DERIVATIVES OF ELLIPTIC ORBITAL INTEGRALS ON A SYMPLECTIC SPACE

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Abstract. For a real reductive dual pair with one member compact we study the orbital integrals on the corresponding symplectic space that occur in the Weyl–Harish-Chandra integration formula on that space. We obtain estimates of the derivatives of such integrals. These estimates are needed for expressing the intertwining distribution attached to a pair of representations in Howe's correspondence in terms of the orbital integrals. This is in analogy to Harish-Chandra's theory, where the distribution character of an irreducible admissible representation of a real reductive group factors through the semisimple orbital integrals on the group.

1. Introduction

Let W be a finite dimensional real vector space with a non-degenerate symplectic form $\langle\cdot,\cdot\rangle$, Sp the corresponding symplectic group and $\widetilde{\mathrm{Sp}}$ the metaplectic group. Let $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$ respectively denote the preimages in $\widetilde{\mathrm{Sp}}$ of a real reductive dual pair G, G' in Sp. Moreover, let $\Pi \otimes \Pi'$ be an irreducible admissible representation of $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}}'$ in Howe's correspondence. Such a representation $\Pi \otimes \Pi'$ is attached to a tempered distribution $f_{\Pi\otimes\Pi'}$ on W, called the intertwining distribution, which is uniquely determined up to a scalar multiple, see [11]. It is the Weyl symbol, in the sense of [5, Chapter XVIII], of the operator realizing $\Pi \otimes \Pi'$ as a quotient of the Weil representation. The asymptotic properties of $f_{\Pi\otimes\Pi'}$ determine the associated varieties of the primitive ideals of Π and Π' and, under some more assumptions, the wave front sets of these representations, see [11] and [10].

In the cases that are usually studied (for instance, when Π is unitary, when the dual pair is in the stable range with G the smaller member, or when the group G is compact – and more generally under the assumptions of [11, Theorem 3.1]), the intertwining distribution $f_{\Pi\otimes\Pi'}$ may be expressed as an integral involving the distribution character Θ_Π of Π. Moreover, if the group G is compact, then the distribution character $\Theta_{\Pi'}$ may also be recovered from $f_{\Pi\otimes\Pi'}$ via an explicit formula, [10, (5.27)]. Thus we have a diagram

$$\Theta_\Pi \longrightarrow f_{\Pi\otimes\Pi'} \longrightarrow \Theta_{\Pi'}. \tag{1}$$

The problem of computing the intertwining distribution explicitly is our main motivation for the present article.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})^{\mathrm{G}}$ the subalgebra of G-invariants in $\mathcal{U}(\mathfrak{g})$. Similarly, define $\mathfrak{g}'$, $\mathcal{U}(\mathfrak{g}')$ and $\mathcal{U}(\mathfrak{g}')^{\mathrm{G}'}$ for G'. Then

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$f_{\Pi\otimes\Pi'}$ turns out to be an invariant eigendistribution on the symplectic space W, i.e. $\mathrm{G}\times\mathrm{G}'$-invariant and an eigendistribution of $\mathcal{U}(\mathfrak{g})^{\mathrm{G}}$ and $\mathcal{U}(\mathfrak{g}')^{\mathrm{G}'}$. The corresponding eigenvalues are the infinitesimal characters of Π and Π', respectively. See [11].

Harish-Chandra's method of descent is one of the main tools for studying invariant eigendistributions on a real reductive Lie algebra $\mathfrak{g}$. Using the adjoint action of G on its Lie algebra $\mathfrak{g}$, it takes an invariant eigendistribution on $\mathfrak{g}$ to a distribution defined on the Cartan subalgebras of $\mathfrak{g}$. See for instance [2]. In [8], we started developing "a method of descent" to study invariant eigendistributions on symplectic spaces. The key fact for this is that the symplectic space W is the odd part $\mathfrak{s}_{\bar{1}}$ of a classical real Lie superalgebra $\mathfrak{s}$ constructed from the dual pair (G,G'). The adjoint action of the Lie group on its Lie algebra of Harish-Chandra's method is replaced by the adjoint action of $\mathrm{S}=\mathrm{G}\times\mathrm{G}'$ on $\mathfrak{s}_{\bar{1}}$. The "descent" of an invariant eigendistribution is obtained by an analog of the Weyl–Harish-Chandra formula on the symplectic space, proved in [8, Theorem 21]. It gives the integral on W of a continuous compactly supported function in terms of almost-semisimple orbital integrals parametrized by mutually non-conjugate Cartan subspaces $\mathfrak{h}_{\bar{1}}$ of $\mathrm{W}=\mathfrak{s}_{\bar{1}}$. Considering the orbital integrals as functions of their parameters leads to a distribution-valued map, defined on a suitable subset of the union of the non-conjugate $\mathfrak{h}_{\bar{1}}$'s, which we called the Harish-Chandra regular almost-semisimple orbital integral on W; see Definition 3.2.

Unlike the Lie algebra case, the Harish-Chandra regular almost-semisimple orbital integral of a Schwartz function on W need not be a Schwartz function on the Cartan subspaces. It was proved that, applied to a rapidly decreasing function on W, any such orbital integral is rapidly decreasing at infinity on the corresponding $\mathfrak{h}_{\bar{1}}$. The question of differentiability, which is difficult in general, was left open. In this paper we answer this question when G is compact. This assumption simplifies the structure of the orbital integral on W. Indeed, unless G is a compact unitary group, there is only one conjugacy class of Cartan subspaces in W, and the corresponding orbital integral is always almost elliptic; see subsection 2.3.

Originally, our orbital integral is defined as a map on the regular elements of the Cartan subspaces of W. It turns out that it can also be defined in terms of regular elements of elliptic Cartan subalgebras $\mathfrak{h}\subseteq\mathfrak{g}$ (or equivalently, $\mathfrak{h}'\subseteq\mathfrak{g}'$). Let l be the rank of G and l' the rank of G'. The regularity properties of the orbital integral are simpler if $l>l'$. In this case, the orbital integral extends to a smooth function on a cone in $\mathfrak{h}'$ which is a union of closed Weyl chambers. In fact, it may be viewed as a pullback of the classical Harish-Chandra orbital integral.

On the other hand, if $l\leq l'$, then our orbital integral extends to the entire $\mathfrak{h}$. However it is not smooth. On the kernels of non-compact imaginary roots of $(\mathfrak{g}',\mathfrak{h}')$ transferred to $\mathfrak{h}$, it has only finitely many derivatives, and we describe them precisely. This order of differentiability is determined by the structure of the dual pair and is enough to express the intertwining distribution $f_{\Pi\otimes\Pi'}$ in terms of our orbital integral and hence describe explicitly the left arrow in (1). This will be done in the forthcoming paper [9], to which we refer for additional information: the explicit formula for the intertwining distributions $f_{\Pi\otimes\Pi'}$ would require too much of additional notation to be included in this short article.

The main theorems of this paper are Theorems 3.3 and 3.5. Their proofs are based on classical results of Harish-Chandra, Rossmann and Wallach, and a deep relatively recent result concerning compact Lie group invariants, recalled in subsection 2.4 below.

It is not surprising that the regularity properties of the orbital integral are simpler when $l > l'$. In fact, it is known that in this case the representations Π' are in the holomorphic (or anti-holomorphic) discrete series of G', and their characters are pretty well understood. This parallels the fact that our computations are much easier than when $l \leq l'$. On the other hand, unless both groups G and G' are compact, the representations Π' one gets in the complementary case $l \leq l'$ are singular unitary highest weight representations. They are classified, but their characters are still murky.

2. Notation and preliminaries

2.1. **The Lie superalgebra associated with a type I dual pair.** In this paper we consider the real reductive dual pairs $(\mathrm{G}, \mathrm{G}')$ which are irreducible (i.e. no nontrivial direct sum decomposition of the symplectic space W is simultaneously preserved by G and G') and for which G is compact. According to Howe's classification [6], they are the following pairs of type I:

$$(\mathrm{O}_d, \mathrm{Sp}_{2m}(\mathbb{R}))\,, \quad (\mathrm{U}_d, \mathrm{U}_{p,q})\,, \quad (\mathrm{Sp}_d, \mathrm{O}^*_{2m})\,.$$

By [12, section 2], we may view such a pair $(\mathrm{G}, \mathrm{G}')$ acting on the symplectic space W as a supergroup $(\mathrm{S}, \mathfrak{s})$. Here S is a Lie group isomorphic to the direct product $\mathrm{G} \times \mathrm{G}'$, and $\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$ is a Lie superalgebra with even part $\mathfrak{s}_{\overline{0}}$ equal to the Lie algebra of S and odd part $\mathfrak{s}_{\overline{1}}$ equal to W. The Lie superalgebra $\mathfrak{s}$ can be realized as a subalgebra of the Lie superalgebra $\mathrm{End}(\mathsf{V})$ of the endomorphisms of a finite dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded vector space $\mathsf{V} = \mathsf{V}_{\overline{0}} \oplus \mathsf{V}_{\overline{1}}$ over $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We recall the construction of $\mathfrak{s}$.

Let $\mathsf{V}_{\overline{0}}$ and $\mathsf{V}_{\overline{1}}$ be finite dimensional left vector spaces over $\mathbb{D}$, and let $d = \dim_{\mathbb{D}} \mathsf{V}_{\overline{0}}$ and $d' = \dim_{\mathbb{D}} \mathsf{V}_{\overline{1}}$. Set $\mathsf{V} = \mathsf{V}_{\overline{0}} \oplus \mathsf{V}_{\overline{1}}$ and define an element $\mathsf{S} \in \mathrm{End}(\mathsf{V})$ by

$$\mathsf{S}(v_0 + v_1) = v_0 - v_1 \qquad (v_0 \in \mathsf{V}_{\overline{0}}, v_1 \in \mathsf{V}_{\overline{1}}).$$

Let

$$\begin{aligned}
&\mathrm{End}(\mathsf{V})_{\overline{0}} = \{x \in \mathrm{End}(\mathsf{V});\ \mathsf{S}x = x\mathsf{S}\},\\
&\mathrm{End}(\mathsf{V})_{\overline{1}} = \{x \in \mathrm{End}(\mathsf{V});\ \mathsf{S}x = -x\mathsf{S}\},\\
&\mathrm{GL}(\mathsf{V})_{\overline{0}} = \mathrm{End}(\mathsf{V})_{\overline{0}} \cap \mathrm{GL}(\mathsf{V}).
\end{aligned}$$

Let # be an involution on $\mathbb{D}$ (non-trivial if $\mathbb{D} \neq \mathbb{R}$). Let $(\cdot,\cdot)$ be a positive-definite #-Hermitian form on $\mathsf{V}_{\overline{0}}$ and $(\cdot,\cdot)'$ a non-degenerate #-skew-Hermitian form on $\mathsf{V}_{\overline{1}}$. Denote by $(\cdot,\cdot)''$ the direct sum of the two forms $(\cdot,\cdot)$ and $(\cdot,\cdot)'$. Let

$$\begin{aligned}
&\mathfrak{s}_{\overline{0}} = \{x \in \mathrm{End}(\mathsf{V})_{\overline{0}};\ (xu, v)'' = -(u, xv)'',\ u, v \in \mathsf{V}\}, && (2)\\
&\mathfrak{s}_{\overline{1}} = \{x \in \mathrm{End}(\mathsf{V})_{\overline{1}};\ (xu, v)'' = (u, \mathsf{S}xv)'',\ u, v \in \mathsf{V}\}, && (3)\\
&\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}},\\
&\mathrm{S} = \{s \in \mathrm{GL}(\mathsf{V})_{\overline{0}};\ (su, sv)'' = (u, v)'',\ u, v \in \mathsf{V}\},\\
&\langle x, y\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathsf{S}xy).
\end{aligned}$$

Then $(\mathrm{S}, \mathfrak{s})$ is a real Lie supergroup, i.e. a real Lie group S together with a real Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$, whose even component $\mathfrak{s}_{\overline{0}}$ is the Lie algebra of S. By restriction, we have the identification

$$\mathfrak{s}_{\overline{1}} = \mathrm{Hom}_{\mathbb{D}}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0}})\,. \qquad (4)$$

We shall write $\mathfrak{s}(\mathsf{V})$ instead of $\mathfrak{s}$ whenever we want to specify the Lie superalgebra $\mathfrak{s}$ constructed as above from a given V and $(\cdot,\cdot)''$.

The group S acts on $\mathfrak{s}$ by conjugation and $\langle\cdot,\cdot\rangle$ is a non-degenerate S-invariant form on the real vector space $\mathfrak{s}$, whose restriction to $\mathfrak{s}_{\overline{0}}$ is symmetric and to $\mathfrak{s}_{\overline{1}}$ is skew-symmetric. We shall employ the notation $s.x = sxs^{-1}$ for the action of $s \in \mathrm{S}$ on $x \in \mathfrak{s}$. In terms of our previous notation,

$$\mathfrak{g} = \mathfrak{s}_{\overline{0}}|_{\mathsf{V}_{\overline{0}}}, \quad \mathfrak{g}' = \mathfrak{s}_{\overline{0}}|_{\mathsf{V}_{\overline{1}}}, \quad \mathrm{W} = \mathfrak{s}_{\overline{1}}, \quad \mathrm{G} = \mathrm{S}|_{\mathsf{V}_{\overline{0}}}, \quad \mathrm{G}' = \mathrm{S}|_{\mathsf{V}_{\overline{1}}},$$

so that

$$\mathfrak{s}_{\overline{0}} = \mathfrak{g} \oplus \mathfrak{g}' \quad \text{and} \quad \mathrm{S} = \mathrm{G} \times \mathrm{G}'.$$

Notice that the action of $\mathrm{S} = \mathrm{G} \times \mathrm{G}'$ on $\mathfrak{s}_{\overline{1}} = \mathrm{W}$ by conjugation corresponds to the action of G on W by left multiplication and of G′ on W via right multiplication by the inverse. Also, we have the unnormalized moment maps

$$\tau : \mathrm{W} \ni w \to w^2|_{\mathsf{V}_{\overline{0}}} \in \mathfrak{g}, \qquad \tau' : \mathrm{W} \ni w \to w^2|_{\mathsf{V}_{\overline{1}}} \in \mathfrak{g}'. \tag{5}$$

2.2. **Cartan subspaces of** $\mathrm{W} = \mathfrak{s}_{\overline{1}}$**.** An element $x \in \mathfrak{s}$ is called semisimple (resp., nilpotent) if x is semisimple (resp., nilpotent) as an endomorphism of V. We say that a semisimple element $x \in \mathfrak{s}_{\overline{1}}$ is regular if it is nonzero and $\dim(S.x) \geq \dim(S.y)$ for all semisimple $y \in \mathfrak{s}_{\overline{1}}$. For $x, y \in \mathfrak{s}_{\overline{1}}$, let $\{x, y\} = xy + yx \in \mathfrak{s}_{\overline{0}}$ denote their anticommutator. Let $x \in \mathfrak{s}_{\overline{1}}$ be fixed. The anticommutant and the double anticommutant of x in $\mathfrak{s}_{\overline{1}}$ are

$$\begin{aligned} {}^{x}\mathfrak{s}_{\overline{1}} &= \{y \in \mathfrak{s}_{\overline{1}} : \{x, y\} = 0\}, \\ {}^{{}^{x}\mathfrak{s}_{\overline{1}}}\mathfrak{s}_{\overline{1}} &= \bigcap_{y \in {}^{x}\mathfrak{s}_{\overline{1}}} {}^{y}\mathfrak{s}_{\overline{1}}, \end{aligned}$$

respectively. A Cartan subspace $\mathfrak{h}_{\overline{1}}$ of $\mathfrak{s}_{\overline{1}}$ is defined as the double anticommutant of a regular semisimple element $x \in \mathfrak{s}_{\overline{1}}$. We denote by $\mathfrak{h}_{\overline{1}}^{reg}$ the set of regular elements in $\mathfrak{h}_{\overline{1}}$.

Next we describe the Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathfrak{s}_{\overline{1}}$ for the supergroups associated with the irreducible dual pairs $(\mathrm{G}, \mathrm{G}')$ with G compact. We refer to [12, §6] and [8, §4] for the proofs omitted here. Recall that l is the rank of $\mathfrak{g}$ and l' the rank of $\mathfrak{g}'$. Set

$$l'' = \min\{l, l'\}. \tag{6}$$

Given a Cartan subspace $\mathfrak{h}_{\overline{1}}$, there are $\mathbb{Z}/2\mathbb{Z}$-graded subspaces $\mathsf{V}^j \subseteq \mathsf{V}$ such that the restriction of the form $(\cdot,\cdot)''$ to each V^j is non-degenerate, V^j is orthogonal to V^k for $j \neq k$ and

$$\mathsf{V} = \mathsf{V}^0 \oplus \mathsf{V}^1 \oplus \mathsf{V}^2 \oplus \cdots \oplus \mathsf{V}^{l''}. \tag{7}$$

The subspace V^0 coincides with the intersection of the kernels of the elements of $\mathfrak{h}_{\overline{1}}$ (equivalently, $\mathsf{V}^0 = \mathrm{Ker}(x)$ if $\mathfrak{h}_{\overline{1}} = {}^{{}^{x}\mathfrak{s}_{\overline{1}}}\mathfrak{s}_{\overline{1}}$). For $1 \leq j \leq l''$, the subspaces $\mathsf{V}^j = \mathsf{V}^j_{\overline{0}} \oplus \mathsf{V}^j_{\overline{1}}$ are described as follows.

Suppose $\mathbb{D} = \mathbb{R}$. Then there is a basis v_0, v_0' of $\mathsf{V}^j_{\overline{0}}$ and basis v_1, v_1' of $\mathsf{V}^j_{\overline{1}}$ such that

$$\begin{aligned} (v_0, v_0)'' = (v_0', v_0')'' = 1, \qquad (v_0, v_0')'' = 0, \\ (v_1, v_1)'' = (v_1', v_1')'' = 0, \qquad (v_1, v_1')'' = 1. \end{aligned}$$

Let $\mathfrak{s}_{\bar{1}}(\mathsf{V}^j)$ be defined as in (2) with V replaced by V_j. The following formulas define an element $u_j \in \mathfrak{s}_{\bar{1}}(\mathsf{V}^j)$,

$$u_j(v_0) = \frac{1}{\sqrt{2}}(v_1 - v_1'), \qquad u_j(v_1) = \frac{1}{\sqrt{2}}(v_0 - v_0'),$$
$$u_j(v_0') = \frac{1}{\sqrt{2}}(v_1 + v_1'), \qquad u_j(v_1') = \frac{1}{\sqrt{2}}(v_0 + v_0').$$

Suppose $\mathbb{D} = \mathbb{C}$. Then $\mathsf{V}^j_{\bar{0}} = \mathbb{C}v_0$, $\mathsf{V}^j_{\bar{1}} = \mathbb{C}v_1$, where $(v_0, v_0)'' = 1$ and $(v_1, v_1)'' = \delta_j i$, with $\delta_j = \pm 1$. The following formulas define an element $u_j \in \mathfrak{s}_{\bar{1}}(\mathsf{V}^j)$,

$$u_j(v_0) = e^{-i\delta_j \frac{\pi}{4}} v_1, \qquad u_j(v_1) = e^{-i\delta_j \frac{\pi}{4}} v_0. \tag{8}$$

Suppose $\mathbb{D} = \mathbb{H}$. Then $\mathsf{V}^j_{\bar{0}} = \mathbb{H}v_0$, $\mathsf{V}^j_{\bar{1}} = \mathbb{H}v_1$, where $(v_0, v_0)'' = 1$ and $(v_1, v_1)'' = i$. The following formulas define an element $u_j \in \mathfrak{s}_{\bar{1}}(\mathsf{V}^j)$,

$$u_j(v_0) = e^{-i\frac{\pi}{4}} v_1, \qquad u_j(v_1) = e^{-i\frac{\pi}{4}} v_0.$$

In any case, by extending each u_j by zero outside V^j, we have

$$\mathfrak{h}_{\bar{1}} = \sum_{j=1}^{l''} \mathbb{R} u_j\,. \tag{9}$$

Formula (9) describes all Cartan subspaces in $\mathfrak{s}_{\bar{1}}$, up to conjugation by S. In other words it describes a maximal family of mutually non-conjugate Cartan subspaces.

Notice also that there is only one such subspace unless the dual pair $(\mathrm{G}, \mathrm{G}')$ is isomorphic to $(\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l'' = l < p + q$. In the last case there are $\min(l, p) - \max(l - q, 0) + 1$ such subspaces, assuming $p \le q$. In fact, for each m such that $\max(l - q, 0) \le m \le \min(p, l)$ there is a Cartan subspace $\mathfrak{h}_{\bar{1},m}$ determined by the condition that m is the number of the positive δ_j in (8). We may assume that $\delta_1 = \cdots = \delta_m = 1$ and $\delta_{m+1} = \cdots = \delta_l = -1$. By the above description of the spaces $\mathsf{V}^j_{\bar{0}}$ and $\mathsf{V}^j_{\bar{1}}$ for $\mathbb{D} = \mathbb{C}$, we see that the choice of the spaces $\mathsf{V}^j_{\bar{0}}$ may be done independently of m, whereas the spaces $\mathsf{V}^j_{\bar{1}}$ depend on m.

The Weyl group $W(\mathrm{S}, \mathfrak{h}_{\bar{1}})$ is the quotient of the stabilizer of $\mathfrak{h}_{\bar{1}}$ in S by the subgroup $\mathrm{S}^{\mathfrak{h}_{\bar{1}}}$ fixing each element of $\mathfrak{h}_{\bar{1}}$. If $\mathbb{D} \ne \mathbb{C}$, then the group $W(\mathrm{S}, \mathfrak{h}_{\bar{1}})$ acts by all the sign changes and all permutations of the u_j's. If $\mathbb{D} = \mathbb{C}$, then the group $W(\mathrm{S}, \mathfrak{h}_{\bar{1}})$ acts by all the sign changes of the u_j's and all permutations which preserve $(\delta_1, \dots, \delta_{l''})$, see [12, (6.3)].

If $\mathbb{D} \ne \mathbb{C}$ we set $\delta_j = 1$ for all $1 \le j \le l''$. Define

$$J_j = \delta_j \tau(u_j), \qquad J_j' = \delta_j \tau'(u_j) \qquad (1 \le j \le l''). \tag{10}$$

Then J_j, J_j' are complex structures on $\mathsf{V}^j_{\bar{0}}$ and $\mathsf{V}^j_{\bar{1}}$ respectively. Explicitly,

$$\begin{aligned} &J_j(v_0) = -v_0', \quad J_j(v_0') = v_0, \quad J_j'(v_1) = -v_1', \quad J_j'(v_1') = v_1, \qquad \text{if } \mathbb{D} = \mathbb{R},\\ &J_j(v_0) = -iv_0, \quad J_j'(v_1) = -iv_1, \qquad \text{if } \mathbb{D} = \mathbb{C} \text{ or } \mathbb{D} = \mathbb{H}. \end{aligned} \tag{11}$$

(The point of the multiplication by the δ_j in (10) is that the complex structures J_j, J_j' do not depend on the Cartan subspace $\mathfrak{h}_{\bar{1}}$.) In particular, if

$$w = \sum_{j=1}^{l''} w_j u_j \in \mathfrak{h}_{\bar{1}}\,, \tag{12}$$

then

$$\tau(w)=\sum_{j=1}^{l''} w_j^2\delta_j J_j \qquad \text{and} \qquad \tau'(w)=\sum_{j=1}^{l''} w_j^2\delta_j J_j'. \tag{13}$$

Let $\mathfrak{h}_{\overline{1}}^2 \subseteq \mathfrak{s}_{\overline{0}}$ be the subspace spanned by all the squares w^2, $w \in \mathfrak{h}_{\overline{1}}$. Then

$$\mathfrak{h}_{\overline{1}}^2=\sum_{j=1}^{l''}\mathbb{R}(J_j+J_j'). \tag{14}$$

We shall use the identification

$$\mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{0}}} \ni \sum_{j=1}^{l''} y_j J_j=\sum_{j=1}^{l''} y_j J_j' \in \mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{1}}}. \tag{15}$$

Recall that $l''=\min\{l,l'\}$, where l and l' are the ranks of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. If $l''=l$, then $\mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{0}}}$ is an elliptic Cartan subalgebra of $\mathfrak{g}$ which we denote by $\mathfrak{h}$. (Recall that this means that all the roots of $\mathfrak{h}$ in $\mathfrak{g}_{\mathbb{C}}$ are purely imaginary.) The identification (15) embeds $\mathfrak{h}$ diagonally in $\mathfrak{g}$ and in $\mathfrak{g}'$. Similarly, if $l''=l'$, then $\mathfrak{h}_{\overline{1}}^2|_{\mathsf{V}_{\overline{1}}}$ is an elliptic Cartan subalgebra of $\mathfrak{g}'$ which we denote by $\mathfrak{h}'$ and we diagonally embed it in $\mathfrak{g}$ and in $\mathfrak{g}'$ by (15). If $l\leq l'$ we denote by $\mathfrak{z}'\subseteq\mathfrak{g}'$ the centralizer of $\mathfrak{h}$. Similarly, if $l'\leq l$ we denote by $\mathfrak{z}\subseteq\mathfrak{g}$ the centralizer of $\mathfrak{h}'$. In particular, if $l'=l$, then $\mathfrak{z}'=\mathfrak{h}'=\mathfrak{h}=\mathfrak{z}$, where the first equality is in $\mathfrak{g}$, the second is (15) and the last is in $\mathfrak{g}'$.

Notice that when the dual pair is $(\mathrm{U}_l,\mathrm{U}_{p,q})$ with $l=l''<p+q$, then $\mathfrak{h}_{\overline{1},m}^2|_{\mathsf{V}_{\overline{0}}}=\mathfrak{h}_{\overline{1},m'}^2|_{\mathsf{V}_{\overline{0}}}$ for all m,m'. There is a unique Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing all $\tau(\mathfrak{h}_{\overline{1},m})$.

Let $\mathfrak{s}_{\overline{0}\mathbb{C}}=\mathfrak{g}_{\mathbb{C}}\oplus\mathfrak{g}'_{\mathbb{C}}$ be the complexification of $\mathfrak{s}_{\overline{0}}$. Fix a system of positive roots for the adjoint action of $\mathfrak{h}_{\overline{1}}^2$ on $\mathfrak{s}_{\overline{0}\mathbb{C}}$ and let $\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}$ denote their product. Suppose first that $l\leq l'$. Then $\mathfrak{h}$ is an elliptic Cartan subalgebra of $\mathfrak{g}$ and, using the identification (15), it is contained in an elliptic Cartan subalgebra of $\mathfrak{g}'$, say $\mathfrak{h}'$. Since $\mathfrak{h}$ preserves both $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$, our choice of positive roots for $(\mathfrak{h}_{\overline{1}\mathbb{C}}^2,\mathfrak{s}_{\overline{0}\mathbb{C}})$ fixes a positive root system of $(\mathfrak{h}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}})$ and extends to a compatible positive root system for $(\mathfrak{h}'_{\mathbb{C}},\mathfrak{g}'_{\mathbb{C}})$. Let $\pi_{\mathfrak{g}/\mathfrak{h}}$ be the product of positive roots of $(\mathfrak{h}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}})$ and let $\pi_{\mathfrak{g}'/\mathfrak{z}'}$ be the product of positive roots of $(\mathfrak{h}'_{\mathbb{C}},\mathfrak{g}'_{\mathbb{C}})$ such that the corresponding root spaces do not occur in $\mathfrak{z}'_{\mathbb{C}}$. If $l'<l$, then $\pi_{\mathfrak{g}'/\mathfrak{h}'}$ and $\pi_{\mathfrak{g}/\mathfrak{z}}$ can be similarly defined. Then for all $w\in\mathfrak{h}_{\overline{1}}$

$$\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)=\begin{cases}\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w)) & \text{if } l\leq l',\\ \pi_{\mathfrak{g}/\mathfrak{z}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{h}'}(\tau'(w)) & \text{if } l\geq l'.\end{cases} \tag{16}$$

Lemma 2.1. *There is a constant $C(\mathfrak{h}_{\overline{1}})$, which depends on $\mathfrak{h}_{\overline{1}}$, such that $|C(\mathfrak{h}_{\overline{1}})|=1$ and*

$$|\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2)|=C(\mathfrak{h}_{\overline{1}})\,\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) \qquad (w\in\mathfrak{h}_{\overline{1}}).$$

Proof. Suppose that $l\leq l'$. The root systems of the two members in each dual pair are such that for $w=\sum_{j=1}^{l} w_j u_j\in\mathfrak{h}_{\overline{1}}$, the functions $\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))$ and $\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))$ differ by a constant multiple of a product of powers of the w_j^2's. More precisely, the right-hand side of (16) is equal to

$$\begin{array}{lr}
\left(\prod_{1\le j<k\le l} i(-\delta_j w_j^2+\delta_k w_k^2)\right)^2\cdot\prod_{j=1}^l(-i\delta_j w_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{C},\\
\left(\prod_{1\le j<k\le l}(-w_j^4+w_k^4)\right)^2\cdot\prod_{j=1}^l 2iw_j^2\cdot\prod_{j=1}^l(-w_j^4)^{d'-d} & \text{if } \mathbb{D}=\mathbb{H},\\
\left(\prod_{1\le j<k\le l}(-w_j^4+w_k^4)\right)^2\cdot\prod_{j=1}^l 2iw_j^2\cdot\prod_{j=1}^l(iw_j^2)^{d'-d} & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l},\\
\left(\prod_{1\le j<k\le l}(-w_j^4+w_k^4)\right)^2\cdot\prod_{j=1}^l iw_j^2\cdot\prod_{j=1}^l 2iw_j^2\cdot\prod_{j=1}^l(iw_j^2)^{d'-d+1} & \\
 & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathfrak{g}=\mathfrak{so}_{2l+1}.
\end{array}$$

The lemma is an immediate consequence of the above formulas. The case $l>l'$ is similar. □

We notice that that $|\pi_{\mathfrak{s}_{\bar{0}}/\mathfrak{h}_{\bar{1}}^2}|$ is the non-negative Jacobian which occurs in the Weyl–Harish-Chandra integration formula on $\mathfrak{s}_{\bar{1}}$, see [8, Theorem 20] and (23), (22) below. If $\mathfrak{h}_{\bar{1}}$ is a Cartan subspace of W, then

$$\mathfrak{h}_{\bar{1}}^{reg}=\{w\in\mathfrak{h}_{\bar{1}}:\pi_{\mathfrak{s}_{\bar{0}}/\mathfrak{h}_{\bar{1}}^2}(w^2)\neq 0\}\,. \tag{17}$$

2.3. **The Weyl–Harish-Chandra integration formula on** $\mathrm{W}=\mathfrak{s}_{\bar{1}}$**.** This section is based on [8], to which we refer for a detailed discussion.

Let $\mathcal{S}(\mathrm{W})$ and $\mathcal{S}^*(\mathrm{W})$ be the Schwartz space on W and the space of tempered distributions on W, respectively, and let $\mathcal{S}(\mathrm{W})^{\mathrm{S}}$ and $\mathcal{S}^*(\mathrm{W})^{\mathrm{S}}$ be their subspaces of S-invariant elements. Fix a Cartan subspace $\mathfrak{h}_{\bar{1}}\subseteq\mathrm{W}$ and an element $w\in\mathfrak{h}_{\bar{1}}^{reg}$. Let $\mathrm{S}^{\mathfrak{h}_{\bar{1}}}$ denote the centralizer of $\mathfrak{h}_{\bar{1}}$ in S and let $d(s\mathrm{S}^{\mathfrak{h}_{\bar{1}}})$ be an invariant measure on the quotient space $\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}$. We first exclude the case in which

$$\mathrm{G}=\mathrm{O}_{2l+1}\quad\text{with}\quad l<l'. \tag{18}$$

The orbital integral attached to the orbit $\mathcal{O}(w)=\mathrm{S}.w$ is the element of $\mathcal{S}^*(\mathrm{W})^{\mathrm{S}}$ defined for $\phi\in\mathrm{S}(\mathrm{W})$ by

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}}(\phi)=\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}}\phi(s.w)\,d(s\mathrm{S}^{\mathfrak{h}_{\bar{1}}}). \tag{19}$$

Suppose now G is as in (18). One needs to modify (19) because the union of the orbits $\mathrm{S}.w$ over all $w\in\mathfrak{h}_{\bar{1}}^{reg}$ would not be dense in W; see [8, Theorem 20]. For instance, for $\mathrm{S}=\mathrm{O}_1\times\mathrm{Sp}_{2n}=\{\pm1\}\times\mathrm{Sp}(\mathrm{W})$ there are no non-zero semisimple elements in W and (19) would reduce to evaluation at 0. So, let $w_0\in\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)$ be a non-zero element and let $w\in\mathfrak{h}_{\bar{1}}^{reg}$ be as in (12). Then $w+w_0$ is called a regular almost semisimple element. Its centralizer in S is denoted by $\mathrm{S}^{\mathfrak{h}_{\bar{1}}+w_0}$. Set $\mathcal{O}(w)=\mathrm{S}.(w+w_0)$ and define

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}}(\phi)=\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}+w_0}}\phi(s.(w+w_0))\,d(s\mathrm{S}^{\mathfrak{h}_{\bar{1}}+w_0}). \tag{20}$$

Then this is independent of the choice of $w_0\in\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)$ and, up to a constant multiple which does not depend on w_0,

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}}(\phi)=\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}}\int_{\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)}\phi(s.(w+w^0))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}_{\bar{1}}}). \tag{21}$$

In passing, it should be noted that (21) can be used as a unified definition of the orbital integrals for all compact pairs (G,G') because $\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)=\{0\}$ when G is compact and different from (18); see [8, Proposition 10]. The orbital integrals (19) and (21) are well defined, tempered distribution on W, which depend only on $\tau(w)$, or equivalently $\tau'(w)$ via the identification (15).

Fix a compatible positive complex structure J on W. This means that $J \in \mathfrak{sp}$ is such that $J^2 = -1$ (minus the identity) and the symmetric bilinear form $\langle J\cdot, \cdot\rangle$ is positive definite on W. Let μ_{W} be the Lebesgue measure on W normalized so that the volume of the unit cube with respect to this form is 1. Choose a positive Weyl chamber $\mathfrak{h}_{\overline{1}}^{+} \subseteq \mathfrak{h}_{\overline{1}}{}^{reg}$, i.e. an open fundamental domain for the action of the Weyl group, $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$. We shall normalize the above orbital integrals so that the Weyl–Harish-Chandra integration formula on $\mathfrak{s}_{\overline{1}}$, [8, Theorem 21], reads for all $\phi \in \mathcal{S}(\mathrm{W})$

$$\mu_{\mathrm{W}}(\phi) = \int_{\tau'(\mathfrak{h}_{\overline{1}}^{+})} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^{2}}(w^2)| \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi)\, d\tau'(w) \tag{22}$$

if $l \geq l'$, and

$$\mu_{\mathrm{W}}(\phi) = \sum_{\mathfrak{h}_{\overline{1}}} \int_{\tau(\mathfrak{h}_{\overline{1}}^{+})} |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^{2}}(w^2)| \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi)\, d\tau(w) \tag{23}$$

if $l \leq l'$. The sum in (23) is over the family of mutually non-conjugate Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathrm{W} = \mathfrak{s}_{\overline{1}}$. It reduces to a single term except when the dual pair $(\mathrm{G}, \mathrm{G}')$ is isomorphic to $(\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l < p + q$.

Since G is compact, $\tau(\mathfrak{h}_{\overline{1}})$ and $\tau'(\mathfrak{h}_{\overline{1}})$ always consist of elliptic elements. So the corresponding orbital integrals in (22) and (23) will be elliptic, except in the case (18), where it will be almost elliptic because of the additional nilpotent part.

2.4. **A theorem of Schwarz, Mather and Astengo-Di Blasio-Ricci.** The unnormalized moment map

$$\tau' : \mathrm{W} \to \mathfrak{g}'^{*},\ \tau'(w)(y) = \langle yw, w\rangle \qquad (w \in \mathrm{W},\ y \in \mathfrak{g}') \tag{24}$$

is a quadratic polynomial map with compact fibers. Hence the pull-back

$$\tau'^{*} : \mathcal{S}(\mathfrak{g}') \ni \psi \to \psi \circ \tau' \in \mathcal{S}(\mathrm{W})^{\mathrm{G}} \tag{25}$$

is well defined and continuous, [10, Lemma 6.1]. The fact that τ'^{*} admits a continuous inverse is a deep result. It is a special instance of a theorem of Astengo, Di Blasio and Ricci, [1, Theorem 6.1], which extends to the space of Schwartz functions previous results proved by G. Schwarz, [13], and Mather, [7] for the smooth case. In our situation, this theorem states that there is a continuous map

$$\tau'_{*} : \mathcal{S}(\mathrm{W})^{\mathrm{G}} \to \mathcal{S}(\mathfrak{g}') \tag{26}$$

such that

$$\tau'^{*} \circ \tau'_{*}(\phi) = \phi \qquad (\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}})\,. \tag{27}$$

In particular, τ'^{*} is surjective and, by dualizing (25), we get a continuous injective push-forward of distributions

$$\tau'^{*}_{*} : \mathcal{S}^{*}(\mathrm{W})^{\mathrm{G}} \to \mathcal{S}^{*}(\mathfrak{g}')$$

given by

$$\tau'^{*}_{*}(u)(\psi) = u(\psi \circ \tau') \qquad (u \in \mathcal{S}^{*}(\mathrm{W})^{\mathrm{G}},\ \psi \in \mathcal{S}(\mathfrak{g}'))\,. \tag{28}$$

Since $\tau'(\mathrm{W}) \subseteq \mathfrak{g}'$ may be a proper subset, the map (26) is not unique. However, the map $\tau_*'^*$ is independent of the choice of τ_*' in (27) and satisfies

$$u(\phi) = \tau_*'^*(u)(\tau_*'(\phi)) = u(\tau_*'(\phi) \circ \tau') \qquad (u \in \mathcal{S}^*(\mathrm{W})^{\mathrm{G}},\ \phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}})\,. \tag{29}$$

Notice that any distribution in the range of $\tau_*'^*$ is supported in $\tau'(\mathrm{W})$. Hence the restriction of $\tau_*'^*$ to $\mathcal{S}(\mathrm{W})^{\mathrm{G}} \subseteq \mathcal{S}^*(\mathrm{W})^{\mathrm{G}}$ does not coincide with τ_*'.

The following lemma shows that τ_*' is G'-equivariant on $\tau'(\mathrm{W})$. For a function ψ on W and $g' \in \mathrm{G}'$, we shall denote by $\psi^{g'}$ the function on W defined by $\psi^{g'}(w) = \psi(g'.w)$.

Lemma 2.2. *Let $\phi \in \mathcal{S}^*(\mathrm{W})^{\mathrm{G}}$ and $g' \in \mathrm{G}'$. Then $\phi^{g'} \in \mathcal{S}^*(\mathrm{W})^{\mathrm{G}}$ and*

$$\tau_*'(\phi^{g'}) = \tau_*'(\phi)^{g'} \qquad \text{on } \tau'(\mathrm{W}).$$

Proof. Let $w \in \mathrm{W}$. By (27), $\tau_*'(\phi) \circ \tau' = \phi$. Hence, since τ' is G'-equivariant,

$$\tau_*'(\phi)^{g'}(\tau'(w)) = \tau_*'(\phi)\big(g'.\tau'(w)\big) = \tau_*'(\phi)\big(\tau'(g'.w)\big) = \phi(g'.w) = \phi^{g'}(w)\,.$$

On the other hand, (27) applied to $\phi^{g'}$ gives $\tau_*'(\phi^{g'})(\tau'(w)) = \phi^{g'}(w)\,.$ □

3. An almost-elliptic orbital integral on the symplectic space

In this section we define the orbital integrals we are concerned with in this paper and study their differentiability properties. We first need a lemma.

Lemma 3.1. *Suppose $l < l'$ and $\mathbb{D} = \mathbb{C}$. Then for* $\max(l-q, 0) \le m < m' \le \min(p, l)$,

$$\tau(\mathfrak{h}_{\bar{1},m}^{reg}) \cap \tau(\mathfrak{h}_{\bar{1},m'}^{reg}) = \emptyset.$$

Thus in the only case when there is more than one Cartan subspace $\mathfrak{h}_{\bar{1}}$, the union

$$\bigcup_{\mathfrak{h}_{\bar{1}}} \tau(\mathfrak{h}_{\bar{1},m}^{reg})$$

over the family of mutually non-conjugate Cartan subspaces $\mathfrak{h}_{\bar{1}} \subseteq \mathrm{W} = \mathfrak{s}_{\bar{1}}$ is disjoint.

Proof. We see from (13) and (17) that

$$\tau(\mathfrak{h}_{\bar{1},m}^{reg}) = \{\sum_{j=1}^{l} y_j J_j;\ y_1, \dots, y_m > 0 > y_{m+1}, \dots, y_l, \quad y_j \neq y_k \text{ for } j \neq k\}. \tag{30}$$

If $m' > m$, then the $(m+1)$-th component of an element of $\tau(\mathfrak{h}_{\bar{1},m}^{reg})$ is strictly negative, whereas it is strictly positive for an element of $\tau(\mathfrak{h}_{\bar{1},m'}^{reg})$. □

Lemma 3.1 shows that the following definition makes sense when there is more than one conjugacy class of Cartan subspaces.

Definition 3.2. Let $C_{\mathfrak{h}_{\bar{1}}} = C(\mathfrak{h}_{\bar{1}}) \cdot i^{\dim \mathfrak{g}/\mathfrak{h}}$, where $C(\mathfrak{h}_{\bar{1}})$ is as in Lemma 2.1. The *Harish-Chandra regular almost-elliptic orbital integral on* W is the function F defined as follows. For $l \le l'$,

$$F : \bigcup_{\mathfrak{h}_{\bar{1}}} \tau(\mathfrak{h}_{\bar{1}}{}^{reg}) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{31}$$

is given by

$$F(y) = \sum_{\mathfrak{h}_{\bar{1}}} C_{\mathfrak{h}_{\bar{1}}} \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}} \qquad (y \in \bigcup_{\mathfrak{h}_{\bar{1}}} \tau(\mathfrak{h}_{\bar{1}}{}^{reg}),\ y = \tau(w) = \tau'(w)), \tag{32}$$

(where we are using the identifications (13) and (15).) If $l > l'$, then all Cartan subspaces are conjugate to $\mathfrak{h}_{\bar{1}}$ and

$$F : \tau(\mathfrak{h}_{\bar{1}}{}^{reg}) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{33}$$

is given by

$$F(y) = C_{\mathfrak{h}_{\bar{1}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}} \qquad (y \in \tau(\mathfrak{h}_{\bar{1}}{}^{reg}),\ y = \tau(w) = \tau'(w)). \tag{34}$$

Following Harish-Chandra's notation, we shall write $F_\phi(y)$ for $F(y)(\phi)$.

Remarks. (a) Given $y \in \bigcup_{\mathfrak{h}_{\bar{1}}} \tau(\mathfrak{h}_{\bar{1}}{}^{reg})$, two elements w, w' for which $y = \tau(w) = \tau(w')$ differ by ± 1 on each component w_j in (12). Since all sign changes are in the Weyl group $W(S, \mathfrak{h}_{\bar{1}})$, we have $\mathcal{O}(w) = \mathcal{O}(w')$. So F is well-defined as a function of y.

(b) Both definitions (31) and (33) could be unified into the first one because $\mathfrak{h}' = \mathfrak{z}'$ if $l \geq l'$.

(c) The definition of F is motivated by (22) and (23). Our final goal will be to use F to "transfer" objects defined on the compact group G to similar objects defined on the noncompact group G', as in the diagram (1). F is therefore defined on a subset of the Cartan subalgebra of $\mathfrak{g}$ and involves as a regularizing factor the part of the Jacobian (16) containing the product of roots for $\mathfrak{g}'$. This lack of symmetry with respect to (22) and (23) explains the different regularity properties of F we shall prove in the cases $l \leq l'$ and $l > l'$.

(d) Explicit formulas for $F(y)$ in terms of Harish-Chandra's orbital integral will be given in (39) and (72) below.

Theorems 3.3 and 3.5 are the main results of this paper.

Theorem 3.3. *Suppose $l > l'$. Let $\mathfrak{h}'^{In-reg} \subseteq \mathfrak{h}'$ be the subset where no non-compact roots vanish. Identify $\tau(\mathfrak{h}_{\bar{1}}^{reg}) = \tau'(\mathfrak{h}_{\bar{1}}^{reg}) \subseteq \mathfrak{h}'^{\,reg}$ as in (15). Then F extends to a distribution-valued map on $\mathfrak{h}'^{In-reg} \cap \tau'(\mathrm{W})$ that is $W(\mathrm{G}', \mathfrak{h}')$-skew-invariant. The function*

$$F : \mathfrak{h}'^{In-reg} \cap \tau'(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{35}$$

is smooth in the sense that it is differentiable in the interior of $\mathfrak{h}'^{In-reg} \cap \tau'(\mathrm{W})$ and any derivative of F extends to a continuous function on the closure in $\mathfrak{h}'$ of any connected component of $\mathfrak{h}'^{In-reg} \cap \tau'(\mathrm{W})$.

Proof. Let $\mathrm{H}' \subseteq \mathrm{G}'$ be the Cartan subgroup with Lie algebra $\mathfrak{h}'$. Denote by $\Delta(\mathrm{H}') \subseteq \mathrm{S} = \mathrm{G} \times \mathrm{G}'$ the diagonal embedding. Then,

$$\mathrm{S}^{\mathfrak{h}_{\bar{1}}} = \Delta(\mathrm{H}')(\mathrm{Z} \times \{1\}), \tag{36}$$

where $\mathrm{Z} \subseteq \mathrm{G}$ is the centralizer of $\mathfrak{h} \subseteq \mathfrak{g}$. Fix a function $\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}$ and let $\psi = \tau'_*(\phi) \in \mathcal{S}(\mathfrak{g}')$, see (26). Then, by (29),

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}}(\phi) = \tau'^{*}_{*}(\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}})(\psi) = \mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}}(\tau'_*(\phi) \circ \tau') = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}} \psi(\tau'(s.w))\, d(s\mathrm{S}^{\mathfrak{h}_{\bar{1}}}). \tag{37}$$

Observe that, if $y = \tau'(w)$ and $s = (g, g') \in \mathrm{S}$, then

$$\tau'(s.w) = \big((g, g').w\big)^2|_{\mathrm{V}_{\overline{1}}} = g' y g'^{-1} = g'.y$$

does not depend on $g \in \mathrm{G}$. Since G is compact, (37) is a constant multiple of

$$\int_{\mathrm{G}'/\mathrm{H}'} \psi(g'.y)\, d(g'\mathrm{H}')\,, \tag{38}$$

where $y = \tau'(w)$. Therefore, for some constant $C'_{\mathfrak{h}_{\overline{1}}}$,

$$F_\phi(y) = C_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}(\phi) = C'_{\mathfrak{h}_{\overline{1}}}\, \Phi_{\tau'_*(\phi)}(y)\,. \tag{39}$$

where

$$\Phi_\psi(y) = \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \int_{\mathrm{G}'/\mathrm{H}'} \psi(g'.y)\, d(g'\mathrm{H}') \tag{40}$$

is Harish-Chandra's orbital integral of $\psi \in \mathcal{S}(\mathfrak{g}')$. If $\phi \in \mathcal{S}(\mathrm{W})$ is arbitrary, let $\phi^{\mathrm{G}} \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}$ be defined by

$$\phi^{\mathrm{G}}(w) = \frac{\int_{\mathrm{G}} \phi(g.w)\, dg}{\int_{\mathrm{G}} dg}\,. \tag{41}$$

If $w \in \mathfrak{h}_{\overline{1}}^{reg}$ then $F_\phi(y) = F_{\phi^{\mathrm{G}}}(y)$ because $F(y) \in \mathcal{S}^*(\mathrm{W})^{\mathrm{S}}$. Thus

$$F_\phi(y) = C'_{\mathfrak{h}_{\overline{1}}}\, \Phi_{\tau'_*(\phi^{\mathrm{G}})}(y) \qquad (w \in \mathfrak{h}_{\overline{1}}^{reg}, \phi \in \mathcal{S}(\mathrm{W}))\,. \tag{42}$$

By [3, Theorem 2, page 207], the right-hand side of (42) provides a G-invariant extension of F_ϕ to $\mathfrak{h}'^{In-reg}$ that is $W(\mathrm{G}', \mathfrak{h}')$-skew-invariant. We now claim that $F(y)$ is also G'-invariant for $y \in \mathfrak{h}'^{In-reg} \cap \tau'(\mathrm{W})$. By possibly replacing ϕ by ϕ^{G}, we can suppose $\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}$. By Lemma 2.2, for every $y \in \mathfrak{h}'^{In-reg} \cap \tau'(\mathrm{W})$,

$$\Phi_{\tau'_*(\phi)}(y) = \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \int_{\mathrm{G}'/\mathrm{H}'} \tau'_*(\phi)^{g'}(y)\, d(g'\mathrm{H}') = \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \int_{\mathrm{G}'/\mathrm{H}'} \tau'_*(\phi^{g'})(y)\, d(g'\mathrm{H}')\,.$$

Hence, for any $x' \in G'$,

$$\Phi_{\tau'_*(\phi^{x'})}(y) = \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \int_{\mathrm{G}'/\mathrm{H}'} \tau'_*(\phi^{x'g'})(y)\, d(g'\mathrm{H}') = \Phi_{\tau'_*(\phi)}(y)\,,$$

which proves the claim. The regularity properties of F follow from [3, Theorem 2, page 207] and the fact that the map τ'_* is continuous.

An element of $\mathfrak{h}'$ which is a boundary point of a connected component of $\mathfrak{h}'^{In-reg} \cap \tau'(\mathrm{W})$ is either in $\mathfrak{h}'^{In-reg}$ or a boundary point of a connected component of $\mathfrak{h}'^{In-reg}$ in $\mathfrak{h}'$. In the first case, the continuous extension F at y follows from Harish-Chandra's result mentioned above. Suppose then y is a boundary point of a connected component of $\mathfrak{h}'^{In-reg}$ in $\mathfrak{h}'$. It follows from [3, Lemma 25, page 232], that, for every fixed $\phi \in \mathcal{S}(\mathrm{W})$, the function $F_\phi = C'_{\mathfrak{h}_{\overline{1}}}\, \Phi_{\tau'_*(\phi^{\mathrm{G}})}$ continuously extends at y. Since the space of distributions is weakly complete, [4, Theorem 2.1.8], these limits define an element $F(y) \in \mathcal{S}^*(\mathrm{W})$. Thus F continuously extends at y. Finally, by continuity, $F(y) \in \mathcal{S}^*(\mathrm{W})^{\mathrm{S}}$. □

In the rest of this paper we suppose that $l \leq l'$.

To state the regularity properties of the almost elliptic orbital integral in this case, we need a few more definitions. Let Σ_l be the l-th symmetric group and let

$$W(\mathrm{G},\mathfrak{h}) = \begin{cases} \Sigma_l & \text{if } \mathbb{D}=\mathbb{C},\\ \Sigma_l \ltimes \{\pm 1\}^l & \text{otherwise.}\end{cases} \tag{43}$$

Denote the elements of Σ_l by σ and the elements of $\{\pm1\}^l$ by $\epsilon=(\epsilon_1,\epsilon_2,\dots,\epsilon_l)$, so that an arbitrary element of the group (43) looks like $\epsilon\sigma$, with $\epsilon=(1,1,\dots,1)$, if $\mathbb{D}=\mathbb{C}$. This group acts on $\mathfrak{h}$ as follows:

$$(\epsilon\sigma)\sum_{j=1}^{l} y_j J_j = \sum_{j=1}^{l}\epsilon_j y_{\sigma^{-1}(j)} J_j \tag{44}$$

and coincides with the Weyl group, equal to the normalizer of $\mathfrak{h}$ in G modulo the centralizer of $\mathfrak{h}$ in G, as the indicated by the notation. Recall the sign character $\mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}$ of the Weyl group, defined by

$$\pi_{\mathfrak{g}/\mathfrak{h}}(sy) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\,\pi_{\mathfrak{g}/\mathfrak{h}}(y) \qquad (s\in W(\mathrm{G},\mathfrak{h}),\ y\in\mathfrak{h})\,. \tag{45}$$

Recall from Lemma 3.1 that the union $\bigcup_{\mathfrak{h}_{\bar 1}}\tau(\mathfrak{h}_{\bar 1}{}^{reg})$ over the mutually non-conjugate Cartan subspaces $\mathfrak{h}_{\bar 1}$ of W is disjoint whenever there is more than one such conjugacy class.

Lemma 3.4. *Suppose $l\le l'$. Then the closure of $W(\mathrm{G},\mathfrak{h})\big(\bigcup_{\mathfrak{h}_{\bar 1}}\tau(\mathfrak{h}_{\bar 1}{}^{reg})\big)\subseteq\mathfrak{h}$ is equal to*

$$\mathfrak{h}\cap\tau(\mathrm{W}) = \begin{cases} \mathfrak{h} & \text{if } \mathbb{D}\neq\mathbb{C},\\ W(\mathrm{G},\mathfrak{h})\big(\{\sum_{j=1}^{l} y_jJ_j;\ y_1,\dots,y_{\max(l-q,0)}\ge 0\ge y_{\min(p,l)+1},\dots,y_l\}\big) & \text{if } \mathbb{D}=\mathbb{C}.\end{cases}$$

As a consequence, for a fixed Weyl chamber $\mathfrak{h}^+$, we may choose $\mathfrak{h}_{\bar 1}^+$ so that

$$\textstyle\bigcup_{\mathfrak{h}_{\bar 1}}\tau(\mathfrak{h}_{\bar 1}^+) = \mathfrak{h}^+\cap\tau(\mathrm{W})\,. \tag{46}$$

Proof. The proof that $W(\mathrm{G},\mathfrak{h})\big(\bigcup_{\mathfrak{h}_{\bar 1}}\tau(\mathfrak{h}_{\bar 1}{}^{reg})\big)$ is equal to the sets on the right-hand side of the first displayed formula follows from (13), (30) and (44). To show that this is equal to $\mathfrak{h}\cap\tau(\mathrm{W})$, notice first that $W(\mathrm{G},\mathfrak{h})\big(\mathfrak{h}\cap\tau(\mathrm{W})\big)\subseteq\mathfrak{h}\cap\tau(\mathrm{W})$. Indeed, if $X=\tau(w)\in\mathfrak{h}\cap\tau(\mathrm{W})$ and $g\in\mathrm{G}$ represents an element of $W(\mathrm{G},\mathfrak{h})$, then $gXg^{-1}=g\tau(w)g^{-1}=\tau(g.w)\in\mathfrak{h}\cap\tau(\mathrm{W})$. Therefore

$$W(\mathrm{G},\mathfrak{h})\big(\textstyle\bigcup_{\mathfrak{h}_{\bar 1}}\tau(\mathfrak{h}_{\bar 1}{}^{reg})\big)\subseteq W(\mathrm{G},\mathfrak{h})\big(\mathfrak{h}\cap\tau(\mathrm{W})\big)\subseteq\mathfrak{h}\cap\tau(\mathrm{W})\,.$$

If $\mathbb{D}\neq\mathbb{C}$, then $W(\mathrm{G},\mathfrak{h})\big(\bigcup_{\mathfrak{h}_{\bar 1}}\tau(\mathfrak{h}_{\bar 1}{}^{reg})\big)=\mathfrak{h}$ implies that they must be equal to $\mathfrak{h}\cap\tau(\mathrm{W})$.

Suppose then $\mathbb{D}=\mathbb{C}$. We claim that $\mathfrak{h}\cap\tau(\mathrm{W})$ is contained in the set on the right-hand side of the displayed formula. For this, recall the notation introduced in subsection 2.1 and observe that any $x\in\mathfrak{g}$ defines a skew-hermitian form β_x on $\mathrm{V}_{\bar 0}$ by $\beta_x(u,u')=(xu,u')$. Let $y=\tau(w)\in\mathfrak{h}\cap\tau(\mathrm{W})$. If $w=\begin{pmatrix}0 & a^*\\ a & 0\end{pmatrix}$, then $\tau(w)=a^*a$. In this notation, the defining equation for $\mathfrak{s}_{\bar 1}$ in (3) becomes $(au,v)'=(v,a^*v)$ for all $u\in\mathrm{V}_{\bar 0}, v\in\mathrm{V}_{\bar 1}$. So,

$$\beta_{a^*a}(u,u')=(a^*au,u')=(au,au')' \qquad (u,u'\in\mathrm{V}_{\bar 0})\,,$$

i.e. β_{a^*a} is the pullback to $\mathrm{V}_{\bar 0}$ of the defining form $(\cdot,\cdot)'$ of $\mathrm{U}_{p,q}$ and hence it has signature (r,s) with $r\le p$, $s\le q$, $r+s\le l=\dim\mathrm{V}_{\bar 0}$. Since $a^*a=y\in\mathfrak{h}$ is diagonal, r and s

must be the number of positive and negative components of y, respectively. This proves the claim. $\square$

Let

$$r = \frac{2\,\dim(\mathfrak{g})}{\dim(\mathsf{V}_{\overline{0}})}, \tag{47}$$

where we view both $\mathfrak{g}$ and $\mathsf{V}_{\overline{0}}$ as vector spaces over $\mathbb{R}$. Explicitly,

$$r = \begin{cases} 2l-1 & \text{if } \mathrm{G} = \mathrm{O}_{2l}, \\ 2l & \text{if } \mathrm{G} = \mathrm{O}_{2l+1}, \\ l & \text{if } \mathrm{G} = \mathrm{U}_l, \\ l+\frac{1}{2} & \text{if } \mathrm{G} = \mathrm{Sp}_l. \end{cases} \tag{48}$$

Recall also that $d' = \dim_{\mathbb{D}} \mathrm{V}_{\overline{1}}$.

Theorem 3.5. *Suppose $l \leq l'$. There is a unique extension of the function*

$$F : \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg}) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{49}$$

to

$$F : \mathfrak{h} \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{50}$$

so that

$$F(sy) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) F(y) \qquad (s \in W(\mathrm{G}, \mathfrak{h}),\ y \in \mathfrak{h})\,. \tag{51}$$

This extension is supported in $\mathfrak{h} \cap \tau(\mathrm{W})$. The map (50) is smooth on the subset where each $y_j \neq 0$ and, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_l)$ with

$$\max(\alpha_1, \dots, \alpha_l) \leq \begin{cases} d' - r - 1 & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C}, \\ 2(d'-r) & \text{if } \mathbb{D} = \mathbb{H}, \end{cases}$$

the function $\partial(J_1^{\alpha_1} J_2^{\alpha_2} \dots J_l^{\alpha_l}) F(y)$ extends to a continuous function on $\mathfrak{h} \cap \tau(\mathrm{W})$. This extension is equal to zero on the boundary of $\mathfrak{h} \cap \tau(\mathrm{W})$.

The following section will be devoted to the proof of this theorem. In particular, the extension of $\partial(J^\alpha)F(y)$ is a consequence of Lemma 4.8.

4. Proof of Theorem 3.5 when $l \leq l'$

From now on we assume that $l \leq l'$.

Let $\mu_{\mathfrak{g}}$ be a Lebesgue measure on $\mathfrak{g}$. Let us normalize the orbital integrals $\mu_{\mathrm{G}.y} \in \mathcal{S}^*(\mathfrak{g})$, $y \in \mathfrak{h}^{reg}$, so that

$$\mu_{\mathfrak{g}} = \int_{\mathfrak{h}^+} |\pi_{\mathfrak{g}/\mathfrak{h}}(y)|^2 \mu_{\mathrm{G}.y}\, dy\,, \tag{52}$$

where $\mathfrak{h}^+ \subseteq \mathfrak{h}^{reg}$ is a Weyl chamber. Explicitly, for $\psi \in \mathcal{S}(\mathfrak{g})$

$$\int_{\mathfrak{g}} \psi(x) d\mu_{\mathrm{G}.y}(x) = \int_{\mathrm{G}/\mathrm{H}} \psi(gyg^{-1})\, d(g\mathrm{H})$$

so that

$$\int_{\mathfrak{g}} \psi(x)\, d\mu_{\mathfrak{g}}(x) = \int_{\mathfrak{h}^+} |\pi_{\mathfrak{g}/\mathfrak{h}}(y)|^2 \int_{\mathrm{G}/\mathrm{H}} \psi(gyg^{-1})\, d(g\mathrm{H})\, dy\,.$$

Let $\mathrm{W}_{\mathfrak{g}} \subseteq \mathrm{W}$ be the maximal subset such that $\tau|_{\mathrm{W}_{\mathfrak{g}}} : \mathrm{W}_{\mathfrak{g}} \to \mathfrak{g}$, that is the restriction of τ to $\mathrm{W}_{\mathfrak{g}}$, is a submersion. Then [10, Lemma 2.6] shows that $\mathrm{W}_{\mathfrak{g}}$ consists of all the elements $w \in \mathrm{W}$ such that for any $x \in \mathfrak{g}$,

$$xw = 0 \ \text{ implies } \ x = 0\,. \tag{53}$$

The hypothesis in the implication (53) means that x restricted to the image $\mathrm{Im}(w) \subseteq \mathsf{V}_{\overline{0}}$ of w is zero. In this case, x preserves the orthogonal complement $\mathrm{Im}(w)^{\perp}$ of $\mathrm{Im}(w)$ in $\mathsf{V}_{\overline{0}}$. Thus $w \in \mathrm{W}_{\mathfrak{g}}$ if and only if the Lie algebra of the isometries of $\mathrm{Im}(w)^{\perp}$ is zero. This happens if w is surjective or if $w \in \mathrm{G} = \mathrm{O}_{2l+1}$ and the dimension of $\mathrm{Im}(w)$ is $\dim(\mathsf{V}_{\overline{0}}) - 1$. In particular,

$$\mathrm{W}_{\mathfrak{g}} \neq \emptyset \ \text{ if and only if } \ l \leq l'. \tag{54}$$

Lemma 4.1. *Let $\mathfrak{h}_{\overline{1}}$ be a Cartan subspace in* W. *Suppose $w = \sum_{j=1}^{l} w_j u_j \in \mathfrak{h}_{\overline{1}}$ is as in* (12). *If $w_j \neq 0$ for all $1 \leq j \leq l$, then $w \in \mathrm{W}_{\mathfrak{g}}$. In particular, $\mathfrak{h}_{\overline{1}}^{reg} \subseteq \mathrm{W}_{\mathfrak{g}}$. Moreover,*

$$\bigcup\nolimits_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{+}) = \mathfrak{h}^{+} \cap \tau(\mathrm{W}_{\mathfrak{g}})\,. \tag{55}$$

Proof. According to (12), w is a block-diagonal matrix with $j-$th block equal to u_j. Suppose $w_j \neq 0$ for all $1 \leq j \leq l$. Since $l \leq l'$, then w is surjective if $\mathrm{G} \neq \mathrm{O}_{2l+1}$ and $\mathrm{Im}(w)$ has codimension ≤ 1 if $\mathrm{G} = \mathrm{O}_{2l+1}$.

Suppose now that $w \in \mathfrak{h}_{\overline{1}}^{reg}$. By (17), this means that $\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}^2}(w^2) \neq 0$. In turn, this implies that $w_j \neq 0$ for all $1 \leq j \leq l$, as one can check from the explicit formulas in the proof of Lemma 2.1.

The last equality follows from the first part of this lemma together with Lemma 3.4. ☐

Recall [4, Theorem 6.1.2] that a smooth submersion $f : X \to Y$ between open subsets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ induces a pullback f^* of distributions on Y to distributions on X that generalizes the pullback of functions. Let us fix Lebsegue measures μ_X and μ_Y on X and Y, respectively. For any locally integrable function $u : Y \to \mathbb{C}$, the pullback of the distribution $u\mu_Y$ is defined by

$$f^*(u\mu_Y) = f^*(u)\mu_X, \quad \text{where} \quad f^*(u) = u \circ f : X \to \mathbb{C}\,.$$

By choosing $u = 1$, the constant function, we see that $f^*(\mu_Y) = \mu_X$. So, the pullback of distributions depends on the normalization of the Lebesgue measures involved. In the situation we consider, let $\tau|_{\mathrm{W}_{\mathfrak{g}}}^{*}\mu_{\mathfrak{g}}$ denote the pullback measure of $\mu_{\mathfrak{g}}$ under $\tau|_{\mathrm{W}_{\mathfrak{g}}}$. We shall assume that the measures are normalized so that

$$\tau|_{\mathrm{W}_{\mathfrak{g}}}^{*}\mu_{\mathfrak{g}} = \mu_{\mathrm{W}}|_{\mathrm{W}_{\mathfrak{g}}}\,. \tag{56}$$

Lemma 4.2 below shows that $F(y)$, (49), restricted to $\mathrm{W}_{\mathfrak{g}}$ is the pullback under $\tau|_{\mathrm{W}_{\mathfrak{g}}}$ of the Harish-Chandra orbital integral on $\mathfrak{g}$, and thus provides some justification for the name.

Lemma 4.2. *The following formula holds,*

$$\tau|_{\mathrm{W}_{\mathfrak{g}}}^{*}(\pi_{\mathfrak{g}/\mathfrak{h}}(y)\mu_{\mathrm{G}.y}) = F(y)|_{\mathrm{W}_{\mathfrak{g}}} \qquad (y = \tau(w), w \in {\mathfrak{h}_{\overline{1}}}^{reg})\,. \tag{57}$$

Proof. Notice that both sides of (57) are orbital integrals on W. Hence they have to be multiples of each other. The point is to show that they are equal.

Let $\mathcal{D}'(\mathfrak{g})$ and $\mathcal{D}'(\mathrm{W}_\mathfrak{g})$ denote the spaces of distributions on $\mathfrak{g}$ and $\mathrm{W}_\mathfrak{g}$, respectively. Then $\tau|_{\mathrm{W}_\mathfrak{g}}^* : \mathcal{D}'(\mathfrak{g}) \to \mathcal{D}'(\mathrm{W}_\mathfrak{g})$ is a continuous linear map. Hence, by (52),

$$\tau|_{\mathrm{W}_\mathfrak{g}}^* \mu_\mathfrak{g} = \int_{\mathfrak{h}^+} |\pi_{\mathfrak{g}/\mathfrak{h}}(y)|^2 \tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.y})\, dy\,.$$

Let $w \in \mathfrak{h}_{\overline{1}}^+$ and $y = \tau(w)$. The set $\tau|_{\mathrm{W}_\mathfrak{g}}^{-1}(\mathrm{G}.y) = \mathrm{S}.w$ is a single S-orbit. It is the support of the S-invariant positive measure $\tau|_{\mathrm{W}_\mathfrak{g}}^* \mu_{\mathrm{G}.y}$. It is also the support of the measure $\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}$ appearing in the decomposition of $\mu_\mathrm{W}|_{\mathrm{W}_\mathfrak{g}}$, see (23). Since $\tau|_{\mathrm{W}_\mathfrak{g}}^* \mu_\mathfrak{g} = \mu_\mathrm{W}|_{\mathrm{W}_\mathfrak{g}}$ by (56) and because of (55), the contributions of these two measures on each of the disjoint orbits must agree. So

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|^2 \tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.\tau(w)}) = |\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))| \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}|_{\mathrm{W}_\mathfrak{g}} \qquad (w \in \mathfrak{h}_{\overline{1}}^+).$$

Hence,

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|\; \tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.\tau(w)}) = |\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))|\; \mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}|_{\mathrm{W}_\mathfrak{g}} \qquad (w \in \mathfrak{h}_{\overline{1}}{}^{reg}),$$

because both sides are $W(\mathrm{S}, \mathfrak{h}_{\overline{1}})$-invariant. Thus,

$$\begin{aligned}&\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))\tau|_{\mathrm{W}_\mathfrak{g}}^*(\mu_{\mathrm{G}.\tau(w)})\\ &\qquad = \left(\frac{|\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))|}{\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))}\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))}{|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|}\right)\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))\mu_{\mathcal{O}(w),\mathfrak{h}_{\overline{1}}}|_{\mathrm{W}_\mathfrak{g}} \qquad (w \in \mathfrak{h}_{\overline{1}}{}^{reg}).\end{aligned}$$

Let $C(\mathfrak{h}_{\overline{1}})$ be the constant in Lemma 2.1. Then

$$\frac{|\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))|}{\pi_{\mathfrak{g}'/\mathfrak{z}'}(\tau'(w))}\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))}{|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|} = C(\mathfrak{h}_{\overline{1}})\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))^2}{|\pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w))|^2} = C(\mathfrak{h}_{\overline{1}}) i^{\dim \mathfrak{g}/\mathfrak{h}}.$$

Hence, the lemma follows. □

The distribution on the right-hand side of (57) extends to $\mathfrak{h}$ by skew-invariance with respect to the Weyl group $W(\mathrm{G}, \mathfrak{h})$. This motivates the $W(\mathrm{G}, \mathfrak{h})$-skew invariant extension of F in the next lemma.

Lemma 4.3. *There is a unique extension of the function*

$$F : \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg}) \to \mathcal{S}^*(\mathrm{W})^\mathrm{S} \tag{58}$$

to

$$F : W(\mathrm{G}, \mathfrak{h})\Big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg})\Big) \to \mathcal{S}^*(\mathrm{W})^\mathrm{S} \tag{59}$$

so that

$$F(sy) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)F(y) \qquad (s \in W(\mathrm{G}, \mathfrak{h}),\ y \in W(\mathrm{G}, \mathfrak{h})\Big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg})\Big)). \tag{60}$$

Proof. Let $W(\mathrm{S}, \mathfrak{h}_{\overline{1}}, \mathfrak{h}) \subseteq \Sigma_l \subseteq W(\mathrm{G}, \mathfrak{h})$ be the subgroup leaving the sequence $\delta_1,\ \delta_2,\ \dots,\ \delta_l$ fixed. So $W(\mathrm{S}, \mathfrak{h}_{\overline{1}}, \mathfrak{h}) = \Sigma_l$ if $\mathbb{D} \neq \mathbb{C}$ and, if $\mathbb{D} = \mathbb{C}$, then $W(\mathrm{S}, \mathfrak{h}_{\overline{1},m}, \mathfrak{h}) = \Sigma_{m,l-m}$ is the subgroup of Σ_l separately permuting the first m elements and the last $l-m$ elements. Each $\sigma \in W(\mathrm{S}, \mathfrak{h}_{\overline{1}}, \mathfrak{h})$ commutes with τ by (12) and (13). Moreover, by (17) and the formulas in the proof of Lemma 2.1, for every $\sigma \in W(\mathrm{S}, \mathfrak{h}_{\overline{1}}, \mathfrak{h})$ we have $\sigma(\mathfrak{h}_{\overline{1}}^{reg}) = \mathfrak{h}_{\overline{1}}^{reg}$. Thus $\sigma\tau(\mathfrak{h}_{\overline{1}}^{reg}) = \tau(\mathfrak{h}_{\overline{1}}^{reg})$. By its definition in (32), F satisfies

$$F(\sigma y) = \mathrm{sgn}_{\mathfrak{g}/\mathfrak{h}}(\sigma) F(y) \qquad (\sigma \in W(\mathrm{S}, \mathfrak{h}_{\overline{1}}, \mathfrak{h}),\ y \in \tau({\mathfrak{h}_{\overline{1}}}^{reg})).$$

Since

$$W(\mathrm{G}, \mathfrak{h})\Big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau({\mathfrak{h}_{\overline{1}}}^{reg})\Big) = \bigcup_{\mathfrak{h}_{\overline{1}}} (W(\mathrm{G}, \mathfrak{h})/W(\mathrm{S}, \mathfrak{h}_{\overline{1}}, \mathfrak{h}))\tau({\mathfrak{h}_{\overline{1}}}^{reg}), \tag{61}$$

where the union on the right-hand side is disjoint, the $W(\mathrm{G}, \mathfrak{h})$-skew-invariant extension of F in (60) is compatible with its original definition. Hence the claim follows. □

Recall from Lemma 3.4 that $\mathfrak{h} \cap \tau(\mathrm{W})$ is the closure of $W(\mathrm{G}, \mathfrak{h})\big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau({\mathfrak{h}_{\overline{1}}}^{reg})\big)$ inside $\mathfrak{h}$. We would like to extend the function F from the set $W(\mathrm{G}, \mathfrak{h})\big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau({\mathfrak{h}_{\overline{1}}}^{reg})\big)$ to $\mathfrak{h} \cap \tau(\mathrm{W})$. This will require some more work.

Fix an elliptic Cartan subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}'$ containing $\tau'(\mathfrak{h}_{\overline{1}})$. When the dual pair $(\mathrm{G}, \mathrm{G}')$ is isomorphic to $(\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l < p + q$, we listed more than one $\mathfrak{h}_{\overline{1}}$. We may assume that $\mathfrak{h}'$ contains $\tau'(\mathfrak{h}_{\overline{1}})$ for all of them.

Let ${\mathfrak{h}'}^{In-reg} \subseteq \mathfrak{h}'$ be the subset where no non-compact roots of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ vanish. Set $\mathfrak{h}_{\overline{1}}^{In-reg} = \tau'^{-1}({\mathfrak{h}'}^{In-reg}) \cap \mathfrak{h}_{\overline{1}}$. Then $\tau(\mathfrak{h}_{\overline{1}}^{In-reg})$ is the set of the elements $y \in \tau(\mathfrak{h}_{\overline{1}})$ such that, under the identification (15), no non-compact root of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ vanishes at y. The following lemma describes this set explicitly.

Lemma 4.4. *For our specific Cartan subspace (9), the set $\tau(\mathfrak{h}_{\overline{1}}^{In-reg})$ consists of elements $y = \sum_{j=1}^{l} y_j J_j$, such that*

$$\begin{array}{ll} y_j > 0 \text{ for all } j, & \text{if } \mathrm{G} = \mathrm{O}_{2l} \text{ or } \mathrm{G} = \mathrm{O}_{2l+1} \text{ or } \mathrm{G} = \mathrm{Sp}_l \text{ with } l < l' \text{ or } 1 = l = l', \\ y_j \geq 0 \text{ for all } j\,,\ y_j + y_k > 0 \text{ for all } j \neq k, & \text{if } \mathrm{G} = \mathrm{Sp}_l \text{ and } 1 < l = l' \end{array} \tag{62}$$

and

$$\left.\begin{array}{ll} y_j > 0 & \text{if } j \leq m \text{ and } l - m < q, \\ y_j \geq 0 & \text{if } j \leq m \text{ and } l - m = q \text{ when } l \geq q, \\ y_j < 0 & \text{if } m < j \text{ and } m < p, \\ y_j \leq 0 & \text{if } m < j \text{ and } m = p \text{ when } l \geq p, \\ y_j - y_k > 0 & \text{if } j \leq m < k\,. \end{array}\right\} \text{if } \mathrm{G}' = \mathrm{U}_{p,q} \text{ and } \mathfrak{h}_{\overline{1}} = \mathfrak{h}_{\overline{1},m}\,. \tag{63}$$

In particular, in the last case,

$$\tau(\mathfrak{h}_{\overline{1},m}^{In-reg}) \cap \tau(\mathfrak{h}_{\overline{1},m'}^{In-reg}) \neq \emptyset \quad \text{implies} \quad |m - m'| \leq 1, \tag{64}$$

$$\tau(\mathfrak{h}_{\overline{1},m}^{In-reg}) \cap \tau(\mathfrak{h}_{\overline{1},m+1}^{In-reg}) \subseteq \{\sum_{j=1}^{l} y_j J_j;\ y_1, \dots, y_m \geq 0 = y_{m+1} \geq y_{m+2}, \dots, y_l\}.$$

Proof. We see from (13) that the set $\tau(\mathfrak{h}_{\overline{1}})$ consists of elements $y = \sum_{j=1}^{l} y_j J_j$, such that $\delta_j y_j \geq 0$ for all $1 \leq j \leq l$. Hence $\sum_{j=1}^{l} y_j J'_j \in \mathfrak{h}'$ not annihilated by any imaginary non-compact root of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ implies (62) when $\mathbb{D} \neq \mathbb{C}$.

If $\mathrm{G}' = \mathrm{U}_{p,q}$, then the non-compact roots of $\mathfrak{h}'$ in $\mathfrak{g}'_{\mathbb{C}}$ acting on elements of $\mathfrak{h} \subseteq \mathfrak{h}'$ are given by

$$\mathfrak{h} \ni \sum_{j=1}^{l} y_j J'_j \to \pm i(y_j - y_k) \in i\mathbb{R}, \qquad \text{if } \ j \leq m < k \ \text{ or } \ k \leq m < j,$$
$$\mathfrak{h} \ni \sum_{j=1}^{l} y_j J'_j \to \pm i y_j \in i\mathbb{R}, \qquad \text{if } \ j \leq m \text{ and } l-m < q \quad \text{or} \ \ m < j \text{ and } m < p.$$

Hence, (63) follows. The last statement follows from the equality

$$\tau(\mathfrak{h}_{\bar{1},m}) \cap \tau(\mathfrak{h}_{\bar{1},m+k}) = \{\sum_{j=1}^{l} y_j J_j;\ y_1, \dots, y_m \geq 0 = y_{m+1} = \cdots = y_{m+k} \geq y_{m+k+1}, \dots, y_l\}, \tag{65}$$

which is a consequence of (30). ☐

Lemma 4.5. *For a fixed Cartan subspace $\mathfrak{h}_{\bar{1}}$, the function*

$$F : \tau(\mathfrak{h}_{\bar{1}}^{reg}) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{66}$$

extends to a smooth function

$$F : \tau(\mathfrak{h}_{\bar{1}}^{In-reg}) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \tag{67}$$

whose all derivatives are bounded. Further, any derivative of (67) extends to a continuous function on the closure of any connected component of $\tau(\mathfrak{h}_{\bar{1}}^{In-reg})$.

Proof. The proof of this lemma is similar to that of Theorem 3.3. So, we just indicate the points that need to be modified.

Let $\mathrm{H} \subseteq \mathrm{G}$ be the Cartan subgroup with Lie algebra $\mathfrak{h}$. Denote by $\Delta(\mathrm{H}) \subseteq \mathrm{G} \times \mathrm{G}'$ be the diagonal embedding. Then,

$$\mathrm{S}^{\mathfrak{h}_{\bar{1}}} = \Delta(\mathrm{H})(\{1\} \times \mathrm{Z}'), \tag{68}$$

where $\mathrm{Z}' \subseteq \mathrm{G}'$ is the centralizer of $\mathfrak{h} \subseteq \mathfrak{g}'$. Fix a function $\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}$ and let $\psi = \tau'_*(\phi) \in \mathcal{S}(\mathfrak{g}')$, see (26).

For a moment, let us exclude the case $\mathrm{G} = \mathrm{O}_{2l+1}$ with $l < l'$. Then, by (37), for $w \in \mathfrak{h}_{\bar{1}}^{reg}$ and $y = \tau(w)$,

$$\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}}(\phi) = \tau'^{*}_{*}(\mu_{\mathcal{O}(w),\mathfrak{h}_{\bar{1}}})(\psi) = \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}} \psi(\tau'(s.w))\, d(s\mathrm{S}^{\mathfrak{h}_{\bar{1}}}). \tag{69}$$

Since G is compact, (69) is a constant multiple of

$$\int_{\mathrm{G}'/\mathrm{Z}'} \psi(g'.y)\, d(g'\mathrm{Z}'). \tag{70}$$

As checked in [8, (23)], there is a positive constant C such that

$$\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}'/\mathrm{Z}'}\psi(g'.y)\,d(g'\mathrm{Z}') = C\partial(\pi_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'/\mathrm{H}'}\psi(g'.(y+y'')\,d(g'\mathrm{H}')\right)\Big|_{y''=0}, \quad (71)$$

where $y\in\mathfrak{h}$ and $y''\in\mathfrak{h}'\cap[\mathfrak{z}',\mathfrak{z}']$. Therefore, if $y=\tau(w)$ with $w\in\mathfrak{h}_{\overline{1}}^{reg}$, then for some constant $C'_{\mathfrak{h}_{\overline{1}}}$,

$$F_\phi(y)=C'_{\mathfrak{h}_{\overline{1}}}\partial(\pi_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\tau'_*(\phi)(g'.(y+y''))\,dg'\right)\Big|_{y''=0} \quad (72)$$

For arbitrary $\phi\in\mathcal{S}(\mathrm{W})$ and $y\in\tau(\mathfrak{h}_{\overline{1}}^{reg})$ we have by S-invariance, $F_\phi(y)=F_{\phi^{\mathrm{G}}}(y)$, where ϕ^{G} is as in (41).

Hence, the lemma follows from [3, Theorem 2, page 207 and Lemma 25, page 232], the fact that the map τ'_*, (26), is continuous and the fact the space of the distributions is weakly complete, [4, Theorem 2.1.8]. Since $\tau(\mathfrak{h}_{\overline{1}}^{In-reg})=\tau'(\mathfrak{h}_{\overline{1}}^{In-reg})\subseteq\tau'(\mathrm{W})$ by the identification (15), the proof that the distribution $F(y)$ is also G′-invariant is as in Theorem 3.3.

Suppose now that $\mathrm{G}=\mathrm{O}_{2l+1}$ with $l<l'$. Let $w_0\in\mathfrak{s}_{\overline{1}}(\mathsf{V}^0)$ be as in (20). Then $(w+w_0)^2=w^2+w_0^2$. Hence,

$$\begin{aligned}\tau'^*_*(\mu_{\mathcal{O}(w)})(\psi) &= \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}}\psi(\tau'(s.(w+w_0)))\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}) \quad (73)\\ &= \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0}}\psi(s.(\tau'(w)+\tau'(w_0)))\,d(s\mathrm{S}^{\mathfrak{h}_{\overline{1}}+w_0})\\ &= C_1\int_{\mathrm{G}'/\mathrm{Z}'^n}\psi(g.(y+n))\,d(g\mathrm{Z}'^n),\end{aligned}$$

where C_1 is a positive constant, $y=\tau'(w)$, $n=\tau'(w_0)$ and Z'^n is the centralizer of n in Z'.

Let $\pi^{short}_{\mathfrak{z}'/\mathfrak{h}'}$ denote the product of the positive short roots of $\mathfrak{h}'$ in $\mathfrak{z}'_{\mathbb{C}}$. As checked in [8, (35)], there is a positive constant C such that

$$\partial(\pi^{short}_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+x)\int_{\mathrm{G}'/\mathrm{H}'}\psi(g.(y+y''))\,d(g\mathrm{H}')\right)\Big|_{y''=0} = C\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{G}'/\mathrm{Z}'^n}\psi(g.(y+n))\,d(g\mathrm{Z}'^n). \quad (74)$$

Therefore, as in (72),

$$F_\phi(y)=C'_{\mathfrak{h}_{\overline{1}}}\partial(\pi^{short}_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\tau'_*(\phi)(g'.(y+y''))\,dg'\right)\Big|_{y''=0}. \quad (75)$$

Hence the lemma follows from the theorems of Harish-Chandra, as before. □

For a test function ϕ on a finite dimensional vector space U set

$$\phi_t(u)=t^{-\dim \mathrm{U}}\phi(t^{-1}u)\qquad (t>0,\ u\in\mathrm{U}). \quad (76)$$

A distribution Φ on U is said to be homogeneous of degree $a \in \mathbb{C}$ provided

$$\Phi(\phi_t) = t^a \Phi(\phi) \qquad (t > 0,\ \phi \in C_c^\infty(\mathrm{U})).$$

In particular, if P is a homogeneous polynomial function on U, then we may view P as a distribution homogeneous of degree $\deg P$, which satisfies

$$P(tu) = t^{\deg P} P(u) \qquad (t > 0\,, \quad u \in \mathrm{U})\,.$$

The continuous extension of F and its derivatives from $W(\mathrm{G}, \mathfrak{h})\big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}{}^{reg})\big)$ to its closure $\mathfrak{h} \cap \tau(\mathrm{W})$ will be made using a rank-one reduction. So we first look at the rank-one case $l = 1$.

Lemma 4.6. *Suppose $l = 1$. Set*

$$F^{(k)} = \lim_{y \to 0} \partial(J_1^k) F(y) \qquad (k = 0, 1, \dots).$$

(We know from Lemma 4.5 that these limits exist.) Then $\tau_*'^*(F^{(k)})$ is homogeneous of degree

$$\begin{array}{ll} -\dim \mathfrak{g}' + \deg \pi_{\mathfrak{g}'/\mathfrak{z}'} + l' - 1 - k, & \textit{if } \mathrm{G} = \mathrm{O}_{2l+1} \textit{ and } l < l', \\ -\dim \mathfrak{g}' + \deg \pi_{\mathfrak{g}'/\mathfrak{z}'} - k, & \textit{otherwise.} \end{array} \tag{77}$$

Furthermore,

$$\mathrm{supp}(\tau_*'^*(F^{(k)})) \subseteq \tau'(\tau^{-1}(0)). \tag{78}$$

Proof. It suffices to consider the restriction of F to $\tau(\mathfrak{h}_{\overline{1}}{}^{reg})$ for one of the Cartan subspaces $\mathfrak{h}_{\overline{1}}$. Let $\phi \in \mathcal{S}(\mathrm{W})^{\mathrm{G}}$ and let $\psi = \tau_*'(\phi) \in \mathcal{S}(\mathfrak{g}')$. For a moment, let us exclude the case $\mathrm{G} = \mathrm{O}_{2l+1}$, $l < l'$. As we have seen in the proof of Lemma 4.5, there is a non-zero constant C, such that for $t > 0$

$$\begin{aligned} \tau_*'^*(F^{(0)})(\psi_t) &= C \lim_{y \to 0} \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{\mathrm{G}'/\mathrm{Z}'} \psi_t(g.y)\, d(g\mathrm{Z}') \\ &= C \lim_{y \to 0} \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{\mathrm{G}'/\mathrm{Z}'} t^{-\dim(\mathfrak{g}')} \psi(g.t^{-1}y)\, d(g\mathrm{Z}') \\ &= t^{-\dim(\mathfrak{g}') + \deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})} \tau_*'^*(F^{(0)})(\psi). \end{aligned}$$

For the derivatives, the computation is similar since, by continuity of $\tau_*'^*$, we have

$$\tau_*'^*(F^{(k)})(\psi_t) = \lim\nolimits_{t \to 0} \partial(J_1^k)\big(\tau_*'^*(F(y))(\psi_t)\big)\,.$$

This proves (77).

Let $U \subseteq \mathrm{W}$ be an open subset with compact closure $\overline{U}$ such that $\overline{U} \cap \tau^{-1}(0) = \emptyset$. For $w' \in \overline{U}$ let $w' = w_s' + w_n'$ be its Jordan decomposition and let ϵ be the minimum of all the $|w_s'|$ (for some fixed norm $|\cdot|$ on W) such that $w' \in \overline{U}$. Then $\epsilon > 0$ because otherwise there would be a non-zero nilpotent element of W outside of $\tau^{-1}(0)$, which is impossible. Hence

$$\mathrm{S}.w \cap U = \emptyset \qquad (w \in \mathfrak{h}_{\overline{1}},\ |w| < \epsilon). \tag{79}$$

Since $\mathrm{supp}\, F(\tau(w)) = \mathrm{S}.w$, this implies (78).

Suppose now $\mathrm{G} = \mathrm{O}_{2l+1}$ and $l < l'$. Then for $t > 0$,

$$\tau_*'^*(F^{(0)})(\psi_t) = C \lim_{y \to 0} \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\overline{1}}}} \int_{\mathfrak{s}_{\overline{1}}(\mathrm{V}^0)} \psi_t(\tau'(s.(w + w^0)))\, dw^0\, d(s\mathrm{S}^{\mathfrak{h}}_{\overline{1}})$$

A linear change of variables in $\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)$ gives

$$\begin{aligned}
&\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}}\int_{\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)}\psi_t(\tau'(s.(w+w^0)))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}}_{\bar{1}})\\
&\quad = t^{-\dim(\mathfrak{g}')}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}}\int_{\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)}\psi(\tau'(s.(t^{-1/2}w+t^{-1/2}w^0)))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}}_{\bar{1}})\\
&\quad = t^{-\dim(\mathfrak{g}')+\frac{1}{2}\dim(\mathfrak{s}_{\bar{1}}(\mathsf{V}^0))}\pi_{\mathfrak{g}'/\mathfrak{z}'}(y)\int_{\mathrm{S}/\mathrm{S}^{\mathfrak{h}_{\bar{1}}}}\int_{\mathfrak{s}_{\bar{1}}(\mathsf{V}^0)}\psi(\tau'(s.(t^{-1/2}w+w^0)))\,dw^0\,d(s\mathrm{S}^{\mathfrak{h}}_{\bar{1}})\\
&\quad = t^{-\dim(\mathfrak{g}')+\deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})+\frac{1}{2}\dim(\mathfrak{s}_{\bar{1}}(\mathsf{V}^0))}\pi_{\mathfrak{g}'/\mathfrak{z}'}(t^{-1}y)\int_{\mathrm{G}'/\mathrm{Z}'^n}\psi(g.(t^{-1}y+n))\,d(g\mathrm{Z}'^n)).
\end{aligned}$$

Hence, by taking the limit for $y\to 0$, we conclude that

$$\tau'^*_*(F^{(0)})(\psi_t)=t^{-\dim(\mathfrak{g}')+\deg(\pi_{\mathfrak{g}'/\mathfrak{z}'})+\frac{1}{2}\dim(\mathfrak{s}_{\bar{1}}(\mathsf{V}^0))}\tau'^*_*(F^{(0)})(\psi).$$

Since $\dim(\mathfrak{s}_{\bar{1}}(\mathsf{V}^0))=2l'-2$, (77) follows.

Also, with the above notation, $w+w_0$ is a Jordan sum with w, the semisimple part, and w_0, the nilpotent part. Hence, as in (79), we have

$$\mathrm{S}.(w+w_0)\cap U=\emptyset \qquad (w\in\mathfrak{h}_{\bar{1}},\ |w|<\epsilon).$$

Since $\operatorname{supp} F(\tau(w)))=\mathrm{S}.(w+w_0)$, (78) follows. □

Lemma 4.7. *Let $l=1$. Then $\mathfrak{h}=\mathbb{R}J_1$ and*

$$W(\mathrm{G},\mathfrak{h})\Big(\bigcup_{\mathfrak{h}_{\bar{1}}}\tau(\mathfrak{h}_{\bar{1}}^{reg})\Big)=\begin{cases}\mathbb{R}^+J_1 & \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{U}_1,\mathrm{U}_{l'}=\mathrm{U}_{l',0}),\\ \mathbb{R}^-J_1 & \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{U}_1,\mathrm{U}_{l'}=\mathrm{U}_{0,l'}),\\ \mathbb{R}^\times J_1 & \text{if } (\mathrm{G},\mathrm{G}')=(\mathrm{O}_3,\mathrm{Sp}_{2l'}),\ (\mathrm{O}_2,\mathrm{Sp}_{2l'}),\ (\mathrm{Sp}_1,\mathrm{O}^*_{2l'})\\ & \text{or } (\mathrm{U}_1,\mathrm{U}_{p,q}) \text{ with } 1\le p\le q.\end{cases}$$

Let $F(y)$ denote the function (59). For an integer $k=0,1,2,\dots$ define

$$\langle F^{(k)}\rangle=\lim_{y\to 0\pm}\partial(J_1^k)F(yJ_1)$$

if $(\mathrm{G},\mathrm{G}')=(\mathrm{U}_1,\mathrm{U}_{l'})$ and

$$\langle F^{(k)}\rangle=\lim_{y\to 0+}(\partial(J_1^k)F(yJ_1)-\lim_{y\to 0-}(\partial(J_1^k)F(yJ_1)$$

in the remaining cases. Assume that $1<l'$. Then

$$\langle F^{(k)}\rangle=0 \text{ if } 0\le k<\begin{cases}2l'-2 & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathrm{G}=\mathrm{O}_3,\\ 2l'-1 & \text{if } \mathbb{D}=\mathbb{R} \text{ and } \mathrm{G}=\mathrm{O}_2,\\ l'-1 & \text{if } \mathbb{D}=\mathbb{C},\\ 2(l'-1) & \text{if } \mathbb{D}=\mathbb{H}.\end{cases}$$

Proof. Suppose $(\mathrm{G},\mathrm{G}')=(\mathrm{O}_3,\mathrm{Sp}_{2l'})$. We know from Lemma 4.6 that the distribution $\tau'_*(\langle F^{(k)}\rangle)$ is supported in $\tau'(\tau^{-1}(0))$. However Lemma 4.2 shows that for any $\phi\in C_c^\infty(\mathrm{W}_\mathfrak{g})$, $F(y)(\phi)$ is a smooth function of $y\in\mathfrak{h}$. Therefore, $\langle F^{(k)}\rangle|_{\mathrm{W}_\mathfrak{g}}=0$. Hence,

$$\operatorname{supp}(\tau'^*_*(\langle F^{(k)}\rangle))\subseteq\tau'(\tau^{-1}(0)\setminus\mathrm{W}_\mathfrak{g}).$$

As mentioned at the beginning of this section, $\tau^{-1}(0)\setminus\mathrm{W}_\mathfrak{g}$ consists of elements of rank at most 1. They form a G-orbit of a 1-dimensional subspace $\mathsf{V}_{\bar{0},\star}$ of $\mathsf{V}_{\bar{0}}$. For $\mathsf{V}_\star=\mathsf{V}_{\bar{0},\star}+\mathsf{V}_{\bar{1}}$, the corresponding dual pair is $(\mathrm{O}, \mathrm{Sp}_{2l'}-$. Then $\tau'(\mathfrak{s}_{\overline{1}}(\mathsf{V}_\star))$ is a minimal nilpotent orbit $\mathcal{O}_{min}$. Since τ' is constant on G-orbits, $\tau'(\tau^{-1}(0) \setminus \mathrm{W}_{\mathfrak{g}}) = \tau'(\mathfrak{s}_{\overline{1}}(\mathsf{V}_\star)) = \mathcal{O}_{min}$. Furthermore, $\dim(\mathcal{O}_{min}) = 2l'$. Lemma 4.6 shows that $\tau'^*_*(\langle F^{(k)}\rangle)$ is a homogeneous distribution of degree

$$-\dim\,\mathfrak{g}' + \deg(\pi_{\mathfrak{g}'/\mathfrak{z}'}) + l' - 1 - k = -\dim\,\mathfrak{g}' + 3l' - 2 - k$$

However, as shown in [14, Lemma 6.2], $\tau'^*_*(\langle F^{(k)}\rangle) = 0$ if the homogeneity degree is greater than $-\dim\,\mathfrak{g}' + \frac{1}{2}\dim\,\mathcal{O}_{min}$. Hence the claim follows.

Exactly the same argument works if $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_{2l'})$, or $(\mathrm{Sp}_1, \mathrm{O}^*_{2l'})$, or $(\mathrm{U}_1, \mathrm{U}_{p,q})$ with $1 \le p \le q$, except that $\tau'(\tau^{-1}(0) \setminus \mathrm{W}_{\mathfrak{g}}) = \{0\}$, because $\tau^{-1}(0) = \mathrm{W}_{\mathfrak{g}} \cup \{0\}$. So, instead of relying on [14, Lemma 6.2], we may use the classical description of distributions supported at $\{0\}$, [4, Theorem 2.3.4.].

Suppose $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_1, \mathrm{U}_{l'})$. Then (69), (70) and (72) show that for $\psi \in C_c^\infty(\mathfrak{g}')$, $0 \ne y = \tau(w) = \tau'(w)$ ans a constant C,

$$\tau'^*_*(F(y))(\psi) = C\,\pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{\mathrm{G}'} \psi(g'.y)\,dg'.$$

Since the group G' is compact, the last integral defines a smooth function of $y = y'J_1$. Also, in this case, $\pi_{\mathfrak{g}'/\mathfrak{z}'}(y) = (iy')^{l'-l}$. Hence, the claim follows. □

Lemma 4.8. *Let $F(y)$ denote the function (59), with*

$$y = \sum_{j=1}^{l} y_j J_j \in W(\mathrm{G}, \mathfrak{h})\Big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})\Big).$$

For any multiindex $\alpha = (\alpha_1, \dots, \alpha_l)$ set $\partial(J)^\alpha = \partial(J_1)^{\alpha_1} \dots \partial(J_l)^{\alpha_l}$. For $1 \le j \le l$ define

$$\langle \partial(J)^\alpha F\rangle_{y_j=0} = \lim_{y_j \to 0\pm} \partial(J)^\alpha F(y)$$

if $\{y_j \ne 0;\ y \in W(\mathrm{G}, \mathfrak{h})\Big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})\}\Big) = \mathbb{R}^\pm$, and

$$\langle \partial(J)^\alpha F\rangle_{y_j=0} = \lim_{y_j \to 0+} \partial(J)^\alpha F(y) - \lim_{y_j \to 0-} \partial(J)^\alpha F(y)$$

if $\{y_j \ne 0;\ y \in W(\mathrm{G}, \mathfrak{h}) \bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})\} = \mathbb{R}^\times$. Then for $1 \le j \le l$

$$\langle \partial(J)^\alpha F\rangle_{y_j=0} = 0 \text{ if } 0 \le \alpha_j < \begin{cases} 2(l'-l)+1 & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathrm{G} = \mathrm{O}_{2l}, \\ 2l' - 2l & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathrm{G} = \mathrm{O}_{2l+1}, \\ l' - l & \text{if } \mathbb{D} = \mathbb{C}, \\ 2(l'-l) & \text{if } \mathbb{D} = \mathbb{H}. \end{cases}$$

(Here $\langle \partial(J)^\alpha F\rangle_{y_j=0}$ is a function of the y with $y_j \ne 0$.)

Proof. Without any loss of generality we may assume that $j = l$. Let $w = \sum_{j=1}^{l-1} w_j u_j$, where $\delta_j w_j^2 = y_j$, $1 \le j \le l-1$. Recall the decomposition (7). The centralizer of w in $\mathrm{W} = \mathfrak{s}_{\overline{1}}$ is equal to

$$\mathfrak{s}_{\overline{1}}^w = \mathfrak{s}_{\overline{1}}(\mathsf{V})^w = \mathfrak{s}_{\overline{1}}(\mathsf{V}^1)^w \oplus \cdots \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^{l-1})^w \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l). \qquad (80)$$

As checked in the proof of [12, Theorem 4.5] [1], there is a slice through w equal to

$$U_w = (w_1-\epsilon, w_1+\epsilon)u_1 + \cdots + (w_{l-1}-\epsilon, w_{l-1}+\epsilon)u_{l-1} + \mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l),$$

where $\epsilon > 0$ is sufficiently small. To underline its dependence on the graded space V, let us denote the function (59) by $F_{\mathsf{V}}(y)$. Recall that $y = \sum_{j=1}^{l} y_j J_j \in W(\mathrm{G},\mathfrak{h})\big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})\big)$ and let w_y be such that $\tau'(w_y) = y$. The Lebesgue measure on $\mathfrak{s}_{\overline{1}}(\mathsf{V})$ is fixed and the orbital integral $\mu_{\mathcal{O}(w_y),\mathfrak{h}_{\overline{1}}}$ is normalized as in (23). We normalize the Lebesgue measure on each $\mathfrak{s}_{\overline{1}}(\mathsf{V}^j)$ and on $\mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l)$ so that via the direct sum decomposition

$$\mathfrak{s}_{\overline{1}}(\mathsf{V}) = \mathfrak{s}_{\overline{1}}(\mathsf{V}^1) \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^2) \oplus \cdots \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^{l-1}) \oplus \mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l)$$

we get the same measure on $\mathrm{W} = \mathfrak{s}_{\overline{1}}(\mathsf{V})$. Then the $\mathrm{S}(\mathsf{V})$-orbital integral $\mu_{\mathcal{O}(w_y),\mathfrak{h}_{\overline{1}}}$ restricts to U_w and, since U_w is a slice, the result is the tensor product of $\mathrm{S}(\mathsf{V}^j)^w$-orbital integrals and the $\mathrm{S}(\mathsf{V}^0 \oplus \mathsf{V}^l)$-orbital integral. Therefore,

$$\begin{aligned} F_{\mathsf{V}}(y)|_{\mathrm{U}_w} = P(y)(F_{\mathsf{V}^1}(y_1J_1)|_{(w_1-\epsilon,w_1+\epsilon)u_1} \otimes \cdots \\ \cdots \otimes F_{\mathsf{V}^{l-1}}(y_{l-1}J_{l-1})|_{(w_{l-1}-\epsilon,w_{l-1}+\epsilon)u_{l-1}} \otimes F_{\mathsf{V}^0\oplus\mathsf{V}^l}(y_lJ_l)), \end{aligned} \quad (81)$$

where $P(y)$ is a polynomial. In (81)

$$F_{\mathsf{V}^j}(y_jJ_j)|_{(w_j-\epsilon,w_j+\epsilon)u_j} \in \mathcal{D}'((w_j-\epsilon, w_j+\epsilon)u_j) \qquad (1 \le j \le l-1)$$

and

$$F_{\mathsf{V}^0\oplus\mathsf{V}^l}(y_lJ_l) \in \mathcal{D}'(\mathfrak{s}_{\overline{1}}(\mathsf{V}^0 \oplus \mathsf{V}^l)). \quad (82)$$

Here $\mathcal{D}'(\mathrm{X})$ denotes the space of distributions on X. Since the dimension of a Cartan subalgebra of $\mathrm{S}(\mathsf{V}^0 \oplus \mathsf{V}^l)|_{\mathsf{V}_{\overline{1}}}$ is equal to $l' - l + 1$, Lemma 4.8 follows from (81), (82) and Lemma 4.7. This verifies the claim for $\alpha = (0,\dots,0,k,0,\dots,0)$ with k on the place j. To complete the proof, we repeat the same argument with F replaced by $\partial(J)^\beta F$, where $\beta_j = 0$. □

Lemmas 4.5 and 4.8 provide a further extension of the function F, (59) to a continuous function

$$F : W(\mathrm{G},\mathfrak{h})\Big(\bigcup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{In-reg})\Big) \to \mathcal{S}^*(\mathrm{W})^{\mathrm{S}} \quad (83)$$

which satisfies the symmetry condition (60). Now Lemma 3.4 completes the proof of Theorem 3.5.

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[1] The statement of that theorem needs to be modified as follows. "Let $x \in \mathfrak{g}_1$ be semisimple. Then $\mathfrak{g}_{\overline{1}}^x$ has a basis of G^x-invariant neighborhoods of x consisting of admissible slices U_x through x. If $\ker(x) = 0$ then one may choose the U_x so that, for $i = \overline{0}, \overline{1}$,

$$\mathrm{U}_x \ni y \to y^2|_{\mathsf{V}_i} \in \mathfrak{g}_0(\mathsf{V}_i)^{x^2}$$

is an (injective) immersion."

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