y}(x) \int_{\mathrm{W}} \chi_x(w)\phi(w)\, dw\, dx \tag{255}$$
$$= \int_{\mathfrak{g}} \hat{\mu}_0(x) \int_{\mathrm{W}} \chi_x(w)\phi(w)\, dw\, dx = \int_{\mathfrak{g}} \int_{\mathrm{W}} \chi_x(w)\phi(w)\, dw\, dx.$$

By definition of f_ϕ, (255) is a limit of S-invariant measures. Hence it is an S-invariant measure. The last expression shows that this measure is homogeneous of degree $-2 \dim \mathfrak{g}$.

Hence, by (129), the image of it under the map τ'_* is G'-invariant and homogeneous of degree $\frac{1}{2}(\dim \mathrm{W} - 2\dim \mathfrak{g}) - \dim \mathfrak{g}'$. Therefore that image is a multiple of a nilpotent orbital integral over a nilpotent orbit of dimension $\dim \mathrm{W} - 2\dim \mathfrak{g}$, (see [BV80]).

Under the assumption $d' \geq 2r$, if our pair is not in the stable range then $\mathbb{D} = \mathbb{C}$, i.e. the pair consists of the unitary groups. In this case Lemma 61 shows that

$$\dim \mathrm{W} - 2\dim \mathfrak{g} = \dim \mathcal{O}'_k \quad \text{where} \quad k = d \quad \text{or} \quad d' - d.$$

But the case $k = d$ is impossible, because $k \leq m$ and we assume that $m < d$. Hence the limit is a possibly zero multiple of the orbital integral over the orbit $\mathcal{O}_{d'-d}$, assuming $d' - d \leq m$. But the assumption $d' \geq 2r$ means that $d' \geq 2d$. Hence, $d' - d \geq d > m$. Thus the limit is zero.

Suppose our pair is in the stable range. Then $0 \in \mathrm{W}_{\mathfrak{g}}$. Furthermore, by Lemma 11 and the continuity of the pull-back τ^*,

$$\frac{1}{C_{\mathfrak{h}_{\bar{1}}}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)} f_\phi(y) = \tau^*(\mu_{\mathrm{G}.y})(\phi) \underset{y\to 0}{\to} \tau^*(\delta_0)(\phi) = \mu_{\mathcal{O}_m}(\phi) \qquad (\phi \in C_c^\infty(\mathrm{W}_{\mathfrak{g}})). \tag{256}$$

Clearly, (256) implies that the restriction of that measure to $\mathrm{W}_{\mathfrak{g}}$ is a multiple of $\mu_{\mathcal{O}}|_{\mathrm{W}_{\mathfrak{g}}}$. Also, the map τ'_* is injective (on invariant distributions). Hence, (256) is a non-zero multiple of $\mu_{\mathcal{O}_m}$. Notice that the left hand side of (256) is a non-negative measure. Thus our lemma follows. ☐

The two lemmas below shed some light at the connection of our limit formula in Lemma 62 and Rossmann's result [Ros90] concerning limts of nilpotent orbital integrals.

Lemma 63. *Suppose the pair* $(\mathrm{G}, \mathrm{G}')$ *is in the stable range with* G*-the smaller member. Define a polynomial* $p_{\mathcal{O}'} \in \mathbb{C}[\mathfrak{h}']$ *by*

$$p_{\mathcal{O}'}(y + y'') = \begin{cases} \pi_{\mathfrak{g}/\mathfrak{h}}(y)\pi_{\mathfrak{z}''/\mathfrak{h}''}(y'') & \text{if } \mathrm{G} \neq \mathrm{O}_{2l+1}, \\ \pi_{\mathfrak{g}/\mathfrak{h}}(y)\pi^{short}_{\mathfrak{z}''/\mathfrak{h}''}(y'') & \text{if } \mathrm{G} = \mathrm{O}_{2l+1}, \end{cases}$$

where $y \in \mathfrak{h}$, $y'' \in \mathfrak{h}''$ *and* $\pi^{short}_{\mathfrak{z}''/\mathfrak{h}''}$ *is the product of the short roots of* $\mathfrak{h}''$ *in* $\mathfrak{z}''_{\mathbb{C}}$*. Then* $p_{\mathcal{O}'}$ *generates the representation of the Weyl group* $W(\mathrm{G}'_{\mathbb{C}}, \mathfrak{h}'_{\mathbb{C}})$ *corresponding to the orbit* $\mathcal{O}'_{\mathbb{C}}$ *via Springer correspondence, as explained in* [AKP13, sec.4].

Proof. We see from (58) that the polynomial in question is equal to

$$\begin{cases} \left(\prod_{1\leq j<k\leq l} i(-y_j + y_k)\right) \cdot \left(\prod_{l<j<k\leq l'} i(-y_j + y_k)\right) & \text{if } \mathbb{D} = \mathbb{C}, \\ \left(\prod_{1\leq j<k\leq l}(-y_j^2 + y_k^2) \cdot \prod_{j=1}^{l} 2iy_j\right) \cdot \left(\prod_{l<j<k\leq l'}(-y_j^2 + y_k^2)\right) & \text{if } \mathbb{D} = \mathbb{H}, \\ \left(\prod_{1\leq j<k\leq l}(-y_j^2 + y_k^2)\right) \cdot \left(\prod_{l<j<k\leq l'}(-y_j^2 + y_k^2) \cdot \prod_{j=l+1}^{l'} 2iy_j\right) & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l}, \\ \left(\prod_{1\leq j<k\leq l}(-y_j^2 + y_k^2) \cdot \prod_{j=1}^{l} iy_j\right) \cdot \left(\prod_{l<j<k\leq l'}(-y_j^2 + y_k^2)\right) & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l+1}, \end{cases} \tag{257}$$

where the parenthesis separate the factors in the definition of $p_{\mathcal{O}'}$. In terms of [AKP13, sec.3] the above products may be rewritten, up to a constant multiple, as

$$\begin{array}{ll} \Delta_{(1^l)}(y_1,\dots,y_l)\Delta_{(1^{l'-l})}(y_{l+1},\dots,y_{l'}) & \text{if } \mathbb{D}=\mathbb{C},\\ \Delta_{(\emptyset,1^l)}(y_1,\dots,y_l)\Delta_{(1^{l'-l},\emptyset)}(y_{l+1},\dots,y_{l'}) & \text{if } \mathbb{D}=\mathbb{H},\\ \Delta_{(1^l,\emptyset)}(y_1,\dots,y_l)\Delta_{(\emptyset,1^{l'-l})}(y_{l+1},\dots,y_{l'}) & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l},\\ \Delta_{(\emptyset,1^l)}(y_1,\dots,y_l)\Delta_{(1^{l'-l},\emptyset)}(y_{l+1},\dots,y_{l'}) & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}, \end{array} \tag{258}$$

Hence the lemma follows from [AKP13, Theorem 12]. □

Observe that for dual pairs in the stable range and for $\psi\in\mathcal{S}(\mathfrak{g}')$ Rossmann's formula, [Ros90] indicates that

$$\lim_{y'\to 0}\partial(p_{\mathcal{O}'})\pi_{\mathfrak{g}'/\mathfrak{h}'}(y')\int_{\mathrm{G}'}\psi(g.y')\,dy = C_1\mu_{\mathcal{O}'}(\psi) \tag{259}$$

and Lemma 62 shows that

$$\lim_{y\to 0}\frac{1}{\pi_{\mathfrak{g}/\mathfrak{h}}(y)}\lim_{y''\to 0}\partial(\tilde\pi_{\mathfrak{z}''/\mathfrak{h}''})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\psi(g.(y+y''))\,dy\right) = C_2\mu_{\mathcal{O}'}(\psi), \tag{260}$$

where $y\in\tau(\mathfrak{h}_{\overline{1}}^{reg})$, $\tilde\pi_{\mathfrak{z}''/\mathfrak{h}''}=\pi_{\mathfrak{z}''/\mathfrak{h}''}$ if $\mathrm{G}\neq\mathrm{O}_{2l+1}$, and $\tilde\pi_{\mathfrak{z}''/\mathfrak{h}''}=\pi^{short}_{\mathfrak{z}''/\mathfrak{h}''}$ if $\mathrm{G}=\mathrm{O}_{2l+1}$. In fact (260) is a stronger version of (259) because, in general, if p is a product of linear factors vanishing at 0 and F has limit $F(0)$ at 0, then $[\partial(p)(pF)](0)=\partial(p)(p)\,F(0)$.

Appendix A: A few facts about nilpotent orbits

Let $\mathfrak{g}'$ be a semisimple Lie algebra over $\mathbb{C}$. Then there is a unique non-zero nilpotent orbit in $\mathfrak{g}'$ of minimal dimension which is contained in the closure of any non-zero nilpotent orbit, [CM93, Theorem 4.3.3, Remark 4.3.4]. The dimension of that orbit is equal to one plus the number of positive roots not orthogonal to the highest root, relative to a choice of a Cartan subalgebra and a choice of positive roots, [CM93, Lemma 4.3.5]. Thus in the case $\mathfrak{g}'=\mathfrak{sp}_{2l}(\mathbb{C})$, the dimension of the minimal non-zero nilpotent orbit is equal to $2l$. This is precisely the dimension of the defining module for the symplectic group $\mathrm{Sp}_{2l}(\mathbb{C})$, which may be viewed as the symplectic space for the dual pair $(\mathrm{O}_1,\mathrm{Sp}_{2l}(\mathbb{C}))$.

Consider the dual pair $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_1,\mathrm{Sp}_{2l}(\mathbb{R}))$, with the symplectic space W and the unnormalized moment map $\tau':\mathrm{W}\to\mathfrak{g}'$. Since $\mathrm{W}\setminus\{0\}$ is a single G'-orbit, so is $\tau'(\mathrm{W}\setminus\{0\})$. Further, $\dim(\tau'(\mathrm{W}\setminus\{0\}))=\dim(\mathrm{W})=2l$. Hence, $\tau'(\mathrm{W}\setminus\{0\})\subseteq\mathfrak{g}'$ is a minimal non-zero G'-orbit. In fact, there are only two such orbits, [CM93, Theorem 9.3.5]. In terms of dual pairs, the second one is obtained from the same pair, with the symplectic form replaced by its negative (or equivalently the symmetric form on the defining module for O_1 replaced by its negative).

Consider an irreducible dual pair (G,G') with G compact. Denote by l the dimension of a Cartan subalgebra of $\mathfrak{g}$ and by l' the dimension of a Cartan subalgebra of $\mathfrak{g}'$. Let us identify the corresponding symplectic space W with $\mathrm{Hom}(\mathsf{V}_{\overline{1}},\mathsf{V}_{\overline{0}})$ as in [Prz91, sec.2].

Recall that $\mathrm{W}_{\mathfrak{g}}$ denotes the maximal subset of W on which the restriction of the unnormalized moment map $\tau:\mathrm{W}\to\mathfrak{g}$ is a submersion. Then [Prz91, Lemma 2.6] shows that $\mathrm{W}_{\mathfrak{g}}$ consists of all the elements $w \in \mathrm{W}$ such that for any $x \in \mathfrak{g}$,

$$xw = 0 \text{ implies } x = 0\,. \tag{A.1}$$

The condition (A.1) means that x restricted to the image of w is zero. But in that case x preserves the orthogonal complement of that image. Thus we need to know that the Lie algebra of the isometries of that orthogonal complement is zero. This happens if w is surjective or if G is the orthogonal group and the dimension of the image of w in $\mathsf{V}_{\overline{0}}$ is $\geq \dim(\mathsf{V}_{\overline{0}}) - 1$. Thus

$$\mathrm{W}_{\mathfrak{g}} \neq \emptyset \text{ if and only if } l \leq l'. \tag{A.2}$$

Consider in particular the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_3, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $1 \leq l'$. We see from the above discussion that $\mathrm{W}_{\mathfrak{g}}$ consists of elements of rank ≥ 2. Hence, $\mathrm{W} \setminus (\mathrm{W}_{\mathfrak{g}} \cup \{0\})$ consists of elements w or rank equal 1. By replacing $\mathsf{V}_{\overline{0}}$ with the image of w we may consider w as an element of the symplectic space for the pair $(\mathrm{O}_1, \mathrm{Sp}_{2l'})$. Hence the image of w under the moment map generates a minimal non-zero nilpotent orbit in $\mathfrak{g}'$.

If $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$, with $1 \leq l'$, then $\mathrm{W}_{\mathfrak{g}}$ consists of elements of rank ≥ 1. Therefore $\mathrm{W} \setminus \mathrm{W}_{\mathfrak{g}} = \{0\}$.

Appendix B: Pull-back of a distribution via a submersion

We collect here some textbook results which are attributed to Ranga Rao in [BV80] and in [Har11]. These results date back to the time before the textbook [Hör83] was available.

We shall use the definition of a smooth manifold and a distribution on a smooth manifold as described in [Hör83, sec. 6.3]. Thus, if M is a smooth manifold of dimension m and

$$M \supseteq M_\kappa \overset{\kappa}{\to} \tilde{M}_\kappa \subseteq \mathbb{R}^m$$

is any coordinate system on M, then a distribution u on M is the collection of distributions $u_\kappa \in \mathcal{D}'(\tilde{M}_\kappa)$ such that

$$u_{\kappa_1} = (\kappa \circ \kappa_1^{-1})^* u_\kappa. \tag{B.1}$$

Suppose W is another smooth manifold of dimension n and v is a distribution on W. Thus for any coordinate system

$$W \supseteq W_\lambda \overset{\lambda}{\to} \tilde{W}_\lambda \subseteq \mathbb{R}^n$$

we have a distribution $v_\lambda \in \mathcal{D}'(\tilde{W}_\lambda)$ such that the condition (B.1) holds. Suppose

$$\sigma : M \to W$$

is a submersion. Then for every κ there is a unique distribution $u_\kappa \in \mathcal{D}'(\tilde{M}_\kappa)$ such that

$$u_\kappa|_{(\lambda\circ\sigma\circ\kappa^{-1})^{-1}(\tilde{W}_\lambda)} = (\lambda \circ \sigma \circ \kappa^{-1})^* v_\lambda. \tag{B.2}$$

Since

$$(\kappa \circ \kappa_1^{-1})^* (\lambda \circ \sigma \circ \kappa^{-1})^* v_\lambda = (\lambda \circ \sigma \circ \kappa^{-1} \circ \kappa \circ \kappa_1^{-1})^* v_\lambda = (\lambda \circ \sigma \circ \kappa_1^{-1})^* v_\lambda$$

the u_κ satisfy the condition (B.1). The resulting distribution u is denoted by $\sigma^* v$ and is called the pullback of v from W to M via σ.

Proposition B.1. *Let M and W be smooth manifolds and let $\sigma : M \to W$ be a surjective submersion. Suppose $u_n \in \mathcal{D}'(W)$ is a sequence of distributions such that*

$$\lim_{n\to\infty} \sigma^* u_n = 0 \qquad \text{in } \mathcal{D}'(M). \tag{B.3}$$

Then

$$\lim_{n\to\infty} u_n = 0 \qquad \text{in } \mathcal{D}'(W). \tag{B.4}$$

In particular the map $\sigma^* : \mathcal{D}'(W) \to \mathcal{D}'(M)$ is injective.

More generally, if $u_n \in \mathcal{D}'(W)$ and $\tilde{u} \in \mathcal{D}'(M)$ are such that

$$\lim_{n\to\infty} \sigma^* u_n = \tilde{u} \qquad \text{in } \mathcal{D}'(M), \tag{B.5}$$

then there is a distribution $u \in \mathcal{D}'(W)$ such that

$$\lim_{n\to\infty} u_n = u \qquad \text{in } D'(W) \tag{B.6}$$

and $\tilde{u} = \sigma^* u$.

Proof. By the definition of a distribution on a manifold, as in [Hör83, sec.6.3], we may assume that M is an open subset of $\mathbb{R}^m$ and W is an open subset of $\mathbb{R}^n$.

We recall the definition of the pull-back

$$\sigma^* : \mathcal{D}'(W) \to \mathcal{D}'(M) \tag{B.7}$$

from the proof of Theorem 6.1.2 in [Hör83]. Fix a point $x_0 \in M$ and a smooth map $g : M \to \mathbb{R}^{m-n}$ such that

$$\sigma \oplus g : M \to \mathbb{R}^n \times \mathbb{R}^{m-n}$$

has a bijective differential at x_0. By Inverse Function Theorem there is an open neighborhood M_0 of x_0 in M such that

$$(\sigma \oplus g)\,|_{M_0} : M_0 \to Y_0$$

is a diffeomerphism onto an open neighborhood Y_0 of $\sigma \oplus g(x_0) = (\sigma(x_0), g(x_0))$ in $\mathbb{R}^n \times \mathbb{R}^{m-n}$. Let

$$h : Y_0 \to M_0$$

denote the inverse. For $\phi \in C_c^\infty(M_0)$ define $\Phi \in C_c^\infty(Y_0)$ by

$$\Phi(y) = \phi(h(y))|\det h'(y)| \qquad (y \in Y_0). \tag{B.8}$$

Then

$$\sigma^* u(\phi) = u \otimes 1(\Phi) \qquad (u \in \mathcal{D}'(W),\ \phi \in C_c^\infty(M_0)). \tag{B.9}$$

By localization this gives the pull-back (B.7).

Let W_0 be an open neighborhood of $\sigma(x_0)$ in W and let X_0 be an open neighborhood of $g(x_0)$ in $\mathbb{R}^{m-n}$ such that

$$W_0 \times X_0 \subseteq Y_0.$$

Fix a function $\eta \in C_c^\infty(X_0)$ such that

$$\int_{X_0} \eta(x)\, dx = 1.$$

Given $\psi \in C_c^\infty(W_0)$ define $\phi \in C_c^\infty(M_0)$ by

$$\Phi(x', x'') = \psi(x')\eta(x'') \qquad (x' \in W_0,\ x'' \in X_0),$$

where Φ is related to ϕ via (B.8). Then

$$\sigma^* u(\phi) = u(\psi).$$

Hence the assumption (B.3) implies

$$\lim_{n\to\infty} u_n(\psi) = 0 \qquad (\psi \in C_c^\infty(W_0)).$$

Thus, by [Hör83, Theorem 2.1.8],

$$\lim_{n\to\infty} u_n|_{W_0} = 0$$

in $\mathcal{D}'(W_0)$. Since the point $x_0 \in M$ is arbitrary, the claim (B.4) follows by localization.

Similarly, the assumption (B.5) implies that for any $\psi \in C_c^\infty(W_0)$

$$\lim_{n\to\infty} u_n(\psi) = \lim_{n\to\infty} \sigma^* u_n(\phi) = \tilde{u}(\phi)$$

exists. Thus, by [Hör83, Theorem 2.1.8], there is $u \in \mathcal{D}'(W_0)$ such that

$$\lim_{n\to\infty} u_n|_{W_0} = u.$$

By the continuity of σ^*, $\sigma^* u = \tilde{u}$. Again, since the point $x_0 \in M$ is arbitrary, the claim follows by localization. □

Lemma B.2. *Let M and W be smooth manifolds and let $\sigma : M \to W$ be a surjective submersion. Then for any smooth differential operator D on W there is, not necessary unique, smooth differential operator $\sigma^* D$ on M such that*

$$\sigma^*(u \circ D) = (\sigma^* u) \circ (\sigma^* D) \qquad (u \in \mathcal{D}'(W)).$$

If D annihilates constants then so does $\sigma^* D$. The operator $\sigma^* D$ is unique if σ is a diffeomerphism.

Proof. Suppose σ is a diffeomerphism between two open subsets of $\mathbb{R}^n$. Then

$$\sigma^* u(\phi) = u(\phi \circ \sigma^{-1} |\det((\sigma^{-1})')|) \qquad (\phi \in C_c^\infty(M)).$$

Let

$$(\sigma^* D)(\phi) = (D(\phi \circ \sigma^{-1})) \circ \sigma \qquad (\phi \in C_c^\infty(M)).$$

Hence

$$\begin{aligned}
\sigma^*(u \circ D)(\phi) &= (u \circ D)(\phi \circ \sigma^{-1} |\det((\sigma^{-1})')|) \\
&= u(D(\phi \circ \sigma^{-1} |\det((\sigma^{-1})')|)) \\
&= u(((D(\phi \circ \sigma^{-1}) \circ \sigma) \circ \sigma^{-1} |\det((\sigma^{-1})')|))
\end{aligned}$$

Using the local classification of the submersions modulo the diffeomerphism [Die71, 16.7.4], we may assume that σ is a linear projection

$$\sigma : \mathbb{R}^{m+n} \ni (x, y) \to x \in \mathbb{R}^n,$$

in which case the lemma is obvious. □

Suppose M is a Lie group. Then there are functions $m_\kappa \in C^\infty(\tilde{M}_\kappa)$ such that the formula

$$\int_M \phi \circ \kappa(y)\, d\mu_M(y) = \int_{\tilde{M}_\kappa} \phi(x) m_\kappa(x)\, dx \qquad (\phi \in C_c^\infty(\tilde{M}_\kappa)) \tag{B.10}$$

defines a left invariant Haar measure on M. We shall tie the normalization of the Haar measure $d\mu_M(y)$ on M to the normalization of the Lebesgue measure dx on $\mathbb{R}^m$ by requiring that near the identity,

$$m_{\exp^{-1}}(x) = \det\left(\frac{1 - e^{-\operatorname{ad}(x)}}{\operatorname{ad}(x)}\right), \tag{B.11}$$

as in [Hel84, Theorem 1.14, page 96]. Collectively, the distributions $m_\kappa(x)\, dx \in \mathcal{D}'(\tilde{M}_\kappa)$ form a distribution density on M. (See [Hör83, sec. 6.3] for the definition of a distribution density.)

Suppose W is another Lie group with the left Haar measure given by

$$\int_W \psi \circ \lambda(y)\, d\mu_W(y) = \int_{\tilde{W}_\lambda} \phi(x) w_\lambda(x)\, dx \qquad (\psi \in C_c^\infty(\tilde{W}_\lambda)),$$

and let $\sigma : M \to W$ be a submersion. Given any distribution density $v_\lambda \in \mathcal{D}'(\tilde{W}_\lambda)$ we associate to it a distribution on W given by $\frac{1}{w_\lambda} v_\lambda \in \mathcal{D}'(\tilde{W}_\lambda)$. We may pullback this distribution to M and obtain another distribution. Then we multiply by the m_κ and obtain a distribution density. Thus, if $\sigma : M_\kappa \to W_\lambda$ then

$$(\sigma^* v)_\kappa = m_\kappa(\lambda \circ \sigma \circ \kappa^{-1})^*(\frac{1}{w_\lambda} v_\lambda). \tag{B.12}$$

Distribution densities on W are identified with the continuous linear forms on $C_c^\infty(W)$ by

$$v(\psi \circ \lambda) = v_\lambda(\psi) \qquad (\psi \in C_c^\infty(\tilde{W}_\lambda)).$$

(Here v stands for the corresponding continuous linear form.) In particular if $F \in C(W)$. then $F\mu_W$ is a continuous linear form on $C_c^\infty(W)$ and for $\psi \in C_c^\infty(\tilde{W}_\lambda)$,

$$\begin{aligned}(F\mu_W)_\lambda(\psi) &= (F\mu_W)(\psi \circ \lambda) = \int_W \psi \circ \lambda(y) F(y)\, d\mu_W(y) \\ &= \int_{\tilde{W}_\lambda} \psi(x) F \circ \lambda^{-1}(x) w_\lambda(x)\, dx.\end{aligned}$$

Hence, for $\phi \in C_c^\infty(\tilde{M}_\kappa)$, with $\sigma : M_\kappa \to W_\lambda$,

$$\begin{aligned}(\sigma^*(F\mu_W))_\kappa(\phi) &= (\lambda \circ \sigma \circ \kappa^{-1})^*(\frac{1}{w_\lambda}(F\mu_W)_\lambda)(m_\kappa \phi) \\ &= \int_{\tilde{M}_\kappa} m_\kappa(x)\phi(x) F \circ \lambda^{-1} \circ (\lambda \circ \sigma \circ \kappa^{-1})(x)\, dx \\ &= \int_{\tilde{M}_\kappa} \phi(x)(F \circ \sigma) \circ \kappa^{-1}(x) m_\kappa(x)\, dx \\ &= \int_M \phi \circ \kappa(y)(F \circ \sigma)(y)\, d\mu_M(y)\end{aligned}$$

Thus

$$\sigma^*(F\mu_W) = F \circ \sigma \mu_M. \tag{B.13}$$

As explained above, we identify $\mathcal{D}'(M)$ with the space of the continuous linear forms on $C_c^\infty(M)$ and similarly for W and obtain

$$\sigma^* : \mathcal{D}'(M) \to \mathcal{D}'(W) \tag{B.14}$$

as the unique continuous extension of (B.13). Our identification of distribution densities with continuous linear forms on on the space of the smooth compactly supported functions applies also to submanifolds of Lie groups.

Let S be a Lie group acting on another Lie group W and let $U \subseteq W$ be a submanifold. (In our applications W is going to be a vector space.) We shall consider the following function

$$\sigma : \mathrm{S} \times U \ni (s, u) \to su \in W. \tag{B.15}$$

The following fact is easy to check

Lemma B.3. *If $\mathcal{O} \subseteq W$ is an S-orbit then $\sigma^{-1}(\mathcal{O}) = \mathrm{S} \times (\mathcal{O} \cap U)$.*

Assume that the map (B.15) is submersive. Let us fix Haar measures on S and on W so that the pullback

$$\sigma^* : \mathcal{D}'(W) \to \mathcal{D}'(\mathrm{S} \times U)$$

is well defined, as in (B.14). Denote by $\mathrm{S}^U \subseteq \mathrm{S}$ the stabilizer of U.

Lemma B.4. *Assume that the map (B.15) is submersive and surjective. Let $\mathcal{O} \subseteq W$ be an S-orbit and let $\mu_\mathcal{O} \in \mathcal{D}'(W)$ be an S-invariant positive measure supported on the closure on $\mathcal{O}$. Let $\mu_\mathcal{O}|_U \in \mathcal{D}'(U)$ be the restriction of $\mu_\mathcal{O}$ to U in the sense of* [Hör83, Cor. 8.2.7]. *Then $\mu_\mathcal{O}|_U$ is a positive S^U-invariant measure supported on the closure of $\mathcal{O} \cap U$ in U. Moreover,*

$$\sigma^*\mu_\mathcal{O} = \mu_\mathrm{S} \otimes \mu_\mathcal{O}|_U. \tag{B.16}$$

Proof. Let $s \in \mathrm{S}^U$. Then

$$s^*(\mu_\mathcal{O}|_U) = (s^*\mu_\mathcal{O})|_U = \mu_\mathcal{O}|_U.$$

Hence the distribution $\mu_\mathcal{O}|_U$ is S^U-invariant. Lemma B.1 implies that $\mu_\mathcal{O}|_U \neq 0$ and Lemma B.3 that $\mu_\mathcal{O}|_U$ is supported in the closure of $\mathcal{O} \cap U$ in U. Since the pullback of a positive measure is a non-negative measure, $\mu_{\mathcal{O}}|_U$ is a positive S^U-invariant measure supported on the closure of $\mathcal{O} \cap U$ in U.

Theorem 3.1.4' in [Hör83] implies that there is a positive measure $\mu_{\mathcal{O}\cap U}$ on U such that

$$\sigma^*\mu_{\mathcal{O}} = \mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U}.$$

Consider the embedding

$$\sigma_1 : U \ni u \to (1, u) \in \mathrm{S} \times U.$$

Then $\sigma \circ \sigma_1 : U \to W$ is the inclusin of U into W. Hence,

$$(\sigma \circ \sigma_1)^*\mu_{\mathcal{O}} = \mu_{\mathcal{O}}|_U.$$

The conormal bundle to σ_1, as defined in [Hör83, Theorem 8.2.4], is equal to

$$N_{\sigma_1} = T^*_{\{1\}\times U} = T^*(\mathrm{S}) \times 0 \subseteq T^*(\mathrm{S}) \times T^*(U) = T^*(\mathrm{S} \times U).$$

By the S-invariance of $\sigma^*\mu_{\mathcal{O}}$,

$$WF(\mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U}) \subseteq 0 \times T^*(U) \subseteq T^*(\mathrm{S} \times U).$$

Hence

$$N_{\sigma_1} \cap WF(\mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U}) = 0.$$

Therefore

$$\mu_{\mathcal{O}}|_U = (\sigma \circ \sigma_1)^*\mu_{\mathcal{O}} = \sigma_1^* \circ \sigma^*\mu_{\mathcal{O}} = \sigma_1^*(\mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U}) = \mu_{\mathcal{O}\cap U}.$$

This implies (B.16). □

Appendix C: Some confluent hypergeometric polynomials

For two integers a and b define the following functions in the real variable ξ,

$$P_{a,b,2}(\xi) = \begin{cases} \sum_{k=0}^{b-1} \frac{a(a+1)...(a+k-1)}{k!(b-1-k)!} 2^{-a-k}\xi^{b-1-k} & \text{if } b \geq 1 \\ 0 & \text{if } b \leq 0, \end{cases} \tag{C.1}$$

$$P_{a,b,-2}(\xi) = \begin{cases} (-1)^{a+b-1} \sum_{k=0}^{a-1} \frac{b(b+1)...(b+k-1)}{k!(a-1-k)!} (-2)^{-b-k}\xi^{a-1-k} & \text{if } a \geq 1 \\ 0 & \text{if } a \leq 0, \end{cases} \tag{C.2}$$

where $a(a+1)...(a+k-1) = 1$ if $k = 0$. Notice that

$$P_{a,b,-2}(\xi) = P_{b,a,2}(-\xi) \qquad (\xi \in \mathbb{R},\ a, b \in \mathbb{Z}). \tag{C.3}$$

Set

$$\begin{aligned} P_{a,b}(\xi) &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi) + P_{a,b,-2}(\xi)\mathbb{I}_{\mathbb{R}^-}(\xi)) \\ &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi) + P_{b,a,2}(-\xi)\mathbb{I}_{\mathbb{R}^+}(-\xi)), \end{aligned} \tag{C.4}$$

where $\mathbb{I}_S$ denotes the indicator function of the set S. Also, let

$$Q_{a,b}(iy) = 2\pi \begin{cases} 0 & \text{if } a + b \geq 1, \\ \sum_{k=b}^{-a} \frac{a(a+1)...(a+k-1)}{k!} 2^{-a-k}(1 - iy)^{k-b} & \text{if } -a > b - 1 \geq 0, \\ \sum_{k=a}^{-b} \frac{b(b+1)...(b+k-1)}{k!} 2^{-b-k}(1 + iy)^{k-a} & \text{if } -b > a - 1 \geq 0, \\ (1 + iy)^{-a}(1 - iy)^{-b} & \text{if } a \leq 0 \text{ and } b \leq 0. \end{cases} \tag{C.5}$$

Proposition C.1. *For any $a,\ b \in \mathbb{Z}$, the formula*

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b}\phi(y)\,dy \qquad (\phi \in \mathcal{S}(\mathbb{R})) \tag{C.6}$$

defines a tempered distribution on $\mathbb{R}$. The restriction of the Fourier transform of this distribution to $\mathbb{R}\setminus\{0\}$ is a function given by

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b}e^{-iy\xi}\,dy = P_{a,b}(\xi)e^{-|\xi|}. \tag{C.7}$$

The right hand side of (C.7) is an absolutely integrable function on the real line and thus defines a tempered distribution on $\mathbb{R}$. Furthermore,

$$(1+iy)^{-a}(1-iy)^{-b} = \frac{1}{2\pi}\int_{\mathbb{R}} P_{a,b}(\xi)e^{-|\xi|}e^{iy\xi}\,dy + \frac{1}{2\pi}Q_{a,b}(iy) \tag{C.8}$$

and hence,

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b}e^{-iy\xi}\,dy = P_{a,b}(\xi)e^{-|\xi|} + Q_{a,b}(-\frac{d}{d\xi})\delta_0(\xi). \tag{C.9}$$

Proof. Since, $|1\pm iy| = \sqrt{1+y^2}$, (C.6) is clear. The integral (C.7) is equal to

$$\begin{aligned} &\frac{1}{i}\int_{i\mathbb{R}} (1+z)^{-a}(1-z)^{-b}e^{-z\xi}\,dz && \text{(C.10)}\\ =\ & 2\pi(-\mathbb{I}_{\mathbb{R}^+}(\xi)\,res_{z=1}(1+z)^{-a}(1-z)^{-b}e^{-z\xi} + \mathbb{I}_{\mathbb{R}^-}(\xi)\,res_{z=-1}(1+z)^{-a}(1-z)^{-b}e^{-z\xi}). \end{aligned}$$

The computation of the two residues is straightforward and (C.7) follows.

Since

$$\int_0^\infty e^{-\xi}e^{i\xi y}\,d\xi = (1-iy)^{-1},$$

we have

$$\int_0^\infty \xi^m e^{-\xi}e^{i\xi y}\,d\xi = \left(\frac{d}{d(iy)}\right)^m (1-iy)^{-1} = m!(1-iy)^{-m-1} \qquad (m=0,1,2,...). \tag{C.11}$$

Thus, if $b \geq 1$, then

$$\begin{aligned} \int_0^\infty P_{a,b,2}(\xi)e^{-\xi}e^{i\xi y}\,d\xi &= \sum_{k=0}^{b-1} \frac{a(a+1)...(a+k-1)}{k!}2^{-a-k}(1-iy)^{-b+k}\\ =\ & (1-iy)^{-b}2^{-a}\sum_{k=0}^{b-1}\frac{(-a)(-a-1)...(-a-k+1)}{k!}\left(-\frac{1}{2}(1-iy)\right)^k. \end{aligned}$$

Also, if $a \leq 0$, then

$$\begin{aligned} 2^a(1+iy)^{-a} &= \left(1-\frac{1}{2}(1-iy)\right)^{-a} = \sum_{k=0}^{-a}\binom{-a}{k}\left(-\frac{1}{2}(1-iy)\right)^k\\ =\ & \sum_{k=0}^{-a}\frac{(-a)(-a-1)...(-a-k+1)}{k!}\left(-\frac{1}{2}(1-iy)\right)^k. \end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^\infty P_{a,b,2}(\xi)e^{-\xi}e^{i\xi y}\,d\xi - (1+iy)^{-a}(1-iy)^{-b} \qquad \text{(C.12)}\\
= &\ (1-iy)^{-b}2^{-a}\left(\sum_{k=0}^{b-1}\frac{(-a)(-a-1)...(-a-k+1)}{k!}\left(-\frac{1}{2}(1-iy)\right)^k\right.\\
&\left. -\sum_{k=0}^{-a}\frac{(-a)(-a-1)...(-a-k+1)}{k!}\left(-\frac{1}{2}(1-iy)\right)^k\right).
\end{aligned}$$

Recall that $P_{a,b,-2}=0$ if $a\leq 0$. Hence, (C.7) shows that (C.12) is the inverse Fourier transform of a distribution supported at $\{0\}$ - a polynomial.

Suppose $-a<b-1$. Then (C.12) is equal to

$$2^{-a}\sum_{k=-a+1}^{b-1}\frac{(-a)(-a-1)...(-a-k+1)}{k!}\left(-\frac{1}{2}\right)^k(1-iy)^{k-b},$$

which tends to zero if y goes to infinity. The only polynomial with this property is the zero polynomial. Thus in this case (C.12) is zero. If $-a=b-1$, then (C.12) is obviously zero.

Suppose $-a>b-1$. Then (C.12) is equal to

$$-2^{-a}\sum_{k=b}^{-a}\frac{(-a)(-a-1)...(-a-k+1)}{k!}\left(-\frac{1}{2}\right)^k(1-iy)^{k-b}. \qquad \text{(C.13)}$$

As in (C.11) we have

$$\int_{-\infty}^0 \xi^m e^{\xi}e^{i\xi y}\,d\xi = \left(\frac{d}{d(iy)}\right)^m(1+iy)^{-1} = (-1)^m m!(1+iy)^{-m-1} \qquad (m=0,1,2,...).$$

Suppose $a\geq 1$. Then

$$\begin{aligned}
&\int_{-\infty}^0 P_{a,b,-2}(\xi)e^{\xi}e^{i\xi y}\,d\xi = (-1)^{a+b-1}\sum_{k=0}^{a-1}\frac{b(b+1)...(b+k-1)}{k!}(-2)^{-b-k}(-1)^{a-1+k}(1+iy)^{-a+k}\\
= &\ (1+iy)^{-a}2^{-b}\sum_{k=0}^{a-1}\frac{(-b)(-b-1)...(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^k.
\end{aligned}$$

Also, if $b\leq 0$, then

$$2^b(1-iy)^{-b} = \sum_{k=0}^{-b}\frac{(-b)(-b-1)...(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^k.$$

Hence,

$$\begin{aligned}
&\int_{-\infty}^{0} P_{a,b,-2}(\xi)e^{\xi}e^{i\xi y}\,d\xi - (1+iy)^{-a}(1-iy)^{-b} \qquad \text{(C.14)}\\
= \;& (1+iy)^{-a}2^{-b}\left(\sum_{k=0}^{a-1}\frac{(-b)(-b-1)...(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^{k}\right.\\
&\left. - \sum_{k=0}^{-b}\frac{(-b)(-b-1)...(-b-k+1)}{k!}\left(-\frac{1}{2}(1+iy)\right)^{k}\right).
\end{aligned}$$

As before, we show that (C.14) is zero if $-b \leq a-1$. If $-b > a-1$, then (C.14) is equal to

$$-2^{-b}\sum_{k=a}^{-b}\frac{(-b)(-b-1)...(-b-k+1)}{k!}\left(-\frac{1}{2}\right)^{k}(1+iy)^{k-a}.$$

If $a \geq 1$ and $b \geq 1$, then our computations show that

$$\int_{0}^{\infty} P_{a,b,2}(\xi)e^{-\xi}e^{i\xi y}\,d\xi + \int_{-\infty}^{0} P_{a,b,-2}(\xi)e^{\xi}e^{i\xi y}\,d\xi - (1+iy)^{-a}(1-iy)^{-b} \qquad \text{(C.15)}$$

is a polynomial which tends to zero if y goes to infinity. Thus (C.15) is equal zero. This completes the proof of (C.8). The statement (C.9) is a direct consequence of (C.8). $\square$

The test functions which occur in Proposition C.1 don't need to be in the Schwartz space. In fact the test functions we shall use in our applications are not necessarily smooth. Therefore we'll need a more precise version of the formula (C.9). This requires a definition and two well known lemmas.

Following Harish-Chandra denote by $\mathcal{S}(\mathbb{R}^{\times})$ the space of the smooth complex valued functions defined on $\mathbb{R}^{\times}$ whose all derivatives are rapidly decreasing at infinity and have limits at zero from both sides. For $\psi \in \mathcal{S}(\mathbb{R}^{\times})$ let

$$\psi(0+) = \lim_{x\to 0+}\psi(\xi),\ \psi(0+) = \lim_{x\to 0-}\psi(\xi),\ \langle\psi\rangle_0 = \psi(0+) - \psi(0-).$$

In particular the condition $\langle\psi\rangle_0 = 0$ means that ψ extends to a continuous function on $\mathbb{R}$.

Lemma C.2. *Let $c = 0, 1, 2, \dots$ and let $\psi \in \mathcal{S}(\mathbb{R}^{\times})$. Suppose*

$$\langle\psi\rangle_0 = \cdots = \langle\psi^{(c-1)}\rangle_0 = 0. \qquad \text{(C.16)}$$

(The condition (C.16) is empty if $c = 0$.) Then

$$\left|\int_{\mathbb{R}^{\times}} e^{-iy\xi}\psi(\xi)\,d\xi\right| \leq \min\{1, |y|^{-c-1}\}(|\langle\psi^{(c)}\rangle_0| + \parallel \psi^{(c+1)} \parallel_1 + \parallel \psi \parallel_1) \qquad \text{(C.17)}$$

Proof. Integration by parts shows that for $z \in \mathbb{C}^{\times}$

$$\int_{\mathbb{R}^{+}} e^{-z\xi}\psi(\xi)\,d\xi = z^{-1}\psi(0+) + \cdots + z^{-c-1}\psi^{(c)}(0+) + z^{-c-1}\int_{\mathbb{R}^{+}} e^{-z\xi}\psi^{(c+1)}(\xi)\,d\xi,$$

$$\int_{\mathbb{R}^{-}} e^{-z\xi}\psi(\xi)\,d\xi = -z^{-1}\psi(0-) - \cdots - z^{-c-1}\psi^{(c)}(0-) + z^{-c-1}\int_{\mathbb{R}^{-}} e^{-z\xi}\psi^{(c+1)}(\xi)\,d\xi.$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^\times} e^{-z\xi}\psi(\xi)\,d\xi\\ =\;& z^{-1}\langle\psi\rangle_0+\cdots+z^{-c}\langle\psi^{(c-1)}\rangle_0+z^{-c-1}\langle\psi^{(c)}\rangle_0+z^{-c-1}\int_{\mathbb{R}^\times} e^{-z\xi}\psi^{(c+1)}(\xi)\,d\xi\end{aligned}$$

and (C.17) follows. □

Lemma C.3. *Under the assumptions of Lemma C.2, with* $1\le c$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^\times}(iy)^k e^{-iy\xi}\psi(\xi)\,d\xi\,dy=2\pi\psi^{(k)}(0)\qquad(0\le k\le c-1),$$

where each consecutive integral is absolutely convergent.

Proof. Since

$$\int_{\mathbb{R}}|y|^{c-1}\min\{1,|y|^{-c-1}\}\,dy<\infty,$$

the absolute convergence follows from Lemma C.2. Since the Fourier transform of ψ is absolutely integrable and since ψ is continuous at zero, Fourier inversion formula [Hör83, (7.1.4)] shows that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^\times} e^{-iy\xi}\psi(\xi)\,d\xi\,dy=2\pi\psi(0). \tag{C.18}$$

Also, for $0<k$,

$$\begin{aligned}&\int_{\mathbb{R}^\times}(iy)^k e^{-iy\xi}\psi(\xi)\,d\xi=\int_{\mathbb{R}^\times}(-\partial_\xi)\left((iy)^{k-1}e^{-iy\xi}\right)\psi(\xi)\,d\xi\\ =\;&\int_{\mathbb{R}^+}(-\partial_\xi)\left((iy)^{k-1}e^{-iy\xi}\right)\psi(\xi)\,d\xi+\int_{\mathbb{R}^-}(-\partial_\xi)\left((iy)^{k-1}e^{-iy\xi}\right)\psi(\xi)\,d\xi\\ =\;&(iy)^{k-1}\psi(0+)+\int_{\mathbb{R}^+}(iy)^{k-1}e^{-iy\xi}\psi'(\xi)\,d\xi\\ -\;&(iy)^{k-1}\psi(0-)+\int_{\mathbb{R}^-}(iy)^{k-1}e^{-iy\xi}\psi'(\xi)\,d\xi\\ =\;&(iy)^{k-1}\langle\psi\rangle_0+\int_{\mathbb{R}^\times}(iy)^{k-1}e^{-iy\xi}\psi'(\xi)\,d\xi.\end{aligned}$$

Hence, by induction on k and by our assumption

$$\begin{aligned}\int_{\mathbb{R}^\times}(iy)^k e^{-iy\xi}\psi(\xi)\,d\xi\;&=\;(iy)^{k-1}\langle\psi\rangle_0+(iy)^{k-2}\langle\psi'\rangle_0+\cdots+\langle\psi^{(k-1)}\rangle_0\\ &+\;\int_{\mathbb{R}^\times} e^{-iy\xi}\psi^{(k)}(\xi)\,d\xi\\ &=\;\int_{\mathbb{R}^\times} e^{-iy\xi}\psi^{(k)}(\xi)\,d\xi.\end{aligned}$$

Therefore our lemma follows from (C.18). □

The following proposition is an immediate consequence of Lemmas C.2, C.3, and the formula (C.8).

Proposition C.4. *Fix two integers $a,\ b \in \mathbb{Z}$ and a function $\psi \in \mathcal{S}(\mathbb{R}^\times)$. Let $c = -a - b$. If $c \geq 0$ assume that*

$$\langle \psi \rangle_0 = \cdots = \langle \psi^{(c)} \rangle_0 = 0. \tag{C.19}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}^\times} (1+iy)^{-a}(1-iy)^{-b} e^{-iy\xi} \psi(\xi)\, d\xi\, dy \qquad \text{(C.20)} \\ = &\int_{\mathbb{R}^\times} P_{a,b}(\xi) e^{-|\xi|} \psi(\xi)\, d\xi + Q_{a,b}(\partial_\xi)\psi(\xi)|_{\xi=0} \\ = &\int_{\mathbb{R}} \left(P_{a,b}(\xi) e^{-|\xi|} + Q_{a,b}(-\partial_\xi)\delta_0(\xi) \right) \psi(\xi)\, d\xi\,,. \end{aligned}$$

where δ_0 denotes the Dirac delta at 0.
(Recall that $Q_{a,b} = 0$ if $c < 0$ and $Q_{a,b}$ is a polynomial of degree if c, if $c \geq 0$.)

Let $\mathcal{S}(\mathbb{R}^+)$ be the space of the smooth complex valued functions whose all derivatives are rapidly decreasing at infinity and have limits at zero. Then $\mathcal{S}(\mathbb{R}^+)$ may be viewed as the subspace of the functions in $\mathcal{S}(\mathbb{R}^\times)$ which are zero on $\mathbb{R}^-$. Similarly we define $\mathcal{S}(\mathbb{R}^-)$. The following proposition is a direct consequence of Proposition C.4. We sketch an independent proof below.

Proposition C.5. *There is a seminorm p on the space $\mathcal{S}(\mathbb{R}^+)$ such that*

$$\left| \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi)\, d\xi \right| \leq \min\{1, |z|^{-1}\} p(\psi) \qquad (\psi \in \mathcal{S}(\mathbb{R}^+),\ Re\, z \geq 0). \tag{C.21}$$

For any integers $a, b \in \mathbb{Z}$ such that $a + b \geq 1$ and any function $\psi \in \mathcal{S}(\mathbb{R}^+)$

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b} \int_{\mathbb{R}^+} e^{-iy\xi} \psi(\xi)\, d\xi\, dy = 2\pi \int_{\mathbb{R}^+} P_{a,b,2}(\xi) e^{-\xi} \psi(\xi)\, d\xi, \tag{C.22}$$

and

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b} \int_{\mathbb{R}^-} e^{-iy\xi} \psi(\xi)\, d\xi\, dy = 2\pi \int_{\mathbb{R}^-} P_{a,b,-2}(\xi) e^{\xi} \psi(\xi)\, d\xi, \tag{C.23}$$

where each consecutive integral is absolutely convergent.

Let $a, b, c \in \mathbb{Z}$, $c \geq 0$, be such that $a + b + c \geq 0$. Suppose $\psi \in \mathcal{S}(\mathbb{R}^+)$ is such that

$$\psi(0) = \psi'(0) = \cdots = \psi^{(c)}(0). \tag{C.24}$$

Then the equalities (C.22) and (C.23) hold too.

Proof. Clearly

$$\left| \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi)\, d\xi \right| \leq \int_{\mathbb{R}^+} e^{-\operatorname{Re} z\xi} |\psi(\xi)|\, d\xi \leq \| \psi \|_1 .$$

Integration by parts shows that for $z \neq 0$,

$$\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi = z^{-1}\psi(0) + z^{-1}\int_{\mathbb{R}^+} e^{-z\xi}\psi'(\xi)\,d\xi.$$

Hence (C.21) follows with $p(\psi) = |\psi(0)| + \parallel \psi \parallel_1 + \parallel \psi' \parallel_1$.

Let $a, b \in \mathbb{Z}$ be such that $a + b \geq 1$. Then the function

$$(1+z)^{-a}(1-z)^{-b}\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi$$

is continuous on $\mathrm{Re}\, z \geq 0$ and meromorphic on $\mathrm{Re}\, z > 0$ and (C.21) shows that it is dominated by $|z|^{-2}$. Therefore Cauchy's Theorem implies that the left hand side of (C.22) is equal to

$$-2\pi\, \mathrm{res}_{z=1}\left((1+z)^{-a}(1-z)^{-b}\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi\right).$$

The computation of this residue is straightforward. This verifies (C.22) The proof of (C.23) is entirely analogous.

Integration by parts shows that under the assumption (C.24)

$$\int_{\mathbb{R}^+} e^{-z\xi}\psi(\xi)\,d\xi = z^{-c-1}\int_{\mathbb{R}^+} e^{-z\xi}\psi^{(c+1)}(\xi)\,d\xi \qquad (\mathrm{Re}\, z \geq 0,\ z \neq 0).$$

Hence, the above argument caries over and verifies the equalities (C.22) and (C.23). □

Proposition C.6. *Suppose $a, b, c \in \mathbb{Z}$ are such that*

$$b \geq 1,\ a + b + c = 1 \text{ and } c \geq 0. \tag{C.25}$$

Then

$$P_{a,b,2}(\xi)\xi^c = \frac{(b+c-1)!}{(b-1)!2^c} P_{a+c,b+c,2}(\xi). \tag{C.26}$$

Suppose now that

$$a \geq 1,\ a + b + c = 1 \text{ and } c \geq 0. \tag{C.27}$$

Then

$$P_{a,b,-2}(\xi)\xi^c = (-1)^c \frac{(a+c-1)!}{(a-1)!2^c} P_{a+c,b+c,-2}(\xi). \tag{C.28}$$

Proof. Because of (C.3), it is enough to prove (C.30). We compute

$$\begin{aligned}
P_{a,b,2}(\xi)\xi^c &= \sum_{k=0}^{b-1} \frac{a(a+1)\dots(a+k-1)}{k!(b-1-k)!} 2^{-a-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} (-1)^k \frac{(-a)(-a-1)\dots(-a-k+1)}{k!(b-1-k)!} 2^{-a-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} (-1)^k \frac{(-a)!}{k!(b-1-k)!(-a-k)!} 2^{-a-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} (-1)^k \frac{(b+c-1)!}{k!(b-1-k)!(b+c-1-k)!} 2^{-a-k}\xi^{b+c-1-k}
\end{aligned}$$

and

$$\begin{aligned}
P_{a+c,b+c,2}(\xi) &= \sum_{k=0}^{b+c-1} \frac{(a+c)(a+c+1)\dots(a+c+k-1)}{k!(b+c-1-k)!} 2^{-a-c-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} \frac{(a+c)(a+c+1)\dots(a+c+k-1)}{k!(b+c-1-k)!} 2^{-a-c-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} (-1)^k \frac{(-a-c)(-a-c-1)\dots(-a-c-k+1)}{k!(b+c-1-k)!} 2^{-a-c-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} (-1)^k \frac{(b-1)(b-1-1)\dots(b-k)}{k!(b+c-1-k)!} 2^{-a-c-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} (-1)^k \frac{(b-1)(b-1-1)\dots(b-k)}{k!(b+c-1-k)!} 2^{-a-c-k}\xi^{b+c-1-k} \\
&= \sum_{k=0}^{b-1} (-1)^k \frac{(b-1)!}{k!(b+c-1-k)!(b-1-k)!} 2^{-a-c-k}\xi^{b+c-1-k}
\end{aligned}$$

and the claim follows. □

By combining Proposition C.6 with (C.3) we obtain the following proposition.

Proposition C.7. *Suppose $a, b, c \in \mathbb{Z}$ are such that*

$$a \geq 1,\ a+b+c = 1 \text{ and } c \geq 0. \tag{C.29}$$

Then

$$P_{a,b,-2}(\xi)(-\xi)^c = \frac{(a+c-1)!}{(a-1)!2^c} P_{a+c,b+c,-2}(\xi). \tag{C.30}$$

The following lemma is an immediate consequence of the definition of $P_{a,b,2}$, $P_{a,b,-2}$ and their relation (C.3).

Lemma C.8. *Suppose $a, b \in \mathbb{Z}$. Then*

$$P'_{a,b,2}(\xi) = P_{a,b-1,2}(\xi) \quad \text{and} \quad P'_{a,b,-2}(\xi) = P_{a-1,b,-2}(\xi)\,. \tag{C.31}$$

If $b \geq 1$ then

$$P_{a,b,2}(0) = 2^{1-a-b}\,\frac{a(a+1)\dots(a+b-2)}{(b-1)!}\,. \tag{C.32}$$

If, moreover, $a \leq 0$ and $a+b \leq 1$, then

$$P_{a,b,2}(0) = (-1)^b\,2^{1-a-b}\binom{-a}{-a-b+1}. \tag{C.33}$$

If $a \geq 1$ then

$$P_{a,b,-2}(0) = 2^{1-a-b}\,\frac{b(b+1)\dots(b+a-2)}{(a-1)!}\,. \tag{C.34}$$

If, moreover, $b \leq 0$ and $a+b \leq 1$, then

$$P_{a,b,-2}(0) = (-1)^a\,2^{1-a-b}\binom{-b}{-a-b+1}. \tag{C.35}$$

Appendix D: Wave front set of an asymptotically homogeneous distribution

Let

$$\mathcal{F}f(x) = \int_{\mathbb{R}^n} f(y)e^{-2\pi i x\cdot y}\,dy$$

denote the usual Fourier transform on $\mathbb{R}^n$. Recall that for $t > 0$ the function $M_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $M_t(x) = tx$.

Lemma D.1. *Suppose $f, u \in \mathcal{S}^*(\mathbb{R}^n)$ and u is homogeneous of degree $d \in \mathbb{C}$. Suppose*

$$t^d M^*_{t^{-1}} f(\psi) \underset{t\to 0+}{\to} u(\psi) \qquad (\psi \in \mathcal{S}(\mathbb{R}^n)). \tag{D.1}$$

Then

$$WF_0(\mathcal{F}^{-1}f) \supseteq \operatorname{supp} u. \tag{D.2}$$

Proof. Suppose $\Phi \in C_c^\infty(\mathbb{R}^n)$ is such that $\Phi(0) \neq 0$. We need to show that the localized Fourier transform

$$\mathcal{F}((\mathcal{F}^{-1}f)\Phi)$$

is not rapidly decreasing in any open cone Γ which has a non-empty intersection with $\operatorname{supp} u$. (See [Hör83, Definition 8.1.2].) In order to do it, we'll choose a function $\psi \in C_c^\infty(\Gamma)$ such that $u(\psi) \neq 0$ and show that

$$\int_{\mathbb{R}^n} (t^{-1})^{-d}\mathcal{F}((\mathcal{F}^{-1}f)\Phi)(t^{-1}x)\psi(x)\,dx \underset{t\to 0+}{\to} u(\psi), \tag{D.3}$$

assuming $\Phi(0) = 1$. Let $\phi = \mathcal{F}\Phi$. Then $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. Notice that

$$t^d M^*_{t^{-1}}(f * \phi) = (t^d M^*_{t^{-1}} f) * (t^{-n} M^*_{t^{-1}} \phi), \tag{D.4}$$

so that, by setting $\check{\psi}(x) = \psi(-x)$, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} (t^{-1})^{-d} \mathcal{F}((\mathcal{F}^{-1} f)\Phi)(t^{-1}x)\psi(x)\, dx \\ = \ & t^d M^*_{t^{-1}}(f * \phi) * \check{\psi}(0) = (t^d M^*_{t^{-1}} f) * \left((t^{-n} M^*_{t^{-1}} \phi) * \check{\psi}\right)(0). \end{aligned} \tag{D.5}$$

We'll check that for an arbitrary $\psi \in \mathcal{S}(\mathbb{R}^n)$

$$(t^{-n} M^*_{t^{-1}} \phi) * \psi \underset{t \to 0+}{\to} \psi \tag{D.6}$$

in the topology of $\mathcal{S}(\mathbb{R}^n)$. This, together with (D.5) and Banach-Steinhaus Theorem, [Rud91, Theorem 2.6], will imply (D.3). Explicitly,

$$\left((t^{-n} M^*_{t^{-1}} \phi) * \psi\right)(x) - \psi(x) = \int_{\mathbb{R}^n} \phi(y)(\psi(x - ty) - \psi(x))\, dy. \tag{D.7}$$

Fix $N = 0, 1, 2, \dots$ and $\epsilon > 0$. Choose $R > 0$ so that

$$\int_{|y| \geq R} |\phi(y)|\, dy \cdot \left((|(1 + |y|)^N + 1\right) \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\psi(x)| < \epsilon. \tag{D.8}$$

Let $0 < t \leq 1$. Then

$$\begin{aligned} & (1 + |x|)^N \int_{|y| \geq R} |\phi(y)||\psi(x - ty)|\, dy \\ \leq \ & \int_{|y| \geq R} |\phi(y)|(1 + |ty|)^N (1 + |x - ty|)^N |\psi(x - ty)|\, dy \\ \leq \ & \int_{|y| \geq R} |\phi(y)|(1 + |y|)^N\, dy \cdot \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\psi(x)| \end{aligned} \tag{D.9}$$

and

$$\begin{aligned} & (1 + |x|)^N \int_{|y| \geq R} |\phi(y)||\psi(x)|\, dy \\ \leq \ & \int_{|y| \geq R} |\phi(y)|\, dy \cdot \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\psi(x)| \end{aligned} \tag{D.10}$$

so that, by (D.8),

$$(1 + |x|)^N \left| \int_{|y| \geq R} \phi(y)(\psi(x - ty) - \psi(x))\, dy \right| < \epsilon \qquad (0 < t \leq 1,\ x \in \mathbb{R}^n). \tag{D.11}$$

Choose $r > 0$ so that

$$(1 + |x|)^N \left| \int_{|y| \leq R} \phi(y)(\psi(x - ty) - \psi(x))\, dy \right| < \epsilon \qquad (0 < t \leq 1,\ |x| \geq r). \tag{D.12}$$

Since the function ψ is uniformly continuous,

$$\limsup_{t\to 0+} \sup_{|x|\le r} \left| \int_{|y|\le R} \phi(y)(\psi(x-ty)-\psi(x))\,dy \right| = 0. \tag{D.13}$$

Hence,

$$\limsup_{t\to 0+} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left| \int_{|y|\le R} \phi(y)(\psi(x-ty)-\psi(x))\,dy \right| \le \epsilon. \tag{D.14}$$

By combining (D.11) and (D.14) we see that

$$\limsup_{t\to 0+} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left| \int_{\mathbb{R}^n} \phi(y)(\psi(x-ty)-\psi(x))\,dy \right| \le 2\epsilon. \tag{D.15}$$

Since the $\epsilon > 0$ is arbitrary, (D.15) and (D.7) show that

$$\limsup_{t\to 0+} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|(t^{-n}M^*_{t^{-1}}\phi)*\psi(x)-\psi(x)\right| = 0. \tag{D.16}$$

Since the differentiation commutes with the convolution, (D.16) implies (D.6) and we are done. $\square$

Appendix E: A proof of a cocycle property

Here we proof the formula (38). Consider $g_1, g_2 \in \mathrm{Sp}^J$ as elements of $\mathrm{End}(\mathrm{W}_{\mathbb{C}}^+)$ by restriction. They preserve the positive definite hermitian form $H(\cdot,\cdot)$, (27). Let $K_1 = \mathrm{Ker}(g_1-1)$, $K_2 = \mathrm{Ker}(g_2-1)$, $K_{12} = \mathrm{Ker}(g_1g_2-1)$, $\mathrm{U}_1 = (g_1-1)\mathrm{W}_{\mathbb{C}}^+$, $\mathrm{U}_2 = (g_2-1)\mathrm{W}_{\mathbb{C}}^+$, $\mathrm{U}_{12} = (g_1g_2-1)\mathrm{W}_{\mathbb{C}}^+$ and $\mathrm{U} = \mathrm{U}_1\cap\mathrm{U}_2$.

We assume in this appendix that $K_1 = \{0\}$. In this case $\mathrm{U} = \mathrm{U}_2$. Moreover,

$$K_2\cap K_{12} = K_1\cap K_2 = \{0\}.$$

Hence there is a subspace $\mathrm{W}_2 \subseteq \mathrm{W}_{\mathbb{C}}^+$ such that

$$\mathrm{W}_{\mathbb{C}}^+ = K_{12}\oplus \mathrm{W}_2\oplus K_2. \tag{E.1}$$

Recall that

$$\mathrm{W}_{\mathbb{C}}^+ = \mathrm{U}\oplus K_2 \tag{E.2}$$

is an orthogonal direct sum decomposition. Define an element $h\in \mathrm{GL}(\mathrm{W}_{\mathbb{C}}^+)$ by

$$h|_{K_{12}\oplus \mathrm{W}_2} = (g_1^{-1}-1)^{-1}(g_2-1), \quad h|_{K_2} = (g_1^{-1}-1)^{-1}. \tag{E.3}$$

Fix a basis w_i of $\mathrm{W}_{\mathbb{C}}^+$ so that $w_i\in K_{12}$ if $i\le a$, $w_i\in \mathrm{W}_2$ if $a<i\le b$ and $w_{b+1}, w_{b+2}, \dots$ is a basis of K_2 that is orthonormal with respect to H. Then

$$hw_i = w_i \qquad (i\le a). \tag{E.4}$$

Lemma E.1. *The following equalities hold:*

$$\begin{aligned} & \det(H((g_1g_2-1)w_i, hw_j)_{a<i,j}) \qquad\qquad (E.5)\\ = \;& \det(H(\tfrac{1}{2}(c(g_1)+c(g_2))(g_2-1)w_i, (g_2-1)w_j)_{a<i,j\le b})\\ = \;& \det(H((g_1g_2-1)w_i, w_j)_{a<i,j})\overline{\det(h)}. \end{aligned}$$

Moreover, we have

$$\det(H(w_i,(g_1^{-1}-1)hw_j)_{i,j}) = \overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}, \tag{E.6}$$

so that

$$\overline{\det(h)} = \frac{\overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}}{\det(H(w_i,(g_1^{-1}-1)w_j)_{i,j})}. \tag{E.7}$$

Proof. Notice that both

$$\begin{aligned} c(g_1) &= (g_1+1)(g_1-1)^{-1} : \mathrm{W}_{\mathbb{C}}^+ \to \mathrm{W}_{\mathbb{C}}^+, \\ c(g_2) &= (g_2+1)(g_2-1)^{-1} : \mathrm{U} \to \mathrm{W}_{\mathbb{C}}^+ \end{aligned}$$

are well defined on the space U and

$$\begin{aligned} & (g_1-1)\frac{1}{2}(c(g_1)+c(g_2))(g_2-1) \\ = \ & \frac{1}{2}((g_1+1)(g_2-1)+(g_1-1)(g_2+1)) \\ = \ & g_1g_2-1. \end{aligned} \tag{E.8}$$

Suppose $a < i, j \le b$. Then (E.8) and (E.2) show that

$$\begin{aligned} H((g_1g_2-1)w_i,hw_j) &= H((g_1g_2-1)w_i,(g_1^{-1}-1)^{-1}(g_2-1)w_j) \\ &= H((g_1-1)^{-1}(g_1g_2-1)w_i,(g_2-1)w_j) \\ &= H((g_1-1)^{-1}(g_1-1)\frac{1}{2}(c(g_1)+c(g_2))(g_2-1)w_i,(g_2-1)w_j) \\ &= H(\frac{1}{2}(c(g_1)+c(g_2))(g_2-1)w_i,(g_2-1)w_j). \end{aligned} \tag{E.9}$$

Suppose $j \le b < i$. Then $(g_1g_2-1)w_i = (g_1-1)w_i$. Hence,

$$\begin{aligned} H((g_1g_2-1)w_i,hw_j) &= H((g_1-1)w_i,(g_1^{-1}-1)^{-1}(g_2-1)w_j) \\ &= H(w_i,(g_2-1)w_j) \\ &= H((g_2^{-1}-1)w_i,w_j) \\ &= H(-g_2^{-1}(g_2-1)w_i,w_j) \\ &= H(0,w_j) \\ &= 0. \end{aligned} \tag{E.10}$$

If $b < i, j$, then

$$H((g_1g_2-1)w_i,hw_j) = H((g_1-1)w_i,hw_j). \tag{E.11}$$

Notice that

$$\begin{aligned} & \det(H((g_1-1)w_i,hw_j)_{b<i,j}) = \det(H(w_i,(g_1^{-1}-1)hw_j)_{b<i,j}) \\ = \ & \det(H(w_i,w_j)_{b<i,j}) = 1. \end{aligned} \tag{E.12}$$

The first equality in (E.5) follows from relations (E.9), (E.10), (E.11) and (E.12).

Since h preserves the subspace K_{12}, it makes sense to define $\tilde{h} \in \mathrm{GL}(\mathrm{W}_{\mathbb{C}}^{+}/K_{12})$ by

$$\tilde{h}(w + K_{12}) = hw \qquad (w \in \mathrm{W}_{\mathbb{C}}^{+}).$$

Then

$$\det(H((g_1g_2-1)w_i, hw_j)_{a<i,j}) = \det(H((g_1g_2-1)w_i, w_j)_{a<i,j})\overline{\det(\tilde{h})}.$$

But (E.4) implies $\det(\tilde{h}) = \det(h)$. Hence the second equality in (E.5) follows.

Also, if $j \le b < i$, then

$$H(w_i, (g_1^{-1}-1)hw_j) = H(w_i, (g_2-1)w_j) = 0$$

because of (E.2). Hence,

$$\begin{aligned}\det(H(w_i,(g_1^{-1}-1)hw_j)_{i,j}) &= \det(H(w_i,(g_1^{-1}-1)hw_j)_{i,j\le b})\det(H(w_i,(g_1^{-1}-1)hw_j)_{b<i,j})\\ &= \det(H(w_i,(g_1^{-1}-1)hw_j)_{i,j\le b})\\ &= \det(H(w_i,(g_2-1)w_j)_{i,j\le b})\\ &= \overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}.\end{aligned}$$

This verifies (E.6). The formula (E.7) follows immediately from (E.6). □

Corollary E.2. *With the notation of Lemma E.1*

$$\begin{aligned}&\det(H(\frac{1}{2}(c(g_1)+c(g_2))(g_2-1)w_i,(g_2-1)w_j)_{a<i,j\le b})\\ &= \frac{\det(H((g_1g_2-1)w_i,w_j)_{a<i,j})\overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}}{\det(H((g_1-1)w_i,w_j)_{i,j})}\end{aligned}$$

Lemma E.3. *Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and assume that $K_1 = \{0\}$. Let $\mathsf{V} \subseteq \mathrm{U}$ denote the radical of the form $H(\frac{1}{2}(c(g_1)+c(g_2))\cdot,\cdot)_{\mathrm{U}}$ and let $H(\frac{1}{2}(c(g_1)+c(g_2))\cdot,\cdot)_{\mathrm{U}/\mathsf{V}}$ denote the resulting non-degenerate form on the quotient U/V. Then $\mathsf{V} = (g_2-1)K_{12}$ and*

$$\begin{aligned}&\frac{\det(g_1g_2-1)_{\mathrm{U}_{12}}}{\det(g_1-1)_{\mathrm{W}_{\mathbb{C}}^{+}}\det(g_2-1)_{\mathrm{U}}}\\ &= \frac{\det(H(\frac{1}{2}(c(g_1)+c(g_2))\cdot,\cdot)_{\mathrm{U}/\mathsf{V}})}{|\det(g_2-1\colon K_{12}\to\mathsf{V})|^2},\end{aligned} \tag{E.13}$$

where $|\det(g_2-1\colon K_{12}\to\mathsf{V})|$ is the absolute value of the determinant of the matrix of g_2-1 with respect to any orthonormal basis of K_{12} and V.

Proof. We use the notation of Lemma E.1 and make the following additional assumptions: W_2 is the orthogonal complement of $K_{12}+K_2$ in $\mathrm{W}_{\mathbb{C}}^{+}$ with respect to H, w_1, w_2, ... is a basis of $\mathrm{W}_{\mathbb{C}}^{+}$ such that w_1, w_2, ..., w_a is an orthonormal basis of K_{12}, and w_{a+1}, w_{a+2}, ..., w_b is an orthonormal basis of W_2.

Let $Q \in \mathrm{GL}(\mathrm{W}_{\mathbb{C}}^{+})$ be such that

$Qw_1, Qw_2, ...$ is an orthonormal basis of $\mathrm{W}_{\mathbb{C}}^{+}$,
$Qw_i = w_i$ if $i \le b$,
Qw_i is orthogonal to $K_{12}+\mathrm{W}_2$ if $b < i$.

Define the matrix elements $Q_{j,i}$ by

$$Qw_i = \sum_j Q_{j,i} w_j.$$

Then

$$Q_{j,i} = \delta_{j,i} \text{ if } i \le b.$$

Hence,

$$\det(Q) = \det((Q_{j,i})_{1\le j,i}) = \det((Q_{j,i})_{b<j,i}) = \det((Q_{j,i})_{a<j,i})$$

and

$$1 = \det(H(Qw_i, Qw_j)_{1\le i,j}) = |\det(Q)|^2 \det(H(w_i, w_j)_{1\le i,j}).$$

Therefore

$$|\det((Q_{j,i})_{a<j,i}|^2 \det(H(w_i, w_j)_{1\le i,j}) = 1. \tag{E.14}$$

It is easy to check from (E.8) that $(g_2-1)K_{12} = \mathsf{V}$. In particular, $\dim \mathsf{V} = \dim K_{12} = a$. Let u_1, u_2, ..., u_b be an orthogonal basis of U such that u_1, u_2, ..., u_a span V. Define the matrix elements $(g_2-1)_{k,i}$ by

$$(g_2-1)w_i = \sum_{k=1}^{b} (g_2-1)_{k,i} u_k \qquad (1 \le i \le b).$$

Hence,

$$(g_2-1)_{k,i} = 0 \text{ if } i \le a < k.$$

Therefore

$$\begin{aligned} & \det(((g_2-1)_{k,i})_{1\le k,i\le b}) \\ = \ & \det(((g_2-1)_{k,i})_{1\le k,i\le a}) \det(((g_2-1)_{k,i})_{a<k,i\le b}). \end{aligned} \tag{E.15}$$

Define $h \in \mathrm{GL}(\mathrm{W}_{\mathbb{C}}^+)$ as in (E.3). Then, by (E.6),

$$\begin{aligned} \overline{\det(h)} &= \overline{\det((g_1^{-1}-1)^{-1}(g_1^{-1}-1)h)} \\ &= \overline{\det(g_1^{-1}-1)}^{-1} \overline{\det((g_1^{-1}-1)h)} \\ &= \overline{\det(g_1^{-1}-1)}^{-1} \det(H(w_i, (g_1^{-1}-1)hw_j)_{1\le i,j}) \det(H(w_i, w_j)_{1\le i,j})^{-1} \\ &= \overline{\det(g_1^{-1}-1)}^{-1} \overline{\det(H((g_2-1)w_i, w_j)_{i,j\le b})} \det(H(w_i, w_j)_{1\le i,j})^{-1} \end{aligned} \tag{E.16}$$

Also,

$$\begin{aligned} & \det(H(\frac{1}{2}(c(g_1)+c(g_2))(g_2-1)w_i, (g_2-1)w_j)_{a<i,j\le b}) \\ = \ & \det(H(\frac{1}{2}(c(g_1)+c(g_2))u_k, u_l)_{a<k,l\le b}) \, |\det(((g_2-1)_{k,i})_{a<k,i\le b}|^2. \end{aligned} \tag{E.17}$$

Since $\mathrm{W}_{\mathbb{C}}^{+}$ is the orthogonal direct sum of K_{12} and U_{12}, the vectors Qw_j, $a < j$, form an orthonormal basis of U_{12}, so that

$$\begin{aligned} \det(g_1g_2 - 1)_{\mathrm{U}_{12}} &= \det(H((g_1g_2-1)Qw_i, Qw_j)_{a<i,j}) \qquad \text{(E.18)}\\ = \quad &|\det((Q_{i,j})_{a<i,j})|^2 \det(H((g_1g_2-1)w_i, w_j)_{a<i,j}). \end{aligned}$$

Define an element $q \in \mathrm{GL}(\mathrm{W}_{\mathbb{C}}^{+})$ by

$$\begin{aligned} qw_i &= u_i \text{ if } i \le b, \\ qw_i &= w_i \text{ if } b < i. \end{aligned}$$

Then qw_1, qw_2, …, qw_b is an orthonormal basis of U so that

$$\det(g_2 - 1)_{\mathrm{U}} = \det(H((g_2-1)qw_i, qw_j)_{i,j\le b}).$$

Define the coefficients $q_{i,j}$ by

$$qw_i = \sum_j q_{j,i} w_j.$$

Then

$$q_{j,i} = \delta_{j,i} \text{ if } b < i$$

so that

$$\det(q) = \det((q_{j,i})_{1\le i,j}) = \det((q_{j,i})_{1\le i,j\le b}).$$

Also,

$$(g_2 - 1)qw_i = \sum_j q_{j,i}(g_2-1)w_j = \sum_{j\le b} q_{j,i}(g_2-1)w_j \qquad (i \le b).$$

Therefore,

$$\det(H((g_2-1)qw_i, qw_j)_{i,j\le b}) = |\det(q)|^2 \det(H((g_2-1)w_i, w_j)_{i,j\le b}).$$

Define the coefficients $q_{i,j}^{-1}$ of the inverse map q^{-1} by

$$w_i = q^{-1}(qw_i) = \sum_j q_{i,j}^{-1} qw_j.$$

Since, the qw_i form an orthonormal basis of $\mathrm{W}_{\mathbb{C}}^{+}$,

$$q_{i,j}^{-1} = H(q^{-1}qw_i, qw_j) = H(w_i, qw_j) = \overline{H(qw_j, w_i)},$$

so that

$$q_{i,j}^{-1} = \begin{cases} \overline{H(u_j, w_i)} \text{ if } j \le b, \\ \overline{H(w_j, w_i)} \text{ if } j > b, \\ \overline{H(w_j, w_i)} = \delta_{i,j} \text{ if } i, j > b. \end{cases}$$

In particular,

$$q_{i,j}^{-1} = 0 \text{ if } j \le b < i$$

so that

$$\det(q)^{-1} = \det(q^{-1}) = \det((q_{i,j}^{-1})_{i,j\le b}) = \det(\overline{H(u_j, w_i)}_{i,j\le b}).$$

Thus

$$\begin{aligned}
&\overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}\,\det(g_2-1\colon \mathrm{U}\to\mathrm{U}) \qquad\qquad (E.19)\\
=\ &\overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}\,\det(H((g_2-1)w_i,w_j)_{i,j\le b})\,|\det(q)|^2\\
=\ &|\det(H((g_2-1)w_i,w_j)_{i,j\le b})|^2|\det(q)|^2\\
=\ &|\det(H(\sum_{k=1}^{b}(g_2-1)_{k,i}u_k,w_j)_{i,j\le b})|^2\,|\det(q)|^2\\
=\ &|\det((g_2-1)_{k,i})_{k,i\le b})\det(H(u_k,w_j)_{k,j\le b})|^2\,|\det(q)|^2\\
=\ &|\det((g_2-1)_{k,i})_{k,i\le b})|^2\\
=\ &|\det((g_2-1)_{k,i})_{k,i\le a})|^2|\det((g_2-1)_{k,i})_{a<k,i\le b})|^2,
\end{aligned}$$

where the last equality follows from (E.15). Notice also that

$$\begin{aligned}
\overline{\det(g_1^{-1}-1)} &= \overline{\det(H((g_1^{-1}-1)Qw_j,Qw_k)_{1\le j,k})} \qquad\qquad (E.20)\\
&= \overline{\det(H(Qw_j,(g_1-1)Qw_k)_{1\le j,k})}\\
&= \det(H((g_1-1)Qw_k,Qw_j)_{1\le j,k})\\
&= \det(g_1-1).
\end{aligned}$$

The formula (E.13) follows from (E.5) and (E.14) - (E.19) via a straightforward computation:

$$\begin{aligned}
&\frac{\det(g_1g_2-1\colon \mathrm{U}_{12}\to\mathrm{U}_{12})}{\det(g_1-1\colon \mathrm{W}_{\mathbb{C}}^+\to\mathrm{W}_{\mathbb{C}}^+)\det(g_2-1\colon \mathrm{U}\to\mathrm{U})}\\
=\ &\frac{|\det((Q_{i,j})_{a<i,j})|^2\,\det(H((g_1g_2-1)w_i,w_j)_{a<i,j})}{\det(g_1-1\colon \mathrm{W}_{\mathbb{C}}^+\to\mathrm{W}_{\mathbb{C}}^+)\det(g_2-1\colon \mathrm{U}\to\mathrm{U})}\\
=\ &\frac{|\det((Q_{i,j})_{a<i,j})|^2\,\det(H(\frac{1}{2}(c(g_1)+c(g_2))u_k,u_l)_{a<k,l\le b})\,|\det(((g_2-1)_{k,i})_{a<k,i\le b})|^2}{\overline{\det(h)}\det(g_1-1)\det(g_2-1\colon \mathrm{U}\to\mathrm{U})}\\
=\ &\frac{|\det((Q_{i,j})_{a<i,j})|^2\,\det(H(\frac{1}{2}(c(g_1)+c(g_2))u_k,u_l)_{a<k,l\le b})\,|\det(((g_2-1)_{k,i})_{a<k,i\le b})|^2}{\overline{\det(g_1^{-1}-1)}^{-1}\,\overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}\,\det(H(w_i,w_j)_{1\le i,j})^{-1}\det(g_1-1)\det(g_2-1\colon \mathrm{U}\to\mathrm{U})}\\
=\ &\frac{\det(H(\frac{1}{2}(c(g_1)+c(g_2))u_k,u_l)_{a<k,l\le b})\,|\det(((g_2-1)_{k,i})_{a<k,i\le b})|^2}{\overline{\det(g_1^{-1}-1)}^{-1}\,\overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}\,\det(g_1-1)\det(g_2-1\colon \mathrm{U}\to\mathrm{U})}\\
=\ &\frac{\det(H(\frac{1}{2}(c(g_1)+c(g_2))u_k,u_l)_{a<k,l\le b})\,|\det(((g_2-1)_{k,i})_{a<k,i\le b})|^2}{\overline{\det(H((g_2-1)w_i,w_j)_{i,j\le b})}\,\det(g_2-1\colon \mathrm{U}\to\mathrm{U})}\\
=\ &\frac{\det(H(\frac{1}{2}(c(g_1)+c(g_2))u_k,u_l)_{a<k,l\le b})\,|\det(((g_2-1)_{k,i})_{a<k,i\le b})|^2}{|\det((g_2-1)_{k,i})_{k,i\le a})|^2|\det((g_2-1)_{k,i})_{a<k,i\le b})|^2}\\
=\ &\frac{\det(H(\frac{1}{2}(c(g_1)+c(g_2))u_k,u_l)_{a<k,l\le b})}{|\det((g_2-1)_{k,i})_{k,i\le a})|^2}.
\end{aligned}$$

□

Since, in terms of (37),

$$\frac{\det(H(\frac{1}{2}(c(g_1)+c(g_2))\cdot,\cdot)_{\mathsf{U}/\mathsf{V}})}{|\det(H(\frac{1}{2}(c(g_1)+c(g_2))\cdot,\cdot)_{\mathsf{U}/\mathsf{V}})|} = \frac{\det(H(i(-i)\frac{1}{2}(c(g_1)+c(g_2))\cdot,\cdot)_{\mathsf{U}/\mathsf{V}})}{|\det(H(i(-i)\frac{1}{2}(c(g_1)+c(g_2))\cdot,\cdot)_{\mathsf{U}/\mathsf{V}})|}$$
$$= \quad i^{\operatorname{sgn} h_{g_1,g_2}},$$

the formula (38) follows from (E.13).

References

[AB95] J. Adams and D. Barbasch. Reductive dual pairs correspondence for complex groups. *J. Funct. Anal.*, 132:1–42, 1995.

[ABP+07] J. Adams, D. Barbasch, A. Paul, P. Trapa, and D. A. Vogan. Unitary Shimura corresondence for split real groups. *Journal of the AMS*, 20:701–751, 2007.

[Ada83] J. Adams. Discrete spectrum of the dual pair $(O(p,q), Sp(2m,\mathbb{R})$. *Invent. Math*, 74:449–475, 1983.

[Ada87] J. Adams. Unitary highest weight modules. *Adv. in Math*, 63:113–137, 1987.

[Ada98] J. Adams. Lifting of characters on orthogonal and metaplectic groups. *Duke Math. J.*, 92(1):129–178, 1998.

[AKP13] A.-M. Aubert, W. Kraśkiewicz, and T. Przebinda. Howe correspondence and Springer correspondence for real reductive dual pairs. *Manuscripta Math.*, 143:81–130, 2013.

[AP14] A.-M. Aubert and T. Przebinda. A reverse engineering approach to the Weil Representation. *Central Eur. J. Math.*, 12:1500–1585, 2014.

[BP14] F. Bernon and T. Przebinda. The Cauchy Harish-Chandra integral and the invariant eigendistributions. *Internat. Math. Res. Notices*, 14:3818–3862, 2014.

[BV80] D. Barbasch and D. Vogan. The local structure of characters. *J. Funct. Anal.*, 37(1):27–55, 1980.

[CM93] D. Collingwood and W. McGovern. *Nilpotent orbits in complex semisimple Lie algebras.* Reinhold, Van Nostrand, New York, 1993.

[Dad82] J. Dadok. On the C^∞ Chevalley Theorem. *Advances in Math.*, 44:121–131, 1982.

[Die71] J. Dieudonné. *Élements d'Analyse.* Gauthier-Villars Éditeur, 1971.

[DKP97] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda. Nilpotent Orbits and Complex Dual Pairs. *J. Algebra*, 190:518–539, 1997.

[DKP05] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda. Dual Pairs and Kostant-Sekiguchi Correspondence. II. Classification of Nilpotent Elements. *Central Eur. J. Math.*, 3:430–464, 2005.

[DP96] A. Daszkiewicz and T. Przebinda. The oscillator character formula, for isometry groups of split forms in deep stable range. *Invent. Math.*, 123(2):349–376, 1996.

[DV90] M. Duflo and M. Vergne. Orbites coadjointes et cohomologie équivariante. In *The orbit method in representation theory (Copenhagen, 1988)*, volume 82 of *Progr. Math.*, pages 11–60. Birkhäuser Boston, Boston, MA, 1990.

[EHW83] T. J. Enright, R. Howe, and N. R. Wallach. A classification of unitary highest weight modules. *Proceedings of Utah Conference, 1982*, pages 97–143, 1983.

[Enr88] T. Enright. Analogues of Kostant's $\mathfrak{u}$-cohomology formulas for unitary highest weight modules. *J. reine angew. math.*, 392, 1988.

[EW04] T. Enright and J. Willebbring. Hilbert series, Howe duality and branching for classical groups. *Ann. of Math.*, 159, 2004.

[GZ13] R. Gomez and C. Zhu. Local theta lifting of generalized Whittaker models associated to nilpotent orbits. *Preprint to appear in Geom. Funct. Ana.. arXiv:1302.3744*, 2013.

[Har55] Harish-Chandra. Representations of Semisimple Lie Groups IV. *Amer. J. Math.*, 77:743–777, 1955.

[Har57a] Harish-Chandra. Differential operators on a semisimple Lie algebra. *Amer. J. Math.*, 79:87–120, 1957.

[Har57b] Harish-Chandra. Fourier Transform on a semsimple Lie algebra I. *Amer. J. Math.*, 79:193–257, 1957.

[Har63] Harish-Chandra. Invariant eigendistributions on semisimple Lie groups. *Bull. Amer. Math. Soc.*, 69:117–123, 1963.

[Har65] Harish-Chandra. Invariant Eigendistributions on a Semisimple Lie algebra. *Pubi. Math. IHES*, 27:5–54, 1965.

[Har11] B. Harris. Fourier transforms of nilpotent orbits, limit formulas for reductive Lie groups and wave fron cycles for tempered representations. *MIT thesis*, 2011.

[HC64] Harish-Chandra. Invariant differential operators and distributions on a semisimple Lie algebra. 86:534–564, 1964.

[He03] H. He. Unitary Representations and Theta Correspondence for Type I Classical Groups. *J. Funct. Anal.*, 199:92–121, 2003.

[Hel84] S. Helgason. *Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators, and Spherical Functions.* Academic Press, 1984.

[Hör83] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer Verlag, 1983.

[How79] R. Howe. θ-series and invariant theory. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 275–285. Amer. Math. Soc., Providence, R.I., 1979.

[How80] R. Howe. Quantum mechanics and partial differential equations. *J. Funct. Anal.*, 38:188–254, 1980.

[How81] R. Howe. Wave Front Sets of Representations of Lie Groups. In *Automorphic forms, Representation Theory and Arithmetic*, pages 117–140. Tata Institute of Fundamental Research, Bombay, 1981.

[How89a] R. Howe. Remarks on Classical Invariant Theory. *Trans. Amer. Math. Soc.*, 313:539–570, 1989.

[How89b] R. Howe. Transcending Classical Invariant Theory. *J. Amer. Math. Soc. 2*, 2:535–552, 1989.

[Kna86] A. Knapp. *Representation Theory of Semisimple groups, an overview based on examples.* Princeton Mathematical Series. Princeton University Press, Princeton, New Jersey, 1986.

[Li89] Jian-Shu Li. Singular unitary representations of classical groups. *Invent. Math.*, 97(2):237–255, 1989.

[LM15] H. Y. Loke and J.J. Ma. Invariants and K-spectrum of local theta lifts. *Compos. Math.*, 151:179–206, 2015.

[LPTZ03] J. S. Li, A. Paul, E. C. Tan, and C. B. Zhu. The explicit duality correspondence of $(\mathrm{Sp}(p,q), \mathrm{O}^*(2n))$. *J. Funct. Anal.*, 200():71–100, 2003.

[Mac80] I. G. Macdonald. The volume of a compact Lie group. *Invent. Math.*, 56:93–95, 1980.

[Mar75] S. Martens. The characters of the holomorphic discrete series. *Proceedings of the National Academy of Sciences*, 72:3275–3276, 1975.

[Moe89] Moeglin C. Correspondance de Howe pour les paires duales réductives duales: quelques calculs dans la cas archimdien. *J. Funct. Anal.*, 85:1–85, 1989.

[Moe98] Moeglin, C. Correspondance de Howe et front d'onde. *Adv. in Math.*, 133:224–285, 1998.

[MPP15] M. McKee, A. Pasquale, and T. Przebinda. Semisimple orbital integrals on the symplectic space for a real reductive dual pair. *J. Funct. Anal.*, 268:275–335, 2015.

[NOT+01] Kyo Nishiyama, Hiroyuki Ochiai, Kenji Taniguchi, Hiroshi Yamashita, and Shohei Kato. *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*, volume 273. Astérisque, 2001.

[Pan10] Shu-Yen Pan. Orbit correspondence for real reductive dual pairs. *Pacific J. Math.*, 248(2):403–427, 2010.

[Pau98] A. Paul. Howe correpondence for real unitary groups I. *J. Funct. Anal.*, 159:384–431, 1998.

[Pau00] A. Paul. Howe correspondence for real unitary groups II. *Proc. Amer. Math. Soc.*, 128:3129–3136, 2000.

[Pau05] A. Paul. On the Howe correspondence for symplectic-orthogonal dual pairs. *J. Funct. Anal.*, 228:270–310, 2005.

[Prz91] T. Przebinda. Characters, dual pairs, and unipotent representations. *J. Funct. Anal.*, 98(1):59–96, 1991.

[Prz93] T. Przebinda. Characters, dual pairs, and unitary representations. *Duke Math. J.*, 69(3):547–592, 1993.

[Prz96] T. Przebinda. The duality correspondence of infinitesimal characters. *Coll. Math*, 70:93–102, 1996.

[Prz06] T. Przebinda. Local Geometry of Orbits for an Ordinary Classical Lie Supergroup. *Central Eur. J. Math.*, 4:449–506, 2006.

[Rao93] R. Ranga Rao. On some explicit formulas in the theory of Weil representations. *Pacific J. Math.*, 157:335–371, 1993.

[Ren98] D. Renard. Transfer of orbital integrals between $\mathrm{Mp}(2n, \mathbb{R})$ and $\mathrm{SO}(n, n+1)$. *Duke Math J.*, 95:125–450, 1998.

[Ros90] W. Rossmann. *Nilpotent Orbital Integrals in a Real Semisimple Lie Algebra and Representations of Weyl Groups*, volume 92 of *Progress in Math.* Birkhäuser Boston, Boston, MA, 1990.

[Ros95] W. Rossmann. Picard-Lefschetz theory and characters of semisimple a Lie group. *Invent. Math.*, 121:579–611, 1995.

[Rud91] Rudin, W. *Functional Analysis.* McGraw-Hill, Inc, 1991.

[Sch74] G. Schwarz. Smooth functions invariant under the action of a compact Lie group. *Topology*, 14:63–68, 1974.

[Ste93] E.M. Stein. *Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton, NJ, 1993.

[SV00] W. Schmid and K. Vilonen. Characteristic cycles and wave front cycles of representations of reductive lie groups. *Ann. of Math. (2)*, 151(3):1071 – 1118, 2000.

[Tho09] T. Thomas. Weil representation, Weyl transform, and transfer factor. *preprint on author's webpage*, 2009.

[Vog78] D. Vogan. Gelfand-Kirillov dimension for Harish-Chandra modules. *Invent. Math.*, 48:75–98, 1978.

[Vog89] D. Vogan. *Associated varieties and unipotent representations.* Harmonic analysis on reductive groups (Brunswick, ME, 1989). Birkhäuser Boston Inc., Boston, MA, 1989.

[Wal84] N. Wallach. The asymptotic behavior of holomorphic representations. *Mém. Soc. Math. France (N.S.)*, (15):291–305, 1984.

[Wal93] N. Wallach. Invariant Differential Operators on a Reductive Lie Algebra and Weyl Group Representations. *J. Amer. Math. Soc.6*, 4:779–816, 1993.

[Wey46] Weyl, H. *The classical groups.* Princeton Univ. Press, Princeton, N.J., 1946.

Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA
E-mail address: mark-mckee@uiowa.edu

Institut Elie Cartan de Lorraine (IECL UMR CNRS 7502), Université de Lorraine, F-57045 Metz, France
E-mail address: angela.pasquale@univ-lorraine.fr

Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA
E-mail address: przebinda@gmail.com