PLANCHEREL FORMULA FOR A REAL REDUCTIVE LIE GROUP, XIAMEN, JULY 29 - AUGUST 2, 2019

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Contents

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1. **Introduction**

[1] Classical Harmonic Analysis concerns a decomposition of a function (signal) into a superposition of components corresponding to *simple harmonics*. The analysis of the signal aims at finding these components and the synthesis is the reconstruction of the signal out of them. There is often some "signal processing", dictated by applications, between the analysis and synthesis.

The *simple harmonics* behave well under various symmetries and this is the reason for the decomposition. The fundamental results are Parseval's Theorem (1806) for the Fourier series, [Par06], and Plancherel's Theorem (1910) for the Fourier transform, [Pla10]. Among the best known applications is the Magnetic Resonance Imaging, for which Peter Mansfield and Paul Lauterbur were awarded a Nobel prize in 2003.

If the function is defined on a commutative group, such as the additive group of the real numbers or the multiplicative group of the complex numbers of absolute value one, then the *simple harmonics* are the eigenvectors under the translations. This is the ultimate symmetry one could expect.

Problems arising in Physics and Number Theory motivated a rapid growth of Harmonic Analysis on non-commutative groups. The earliest examples were the Heisenberg group, necessary for a formulation of the principles of Quantum Mechanics (J. Von Neumann 1926, [vN26]), and the compact Lie groups (Peter-Weyl 1927, [PW27]). Here the *simple harmonics* are replaced by *irreducible unitary representations.* All of them may be found by analyzing the square integrable functions on these groups, so that an analog of Plancherel's Theorem may be viewed as the top achievement of the theory.

However, there are plenty of other groups of interest which have irreducible unitary representations occurring outside the space of the square integrable functions on the group. The main class are the non-compact semisimple Lie groups, such as $\mathrm{SL}_2(\mathbb{R})$. Though the irreducible unitary representations of most of them are not understood yet, the representations that can be found in the space of the square integrable functions on the group are known and the decomposition of an arbitrary such function in terms of these representations is known as the *Plancherel formula*. For the group $\mathrm{SL}_n(\mathbb{C})$ this formula was first found by Gelfand and Naimark in 1950, and for $\mathrm{SL}_n(\mathbb{R})$ by Gelfand and Graev in 1953, [GG53]. The Plancherel formula on an arbitrary Real Reductive Group was published by Harish-Chandra in 1976, [Har76], and is considered as one of the greatest achievements of Mathematics of the 20th century.

A goal of these lectures is to explain the ingredients of Harish-Chandra's Plancherel formula, explain how they fit together, study particular cases and go through all the details for the group of the real unimodular matrices of size two.

All the necessary information in its original nearly perfect form is contained in Harish-Chadra's articles [HC14a], [HC14b], [HC14c], , [HC14d], [HC18]. The example $\mathrm{SL}_2(\mathbb{R})$ is explained in classical books such as [Lan75]. For all of that, a good understanding of the Fourier Transform and Distribution theory on an Euclidean space is needed. Here

[1]This version was corrected/improved with the help of Muna Naik. I would like to thank him for a careful reading.

Hörmender's "The Analysis of Linear Partial Differential Operators I" is one of the best references, [Hör83].

2. The Fourier Transform on the Schwartz space $\mathcal{S}(\mathbb{R})$

Here we follow [Hör83, section 7.1]. Recall that the Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all infinitely many times differentiable functions $f:\mathbb{R}\to\mathbb{C}$ such that for any two integers $n,k\geq 0$

$$\sup_{x\in\mathbb{R}}|x^n\partial_x^k f(x)|<\infty\,.$$

In particular $\mathcal{S}(\mathbb{R})\subseteq \mathrm{L}^1(\mathbb{R})$ and we have the well defined

Theorem 1. *The Fourier transform*

$$\hat{f}(y)=\mathcal{F}f(y)=\int_{\mathbb{R}}e^{-2\pi ixy}f(x)\,dx \qquad (y\in\mathbb{R},\ f\in\mathcal{S}(\mathbb{R})) \tag{1}$$

maps the Schwartz space $\mathcal{S}(\mathbb{R})$ into itself, is invertible, and the inverse is given by

$$f(x)=\int_{\mathbb{R}}e^{2\pi ixy}\mathcal{F}f(y)\,dx \qquad (x\in\mathbb{R},\ f\in\mathcal{S}(\mathbb{R}))\,. \tag{2}$$

Since

$$\frac{d}{dy}\mathcal{F}f(y)=\int_{\mathbb{R}}e^{-2\pi ixy}(-2\pi ix)f(x)\,dx \tag{3}$$

and

$$\int_{\mathbb{R}}e^{-2\pi ixy}f'(x)\,dx=2\pi iy\mathcal{F}f(y)\,, \tag{4}$$

the inclusion $\mathcal{F}\mathcal{S}(\mathbb{R})\subseteq\mathcal{S}(\mathbb{R})$ is easy to check. For the rest we need a few lemmas.

Lemma 2. *Let $f\in\mathcal{S}(\mathbb{R})$ be such that $f(0)=0$. Set $g(x)=\int_0^1 f'(tx)\,dt$. Show that $f(x)=xg(x)$ and $g\in\mathcal{S}(\mathbb{R})$.*

Proof. The equality $f(x)=xg(x)$ is immediate from the Fundamental Theorem of Calculus, via a change of variables $y=tx$.

Fix two non-negative integers n and k. Suppose $|x|\leq 1$. Then

$$\left|x^n g^{(k)}(x)\right|=\left|x^n\int_0^1 t^k f^{(k+1)}(tx)\,dt\right|\leq\int_0^1|f^{(k+1)}(tx)|\,dt\leq\max_{y\in\mathbb{R}}|f(y)|<\infty.$$

Notice that

$$\begin{aligned} x^n g^{(k)}(x) &= x^n\left(\frac{d}{dx}\right)^k(x^{-1}f(x)) \\ &= x^n\sum_{p=0}^{k}\frac{k!}{p!(k-p)!}(-1)(-2)\dots(-p)x^{-p-1}f^{(k-p)}(x) \end{aligned}$$

and that

$$\max_{|x|\geq 1}\left|x^{n-p-1}f^{(k-p)}(x)\right|<\infty.$$

Hence

$$\max_{x\in\mathbb{R}} |x^n g^{(k)}(x)| < \infty.$$

□

Corollary 3. *Fix $y \in \mathbb{R}$. Let $\phi \in \mathcal{S}(\mathbb{R})$ be such that $\phi(y) = 0$. Then that there is $\psi \in \mathcal{S}(\mathbb{R})$ such that*

$$\phi(x) = (x - y)\psi(x).$$

Lemma 4. *Let $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be a linear map with the property that if $\phi(y) = 0$ for some $y \in \mathbb{R}$ then $T\phi(y) = 0$ for the same y. Then there is a function $c(x)$ such that*

$$T\phi(x) = c(x)\phi(x) \qquad (\phi \in \mathcal{S}(\mathbb{R}).$$

(In other words, T is the multiplication by the function c.)

Proof. Let $\phi_1(x) = e^{-x^2}$. As we know this function belongs to $\mathcal{S}(\mathbb{R})$. Fix $x \in \mathbb{R}$. Then for any $\phi_2 \in \mathcal{S}(\mathbb{R})$

$$(\phi_2(x)\phi_1 - \phi_1(x)\phi_2)(x) = \phi_2(x)\phi_1(x) - \phi_1(x)\phi_2(x) = 0.$$

Hence, by the assumption on T,

$$0 = T(\phi_2(x)\phi_1 - \phi_1(x)\phi_2)(x). \tag{5}$$

Since T is linear

$$T(\phi_2(x)\phi_1) = \phi_2(x)T(\phi_1) \text{ and } T(\phi_1(x)\phi_2) = \phi_1(x)T(\phi_2).$$

Thus evaluation at x and using (5) we see that

$$0 = \phi_2(x)T(\phi_1)(x) - \phi_1(x)T(\phi_2)(x).$$

Therefore

$$T(\phi_2)(x) = \frac{T(\phi_1)(x)}{\phi_1(x)}\phi_2(x).$$

Thus the claim holds with

$$c(x) = \frac{T(\phi_1)(x)}{\phi_1(x)}.$$

□

Lemma 5. *Suppose $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is a linear map which commutes with the multiplication by x:*

$$T(xf(x)) = xT(f)(x) \qquad (f \in \mathcal{S}(\mathbb{R})).$$

Then there is a function $c(x)$ such that

$$T\phi(x) = c(x)\phi(x) \qquad (\phi \in \mathcal{S}(\mathbb{R})).$$

Proof. It'll suffice to show that T satisfies the assumptions of Lemma 4. Fix $y \in \mathbb{R}$. Let $\phi \in \mathcal{S}(\mathbb{R})$ be such that $\phi(y) = 0$. Then, by Corollary 3 there is $\psi \in \mathcal{S}(\mathbb{R})$ such that

$$\phi(x) = (x - y)\psi(x).$$

Hence

$$T(\phi)(x) = T((x - y)\psi(x)) = (x - y)T(\phi)(x),$$

which is zero if $x = y$. □

Proposition 6. *Suppose $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is a linear map which commutes with the multiplication by x:*

$$T(xf(x)) = xT(f)(x) \qquad (f \in \mathcal{S}(\mathbb{R}))$$

and with the derivative

$$T(f') = T(f)' \qquad (f \in \mathcal{S}(\mathbb{R})).$$

Then there is a constant c such that

$$T\phi(x) = c\phi(x) \qquad (\phi \in \mathcal{S}(\mathbb{R})).$$

Proof. We know from Lemma 5. that T coincides with the multiplication by a function $c(x)$. Since T commutes with the derivative we see that for any $f \in \mathcal{S}(\mathbb{R})$

$$c(x)f'(x) = (c(x)f(x))'.$$

Since the multiplication by $c(x)$ prserves the Schwartz space, $c(x)$ is differentiable and

$$(c(x)f(x))' = c'(x)f(x) + c(x)f'(x)\,.$$

Hence $c'(x) = 0$. therefore $c(x)$ is a constant. □

Lemma 7. *Let $Rf(x) = f(-x)$. The map $T = R\mathcal{F}^2 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ commutes with the multiplication by x and with the derivative.*

Proof. Since

$$R(xf(x)) = -xf(-x) = -xR(f)(x) \text{ and } R(f') = -R(f)\,,$$

it'll suffice to check that

$$\mathcal{F}^2(xf(x)) = -x\mathcal{F}^2(f)(x) \text{ and } \mathcal{F}^2(f') = -\left(\mathcal{F}^2(f)\right)'\,,$$

which follows from the relations (3) and (4). □

Lemma 8. *Let $f(x) = e^{-\pi x^2}$. Then $\mathcal{F}f = f$. (Fourier transform of the normalized Gaussian is the same Gaussian.)*

Proof. Since

$$\frac{d}{dx}f(x) = -2\pi x f(x)$$

the formulas (3) and (4) show that

$$\frac{d}{dx}\left(\mathcal{F}f(x) \cdot f(x)\right) = 0\,.$$

Hence

$$\mathcal{F}f(x) = const f(x)\,.$$

Evaluating at $x = 0$ gives

$$\int_{\mathbb{R}} f(y)\,dy = const\,.$$

By squaring the integral and using polar coordinates we show that $const = 1$. □

Now we are ready to prove the inversion formula in Theorem 1. We see from that the map $R\mathcal{F}^2 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is the identity,

$$R\mathcal{F}^2 f = f \qquad (f \in \mathcal{S}(\mathbb{R})).$$

We know from Lemma 7 and Proposition 6 that the map $R\mathcal{F}^2$ is a constant multiple of the identity: $cI = R\mathcal{F}^2$. Now Lemma 8 shows that with $f(x) = e^{-\pi x^2}$

$$cf = R\mathcal{F}^2 f = Rf = f.$$

Thus $c = 1$. Hence

$$R\mathcal{F} = \mathcal{F}^{-1}$$

and the formula2 follows. This completes the proof of Theorem 1.

3. **Magnetic Resonance Imaging**

Suppose a source at $s \in \mathbb{R}$ is emitting a signal with frequency $ks \in \mathbb{R}$ and amplitude $A(s)$. The collective signal received is

$$B(x) = \int_{\mathbb{R}} A(s) e^{2\pi i x k s}\,ds\,.$$

By Fourier inversion,

$$A(s) = k \int_{\mathbb{R}} B(x) e^{-2\pi i x k s}\,dx\,.$$

Hence we can recover $A(s)$ from $B(x)$. In particular, if we the function $A(s)$ is linear (in some large interval contained in $[0, \infty)$) and if we know $A(s)$ then we know s, i.e. the location of the source. For the related physics see
$youtube.com/watch?v = pGcZvSG805Y \quad youtube.com/watch?v = djAxjtN\_7VE.$

4. **The Fourier Transform on the space** $C^\infty(\mathrm{U}_1)$

Here $\mathrm{U}_1 = \{u \in \mathbb{C};\ |u| = 1\}$ is the group of the unitary matrices of size 1. We shall use the following identification of groups

$$\mathbb{R}/2\pi\mathbb{Z} \ni \theta + 2\pi\mathbb{Z} \to e^{i\theta} \in \mathrm{U}_1\,.$$

Here we follow [SS03, section 2.2]. Recall that a function $f : \mathbb{Z} \to \mathbb{C}$ is called rapidly decreasing if

$$\sup_{n\in\mathbb{Z}} |(1 + |n|)^k |f(n)| < \infty \qquad (n \in \mathbb{Z})\,.$$

Theorem 9. *The Fourier transform*

$$\hat{f}(n) = \mathcal{F}f(n) = \int_0^1 e^{-2\pi i x n} f(x)\, dx \qquad (n \in \mathbb{Z},\ f \in C^\infty(\mathrm{U}_1)) \tag{6}$$

maps the space $C^\infty(\mathrm{U}_1)$ into the space of the rapidly decreasing functions on $\mathbb{Z}$, is invertible, and the inverse given by

$$f(x) = \sum_{n\in\mathbb{Z}} e^{2\pi i x n} \mathcal{F}f(n)\, dx \qquad (x \in \mathbb{R},\ f \in C^\infty(\mathrm{U}_1))\,. \tag{7}$$

(Notice that the integral (6) converges as long as the function f is absolutely integrable. Hence we have the Fourier $\mathcal{F}f$ transform for any $f \in \mathrm{L}^1(\mathrm{U}_1)$.) Since

$$\int_0^1 e^{-2\pi i x n} f'(x)\, dx = 2\pi i n \mathcal{F}f(y)\,, \tag{8}$$

the rapid decrease of $\mathcal{F}f$ is easy to check. For the inversion formula we need the following Lemma.

Lemma 10. *If $f : \mathrm{U}_1 \to \mathbb{C}$ is continuous and $\mathcal{F}f = 0$, then $f = 0$.*

Proof. Since

$$\int_0^1 e^{-2\pi i x n} f(x+y)\, dx = e^{2\pi i y n} \int_0^1 e^{-2\pi i x n} f(x)\, dx$$

it'll suffice to show that $f(0) = 0$.

Suppose the claim is false. We may assume that f is real valued and that $f(0) > 0$. We shall arrive at a contradiction. Choose $0 < \delta \leq \frac{\pi}{2}$ so that

$$f(x) > \frac{f(0)}{2} \qquad (|x| < \delta)\,.$$

Let $\epsilon > 0$ be so small that the function

$$p(x) = \epsilon + \cos x$$

satisfies

$$|p(x)| < 1 - \frac{\epsilon}{2} \qquad (\delta \leq |x| \leq \pi)\,.$$

Choose $0 < \eta < \delta$ so that

$$|p(x)| \geq 1 + \frac{\epsilon}{2} \qquad (|x| \leq \eta)\,.$$

Then for $k = 0, 1, 2, ...,$

$$\left| \int_{\delta \leq |x| \leq \pi} f(x) p(x)^k\, dx \right| \leq 2\pi \parallel f \parallel_\infty \left(1 - \frac{\epsilon}{2}\right)^k\,,$$

$$\int_{\eta \leq |x| < \delta} f(x) p(x)^k\, dx \geq 0\,,$$

$$\int_{|x| < \eta} f(x) p(x)^k\, dx \geq 2\eta \frac{f(0)}{2} \left(1 + \frac{\epsilon}{2}\right)^k\,.$$

Therefore

$$\lim_{k\to\infty} \int_{|x|\leq\pi} f(x)p(x)^k\, dx = +\infty\,.$$

However $p(x)^k$ is a trigonometric polynomial (linear combination of powers of exponentials). Hence the assumption $\mathcal{F}f = 0$ implies

$$\int_{|x|\leq\pi} f(x)p(x)^k\, dx = 0\,.$$

□

5. **Some Functional Analysis on a Hilbert space**

Here we follow [Lan85, sections 1.2 and 1.3]. Let V be a Hilbert space. A continuous linear map $A : \mathsf{V} \to \mathsf{V}$ is called compact if it any bounded sequence $v_n \in \mathsf{V}$ to a sequence Av_n that has a convergent subsequence.

Theorem 11. *Let A be a compact hermitian operator on the Hilbert space V. Then the family of eigenspaces V_λ, where λ ranges over all eigenvalues (including 0), is an orthogonal decomposition of E:*

$$\mathsf{V} = \bigoplus_\lambda \mathsf{V}_\lambda\,.$$

For $\lambda \neq 0$ the eigenspace V_λ is finite dimensional.

Let S be a set of operators (continuous linear maps) on V. We say that V is S-irreducible if V has no closed S-invariant subspace other than $\{0\}$ and V itself. We say that V is completely reducible for S if V is the orthogonal direct sum of S-irreducible subspaces. Two subspaces $\mathsf{V}_1, \mathsf{V}_2 \subseteq E$ are called S-isomorphic if there is an isometry from V_1 onto V_2 which intertwines the action of S. The number of elements in such an isomorphism class is called the multiplicity of that isomorphism class in V.

A subalgebra $\mathcal{A}$ of operators on V is said to be $*$-closed if whenever $A \in \mathcal{A}$, then $A^* \in \mathcal{A}$. As explained in [Lan85, section 1.2], the following Theorem is a consequence of Theorem 11.

Theorem 12. *Let $\mathcal{A}$ be a $*$-closed subalgebra of compact operators on a Hilbert space V. Then V is completely reducible for $\mathcal{A}$, and each irreducible subspace occurs with finite multiplicity.*

The following two theorems and the corollary below are known as Schur's Lemma.

Theorem 13. *Let S be a set of operators acting irreducibly on a finite dimensional vector space V. Let A be an operator such that $AB = BA$ for all $B \in S$. Then $A = \lambda I$ for some complex number λ.*

Proof. Let λ be an eigenvalue of A and W denotes the null space of $A - \lambda I$. Since $AB = BA$ for all $B \in S$, it is easy to see that W is S-invariant. As S acts irreducibly on V and $W \neq 0$, it follows that $W = V$. Hence $A = \lambda I$. □

Theorem 14. *Let S be a set of operators acting irreducibly on the Hilbert space* V*. Let A be a hermitian operator such that $AB = BA$ for all $B \in S$. Then $A = cI$ for some real number c.*

Proof. See [Lan85, Appendix 1, Theorem 4]. □

Corollary 15. *Let S be a set of operators acting irreducibly on the Hilbert space* V*. Let A be an operator such that $AB = BA$ and $A^*B = BA^*$ for all $B \in S$. Then $A = cI$ for some $c \in \mathbb{C}$.*

Proof. We write $A = B + iC$, where

$$B = \frac{1}{2}(A + A^*)\,, \quad C = \frac{1}{2i}(A - A^*)\,.$$

Then B and C commute with S. Hence there are real numbers b and c such that $B = bI$ and $C = cI$. Hence $A = (b + ic)I$. □

We shall also use a few facts concerning integral kernel operators. The proofs may be found in [Lan85, section 1.3]

Theorem 16. *Let (X, M, dx) and (Y, N, dy) be measured spaces, and assume that $L^2(X)$, $L^2(Y)$ have countable orthogonal bases. Let $q \in L^2(X \times Y)$. Then the the formula*

$$Qf(x) = \int_Y q(x, y)f(y)dy$$

defines a bounded, compact operator from $L^2(Y)$ into $L^2(X)$ with $\| Q \| \leq \| q \|_2$. If $(X, M, dx) = (Y, N, dy)$ then Q is of trace class. If X *is a topological space, dx is a Borel measure and in addition q is continuous, then*

$$\operatorname{tr} Q = \int_X q(x, x)\, dx\,.$$

6. **Representations of locally compact groups**

Let G be a locally compact group and let V be a topological vector space. We shall always assume that $\mathsf{V} \neq \{0\}$. Let $\mathrm{GL}(\mathsf{V})$ denote the group of the invertible continuous endomorphisms of V. A representation of G on V is a pair (π, V), where $\pi : \mathrm{G} \to \mathrm{GL}(\mathsf{V})$ is a group homomorpism such that the map

$$\mathrm{G} \times \mathsf{V} \ni (g, v) \to \pi(g)v \in \mathsf{V}$$

is continuous. A subspace $\mathsf{W} \subseteq \mathsf{V}$ is called invariant if $\rho(\mathrm{G})\mathsf{W} \subseteq \mathsf{W}$. The representation (ρ, V) is called irreducible if V does not contain any closed invariant subspaces other than $\{0\}$ an V.

If (π_1, V_1), (π_2, V_2) are representations of G, then a continuous linear map $T\mathsf{V}_1 \to \mathsf{V}_2$ such that

$$T\pi_1(g) = \pi_2(g)T \qquad (g \in \mathrm{G})$$

is called an intertwining operator, or a G-homomorphism. The vector space of all the intertwining maps will be denoted $\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}_1, \mathsf{V}_2)$ or more precisely, $\mathrm{Hom}_{\mathrm{G}}((\pi_1, \mathsf{V}_1), (\pi_2, \mathsf{V}_2))$. The representations (π_1, V_1), (π_2, V_2) are equivalent if there exists a bijective $T \in \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}_1, \mathsf{V}_2)$.

A basic example of a representation of G is the right regular representation $(R, C(\mathrm{G})$. Here $C(\mathrm{G})$ is the space of the continuous, complex valued, functions on G (with the topology of uniform convergence on compact sets) and

$$R(g)f(x) = f(xg) \qquad (g, x \in \mathrm{G}, f \in C(\mathrm{G}))\,.$$

Similarly, we have the left regular representation $(L, C(\mathrm{G}))$,

$$L(g)f(x) = f(g^{-1}x) \qquad (g, x \in \mathrm{G}, f \in C(\mathrm{G}))\,.$$

Given an irreducible representation (π, V) and an element $\lambda \in \mathsf{V}'$ (a continuous linear functional on V) the forula

$$Tv(x) = \lambda(\pi(x)v) \qquad (x \in \mathrm{G})$$

defines an injective map $T \in \mathrm{Hom}_{\mathrm{G}}((\pi, \mathsf{V}), (R, C(\mathrm{G})))$. Thus every abstract representation of G may be viewed as a subrepresentation of the right regular representation on the continuous functions on G. The function

$$\mathrm{G} \ni x \to \lambda(\pi(x)v) \in \mathbb{C}$$

is called a matrix coefficient of the representation (π, V).

If V is a Hilbert space, then a representation (π, V) is called unitary if every operator $\pi(g)$, $g \in \mathrm{G}$, is unitary. Two unitary representations (π_1, V_1), (π_2, V_2) are called unitarili equivalen if there is a bjiective and isometric G-homomorphism $T : \mathsf{V}_1 \to \mathsf{V}_2$.

Here is the classical version of Schur's Lemma, which is an immediate consequence of Corollary 15.

Theorem 17. *Let (π, V) be an irreducible unitary representation of* G. *Then*

$$\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}) = \mathbb{C}I\,.$$

Corollary 18. *For two irreducible unitary representations (ρ, V) and (ρ', V') of* G,

$$dim\ \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}') = \begin{cases} 1 \ \text{if } (\rho, \mathsf{V}) \simeq (\rho', \mathsf{V}') \\ 0 \ \text{if } (\rho, \mathsf{V}) \not\simeq (\rho', \mathsf{V}') \end{cases}$$

(Here $\simeq$ stands for unitary equivalence.)

Proposition 19. *If two unitary representations (π, V), (π', V') are equivalent then they are unitarily equivalent.*

Proof. Let $(\ ,\ )$ be the invariant scalar product on V and let $(\ ,\ )'$ be the invariant scallar product on V'. Pick an isomorphism $T : \mathsf{V} \to \mathsf{V}'$. Define $T^* : \mathsf{V}' \to \mathsf{V}$ by

$$(Tu, v)' = (u, T^*v) \qquad (u \in \mathsf{V}, v \in \mathsf{V}).$$

Then $T^*\mathsf{V}' \to \mathsf{V}$ is also a morphism. Hence, $T^*T : \mathsf{V} \to \mathsf{V}$ commutes with the action of G. Hence, there is $\lambda \in \mathbb{C}$ such that $T^*T = \lambda I$. Thus, for any $u, v \in \mathsf{V}$,

$$(Tu, Tv)' = (u, T^*Tv) = \overline{\lambda}(u, v).$$

In particular, by taking $u = v \neq 0$ we see that $\lambda > 0$. Hence,

$$\frac{1}{\sqrt{\lambda}} T : \mathsf{V} \to \mathsf{V}'$$

is an isometry and a morphism. □

Here is one more version of Schur's Lemma. For a proof see [Wal88a, 1.2.2].

Theorem 20. *Let (π, V) be an irreducible unitary representation of G. Let $\mathsf{V}_0 \subseteq \mathsf{V}$ be a dense G-invariant subspace and let $A : \mathsf{V}_0 \to \mathsf{V}$ be a linear G-intertwining map.*

Suppose $\mathsf{V}_1 \subseteq \mathsf{V}$ is a dense subspace and let $B : \mathsf{V}_1 \to \mathsf{V}$ a linear map such that

$$(Av_0, v_1) = (v_0, Bv_1) \qquad (v_0 \in \mathsf{V}_0, v_1 \in \mathsf{V}_1)\,.$$

Then A is a scalar multiple of the identity on V, restricted to V_0.

7. Haar measures and extension of a representation of G to a representation of $L^1(\mathrm{G})$

A proof of the following theorem may be found in [HR63, (15.5)-(15.11)]

Theorem 21. *There is a positive Borel measure dx on G such that*

$$\int_{\mathrm{G}} f(gx)\,dx = \int_{\mathrm{G}} f(x)\,dx \qquad (g \in \mathrm{G}, f \in C_c(\mathrm{G}))\,.$$

This measure is unique up to a constant multiple. Furthermore there is a group homomrphism $\Delta : \mathrm{G} \to \mathbb{R}^+$ such that

$$\int_{\mathrm{G}} f(xg)\,dx = \Delta(g) \int_{\mathrm{G}} f(x)\,dx \qquad (g \in \mathrm{G}, f \in C_c(\mathrm{G}))\,.$$

The measure dx is called the left invariant Haar measure on G. The group G is called unimodular if $\Delta = 1$. This is certainly the case if G is compact. The Lebesgue measures we used in sections 2 and 4 are Haar measures and the groups are unimodular. An example of a non-unimodular group is

$$P = \left\{ p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} a \in R^\times, b \in \mathbb{R} \right\}\,.$$

Here $|a|^{-2} da\, db$ is the left invariant Haar measure and $\Delta(p) = a^2$.

The convolution $\phi * \psi$ of two functions $\phi, \psi \in \mathrm{L}^1(\mathrm{G})$ is defined by

$$\phi * \psi(y) = \int_{\mathrm{G}} \phi(x)\psi(x^{-1}y)\,dx\,.$$

The Banach space $\mathrm{L}^1(\mathrm{G})$ with the product defined by the convolution is a Banach algebra. This algebras may have no identity, but they always has an approximate identity.

Theorem 22. *There is a sequence $\phi_n \in C_c(\mathrm{G})$ such that*

(1) $\phi_n \geq 0 \qquad (n = 0, 1, 2, \ldots)$,

(2) $\int_{\mathrm{G}} \phi_n(x)\, dx = 1 \qquad (n = 0, 1, 2, \ldots)$,

(3) *Given a neighborhood U of the identity in* G,
the support of ϕ_n is contained in U for all n sufficiently large.

(See [Lan85, secion 1.1] for a proof.) Given a representation (π, V) the formula

$$\pi(\phi) = \int_{\mathrm{G}} \phi(x)\pi(x) \qquad (\phi \in \mathrm{L}^1(\mathrm{G}))$$

defines a representation of the Banach algebra $\mathrm{L}^1(\mathrm{G})$ on V, i.e.

$$\pi(\phi * \psi) = \pi(\phi)\pi(\psi) \qquad (\phi, \psi \in \mathrm{L}^1(\mathrm{G}))\,.$$

By restriction, (π, V) is also a representation of the convolution algebra $C_c(\mathrm{G})$. As straightforward consequence of Theorem 22 we see that a subspace $\mathsf{W} \subseteq \mathsf{V}$ is G-invariant if and only if it is $C_c(\mathrm{G})$-invariant. Also, (π, V) is G-irreducible if and only if it is $C_c(\mathrm{G})$-irreducible.

8. **Representations of a compact group**

Let G be a compact group with the Haar measure of total mass equal to 1. If (π, V) is a representation of G on a Hilbert space V, then, by averaging the norm on V over G, we obtain another norm with respect to which the representation is unitary.

Hence, while considering representations of G on Hilbert spaces, we may assume that they are unitary.

Theorem 23. *Any irreducible unitary representation (π, V) of* G *is finite dimensional.*

Proof. Let $v \in \mathsf{V}$ be a unit vector and let P be the orthogonal projection on the one-dimensional space $\mathbb{C}v$. Let Q be the continuous linear map defined by

$$Q = \int_{\mathrm{G}} \pi(x)^{-1} P \pi(x)\, dx\,.$$

Then $Q = Q^*$ commutes with the action of G. Hence Schur's Lemma, Theorem 16, implies that there is a constant $c \in \mathbb{R}$ such that $Q = cI$. Since π is unitary,

$$(Qv, v) = \int_{\mathrm{G}} (P\pi(x)v, \pi(x)v)\, dx = \int_{\mathrm{G}} (\pi(x)v, v)(v, \pi(x)v)\, dx = \int_{\mathrm{G}} |(\pi(x)v, v)|^2\, dx > 0\,.$$

Hence $c > 0$.

Let v_1, v_2, ... be an orthonormal basis of V. Then for each $x \in \mathrm{G}$, $\pi(x)v_1$, $\pi(x)v_2$, ...is also an orthonormal basis of V. Hence

$$\sum_{n=1}^{\infty} (P\pi(x)v_n, \pi(x)v_n) = \sum_{n=1}^{\infty} ((\pi(x)v_n, v)v, \pi(x)v_n) = \sum_{n=1}^{\infty} |(\pi(x)v_n, v)|^2 = \| v \| = 1\,.$$

But $(P\pi(x)v_n, \pi(x)v_n) = (\pi(x)^{-1}P\pi(x)v_n, v_n)$. Hence, after integration over G we see that

$$\sum_{n=1}^{\infty}(Qv_n, v_n) = 1\,.$$

Thus c times the number of the elements in the basis is equal to 1. Hence, $\dim \mathsf{V} < \infty$. $\square$

By combining Theorem 23 with the Spectral Theorem from Linear Algebra we deduce the following corollary.

Corollary 24. *An irreducible unitary representation of a compact abelian group is one dimensional.*

If $\dim \mathsf{V} < \infty$, we let V^c denote the vector space dual to V. The contragredient representation (π^c, V^c) is defined by

$$\pi^c(g)v^c(v) = v^c(\pi(g^{-1})v) \qquad (v \in \mathsf{V},\ v^c \in \mathsf{V}^c, g \in \mathrm{G}).$$

Given two finite dimensional representations (π, V) and (π', V') define their direct sum $(\pi \oplus \pi', \mathsf{V} \oplus \mathsf{V}')$ by

$$(\pi \oplus \pi')(g)(v, v') = (\pi(g)(v), \pi'(g)(v')) \quad (g \in \mathrm{G}, v \in \mathsf{V}, v' \in \mathsf{V}')$$

and the tensor product $(\pi \otimes \pi', \mathsf{V} \otimes \mathsf{V}')$ by

$$(\pi \otimes \pi')(g)[v \otimes v'] = [\pi(g)(v)] \otimes [\pi'(g)(v')] \quad (g \in \mathrm{G}, v \in \mathsf{V}, v' \in \mathsf{V}').$$

By definition the character $\Theta_{\mathsf{V}} = \Theta_\pi$ of a finite dimensional representation (π, V) is the following complex valued function on the group:

$$\Theta_{\mathsf{V}}(g) = tr(\pi(g)) \qquad (g \in \mathrm{G}).$$

This function is invariant under conjugation

$$\Theta(hgh^{-1}) = \Theta(g) \qquad (h, g \in \mathrm{G}).$$

Also, we have

$$\Theta_{\mathsf{V}\oplus\mathsf{V}'} = \Theta_{\mathsf{V}} + \Theta_{\mathsf{V}'},\ \Theta_{\mathsf{V}\otimes\mathsf{V}'} = \Theta_{\mathsf{V}}\Theta_{\mathsf{V}'} \text{ and } \Theta_{\pi^c} = \overline{\Theta_\pi}. \tag{9}$$

Denote by $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} \subseteq \mathrm{L}^2(\mathrm{G})$ the subspace of the functions invariant by the conjugation by all the elements of G. Our characters live in $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}}$.

Lemma 25. *Suppose (π, V) is a non-trivial irreducible unitary representation of* G*. Then*

$$\int_{\mathrm{G}} \pi(x)v\, dx = 0 \qquad (v \in \mathsf{V}).$$

Proof. Notice that the integral defines a G-invariant vector $u \in \mathsf{V}$. Since π is irreducible, either $u = 0$ or $\mathsf{V} = \mathbb{C}u$. Since π is non-trivial, the second option is impossible. $\square$

Corollary 26. *Suppose (π, V) is a non-trivial irreducible unitary representation of* G*. Then*

$$\int_{\mathrm{G}} \pi(x)\, dx = 0\,.$$

Corollary 27. *Suppose* (π, V) *is a finite dimensional representation of* G. *Then*

$$\int_{\mathrm{G}} \Theta_\pi(x)\, dx = dim\, \mathsf{V}^{\mathrm{G}},$$

where $\mathsf{V}^{\mathrm{G}} \subseteq \mathsf{V}$ *is the space of the* G*-invariant vectors.*

Proposition 28. *The characters of irreducible representations form an orthonormal set in* $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}}$.

Proof. Consider two such representations (π, V) and (π', V'). The group G acts on this vector space $\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}')$ by

$$gT(v) = \pi'(g)T\pi(g^{-1})v \qquad (g \in \mathrm{G}, T \in \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}'), v \in \mathsf{V}).$$

This way $\mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}')$ becomes a representation of G. It is easy to check that as such it is isomorphic to $(\pi^c \otimes \pi', \mathsf{V}^c \otimes \mathsf{V}')$. Hence, by (9),

$$\Theta_{\mathrm{Hom}(\mathsf{V},\mathsf{V}')}(g) = \overline{\Theta_\pi(g)}\Theta_{\pi'}(g) \qquad (g \in \mathrm{G}).$$

Therefore,

$$\begin{aligned}(\Theta_\pi', \Theta_\pi) &= \int_{\mathrm{G}} \Theta_{\mathrm{Hom}(\mathsf{V},\mathsf{V}')}(g)\, dg \\ &= \text{dimension of the space of the G-invariants in } \mathrm{Hom}_{\mathrm{G}}(\mathsf{V}, \mathsf{V}').\end{aligned}$$

Thus the formula follows from Corollary 18. □

Proposition 29. *Any finite dimensional representation of* G *decomposes into the direct sum of irreducible representations.*

Proof. This follows from the fact that the orthogonal complement of a G-invariant subspace is G-invariant. □

Let (π, V) be a finite dimensional representation of G and let

$$\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2 \oplus ... \oplus \mathsf{V}_n$$

be the decomposition into irreducibles. The number of the V_j which are isomorphic to V_1 is called the multiplicity of V_1 in V, denoted m_1. We may collect the isomorphic representations in the above formula and (after changing the indecies appropriately) get the following decomposition

$$\mathsf{V} = \mathsf{V}_1^{\oplus m_1} \oplus \mathsf{V}_2^{\oplus m_2} \oplus ... \oplus \mathsf{V}_k^{\oplus m_k},$$

where the V_j are irreducible and mutually non-isomorphic and

$$\mathsf{V}_j^{\oplus m_j} = \mathsf{V}_j \oplus \mathsf{V}_j \oplus ... \oplus \mathsf{V}_j \ (m_j \text{ summands}).$$

This is the primary decomposition of V, which is also denoted by

$$\mathsf{V} = m_1\mathsf{V}_1 \oplus m_2\mathsf{V}_2 \oplus ... \oplus m_k\mathsf{V}_k = m_1 \cdot \mathsf{V}_1 \oplus m_2 \cdot \mathsf{V}_2 \oplus ... \oplus m_k \cdot \mathsf{V}_k,$$

for brevity.

Proposition 30. *Let*

$$\mathsf{V} = m_1\mathsf{V}_1 \oplus m_2\mathsf{V}_2 \oplus ... \oplus m_k\mathsf{V}_k,$$

be the primary decomposition of V. *Then*

$$\Theta_{\mathsf{V}} = m_1\Theta_{\mathsf{V}_1} + m_2\Theta_{\mathsf{V}_2} + ..m_k\Theta_{\mathsf{V}_k}$$

and

$$m_j = (\Theta_{\mathsf{V}}, \Theta_{\mathsf{V}_j}).$$

In particular V *is irreducible if and only if* $(\Theta_{\mathsf{V}}, \Theta_{\mathsf{V}}) = 1$.

Proof. This follows from (9) and from Lemma 28. □

Given a finite dimensional representation (π, V) define a map

$$M_\pi : \mathrm{End}(\mathsf{V}) \to \mathrm{L}^2(\mathrm{G}),\ M_\pi(T)(g) = tr(\pi(g)T) = tr(T\pi(g)).$$

In particular, if $I \in \mathrm{End}(\mathsf{V})$ is the identity, then

$$M_\pi(I) = \Theta_\pi.$$

Also, it is easy to see that the subspace $M_\pi(\mathrm{End}(\mathsf{V})) \subseteq \mathrm{L}^2(\mathrm{G})$ depends only on the equivalence class of π.

Theorem 31. *Suppose* (π, V) *and* (π', V') *are two irreducible unitary representations of* G. *Then*

a) *if* (π, V) *is not equivalent to* (π', V'), *then* $M_\pi(\mathrm{End}(\mathsf{V})) \perp M_{\pi'}(\mathrm{End}(\mathsf{V}'))$,
b) *for any* $S, T \in \mathrm{End}(\mathsf{V})$, $(M_\pi(S), M_\pi(T)) = \frac{1}{dim\,\mathsf{V}} tr(ST^*)$.

Proof. Define a representation $(\Pi, \mathrm{End}(\mathsf{V}))$ of the group $\mathrm{G}\times\mathrm{G}$ on the vetor space $\mathrm{End}(\mathsf{V})$ by

$$\Pi(g_1, g_2)T = \pi(g_2)T\pi(g_1^{-1}) \qquad (g_1, g_2 \in \mathrm{G},\ T \in \mathrm{End}(\mathsf{V})).$$

Notice that $(\Pi, \mathrm{End}(\mathsf{V}))$ is isomorphic to outer tensor product $(\pi^c \otimes \pi, \mathsf{V}^c \otimes \mathsf{V})$. Hence,

$$\int_{\mathrm{G}\times\mathrm{G}} |\Theta_\Pi(x_1, x_2)|^2\, dx_1\, dx_2 = \int_{\mathrm{G}\times\mathrm{G}} |\overline{\Theta_\pi(x_1)}\Theta_\pi(x_2)|^2\, dx_1\, dx_2$$
$$= \int_{\mathrm{G}} |\Theta_\pi(x_1)|^2\, dx_1 \int_{\mathrm{G}} |\Theta_\pi(x_2)|^2\, dx_2 = 1.$$

Hence, by Proposition 30, $(\Pi, \mathrm{End}(\mathsf{V}))$ is irreducible.

We view $\mathrm{End}(\mathsf{V})$ as a Hilbert space with the following scalar product

$$(S_1, S_2) = tr(S_1S_2^*) \qquad (S_1, S_2 \in \mathrm{End}(\mathsf{V})),$$

and similarly for V'. In particular, we have the adjoint map

$$M_{\pi'}^* : \mathrm{L}^2(\mathrm{G}) \to \mathrm{End}(\mathsf{V}).$$

Notice also that

$$M_{\pi'}^* M_\pi : \mathrm{End}(\mathsf{V}) \to \mathrm{End}(\mathsf{V}')$$

is a $\mathrm{G}\times\mathrm{G}$-intertwining map. If the two representations (π, V) and (π', V') are not isomorphic then The $\mathrm{G}\times\mathrm{G}$-modules $\mathrm{End}(\mathsf{V})$ and $\mathrm{End}(\mathsf{V}')$ are not isomorphic. Hence Corollary 18 shows that $M_{\pi'}^* M_\pi = 0$. Thus for $S \in \mathrm{End}(\mathsf{V})$ and $T \in \mathrm{End}(\mathsf{V}')$,

$$(M_\pi(S), M_{\pi'}(T)) = (S, M_{\pi'}^* M_\pi(T)) = 0$$

and a) follows.

Similarly, there is $\lambda \in \mathbb{C}$ such that $M_\pi^* M_\pi = \lambda I$. Furtheremore,

$$\lambda \cdot dim\, \mathsf{V} = (I, M_\pi^* M_\pi(I)) = (M_\pi(I), M_\pi(I)) = (\Theta_\pi, \Theta_\pi) = 1$$

and b) follows. □

Theorem 32. *There are the following direct sum orthogonal decompositions of Hilbert spaces*

a) $\mathrm{L}^2(\mathrm{G}) = \sum_{\pi\in\hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$,
b) $\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} = \sum_{\pi\in\hat{\mathrm{G}}} \mathbb{C}\Theta_\pi$.

Proof. Part b) follows from part a), because, by Theorem 17, $\mathrm{End}(\mathsf{V})^{\mathrm{G}} = \mathbb{C}I$. Indeed,

$$\mathrm{L}^2(\mathrm{G})^{\mathrm{G}} = \sum_{\pi\in\hat{\mathrm{G}}} (M_\pi(\mathrm{End}(\mathsf{V})))^{\mathrm{G}} = \sum_{\pi\in\hat{\mathrm{G}}} \mathrm{M}_\pi(\mathrm{End}(\mathsf{V})^{\mathrm{G}}), = \sum_{\pi\in\hat{\mathrm{G}}} \mathbb{C}\Theta_\pi.$$

where only the middle equation

$$(M_\pi(\mathrm{End}(\mathsf{V})))^{\mathrm{G}} = \mathrm{M}_\pi(\mathrm{End}(\mathsf{V})^{\mathrm{G}})$$

requires an explanation. We need to show that if

$$tr(T\pi(ghg^{-1})) = tr(T\pi(h))$$

for all $h \in \mathrm{G}$, then $\pi(g^{-1})T\pi(g) = T$. Since $tr(T\pi(ghg^{-1}) = tr(\pi(g^{-1})T\pi(g)\pi(h))$, we'll be done as soon as we show that if $tr(S\pi(h)) = 0$ for all $h \in \mathrm{G}$, then $S = 0$. But, since the map M_π is injective (because π is irreducible), this is indeed the case.

Part a) requires some work. We follow the argument in [Kna86, Theorem 1.12]. Let $\mathrm{U} = \sum_{\pi\in\hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$. This is a subspace of $\mathrm{L}^2(\mathrm{G})$ closed under the left and right translations and under the $*$ operation, $\phi \to \phi^*$, $\phi^*(x) = \overline{\phi(x^{-1})}$. Hence so is the orthogonal complement $\mathrm{U}^\perp \subseteq \mathrm{L}^2(\mathrm{G})$.

Suppose $\mathrm{U}^\perp \neq \{0\}$. We shall arrive at a contradiction. Let $\phi \in \mathrm{U}^\perp$ be non-zero. Let $\phi_n \in C_c(\mathrm{G})$ be an approximate identity, as in Theorem 22. Then a straightforward argument shows that

$$\lim_{n\to\infty} \parallel \phi_n * \phi - \phi \parallel_2 = 0\,.$$

Hence there is n such that $\phi_n * \phi \neq 0$. But this function is continuous. Thus we may assume that ϕ is continuous. Furthermore, applying the translations and the $*$ operation we may assume that $\phi = \phi^*$ and $\phi(1) > 0$. Replacing ϕ by the integral

$$\int_{\mathrm{G}} L(x)R(x)\phi\, dx$$

we may assume that ϕ is invariant under conjugation. Let

$$T\psi(x) = \int_{\mathrm{G}} \phi(x^{-1}y)\psi(y)\, dy\,.$$

Since the integral kernel $\phi(x^{-1}y)$ is continuous on $\mathrm{G} \times \mathrm{G}$, T is a compact operator on $\mathrm{L}^2(\mathrm{G})$. Furthermore, $T = T^* \neq 0$. Hence, by Theorem 11, T has a non-zero finite dimensional eigenspace $\mathsf{V}_\lambda \subseteq \mathrm{L}^2(\mathrm{G})$. Since T commutes with $L(\mathrm{G})$, V_λ is closed under $L(\mathrm{G})$. Hence Theorem 29 shows that there is a G-irreducible subspace $\mathsf{W}_\lambda \subseteq \mathsf{V}_\lambda$. Let f_j be an orthonarmal basis of W_λ. Set

$$h_{i,j}(x) = (L(x)f_i, f_j) = \int_{\mathrm{G}} f_i(x^{-1}y)\overline{f(j(y)}\, dy\,.$$

This is a matrix coefficient of an irreducible representation of G, thus it belongs to U. Therefore,

$$0 = (\phi, h_{i,i}) = \int_{\mathrm{G}}\int_{\mathrm{G}} \phi(x^{-1}y)\overline{f_i(y)}\, dy\, dx = \lambda \int_{\mathrm{G}} f_i(x)\overline{f_i(y)}\, dx\,,$$

which is a contradiction. □

Theorem 33. *Any unitary representation (ρ, W) of* G *is completely reducible, i.e. the Hilbert space* W *is the orthogonal sum of irreducible finite dimensional representations of* G*.*

The converse is not true, i.e. there are locally compact but not compact groups for which the right regular representation decomposes into a direct sum of irreducible, see [Bag72] or [Knu17]. They are called "Fell groups".

Proof. We follow [Kna86, Theorem 1.12]. Suppose (ρ, W) is not completely reducible. By Zorn's Lemma we may choose a maximal orthogonal set of finite dimensional irreducible invariant subspaces. Let U denote the closure of their sum. Suppose $0 \neq v \in \mathrm{U}^\perp$. We'll arrive at a contradiction.

Let $\phi_n \in C_c(\mathrm{G})$ be an approximate identity, as in Theorem 22. Then, as we have seen before, there is n such that $\rho(\phi_n)v \neq 0$. We fix this n.

Theorem 32 a) implies that there is a finite set $F \subseteq \hat{\mathrm{G}}$ and $\phi \in \bigoplus_{\pi\in F} M_\pi(\mathrm{End}(\mathsf{V}))$, such that

$$\| \phi_n - \phi \|_2 \leq \frac{1}{2\| v \|} \| \rho(\phi_n)v \| \ .$$

Since the total mass of G is 1, we have

$$\| \phi_n - \phi \|_1 \leq \| \phi_n - \phi \|_2 \ .$$

Hence

$$\| \rho(\phi_n)v - \rho(\phi)v \| \leq \| \phi_n - \phi \|_1 \| v \| \leq \frac{1}{2} \| \rho(\phi_n)v \| \ .$$

Therefore

$$\| \rho(\phi)v \| \geq \| \rho(\phi_n)v \| - \| \rho(\phi_n)v - \rho(\phi)v \| \geq \frac{1}{2} \| \rho(\phi_n)v \| > 0\,.$$

This is a contradiction because $\rho(\phi)v$ lies in a finite dimensional invariant subspace of W. □

For a function $\phi \in \mathrm{L}^1(\mathrm{G})$ define the Fourier transform

$$\mathcal{F}\phi(\pi) = \int_{\mathrm{G}} \phi(x)\pi(x)\,dx = \int_{\mathrm{G}} \phi(x)\pi(x)\,dx = \pi(\phi)\,. \tag{10}$$

Notice that in order to be consistent with the theory of Fourier series, section 4, we should replace π by π^c in this definition. However we shall follow the tradition and not do that.

Thus for a representation (π, V), $\mathcal{F}\phi(\pi) \in \mathrm{End}(\mathsf{V})$. Notice that

$$\pi(g)\mathcal{F}\phi(\pi) = \mathcal{F}(L(g)\phi) \qquad (g \in \mathrm{G})\,. \tag{11}$$

Set $d(\pi) = \dim \mathsf{V}) = \Theta_\pi(1)$. This is the degree of the representation (π, V).

Theorem 34. *(Fourier inversion for* G*) For any* $\phi \in \bigoplus_{\pi\in\hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$ *(algebraic sum),*

$$\phi(g) = \sum_{\pi\in\hat{\mathrm{G}}} d(\pi)\cdot tr(\mathcal{F}\phi(\pi)\pi(g^{-1})) \qquad (g \in \mathrm{G})\,, \tag{12}$$

or equivalently

$$\phi(1) = \sum_{\pi\in\hat{\mathrm{G}}} d(\pi)\cdot \Theta_\pi(\phi) \qquad (g \in \mathrm{G})\,, \tag{13}$$

Notice that in the case of the classical Fourier series, section 4, the above formula (12) refers only to trigonometric polynomials, ϕ.

Proof. Clearly (13) is a particular case of (12). Also, (12) follows from (13) and (11).

Let us write (π, V_π) for (π, V) in order to indicate the dependence of the vector space on π. By definition, there is a finite set $F \subseteq \hat{\mathrm{G}}$ and operators $T_\pi \in \mathrm{End}(\mathsf{V}_\pi)$ such that

$$\phi(g) = \sum_{\rho\in F} \mathrm{tr}(T_\rho\rho(g)) \qquad (g \in \mathrm{G})\,.$$

Hence

$$\mathcal{F}\phi(\pi) = \sum_{\rho\in F}\int_{\mathrm{G}} \mathrm{tr}(T_\rho\rho(g))\pi(g)\,dg\,.$$

Therefore, by Theorem 31

$$\begin{aligned}\mathrm{tr}\,\mathcal{F}\phi(\pi) &= \sum_{\rho\in F}\int_{\mathrm{G}} \mathrm{tr}(T_\rho\rho(g))\,\mathrm{tr}\,\pi(g)\,dg = \int_{\mathrm{G}} \mathrm{tr}(T_\pi\pi(g))\,\mathrm{tr}\,\pi(g)\,dg\\ &= (T_\pi\pi(g)), \pi(g)) = \frac{1}{d_\pi}\,\mathrm{tr}(T_\pi\pi(g)\pi(g)^*) = \frac{1}{d_\pi}\,\mathrm{tr}(T_\pi)\,.\end{aligned}$$

Hence

$$\sum_{\pi\in\hat{\mathrm{G}}} d(\pi)\cdot\Theta_\pi(\phi) = \sum_{\pi\in\hat{\mathrm{G}}} d(\pi)\cdot\mathrm{tr}\,\mathcal{F}\phi(\pi) = \sum_{\pi\in\hat{\mathrm{G}}} tr(T_\pi) = \phi(1).$$

□

Theorem 35. *(Parseval formula) For any* $\phi \in \bigoplus_{\pi\in\hat{\mathrm{G}}} M_\pi(\mathrm{End}(\mathsf{V}))$,

$$\| \phi \|_2^2 = \sum_{\pi\in\hat{\mathrm{G}}} d_\pi \ \| \mathcal{F}\phi(\pi) \|^2 \,.$$

Proof. That the left hand side is equal to

$$\phi * \phi^*(1) = \sum_{\pi\in\hat{\mathrm{G}}} d_{\pi^c}\, tr(\mathcal{F}\phi(\pi)\mathcal{F}\phi(\pi)^*),$$

which coincides with the right hand side. □

9. Square integrable representations of a locally compact group

In this section G is a locally compact unimodular group. We follow [Wal88a, section 1.3].

Lemma 36. *Let* (τ, V) *be an irreducible unitary representation of* G*. Suppose that for some non-zero vectors* $u_0, v_0 \in \mathsf{V}$

$$\int_{\mathrm{G}} |(\tau(x)u_0, v_0)|^2\, dx < \infty \,. \tag{14}$$

Then for arbitrary $u, v \in \mathsf{V}$,

$$\int_{\mathrm{G}} |(\tau(x)u, v)|^2\, dx < \infty \,. \tag{15}$$

Moreover, the map $T : \mathsf{V} \to \mathrm{L}^2(\mathrm{G})$ *defined by*

$$Tu(x) = (\tau(x)u, v_0) \qquad (u \in \mathsf{V},\ x \in \mathrm{G})$$

is G*-intertwining and has the property that there is* $t > 0$ *such that*

$$(Tu, Tv) = t(u, v) \qquad (u, v \in \mathsf{V}) \,. \tag{16}$$

Proof. Let W_0 be the linear span of all the vectors $\pi(g)u_0$, $g \in \mathrm{G}$. This is a G-invariant subspace of V. Since V is irreducible, W_0 is dense in V. Let

$$\mathsf{W} = \{u \in \mathsf{V};\ \int_{\mathrm{G}} |(\tau(x)u, v_0)|^2\, dx < \infty\} \,.$$

Then W is also a G-invariant subspace of V and W contains W_0. Hence W is dense in V.

Clearly

$$T\tau(g)u = R(g)Tu \qquad (g \in \mathrm{G}) \,.$$

Define an inner product on W by

$$\langle w_1, w_2\rangle = (w_1, w_2) + (Tw_1, Tw_2) \qquad (w_1, w_2 \in \mathsf{W}) \,.$$

One checks that W with this inner product is complete. (Every Cauchy sequence in W has a limit in W.) Thus (τ, W), with this new inner product, is a unitary representation of G.

The inclusion

$$I_{\mathsf{W}} : \mathsf{W} \ni w \to w \in \mathsf{V}$$

is a bounded linear map which intertwines the actions of G. The adjoint

$$I_{\mathsf{W}}^* : \mathsf{V} \to \mathsf{W}$$

is also a bounded linear map which intertwines the actions of G. Now we apply Theorem 20 with $A = I_{\mathsf{W}}^*$, $\mathsf{V}_0 = \mathsf{V}$, $B = I_{\mathsf{W}}$ and $\mathsf{V}_1 = \mathsf{W}$. The conclusion is that I_{W}^* is a scalar multiple of the identity. Hence $\mathsf{W} = \mathsf{V}$. This means that (14) holds for the fixed $v_0 \in \mathsf{V}_0$ and arbitrary $u \in \mathsf{V}$ in place of the u_0. Now we fix arbitrary $u \in \mathsf{V}$ and, since the group is unimodular, we may apply the same argument to see that (14) holds with the u_0 replaced by u and v_0 replaced by an arbitrary $v \in \mathsf{V}$. This verifies (15).

Now V is equipped with two inner products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ preserved by G. Define a map $S : \mathsf{V} \to \mathsf{V}$ by

$$\langle u, v\rangle = (Su, v)\,.$$

Then S commutes with the action of G. Hence $S = sI$ for some $s \in \mathbb{C}$. Explicitly,

$$(u, v) + (Tu, Tv) = s(u, v)\,.$$

Since $T \neq 0$, by taking $u = v$, we see that $s = 1 + t$ with $t > 0$. This implies (16). □

An irreducible unitary representation of G is called square integrable if one, or equivalently all, its matrix coefficients are square integrable.

Remark 1. *The assumption that the group* G *is unimodular is essential. Lemma 36 fails for example for* G = "$ax + b$" *group. Recall,* [Fol95, page 188 and 242]*, that $G = \mathbb{R}^+ \times \mathbb{R}$ with the multiplication*

$$(a, b)(a', b') = (aa', b + ab')$$

and the left invariant Haar measure

$$a^{-2} da\, db\,.$$

The following formula defines an irreducible unitary representation of G *on* $\mathrm{L}^2(0,\infty)$*,*

$$\pi(a, b)u(s) = a^{\frac{1}{2}} e^{2\pi i b s} u(as) \qquad (a \in \mathbb{R}^+\,,\ b \in \mathbb{R}\,,\ s \in (0,\infty)\,,\ u \in \mathrm{L}^2(0,\infty))\,,$$

see [Fol95, page 242]*. Hence for two functions $u, v \in \mathrm{L}^2(0,\infty)$ the corresponding matrix coefficient has the* L^2 *norm squared equal to*

$$\begin{aligned}
\int |(\pi(a,b)u, v)|^2 a^{-2} da\, db &= \int \left|\int a^{\frac{1}{2}} e^{2\pi i b s} u(as)\overline{v}(s)\, ds\right|^2 a^{-2} da\, db \\
&= \int \left(\left|\int e^{2\pi i b s} u(as)\overline{v}(s)\, ds\right|^2 db\right) a^{-1} da = \int \left(\int |u(as)\overline{v}(s)|^2\, ds\right) a^{-1} da \\
&= \int\int |u(as)\overline{v}(s)|^2\, a^{-1} da\, ds = \int |u(a)|^2\, a^{-1} da \int |v(s)|^2\, ds\,.
\end{aligned}$$

This expression is finite if and only if

$$\int |u(a)|^2\, a^{-1} da < \infty\,,$$

which is the case if $u \in C_c(0,\infty)$ *but is not the case if* u *is the indicator function of the interval* $(0,1)$*.*

Proposition 37. *Let* (σ, U) *and* (τ, V) *be irreducible unitary square integrable representations of* G*. Then*

$$\int_{\mathrm{G}} (\sigma(x)u_1, u_2)\overline{(\tau(x)v_1, v_2)}\, dx = 0, \quad (u_1, u_2 \in \mathrm{U}\,,\ v_1, v_2 \in \mathsf{V}) \ \ \textit{if} \ \ \sigma \not\simeq \tau. \tag{17}$$

There is a positive number $d(\tau)$ *such that*

$$\int_{\mathrm{G}} (\tau(x)u_1, u_2)\overline{(\tau(x)v_1, v_2)}\, dx = \frac{1}{d(\tau)}(u_1, v_1)(v_2, u_2), \qquad (u_1, u_2, v_1, v_2 \in \mathsf{V})\,. \tag{18}$$

Proof. For $u \in \mathrm{U}, v \in \mathsf{V}$, we define a sesquilinear form $S_{u,v} : \mathrm{U} \times \mathsf{V} \to \mathbb{C}$ given by

$$S_{u,v}(u_1, v_1) := \int_{\mathrm{G}} (\sigma(x)u_1, u)\overline{(\tau(x)v_1, v)}\, dx, \quad u_1 \in \mathrm{U}, v_1 \in \mathsf{V}.$$

Then (16) shows that there is a positive constant $C_{u,v}$ such that

$$|S_{u,v}(u_1, v_1)| \leq \left(\int_{\mathrm{G}} |(\sigma(x)u_1, u)|^2\, dx\right)^{1/2} \left(\int_{\mathrm{G}} |(\tau(x)v_1, v)|^2\, dx\right)^{1/2} \leq C_{u,v} \parallel u_1 \parallel \parallel v_1 \parallel .$$

Hence there is a bounded linear map $A : \mathsf{V} \to \mathrm{U}$ such that

$$S_{u,v}(u_1, v_1) = (u_1, Av_1), \qquad (u_1 \in \mathrm{U}, v_1 \in \mathsf{V})\,.$$

It is also easy to see that A is G-intertwining. A is zero if the representations σ and τ are not equivalent. Hence (17) follows.

Suppose $\sigma = \tau$. In this case $A = \lambda_{u,v} I$ for some scalar $\lambda_{u,v} \in \mathbb{C}$. Hence we get

$$S_{u,v}(u_1, v_1) = \bar{\lambda}_{u,v}(u_1, v_1). \tag{19}$$

Since G is unimodular, we have

$$\begin{aligned} S_{u,v}(u_1, v_1) &= \int_{\mathrm{G}} (\tau(x)u_1, u)\overline{(\tau(x)v_1, v)}\, dx \\ &= \int_{\mathrm{G}} (\tau(x^{-1})u_1, u)\overline{(\tau(x^{-1})v_1, v)}\, dx \\ &= \int_{\mathrm{G}} (u_1, \tau(x)u)\overline{(v_1, \tau(x)v)}\, dx \\ &= \int_{\mathrm{G}} (\tau(x)v, v_1)\overline{(\tau(x)u, u_1)}\, dx \\ &= S_{v_1,u_1}(v, u). \end{aligned}$$

Owing to (19) we get

$$\bar{\lambda}_{u,v}(u_1, v_1) = \bar{\lambda}_{v_1,u_1}\overline{(u, v)}.$$

Let w be a fixed vector of norm one. By choosing $u_1 = v_1 = w$, we get

$$\bar{\lambda}_{u,v} = \lambda\overline{(u, v)},$$

where $\lambda = \bar{\lambda}_{w,w}$. Thus from (19) we get

$$\int_G (\tau(x)u_1, u)\overline{(\tau(x)v_1, v)}\, dx = \lambda(u_1, v_1)\overline{(u, v)}.$$

By taking $u_1 = v_1 = u = v \neq 0$, we see that $\lambda > 0$. □

The constant $d(\tau)$ is called the formal degree of τ. It depends on the choice of the Haar measure on G. If G is compact with the Haar measure of total mass 1 then, as we have seen in Theorem 31, $d(\tau) = \dim \mathsf{V}$. Hence the name.

10. $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$

10.1. **The maximal compact subgroup** $\mathrm{K} = \mathrm{SO}_2(\mathbb{R})$.

We let

$$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and define

$$\mathrm{K} = \mathrm{SO}_2(\mathbb{R}) = \{k(\theta);\ \theta \in \mathbb{R}\}\,.$$

This is a maximal compact subgroup of G, which is unique up to conjugation. The group K is commutative. Define the characters

$$\chi_n(k(\theta)) = e^{in\theta} \qquad (\theta \in \mathbb{R}, n \in \mathbb{Z})\,.$$

For two integers n and m define

$$S_{n,m} = \{f \in C_c(\mathrm{G});\ f(k(\theta_1)gk(\theta_2) = e^{-in\theta_1}f(g)e^{-im\theta_2}\}\,.$$

Since $k(\pi) = -I$, we see that $S_{n,m}$ is zero unless m and n have the same parity. In order to see that $S_{n,m}$ is not zero if n and m have the same parity we need to recall Cartan decomposition of G, [Lan85, page 139]. Let

$$\mathrm{A}^+ = \left\{ a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 1 \right\}\,.$$

The Spectral Theorem for symmetric matrices of size two implies that the map

$$\mathrm{K} \times \mathrm{A}^+ \times \mathrm{K} \ni (k_1, a, k_2) \to k_1 a k_2 \in \mathrm{G},$$

is a smooth double covering of a dense open subset of G. Two elements (k_1, a, k_2) and (k_1', a', k_2') are in the same fiber if and only if $(k_1, a, k_2) = (k_1', a', k_2')$ or $(k_1, a, k_2) = (-k_1', a', -k_2')$. If $f \in C_c(\mathrm{A}^+)$ is a non-zero function and m and n have the same parity then the formula

$$\tilde{f}(k(\theta_1), a, k(\theta_2)) = e^{-in\theta_1}f(a)e^{-im\theta_2},$$

defines a non-zero continuous function on the product, which is constant on the fibers of covering map. Hence this function descends to a non-zero element of $S_{n,m}$.

Lemma 38. *The algebraic sum* $\bigoplus_{n,m} S_{n,m}$ *is* L^1*-dense in* $C_c(\mathrm{G})$*. In fact, given* $\epsilon > 0$ *and* $f \in C_c(\mathrm{G})$*, there exists a function* $g \in \bigoplus_{n,m} S_{n,m}$ *such that the support of* g *is contained in* $\mathrm{K}(\operatorname{supp} f)\mathrm{K}$*, and such that* $\| f - g \|_\infty < \epsilon$*.*

Proof. This follows from the properties of the Fourier series. See [Lan85, page 20]. □

Lemma 39. *The following formulas hold.*

$$\begin{aligned} S_{n,m} * S_{l,q} &= \{0\} \quad \text{if} \quad m \neq l\,, \\ S^*_{n,m} &= S_{m,n}\,, \\ S_{n,m} * S_{m,q} &\subseteq S_{n,q}\,. \end{aligned}$$

Proof. This is straightforward [Lan85, page 22]. □

Lemma 40. *Let* $\mathrm{S} = \{x \in \mathrm{G};\ x = x^t\}$. *Then the map*

$$\mathrm{K} \times \mathrm{S} \ni (k, s) \to ks \in \mathrm{G}$$

is a bijective diffeomrophism.

Proof. This is well known from Linear Algebra. See straightforward [Lan85, page 22]. □

Lemma 41. *The algebra* $S_{0,0}$ *is commutative.*

Proof. The argument is due to I. M. Gelfand. Consider an element $x \in \mathrm{G}$. Then there is $k \in \mathrm{K}$ and $s \in \mathrm{S}$ such that $x = ks$. Hence the transpose

$$x^t = k^t s^t = k^{-1} s = k^{-1} x k^{-1}\,. \tag{20}$$

Also, the group G is unimodular. Hence the Haar measure is invariant under the change of variables $x \to x^t$. Let $f, g \in S_{0,0}$. Set $f^t(x) = f(x^t)$. It is easy to check that

$$(f * g)^t = g^t * f^t\,.$$

Notice that $f^t = f$. Indeed,

$$f^t(x) = f(x^t) = f(k^{-1} x k^{-1}) = f(x)\,,$$

because elements of $S_{0,0}$ are K-bi-invariant. Hence

$$g^t * f^t = g * f\,.$$

By Lemma 39, $f * g \in S_{0,0}$. Hence $(f * g)^t = f * g$. The conclusion follows. □

Lemma 42. *For any* $n \in \mathbb{Z}$, *the algebra* $S_{n,n}$ *is commutative.*

Proof. The argument is due to S. Lang, see [Lan85, pages 21-23]. Let

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly $\gamma = \gamma^{-1}$. Notice that $\gamma x \gamma \in \mathrm{SL}_2(\mathbb{R})$ whenever $x \in \mathrm{SL}_2(\mathbb{R})$. In fact the map

$$\mathrm{SL}_2(\mathbb{R}) \ni x \to \gamma x \gamma \in \mathrm{SL}_2(\mathbb{R})$$

is an involutive group automorphism. In particular it preserves the Haar measure. Furthermore,

$$\gamma k \gamma = k^{-1}, \qquad (k \in \mathrm{K}).$$

Let $s = s^t \in \mathrm{G}$. Then there is $k \in \mathrm{K}$ such that $ksk^{-1} = d$ is diagonal. Therefore

$$\gamma s \gamma = \gamma k^{-1} d k \gamma = \gamma k^{-1} \gamma \gamma d \gamma \gamma k \gamma = k d \gamma k^{-1} = k^2 s k^{-2}.$$

Thus there is $k_s \in \mathrm{K}$ such that

$$\gamma s \gamma = k_s^{-1} s k_s,$$

where $k_s = k^{-2}$. Recall that $f^t(x) = f(x^t)$ and let $f^\gamma(x) = f(\gamma x \gamma)$. Then, for $f \in S_{n,n}$

$$f^\gamma(x) = f^t(x)$$

Indeed, write $x = ks$. Then

$$\begin{aligned} f^\gamma(ks) &= f(\gamma k s \gamma) = f(k^{-1}\gamma s \gamma) = \chi_n(k) f(\gamma s \gamma) \\ &= \chi_n(k) f(k_s^{-1} s k_s) = \chi_n(k)\chi_n(k_s) f(s) \chi_n(k_s)^{-1} \\ &= \chi_n(k) f(s) = f(s)\chi_n(k) = f(sk^{-1}) = f((ks)^t) \\ &= f^t(ks)\,. \end{aligned}$$

Therefore for any $f, g \in S_{n,n}$,

$$f^\gamma * g^\gamma = f^t * g^t = (g * f)^t = (g * f)^\gamma = g^\gamma * f^\gamma$$

and we are done. □

For a representation (π, V) of G on a Banach space V define

$$\mathsf{V}_n = \{v \in \mathsf{V};\ \pi(k(\theta))v = e^{in\theta} v, \theta \in \mathbb{R}\}\,.$$

Then V_n is called the n-th isotypic component of V. Clearly V_n is a closed subspace of V.

We define an operator $P_n : V \to V$ by

$$P_n v := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \pi(k_\theta) v \, d\theta, \quad (v \in V).$$

Since the representation π is locally bounded (i.e. given a compact subset $\mathcal{K}$ of G, the set $\pi(\mathcal{K})$ is bounded in $\mathrm{GL}(\mathsf{V})$ (see [Lan85, page 2]), it follows that P_n is a bounded operator from V to V. It can be also easily checked that P_n is a continuous projection from V onto V_n. For $v \in V$, $P_n v$ is called the n-component of v.

Lemma 43. *Let (π, V) be a representation of* G *on a Banach space* V. *Then*

$$\pi(S_{n,m})\mathsf{V} \subseteq \mathsf{V}_n$$

and for $q \neq m$,

$$\pi(S_{n,m})\mathsf{V}_q = 0\,.$$

Proof. See [Lan85, page 23]. □

Lemma 44. *Let (π, V) be an irreducible representation of* G *on a Banach space* V. *Then for any q such that $\mathsf{V}_q \neq 0$, the space V_q is $S_{q,q}$-irreducible. Also, $\pi(S_{q,q})\mathsf{V}_q \neq 0$.*

Proof. This follows from the fact that the $*$ algebra

$$\bigoplus_{n,m} S_{n,m} \tag{21}$$

is L^1-dense in $C_c(\mathrm{G})$ (see [Lan85, page 24]). We recall the argument for reader's convenience.

Let $0 \neq W \subseteq \mathsf{V}_q$ be a closed $S_{q,q}$-invariant subspace. Lemma 43 implies that for any $w \in W$ and any f in the algebra (21) the q-component of $\pi(f)w$ is in W. Since the algebra (21) is L^1-dense in $C_c(\mathrm{G})$ we see that the q-component of $\pi(f)w$ is in W for any $f \in C_c(\mathrm{G})$. Since the representation (π, V) of G is irreducible, $\pi(C_c(\mathrm{G}))W$ is dense in V. (This follows from Theorem 22.) In particular the q-component of $\pi(C_c(\mathrm{G}))W$ is dense in V_q. Therefore, W is dense in V_q. Hence $W = \mathsf{V}_q$. □

Theorem 45. *Let (π, V) be an irreducible representation of* G *on a Banach space* V*. Fix an integer n. If* $\dim \mathsf{V}_n < \infty$ *then* $\dim \mathsf{V}_n = 1$ *or* 0.

Proof. Let $V_n \neq 0$. We know from Lemma 44 that the commutative algebra $\pi(S_{n,n})$ acts irreducibly on V_n. Hence if $f \in S_{n,n}$, owing to Theorem 13 it follows that, $\pi(f) = \lambda_f I$ for some $\lambda_f \in \mathbb{C}$. Now it is clear that $\dim \mathsf{V}_n = 1$. □

Warning: there exists a Banach space V of infinite dimension and a linear operator $T \in \mathrm{End}(\mathsf{V})$ acting irreducibly on V, see [Enf87].

Definition 46. *A representation (π, V) of* G *on a Banach space* V *is admissible if* $\dim \mathsf{V}_n < \infty$ *for every integer n.*

Theorem 47. *Let (π, V) be an irreducible unitary representation of* G *on a Hilbert space* V*. Fix an integer n. Then* $\dim \mathsf{V}_n = 1$ *or* 0.

Proof. In this case $\pi(S_{n,n})$ is a commutative $*$ algebra. Hence the claim follows from Corollary 14. □

Theorem 48. *Let (π, V) be an irreducible representation of* G *on a Banach space* V*. Then the space of finite sums*

$$\bigoplus_{n \in \mathbb{Z}} \mathsf{V}_n \subseteq \mathsf{V}$$

is dense.

If (π, V) be an irreducible representation of G *on a Hilbert space space* V *such that π is unitary on K, then*

$$\mathsf{V} = \sum_{n \in \mathbb{Z}} \mathsf{V}_n$$

is a Hilbert space direct sum orthogonal decomposition.

Proof. This follows from the fact that the $*$ algebra

$$\bigoplus_{n,m} S_{n,m}$$

is dense in $C_c(\mathrm{G})$ see [Lan85, page 25]. □

10.2. **Induced representations.** Recall the following subgroups of G:

$$\mathrm{K} = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\ \theta \in \mathbb{R} \right\},\ \mathrm{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a > 0 \right\},\ \mathrm{N} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{R} \right\}.$$

The following map is a bijective diffeomorphism

$$\mathrm{A} \times \mathrm{N} \times \mathrm{K} \ni (a, n, k) \to ank \in \mathrm{G},$$

giving the Iwasawa decomposition $\mathrm{G} = \mathrm{ANK}$. Let us fix Haar measures on these groups

$$d\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{2\pi} d\theta,\quad d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{d^+a}{a},\quad d\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = dn\,.$$

where d^+a is the Lebesgue measure on A when we view A as an open subset of $\mathbb{R}$. Then in terms of the Iwasawa decomposition, the formula

$$dx = da\, dn\, dk \qquad (x = ank,\ a \in \mathrm{A}, n \in \mathrm{N}, k \in \mathrm{K}),$$

defines a Haar measure on G. Let

$$\alpha\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} := a^2\,,\quad \rho\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} := a\,.$$

Let $\mathrm{P} = \mathrm{AN}$. This is a subgroup of G with the modular function

$$\Delta\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) = a^2\,.$$

Theorem 49. *Let (σ, V) be a representation of* P *on a Hilbert space* V. *Denote by* $\mathsf{V}(\sigma)$ *the space of functions* $f : \mathrm{G} \to \mathsf{V}$ *such that*

$$f(px) = \Delta(p)^{\frac{1}{2}}\sigma(p)f(x) \qquad (x \in \mathrm{G})$$

whose restriction to K *is square integrable:*

$$\int_{\mathrm{K}} \| f(k) \|^2\ dk < \infty\,.$$

Set

$$\pi(g)f(x) = f(xg) \qquad (g, x \in \mathrm{G})$$

Then $(\pi, \mathsf{V}(\sigma))$ is a representation of G. *Moreover, π is bounded if σ is bounded and is unitary if σ is unitary.*

Proof. See [Lan85, Theorem 2, page 44]. □

Our main example is going to be the case when $\mathsf{V} = \mathbb{C}$ and for some fixed complex number s,

$$\sigma(an) = \rho(a)^s \qquad (a \in \mathrm{A}, n \in \mathrm{N})\,.$$

Then we shall write $\mathsf{V}(s) = \mathsf{V}(\sigma)$ and $\pi_s = \pi$. Then the transformation property of functions in $\mathsf{V}(s)$ looks as follows

$$f(anx) = \rho(a)^{1+s}f(x) \qquad (a \in \mathrm{A}, n \in \mathrm{N}, x \in \mathrm{G})\,. \tag{22}$$

The resulting representation$(\pi_s, \mathsf{V}(s))$ is called the principal series representation.

Lemma 50. *Let*

$$\rho_s(ank) = \rho(a)^{1+s} \qquad (a \in \mathrm{A}, n \in \mathrm{N}, k \in \mathrm{K})\,.$$

Then $\rho_s \in \mathsf{V}(s)$ *is* K*-ivariant and*

$$(\pi_s(x)\rho_s, \rho_s) = \int_{\mathrm{K}} \rho_s(kx)\,dk = \int_{\mathrm{K}} \rho_0(kx)^{1+s}\,dk\,.$$

Proof. See [Lan85, page 47]. □

The function

$$\phi_s = \int_{\mathrm{K}} \rho_0(kx)^{1+s}\,dk \tag{23}$$

is known as a spherical function.

Lemma 51. *For a function* $\psi \in C_c(\mathrm{G})$, $\pi_s(\psi)$ *is an integral kernel operator with the integral kernel*

$$q_\psi(k,k') = \int_{\mathrm{N}} \int_{\mathrm{A}} \psi(k'^{-1}ank)\rho(a)^{1+s}\,da\,dn\,.$$

Furtheremore the operator $\pi_s(\psi)$ *is of trace class and*

$$\operatorname{tr} \pi_s(\psi) = \int_{\mathrm{K}} q_\psi(k,k)\,dk = \int_{\mathrm{K}} \int_{\mathrm{N}} \int_{\mathrm{A}} \psi(k^{-1}ank)\,\rho(a)^{1+s}\,da\,dn\,dk.$$

Proof. The formula for the integral kernel is obtained via a straightforward computation in [Lan85, page 48]. It is a continuous function. Hence of the trace class. □

By computing a few Jacobians, as in [Lan85, page 68], we obtain the following lemma.

Lemma 52. *Let* $\phi \in C_c(\mathrm{G})$ *and* $a \in \mathrm{A}$ *be such that* $\alpha(a) \neq 1$*. Then we have,*

$$\int_{\mathrm{N}} \phi(ana^{-1}n^{-1})\,dn = \frac{1}{|\alpha(a) - 1|} \int_{\mathrm{N}} \phi(n)\,dn. \tag{24}$$

Moreover the function $x \mapsto \phi(x^{-1}ax)$ *has compact support on* $\mathrm{A}\backslash\mathrm{G}$ *and we have*

$$\int_{\mathrm{A}\backslash\mathrm{G}} \phi(x^{-1}ax)\,d\dot{x} = \frac{\rho(a)}{|D(a)|} \int_{\mathrm{N}} \int_{\mathrm{K}} \phi(kank^{-1})\,dk\,dn\,, \tag{25}$$

where $d\dot{x}$ *is the unique* G*-invarint measure on* $\mathrm{A}\backslash\mathrm{G}$ *(see* [Lan85, Theorem 1, page 37]*) and*

$$D(a) = \rho(a) - \rho(a)^{-1}. \tag{26}$$

Theorem 53. *Let* Θ_{π_s} *be a function on* G *defined as follows.*

$$\Theta_{\pi_s}(x) = \begin{cases} 2\left[\frac{\rho(a)^s + \rho(a)^{-s}}{|D(a)|}\right] & \text{if } x \text{ is conjugate to } a \in \mathrm{A}\,, \\ 0 & \text{if } x \text{ is not conjugate to any element of } \mathrm{A}\,. \end{cases} \tag{27}$$

Then

$$\operatorname{tr} \pi_s(\psi) = \int_{\mathrm{G}} \Theta_{\pi_s}(x)\psi(x)\,dx \qquad (\psi \in C_c(\mathrm{G}))\,. \tag{28}$$

Proof. By combining Lemmas 51 and 52 we see that

$$\operatorname{tr}\pi_s(\psi)=\int_{\mathrm{A}}|D(a)|\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}ax)\,d\dot{x}\,\rho(a)^s\,da\,. \tag{29}$$

Changing the variable a to a^{-1} gives

$$\operatorname{tr}\pi_s(\psi)=\int_{\mathrm{A}}|D(a^{-1})|\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}a^{-1}x)\,d\dot{x}\,\rho(a)^{-s}\,da\,.$$

However $|D(a^{-1})|=|D(a)|$ and

$$\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}a^{-1}x)\,d\dot{x}=\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}ax)\,d\dot{x}\,.$$

Therefore

$$\begin{aligned}\operatorname{tr}\pi_s(\psi)&=\frac{1}{2}\int_{\mathrm{A}}|D(a)|\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}ax)\,d\dot{x}\,(\rho(a)^s+\rho(a)^{-s})\,da\\&=\int_{\mathrm{A}^+}|D(a)|\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}ax)\,d\dot{x}\,(\rho(a)^s+\rho(a)^{-s})\,da\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathrm{G}}\Theta_{\pi_s}(x)\psi(x)\,dx&=\frac{1}{4}\int_{\mathrm{A}}|D(a)|^2\int_{\mathrm{A}\backslash\mathrm{G}}\Theta_{\pi_s}(x^{-1}ax)\psi(x^{-1}ax)\,d\dot{x}\,da\\&=\frac{1}{4}\int_{\mathrm{A}}|D(a)|\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}ax)\,d\dot{x}\,\Theta_{\pi_s}(a)|D(a)|\,da\\&=\frac{1}{2}\int_{\mathrm{A}^+}|D(a)|\int_{\mathrm{A}\backslash\mathrm{G}}\psi(x^{-1}ax)\,d\dot{x}\,\Theta_{\pi_s}(a)|D(a)|\,da\,.\end{aligned}$$

Since for any test function $\phi\in C_c(\mathrm{A}^+)$ we can find $\psi\in C_c(\mathrm{G})$ such that the orbital integral of ψ is equal to ϕ, (27) follows. □

Notice that the trace is given via integration against a G-invariant locally integrable function. Let ${}^0\mathrm{M}=\{1,-1\}\subseteq\mathrm{G}$. This is the centralizer A in K. If δ is a character of the group ${}^0\mathrm{M}$, we define $\mathsf{V}(\delta,s)\subseteq\mathsf{V}(s)$ to be the subspace of functions ϕ such that

$$\phi(mx)=\delta(m)\psi(x)\qquad(m\in{}^0\mathrm{M},x\in\mathrm{G})\,.$$

Denote by $(\pi_{\delta,s},\mathsf{V}(\delta,s))$ the resulting representation of G. Clearly $(\pi_s,\mathsf{V}(s))$ is the direct sum of the two subrepresentations $(\pi_{\delta,s},\mathsf{V}(\delta,s))$.

Theorem 54. *Let $\Theta_{\pi_{\delta,s}}$ be a function on* G *defined as follows.*

$$\Theta_{\pi_s}(mx)=\begin{cases}\delta(m)\frac{\rho(a)^s+\rho(a)^{-s}}{D(a)} & \textit{if } x \textit{ is conjugate to } ma, \textit{ where } m\in{}^0\mathrm{M} \textit{ and } a\in\mathrm{A}\,,\\ 0 & \textit{if } x \textit{ is not conjugate to any element of } \pm\mathrm{A}\,.\end{cases} \tag{30}$$

Then

$$\operatorname{tr}\pi_{\delta,s}(\psi)=\int_{\mathrm{G}}\Theta_{\pi_{\delta,s}}(x)\psi(x)\,dx\qquad(\psi\in C_c(\mathrm{G}))\,. \tag{31}$$

10.3. **Finite dimensional representations.** Define

$$D(k(\theta)) = e^{i\theta} - e^{-i\theta} \qquad (k(\theta) \in \mathrm{K})\,. \tag{32}$$

Lemma 55. *Fix an integer $n \geq 1$. For non-negative integers p,q with $p+q = n-1$ let $f_{p,q}$ be a function of*

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $f_{p,q}(x) = c^p d^q$. Then $f_{p,q} \in \mathsf{V}(\delta, -n)$, where $\delta(-1) = (-1)^{p+q}$. The functions $f_{p,q}$ span a G-invariant finite dimensional subspace

$$\mathrm{U}(\delta, -n) = \bigoplus_{p+q=n-1} \mathbb{C} f_{p,q} \subseteq \mathsf{V}(\delta, -n)$$

of dimension n. Denote the resulting representation by $(\sigma_{\delta,-n}, \mathrm{U}(\delta, -n))$. Then

$$\mathrm{tr}\, \sigma_{\delta,-n}(ma) = \delta(m) \frac{\rho(a)^n - \rho(a)^{-n}}{D(a)} \qquad (m \in {}^0M, a \in \mathrm{A})\,. \tag{33}$$

Also,

$$\mathrm{tr}\, \sigma_{\delta,-n}(k(\theta)) = \frac{e^{in\theta} - e^{-in\theta}}{D(k(\theta))} \qquad (k(\theta) \in \mathrm{K})\,. \tag{34}$$

Proof. Everything till (33) is straightforward. See [Lan85, page 151]. The formula (33) is verified in [Lan85, Lemma 2, page 151].

Let $\mathrm{G}_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$. Then $\mathrm{G} \subseteq \mathrm{G}_{\mathbb{C}}$ is a subgroup and the representation $(\sigma_{\delta,-n}, \mathrm{U}(\delta, -n))$ extends to a representation of $\mathrm{G}_{\mathbb{C}}$ so that $\sigma_{\delta,-n} : \mathrm{G}_{\mathbb{C}} \to \mathrm{GL}(\mathrm{U}(\delta, -n))$ is a polynomial map. Notice that

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Thus $k(\theta)$ is conjugate to

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{35}$$

within $\mathrm{G}_{\mathbb{C}}$. In particular the trace evaluated on $k(\theta)$ is equal to the trace evaluated on the element (35). This last trace is computed in [Lan85, Lemma 3, 152]. Thus (34) follows. □

10.4. **Smooth vectors and analytic vectors.** Let (π, V) be a representation of G on a Banach space V. Denote by $\mathsf{V}^\infty \subseteq \mathsf{V}$ the subspace of all vectors v such that the map

$$\mathrm{G} \ni x \to \pi(x)v \in \mathsf{V} \tag{36}$$

is smooth (infinitely many times differentiable). This subspace is dense because, as is easy to check, $\pi(C_c^\infty(\mathrm{G}))\mathsf{V} \subseteq \mathsf{V}^\infty$. For $X \in \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, the Lie algebra of G, define

$$d\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v|_{t=0} \qquad (v \in \mathsf{V}^\infty)\,. \tag{37}$$

One checks without difficulties that the map

$$d\pi : \mathfrak{g} \to \mathrm{End}(\mathsf{V}^\infty) \tag{38}$$

is a Lie algebra homomorphism. By linearity we extend it to the complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$$d\pi : \mathfrak{g}_\mathbb{C} \to \mathrm{End}(\mathsf{V}^\infty) \tag{39}$$

and obtain a representation $(d\pi, \mathsf{V}^\infty)$ of the Lie algebra $\mathfrak{g}_\mathbb{C}$. One drawback of this representation is that the closure of a $\mathfrak{g}$-invariant subspace $\mathrm{U} \subseteq \mathsf{V}^\infty$ in V does not need to be G-invariant. Indeed, consider the right regular representation $(R, \mathrm{L}^2(\mathrm{G}))$ of G. Let $W \subseteq \mathrm{G}$, $W \neq \mathrm{G}$, be a non-empty open set. Then the space $\mathrm{U} = C_c^\infty(W)$ is closed under the action of $dR(\mathfrak{g})$, but the closure $\mathrm{L}^2(W)$ is not $R(\mathrm{G})$ invariant. For this reason one is lead to study the space $\mathsf{V}^{an} \subseteq \mathsf{V}$ of all vectors v such that the map.

$$\mathrm{G} \ni x \to \pi(x)v \in \mathsf{V} \tag{40}$$

is analytic.

Theorem 56. *Let $\mathsf{X} \subseteq \mathsf{V}^{an}$ be a $\mathfrak{g}$-invariant vector subspace. Then the closure of X in V is* G*-invarant*

Proof. We follow [Var89, Theorem 2, page 108]. It is enough to prove that for any $v \in \mathrm{U}$ and for any $x \in \mathrm{G}$, $\pi(x)u$ belonges to the closure of U. Suppose not. Then by Haan-Banach Theorem there is λ in the dual of V such that λ is equal to zero on the closure of U, but for some $x_0 \in \mathrm{G}$, $\lambda(\pi(x_0)v) \neq 0$.

By assumption the function

$$\mathrm{G} \ni x \to \lambda(\pi(x)v) \in \mathbb{C}$$

is analytic. By the choice of λ, its Taylor series at $x = 1$ is zero. Indeed, for any $X_1, X_2, ..., X_n \in \mathfrak{g}$,

$$d\pi(X_1)d\pi(X_2)...d\pi(X_n)v \in \mathrm{U}.$$

Hence

$$\lambda(d\pi(X_1)d\pi(X_2)...d\pi(X_n)v) = 0$$

Also, $\lambda(v) = 0$. Thus the function is zero because G is connected. This is a contradiction. □

10.5. **The derivative of the right regular representation.** Every element $g \in \mathrm{GL}_2^+(\mathbb{R})$ has a unique decomposition as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{41}$$

where $u > 0$, $y > 0$, $x \in \mathbb{R}$ and $\theta \in [-\pi, \pi)$. We extend any function defined on $\mathrm{SL}_2(\mathbb{R})$ to a function on $\mathrm{GL}_2^+(\mathbb{R})$ by making it independent of the variable u. Also, we extend the right regular representation of $\mathrm{SL}_2(\mathbb{R})$ to act on such defined functions on $\mathrm{GL}_2^+(\mathbb{R})$. Then a straightforward computation, see [Lan85, 113-116] verifies the following formulas,

$$\begin{aligned}
dR\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= y\cos 2\theta\,\partial_x + y\sin 2\theta\,\partial_y + \sin^2\theta\,\partial_\theta\,, \qquad (42)\\
dR\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \partial_\theta\,,\\
dR\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= 2y\cos 2\theta\,\partial_x + 2y\sin 2\theta\,\partial_y - \cos 2\theta\,\partial_\theta\,,\\
dR\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= y\cos 2\theta\,\partial_x + y\sin 2\theta\,\partial_y - \cos^2\theta\,\partial_\theta\,,\\
dR\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= -2y\sin 2\theta\,\partial_x + 2y\cos 2\theta\,\partial_y + \sin 2\theta\,\partial_\theta\,.
\end{aligned}$$

10.6. **The universal enveloping algebra.** Here $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the complex tensor algebra of $\mathfrak{g}$ divided by the ideal generated by element $AB - BA - [A, B]$, $A, B \in \mathfrak{g}$. The Poincare-Birkhoff-Witt Theorem says that if A, B, C form a basis of the vector space $\mathfrak{g}_{\mathbb{C}}$, then the elements

$$A^a B^b C^c \qquad (0 \le a, b, c \in \mathbb{Z})$$

form a basis of the vector space $\mathcal{U}(\mathfrak{g})$. (Here $A^0 = B^0 = C^0 = 1$.) Furthermore, for any $A, B, C \in \mathfrak{g}_{\mathbb{C}}$ satisfying the commutation relations

$$[A, B] = 2B\,,\ [A, C] = -2C\,,\ [B, C] = A,$$

the element $\mathcal{C} = A^2 + 2(BC + CB)$, called the Casimir element, generates the the center of $\mathcal{U}(\mathfrak{g})$ and does not depend on the choice of the A, B, C. For instance one can take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus the center of $\mathcal{U}(\mathfrak{g})$ is equal to $\mathbb{C}[\mathcal{C}]$. Also, any representation of the Lie algebra $\mathfrak{g}$ extends to a representation of the algebra $\mathcal{U}(\mathfrak{g})$. In terms of the coordinates used in section 10.5,

$$dR(\mathcal{C}) = 4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x\partial_\theta \qquad (43)$$

Proofs may be found in [Lan85, pages 191-198].

10.7. K**-multiplicity** 1 **representations.** Here $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$ and we consider only representations (π, V) of G on Banach spaces V such that for each integer n the isotypic component $\mathsf{V}_n \subseteq \mathsf{V}$ has dimension at most 1.

Lemma 57. *For any $n \in \mathbb{Z}$, the space $S_{n,n}^\infty = S_{n,n} \cap C^\infty(\mathrm{G})$ is dense in $S_{n,n}$ in the topology of uniform convergence.*

Proof. See [Lan85, page 101 and proof of Lemma 1, page 19]. ☐

Let V be a complex vector space and a $\mathfrak{g}$ module. Denote the action of $\mathfrak{g}$ by ρ:

$$Xv = \rho(X)v \qquad (X \in \mathfrak{g}, v \in V)\,.$$

Suppose that V is also a K module. Denote the action of K by σ:

$$kv = \sigma(k)v \qquad (k \in \mathrm{K}, v \in V)\,.$$

The representation (σ, V) is called locally finite if the linear span of $\sigma(\mathrm{K})v$ is finite dimensional for every $v \in V$.

The space V is called a $(\mathfrak{g}, \mathrm{K})$ module if it is a locally finite K module and the actions of $\mathfrak{g}$ and K satisfy the following compatibility conditions.
(1) If F is a finite dimensional K stable subspace of V then the representation of K on F is differentiable with $d\sigma|_{\mathfrak{k}} = \rho|_{\mathfrak{k}}$ and
(2) $\rho(k)\sigma(X)v = \sigma(\mathrm{Ad}(k)X)\rho(k)v \qquad (k \in \mathrm{K}, X \in \mathfrak{g}, v \in V)$.
It is known that if (π, V) is an admissible representation of $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$ on a Banach space V, then $\mathsf{V}_{\mathrm{K}} := \bigoplus_{n\in\mathbb{Z}} \mathsf{V}_n$ is a $(\mathfrak{g}, \mathrm{K})$ module. Let (π, V) and (σ, U) be two admissible Banach space representations of G. We shall say that they are infinitesimally isomorphic or infinitesimally equivalent if there exists a linear isomorphism $T : \mathsf{V}_K \to \mathrm{U}_K$ such that

$$Td\pi(X) = d\sigma(X)T \quad \text{on } V_K.$$

Theorem 58. *Let (π, V) be a representation of* G *on a Banach space* V *such that for each integer n the isotypic component $\mathsf{V}_n \subseteq \mathsf{V}$ has dimension at most* 1*. Then $\mathsf{V}_{\mathrm{K}} \subseteq \mathsf{V}^{an}$.*

Proof. It'll suffice to show that for a fixed n and a non-zero vector $v \in \mathsf{V}_n$ the function

$$f_v(g) = \pi(g)v \qquad (g \in \mathrm{G})$$

is analytic. Notice that

$$f_v(k(\theta)g) = e^{in\theta} f_v(g)\,.$$

Hence (43) shows that

$$dR(\mathcal{C})f_v(g) = (4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x in)f_v(g)\,.$$

Since, by assumption, $\dim \mathsf{V}_n = 1$, we see from Lemmas 57, 43 and 44 that

$$\mathsf{V}_n = \pi(S^\infty_{n,n})\mathsf{V}_n\,.$$

In particular, $\mathsf{V}_n \subseteq \mathsf{V}^\infty$. On the other hand $d\pi(\mathcal{C})$ commutes with $\pi(\mathrm{K})$, hence preserves V_n. Since, by assumption, $\dim \mathsf{V}_n = 1$, $d\pi(\mathcal{C})$ acts on V_n via multiplication by a scalar, call it c_n. Thus

$$d\pi(\mathcal{C})v = c_n v\,.$$

Since

$$R(h)f_v(g) = f_{\pi(h)v}(g)\,,$$

this implies

$$dR(\mathcal{C})f_v = c_n f_v\,.$$

Since the characteristic variety of the system of differential equations

$$(4y^2(\partial_x^2 + \partial_y^2) - 4y\partial_x in)f_v = c_n f_v\,, \quad \partial_\theta f_v = in f_v,$$

is zero the function f_v is analytic. □

Theorem 59. *Let* (π, V) *be a representation* G *on a Banach space* V *such that for each integer* n *the isotypic component* $\mathsf{V}_n \subseteq \mathsf{V}$ *has dimension at most* 1*. Then the map*

$$\mathsf{V} \supseteq \mathrm{U} \to \mathrm{U}_{\mathrm{K}} \subseteq \mathsf{V}_{\mathrm{K}}$$

is a bijection between closed G*-invariant subspaces of* V *and* $(\mathfrak{g}, \mathrm{K})$*-submodules of* V_{K}*. The inverse is given by*

$$\mathsf{V}_{\mathrm{K}} \supseteq \mathsf{X} \to Cl(\mathsf{X}) \subseteq \mathsf{V}\,.$$

where $Cl(\mathsf{X})$ *denotes the closure of* X *in* V*. In particular* (π, V) *is irreducible if and only if the* $(\mathfrak{g}, \mathrm{K})$*-module* V_{K} *is irreducible.*

Proof. This is clear from Theorems 56 and 58. □

Lemma 60. *For any irreducible* $(\mathfrak{g}, \mathrm{K})$*-module* X *and any integer* n *such that* $\mathsf{X}_n \neq 0$*, any non-zero vector* $v \in \mathsf{X}_n$ *is cyclic for the action of* $\mathcal{U}(\mathfrak{g})$ *on* X*.*

Proof. Since $\mathcal{U}(\mathfrak{g})v$ is a submodule of X, the claim is obvious. □

Let $\mathsf{X} = \bigoplus_n \mathsf{X}_n$ ba a $(\mathfrak{g}, \mathrm{K})$-module with $\dim \mathsf{X}_n \leq 1$. A hermitian form $(\cdot, \cdot)$ on X is called $\mathfrak{g}$-invariant if

$$(Xu, v) = -(u, Xv) \qquad (u, v \in \mathsf{X}, X \in \mathfrak{g})\,. \tag{44}$$

A $(\mathfrak{g}, \mathrm{K})$ module is called unitarizable if it admits an invariant positive definite hemitian form.

Theorem 61. *Let* $\mathsf{X} = \bigoplus_n \mathsf{X}_n$ *ba an irreducible* $(\mathfrak{g}, \mathrm{K})$*-module with* $\dim \mathsf{X}_n \leq 1$*. Then any two positive definite* $\mathfrak{g}$*-invariant hermitian products on* X *are positive multiples of each other (assuming they exist).*

Proof. Denote the two $\mathfrak{g}$-invariant positive hermitian products on X by $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$. Then the spaces V_n are mutually orthogonal with respect to $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$. Let $P_n : \mathsf{V} \to \mathsf{V}_n$ denote the orthogonal projection. Fix $m \in \mathbb{Z}$ such that $\mathsf{V}_m \neq 0$ and a vector $0 \neq v_m \in \mathsf{V}_m$. Lemma 60 implies that for any n there is $a \in \mathcal{U}(\mathfrak{g})$ such that $P_n a v_m \neq 0$. Then

$$(av_m, av_m) \;=\; (v_m, a^* a v_m) \;=\; (v_m, P_m a^* a v_m) \;=\; c\langle v_m, P_m a^* a v_m \rangle \;=\; c\langle av_m, av_m \rangle\,,$$

where

$$c = \frac{(v_m, v_m)}{\langle v_m, v_m \rangle}\,.$$

□

Theorem 62. *Two irreducible unitary representations of* G*, of* K*-multiplicity at most* 1*, are unitarily isomorphic if and only if their* $(\mathfrak{g}, \mathrm{K})$*modules are isomorphic.*

Proof. Let (π, V), (σ, U) be the two representations. If they are isomorphic, then clearly so are the $(\mathfrak{g}, \mathrm{K})$-modules. Conversely, suppose

$$L : \mathsf{V}_{\mathrm{K}} \to \mathrm{U}_{\mathrm{K}}$$

is a $\mathfrak{g}$-intertwinig map. Then $(\cdot,\cdot)_{\mathsf{V}}$ and the pull-back of $(\cdot,\cdot)_{\mathrm{U}}$ via L are positive definite $\mathfrak{g}$-invariant hermitian products on V_{K}. Theorem 61 shows that there is a constant $c > 0$ such that

$$(v, v')_{\mathsf{V}} = c(Lv, Lv')_{\mathrm{U}} \qquad (v, v' \in \mathsf{V}) .$$

Hence

$$T = \sqrt{c}L : \mathsf{V}_{\mathrm{K}} \to \mathrm{U}_{\mathrm{K}}$$

is a $\mathfrak{g}$-intertwining isometry. We need to check that T is also G-intertwining.

Since K-finite vectors are analytic, we have

$$\pi(\exp(X))v = \sum_{n=0}^{\infty} \frac{1}{n!} (d\pi(X))^n v \qquad (v \in \mathsf{V}_{\mathrm{K}}) .$$

For $X \in \mathfrak{g}$ in some small neighborhood of zero. Hence, in this neighborhood,

$$T\pi(\exp(X))v = \sum_{n=0}^{\infty} \frac{1}{n!} (d\sigma(X))^n Tv = \sigma(\exp(X))Tv .$$

Thus there is an open neighborhood U of $1 \in \mathrm{G}$ such that

$$T\pi(g)v = \sigma(g)Tv \qquad (g \in U) .$$

Since G is connected U generates G, so the proof is complete. □

10.8. **The character of a $(\mathfrak{g}, \mathrm{K})$-module.**

Theorem 63. *Let (π, V) be a representation of G on a Hilbert space V, with $\dim \mathsf{V}_n \leq 1$ for all $n \in \mathbb{Z}$. Then for any $\phi \in C_c^\infty(\mathrm{G})$, the operator $\pi(\phi)$ is of trace class and the map*

$$C_c^\infty(\mathrm{G}) \ni \phi \to \mathrm{tr}\, \pi(\phi) \in \mathbb{C}$$

is a distribution on G.

Proof. Recall that

$$\int_{\mathrm{G}} \phi(g)\pi(g)\, dg = \int_{\mathrm{G}} \phi(g)\pi(gk^{-1}k)\, dg = \int_{\mathrm{G}} R(k)\phi(g)\pi(g)\, dg\, \pi(k) .$$

Hence by taking derivatives at $k = 1$,

$$0 = \int_{\mathrm{G}} dR(J)\phi(g)\pi(g)\, dg + \int_{\mathrm{G}} \phi(g)\pi(g)\, dg\, d\pi(J) .$$

Let $v_n \in \mathsf{V}_n$ be a unit vector. Then

$$0 = \int_{\mathrm{G}} dR(J)\phi(g)\pi(g)\, dg\, v_n + \int_{\mathrm{G}} \phi(g)\pi(g)\, dg\, in\, v_n .$$

By iterating we see that for $m = 0, 1, 2, ...$

$$(-in)^m \pi(\phi)v_n = \pi(dR(J)^m \phi) .$$

Thus

$$|(\pi(\phi)v_n, v_n)| \leq (1+|n|)^{-m} \parallel \pi(dR(J)^m\phi) \parallel \qquad (n \in \mathbb{Z})\,.$$

Since for $m \geq 2$,

$$\sum_{n\in\mathbb{Z}} (1+|n|)^{-m} < \infty\,,$$

the claim follows. □

Theorem 64. *Let (π, V), (σ, U) be representations of G on Hilbert spaces V and U, with $\dim \mathsf{V}_n \leq 1$ and $\dim \mathrm{U}_n \leq 1$ for all $n \in \mathbb{Z}$. Assume that the $(\mathfrak{g}, \mathrm{K})$-modules V_K and U_K are isomorphic. Then,*

$$\operatorname{tr} \pi(\phi) = \operatorname{tr} \sigma(\phi) \qquad (\phi \in C_c^\infty(\mathrm{G}))\,.$$

Proof. Let $v_n \in \mathsf{V}_n$ and $u_n \in \mathrm{U}_n$ be unit vectors. Then they form orthonormal basis. Let

$$T : \mathsf{V}_\mathrm{K} \to \mathrm{U}_\mathrm{K}$$

be a $(\mathfrak{g}, \mathrm{K})$-intertwining isomorphism. Then there are no-zero numbers $t_n \in \mathbb{C}$ such that $Tv_n = t_n u_n$ for all n. Therefore $T^{-1}u_n = t_n^{-1}v_n$. Set $(T^{-1})^* v_n = \overline{t}_n^{-1} u_n$ and extend it to whole of U_K by linearity . Then the map $(T^{-1})^* : \mathsf{V}_\mathrm{K} \to \mathrm{U}_\mathrm{K}$ satisfies

$$(T^{-1}u, v)_\mathsf{V} = (u, (T^{-1})^* v)_\mathrm{U} \qquad (u \in \mathrm{U}_\mathrm{K}, v \in \mathsf{V}_\mathrm{K})\,.$$

Since v_n and u_n are analytic vectors we have for $\mathsf{X} \in \mathfrak{g}$ close enough to zero,

$$\begin{aligned}
(\pi(\exp(X))v_n, v_n)_\mathsf{V} &= (\sum_{k=0}^\infty \frac{1}{k!} d\pi(X)^k v_n, v_n)_\mathsf{V} = \sum_{k=0}^\infty \frac{1}{k!} (d\pi(X)^k v_n, v_n)_\mathsf{V} \\
&= \sum_{k=0}^\infty \frac{1}{k!} (d\pi(X)^k T^{-1}T v_n, v_n)_\mathsf{V} = \sum_{k=0}^\infty \frac{1}{k!} (T^{-1} d\sigma(X)^k T v_n, v_n)_\mathsf{V} \\
&= \sum_{k=0}^\infty \frac{1}{k!} (d\sigma(X)^k T v_n, (T^{-1})^* v_n)_\mathrm{U} = \sum_{k=0}^\infty \frac{1}{k!} (d\sigma(X)^k t_n u_n, \overline{t}_n^{-1} u_n)_\mathrm{U} \\
&= \sum_{k=0}^\infty \frac{1}{k!} (d\sigma(X)^k u_n, u_n)_\mathrm{U} = (\sum_{k=0}^\infty \frac{1}{k!} d\sigma(X)^k u_n, u_n)_\mathrm{U} = (\sigma(\exp(X))u_n, u_n)_\mathrm{U}\,.
\end{aligned}$$

Thus

$$(\pi(g)v_n, v_n)_\mathsf{V} = (\sigma(g)u_n, u_n)_\mathrm{U}$$

in a neighborhood of the identity of G. Since both functions are analytic and G is connected, they are equal everywhere. Hence

$$(\pi(\phi)v_n, v_n)_\mathsf{V} = (\sigma(\phi)u_n, u_n)_\mathrm{U} \qquad (\phi \in C_c(\mathrm{G}))\,.$$

Therefore

$$\operatorname{tr} \pi(\phi) = \sum_n (\pi(\phi)v_n, v_n)_\mathsf{V} = \sum_n (\sigma(\phi)u_n, u_n)_\mathrm{U} = \operatorname{tr} \sigma(\phi)$$

and we are done. □

Given a representation of G (with K-multiplicities at most 1) we define the character of π as

$$\Theta_\pi(\phi) = \operatorname{tr}\pi(\phi) \qquad (\phi \in C_c^\infty(\mathrm{G})\,.$$

This is a distribution on G which, as shown in Theorem 64, does not depend on the infinitesimal equivalence class of (π, V). Therefore we shall also write $\Theta_\pi = \Theta_{\mathsf{V}_{\mathrm{K}}}$.

10.9. **The unitary dual.**

Lemma 65. *Let*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Then

$$[E^+, E^-] = -4iJ\,, \quad [J, E^+] = 2iE^+\,, \quad [J, E^-] = -2iE^-\,.$$

Proof. This is straightforward. See [Lan85, page 102]. ☐

Corollary 66. *For any* $(\mathfrak{g}, \mathrm{K})$ *module* X*, with the* K*-isotypic components* X_n*,*

$$J\mathsf{X}_n \subseteq \mathsf{X}_n\,, \quad E^+\mathsf{X}_n \subseteq \mathsf{X}_{n+2}\,, \quad E^-\mathsf{X}_n \subseteq \mathsf{X}_{n-2}\,.$$

Proof. This is clear from Lemma 65. ☐

Recall the principal series representation $(\pi_s, \mathsf{V}(s))$.

Lemma 67. *In terms of* (41)*, for* $n \in \mathbb{Z}$ *define*

$$v_n\left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = y^{\frac{1+s}{2}} e^{in\theta}\,.$$

Then $v_n \in \mathsf{V}(s)$ *and*

$$\begin{aligned} d\pi_s(J)v_n &= inv_n\,, \\ d\pi_s(E^-)v_n &= (s+1-n)v_{n-2}\,, \\ d\pi_s(E^+)v_n &= (s+1+n)v_{n+2}\,. \end{aligned}$$

Proof. We see from (42) that

$$\begin{aligned} dR(J)v_n &= inv_n\,, \\ dR(E^-)v_n &= (s+1-n)v_{n-2}\,, \\ dR(E^+)v_n &= (s+1+n)v_{n+2}\,. \end{aligned}$$

But the right regular action coincides with π_s, hence the formulas follow. ☐

By combining Lemma 67 with Theorem 58 we deduce the following Corollary.

Corollary 68. *The* $(\mathfrak{g}, \mathrm{K})$ *module of the principal series* $(\pi_s, \mathsf{V}(s))$ *is equal to*

$$\mathsf{V}(s)_{\mathrm{K}} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}v_n\,.$$

Lemma 67 implies the following Proposition, see [Lan85, pages 119-121].

Proposition 69. *If s is not an integer then*

$$\mathsf{V}(s)^+_{\mathrm{K}} = \bigoplus_{n\in 2\mathbb{Z}} \mathbb{C}v_n \quad \textit{znd} \quad \mathsf{V}(s)^-_{\mathrm{K}} = \bigoplus_{n\in 2\mathbb{Z}+1} \mathbb{C}v_n$$

are irreducible submodules of $\mathsf{V}(s)_{\mathrm{K}}$ and $\mathsf{V}(s)_{\mathrm{K}}$ is the direct sum of them.

If $s = 0$, then $\mathsf{V}(0)_{\mathrm{K}}$ is the dirct sum of three irreducible submodules

$$\mathsf{V}(0)_{\mathrm{K}} = \bigoplus_{n\in 2\mathbb{Z}} \mathbb{C}v_n \oplus \bigoplus_{1\le n\in 2\mathbb{Z}+1} \mathbb{C}v_n \oplus \bigoplus_{-1\ge n\in 2\mathbb{Z}+1} \mathbb{C}v_n \,.$$

If $m \geq 2$ is an integer and $s = m-1$, then $\mathsf{V}(m-1)_{\mathrm{K}}$ contains three irreducible submodules

$$\mathsf{X}^m = \bigoplus_{m\le n,\ n-m\in 2\mathbb{Z}} \mathbb{C}v_n\,, \quad \mathsf{X}^{-m} = \bigoplus_{-m\ge n,\ n-m\in 2\mathbb{Z}} \mathbb{C}v_n\,, \quad \bigoplus_{n-m\in 2\mathbb{Z}+1} \mathbb{C}v_n\,.$$

The quotient module, $\mathsf{V}(m-1)_{\mathrm{K}}$ divided by the three submodules is irreducible, finite dimensional of dimension $m-1$. It has a basis represented by the elements

$$v_{-m+2}\,, \quad v_{-m+4}\,, \quad \dots\,, \quad v_{m-2}\,.$$

If $m \geq 2$ is an integer and $s = -m+1$, then $\mathsf{V}(-m+1)_{\mathrm{K}}$ contains the finite dimensional submodule

$$\mathbb{C}v_{-m+2} \oplus \mathbb{C}v_{-m+4} \oplus \dots \oplus \mathbb{C}v_{m-2}\,.$$

The quotient module, $\mathsf{V}(m-1)_{\mathrm{K}}$ divided by this module is isomorphic to the direct sum of modules X^m and X^{-m} plus the sum of all K*-types of parity opposite to m.*

Thus we have the highest weight modules, lowest weight modules, finite dimensional modules and modules with unbounded K-types on both side.

The commutation relations Lemma 65 and the formulas Lemma 67 with some work imply the following theorem, due to Bargmann, [Bar47]. See [Lan85, page 123].

Theorem 70. *Here is a complete list of the irreducible $(\mathfrak{g}, \mathrm{K})$ modules which are unitarizable, up to equivalence.*

1. *Lowest weight module X^m with lowest weight $m \geq 1$ and the highest weight module X^m with highest weight $m \leq -1$*
2. *Principal series $\mathsf{V}(i\tau)^+_{\mathrm{K}}$ and $\mathsf{V}(i\tau)^-_{\mathrm{K}}$, $\tau \in \mathbb{R} \setminus \{0\}$;*
3. *Principal series $\mathsf{V}(0)^+_{\mathrm{K}}$;*
4. *Complementary series $\mathsf{V}(s)^+_{\mathrm{K}}$, $-1 < s < 1$;*
5. *Trivial module.*

A similar result was obtained by Dan Barbasch, [Bar89] for the complex classical groups.

The closures of these modules in the corresponding principal series representation are representations of G on Hilbert spaces. They are unitary representations except the highest and lowest weight representations. In these cases the inner product inherited from the principal series is not G-invariant. Therefore there is a problem of constructing the unitary representations of G whose $(\mathfrak{g}, \mathrm{K})$-modules are the highest and lowest weight modules. This is explained in the theorem below. See [Lan85, page 181].

Theorem 71. *Define*

$$\tilde{\alpha}\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \frac{1}{2}(a+d-ic+ib)\,, \quad \tilde{\beta}\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \frac{1}{2}(c+b-ia+id)\,.$$

Let $m \geq 2$. Then the closure $\mathsf{V}^{(m)}$ in $\mathrm{L}^2(\mathrm{G})$ of the space

$$\bigoplus_{r=0}^{\infty} \mathbb{C}\tilde{\alpha}^{-m-r}\tilde{\beta}^r$$

is invariant under the left action

$$L(g)\phi(x) = \phi(g^{-1}x)\,.$$

The resulting representation $(L, \mathsf{V}^{(m)})$ of G *is irreducible and unitary. Its $(\mathfrak{g}, \mathrm{K})$-module $\mathsf{V}^{(m)}_{\mathrm{K}}$ is isomorphis to the lowest weight module X^m with the lowest weight m. In order to realize the lowest weight representations we take the complex conjugate of $\mathsf{V}^{(m)}$.*

We shall skip the construction of the unitary representations whose $(\mathfrak{g}, \mathrm{K})$-modules are X^1 and X^{-1}. They are not square integrable. One may find them in [Kna86, page 36].

10.10. **The character of the sum of discrete series.** For an integer $m \geq 2$ let $\Theta_{\mathsf{X}^{(m)}}$ denote the the character of the $(\mathfrak{g}, \mathrm{K})$-module $\mathsf{X}^{(m)}$, and $\Theta_{\mathsf{X}^{(-m)}}$ denote the the character of the $(\mathfrak{g}, \mathrm{K})$-module $\mathsf{X}^{(-m)}$. Let

$$h_t = \left(\begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array}\right), \qquad t \in \mathbb{R}\,.$$

Proposition 72. *The character of the sum of the two discrete series $(\mathfrak{g}, \mathrm{K})$-modules $\mathsf{X}^{(m)}$ and $\mathsf{X}^{(-m)}$ is represented by the* G*-conjugation invariant function $\Theta_{\mathsf{X}^{(m)}\oplus\mathsf{X}^{(-m)}}$ given by*

$$\begin{aligned} \Theta_{\mathsf{X}^{(m)}\oplus\mathsf{X}^{(-m)}}(k(\theta)) &= -\frac{e^{i(m-1)\theta} - e^{-i(m-1)\theta}}{e^{i\theta} - e^{-i\theta}} \qquad (\theta \in \mathbb{R})\,, \\ \Theta_{\mathsf{X}^{(m)}+\mathsf{X}^{(-m)}}(zh_t) &= z^m \frac{e^{-t|m-1|}}{|e^t - e^{-t}|} \qquad (z = \pm 1, t \in \mathbb{R})\,. \end{aligned}$$

Proof. From Proposition 69 we know the structure of the principal series $\mathsf{V}(m-1)_{\mathrm{K}}$, from Theorem 53, the character of it, from Lemma 55 the character of the finite dimensional representation which is in the principal series. Now Theorem 64 justifies the formula

character of principal series
= character of the sum of discrete series + character of finite dimensional module.

This completes the proof □

For convenience define the following G-conjugation invariant functions on G

$$\begin{aligned}\Theta_{\mathsf{X}^{(m)}}(k(\theta)) &= -\frac{e^{i(m-1)\theta}}{e^{i\theta}-e^{-i\theta}} \qquad (\theta \in \mathbb{R}), \\ \Theta_{\mathsf{X}^{(m)}}(zh_t) &= z^m \frac{e^{-|t||m-1|}}{|e^t - e^{-t}|} \qquad (z = \pm 1, t \in \mathbb{R}).\end{aligned}$$

and

$$\begin{aligned}\Theta_{\mathsf{X}^{(-m)}}(k(\theta)) &= \frac{e^{i(m-1)\theta}}{e^{i\theta}-e^{-i\theta}} \qquad (\theta \in \mathbb{R}), \\ \Theta_{\mathsf{X}^{(-m)}}(zh_t) &= z^m \frac{e^{-|t||m-1|}}{|e^t - e^{-t}|} \qquad (z = \pm 1, t \in \mathbb{R}).\end{aligned}$$

These are the characters of the individual discrete series representations, but we don't need to know it to explain Harish-Chandra's Plancharel formula in the $\mathrm{SL}_2(\mathbb{R})$ case.

10.11. **Harish-Chandra's Plancherel formula.** It is not difficult to check that K and $\mathrm{H} = \mathrm{A} \cup (-A)$ are the only Cartan subgroups of G up to conjugacy. Let $\mathrm{A}' = \{a \in \mathrm{A};\ a \neq 1\}$ and let $\mathrm{K}' = \{t \in \mathrm{K};\ t \neq \pm 1\}$. We begin by recalling Harish-Chandra's orbital integrals of a function $\phi \in C_c^\infty(\mathrm{G})$, using the notation of [Lan85],

$$H_\phi^{\mathrm{A}}(zh_t) = |D(h_t)| \int_{\mathrm{A}\backslash\mathrm{G}} \phi(x^{-1}zh_t x)\, d\dot{x} \qquad (z = \pm 1, t \in \mathbb{R}, t \neq 0), \tag{45}$$

$$H_\phi^{\mathrm{K}}(k) = D(k) \int_{\mathrm{K}\backslash\mathrm{G}} \phi(x^{-1}kx)\, d\dot{x} \qquad (k \in \mathrm{K}'),$$

The Haar measure on G may be expressed in terms of these integrals by

$$\int_{\mathrm{G}} \phi(x)\, dx = \int_{\mathrm{K}} H_\phi^{\mathrm{K}}(k)\overline{D(k)}\, dk + \frac{1}{4}\int_{\mathrm{A}'} H_\phi^{\mathrm{A}}(a)|D(a)|\, da + \frac{1}{4}\int_{\mathrm{A}'} H_\phi^{\mathrm{A}}(-a)|D(a)|\, da. \tag{46}$$

Theorem 73. *The function H_ϕ^{A} extends to a smooth function on H. The function H_ϕ^{K} is smooth on H' and its derivatives have one sided limits on the boundary. In these terms,*

$$\partial_\theta H_\phi^{\mathrm{K}}(k(\theta))|_{\theta=0} = -i\phi(1). \tag{47}$$

Proof. This is a problem concerning integrals on a 3 dimensional manifold, $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$. Notice that we may replace ϕ by a K conjugation invarian function $\int_{\mathrm{K}} \phi(kxk^{-1})\, dk$. This leads to analysis on the two dimesional manifold G/K. The computations are done in [Lan85, page 164 -167]. □

Recall the character $\chi_n(k(\theta)) = e^{in\theta}$.

Theorem 74. *For any integer $m \geq 2$ and $\phi \in C_c^\infty(\mathrm{G})$*

$$\begin{aligned}\int_{\mathrm{G}} \Theta_{\mathsf{X}^{(m)}}(x)\phi(x)\, dx &= \int_{\mathrm{K}} H_\phi^{\mathrm{K}}(k)\chi_{m-1}(k)\, dk \\ &+ \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t||m-1|}\, dt + \frac{1}{2}\int_{\mathbb{R}} (-1)^m H_\phi^{\mathrm{A}}(-h_t)e^{-|t||m-1|}\, dt\end{aligned}$$

and

$$\int_{\mathrm{G}} \Theta_{\mathsf{X}^{(-m)}}(x)\phi(x)\,dx = -\int_{\mathrm{K}} H_\phi^{\mathrm{K}}(k)\chi_{-m+1}(k)\,dk + \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t||m-1|}\,dt + \frac{1}{2}\int_{\mathbb{R}}(-1)^m H_\phi^{\mathrm{A}}(-h_t)e^{-|t||m-1|}\,dt\,.$$

Proof. This follows from (46) and the formulas for $\Theta_{\mathsf{X}^{(\pm m)}}$ in previous section. □

Theorem 74 gives formulas for the Fourier coefficients of H_ϕ^{K}:

$$\hat{H}_\phi^{\mathrm{K}}(n) = \int_{\mathrm{K}} H_\phi^{\mathrm{K}}(k)\chi_n(k)\,dk \qquad (n \neq 0)\,.$$

Indeed, for $n \geq 1$,

$$\hat{H}_\phi^{K}(n) = \Theta_{\mathsf{X}^{(n+1)}}(\phi) - \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t|n}\,dt - \frac{1}{2}\int_{\mathbb{R}}(-1)^{n+1}H_\phi^{\mathrm{A}}(-h_t)e^{-|t|n}\,dt$$

and

$$\hat{H}_\phi^{K}(-n) = \Theta_{\mathsf{X}^{(-n-1)}}(\phi) - \frac{1}{2}\int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t||n|}\,dt - \frac{1}{2}\int_{\mathbb{R}}(-1)^{n+1}H_\phi^{\mathrm{A}}(-h_t)e^{-|t||n|}\,dt$$

In particular,

$$\hat{H}_\phi^{K}(n) - \hat{H}_\phi^{\mathrm{K}}(-n) = \Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi) - \int_{\mathbb{R}} H_\phi^{\mathrm{A}}(h_t)e^{-|t|n}\,dt - \int_{\mathbb{R}}(-1)^{n+1}H_\phi^{\mathrm{A}}(-h_t)e^{-|t|n}\,dt\,. \quad (48)$$

On the other hand, for $k \neq \pm 1$

$$H_\phi^{\mathrm{K}}(k) = \sum_{n\in\mathbb{Z}} \hat{H}_\phi^{\mathrm{K}}(n)\chi_{-n}(k) = \sum_{n\in\mathbb{Z}} \hat{H}_\phi^{\mathrm{K}}(n)\chi_n(k^{-1}) = \sum_{n\in\mathbb{Z}} \hat{H}_\phi^{\mathrm{K}}(-n)\chi_{-n}(k^{-1})$$

and therefore

$$H_\phi^{\mathrm{K}}(k) - H_\phi^{\mathrm{K}}(k^{-1}) = \sum_{0\neq n\in\mathbb{Z}} (H_\phi^{\mathrm{K}}(n) - H_\phi^{\mathrm{K}}(-n))\chi_{-n}(k)\,.$$

Thus,

$$H_\phi^{\mathrm{K}}(k(\theta)) - H_\phi^{\mathrm{K}}(k(-\theta)) = \sum_{n=1}^{\infty}(H_\phi^{\mathrm{K}}(n) - H_\phi^{\mathrm{K}}(-n))(-i)\sin(n\theta)\,. \quad (49)$$

Continuing this way (and correcting the constants, if neccessary) one obtains the following lemma, see [Lan85, page 174],

Lemma 75.

$$\frac{\pi}{i}(H^{\mathrm{K}}_\phi(k(\theta)) - H^{\mathrm{K}}_\phi(k(-\theta))) = -\sum_{n=1}^{\infty}\Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi)\sin(n\theta)$$
$$+\sum_{n=1}^{\infty}\int_{\mathbb{R}}\frac{1}{2}\left(H^{\mathrm{A}}_\phi(h_t)+(-1)^{n+1}H^{\mathrm{A}}_\phi(-h_t)\right)\sin(n\theta)e^{-|t|n}\,dt\,. \quad (50)$$

Let

$$\phi^+(x)=\frac{\phi(x)+\phi(-x)}{2} \quad \text{and} \quad \phi^-(x)=\frac{\phi(x)-\phi(-x)}{2}\,.$$

Then a straightforward argument shows that

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}}\frac{1}{2}\left(H^{\mathrm{A}}_\phi(h_t)+(-1)^{n+1}H^{\mathrm{A}}_\phi(-h_t)\right)\sin(n\theta)e^{-|t|n}\,dt$$
$$=\int_{\mathbb{R}}H^{\mathrm{A}}_{\phi^+}(h_t)\frac{\sin(\theta)\cosh(t)}{\cosh(2t)-\cos(2\theta)}\,dt+\int_{\mathbb{R}}H^{\mathrm{A}}_{\phi^+}(h_t)\frac{\sin(2\theta)}{\cosh(2t)-\cos(2\theta)}\,dt$$

For $\psi\in C_c^\infty(\mathrm{G})$ we have the Fourier transform

$$\hat{H}^{\mathrm{A}}_\psi(\lambda)=\int_{\mathbb{R}}H^{\mathrm{A}}_\psi(h_t)\,e^{it\lambda}\,dt=\Theta_{\pi_{i\lambda}}(\phi)\,, \quad (51)$$

where the second equality follows from (29). One computes the Fourier transforms of the functions

$$\frac{\sin(\theta)\cosh(t)}{\cosh(2t)-\cos(2\theta)} \quad \text{and} \quad \frac{\sin(2\theta)}{\cosh(2t)-\cos(2\theta)}$$

to deduce from Lemma (75) and (51) the following theorem. See [Lan85, page 173]

Theorem 76.

$$\frac{\pi}{i}(H^{\mathrm{K}}_\phi(k(\theta)) - H^{\mathrm{K}}_\phi(k(-\theta))) = -\sum_{n=1}^{\infty}\Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi)\sin(n\theta)$$
$$+\frac{1}{2}\int_0^\infty\Theta_{\pi_{+,i\lambda}}(\phi)\frac{\cosh((\frac{\pi}{2}-\theta)\lambda)}{\cosh(\frac{\pi\lambda}{2})}\,d\lambda+\frac{1}{2}\int_0^\infty\Theta_{\pi_{-,i\lambda}}(\phi)\frac{\sinh((\frac{\pi}{2}-\theta)\lambda)}{\sinh(\frac{\pi\lambda}{2})}\,d\lambda$$

Now we take the derivative with respect to θ of both sides, go to limit with $\theta\to 0$ and apply (47) to deduce the Harish-Chandra's Plancherel formula.

Theorem 77. *For any $\phi\in C_c^\infty(\mathrm{G})$,*

$$2\pi\phi(1)=\sum_{n=1}^{\infty}n\Theta_{\mathsf{X}^{(n+1)}+\mathsf{X}^{(-n-1)}}(\phi)$$
$$+\frac{1}{2}\int_0^\infty\Theta_{\pi_{+,i\lambda}}(\phi)\,\lambda\tanh(\frac{\pi\lambda}{2})\,d\lambda+\frac{1}{2}\int_0^\infty\Theta_{\pi_{-,i\lambda}}(\phi)\,\lambda\coth(\frac{\pi\lambda}{2})\,d\lambda \quad (52)$$

This Theorem was published by Harish-Chandra in 1952, [HC52]. Notice that knowing the composition series of the principal series was crucial for this proof.

11. Harish-Chandra's Plancherel formula for a real reductive group

Here we follow [Wal88b, page 248].

Theorem 78. *Let* G *be a real reductive group. Then for* $f \in C_c^\infty(\mathrm{G})$

$$f(1) = \sum_{(\mathrm{P},\mathrm{A})\succ(\mathrm{P}_0,\mathrm{A}_0)} C_{\mathrm{A}} \sum_{\omega\in\mathcal{E}_2({}^0\mathrm{M}_\mathrm{P})} d(\omega)\int_{\mathfrak{a}^*} \Theta_{\mathrm{P},\omega,i\nu}(f)\mu(\omega,i\nu)\,d\nu\,, \tag{53}$$

the summation is over standard parabolic subgroups $\mathrm{P} = {}^0\mathrm{M}_\mathrm{P}\mathrm{AN}$ *containing a fixed minimal Parabolic subgroup* $\mathrm{P}_0 = {}^0\mathrm{M}_{\mathrm{P}_0}\mathrm{A}_0\mathrm{N}_0$ *including* $\mathrm{P} = \mathrm{G}$*,* $\mathcal{E}_2({}^0\mathrm{M}_\mathrm{P})$ *is the set of the equivalence classes of discrete series representations of* ${}^0\mathrm{M}_\mathrm{P}$*,* $d(\omega)$ *is the formal degree of* ω*,* $\Theta_{\mathrm{P},\omega,i\nu}$ *is the character of the induced representation with parameters* ω *and* $i\nu$*, which is irreducible, and* $\mu(\omega,i\nu)$ *is described in* [Har76, section 36].

This theorem was proven by Harish-Chandra in 1976, [Har76]. One basic idea was to replace $C_c^\infty(\mathrm{G})$ by a larger space $\mathcal{C}(\mathrm{G})$ where functions decay at infinity sort of like matrix coefficients of discrete series and realize that the character of discrete series applies to them. In fact Harish-Chandra showed, [Har66, Lemma 81, page 93] that if $\mathcal{C}_\omega(\mathrm{G})$ is a space spanned by the matrix coefficients of a discrete series representation ω then for $\phi \in \mathcal{C}_{\omega'}(\mathrm{G})$

$$d(\omega)\Theta_\omega(\phi) = \begin{cases} \phi(1) & \text{if} \quad \omega' \simeq \omega^*\,, \\ 0 & \text{otherwise} \quad . \end{cases} \tag{54}$$

12. A smooth compactly supported positive definite function whose integral is negative

We refer to [Lan85, pages 62-65] for the definition and basic properties of psitive definite functions on a group. By Bochner's theorem, the integral of a positive definite L^1 function on the real line is positive. As we'll see below, this not true for any non-compact semisiple Lie group.

Let G be a real non-compact semisimple Lie group. Fix an Iwasawa decomposition $\mathrm{G} = \mathrm{KAN}$ and let $\mathrm{W} = \mathrm{W}(\mathrm{G},\mathrm{A})$ denote the Weyl group acting on A and on the Lie algebra $\mathfrak{a}$. This action extends to an action on the complexification $\mathfrak{a}_\mathbb{C}$ by $\mathbb{C}$-linearity and dualizes to an action on $\mathfrak{a}_\mathbb{C}^*$. For $g \in \mathrm{G}$ define $H(g) \in \mathfrak{a}$ by $g \in \mathrm{K}\cdot\exp(H(g))\cdot\mathrm{N}$. Let $\mathfrak{n}$ denote the Lie algebra of N and define $\rho \in \mathfrak{a}^*$ by

$$\rho(H) = \frac{1}{2}\,\mathrm{tr}\ \mathrm{ad}(H)_\mathfrak{n} \qquad (H \in \mathfrak{a})\,.$$

In these terms the Haar measure on G may be expressed as follows

$$\int_\mathrm{G} f(g)\,dg = \int_\mathrm{N}\int_\mathrm{A}\int_\mathrm{K} a^{2\rho} f(kan)\,dk\,da\,dn\,.$$

Recall the spherical functions, [GV88, page 104],

$$\phi_\lambda(g) = \int_\mathrm{K} e^{(\lambda-\rho)(H(gk))}\,dk \qquad (\lambda \in \mathfrak{a}_\mathbb{C}^*)\,.$$

The spherical transform of a function $f : \mathrm{G} \to \mathbb{C}$ is defined as

$$\mathcal{H}f(\lambda) = \frac{1}{|\mathrm{W}|} \int_{\mathrm{G}} f(g)\phi_\lambda(g)\, dg \qquad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*)$$

whenever the integral is absolutely convergent, [GV88, page 106]. More explicitly, if the function f is K-bi invariant then, [GV88, page 107],

$$\mathcal{H}f(\lambda) = \int_{\mathrm{A}} a^\lambda \left[a^\rho \int_{\mathrm{N}} f(an)\, dn \right] da \qquad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*)$$

and the result is W-invariant. Moreover, if A is a Cartan subgroup of G, then the expression in the square brackets, usually called the Abel transform of f, denoted $\mathcal{A}f$, is the Harish-Chandra orbital integral of f evaluated at a, [Wal88a, page 249].

Let $C_c(\mathrm{G}//\mathrm{K})$ and $\mathcal{C}(\mathrm{G}//\mathrm{K})$ denote the commutative convolution algebras of the K-bi invariant and continuous compactly supported and Harish-Chandra Schwartz functions on G, respectively. Recall that the convolution of two functions f_1 and f_2 is defined by

$$f_1 * f_2(g) = \int_{\mathrm{G}} f_1(gh^{-1}) f_2(h)\, dh\,.$$

Notice by the way that the convolution of f with $f^*(g) = \overline{f(g^{-1})}$ is equal to the diagonal matrix coefficient of the left regular representation:

$$f * f^*(g) = \int_{\mathrm{G}} f(gh^{-1})\overline{f(h^{-1})}\, dh = (f, L(g)f)_{\mathrm{L}^2(\mathrm{G})}\,.$$

Let $\mathcal{PW}(\mathfrak{a}_{\mathbb{C}}^*)$ be the Paley-Wiener space of all entire functions $F : \mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}$ for which there is a constant $R > 0$ such that

$$|F(\lambda)| \le C_N (1 + |\lambda|)^{-N} e^{R|\mathrm{Re}(\lambda)|} \qquad (\lambda \in \mathfrak{a}_{\mathbb{C}}^*, N = 0, 1, 2, ...)\,.$$

This is a commutative multiplicative algebra and so is the Schwartz space $\mathcal{S}(i\mathfrak{a}^*)$.

The spherical transform is a (bijective) algebra isomorphism

$$\mathcal{H} : C_c(\mathrm{G}//\mathrm{K}) \to \mathcal{PW}(\mathfrak{a}_{\mathbb{C}}^*)^{\mathrm{W}}$$

which extends to a (bijective) algebra isomorphism

$$\mathcal{H} : \mathcal{C}(\mathrm{G}//\mathrm{K}) \to \mathcal{S}(i\mathfrak{a}^*)^{\mathrm{W}}\,.$$

The inverse is given by

$$f(g) = \frac{1}{|\mathrm{W}|} \int_{i\mathfrak{a}^*} \mathcal{H}f(\lambda)\phi_{-\lambda}(g)|c(\lambda)|^{-2}\, d\lambda\,,$$

where c is the Harish-Chandra c-function and $d\lambda$ is a Lebesgue measure on the vector space $i\mathfrak{a}^*$, see [GV88, Proposition 3.1.4, Proposition 3.3.2 and Theorem 6.4.1]. Recall also that $\phi_{-\lambda}$ is positive definite for $\lambda \in i\mathfrak{a}^*$. Hence $f \neq 0$ is positive definite if $\mathcal{H}f$ is non-negative on $i\mathfrak{a}^*$.

Furthermore,

$$\mathcal{H}(f^*)(\lambda) = \overline{\mathcal{H}f(-\overline{\lambda})} = \overline{\mathcal{H}f(\overline{\lambda})}$$

for the λ in the domain of $\mathcal{H}f$. This is because $\phi^*_\lambda(g) = \phi_{-\overline{\lambda}}(g)$ and $-\lambda = w_0\lambda$, where w_0 is the reflection with respect to the longest root.

Recall Harish-Chandra homomorphism, [GV88, Theorem 2.6.7],

$$\gamma : \mathcal{U}(\mathfrak{g})^{\mathrm{K}} \to \mathcal{U}(\mathfrak{a})^{\mathrm{W}} = \mathbb{C}[\mathfrak{a}^*_{\mathbb{C}}]^{\mathrm{W}}\,.$$

Then for $q \in \mathcal{U}(\mathfrak{g})^{\mathrm{K}}$, acting as a differential operator via the right regular representation,

$$\mathcal{H}(qf)(\lambda) = \gamma(q)(\lambda)\mathcal{H}f(\lambda)\,,$$

see [GV88, Proposition 3.2.1] and [Hel84, Theorem 5.18, page 306].

Proposition 79. *Let $q \in \mathcal{U}(\mathfrak{g})^{\mathrm{K}}$ be such that $\gamma(q)(\rho) < 0$ and $\gamma(q)(\lambda) \geq 0$ for all $\lambda \in i\mathfrak{a}^*$ and $\gamma(q)(\lambda) > 0$ for some $\lambda \in ia^*$. Then for any $f \in C_c(\mathrm{G}//\mathrm{K})$ with a non-zero integral over G, the function $q(f * f^*) \in C_c^\infty(\mathrm{G}//\mathrm{K})$ is positive definite and*

$$\int_{\mathrm{G}} q(f * f^*)(g)\, dg < 0\,.$$

Proof. Since $q(f * f^*) = (qf) * f^*$, this function is in $C_c^\infty(\mathrm{G}//\mathrm{K})$. Furthermore,

$$\begin{aligned}
\int_{\mathrm{G}} q(f * f^*)(g)\, dg &= \mathcal{H}(q(f * f^*))(\rho) = \gamma(q)(\rho)\mathcal{H}(f * f^*)(\rho) \\
&= \gamma(q)(\rho)\mathcal{H}(f)(\rho)\mathcal{H}(f^*)(\rho) \\
&= \gamma(q)(\rho)|\mathcal{H}(f)(\rho)|^2 \\
&= \gamma(q)(\rho)\left|\int_{\mathrm{G}} f(g)\, dg\right|^2 < 0\,.
\end{aligned}$$

Since,

$$\mathcal{H}(q(f * f^*))(\lambda) = \gamma(q)(\lambda)|\mathcal{H}f(\lambda)|^2\,,$$

the inversion formula for the spherical transform implies that the function $q(f * f^*)$ is positive definite. □

Let B be a G-invariant non-degenerate symmetric bilinear form on the real vector space $\mathfrak{g}$, [GV88, page 94], and let $C_B \in \mathcal{U}(\mathfrak{g})^{\mathrm{G}}$ and $C_{B,\mathfrak{a}} \in \mathcal{U}(\mathfrak{a})^{\mathrm{W}}$ be the corresponding Casimir elements. Then

$$\gamma(C_B) = C_{B,\mathfrak{a}} - B(\rho, \rho)\,,$$

where we dualize B from $\mathfrak{a}$ to $\mathfrak{a}^*$, see [GV88, Lemma 2.6.10]. Assuming B is positive definite on $\mathfrak{a}^*$, we see that the polynomial $C_{B,\mathfrak{a}}$ has negative values on $i\mathfrak{a}^* \setminus \{0\}$ and is positive on ρ. Hence,

$$q = -C_B - B(\rho, \rho)$$

satisfies the conditions of Proposition 79.

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