Invent. math. 141, 299–363 (2000)
Digital Object Identifier (DOI) 10.1007/s002220000070

Inventiones mathematicae

A Cauchy Harish-Chandra integral, for a real reductive dual pair

Tomasz Przebinda[⋆]

Department of Mathematics, University of Oklahoma, 601 Elm Avenue, Norman, OK 73019-0315, USA

Oblatum 30-III-1999 & 16-XII-1999
Published online: 29 March 2000 – © Springer-Verlag 2000

Contents

Introduction

The Cauchy determinant identity says

$$(1)\qquad det\left(\frac{1}{1-h_ih'_j}\right)=\frac{\prod_{i<j}(h_i-h_j)\prod_{i<j}(h'_i-h'_j)}{\prod_{i,j}(1-h_ih'_j)},$$

where $h_1, h_2, ..., h_n$ and $h'_1, h'_2, ..., h'_n$ are indeterminates, (see [M], [H5], [Wy]). This identity is equivalent to

$$(2)\quad \frac{1}{\prod_{i,j}(1-h_ih'_j)}=\sum_{k_1>k_2>...>k_n>0}\frac{|h^{k_1}\,h^{k_2}\,...h^{k_n}|}{|h^{n-1}\,h^{n-2}\,...h^{0}|}\frac{|h'^{k_1}\,h'^{k_2}\,...h'^{k_n}|}{|h'^{n-1}\,h'^{n-2}\,...h'^{0}|},$$

⋆ Research partially supported by NSF grant DMS 9204488, NSA grant DMA 904-96-1-0023 and NATO linkage grant OUTR.LG 951526.

where

$$|h^{k_1}\, h^{k_2}\, ...h^{k_n}| = det \begin{bmatrix} h_1^{k_1} & h_1^{k_2} & ... & h_1^{k_n} \\ h_2^{k_1} & h_2^{k_2} & ... & h_2^{k_n} \\ .. & .. & ... & .. \\ h_n^{k_1} & h_n^{k_2} & ... & h_n^{k_n} \end{bmatrix},$$

and similarly for the h_j''s, (see [M], [H5], [R3]).

Consider the following action of the group $U_n \times U_n$ on $M_n(\mathbb{C})$, the space of $n \times n$ matrices with complex entries:

$$A \to gAg'^t \qquad (A \in M_n(\mathbb{C});\ g, g' \in U_n).$$

This action extends to an action on $Sym(M_n(\mathbb{C}))$, the symmetric algebra of $M_n(\mathbb{C})$. The formula (2) is equivalent to

$$\text{(3)} \qquad trace((g, g')|_{Sym(M_n(\mathbb{C}))}) = \sum trace(\Pi(g))\, trace(\Pi(g')) \qquad (g, g' \in U_n),$$

where the summation is over all irreducible polynomial representations Π of U_n, [H5], and both sides are understood as distributions on $U_n \times U_n$.

The formula (3) belongs to The Classical Invariant Theory. In a series of works, culminating in [H1], Roger Howe has generalized this theory by introducing the notion of a reductive dual pair G, G' in a symplectic group Sp.

Let ω be an oscillator representation of $\widetilde{Sp}$, the double cover of Sp, and let $\tilde{G}, \tilde{G}' \subseteq \widetilde{Sp}$ be the preimages of G, G' respectively. Howe's correspondence is a bijection $\Pi \leftrightarrow \Pi'$ between certain (in general unknown) subsets of the admissible duals of $\tilde{G}$ and of $\tilde{G}'$, [H1].

Let Θ be the distribution character of ω. If both G and G' are compact, then G is isomorphic to U_m and G' is isomorphic to U_n, for some m and n, and a slight generalization of (3) gives

$$\text{(4)} \qquad \Theta_{\Pi'}(g') = \int_{\tilde{G}} \overline{\Theta_\Pi(g)}\, \Theta(gg')\, dg \qquad (g' \in \tilde{G}'),$$

where $\Theta_\Pi(g) = trace(\Pi(g))$ is the character of Π (and similarly for Π'), and both sides are understood as distributions on $\tilde{G}'$.

The purpose of this paper is to redefine the right hand side of (4), so that it would make sense, uniformly for all real reductive dual pairs, and to provide some evidence for the conjecture that the resulting formula is compatible with Howe's correspondence.

The main problem, along these lines, is that if G is not compact then the integral (4) has no chance to converge - no matter how far we stretch the theory of generalized functions. However, there is a very simple way around it, which by the way exposes the potential role of the structure of G, G' orbits in W, the corresponding symplectic space.

Recall, [H2], that the mataplecitic group $\widetilde{Sp}$, may be realized in the space $S^*(W)$ of temperate distributions on W, via a map $T : \widetilde{Sp} \ni g \to T(g) \in S^*(W)$, (see Sect. 2). In particular,

$$\Theta(\widetilde{(-1)}g) = T(\widetilde{(-1)}g)(0) = \Theta(\widetilde{(-1)}) \int_W T(g)(w)\, dw \qquad (g \in \widetilde{Sp}).$$

where $\widetilde{(-1)}$ is in the preimage of -1.

Let χ_Π denote the central character of Π, and let $H \subseteq G$ be a Cartan subgroup. Then (4) can be rewritten as

$$\begin{aligned} \Theta_{\Pi'}(g') &= \int_{\tilde{G}} \overline{\Theta_\Pi(\widetilde{(-1)}g)}\Theta(\widetilde{(-1)}gg')\, dg \\ &= \overline{\chi_\Pi(\widetilde{(-1)})}\Theta(\widetilde{(-1)}) \int_{\tilde{G}} \overline{\Theta_\Pi(g)} \int_W T(gg')(w)\, dw\, dg \\ &= \overline{\chi_\Pi(\widetilde{(-1)})}\Theta(\widetilde{(-1)}) \int_{\tilde{G}} \overline{\Theta_\Pi(g)} \int_{G\backslash W} T(gg')(w)\, d\dot{w}\, dg \\ &= \overline{\chi_\Pi(\widetilde{(-1)})}\Theta(\widetilde{(-1)}) \int_{\tilde{H}} \overline{\Theta_\Pi(h)\Delta(h)}\Delta(h) \left[\int_{H\backslash W} T(hg')(w)\, d\dot{w}\right] dh, \end{aligned} \tag{5}$$

where Δ is the Weyl denominator, see [W2]. The point is that the integral in brackets admits a generalization, (see Sect. 2). Hence, via Weyl's integration formula, the integral over $G\backslash W$ in the third line of (5), makes sense, (see 2.17), and this is what we have been looking for. The title of this paper refers to our generalization of the integral in brackets in the formula (5).

As usual in the theory of orbital integrals, there is "an infinitesimal version" of the above-mentioned integral, which lives on the Lie algebra. We define it in Sect. 1, deferring most of the technicalities to Sects. 3–11. The integral on the group is defined in Sect. 2, with the technicalities explained to Sects. 12–14.

In Sects. 3–7 we deal with pairs of type II. The calculations here are relatively straightforward, mainly because our object of study is a non-negative invariant measure. The situation becomes more complex for pairs of type I, Sects. 8–11, where we are led to deal with distributions, which are not measures (Sect. 10).

The last three sections (12–14), where we investigate the integral on the group, are a bit sketchy because the proofs of the results there are analogous to the corresponding proofs in the Lie algebra case.

For computational convenience we reverse the roles of G and G in the rest of this paper.

I would like to thank Wulf Rossmann for his wonderful hospitality during my sabbatical stay in Ottawa in the Fall 1996 and during two summer visits in 1997 and 1998. I am indebted to him for many fruitful conversations, and for his insistence that "there should be contours somewhere in this theory". This finally lead to the proof of Theorems 10.3 and 14.12.

I wish to thank Andrzej Daszkiewicz for participating in our continuing project, of micro-local study of Howe's correspondence, and for his warm and cordial hospitality during my frequent visits in Toruń.

It has been my pleasure to report on the initial results of this paper in Tuczno, July 1995, and in Yale, November 1996. I would like to thank Andrzej Hulanicki and Roger Howe for their kind invitations and interest.

1. The integral on the Lie algebra

Let W be a finite dimensional vector space over $\mathbb{R}$, with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Let $Sp = Sp(W) \subseteq End(W)$ denote the corresponding symplectic group, with the Lie algebra $sp = sp(W) \subseteq End(W)$. Let J be a compatible positive definite complex structure on W. Thus $J \in sp$, $J^2 = -I$, and the symmetric bilinear form

$$\langle J\, w, w' \rangle \qquad (w, w' \in W) \tag{1.1}$$

is positive definite. Let dw be the Lebesgue measure on W normalized so that the volume of the unit cube is 1. Since any two positive definite compatible complex structures on W are conjugate by elements of $Sp(W)$, which have determinant 1, the normalization of the measure dw does not depend on the particular choice of J. Let $\chi(x) = e^{2\pi i x},\ x \in \mathbb{R}$. In these terms, Liouville's formula reads

$$\int_W \chi\left(\frac{i}{2}\langle J(w), w\rangle\right) dw = 1. \tag{1.2}$$

The conjugation by J, $x \to JxJ^{-1}$, defines a Cartan involution θ on the group Sp and on the Lie algebra sp. Let $G, G' \subseteq Sp$ be an irreducible dual pair, with the Lie algebras $\mathfrak{g}, \mathfrak{g}' \subseteq sp$, [H1]. We may, and shall, assume that θ preserves G, G', $\mathfrak{g}$ and $\mathfrak{g}'$.

Let $H' = T'A' \subseteq G'$ be a θ stable Cartan subgroup, where T' is the compact part of H' and A' is the vector part of H, as in [W1, 2.3.6]. Let $A'' = Sp^{A'}$ denote the centralizer of A' in Sp, and let A''' denote the centralizer of A'' in Sp. Clearly, $A''' \subseteq G'$. We shall see in (7.4) that (A'', A''') is a reductive dual pair in $Sp(W)$. We shall define an open dense A''' - invariant subset $W_{A'''} \subseteq W$ such that $A'' \setminus W_{A'''}$ is a manifold, with an A'' - invariant measure $d\dot{w}$ determined by

$$\int_{W_{A'''}} \phi(w)\, dw = \int_{A'''\setminus W_{A'''}} \int_{A'''} \phi(aw)\, da d\dot{w} \qquad (\phi \in C_c(W_{A'''})), \tag{1.3}$$

where da indicates a Haar measure on A'''.

For a vector subspace $V \subseteq sp$ define an unnormalized moment map

$$\tau_V : W \to V^*, \quad \tau_V(w)(x) = \langle xw, w\rangle \qquad (x \in V,\ w \in W). \tag{1.4}$$

For simplicity, let

$$\chi_x(w) = \chi\left(\frac{1}{4}\tau_{sp}(w)(x)\right) = \chi\left(\frac{1}{4}\langle xw, w\rangle\right) \qquad (x \in sp,\ w \in W). \tag{1.5}$$

Recall, [Hö], [H3], that for a finite dimensional vector space V over $\mathbb{R}$, and a distribution $u \in D'(V)$, one defines the wave front set of u, $WF(u) \subseteq V \times V^*$, as follows. The complement of $WF(u)$ in $V \times V^*$ is the union of all sets of the form $U \times U'$, where $U \subseteq V$ is an open set, and $U' \subseteq V^* \setminus \{0\}$ is an open cone such that for any $\psi \in C_c^\infty(U)$ and any $\phi \in C^\infty(U)$, with the derivative $\phi'(U) \subseteq U'$

$$|u(\psi \cdot \overline{\chi} \circ \phi)| \le const_{\psi,\phi,N}(1 + \max_{x \in supp\,\psi} |\phi'(x)|)^{-N} \quad (N \ge 0). \tag{1.6}$$

Here, on the right hand side $|\ |$ stands for a norm on V^*. (Unlike in [Hö], we shall consider the zero section $supp\,u \times \{0\}$ to be in the wave front set of u, whenever this is convenient and does not lead to confusion.) Let $S(V)$ denote the Schwartz space of V, as in [Hö, Chapt. 7].

Lemma 1.7. *Let $\mathfrak{a}''$ denote the Lie algebra of A''. For any $\psi \in S(\mathfrak{a}'')$,*

(a) $$\int_{A''' \setminus W_{A'''}} \left| \int_{\mathfrak{a}''} \psi(x)\chi_x(w)\,dx \right| d\dot{w} < \infty.$$

The formula

(b) $$chc(\psi) = \int_{A''' \setminus W_{A'''}} \int_{\mathfrak{a}''} \psi(x)\chi_x(w)\,dx\,d\dot{w} \qquad (\psi \in S(\mathfrak{a}''))$$

defines a temperate distribution on $\mathfrak{a}'$. The wave front set of this distribution

(c) $$WF(chc) = \{(x, \tau_{\mathfrak{a}''}(w));\ x(w) = 0,\ x \in \mathfrak{a}'',\ w \in W\}.$$

Let $W(H') = W(H', G')$ denote the Weyl group of H' in G'. By definition, $W(H')$ is equal to the normalizer of H' in G' divided by H'. This group acts on the Lie algebra $\mathfrak{h}'$ of H'. We shall say that an element $x' \in \mathfrak{h}'$ is regular, if it is regular in the usual sense (the eigenvalue 0 of $ad(x')$ has multiplicity equal to the dimension of $\mathfrak{h}'$, see [W1, 0.2.1]) and the stabilizer of x' in $W(H')$ is trivial. The set of all such elements $x' \in \mathfrak{h}'$ shall be denoted by $\mathfrak{h}'^r$.

Proposition 1.8. *Fix an element $x' \in \mathfrak{h}'^r$. Then the intersection of $WF(chc)$ with the conormal bundle to the embedding*

$$\mathfrak{g} \ni x \to x' + x \in \mathfrak{a}''$$

is empty (contained in the zero section).

Standard micro-local analysis [Hö, 8.2.4] together with Proposition 1.8, justify the following definition.

Definition 1.9. *Let $x' \in \mathfrak{h}'^r$. Then*

$$chc_{x'}(\psi) = \int_{A'''\backslash W_{A'''}} \int_{\mathfrak{g}} \psi(x)\chi_{x'+x}(w)\,dx\,d\dot{w} \qquad (\psi \in S(\mathfrak{g}))$$

is the pullback of the distribution (7.b) from $\mathfrak{d}'$ to $\mathfrak{g}$ via the embedding

$$\mathfrak{g} \ni x \to x' + x \in \mathfrak{a}''.$$

The following lemma is an obvious consequence of [Hö, 8.2.4] and (1.8).

Lemma 1.10. *For any $x' \in \mathfrak{h}'^r$, $WF(chc_{x'}) = \{(x, \tau_{\mathfrak{g}}(w));\ (x'+x)(w) = 0,$ $x \in \mathfrak{g},\ w \in W\}$.*

Recall, [H1] that the group G' has a defining module V'. Specifically, if the pair G, G' is of type I, then V' is a finite dimensional space over $\mathbb{D} = \mathbb{R},\ \mathbb{C}$ or $\mathbb{H}$ (the quaternions). The division algebra $\mathbb{D}$ is equipped with a (possibly trivial) involution, and the space V' with a non-degenerate form $(\ ,\ )'$, which is either hermitian or skew-hermitian. The group G' coincides with the isometry group of that form. If the pair G, G' is of type II, then $G' = GL_{\mathbb{D}}(V')$, is "the isometry group of the zero form".

For a pair G, G' of type I, let $V'_c \subseteq V'$ be the subspace on which the vector part A' of the Cartan subgroup $H' \subseteq G'$ acts trivially. Let $V'_s \subseteq V'$ be the orthogonal complement of V'_c. Since the restriction of the form $(\ ,\ )'$ to V'_s is split, there is a complete polarization $V'_s = X' \oplus Y'$ preserved by H'. Thus

$$V' = V'_c \oplus V'_s,\ \ V'_s = X' \oplus Y'. \tag{1.11}$$

If the pair G, G' is of type II, we have $V'_c = 0$, and the whole space V' is isotropic. For convenience, we identify V' with X' in this case (and set $Y' = 0$).

Similarly, let V be the defining module for G. Set $d = dim_{\mathbb{D}}(V)$ and $d' = dim_{\mathbb{D}}(V')$. Let

$$p = d - d' + 1,\ d - d',\ d - d',\ 2(d - d'),\ 2(d - d') + 1 \text{ if } G' = O_{p,q},\ Sp_{2n}(\mathbb{R}),\ U_{p,q},\ Sp_{p,q},\ O^*_{2n}, \text{ respectively.} \tag{1.12}$$

For an element $x' \in \mathfrak{h}'$ let $\lambda_1(x'), \lambda_2(x'),\ldots,$ denote the eigenvalues of $x'|X' \in End_{\mathbb{R}}(X')$, the restriction of x' to X'. Fix a norm $|\ |$ on the real vector space $End_{\mathbb{D}}(V'_c)$. Let $V'_c = V'_{c1} \oplus V'_{c1} \oplus \ldots$ be a decomposition of V'_c into H' - irreducible subspaces over $\mathbb{D}$. For $x' \in \mathfrak{h}'$ let $\mu_1(x') = \left|x'|_{V'_{c1}}\right|$, $\mu_2(x') = \left|x'|_{V'_{c2}}\right|$, ... Let $\pi_{\mathfrak{h}'}$ be the product of positive roots of $\mathfrak{h}'_{\mathbb{C}}$ in $\mathfrak{g}'_{\mathbb{C}}$, (with respect to some ordering of roots). Recall that the rank of G is the dimension of a Cartan subalgebra of $\mathfrak{g}$.

Theorem 1.13. *Assume that the rank of G' is less or equal to the rank of G. Fix numbers $N, \epsilon > 0$. If $V'_c = 0$ set*

$$P(x') = \prod_j (|\lambda_j(x')| + 1)^N, \quad (x' \in \mathfrak{h}').$$

If $V'_c \neq 0$, then the group G' is isomorphic to one of the groups listed in (1.12) and the number p defined in (1.12) is non-negative. In this case set

$$P(x') = \prod_j (|\lambda_j(x')| + 1)^N \cdot \prod_j (\mu_j(x') + 1)^{p-\epsilon}, \quad (x' \in \mathfrak{h}').$$

Then the following integral converges and defines a continuous seminorm on $S(\mathfrak{g})$:

$$\int_{\mathfrak{h}'^r} P(x') \left| \pi_{\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx \right| dx' \qquad (\psi \in S(\mathfrak{g})).$$

Proof. This Theorem is a direct consequence of Harish-Chandra's Theorem, regarding his semisimple orbital integral, [Va, part I, p. 47], combined with (11.14) and (10.3) for pairs of type I, and (7.21) for pairs of type II. □

For a finite dimensional vector space V over $\mathbb{R}$, with a Lebesgue measure dx, let

$$\hat{\psi}(\xi) = \int_V \chi(\xi(x))\, \psi(x)\, dx \qquad (\xi \in V^*,\ \psi \in S(V)) \tag{1.14}$$

be the Fourier transform, defined with respect to the character $\overline{\chi}$. The Fourier transform of a temperate distribution $u \in S(V)$ is defined by the usual recipe: $\hat{u}(\psi) = u(\hat{\psi})$, $\psi \in S(V)$.

Let $\mathcal{O}' \subseteq \mathfrak{g}'^*$ be a nilpotent coadjoint orbit. Denote by $\mu_{\mathcal{O}'} \in S'(\mathfrak{g}')$ the canonical invariant positive measure on $\mathcal{O}'$, as explained in [RR], [R1, p. 56] or in [W2].

Harish-Chandra's Regularity Theorem, [W1, 8.3.4], implies that the Fourier transform, $\hat{\mu}_{\mathcal{O}'} \in S'(\mathfrak{g}'^*)$ coincides with a locally integrable, conjugation invariant function on $\mathfrak{g}'$. In particular one can restrict this function to $\mathfrak{h}'^r$. For each connected component $\mathfrak{C} \subseteq \mathfrak{h}'^r$ there is a harmonic polynomial h such that $\hat{\mu}_{\mathcal{O}'}(x') = h(x')/\pi_{\mathfrak{h}'}(x')$, $x' \in \mathfrak{C}$, see [W2]. Moreover there is a constant coefficient differential operator D' on $\mathfrak{h}'$ such that $h = D'\pi_{\mathfrak{h}'}$, see [He, Theorem 3.6, p. 361]. By combining these facts with (1.13) and (10.6), it is easy to deduce the following corollary.

Corollary 1.15. *Suppose the rank of G' is less or equal to the rank of G. If G' is one of the groups listed in (1.12), assume in addition, that*

$$2d' \leq d+2,\ 2d' \leq d,\ 2d' \leq d,\ 2d' \leq d,\ 2d' \leq d+1 \text{ if } G' = O_{p,q},\ Sp_{2n}(\mathbb{R}),\ U_{p,q},\ Sp_{p,q},\ O^*_{2n},\ \textit{respectively}.$$

Then for any nilpotent coadjoint orbit $\mathcal{O}' \subseteq \mathfrak{g}'^*$ and any Cartan subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}'$,

$$\int_{\mathfrak{h}'^r} \left| \hat{\mu}_{\mathcal{O}'}(x')\pi_{\mathfrak{h}'}(x')^2 \int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx \right| dx' < \infty \quad (\psi \in S(\mathfrak{g})).$$

Moreover, the above integral defines a continuous seminorm on $S(\mathfrak{g})$.

Since $A' \subseteq A''' \subseteq H'$, we may normalize the measures involved so that the Weyl integration formula looks as follows,

$$\int_{\mathfrak{g}'} \psi(x)\,dx = \sum \frac{1}{|W(H')|} \int_{\mathfrak{h}'} |\pi_{\mathfrak{h}'}(x')|^2 \int_{G'/A'''} \psi(gxg^{-1})\,d\dot{g}\,dx \qquad (\psi \in S(\mathfrak{g}')), \tag{1.16}$$

where the summation is over a maximal family of mutually non-conjugate Cartan subgroups $H' \subseteq G'$, and $|W(H')|$ is the cardinality of the Weyl group $W(H')$.

Definition 1.17. *Under the assumptions of the Corollary 1.15, given a nilpotent coadjoint orbit $\mathcal{O}' \subseteq \mathfrak{g}'^*$ and the canonical invariant measure $\mu_{\mathcal{O}'}$ define an invariant temperate distribution $\hat{\mu}'_{\mathcal{O}'}$ on $\mathfrak{g}$ by the formula*

$$\hat{\mu}'_{\mathcal{O}'}(\psi) = \sum \frac{1}{|W(H')|} \int_{\mathfrak{h}'^r} \overline{\hat{\mu}_{\mathcal{O}'}(x')} |\pi_{\mathfrak{h}'}(x')|^2 \int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx\,dx' \qquad (\psi \in S(\mathfrak{g})),$$

where the summation is as in (1.16).

The point of this definition is that $\hat{\mu}'_{\mathcal{O}'}$ resembles a constant multiple of the Fourier transform of the sum of canonical measures supported on some orbits contained in $\tau_{\mathfrak{g}} \circ \tau_{\mathfrak{g}'}^{-1}(\mathcal{O}')$:

$$\begin{aligned} \hat{\mu}'_{\mathcal{O}'}(\psi) &= \sum \frac{1}{|W(H')|} \int_{\mathfrak{h}'^r} \overline{\hat{\mu}_{\mathcal{O}'}(x')} |\pi_{\mathfrak{h}'}(x')|^2 \\ &\qquad \int_{A'''\backslash W_{A'''}} \int_{\mathfrak{g}} \chi_{x'+x}(w)\psi(x)\,dx\,d\dot{w}\,dx' \\ &= \sum \frac{1}{|W(H')|} \int_{\mathfrak{h}'^r} \overline{\hat{\mu}_{\mathcal{O}'}(x')} |\pi_{\mathfrak{h}'}(x')|^2 \\ &\qquad \int_{A'''\backslash G'} \int_{G'\backslash W} \int_{\mathfrak{g}} \chi_{x'+x}(gw)\psi(x)\,dx\,d\dot{w}\,d\dot{g}\,dx' \end{aligned} \tag{1.18}$$

$$\begin{aligned}
&= \int_{\mathfrak{g}'} \overline{\hat{\mu}_{\mathcal{O}'}(x')} \int_{G'\backslash W} \int_{\mathfrak{g}} \chi_{x'+x}(w)\psi(x)\,dx\,d\dot{w}\,dx' \\
&= \int_{G'\backslash W} \int_{\mathfrak{g}'} \overline{\hat{\mu}_{\mathcal{O}'}(x')} \chi_{x'}(w)\,dx' \int_{\mathfrak{g}} \chi_x(w)\psi(x)\,dx\,d\dot{w} \\
&= const \cdot \int_{G'\backslash W} \mu_{\mathcal{O}'}(\tau_{\mathfrak{g}'}(w))\hat{\psi}(\tau_{\mathfrak{g}}(w))\,d\dot{w} \\
&= const \cdot \hat{\mu}_{\tau_{\mathfrak{g}}\circ\tau_{\mathfrak{g}'}^{-1}(\mathcal{O}')}(\psi).
\end{aligned}$$

The above reasoning can be made precise in some interesting cases.

Recall, [H4], [Li], that a pair G, G' of type I is in the stable range, with G' - the smaller member, if the space V has an isotropic subspace of dimension greater or equal to the dimension of V'. For such pairs it is well known that if $\mathcal{O}' \subseteq \mathfrak{g}'^*$ is a nilpotent coadjoint orbit, then $\tau_{\mathfrak{g}} \circ \tau_{\mathfrak{g}'}^{-1}(\mathcal{O}') \subseteq \mathfrak{g}^*$ contains a single dense nilpotent coadjoint orbit $\mathcal{O}$. The same holds for pairs of type II, with the rank of G' less or equal to the rank of G. We shall refer to $\mathcal{O}$ as to the orbit corresponding to $\mathcal{O}'$.

Theorem 1.19. *Suppose the pair G, G' is of type I in the stable range, with G' - the smaller member, or a pair of type II, with the rank of G less or equal to the rank of G. Let $\mathcal{O}' \subseteq \mathfrak{g}'^*$ be a nilpotent coadjoint orbit, and let $\mathcal{O} \subseteq \mathfrak{g}^*$ be the corresponding nilpotent coadjoint orbit. Then $\hat{\mu}_{\mathcal{O}} = const \cdot \hat{\mu}'_{\mathcal{O}'}$.*

Proof. We shall provide the argument for pairs of type II at the end of Sect. 7, (see (7.22)–(7.24)).

From now on we assume that the pair G, G' is of type I. In [D-P1] we have constructed a dense $G \cdot G'$-invariant set $W^{max} \subseteq W$ and a continuous linear map $\mathcal{A}$ from the space of rapidly decreasing functions on $\mathfrak{g}$ to the space of rapidly decreasing functions on $\mathfrak{g}'$ such that for any $\psi \in S(\mathfrak{g})$

$$\int_{\mathfrak{g}} \psi(x)\,d\mu_{\mathcal{O}}(x) = const \int_{\mathfrak{g}'} \mathcal{A}\psi(x')\,d\mu_{\mathcal{O}'}(x'), \tag{1.20}$$

where the constant, *const*, does not depend on ψ, and

$$(\mathcal{A}\hat{\psi})\hat{}\,(x') = \int_{G'\backslash W^{max}} \int_{\mathfrak{g}} \chi_{x'+x}(w)\psi(x)\,dx\,d\dot{w} \qquad (x' \in \mathfrak{g}'). \tag{1.21}$$

Here we identify $\mathfrak{g}' = \mathfrak{g}'^*$ via an invariant bilinear form B on $\mathfrak{g}'$. Moreover, the group G' acts freely on W^{max}, so that $G'\backslash W^{max}$ is a manifold with the quotient measure $d\dot{w}$ defined as in (1.3) and $G'\backslash W^{max} \ni \dot{w} \to w \in W^{max}$ is a fixed section. Each consecutive integral in (1.21) is absolutely convergent. Furthermore, by (11.5) and (4.2), $W^{max} \subseteq W_{A''}$ for each A' as in (1.3).

Fix a Cartan subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}'$. Let $D(y')$, $y' \in \mathfrak{g}'$, be the Weyl denominator, (see [W2, 2.3.1]). By a theorem of Harish-Chandra, [Va, part I, Proposition 9, p. 108], there is a finite constant C such that for any $x' \in \mathfrak{h}'^r$ and for any regular semisimple element $y' \in \mathfrak{g}'$

$$\left|\int_{G'/A'''} \chi(B(gx'g^{-1}, y'))\, d\dot{g}\right| \leq C|D(x')|^{-1/2}|D(y')|^{-1/2}. \tag{1.22}$$

Moreover, for $N \geq 0$, large enough,

$$\int_{\mathfrak{g}'} |D(y')|^{-1/2}(1+|y'|)^{-N}\, dy' < \infty, \tag{1.23}$$

where $|\ |$ is a norm on the real vector space $\mathfrak{g}'$.

With the notation (1.21), we have

$$\begin{aligned}\int_{\mathfrak{g}'}\int_{G'/A'''} \chi(B(gx'g^{-1}, y'))\, d\dot{g}\mathcal{A}\hat{\psi}(y')\, dy' &= \int_{G'/A'''} (\mathcal{A}\hat{\psi}\hat{)}(gx'g^{-1})\, d\dot{g} \\ &= \int_{A'''\backslash G'}\int_{G'\backslash W^{max}}\int_{\mathfrak{g}} \chi_{x'+x}(gw)\psi(x)\, dx\, d\dot{w}\, d\dot{g}.\end{aligned} \tag{1.24}$$

The last formula in (1.24), with the $\mathfrak{g}$ replaced by $\mathfrak{a}'$ and with $\psi \in S(\mathfrak{a}'')$, defines the distribution (1.7.b). (Here the special section $G\backslash W^{max} \ni \dot{w} \to w \in W^{max}$, constructed in [D-P1], disappears.) Hence, by the uniqueness of the restriction to $\mathfrak{g}$,

$$\begin{aligned}&\int_{\mathfrak{g}'}\int_{G'/A'''} \chi(B(gx'g^{-1}, y'))\, d\dot{g}\mathcal{A}\hat{\psi}(y')\, dy' \\ &= \int_{A'''\backslash W^{max}}\int_{\mathfrak{g}} \chi_{x'+x}(w)\psi(x)\, dx\, d\dot{w} = \int_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx,\end{aligned} \tag{1.25}$$

where, by (1.22) and (1.23), the integral over $\mathfrak{g}'$ is absolutely convergent. Hence, the distribution $\hat{\mu}'_{\mathcal{O}'}$, defined in (1.17), may be calculated as follows:

$$\begin{aligned}\hat{\mu}'_{\mathcal{O}'}(\psi) &= \sum \frac{1}{|W(H')|}\int_{\mathfrak{h}'^r} \overline{\hat{\mu}_{\mathcal{O}'}(x')\pi_{\mathfrak{h}'}(x')}\pi_{\mathfrak{h}'}(x') \\ &\qquad \int_{\mathfrak{g}'}\int_{G'/A'''} \chi(B(gx'g^{-1}, y'))\, d\dot{g}\mathcal{A}\hat{\psi}(y')\, dy'\, dx' \\ &= \int_{\mathfrak{g}'} \overline{\hat{\mu}_{\mathcal{O}'}(x')}(\mathcal{A}\hat{\psi}\hat{)}(x')\, dx' = \int_{\mathfrak{g}'} \mathcal{A}\hat{\psi}(x')\, d\mu_{\mathcal{O}'}(x') \\ &= const \int_{\mathfrak{g}} \hat{\psi}(x)\, d\mu_{\mathcal{O}}(x),\end{aligned}$$

where, by (1.13) and (1.25), the integral over $\mathfrak{h}'^r$ is absolutely convergent, and the last equation follows from (1.20). □

2. The integral on the group

We shall use the superscript c to indicate the domain of the Cayley transform c, ($c(x) = (x+1)(x-1)^{-1}$, as in [H2]). Thus Sp^c is the domain of c in Sp and sp^c is the domain of c in sp. Define the following set

$$\widetilde{Sp}^c = \{(g,\xi);\ g \in Sp^c,\ \xi^2 = det(i(g-1))^{-1}\}.$$

This is a real analytic manifold, and a two fold cover of Sp^c via the map

$$\widetilde{Sp}^c \ni \tilde{g} = (g,\xi) \to g \in Sp^c. \tag{2.1}$$

For $x \in sp$ the formula $\langle x(w), w'\rangle$ defines a symmetric bilinear form $\langle x\ ,\ \rangle$ on W. The signature of this form, $sgn\langle x\ ,\ \rangle$, is the difference between the dimension of the maximal subspace on which the form is positive definite, and the dimension of the maximal subspace on which the form is negative definite. Let

$$\gamma(x) = |det(x)|^{1/2}\, exp\left(-\frac{\pi}{4} i\, sgn\langle x\ ,\ \rangle\right) \qquad (x \in sp,\ det(x) \neq 0). \tag{2.2}$$

For $(g_1,\xi_1),\ (g_2,\xi_2) \in \widetilde{Sp}^c$ with $c(g_1)+c(g_2)$ invertible in $End(W)$, set

$$(g_1,\xi_1)\cdot(g_2,\xi_2) = (g_1g_2, 2^n\xi_1\xi_2\gamma(c(g_1)+c(g_2))^{-1}), \tag{2.3}$$

where $n = \frac{1}{2}dim(W)$.

Theorem 2.4 [H2]. *(a) Up to a group isomorphism there is a unique connected group $\widetilde{Sp}$ containing $\widetilde{Sp}^c$ with the multiplication given by (2.3) on the indicated subset of $\widetilde{Sp}^c \times \widetilde{Sp}^c$.*
(b) The group $\widetilde{Sp}$ is a connected Lie group which contains $\widetilde{Sp}^c$ as an open submanifold.
(c) The map (2.1) extends to a double covering homomorphism of Lie groups: $\widetilde{Sp} \to Sp$.

The metaplectic group $\widetilde{Sp}$ may be realized as a subset of $S^*(W)$, the space of temperate distributions on W, as follows.

For $\phi, \phi' \in S(W)$, the Schwartz space of W, define the twisted convolution $\phi\natural\phi'$ and ϕ^* by

$$\begin{aligned} \phi\natural\phi'(w') &= \int_W \phi(w)\phi'(w'-w)\chi\left(\frac{1}{2}\langle w, w'\rangle\right) dw, \\ \phi^*(w) &= \overline{\phi(-w)} \quad (w, w' \in W). \end{aligned} \tag{2.5}$$

For a temperate distribution $f \in S^*(W)$ define $f^* \in S^*(W)$ by $f^*(\phi) = f(\phi^*)$. The functions χ_x (1.5) do not belong to $S(W)$, but we may convolve them in the sense analogous to the formula of (2.5). Indeed, let $\phi \in S(W)$. Then for $w' \in W$ and $y \in sp$

$$\int_W \chi_y(w)\phi(w'-w)\chi\left(\frac{1}{2}\langle w, w'\rangle\right) dw$$
$$= \chi_y(w') \int_W \chi\left(\frac{1}{2}\langle (1-y)(w'), w\rangle\right) \chi_y(w)\phi(w)\, dw.$$

Thus, for $1-y$ invertible, the above expression is a Schwartz function of w'. Denote this function by $\chi_y \natural \phi(w')$. Suppose $x \in sp^c$. Then by the same argument $\chi_x \natural (\chi_y \natural \phi) \in S(W)$. Suppose moreover that $x+y$ is invertible in $End(W)$. Let $z = (y-1)(x+y)^{-1}(x-1)+1$. Then $z \in sp^c$ and, by [Hö, 3.4], $\chi_x \natural (\chi_y \natural \phi) = 2^n \gamma(x+y)^{-1} \chi_z \natural \phi$. Thus,

$$\chi_x \natural \chi_y = 2^n \gamma(x+y)^{-1} \chi_z. \tag{2.6}$$

Define the following functions

$$\begin{aligned} &\Theta : \widetilde{Sp^c} \ni \tilde{g} = (g, \xi) \to \xi \in \mathbf{C}, \\ &T : \widetilde{Sp^c} \ni \tilde{g} = (g, \xi) \to \Theta(\tilde{g})\chi_{c(g)} \in S^*(W). \end{aligned} \tag{2.7}$$

Theorem 2.8 [H2]. *The map T extends to a unique injective continuous map $T : \widetilde{Sp} \to S^*(W)$, and the following formulas hold*

(a) $T(\tilde{g}_1) \natural T(\tilde{g}_2) = T(\tilde{g}_1 \cdot \tilde{g}_2) \quad (\tilde{g}_1,\ \tilde{g}_2 \in \widetilde{Sp^c},\ det(c(g_1)+c(g_2)) \neq 0)$

(b) $T(\tilde{g})^* = T(\tilde{g}^{-1}) \quad (\tilde{g} \in \widetilde{Sp})$

(c) $T(1) = \delta.$

Here $\delta \in S^*(W)$ is the Dirac delta at the origin.

For any Lie group G we identify $\mathfrak{g}$, the Lie algebra of G, with the Lie algebra of left invariant vector fields on G. This leads to an identification $T^*G = G \times \mathfrak{g}^*$, so that if $\Psi \in C_c^\infty(G)$, then $d\Psi$ is a $\mathfrak{g}^*$-valued function on G given by the following formula

$$d\Psi(g)(x) = \partial_t \Psi(g\, exp(tx))|_{t=0} \qquad (g \in G,\ x \in \mathfrak{g}).$$

Let us fix a norm $|\ |$ on the real vector space $\mathfrak{g}^*$. Let $u \in D'(G)$ be a distribution on G. The wave front set of u is a closed conic subset $WF(u) \subseteq T^*G$ defined as follows.

Let $U \subseteq G$ be an open set and let $U' \subseteq \mathfrak{g}^* \setminus 0$ be an open cone. Then $WF(u)$ is the complement of the union of all the sets $U \times U'$ such that

$$|u(\Psi \cdot \overline{\chi} \circ \phi)| \leq const_{N,\Psi,\phi}(1 + \max_{g \in supp\, \Psi} |d\phi(g)|)^{-N}, \quad (N = 0, 1, 2, ...)$$

for any $\Psi \in C_c^\infty(U)$ and any real valued $\phi \in C^\infty(U)$ with $d\phi(U) \subseteq U'$.

Let G, G' be an irreducible dual pair in $Sp = Sp(W)$. Let $H' = T'A' \subseteq G'$ be a Cartan subgroup and let $A', A''', W_{A'''} \subseteq W$ be as in Sect. 1.

Lemma 2.9. *For any* $\Psi \in C_c^\infty(\widetilde{A}'')$

(a) $$\int_{A'''\backslash W_{A'''}} \left| \int_{\widetilde{A}''} \Psi(g)T(g)(w)\,dg \right| d\dot{w} < \infty.$$

The formula

(b) $$Chc(\Psi) = \int_{A'''\backslash W_{A'''}} \int_{\widetilde{A}''} \Psi(g)T(g)(w)\,dg\,d\dot{w} \qquad (\Psi \in C_c^\infty(\widetilde{A}''))$$

defines a distribution on $\widetilde{A}''$*. The wave front set of this distribution coincides with the set*

(c) $WF(Chc) = \{(\tilde{g}, \tau_{\mathfrak{a}''}(w));\ \tilde{g} \in \widetilde{A}'',\ w \in W,\ g(w) = -w\} \subseteq \widetilde{A}'' \times \mathfrak{a}''^*.$

We shall say that an element $h' \in H'$ is regular if the multiplicity of the eigenvalue 1 of $Ad(h')$ is equal to the dimension of $\mathfrak{h}'$ (as usual), and the stabilizer of h' in the Weyl group $W(H')$ is trivial. The set of all such elements $h' \in H'$ shall be denoted by H'^r.

Proposition 2.10. *Fix an element* $\tilde{h}' \in \tilde{H}'^r$*. The intersection of the wave front set of the distribution Chc, defined in (2.9.b), with the conormal bundle to the embedding*

$$\widetilde{G} \ni \tilde{g} \to \tilde{h}'\,\tilde{g} \in \widetilde{A}''$$

is empty (contained in the zero section).

Standard micro-local analysis [Hö, 8.2.4] together with Proposition 2.10, justify the following definition.

Definition 2.11. *Let* $\tilde{h}' \in \tilde{H}'^r$*. Then*

$$Chc_{\tilde{h}'}(\tilde{g}) = Chc(\tilde{h}'\tilde{g}) = \int_{A'''\backslash W_{A'''}} T(\tilde{h}'\,\tilde{g})(w)\,d\dot{w} \qquad (\tilde{g} \in \widetilde{G})$$

is the pullback of the distribution (2.9.b) to $\widetilde{G}$ *via the embedding*

$$\widetilde{G} \ni \tilde{g} \to \tilde{h}'\,\tilde{g} \in \widetilde{A}''.$$

Lemma 2.12. *For any* $\tilde{h}' \in \tilde{H}'^r$*,*

$$WF(Chc_{\tilde{h}'}) = \{(\tilde{g}, \tau_{\mathfrak{g}}(w));\ h'g(w) = -w,\ \tilde{g} \in \widetilde{G},\ w \in W\} \subseteq \widetilde{G} \times \mathfrak{g}^*.$$

Proof. This is clear from the Definition 2.11, [Hö, 8.2.4], and (2.9.c). □

Theorem 2.13. *The Cauchy Harish-Chandra Integral on Lie algebra is the lowest term in an asymptotic expansion of the Cauchy Harish-Chandra Integral on the group, in the sense of the following formula:*

$$\lim_{t\to 0+} t^n \int_{A'''\backslash W_{A'''}} T(-\tilde{c}(tx)\tilde{c}(tx'))(w)\,d\dot{w} = \Theta(-\tilde{1}) \int_{A'''\backslash W_{A'''}} \chi_{x+x'}(w)\,d\dot{w}$$
$$(x' \in \mathfrak{h}'^r,\ x \in \mathfrak{g}).$$

Proof. Recall that

$$c(-c(x)c(x')) = c(c(x)c(x'))^{-1} = ((x'-1)(x+x')^{-1}(x-1)+1)^{-1}.$$

Hence,

$$c(-c(tx)c(tx')) = t((tx'-1)(x+x')^{-1}(tx-1)+t)^{-1}.$$

Therefore

$$\begin{aligned}
&\int_{A'''\backslash W_{A'''}} \chi_{c(-c(tx)c(tx'))}(w)\, d\dot{w} \\
&\qquad = \int_{A'''\backslash W_{A'''}} \chi_{((tx'-1)(x+x')^{-1}(tx-1)+t)^{-1}}(t^{1/2}w)\, d\dot{w} \\
&\qquad = t^{-n} \int_{A'''\backslash W_{A'''}} \chi_{((tx'-1)(x+x')^{-1}(tx-1)+t)^{-1}}(w)\, d\dot{w}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&t^n \int_{A'''\backslash W_{A'''}} T(-\tilde{c}(tx)\tilde{c}(tx'))(w)\, d\dot{w} \\
&\qquad = \Theta(-\tilde{c}(tx)\tilde{c}(tx')) \int_{A'''\backslash W_{A'''}} \chi_{((tx'-1)(x+x')^{-1}(tx-1)+t)^{-1}}(w)\, d\dot{w} \\
&\qquad \underset{t\to 0}{\longrightarrow} \Theta(-\tilde{1}) \int_{A'''\backslash W_{A'''}} \chi_{x+x'}(w)\, d\dot{w}.
\end{aligned}$$

□

For a system of positive roots Φ of $\mathfrak{h}'_{\mathbb{C}}$ in $\mathfrak{g}'_{\mathbb{C}}$, set

$$\Delta(h) = \prod_{\alpha\in\Phi}(h^{\alpha/2} - h^{-\alpha/2}) \qquad (h \in H'^r),$$

as in [W1, 7.4.5]. Let $\|\ \|$ denote a norm on the group $\tilde{G}'$, as in [W1, 2.A.2].

Theorem 2.14. *Assume that the rank of G' is less or equal to the rank of G. Let $G_1 \subseteq G$ denote the Zariski identity component of G. Then for any $N \geq 0$ and any $\Psi \in C_c^\infty(\tilde{G}_1)$ the following integral is finite and defines a continuous seminorm on $C_c^\infty(\tilde{G}_1)$:*

$$\int_{\tilde{H}'^r} \| h' \|^N \left| \Delta(h') \int_{\tilde{G}'} Chc(h'g)\Psi(g)\, dg \right| dh'.$$

Proof. This Theorem follows directly from (14.2), (14.3) and (14.12). □

Let Π' be an irreducible admissible representation of $\tilde{G}'$, (see [W1, 3.3.5]). Let $\Theta_{\Pi'}$ denote the distribution character of Π', [W1, 8.1.1]. We identify $\Theta_{\Pi'}$ with the corresponding real analytic function on $\tilde{G}'^{rs}$, the set of regular semisimple elements of $\tilde{G}'$, by Harish-Chandra's Regularity Theorem, (see [W1, 8.4.1] and [B, 2.1.1]). Recall that, by Langlands' classification, [W1, 5.4], and by Harish-Chandra's basic inequality, [W1, 8.6.1], there is $N \geq 0$ such that for any Cartan subgroup $\tilde{H}' \subseteq \tilde{G}'$

$$|\Theta_{\Pi'}(h)\Delta(h)| \leq const \parallel h \parallel^N \qquad (h \in \tilde{H}'). \tag{2.15}$$

By combining (2.14) and (2.15) we see that the following formula defines a distribution on $\tilde{G}_1$:

$$\int_{\tilde{H}'^r} \overline{\Theta_{\Pi'}(h')} |\Delta(h')|^2 \int_{\tilde{G}} Chc(h'g)\Psi(g)\, dg\, dh' \quad (\Psi \in C_c^\infty(\tilde{G}_1)). \tag{2.16}$$

Hence the following definition makes sense.

Definition 2.17. *For an irreducible admissible representation Π' of $\tilde{G}'$ define a distribution $\Theta'_{\Pi'}$ on $\tilde{G}_1$ by*

$$\begin{aligned}\Theta'_{\Pi'}(\Psi) = &\overline{\chi_{\Pi'}(\widetilde{(-1)})}\Theta(\widetilde{(-1)}) \\ &\sum \frac{1}{|W(H')|} \int_{\tilde{H}'^r} \overline{\Theta_{\Pi'}(h')} |\Delta(h')|^2 \int_{\tilde{G}} Chc(h'g)\Psi(g)\, dg\, dh',\end{aligned}$$

where $\Psi \in C_c^\infty(\tilde{G}_1)$, and the summation is as in the Weyl integration formula:

$$\int_{\tilde{G}'} \Psi(g)\, dg = \sum \frac{1}{|W(H')|} \int_{\tilde{H}'^r} |\Delta(h')|^2 \int_{\tilde{G}'/\tilde{A}'''} \Psi(gh'g^{-1})\, dg\, dh'.$$

Moreover, $\widetilde{(-1)} \in \tilde{Sp}$ is in the preimage of -1, and $\chi_{\Pi'}(\widetilde{(-1)})$ is the scalar by which $\Pi'(\widetilde{(-1)})$ acts on the Hilbert space of Π'.

We shall see in (14.2), that for pairs of type II, the formula for $\Theta'_{\Pi'}$ coincides with the character formula for a unitary parabolic induction, which is known to be compatible with Howe's correspondence.

Conjecture 2.18. *Generically, if $\Theta_{\Pi'}|_{\tilde{G}'\setminus\tilde{G}'_1} = 0$, the following equation holds*

$$\Theta'_{\Pi'} = \Theta_{\Pi}|_{\tilde{G}_1},$$

where Π is the irreducible admissible representation of $\tilde{G}$ corresponding to Π' via Howe's correspondence for the dual pair G, G.

We shall explain a few facts in support of this conjecture.

Let J be a positive compatible complex structure on W. Then the complexification $W_{\mathbb{C}}$ of W splits into a direct sum of eigenspace for J. Let $W_{\mathbb{C}}^{\pm}$ be the $\pm$-eigenspace for iJ. Set $P_{W_{\mathbb{C}}^{+}} = (1 + iJ)/2 \in End(W_{\mathbb{C}})$. This a projection onto $W_{\mathbb{C}}^{+}$. Let

$$\Omega(\tilde{g}) = T(\tilde{g})(\chi_{iJ}) \qquad (\tilde{g} \in \tilde{Sp}(W)). \tag{2.19}$$

This is a matrix coefficient of the oscillator representation ω of$\widetilde{Sp}(W)$ corresponding to lowest subrepresentation of the maximal compact subgroup $\widetilde{Sp}(W)^J$, in the sense of Vogan. A straightforward calculation shows that

$$\Omega(\tilde{g})^2 = \frac{1}{det_{W_{\mathbb{C}}^{+}}(gP_{W_{\mathbb{C}}^{+}})} \qquad (\tilde{g} \in \tilde{Sp}(W)). \tag{2.20}$$

Let $\mathcal{H}_\omega$ be the Hilbert space where the oscillator representation ω is realized, and let $\mathcal{H}_\omega^\infty \subseteq \mathcal{H}_\omega$ be the subspace of smooth vectors. Recall the following theorem, [P2, 3.1], which generalizes a result of Li, [Li].

Theorem 2.21. *Suppose*

$$\int_{\widetilde{G}'} |\Theta_{\Pi'}(g)|\Omega(g)\, dg < \infty. \tag{a}$$

Then the formula

$$(\omega(\overline{\Theta}_{\Pi'})v, v') = \int_{\widetilde{G}'} \overline{\Theta}_{\Pi'}(g)(\omega(g')v, v')\, dg' \quad (v, v' \in \mathcal{H}_\omega^\infty) \tag{$*$}$$

defines a $\widetilde{G} \cdot \widetilde{G}'$ - invariant hermitian form on H_ω^∞. Let $R \subseteq \mathcal{H}_\omega^\infty$ denote the radical of this form. Suppose that

(b) *the form ($*$) is positive semidefinite and non-trivial.*

Then the $\widetilde{G} \cdot \widetilde{G}'$ - module $\mathcal{H}_\omega^\infty/R$ equipped with the form induced by ($*$), completes to an irreducible unitary representation of$\widetilde{G} \cdot \widetilde{G}'$, infinitesimally equivalent to $\Pi \otimes \Pi'$ for some irreducible unitary representation Π of$\widetilde{G}$. Thus Π' is associated to Π via Howe's correspondence.

For a Hilbert space $\mathcal{H}$ let $Tr(\mathcal{H})$ denote the space of trace class operators on $\mathcal{H}$. In this context we formulate the following conjecture:

Conjecture 2.22. *Under the assumptions of Theorem 2.21 there is a continuous map*

$$S(W) \ni \phi \to \rho_\Pi(\phi) \in Tr(\mathcal{H}_\Pi) \tag{a}$$

such that for all $\phi \in S(W)$ and all $g \in \widetilde{G}$

$$tr(\rho_\Pi(\phi)\Pi(g)) = \int_{\widetilde{G}'} \int_W \overline{\Theta}_{\Pi'}(g')T(g'g)(w)\phi(w)\, dw\, dg'. \tag{b}$$

It is easy to see, via the van der Corput Lemma, [S, 8.1.2, 8.2.3], and (2.21.a), that each consecutive integral in (2.22.b) is absolutely convergent, (see [P2]).

Recall that the pair G, G' of type II is in the stable range with G'-the smaller member, if $d \geq d'/2$. The proof of the following theorem is based on the construction of Li, [Li]. We leave it to the reader.

Theorem 2.23. *The conjecture (2.22) holds for pairs of type I or II in the stable range with G'-the smaller member.*

For a function $\phi \in S(W)$ let $\phi^{G'}(w) = \int_{G'}(\phi(g'w)\, dg'$, whenever this integral converges. It is clear that, under the additional assumption, $dim\ V' \leq dim\ V$, there are sequences $\phi_n \in S(W)$ such that $\phi_n^{G'}(w) \to 1$, as $n \to \infty$, almost everywhere on W. Formally, by chasing through the Weyl integration formula, we see that

$$\begin{aligned} \Theta'_{\Pi'}(g) &= \int_{\tilde{G}'} \int_{G'\backslash W} \overline{\Theta_{\Pi'}(g')} \Theta(\tilde{(-1)}) T((\tilde{(-1)})^{-1} g' g)(w)\, d\dot{w}\, dg' \\ &= \Theta(\tilde{(-1)}) \lim_{n\to\infty} tr(\rho_\Pi(\phi_n)\Pi((\tilde{(-1)})^{-1} g)). \end{aligned} \tag{2.24}$$

Thus $\Theta'_{\Pi'}$ is an invariant distribution on $\tilde{G}_1$, which can formally be approximated by generalized matrix coefficients of Π. The commons sense dictates that $\Theta'_{\Pi'}$ should, be equal to a constant multiple of $\Theta_\Pi|_{\tilde{G}_1}$. In fact the following theorem holds. The proof is standard, left to the reader.

Theorem 2.25. *Suppose Π is an irreducible admissible representation of a real reductive group E, realized on a Hilbert space $\mathcal{H}_\Pi$. Recall the map*

$$tr_\Pi : Tr(\mathcal{H}_\Pi) \to D'(E), \quad tr_\Pi(T)(\Psi) = tr(\Pi(\Psi)T) \qquad (T \in Tr(\mathcal{H}_\Pi),\ \Psi \in C_c^\infty(E)).$$

Let u be an $Ad(E)$-invariant distribution on E, such that for some sequence $T_n \in Tr(\mathcal{H}_\Pi)$, $u = \lim_{n\to\infty} tr_\Pi(T_n)$, in the topology of $D'(E)$. Then u is a constant multiple of the character Θ_Π.

We also leave to the reader the exercise of checking that the Conjecture (2.18) holds under the assumptions of the main theorems in [P1] or in [D-P2]. (Write down the character Θ_Π calculated there, in terms of integrals over various Cartan subgroups, and see that the result agrees with the $\Theta'_{\Pi'}$.) In particular, (by (2.13), (1.19) and [R3]), the following formula (conjectured by Howe) holds

$$WF(\Pi) = \tau_{\mathfrak{g}}(\tau_{\mathfrak{g}'}^{-1}(WF(\Pi'))), \tag{2.26}$$

if the pair G, G' is in the deep stable range (see [D-P2]), with G' - the smaller member, and Π' is any irreducible unitary representation of $\tilde{G}'$, which occurs in Howe's correspondence. Here $WF(\Pi) \subseteq \mathfrak{g}^*$ is the fiber of the wave front set of Θ_Π at the identity, and similarly for Π'.

Pairs of Lie algebras, of type II

3. Notation

Let V, V' be two finite dimensional vector spaces over $\mathbb{D}$. On the real vector space $W = Hom(V', V) \oplus Hom(V, V')$ define a symplectic form $\langle\ ,\ \rangle$ by

$$\langle w, w'\rangle = tr(xy') - tr(yx') \qquad (w = (x, y),\ w' = (x', y') \in W), \tag{3.1}$$

where $tr = tr_{\mathbb{D}/\mathbb{R}}$. Let $G = GL(V)$, $\mathfrak{g} = End(V)$, $G' = GL(V')$, $\mathfrak{g}' = End(V')$. We identify these Lie algebras with their duals via the bilinear form provided by the trace: $B(x, y) = 2\,tr(xy)$. Then the moment maps $\tau_{\mathfrak{g}} : W \to \mathfrak{g}$, $\tau_{\mathfrak{g}'} : W \to \mathfrak{g}'$ are given by

$$\tau_{\mathfrak{g}}(x, y) = xy,\ \ \tau_{\mathfrak{g}'}(x, y) = yx \qquad ((x, y) \in W). \tag{3.2}$$

The groups G, G' act on W by post-multiplication and pre-multiplication by the inverse, respectively. These actions preserve the symplectic form (3.1). The moment maps (3.2) intertwine these actions with the corresponding adjoint actions.

Fix a positive definite $\mathbb{D}$ - valued hermitian form on V and on V'. Then for $x \in Hom(V', V)$, we have the adjoint $x^* \in Hom(V, V')$. Similarly, for $y \in Hom(V, V')$, we have the adjoint $y^* \in Hom(V', V)$. Let $J(x, y) = (y^*, -x^*)$. Then J is a compatible positive definite complex structure on W. The resulting scalar product restricts to any subspace of W, and yields a normalization of the corresponding Lebesgue measure (so that the volume of the unit cube is 1).

4. The pair $\mathfrak{gl}_n(\mathbb{R}), \mathfrak{gl}_1(\mathbb{R})$

Here we consider the simplest case when $\mathbb{D} = \mathbb{R}$, $dim_{\mathbb{D}} V = n$ and $dim_{\mathbb{D}} V' = 1$. We identify $Hom(V', V) = V = \mathbb{R}^n$ - the space of column vectors, and $Hom(V, V') = \mathbb{R}^{n*}$ - the space of row vectors. Then the adjoint of a vector coincides with the transpose.

The scalar product on $\mathbb{R}^n \subseteq W$ yields the Hilbert Schmidt norm on $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$, which gives the usual normalization of the Lebesgue measure on $\mathfrak{g}$: $dx = dx_{11}dx_{12}...dx_{nn}$. The corresponding canonical Haar measure on $G = GL_n(\mathbb{R})$ can be expressed as follows:

$$\begin{aligned}\int_G \Psi(g)\,dg &= \int \Psi(exp(x)) \left| det\left(\frac{exp(-ad\,x) - 1}{-ad\,x}\right)\right| dx \\ &= \int_{\mathfrak{g}} \Psi(x)|det(x)|^{-n} dx,\end{aligned} \tag{4.1}$$

where $\Psi \in C_c(G)$.

Let $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ denote the area of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. Let $K \subseteq G$ be the centralizer of J. Then K coincides with the orthogonal group $O_n(\mathbb{R})$. The invariant integral on S^{n-1} can be expressed in terms of an integral over K as follows

$$\int_{S^{n-1}} \Psi(\sigma)\, d\sigma = |S^{n-1}| \int_K \Psi(ke_1)\, dk/|K|,$$

where $e_1 = (1, 0, 0, ..., 0)^t \in S^{n-1}$ and $|K|$ is the Haar measure of K. Hence,

$$\int_W \phi(w)\, dw = \frac{|S^{n-1}|}{2} \int_{G'} \int_K \int_{\mathbb{R}^{n*}} \phi(ke_1 a, a^{-1}y)\, dy\, \frac{dk}{|K|}\, da,$$

where $G' = GL_1(\mathbb{R}) = \mathbb{R}^\times$. This formula allows us to define a measure μ on $\mathfrak{g}$ which may be viewed as a "push forward" of the Lebesgue measure dw on W to $\mathfrak{g}$, via the moment map $\tau_\mathfrak{g}$:

$$\mu(\psi) = \frac{|S^{n-1}|}{2} \int_K \int_{\mathbb{R}^{n*}} \psi(ke_1 y)\, dy\, \frac{dk}{|K|} = \int_{G'\backslash W^{max}} \psi(\tau_\mathfrak{g}(w))\, d\dot{w}, \tag{4.2}$$

where $W^{max} = W_{G'} = \{(x, y);\ x \neq 0, y \neq 0\} \subseteq W$. Clearly μ is temperate. The Fourier transform of μ, defined with respect to the character χ, is given by:

$$\hat{\mu}(\psi) = \mu(\hat{\psi}) = \int_\mathfrak{g} \int_\mathfrak{g} \psi(x)\chi(-2tr(xy))\, dx\, d\mu(y) \qquad (\psi \in S(\mathfrak{g})).$$

Using the following formula for the Dirac delta, [Hö, (7.8.5)],

$$\delta(x) = \int_{\mathbb{R}^{n*}} \chi(-tr(xy))\, dy \qquad (x \in \mathbb{R}^n),$$

it is easy to calculate $\hat{\mu}$ explicitly:

$$\begin{aligned} \hat{\mu}(\psi) &= \frac{|S^{n-1}|}{2^{n+1}} \int_{M_{n,n-1}(\mathbb{R})} \int_K \psi(k(0, x)k^{-1})\, \frac{dk}{|K|}\, dx \\ &= \frac{|S^{n-1}|}{2^{n+1}} \int_{\mathfrak{gl}_{n-1}(\mathbb{R})} \int_{\mathbb{R}^{(n-1)*}} \int_K \psi\left(k \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}^{-1} k^{-1}\right) \\ &\qquad\qquad \frac{dk}{|K|}\, dy\, |det(x)|\, dx. \end{aligned} \tag{4.3}$$

Since,

$$\begin{aligned} \int_G \phi(g)\, dg = \frac{|S^{n-1}|}{2} \int_{\mathbb{R}^{(n-1)*}} \int_K \int_{GL_1(\mathbb{R})} \int_{GL_{n-1}(\mathbb{R})} \phi\left(k \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) \\ db\, da\, \frac{dk}{|K|}\, dy, \end{aligned}$$

the Weyl integration formula applied to $\mathfrak{gl}_{n-1}(\mathbb{R})$ shows that

$$2^n\hat{\mu}(\psi) = \sum \frac{1}{|W(H_{n-1})|}\int_{\mathfrak{h}_{n-1}} |\pi_{\mathfrak{h}_{n-1}}(x)|^2|det(x)| \int_{G/H}\psi\left(g\begin{bmatrix}0 & 0\\ 0 & x\end{bmatrix}g^{-1}\right)d\dot{g}\,dx, \tag{4.4}$$

where the summation is over a maximal family of mutually non-conjugate Cartan subgroups $H_{n-1} \subseteq GL_{n-1}(\mathbb{R})$, $\mathfrak{h}_{n-1}$ is the Lie algebra of H_{n-1}, $\pi_{\mathfrak{h}_{n-1}}$ is the product of positive roots of $\mathfrak{h}_{n-1}$ in $\mathfrak{gl}_{n-1}(\mathbb{C})$ and $H = GL_1(\mathbb{R}) \times H_{n-1} \subseteq G$ embedded diagonally.

The formula (4.4) suggests a more intrinsic description of$\hat{\mu}$.

The derivative det' of the determinant $det : \mathfrak{gl}_n(\mathbb{R}) \to \mathbb{R}$ coincides with the matrix of minors of size $n-1$. Hence, the set of $x \in \mathfrak{gl}_n(\mathbb{R})$ with $det'(x) \neq 0$ contains each surface $det^{-1}(s)$, $s \in \mathbb{R}^\times$. Therefore $\delta(det(x)-s)$ is a well defined distribution on $\mathfrak{gl}_n(\mathbb{R})$, see [Hö, 6.1.2]. By the Weyl integration formula

$$\begin{aligned}&\int_{\mathfrak{g}}\psi(x)\delta(det(x)-s)\,dx\\ &= \sum\frac{1}{|W(H)|}\int_{\mathfrak{h}}|\pi_{\mathfrak{h}}(x)|^2\delta(det(x)-s)\int_{G/H}\psi(gxg^{-1})\,d\dot{g}\,dx\\ &= \sum\frac{1}{|W(H)|}\int_{\mathfrak{h}\cap det^{-1}(s)}\frac{|\pi_{\mathfrak{h}}(x)|^2}{|det'(x)|}\int_{G/H}\psi(gxg^{-1})\,d\dot{g}\,d\mu_s(x),\\ &\qquad\qquad (\psi \in S(\mathfrak{g}))\end{aligned} \tag{4.5}$$

where μ_s is the Euclidean surface measure on the indicated surface.

For $x \in \mathfrak{h}$, let $x_1, x_2, ..., x_n$ be the eigenvalues of x, and let $\mathfrak{S}_n$ be the permutation group on n letters. We may assume that

$$\pi_{\mathfrak{h}}(x) = \pi_n(x) = \prod_{1\le i<j\le n}(x_i - x_j) = \sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)x_1^{\sigma(n-1)}x_2^{\sigma(n-2)}...x_n^{\sigma(0)}.$$

Hence, $\frac{|\pi_n(x)|}{|det'(x)|} \le$ polynomial(x), $x \in \mathfrak{h}$. Furthermore, if $x = \begin{bmatrix}0 & 0\\ 0 & y\end{bmatrix}$, where $y \in \mathfrak{gl}_{n-1}(\mathbb{R})$ has eigenvalues $x_2, x_3, ..., x_n$, then $\frac{|\pi_n(x)|^2}{|det'(x)|} = |\pi_{n-1}(y)|^2|det(y)|$. Therefore Harish-Chandra's estimate for orbital integrals, [W1, 7.3.8] and a calculation for $n = 2$, show that the limit if $s \to 0$ of the expression (4.5) exists:

$$(4.6)\qquad \int_{\mathfrak{g}} \psi(x)\delta(det(x))\,dx := \lim_{s\to 0}\int_{\mathfrak{g}} \psi(x)\delta(det(x)-s)\,dx$$

$$= \sum \frac{1}{|W(H)|}\int_{\mathfrak{h}\cap det^{-1}(0)} |\pi_n(x)|^2 |det'(x)|^{-1}\int_{G/H}\psi(gxg^{-1})\,d\dot{g}\,d\mu_0(x)$$

$$= \sum \frac{1}{|W(H_{n-1})|}\int_{\mathfrak{h}_{n-1}} |\pi_{n-1}(x)|^2 |det(x)|\int_{G/H}\psi\left(g\begin{bmatrix}0 & 0\\ 0 & x\end{bmatrix}g^{-1}\right)\,d\dot{g}\,dx,$$

where $\psi \in S(\mathfrak{g})$. Thus,

$$(4.7)\qquad 2^n\hat{\mu} = \delta\circ\det,$$

where the right hand side is defined in (4.6).

We shall calculate the wave front set of μ and of $\hat{\mu}$. Since μ is a homogeneous distribution, [Hö, 8.1.8] implies that the fiber of $WF(\mu)$ over $0 \in \mathfrak{g}$ coincides with $supp\,\hat{\mu} = det^{-1}(0) = \mathfrak{g}_{rk\le n-1}$, the set of elements of rank less or equal to $n-1$. The complement of $\{0\}$ in the support of μ is the set of elements of rank one. This set is a single $G\times G$-orbit, under the left-right action. Hence a point $(x,y)\in(supp\,\mu\setminus\{0\})\times\mathfrak{g}$ belongs to $WF(\mu)$ if and only if y is perpendicular to the orbit of x, (see [Hö, 8.2.5]), i.e. $xy = yx = 0$. Hence,

$$(4.8)\qquad WF(\mu) = \{(x,y)\in\mathfrak{g}_{rk\le 1}\times\mathfrak{g}_{rk\le n-1};\ xy = yx = 0\}.$$

It is clear from (4.8) and from [Hö, 8.2.5] that

$$(4.9)\qquad \begin{aligned} WF(\hat{\mu}) &= \{(x,y)\in\mathfrak{g}\times\mathfrak{g}_{rk\le 1};\ xy = yx = 0\}\\ &= \{(x,\tau_{\mathfrak{g}}(w));\ x(w) = 0,\ x\in\mathfrak{g},\ w\in W\}.\end{aligned}$$

Notice, by the way, that the convergence (4.6) does not happen in the space $D'_\Gamma(\mathfrak{g})$, defined in [Hö, 8.2.2], with $\Gamma = WF(\hat{\mu})$. Indeed, by [Hö, 8.2.5], $WF(\delta\circ det - s) = T^*{}_{det^{-1}(s)}(\mathfrak{g})$, the conormal bundle of the surface $det^{-1}(s)$. Hence

$$(s^{1/n}I,\lambda I)\in WF(\delta\circ det - s)\qquad (s>0,\ \lambda\neq 0),$$

where $I = I_n$ is the identity matrix. But $\lim_{s\to 0}(s^{1/n}I,\lambda I) = (0,\lambda I)\notin\Gamma$, a contradiction.

Finally, the Cauchy Harish-Chandra Integral can be calculated as follows:

$$\begin{aligned}
chc(\psi) &= \int_{G'\backslash W^{max}} \int_{\mathfrak{g}} \psi(x)\chi_x(w)\,dx\,d\dot{w} \\
&= \int_{G'\backslash W^{max}} \int_{\mathfrak{g}} \psi(x)\chi\left(\frac{1}{4}\tau_{\mathfrak{g}}(w)(x)\right)\,dx\,d\dot{w} \\
&= \int_{G'\backslash W^{max}} \int_{\mathfrak{g}} \psi(x)\chi\left(\frac{1}{2}tr(x\tau_{\mathfrak{g}}(w))\right)\,dx\,d\dot{w} \\
&= \int_{\mathfrak{g}} \int_{\mathfrak{g}} \psi(x)\chi\left(\frac{1}{2}tr(xy)\right)\,dx\,d\mu(y) \\
&= \int_{\mathfrak{g}} \int_{\mathfrak{g}} \psi(x)\chi\left(-\frac{1}{2}tr(xy)\right)\,dx\,d\mu(y) \\
&= 2^n\hat{\mu}(\psi).
\end{aligned}$$

Thus,

$$\begin{gathered}
chc = \delta \circ det, \text{ and} \\
WF(chc) = \{(x, \tau_{\mathfrak{g}}(w));\ x(w) = 0,\ x \in \mathfrak{g},\ w \in W\}.
\end{gathered} \tag{4.10}$$

Lemma 4.11. *Let $\psi \in S(\mathfrak{g})$. Then there is a unique continuous function $\tilde{\psi}$ on $\tau_{\mathfrak{g}}(W^{max})$, such that*

(a) $$\int_W \psi \circ \tau_{\mathfrak{g}}(w)\chi\left(\frac{1}{2}\langle w, w'\rangle\right)dw = \tilde{\psi} \circ \tau_{\mathfrak{g}}(w') \qquad (w' \in W^{max}).$$

Moreover,

(b) $$\int_{G'\backslash W^{max}} |\tilde{\psi} \circ \tau_{\mathfrak{g}}(w)|\,d\dot{w} < \infty.$$

Proof. Recall [A-S, 9.1.23, 9.6.21] the following Bessel functions

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty cos(x\, cosh\, t)\,dt, \quad K_0(x) = \int_0^\infty cos(x\, sinh\, t)\,dt \ (x > 0).$$

Let

$$F(x) = \begin{cases} 2K_0(2\pi x^{1/2}) & \text{for } x > 0, \\ -\pi Y_0(2\pi|x|^{1/2}) & \text{for } x < 0. \end{cases}$$

Then, by [A-S, 9.1.23, 9.6.21],

$$F(x) = \int_0^\infty 2cos(\pi(a^{-1} - a\,x))\,da/a \qquad (x \in \mathbb{R}^\times).$$

Let $X = Hom(V', V)$, and let $Y = Hom(V, V')$. For an appropriately normalized element $x_0 \in X \setminus \{0\}$, the left hand side of (4.11.a), viewed as a generalized function of $w' = (x', y') \in W^{max}$, is equal to

$$\begin{aligned}
&\int_Y \int_X \psi(xy)\chi\left(\frac{1}{2}(y'x - yx')\right) dx\, dy \\
&= \int_{G'} \int_K \int_Y \psi(kx_0y)\chi\left(\frac{1}{2}(a\, y'kx_0 - a^{-1}\, yx')\right) dy\, dk\, da \\
&= \int_K \int_Y \psi(kx_0y) \int_G \chi\left(\frac{1}{2}(a^{-1} - a\, y'kx_0\, yx')\right) da\, dy\, dk \\
&= \int_K \int_Y \psi(kx_0y) F(y'kx_0\, yx')\, dy\, dk = \tilde{\psi} \circ \tau_{\mathfrak{g}}(w'),
\end{aligned}$$

where

$$\tilde{\psi}(x') = \int_{\mathfrak{g}} \psi(x) F(tr(xx'))\, d\mu(x) \quad (x' \in \mathfrak{g}). \tag{4.12}$$

We shall check that the integral (4.12) is absolutely convergent, which suffices for (4.11.a), and that the following estimate holds:

$$|\tilde{\psi}(x)| \leq const\, (1 + |log|x||) \qquad (x \in \mathfrak{g}). \tag{4.13}$$

Indeed, by [A-S, 9.1.8, 9.2.2, 9.6.8, 9.7.2],

$$|F(x)| \leq const\, (1 + |log|x||) \qquad (x \in \mathbb{R}).$$

Hence

$$\begin{aligned}
&\int |\psi(x) F(tr(xx'))|\, d\mu(x) \leq const \int |\psi(x)(1 + |log|tr(xx')||)\, d\mu(x) \\
&= const(\mu(|\psi|) + \int |\psi(x) log|tr(xx')||\, d\mu(x)) \\
&= const(\mu(|\psi|) + \int |\psi(x) log|tr(x\frac{x'}{|x'|})||\, d\mu(x) + \int |\psi(x) log|x'||\, d\mu(x)) \\
&\leq const((1 + |log|x'||)\mu(|\psi|) + \int |\psi(x) log|tr(x\frac{x'}{|x'|})||\, d\mu(x))
\end{aligned}$$

Hence, it will suffice to see that the last integral is bounded independently of $x' \neq 0$. But, with $x_0 \in X \setminus \{0\}$, as in the calculation preceding (4.12), and with an appropriate $y_0 \in Y \setminus \{0\}$,

$$\int |\psi(x) log|tr(x\frac{x'}{|x'|})|| \, d\mu(x) = \int_Y \int_{K'} |\psi(kx_0y) log|y\frac{x'}{|x'|}kx_0|| \, dk \, dy$$
$$= \int_0^\infty \int_K \int_K |\psi(kx_0y_0lr)||log|y_0l\frac{x'}{|x'|}kx_0r||r^{n-1} \, dk \, dl \, dr$$
$$\leq const_{N,\psi} \int_0^\infty \int_K \int_K (1+r)^{-N}(|log|y_0l\frac{x'}{|x'|}kx_0|| + |log(r)|)r^{n-1} \, dk \, dl \, dr$$
$$\leq const_{N,\psi}(1 + \int_K \int_K |log|y_0l\frac{x'}{|x'|}kx_0|| \, dk \, dl),$$

and it is easy to see, using polar coordinates, that the last integral is finite, and bounded independently of $x' \neq 0$. Hence the absolute convergence of (4.12) and the estimate (4.13) follow.

It is easy to see that for any polynomial function $P(x')$, $x' \in \mathfrak{g}$, there is a polynomial coefficient differential operator D_P on $\mathfrak{g}$ such that

$$P(x)\tilde{\psi}(x) = (D_P\psi\tilde{)}(x) \qquad (x \in \tau_{\mathfrak{g}}(W^{max})). \tag{4.14}$$

Clearly (4.13) and (4.14) imply

$$|\tilde{\psi}(x)| \leq const_{N,\psi}(1 + |log|x||)(1 + |x|)^{-N} \quad (x \in \tau_{\mathfrak{g}}(W^{max}),\ \psi \in S(\mathfrak{g})).$$

Notice that for N large enough

$$\int (1 + |log|x||)(1 + |x|)^{-N} \, d\mu(x)$$
$$= \int_Y \int_K (1 + |log|kx_0y||)(1 + |kx_0y|)^{-N} \, dk \, dy$$
$$= \int_Y (1 + |log|y||)(1 + |y|)^{-N} \, dy$$
$$= const \int_0^\infty (1 + |log(r)|)(1 + r)^{-N} r^{n-1} \, dr < \infty.$$

Hence (4.11.b) follows, and we are done. □

5. The pair $\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_1(\mathbb{C})$

Let us view the complex numbers as matrices of size 2, with real entries: $z = x + iy = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$. This leads to an embedding

$$\mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_{2n}(\mathbb{R}). \tag{5.1}$$

The orthogonal complement of $\mathfrak{gl}_n(\mathbb{C})$ in $\mathfrak{gl}_{2n}(\mathbb{R})$ consists of matrices, with blocks of size 2 of the form $\begin{bmatrix} -x & y \\ y & x \end{bmatrix}$, $x, y \in \mathbb{R}$. In particular the rank of any such matrix is even. Hence, the intersection of the wave front set of the distribution $\delta \circ det \in S^*(\mathfrak{gl}_{2n}(\mathbb{R}))$ with the conormal bundle to the embedding (5.1), is empty. Therefore the restriction $\delta \circ det|_{\mathfrak{gl}_n(\mathbb{C})} \in D'(\mathfrak{gl}_n(\mathbb{C}))$ exists. We shall give an explicit formula for this distribution.

Since the unitary group $K = U_n$ acts transitively on the unit sphere in $\mathbb{C}^n$, the measure μ, (4.2), can be written as follows:

$$\mu(\psi) = \frac{|S^{2n-1}|}{2} \int_K \int_{\mathbb{R}^{2n*}} \psi(ke_1 y)\, dy\, \frac{dk}{|K|}. \tag{5.2}$$

Hence, the calculation leading to (4.3) shows that

$$\begin{aligned} &\hat{\mu}(\psi) = \int_{M_{2n,2n-1}(\mathbb{R})} \tilde{\psi}(0, x)\, dx, \ \text{where} \\ &\tilde{\psi}(y) = \frac{|S^{2n-1}|}{2^{2n+1}} \int_K \psi(kyk^{-1})\, \frac{dk}{|K|} \quad (\psi \in S(\mathfrak{gl}_{2n}(\mathbb{R})), y \in \mathfrak{gl}_{2n}(\mathbb{R})). \end{aligned} \tag{5.3}$$

Lemma 5.4. *For $\psi \in S(\mathfrak{gl}_n(\mathbb{C}))$,*

$$\delta \circ det|_{\mathfrak{gl}_n(\mathbb{C})}(\psi) = \int_{M_{n,n-1}(\mathbb{C})} \tilde{\psi}(0, x)\, dx, \ \textit{where}$$

$$\tilde{\psi}(y) = \frac{|S^{2n-1}|}{2^{2n+1}} \int_K \psi(kyk^{-1})\, \frac{dk}{|K|} \quad (y \in \mathfrak{gl}_n(\mathbb{C})).$$

Proof. Let $\tilde{\mathbb{C}}$ be the space of matrices of the form $\tilde{z} = \begin{bmatrix} -x & y \\ y & x \end{bmatrix}$, $x, y \in \mathbb{R}$. In terms of the coordinates x, y set $d\tilde{z} = dxdy$. Similarly for $z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in \mathbb{C}$, let $dz = dxdy$. Thus

$$\int_{\mathfrak{gl}_2(\mathbb{R})} \psi(x)\, dx = 4 \int_{\mathbb{C}} \int_{\tilde{\mathbb{C}}} \psi(z + \tilde{z})\, d\tilde{z}\, dz, \tag{5.5}$$

and we use the analogous notation for the integrals over spaces of block matrices with the blocks in $\tilde{\mathbb{C}}$ or in $\mathbb{C}$. Let $f, \phi \in C_c^\infty(\mathfrak{gl}_n(\mathbb{R}))$ be two $Ad(K)$-invariant functions, and let $\psi \in C_c^\infty(\mathfrak{gl}_n(\mathbb{C}))$. Then

$$\begin{aligned} &\int_{\mathfrak{gl}_n(\mathbb{C})} \phi(z) * (\delta \circ det \cdot f)\psi(z)\, dz \\ &= \int_{\mathfrak{gl}_n(\mathbb{C})} \int_{\mathfrak{gl}_{2n}(\mathbb{R})} \phi(z - x)\delta(det(x))\, f(x)\psi(z)\, dx\, dz \\ &= \int_{\mathfrak{gl}_n(\mathbb{C})} \int_{M_{2n,2n-1}(\mathbb{R})} \phi(z - (0, x))\, f(0, x)\tilde{\psi}(z)\, dx\, dz, \end{aligned} \tag{5.6}$$

where $\tilde{\psi}$ is defined as in (5.3). The map

$$\tilde{\mathbb{C}} \ni \tilde{z} = \begin{bmatrix} -x & y \\ y & x \end{bmatrix} \to \tilde{\tilde{z}} = z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in \mathbb{C}$$

extends to a map $M_{n,1}(\tilde{\mathbb{C}}) \ni \tilde{z} \to \tilde{\tilde{z}} \in M_{n,1}(\mathbb{C})$. Thus, by (5.5), the last expression in (5.6) may be rewritten as follows:

$$\begin{aligned} &\int_{\mathfrak{gl}_n(\mathbb{C})} \int_{M_{n,1}(\tilde{\mathbb{C}})} \int_{M_{n,n-1}(\tilde{\mathbb{C}})} \int_{M_{n,n-1}(\mathbb{C})} 4^{n^2} \phi(z - (\tilde{\tilde{x}} + \tilde{x}, y + \tilde{y}) \\ &\qquad\qquad f(\tilde{\tilde{x}} + \tilde{x}, y + \tilde{y}) \psi(z)\, dy\, d\tilde{y}\, d\tilde{x}\, dz \\ &= \int_{\mathfrak{gl}_n(\mathbb{C})} \int_{M_{n,1}(\tilde{\mathbb{C}})} \int_{M_{n,n-1}(\tilde{\mathbb{C}})} \int_{M_{n,n-1}(\mathbb{C})} 4^{n^2} \phi(z - (\tilde{x}, \tilde{y}) \\ &\qquad\qquad f(\tilde{\tilde{x}} + \tilde{x}, y + \tilde{y}) \psi(z + (\tilde{\tilde{x}}, y))\, dy\, d\tilde{y}\, d\tilde{x}\, dz. \end{aligned} \tag{5.7}$$

Notice that if $\int_{\mathfrak{gl}_{2n}(\mathbb{R})} \phi(x)\, dx = 1$, then, by (5.5),

$$\int_{\mathfrak{gl}_n(\mathbb{C})} \int_{M_{n,1}(\tilde{\mathbb{C}})} \int_{M_{n,n-1}(\tilde{\mathbb{C}})} 4^{n^2} \phi(z - (\tilde{x}, \tilde{y}))\, d\tilde{y}\, d\tilde{x}\, dz = 1.$$

Therefore,

$$\begin{aligned} &\int_{\mathfrak{gl}_n(\mathbb{C})} \phi(z) * (\delta \circ det \cdot f) \psi(z)\, dz - \int_{M_{n,n-1}(\mathbb{C})} \tilde{\psi}(0, y)\, dy \\ &= \int_{\mathfrak{gl}_n(\mathbb{C})} \int_{M_{n,1}(\tilde{\mathbb{C}})} \int_{M_{n,n-1}(\tilde{\mathbb{C}})} \int_{M_{n,n-1}(\mathbb{C})} 4^{n^2} \phi(z - (\tilde{x}, \tilde{y})) \\ &\qquad (f(\tilde{\tilde{x}} + \tilde{x}, y + \tilde{y}) \psi(z + (\tilde{\tilde{x}}, y)) - \tilde{\psi}(0, y))\, dy\, d\tilde{y}\, d\tilde{x}\, dz. \end{aligned} \tag{5.8}$$

Assume, in addition, that $\psi \geq 0$. Then (5.8) can be dominated by

$$\int_{M_{n,n-1}(\mathbb{C})} \sup_{\{z-(\tilde{x},\tilde{y}) \in supp\, \phi\}} | f(\tilde{\tilde{x}} + \tilde{x}, y + \tilde{y}) \psi(z + (\tilde{\tilde{x}}, y)) - \tilde{\psi}(0, y)|\, dy. \tag{5.9}$$

The quantity (5.9) tends to zero if the function f tends to 1 uniformly, and the support of ϕ shrinks to zero. Thus [Hö, 8.2.4] completes the proof. □

Let $\delta \in D'(\mathbb{C})$ denote the Dirac delta at 0. Then, as in the previous section, we check that for the complex determinant $det : \mathfrak{gl}_h(\mathbb{C}) \to \mathbb{C}$ and any $s \in \mathbb{C}^\times$ the distribution $\delta(det(z) - s) \in S^*(\mathfrak{gl}_n(\mathbb{C}))$ is well defined and that

$$\lim_{s \to 0} \int_{\mathfrak{gl}_n(\mathbb{C})} \delta(det(z) - s) \psi(z)\, dz = \delta \circ det|_{\mathfrak{gl}_n(\mathbb{C})}(\psi), \tag{5.10}$$

where the right hand side is defined in (4). For this reason we shall denote the distribution (5.10) also by $\delta \circ det$, remembering that $\delta \in \mathcal{D}'(\mathbb{C})$ is the Dirac delta at 0.

As in the previous section we check that

$$\begin{aligned} &chc = \delta \circ det \text{ and} \\ &WF(chc) = \{(x, \tau_{\mathfrak{g}}(w));\ x(w) = 0,\ x \in \mathfrak{g},\ w \in W\}, \end{aligned} \tag{5.11}$$

where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

6. The pair $\mathfrak{gl}_n(\mathbb{H}), \mathfrak{gl}_1(\mathbb{H})$

We view the quaternions as matrices of size 2, with complex entries: $z = \begin{bmatrix} x & y \\ -\overline{y} & \overline{x} \end{bmatrix}$, $x, y \in \mathbb{C}$. This leads to an embedding

$$\mathfrak{gl}_n(\mathbb{H}) \to \mathfrak{gl}_{2n}(\mathbb{C}). \tag{6.1}$$

The orthogonal complement of $\mathfrak{gl}_n(\mathbb{H})$ in $\mathfrak{gl}_{2n}(\mathbb{C})$ consists of matrices, with blocks of size 2 of the form $\begin{bmatrix} -\overline{x} & \overline{y} \\ y & x \end{bmatrix}$, $x, y \in \mathbb{C}$. In particular the (complex) rank of any such matrix is even. Hence, the intersection of the wave front set of the distribution $\delta \circ det \in S^*(\mathfrak{gl}_{2n}(\mathbb{C}))$, with the conormal bundle to the embedding (6.1), is empty. Therefore the restriction $\delta \circ det|_{\mathfrak{gl}_n(\mathbb{H})} \in D'(\mathfrak{gl}_n(\mathbb{H}))$ exists.

Let $K = U_n(\mathbb{H}) = Sp_n$. The argument used to prove (5.4) verifies the following lemma.

Lemma 6.2. *For $\psi \in S(\mathfrak{gl}_n(\mathbb{H}))$,*

$$\begin{aligned} &\delta \circ det|_{\mathfrak{gl}_n(\mathbb{H})}(\psi) = \int_{M_{n,n-1}(\mathbb{H})} \tilde{\psi}(0, x)\, dx, \ \textit{where} \\ &\tilde{\psi}(y) = \frac{|S^{4n-1}|}{2^{4n+1}} \int_K \psi(kyk^{-1})\, \frac{dk}{|K|} \quad (y \in \mathfrak{gl}_n(\mathbb{H})). \end{aligned}$$

Similarly,

$$\begin{aligned} &chc = \delta \circ det|_{\mathfrak{g}} \text{ and} \\ &WF(chc) = \{(x, \tau_{\mathfrak{g}}(w));\ x(w) = 0,\ x \in \mathfrak{g},\ w \in W\}, \end{aligned} \tag{6.3}$$

where $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{H})$.

7. A general pair $(\mathfrak{g}, \mathfrak{g}')$ of type II

Here we resume the notation of Sect. 3. Let $H' = T'A' \subseteq G'$ be a Cartan subgroup, as in Sect. 1. Let

$$V' = V_1' \oplus V_2' \oplus \ldots \tag{7.1}$$

be the decomposition of V' into A' - isotypic components. Equivalently, (7.1) is a decomposition into H' isotypic $\mathbb{D}$ subspaces. The symplectic space W decomposes into a direct sum of mutually orthogonal subspaces:

$$W = \bigoplus_j W_j, \quad W_j = Hom(V_j', V) \oplus Hom(V, V_j'). \tag{7.2}$$

The group A'' (= centralizer of A' in Sp) preserves the decomposition (2) and the obvious restrictions yield the following isomorphisms:

$$\begin{aligned} \mathfrak{a}'' &= End_{\mathbb{R}}(Hom(V_1', V)) \oplus End_{\mathbb{R}}(Hom(V_2', V)) \oplus \ldots, \\ A'' &= GL_{\mathbb{R}}(Hom(V_1', V)) \times GL_{\mathbb{R}}(Hom(V_2', V)) \times \ldots . \end{aligned} \tag{7.3}$$

Let A''' be the centralizer of A'' in Sp. Then for any V_j' as in (7.1) the restriction of A''' to V_j' is isomorphic to $GL_1(\mathbb{R})$. Thus, as a reductive dual pair, (A'', A''') is isomorphic to

$$(GL_{n_1}(\mathbb{R}),\ GL_1(\mathbb{R})) \times (GL_{n_2}(\mathbb{R}),\ GL_1(\mathbb{R})) \times \ldots , \tag{7.4}$$

where $n_j = dim_{\mathbb{R}}\, Hom(V_j', V)$, $j = 1, 2, \ldots$. In terms of (7.2) let $W_{j\,A_j'''} = \{(x, y) \in W_j;\ x \neq 0, y \neq 0\}$. Let $W_{A'''} = W_{1\,A_1'''} \times W_{2\,A_2'''} \times \ldots \subseteq W$. Define a measure $d\dot{w}$ on the quotient manifold $A'''\backslash W_{A'''}$ as in (1.3). Then, as a temperate distribution on $\mathfrak{a}''$,

$$\begin{aligned} chc(x) &= \int_{A'''\backslash W_{A'''}} \chi_x(w)\, d\dot{w} \\ &= \int_{A_1'''\backslash W_{A_1'''}} \chi_{x_1}(w_1)\, d\dot{w}_1 \otimes \int_{A_1'''\backslash W_{A_2'''}} \chi_{x_2}(w_2)\, d\dot{w}_2 \otimes \ldots \end{aligned} \tag{7.5}$$

where $x \in \mathfrak{a}''$ and x_j is the restriction of x to $Hom(V_j', V)$. The Lemma 1.7, for pairs of type II, follows easily from (7.5), (4.10), (5.11) and (6.3).

Proof of Proposition 1.8. Suppose $s_j \in End_{\mathbb{R}}(Hom(V_j', V))$ is of rank one. Then there are non-zero elements $x_j \in Hom(V_j', V)$ and $y_j \in Hom(V, V_j')$ such that

$$s_i(u) = tr(uy_j)x_j \qquad (u \in Hom(V_j', V)),$$

where $tr = tr_{\mathbb{D}/\mathbb{R}}$. In other words, in terms of (7.2) and (7.3), $s_1 + s_2 + \ldots = \tau_{\mathfrak{a}''}(w)$ for $w = (x_1, y_1; x_2, y_2; \ldots)$. Suppose $s_1+s_2+\ldots \in \mathfrak{a}''$ is perpendicular to $\mathfrak{g}$. Then by (1.7) and (3.2)

$$x_1y_1 + x_2y_2 + \ldots = 0. \tag{7.6}$$

Recall the regular element $x' \in \mathfrak{h}'^r$. Suppose $x \in \mathfrak{g}$ is such that $(x'+x, s_1 + s_2 + ...)$ is in the wave front set of the distribution (7.5). Then, by (4.9), we have

$$xx_j = x_j x', \text{ and } y_j x = x' y_j. \tag{7.7}$$

By combining (7.6) and (7.7) we see that

$$x_1 x'^k y_1 + x_2 x'^k y_2 + ... = 0 \qquad (k = 0, 1, 2, ...). \tag{7.8}$$

Since the powers of x', $(x'^1, x'^2, ...)$, span $\mathfrak{h}'$ over the center of $\mathbb{D}$, we see that (8) holds with the x'^k replaced by an arbitrary element of $\mathfrak{h}'$. Hence, for all j,

$$x_j y_j = 0. \tag{7.9}$$

If $dim_{\mathbb{D}} V'_j = 1$ then (7.9) implies that $x_j = 0$ or $y_j = 0$, a contradiction. The remaining case is $\mathbb{D} = \mathbb{R}$ and $dim_{\mathbb{D}} V'_j = 2$. Then (7.9) implies that the image of y_j has dimension 1. On the other hand the second equation in (7.7) shows that the image of y_j is preserved by x'. Since x' is regular, this is a contradiction. □

Fix $x' \in \mathfrak{h}'^r$. The distribution $chc(x'+x)$, $x \in \mathfrak{g}$, is $Ad(G)$-invariant. Therefore it has a well defined restriction to $\mathfrak{h}^r$, for any Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. We denote this restriction by the same symbol $chc(x'+x)$, $x \in \mathfrak{h}^r$. Clearly, this is a non-negative measure invariant under the action of the Weyl group $W(H)$.

Lemma 7.10. *The distribution $chc(x'+x)$, $x \in \mathfrak{g}$, is regular, in the sense that for any $\psi \in S(\mathfrak{g})$,*

$$\begin{aligned} &\int_{\mathfrak{g}} \psi(x) chc(x'+x)\, dx \\ &= \sum \frac{1}{|W(H)|} \int_{\mathfrak{h}^r} chc(x'+x)\, |\pi_{\mathfrak{h}}(x)|^2 \int_{G/H} \psi(ghg^{-1})\, d\dot{g}\, dx, \end{aligned}$$

where the integrals are absolutely convergent, and the summation is over a maximal family of mutually non-conjugate Cartan subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$.

Proof. By Harish-Chandra's Method of Descent, [Va, part I], it will suffice to consider the distribution $chc_{x'}(x) = chc(x'+x)$, $x \in \mathfrak{g}$, in an arbitrarily small, completely invariant open neighborhood of a point in the support of $chc_{x'}$. We shall need some additional notation.

Let $\mathfrak{h}''$ be the centralizer of $\mathfrak{h}'$ in $sp(W)$. Clearly $\mathfrak{h}'' \subseteq \mathfrak{a}''$ and, in terms of (3),

$$\mathfrak{h}'' = End_{\mathfrak{h}'}(Hom(V'_1, V)) \oplus End_{\mathfrak{h}'}(Hom(V'_2, V)) \oplus ... \,. \tag{7.11}$$

From the classification of Cartan subalgebras in $\mathfrak{g}'$ we know that for each j, $\mathfrak{h}'|_{V_j'} \subseteq End_{\mathbb{R}}(V_j')$ is a field isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$. Hence the embedding

$$End_{\mathfrak{h}'}(Hom(V_j', V)) \subseteq End_{\mathbb{R}}(Hom(V_j', V))$$

is either an equation, or is of the form

$$End_{\mathbb{C}}(\mathbb{C}^m) \subseteq End_{\mathbb{R}}(\mathbb{C}^m).$$

Hence, by (5.4) and (5.11), the restriction of chc from $\mathfrak{a}'$ to $\mathfrak{h}''$ exists and is given by

$$chc(x) = \delta \circ det(x_1) \otimes \delta \circ det(x_2) \otimes ..., \tag{7.12}$$

where $x \in \mathfrak{h}''$, x_j is the restriction of x to $Hom(V_j', V)$, and $\delta \circ det$ is as in (4.6) or (5.10), depending on the field $\mathfrak{h}'|_{V_j'}$. Furthermore, it is clear that the distribution $chc_{x'} \in S^*(\mathfrak{g})$ coincides with the pullback of the distribution (7.12) from $\mathfrak{a}''$ to $\mathfrak{g}$ via the embedding

$$\mathfrak{g} \ni x \to x' + x \in \mathfrak{h}''. \tag{7.13}$$

Let $\tilde{x} \in \mathfrak{g}$, and let $\tilde{x} = \tilde{x}_s + \tilde{x}_n$ be the Jordan decomposition of $\tilde{x}$. Let

$$V = V_1 \oplus V_2 \oplus ... \tag{7.14}$$

be the decomposition of V into subspaces over $\mathbb{D}$, which are isotypic as $\mathbb{R}[\tilde{x}_s]$-modules. Then the set of eigenvalues of $\tilde{x}_s|_{V_k}$ has empty intersection with the set of eigenvalues of $\tilde{x}_s|_{V_l}$, for $k \neq l$.

For each j, k, $Hom(V_j', V_k)$ is a vector space over the field $\mathfrak{h}'|_{V_j'}$, and an $\mathbb{R}[\tilde{x}_s]$-module. As such, it is either irreducible, or it is a sum of two irreducible pieces. The second possibility occurs if and only if $\mathbb{D} \neq \mathbb{C}$ and both, $\mathfrak{h}'|_{V_j'}$ and $\mathbb{R}[\tilde{x}_s|_{V_k'}]$, are isomorphic to $\mathbb{C}$. In any case we shall write

$$Hom(V_j', V_k) = Hom(V_j', V_k)_1 \oplus Hom(V_j', V_k)_2,$$

keeping in mind the possibility that the second summand could be zero.

Let $\mathfrak{g}^{\tilde{x}_s}$ denote the centralizer of $\tilde{x}_s$ in $\mathfrak{g}$, and similarly, $\mathfrak{h}''^{\tilde{x}_s} \subseteq \mathfrak{h}''$. Then, in terms of (7.11),

$$\mathfrak{g}^{\tilde{x}_s} = End_{\tilde{x}_s}(V_1) \oplus End_{\tilde{x}_s}(V_2) \oplus ...,$$
$$\mathfrak{h}''^{\tilde{x}_s} = \sum_{j,k} \left(End_{\mathfrak{h}'}(Hom(V_j', V_k)_1) \oplus End_{\mathfrak{h}'}(Hom(V_j', V_k)_2)\right),$$

and (7.13) restricts to an embedding

$$\mathfrak{g}^{\tilde{x}_s} \ni x \to x' + x \in \mathfrak{h}''^{\tilde{x}_s}. \tag{7.15}$$

Suppose, from now on, that $\tilde{x} \in supp(chc_{x'})$. Then $det(x'|_{V'_j} + \tilde{x}_s) = 0$ for each j, where $x'|_{V'_j} + \tilde{x}_s$ is viewed as an element of $End_{\mathbb{R}}(Hom(V'_j, V))$. Let $\tilde{x}_{s,j,k,l}$ denote the restriction of $\tilde{x}_s$ to $Hom(V'_j, V_k)_l$. Then

$$det(x'|_{V'_j} + \tilde{x}_s) = \prod_{k,l} det(x'|_{V'_j} + \tilde{x}_{s,j,k,l}).$$

Hence, we may arrange the indices so that

$$det(x'|_{V'_j} + \tilde{x}_{s,j,j,1}) = 0,$$

$$\text{and } det(x'|_{V'_j} + \tilde{x}_{s,j,k,l}) \neq 0 \text{ for } j \neq k, \text{ or } j = k \text{ and } l \neq 1. \tag{7.16}$$

Moreover, it is clear that

$$chc_{x'} \neq 0, \text{ implies } dim\, V' \leq dim\, V. \tag{7.17}$$

Let $U_{\mathfrak{g}} \subseteq \mathfrak{g}^{\tilde{x}_s}$ be a completely invariant open neighborhood of $\tilde{x}_s$, so small that the second line in (7.16) holds with the $\tilde{x}_s$ replaced by any element $x \in U_{\mathfrak{g}}$. We may, and shall, assume that $U_{\mathfrak{g}}$ is contained in the set of regular elements of $\mathfrak{g}^{\tilde{x}_s}$, and that $Ad(G)U_{\mathfrak{g}}$ is a completely invariant open neighborhood of $\tilde{x}_s$ in $\mathfrak{g}$, as in [Va, part I, p. 16]. Then $\tilde{x} \in Ad(G)U_{\mathfrak{g}}$. Similarly, let $U_{\mathfrak{h}''} \subseteq \mathfrak{h}''^{\tilde{x}_s}$ be a completely invariant open neighborhood of $x' + \tilde{x}_s$, such that $Ad(H'')U_{\mathfrak{h}''}$ is a completely invariant open neighborhood of $x' + \tilde{x}_s$ in $\mathfrak{h}''$. We take the $U_{\mathfrak{h}''}$ small enough, so that for each $x \in U_{\mathfrak{h}''}$ and for each j, at most one of the determinants $det(x_{j,k,l})$ is zero. Then the restriction of chc to $U_{\mathfrak{h}''}$ may be written as follows.

$$\begin{aligned} chc(x) &= \prod_j \delta(\prod_{k,l} det(x_{j,k,l})) \\ &= \prod_j (\sum_{k,l} \delta(det(x_{j,k,l})) \prod_{(k',l')\neq(k,l)} |det_{\mathbb{R}}(x_{j,k',l'})|^{-1}). \end{aligned} \tag{7.18}$$

Hence, the pullback of the distribution (7.18) via (7.15) is given by

$$\begin{aligned} chc(x' + x) = &\,\delta \circ det(x'|_{V'_1} + x_{1,1,1}) \prod_{(k,l)\neq(1,1)} |det_{\mathbb{R}}(x'|_{V'_1} + x_{1,k,l})|^{-1} \\ &\delta \circ det(x'|_{V'_2} + x_{2,2,1}) \prod_{(k,l)\neq(2,1)} |det_{\mathbb{R}}(x'|_{V'_2} + x_{2,k,l})|^{-1} \\ &\dots . \end{aligned} \tag{7.19}$$

Notice that the assumption $det(x'|_{V'_j} + \tilde{x}_{s,j,j,1}) = 0$, (7.16), implies that the fields $\mathfrak{h}'$ and $\mathbb{R}[\tilde{x}_s|_{V'_j}]$ are isomorphic. Moreover the map

$$End_{\tilde{x}_s}(V_j) \ni y \to y \in End_{\mathfrak{h}'}(Hom(V'_j, V_j)_1)$$

is an $\mathbb{R}$-linear bijection. The preimage of $x'|_{V_j'} \in End_{\mathfrak{h}'}(Hom(V_j', V_j)_1)$ via this bijection coincides with $\pm\tilde{x}_s|_{V_j}$. This last element is a constant multiple of the identity I_{V_j} on V_j, if V_j is viewed as a vector space over the field $\mathbb{R}[\tilde{x}_s|_{V_j'}]$. Hence, with $\lambda = \pm\tilde{x}_s|_{V_j} \in \mathbb{R}[\tilde{x}_s|_{V_j'}]$, we have for any $x \in U_{\mathfrak{g}}$,

$$\delta \circ det(x'|_{V_j'} + x_{j,j,1}) = \delta \circ det(\lambda\, I_{V_j'} + x|_{V_j}).$$

By (4.6) and (5.10) the translation of the distribution $\delta \circ det$ by any constant multiple of the identity is regular (in the sense of our Lemma 7.10). Therefore, $chc_{x'}|_{U_{\mathfrak{g}}}$ is regular, and we are done. □

Since we are interested only in the case when $chc_{x'} \neq 0$, we may assume, by (7.17), that V' is a subspace of V. Let $U \subseteq V$ be a complementary subspace, so that $V = V' \oplus U$. This gives embeddings

$$\begin{aligned} &End(V') \to End(V),\ a(v'+u) = a(v'),\ a \in End(V'),\\ (7.20)\quad &End(U) \to End(V),\ b(v'+u) = b(u),\ b \in End(U),\ \text{and}\\ &Hom(U, V') \to End(V),\ c(v'+u) = c(u),\ c \in Hom(U, V'), \end{aligned}$$

where $v' \in V'$ and $u \in U$. Then $\mathfrak{n} = Hom(U, V')$ is the nilpotent radical of a parabolic subalgebra of $\mathfrak{g} = End(V)$ with the Levi factor $\mathfrak{m} = End(V') \oplus End(U)$. Let $K \subseteq G = GL(V)$ be the centralizer of J, (see Sect. 3). This is a maximal compact subgroup of G. For $\psi \in S(\mathfrak{g})$ and $x \in \mathfrak{g}$ set

$$\psi^K(x) = \int_K \psi(kxk^{-1})\,\frac{dk}{|K|},\ \text{and}\ \psi^K_{\mathfrak{n}}(x) = \int_{\mathfrak{n}} \psi^K(x+y)\,dy.$$

Proposition 7.21. *For any $x' \in \mathfrak{h}'^r$ and any $\psi \in S(\mathfrak{g})$,*

$$\int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx = \int_{G'/H'}\int_{End(U)} \psi^K_{\mathfrak{n}}(gx'g^{-1}+x)\,dx\,d\dot{g}.$$

Proof. The argument is straightforward. By (7.10), we express the left hand side in terms of explicit integrals over the regular sets of various Cartan subalgebras of $\mathfrak{g}$, and conclude that the resulting expression is equal to the right hand side.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. We shall describe the restriction of the distribution $chc_{x'}$ to $\mathfrak{h}^r$, and thus refine the formula (7.19).

Recall the decomposition (7.1). For each j, $Hom(V_j', V)$ is a vector space over the field $\mathfrak{h}'|_{V_j'}$ and an $\mathfrak{h}$-module. As such, it decomposes into irreducibles: $Hom(V_j', V) = \sum_k X_{j,k}$. Hence

$$det(x'|_{V_j'} + x) = \prod_k det(x'|_{V_j'} + x|_{X_{j,k}}) \qquad (x' \in \mathfrak{h}'^r,\ x \in \mathfrak{h}^r),$$

where the determinant takes values in the field $\mathfrak{h}'|_{V_j'}$. Hence,

$$\delta(det(x'|_{V_j'}+x)) = \sum_k \delta(det(x'|_{V_j'}+x|_{X_{j,k}})) \prod_{l\neq k} |det_{\mathbb{R}}(x'|_{V_j'}+x|_{X_{j,l}})|^{-1}.$$

Therefore,

$$\begin{aligned}\delta(det(x'+x)) &= \prod_j \delta(det(x'|_{V_j'}+x)) \\ &= \sum_{k_1,k_2,\dots} \prod_j \left(\delta(det(x'|_{V_j'}+x|_{X_{j,k_j}})) \prod_{l\neq k_j} |det_{\mathbb{R}}(x'|_{V_j'}+x|_{X_{j,l}})|^{-1} \right).\end{aligned} \tag{7.21.1}$$

If the term $\delta(det(x'|_{V_j'}+x|_{X_{j,k_j}}))$ is non-zero, then the space X_{j,k_j} is of dimension one over the field $\mathfrak{h}'|_{V_j'}$. Furthermore, since x is regular, only one such term may be non-zero, for each j. In particular, if the expression (7.21.1) is non-zero, then $\mathfrak{h}$ is conjugate to $\mathfrak{h}'+\mathfrak{h}_U$, for some Cartan subalgebra $\mathfrak{h}_U \subseteq End(U)$, (see (7.20)). Assuming $\mathfrak{h}=\mathfrak{h}'+\mathfrak{h}_U$, we can rewrite (7.21.1) as follows:

$$chc(x'+x) = \frac{1}{|W(H_U)|} \sum_{\sigma\in W(H_U)} \delta(x'-(\sigma x)|_{V'})|det_{\mathbb{R}}(ad(\sigma x)|_{\mathfrak{g}'/\mathfrak{h}'+\mathfrak{n}})|^{-1},$$

where $\mathfrak{n}=Hom(U,V')$, $x'\in\mathfrak{h}'^r$, $x\in\mathfrak{h}^r$, $(\sigma x)|_{V'}\in\mathfrak{h}'$ is the restriction of $\sigma x\in\mathfrak{h}$ to V', $\delta\in S^*(\mathfrak{h}')$ is the Dirac delta at the origin, $W(H_U)$ is the Weyl group of H_U, the Cartan subgroup of $GL(U)$ with the Lie algebra $\mathfrak{h}_U$, and $W(H)$ is the Weyl group of H, the Cartan subgroup of $G=GL(V)$ with the Lie algebra $\mathfrak{h}$. Hence, by (7.10), for $\psi\in S(\mathfrak{g})$,

$$\begin{aligned}&\int_{Ad(G)\mathfrak{h}^r} chc(x'+x)\psi(x)\,dx \\ &= \frac{1}{|W(H)|}\int_{\mathfrak{h}^r} |det(ad(x)|_{\mathfrak{g}/\mathfrak{h}})|\, chc(x'+x)\int_{G/H}\psi(gxg^{-1})\,d\dot{g}\,dx \\ &= \frac{1}{|W(H_U)|}\int_{\mathfrak{h}_U^r} |\pi_U(x)|^2|det(ad(x'+x)|_{\mathfrak{n}})|\int_{G/H}\psi(g(x'+x)g^{-1})\,d\dot{g}\,dx \\ &= \frac{1}{|W(H_U)|}\int_{\mathfrak{h}_U^r} |\pi_U(x)|^2\int_{G'/H'}\int_{GL(U)/H_U}\psi_{\mathfrak{n}}^K(g'x'g'^{-1}+gxg^{-1})\,d\dot{g}\,d\dot{g}'\,dx \\ &= \int_{Ad(GL(U))\mathfrak{h}_U^r}\int_{G'/H'}\psi_{\mathfrak{n}}^K(g'x'g'^{-1}+x)\,d\dot{g}'\,dx,\end{aligned}$$

where $|\pi_U(x)|^2=|det(ad(x)|_{End(U)/\mathfrak{h}_U})|$, and the third equality follows from the standard integral formulas, [W1, 2.41, 7.3.7]. Since the set of conjugacy classes of Cartan subalgebras $\mathfrak{h}$, for which (7.21.1) is non-zero, is in one to one correspondence with set of conjugacy classes of Cartan subalgebras $\mathfrak{h}_U$, the Proposition follows. □

Define a continuous linear map

$$(7.22) \qquad \mathcal{A} : S(\mathfrak{g}) \to S(\mathfrak{g}'),\ \mathcal{A}\,\psi(x) = \psi_{\mathfrak{n}}^{K}(x),\ x \in \mathfrak{g}'.$$

It is easy to check that

$$(7.23) \qquad \int_{\mathfrak{g}'} \mathcal{A}\,\psi(x)\,dx = \int_{G'\backslash W^{max}} \psi \circ \tau_{\mathfrak{g}}(w)\,d\dot{w}.$$

where $W^{max} = \{(x, y);\ x \text{ and } y \text{ are of maximal rank}\}$.

Let $\mathcal{O}' \subseteq \mathfrak{g}'$ be a G' orbit. Then $\tau_{\mathfrak{g}}(\tau_{\mathfrak{g}'}(\mathcal{O}'))$ contains a single dense G orbit $\mathcal{O} \subseteq \mathfrak{g}$. Let $\mu_{\mathcal{O}'} \in S^*(\mathfrak{g}')$ be the canonical measure on $\mathcal{O}'$. Then $\mu_{\mathcal{O}'} \circ \mathcal{A}$ is a G invariant measure, a positive constant multiple of $\mu_{\mathcal{O}}$. A straightforward calculation based on (7.21) and (7.23) shows that

(7.24)

$$(\mu_{\mathcal{O}'} \circ \mathcal{A})\hat{}(\psi) = \sum \frac{1}{|W(H')|} \int_{\mathfrak{h}'^r} \hat{\mu}_{\mathcal{O}'}(x')|\pi(x')|^2 \int_{\mathfrak{g}} chc(x' + x)\psi(x)\,dx\,dx',$$

where the summation is as in the Weyl integration formula for $\mathfrak{g}'$. This verifies Theorem (1.19) for pairs of type II.

Pairs of Lie algebras, of type I

8. Notation

Let V, V' be two finite dimensional vector spaces over $\mathbb{D}$ with non-degenerate forms $(\ ,\ )$, $(\ ,\ )'$ - one hermitian and the other one skew-hermitian. Define a map $Hom(V', V) \ni w \to w^* \in Hom(V, V')$ by

$$(8.1) \qquad (wv', v) = (v', w^*v)' \qquad (w \in Hom(V', V),\ v \in V,\ v' \in V').$$

Define a symplectic form $\langle\ ,\ \rangle$ on the real vector space $W = Hom(V', V)$ by

$$(8.2) \qquad \langle w, w' \rangle = tr(w'^* w) \qquad (w, w' \in W).$$

Let $G \subseteq GL(V)$ be the isometry group of the form $(\ ,\ )$, with the Lie algebra $\mathfrak{g} \subseteq End(V)$. Similarly we have the isometry group $G' \subseteq GL(V')$ of the form $(\ ,\ )'$, with the Lie algebra $\mathfrak{g}' \subseteq End(V')$. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the bilinear form provided by the trace, and similarly for $\mathfrak{g}'$. Then the moment maps $\tau_{\mathfrak{g}} : W \to \mathfrak{g}^*$ and $\tau_{\mathfrak{g}'} : W \to \mathfrak{g}'^*$ are given by

$$(8.3) \qquad \tau_{\mathfrak{g}}(w) = ww^*,\ \tau_{\mathfrak{g}'}(w) = w^*w, \qquad (w \in W).$$

The groups G, G' act on W by post-multiplication and premultiplication by the inverse, respectively. These actions preserve the symplectic form (8.1). The moment maps (8.3) intertwine these actions with the corresponding adjoint actions.

9. The pair $sp_{2n}(\mathbb{R}), so_1$

In this case the integral over $G'\backslash W$ may be identified with the integral over W. Hence,

$$(9.1)\quad chc(\psi) = \int_W \int_{sp} \chi_x(w)\psi(x)\, dx\, dw \quad (\psi \in S(sp),\ sp = sp(W)),$$

where each consecutive integral is absolutely convergent. Let

$$(9.2)\qquad sp_{\mathbb{C}}^+ = \{z = x + iy;\ x, y \in sp,\ \langle y\ ,\ \rangle|_{Ker(x)} > 0\},$$

where the statement $\langle y\ ,\ \rangle|_{Ker(x)} > 0$ means that $\langle yw, w\rangle > 0$ for every non-zero element $w \in Ker(x)$. Let $n = \frac{1}{2}dim(W)$.

Proposition 9.3. *(a) The set $sp_{\mathbb{C}}^+$ is contractible.*
(b) There is a unique holomorphic function $chc : sp_{\mathbb{C}}^+ \to \mathbb{C}$ such that $chc(iy) = 2^n|det(y)|^{-1/2}$ for $y \in sp$ such that $\langle y\ ,\ \rangle > 0$. Moreover $|chc(z)| = 2^n|det(z)|^{-1/2}$, $z \in sp_{\mathbb{C}}^+$.
(c) As a distribution on sp, $chc(x) = \lim_{y\to 0} chc(x + iy)$, $x \in sp$, where $\langle y\ ,\ \rangle > 0$.
(d) $WF(chc) = \{(x, \tau_{sp}(w));\ x \in sp,\ w \in W,\ x(w) = 0\}$.

In order to prove the proposition we realize W as $\mathbb{R}^{2n}$, with the symplectic form

$$\langle w, w'\rangle = w'^t Jw,\ \ J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},\ \ (w, w' \in W).$$

Then $sp = sp_{2n}(\mathbb{R}) = \{x \in M_{2n}(\mathbb{R});\ \ Jx + x^t J = 0\}$ and similarly for the complexification $sp_{\mathbb{C}} = sp_{2n}(\mathbb{C})$.

Let $m = 2n$ and let $SM_m(\mathbb{D})$ denote the space of symmetric matrices of size m with entries in $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$. Let

$$SM_m^+(\mathbb{C}) = \{A = B + iC;\ \ B, C \in SM_m(\mathbb{R}),\ \ w^t Bw > 0 \\ \text{for } w \neq 0,\ \ Cw = 0\}.$$

The map

$$\alpha : sp_{2n}(\mathbb{C}) \ni z \to -iJz \in SM_{2n}(\mathbb{C})$$

is a linear isomorphism, and $\alpha(sp_{\mathbb{C}}^+) = SM_m^+(\mathbb{C})$. Part (a) of the proposition is immediate from the following lemma.

Lemma 9.4. *The set $SM_m^+(\mathbb{C})$ is contractible.*

Proof (Rossmann). For $a > 0$, consider the map:

$$SM_m(\mathbb{C}) \ni A = B + iC \to \tilde{A} = B + a\,C^2 + iC \in SM_m(\mathbb{C}).$$

Then $A \in SM_m^+(\mathbb{C})$ if and only if there is $a > 0$ such that the real part of $\tilde{A}$ is positive definite. Since the set of complex symmetric matrices with a positive definite real part is convex, we are done. □

In particular, (9.4) implies that there is a unique holomorphic function $SM_m^+(\mathbb{C}) \ni A \to det^{1/2}(A) \in \mathbb{C}$, which coincides with the positive square root of the determinant of A, if A is real and positive definite. Clearly, the function

$$chc(z) = \frac{2^n}{det^{1/2}(\alpha(z))} \qquad (z \in sp_{\mathbb{C}}^+) \tag{9.5}$$

satisfies (9.3.b).

Recall, [Hö, (3.4.1)"] the following integral formula

$$\frac{1}{det^{1/2}(A)} = \int_{\mathbb{R}^m} e^{-\pi w^t A w}\, dw \qquad (A = B + iC;\;\; B, C \in SM_m(\mathbb{R});\;\; B > 0). \tag{9.6}$$

The statement (9.3.c) is a straightforward consequence of (9.5) and (9.6).

It remains to calculate the wave front set, $WF(chc)$. Let

$$\mu(\psi) = \int_{\mathbb{R}^m} \psi(-ww^t)\, dw \qquad (\psi \in S(SM_m(\mathbb{R}))).$$

This integral is absolutely convergent and μ is a temperate measure on $SM_m(\mathbb{R})$.

Lemma 9.7. *Under the identification $SM_m(\mathbb{R})^* = SM_m(\mathbb{R})$ provided by the trace, ($tr(CD)$, $C, D \in SM_m(\mathbb{R})$),*

(a) $WF(\mu) = \{(C, D) \in supp\,\mu \times SM_m(\mathbb{R});\;\; DC = 0\}$,

(b) $WF(\hat{\mu}) = \{(D, C) \in SM_m(\mathbb{R}) \times (-supp\,\mu);\;\; DC = 0\}$.

Proof. We begin with (a). Clearly the fibers of $WF(\mu)$ over the complement of the support of μ are empty (zero). Notice that the Fourier transform of μ,

$$\begin{aligned}\hat{\mu}(C) &= \int_{SM_m(\mathbb{R})} \chi(-tr(CD))\, d\mu(D) = \lim_{B\to 0} \int_{\mathbb{R}^m} e^{-\pi w^t(-2B-2iC)w} dw \\ &= 2^{-m/2} \lim_{B\to 0} \frac{1}{det^{1/2}(-(B+iC))},\end{aligned} \tag{9.8}$$

where $-B > 0$. In particular, by [Hö, (8.1.18)], The fiber of $WF(\mu)$ over 0 coincides with $supp\,\hat{\mu} = SM_m(\mathbb{R})$.

Since $supp\,\mu \setminus \{0\} = \{gCg^t;\;\; g \in SL_m(\mathbb{R})\}$, and since μ is invariant under the action of $SL_m(\mathbb{R})$, [Hö, 8.2.5] implies that $(C, D) \in WF(\mu)$ if and only if D is perpendicular to the tangent space to $supp\,\mu \setminus \{0\}$ at C. Thus for any $X \in sl_m(\mathbb{R})$,

$$0 = tr(D(XC + CX^t)) = 2tr(DCX).$$

Hence DC is a constant multiple of the identity. But C is not invertible. Therefore the constant is zero, i.e. $DC = 0$. By taking the transpose we get $CD = 0$. This verifies (a).

Let $C, D \in SM_m(\mathbb{R}) \setminus \{0\}$. Then, by [Hö, 8.1.8], and (a), $(D, C) \in WF(\hat{\mu})$ if and only if $(-C, D) \in WF(\mu)$, which happens if and only if $C \in -supp\,\mu$, $D \in SM_m(\mathbb{R})$ and $CD = 0$. Also, the fiber of $WF(\hat{\mu})$ over zero coincides with $-supp\,\mu$. This verifies (b). □

Notice that for $D \in SM_m(\mathbb{R})$ and $w \in \mathbb{R}^m$, $Dww^t = 0$ if and only if $Dw = 0$. Indeed, both sides of the equivalence are invariant under the action of the orthogonal group $O_m(\mathbb{R})$. Hence we can assume that D is diagonal. But in this case the statement is obvious. Hence, it is easy to deduce from (9.7) that, with $\beta : \mathbb{R}^m \to SM_m(\mathbb{R})^*$ defined by $\beta(w)(D) = w^t Dw$, $w \in \mathbb{R}^m$, $D \in SM_m(\mathbb{R})$,

$$\text{(9.9)} \quad WF(\hat{\mu}) = \{(D, \beta(w)) \in SM_m(\mathbb{R}) \times SM_m(\mathbb{R})^*;\ w \in \mathbb{R}^m,\ Dw = 0\}.$$

Notice that for $w \in W$ and $x \in sp$, $\tau_{sp}(w)(x) = \langle xw, w\rangle = w^t Jxw = \beta(w)(Jx)$. Hence, by (9.8),

$$chc(x) = \int_W \chi_x(w)\,dw = \int_W e^{2\pi i w^t \frac{1}{4} Jxw} dw = \hat{\mu}\left(\frac{1}{4} Jx\right) \quad (x \in sp).$$

Therefore (9.3.d) follows from (9.9).

10. The case when $H' \subseteq G'$ is compact

In this section the Cartan subgroup $H' \subseteq G'$ is compact. (This forces the pair G, G' to be of type I.) Let

$$\text{(10.1)} \qquad V' = \sum_{j \in \mathcal{J}'} V'_j$$

be a decomposition of V' into H'-irreducible subspaces over $\mathbb{D}$. If G' is isomorphic to the orthogonal group $O_{p,q}$, with $p + q$ odd, then (10.1) contains the trivial component, which shall be denoted by V'_0. There is no (non-zero) trivial component in any other case. For each $j \in \mathcal{J}' \setminus \{0\}$ there is a complex structure i on V'_j, $(i \in End_{\mathbb{R}}(V'_j),\ i^2 = -1)$, and $\mathbb{R}$-linear coordinates x'_j on $\mathfrak{h}'$, such that

$$\text{(10.2)} \qquad x'|_{V'_j} = i x'_j \qquad (x' \in \mathfrak{h}',\ j \in \mathcal{J}' \setminus \{0\}).$$

Theorem 10.3. *Let p be defined as in (1.12). Assume* $p \geq 0$*. Then for any* $\psi \in S(\mathfrak{g})$ *and any* $\epsilon > 0$*, the following integral*

$$\int_{\mathfrak{h}'^r} \prod_{j\in\mathcal{J}'\setminus\{0\}} (|x'_j|+1)^{p-\epsilon} \left|\pi_{\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx\right| dx'$$

is convergent and defines a continuous seminorm on $S(\mathfrak{g})$.

In order to prove Theorem (10.3) we need some preparation. Let $H \subseteq G$ be a compact Cartan subgroup, with the Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$. The symplectic space W decomposes into a direct orthogonal sum of $H \cdot H'$-irreducible subspaces $W_{j,k}$, over $\mathbb{R}$:

$$W = \sum_{j\in\mathcal{J}',k\in\mathcal{J}} W_{j,k}, \quad W_{j,k} \subseteq Hom(V'_j, V). \tag{10.4}$$

The holomorphic function chc defined on $sp_{\mathbb{C}}^{+}$, see (9.3.b), extends and then restricts to a rational function on $\mathfrak{h}'_{\mathbb{C}}+\mathfrak{h}_{\mathbb{C}}$. We denote by $W_{0,k}$ the subspace on which H' acts trivially, and by $W_{j,0}$ the subspace on which H acts trivially. In terms of (10.4) we have

$$chc_W(z'+z) = \prod_{j\in\mathcal{J}',k\in\mathcal{J}} chc_{W_{j,k}}(z'+z) \qquad (z' \in \mathfrak{h}'_{\mathbb{C}},\ z \in \mathfrak{h}_{\mathbb{C}}), \tag{10.5}$$

where the subscript (W or $W_{j,k}$) indicates the symplectic space with respect to which the corresponding function is defined.

Let $\Phi(\mathfrak{h}')$ be a system of positive roots of $\mathfrak{h}'_{\mathbb{C}}$ in $\mathfrak{g}'_{\mathbb{C}}$, and let $\pi_{\mathfrak{h}'}$ denote the product of all the roots $\alpha \in \Phi(\mathfrak{h}')$. Similarly, we define $\pi_{\mathfrak{h}}$. Then, as a polynomial in the coordinates x'_j, (see (10.2)), $\pi_{\mathfrak{h}'}$ has degree

$$\begin{aligned} &d'-2,\ d'-1,\ d'-1,\ 2d'-1,\ 2d'-2 \text{ if} \\ &\qquad G' = O_{p,q},\ Sp_{2n}(\mathbb{R}),\ U_{p,q},\ Sp_{p,q},\ O^*_{2n},\ \text{respectively}. \end{aligned} \tag{10.6}$$

Let $\tilde{p}$ be a non-negative integer, smaller or equal to the number p, defined in (1.12). Set

$$P(x') = \prod_{j\in\mathcal{J}'\setminus\{0\}} (ix'_j+1)^{\tilde{p}} \qquad (x' \in \mathfrak{h}').$$

By (10.6), the degree of the rational function

$$P(z')\pi_{\mathfrak{h}'}(z')chc_W(z'+z) \qquad (z' \in \mathfrak{h}'_{\mathbb{C}},\ z \in \mathfrak{h}_{\mathbb{C}})$$

with respect to each z'_j, $j \in \mathcal{J}' \setminus \{0\}$, is negative. (Moreover, $\tilde{p} = p$ is maximal with this property.) Hence, by partial fractions,

$$P(z')\pi_{\mathfrak{h}'}(z')chc_W(z'+z)\pi_{\mathfrak{h}}(z) = \sum_L F_{L,z'}(z),$$

(10.7)

$$F_{L,z'}(z) = P_L(z)chc_L(z'+z),\ chc_L(z'+z) = \prod_{j\in\mathcal{J}'\setminus\{0\}} chc_{W_{j,L(j)}}(z'+z),$$

where the summation is over all injections $L : \mathcal{J}' \setminus \{0\} \to \mathcal{J} \setminus \{0\}$, each P_L is a polynomial function, $z' \in \mathfrak{h}'_{\mathbb{C}}$, and $z \in \mathfrak{h}_{\mathbb{C}}$.

Let $\Phi^n(\mathfrak{h}) \subseteq \Phi(\mathfrak{h})$ denote the positive system of non-compact (imaginary) roots. Recall, [Sch], that the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$ are parameterized by strongly orthogonal sets $S \subseteq \Phi^n(\mathfrak{h})$. More precisely, for each $\alpha \in \Phi^n(\mathfrak{h})$ one chooses $X_\alpha \in \mathfrak{g}_\alpha$ (the α-root space in $\mathfrak{g}_{\mathbb{C}}$) and $H_\alpha \in i\mathfrak{h}$ such that the following commutation relations hold:

$$[H_\alpha, X_{\pm\alpha}] = \pm X_{\pm\alpha},\ [X_\alpha, X_{-\alpha}] = H_\alpha,\ \overline{X}_\alpha = X_{-\alpha}.$$

Then, the Cayley transform corresponding to α is defined by

$$c_\alpha = exp\left(\frac{-\pi i}{4} ad(X_\alpha + X_{-\alpha})\right) \in End(\mathfrak{g}_{\mathbb{C}}).$$

For a strongly orthogonal set $S \subseteq \Phi^n(\mathfrak{h})$, define $c_S = \prod_{\alpha\in S} c_\alpha$. The Cartan subalgebra corresponding to S is given by

$$\mathfrak{h}_S = \mathfrak{g} \cap c_S(\mathfrak{h}_{\mathbb{C}}). \tag{10.8}$$

Moreover, two Cartan subalgebras $\mathfrak{h}_{S_1}$, $\mathfrak{h}_{S_2}$ are conjugate if and only if some element of the Weyl group $W(H)$ maps $S_1 \cup (-S_1)$ onto $S_2 \cup (-S_2)$.

It is important to realize that the following equation holds

$$chc_W(z'+z) = chc_W(z' + c_S^{-1}(z)) \qquad (z' \in \mathfrak{h}'_{\mathbb{C}},\ z \in \mathfrak{h}_{S,\mathbb{C}}), \tag{10.9}$$

where $\mathfrak{h}_{S,\mathbb{C}}$ stands for the complexification of $\mathfrak{h}_S$.

Indeed, if $G = U_{p,q}$ and $G' = U_1$, then by (9.3)

$$chc_W(z'+z) = (-1)^p 2^{p+q} \frac{1}{det(z'+z)} \quad (z' \in \mathfrak{h}'_{\mathbb{C}} = \mathfrak{g}'_{\mathbb{C}},\ z \in \mathfrak{g}_{\mathbb{C}}). \tag{10.10}$$

Since the determinant is invariant under conjugation, the formula (10.9) follows. The general case reduces to the above one, via the decomposition

$$chc_W(z'+z) = \prod_{j\in\mathcal{J}'} chc_{W_j}(z'+z)$$
$$(W_j = Hom(V'_j, V),\ z' \in \mathfrak{h}'_{\mathbb{C}},\ z \in \mathfrak{g}_{\mathbb{C}}).$$

By combining (10.7) and (10.9) we see that, with the notation of (7), we have for any strongly orthogonal set $S \subseteq \Phi^n(\mathfrak{h})$, and any $z' \in \mathfrak{h}'_{\mathbb{C}}$ and $z \in \mathfrak{h}_{S,\mathbb{C}}$,

$$\begin{aligned} & P(z')\pi_{\mathfrak{h}'}(z')chc_W(z'+z)\pi_{\mathfrak{h}} \circ c_S^{-1}(z) \\ &= P(z')\pi_{\mathfrak{h}'}(z')chc_W(z'+c_S^{-1}(z))\pi_{\mathfrak{h}} \circ c_S^{-1}(z) = \sum_L F_{L,z'} \circ c_S^{-1}(z). \end{aligned} \tag{10.11}$$

Since we are going to use Stokes formula, [Hö, 6.4.5], we introduce the differential forms involved.

As in (10.1), let

$$V = \sum_{l\in\mathcal{L}} V_l \tag{10.12}$$

be a decomposition of V into H-irreducible subspaces over $\mathbb{D}$. We denote by V_0 the trivial component if it occurs (i.e. is non-zero). For each $l \neq 0$ fix a complex structure i on V_l. Then there are linear coordinates x_l on $\mathfrak{h}$ such that

$$x|_{V_l} = ix_l \qquad (x \in \mathfrak{h},\ l \in \mathcal{L} \setminus \{0\}). \tag{10.13}$$

Define an $\mathbb{R}$-linear isomorphism $c'_S : \mathfrak{h} \to \mathfrak{h}_S$ by

$$c'_S(x) = x \text{ for all } x \in \bigcap_{\alpha\in S} Ker\,\alpha,\ \ c'_S(iH_\alpha) = c_S(H_\alpha) \text{ for all } \alpha \in S. \tag{10.14}$$

Let $dim\,\mathfrak{h} = n$. In terms of (10.13) and (10.14) set

$$\mu = dx_1dx_2...dx_n,\ \ \mu_S = (c'_S)^*\mu = i^{|S|}(c_S)^*\mu,$$

where $|S|$ stands for the cardinality of S. We orient $\mathfrak{h}$ and $\mathfrak{h}_S$ by declaring the following charts to be positive:

$$\kappa : \mathfrak{h} \ni x \to (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \text{ and } \kappa \circ {c'_S}^{-1} : \mathfrak{h}_S \to \mathbb{R}^n.$$

Then, for a test function ψ,

$$\int_{\mathfrak{h}_S} \psi\mu_S = \int_{\mathfrak{h}} (\psi \circ c'_S)\mu = \int_{\mathbb{R}^n} \psi \circ c'_S \circ \kappa^{-1}(x_1, x_2, ..., x_n)\, dx_1dx_2...dx_n. \tag{10.15}$$

Let $H_S \subseteq G$ be the Cartan subgroup corresponding to $\mathfrak{h}_S$. Recall the Harish-Chandra integral, defined with respect to negative roots (see [W1, 7.3.5]):

$$\begin{aligned} & \psi_S(x) = \overline{\pi}_{\mathfrak{h}} \circ {c_S}^{-1}(x)\epsilon_S(x) \int_{G/H_S} \psi(gxg^{-1})d\dot{g}, \\ & \epsilon_S(x) = \prod_{\alpha\circ {c_S}^{-1} \text{ real}} sgn(\overline{\alpha} \circ {c_S}^{-1}(x)) \qquad (x \in \mathfrak{h}^r_S,\ \psi \in S(\mathfrak{g})). \end{aligned} \tag{10.16}$$

The Weyl group $W(H_S)$ preserves the Cartan decomposition $\mathfrak{h}_S = \mathfrak{k}\cap\mathfrak{h}_S \oplus \mathfrak{p}\cap\mathfrak{h}_S$. Let $(\mathfrak{p}\cap\mathfrak{h}_S)^+ \subseteq \mathfrak{p}\cap\mathfrak{h}_S$ be a Weyl chamber on which $\epsilon_S(x) > 0$. Let ${\mathfrak{h}_S}^+ = \mathfrak{k}\cap\mathfrak{h}_S \oplus (\mathfrak{p}\cap\mathfrak{h}_S)^+$, and let n_S denote quotient of the cardinality of the restriction of $W(H_S)$ to $\mathfrak{p}\cap\mathfrak{h}_S$ by the cardinality of $W(H_S)$. Then the Weyl integration formula can be written as

$$(10.17)\qquad \int_{\mathfrak{g}} \psi(x)\,dx = \sum n_S \int_{{h_S}^+} \pi_{\mathfrak{h}}\circ c_S^{-1}\,\psi_S\,\mu_S \qquad (\psi\in S(\mathfrak{g})),$$

where the summation is over a maximal family of mutually non-conjugate Cartan subalgebras $\mathfrak{h}_S \subseteq \mathfrak{g}$.

Notice that the Weyl group $W(H)$ acts on the set of indices $\mathcal{J}\setminus\{0\}$ by permuting the summand in (10.4). For each injection $L : \mathcal{J}\setminus\{0\} \to \mathcal{J}\setminus\{0\}$ choose an element $y^L \in \mathfrak{h}$ so that

$$(10.18)\quad \langle y^L\ ,\ \rangle|_{W_{j,L(j)}} > 0 \text{ for } j\in\mathcal{J}'\setminus\{0\} \text{ and } \sigma(y^L) = y^{\sigma L} \text{ for } \sigma\in W(H).$$

This is possible. Indeed, there are unique elements $H_l \in \mathfrak{h}$ such that $x = \sum_l x_l H_l$ for $x\in\mathfrak{h}$, and we may choose

$$y^L = \sum_{j\in\mathcal{J}'\setminus\{0\}} sgn\langle H_{L(j)}\ ,\ \rangle|_{W_{j,L(j)}}\, H_{L(j)}.$$

Let $y_S^L = \sum y_l^L\, H_l$, where the summation is over the l such that $\alpha(H_l) = 0$ for all $\alpha\in S$. Define an $(n+1)$-chain $\mathcal{C}_S^L$ in $\mathfrak{h}_{S,\mathbb{C}}$ as follows:

$$\mathcal{C}_S^L(t,x) = x + ity_S^L \qquad (x\in\mathfrak{h}_S^+,\ 0\le t\le 1).$$

Let $\Phi_S^n = \{\alpha\in\Phi(\mathfrak{h}),\ \alpha\circ c_S^{-1} \text{ is a non-compact imaginary root for } \mathfrak{h}_S\}$. For $\alpha\in\Phi_S^n$, let $\mathcal{C}_S^L|_\alpha$ denote the restriction of $\mathcal{C}_S^L$ to $Ker\,\alpha\circ c_S^{-1}$. Let $\mathcal{C}_S^L(1)$ denote the restriction of $\mathcal{C}_S^L$ to $t=1$. We give orientations to $\mathcal{C}_S^L$, $\mathcal{C}_S^L(1)$, and $\mathcal{C}_S^L|_\alpha$, by declaring the following charts to be positive:

$$\mathcal{C}_S^L \ni x + ity_S^L \to (t, \kappa\circ c_S^{-1}(x)) \in \mathbb{R}^{1+n},$$

$$\mathcal{C}_S^L(1) \ni x + ity_S^L \to \kappa\circ c_S^{-1}(x) \in \mathbb{R}^n,$$

$$\mathcal{C}_S^L|_\alpha \ni x + ity_S^L \to (t,0) + \kappa\circ c_S^{-1}(x) \in \mathbb{R}^n.$$

Let $\mathfrak{h}_S^{ir} = \{x\in\mathfrak{h}_S,\ \alpha\circ c_S^{-1}(x) \neq 0 \text{ for all } \alpha\in\Phi_S^n\}$. By a theorem of Harish-Chandra, [Va, part I, p. 47], each function ψ_S, defined in (10.16), extends to a smooth function on $\mathfrak{h}_S^{ir}$, which shall be denoted by the same symbol ψ_S. For an integer $N \geq 0$, let $\psi_{S,N}$ denote the extension of degree N of ψ_S, to the complexification of $\mathfrak{h}_S^{ir}$, as in [Hö, 3.1.15], (see (A.1)).

Theorem 10.19. *Fix an element $x' \in \mathfrak{h}'^r$. Then for any integer $N \geq 0$, large enough, and for all $\psi \in S(\mathfrak{g})$,*

$$P(x')\pi_{\mathfrak{h}'}(x') \int_{\mathfrak{g}} chc(x'+x)\psi(x)\,dx$$
$$= \sum n_S \left(\int_{\mathfrak{C}_S^L(1)} \nu_{L,x',S,N} - \int_{\mathfrak{C}_S^L} d(\nu_{L,x',S,N}) - \sum_{\alpha \in \Phi_S^n} \int_{\mathfrak{C}_S^L|_\alpha} \nu_{L,x',S,N} \right),$$

where $\nu_{L,x',S,N} = F_{L,x'} \circ c_S^{-1} \cdot \psi_{S,N} \cdot \mu_S$, the unmarked summation is over a maximal family of mutually non-conjugate Cartan subalgebras $\mathfrak{h}_S \subseteq \mathfrak{g}$ and over all injections $L : \mathcal{J}' \setminus \{0\} \to \mathcal{J} \setminus \{0\}$. Moreover, each integral on the right hand side of the equation is absolutely convergent. Furthermore, the integrals over $\mathfrak{C}_S^L|_\alpha$ are equal to each other.

Proof. The decompositions (10.1), (10.4), and (10.12) are of course related, as follows. If $\mathbb{D} = \mathbb{C}$, then $W_{j,k} = Hom(V_j', V_k)$. For $\mathbb{D} \neq \mathbb{C}$ the space $Hom(V_j', V_k)$, $(j \neq 0, k \neq 0)$ decomposes into two irreducible subspaces. Thus, with $n = dim\,\mathfrak{h}$, we may define the index set $\mathcal{J}$ so that

$$\mathcal{J} \setminus \{0\} = \begin{cases} \{1, 2, ..., n\} & \text{if } \mathbb{D} = \mathbb{C}, \\ \{1, 2, ..., 2n\} & \text{if } \mathbb{D} \neq \mathbb{C}, \end{cases}$$

and choose the complex structures i on V_j' and on V_k so that, for $x \in \mathfrak{h}$,

$$(x'+x)|_{W_{j,k}} = \begin{cases} i(x_k - x_j') & \text{if } k \leq n, \\ i(-x_{k-n} - x_j') & \text{if } k > n. \end{cases} \tag{10.20}$$

It is an exercise to see that one can introduce linear coordinates u_l, v_l, x_l on $\mathfrak{h}_S$ as follows (see (10.13)): if $\mathbb{D} = \mathbb{R}$ then there are numbers $0 \leq a, b$, $a + 2b \leq n$, such that for $x \in \mathfrak{h}$,

$$c_S^{-1}(x)_l = \begin{cases} v_l & \text{if } 1 \leq l \leq a, \\ iu_l + v_l & \text{if } a < l \leq a+b, \\ -iu_{l-b} + v_{l-b} & \text{if } a+b < l \leq a+2b, \\ ix_l & \text{if } a+2b < l \leq n; \end{cases} \tag{10.21}$$

if $\mathbb{D} = \mathbb{C}$, then there is an integer $b \geq 0$, with $2b \leq n$, such that for $x \in \mathfrak{h}_S$

$$c_S^{-1}(x)_l = \begin{cases} iu_l + v_l & \text{if } 1 < l \le b, \\ iu_{l-b} - v_{l-b} & \text{if } b < l \le 2b, \\ ix_l & \text{if } 2b < l \le n; \end{cases} \tag{10.22}$$

if $\mathbb{D} = \mathbb{H}$, then there is an integer $b \geq 0$, with $2b \leq n$, such that for $x \in \mathfrak{h}_S$

$$c_S^{-1}(x)_l = \begin{cases} iu_l + v_l & \text{if } 1 < l \le b, \\ -iu_{l-b} + v_{l-b} & \text{if } b < l \le 2b, \\ ix_l & \text{if } 2b < l \le n. \end{cases} \tag{10.23}$$

Let $L : \mathcal{J}' \to \mathcal{J}$ be an injection. We see from (10.20) that, in terms of (21–23), the function $\left|chc_L(x' + c_S^{-1}(\mathfrak{C}_S^L(t,x)))\right|$, $(x' \in \mathfrak{h}'^r, x \in \mathfrak{h}_S)$, is a constant multiple of the following expression, for $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively:

$$\begin{aligned} &\prod_{1\le L(j)\le a} |v_{L(j)} - ix'_j|^{-1} \prod_{n<L(j)\le n+a} |v_{L(j)-n} + ix'_j|^{-1} \\ &\prod_{a<L(j)\le a+b} |i(u_{L(j)} - x'_j) + v_{L(j)}|^{-1} \\ &\prod_{n+a<L(j)\le n+a+b} |i(u_{L(j)-n} + x'_j) + v_{L(j)-n}|^{-1} \\ &\prod_{a+b<L(j)\le a+2b} |i(-u_{L(j)-b} - x'_j) - v_{L(j)-b}|^{-1} \\ &\prod_{n+a+b<L(j)\le n+a+2b} |i(-u_{L(j)-b-n} + x'_j) - v_{L(j)-b-n}|^{-1} \\ &\prod_{a+2b<L(j)\le n} |i(x_{L(j)} - x'_j) + ty^L_{L(j)}|^{-1} \\ &\prod_{n+a+2b<L(j)\le 2n} |i(x_{L(j)-n} + x'_j) - ty^L_{L(j)-n}|^{-1}, \end{aligned} \tag{10.24}$$

$$\begin{aligned} &\prod_{1\le L(j)\le b} |i(u_{L(j)} - x'_j) + v_{L(j)}|^{-1} \\ &\prod_{b<L(j)\le 2b} |i(u_{L(j)-b} - x'_j) - v_{L(j)-b}|^{-1} \\ &\prod_{2b<L(j)\le n} |i(x_{L(j)} - x'_j) - ty^L_{L(j)}|^{-1}, \end{aligned} \tag{10.25}$$

$$\prod_{1\le L(j)\le b} |i(u_{L(j)} - x'_j) + v_{L(j)}|^{-1}$$
$$\prod_{n<L(j)\le n+b} |i(u_{L(j)-n} + x'_j) + v_{L(j)-n}|^{-1}$$
$$\prod_{b<L(j)\le 2b} |i(-u_{L(j)-b} - x'_j) + v_{L(j)-b}|^{-1}$$
$$\prod_{n+b<L(j)\le n+2b} |i(-u_{L(j)-b-n} + x'_j) + v_{L(j)-b-n}|^{-1} \tag{10.26}$$
$$\prod_{2b<L(j)\le n} |i(x_{L(j)} - x'_j) - ty^L_{L(j)}|^{-1}$$
$$\prod_{n+2b<L(j)\le 2n} |i(x_{L(j)-n} + x'_j) - ty^L_{L(j)-n}|^{-1}.$$

The expressions (10.24–10.26) are products of independent terms of the form

$$|v - ix'|^{-1},\ |v - ix'_1|^{-1}|v - ix'_2|^{-1} \qquad (x' \ne 0,\ x'_1 \ne x'_2,\ x'_1 \ne 0,\ x'_2 \ne 0), \tag{10.27}$$

$$|i(u - x') - v|^{-1},\ |i(u - x'_1) - v|^{-1}|i(u - x'_2) - v|^{-1} \qquad (x'_1 \ne x'_2), \tag{10.28}$$

$$|i(x - x') - t|^{-1},\ |i(x - x'_1) - t|^{-1}|i(x - x'_2) - t|^{-1} \qquad (x'_1 \ne x'_2). \tag{10.29}$$

Since the expressions (10.27–10.29) are locally integrable with respect to $v \in \mathbb{R}$, $(u, v) \in \mathbb{R}^2$, or $(x, t) \in \mathbb{R}^2$, the absolute convergence of the integrals in the Theorem (10.19) follows, (see also (A.3), (A.4)).

In order to prove the formula (10.19) we may assume that $\psi \in C_c^\infty(\mathfrak{g})$. Suppose first that the support of ψ is disjoint with the singular support of the distribution $chc_{x'}$. Then, by (10.11) and by the Weyl integration formula (10.17),

$$\begin{aligned}
&\int_{\mathfrak{g}} P(x')\pi_{\mathfrak{h}'}(x')chc(x' + x)\psi(x)\,dx\\
&= \sum n_S \int_{\mathfrak{h}_S^+} P(x')\pi_{\mathfrak{h}'}(x')chc(x' + x)\pi_{\mathfrak{h}} \circ c_S^{-1}(x)\psi_S(x)\,\mu_S(x)\\
&= \sum n_S \int_{\mathfrak{h}_S^+} F_{L,x'} \circ c_S^{-1} \cdot \psi_S \cdot \mu_S.
\end{aligned} \tag{10.30}$$

Let us fix L and view $\mathfrak{h}_S^+$ as $\mathcal{C}_S^L(0)$, the restriction of $\mathcal{C}_S^L$ to $t = 0$. Also, let $\nu_{L,x',S,N} = F_{L,x'} \circ c_S^{-1} \cdot \psi_{S,N} \cdot \mu_S$. Then, by Stokes formula, [Hö, (6.5.4)], and a Theorem of Harish-Chandra, regarding the singularities of his integral (10.16), [Va, part I, p.47],

$$\begin{aligned} \int_{\mathcal{C}_S^L} d(\nu_{L,x',S,N}) &= \int_{\mathcal{C}_S^L(1)} \nu_{L,x',S,N} - \int_{\mathcal{C}_S^L(0)} \nu_{L,x',S,N} \\ &\quad - \sum_{\alpha\in\Phi_S^n} \int_{\mathcal{C}_S^L|_\alpha} \nu_{L,x',S,N}. \end{aligned} \tag{10.31}$$

Clearly, (10.30) and (10.31) verify the formula (10.19).

For a general $\psi \in C_c^\infty(\mathfrak{g})$ the same argument applies, via a partition of unity and a reduction to the case $G = U_{p,q},\ G' = U_1$, considered in [D-P3]. We explain the details.

Let $\tilde{x} \in \mathfrak{g}$ be in the singular support of the distribution $chc_{x'}$. Let $\tilde{x}_s$ be the semisimple part in the Jordan decomposition of $\tilde{x}$. Let

$$V = \sum_k \tilde{V}_k \tag{10.32}$$

be the decomposition of V into $\mathbb{R}[\tilde{x}_s]$-isotypic subspaces over $\mathbb{D}$. The decomposition (10.32) is direct, orthogonal, and the sets of eigenvalues of $\tilde{x}_s|_{\tilde{V}_k}$ are disjoint, as k varies. Let $\tilde{W}_{j,k} = Hom(V_j', \tilde{V}_k)$. Since the $x' \in \mathfrak{h}'^r$ is regular, we may arrange the indices so that

$$Ker(x' + \tilde{x}_s) \cap \tilde{W}_{j,k} \neq \emptyset \text{ if and only if } j = k = 1, 2, ..., m. \tag{10.33}$$

A straightforward, case by case, verification shows that for $1 \leq k \leq m$,

$$\tilde{W}_{k,k} = \begin{cases} Ker(x' + \tilde{x}_s) \cap \tilde{W}_{k,k} & \text{if } \mathbb{D} = \mathbb{C}, \\ Ker(x' + \tilde{x}_s) \cap \tilde{W}_{k,k} \oplus (Ker(x' + \tilde{x}_s) \cap \tilde{W}_{k,k})^\perp & \text{if } \mathbb{D} \neq \mathbb{C}. \end{cases}$$

In particular,

$$W = Ker(x' + \tilde{x}_s) \oplus (Ker(x' + \tilde{x}_s))^\perp. \tag{10.34}$$

Using the complex structure i on V_k' we view $Ker(x'+\tilde{x}_s) \cap \tilde{W}_{k,k}$ as a complex vector space. The Hermitian form $\langle\ ,\ \rangle + \langle i\ ,\ \rangle$ on this space is preserved by $G^{\tilde{x}_s}|_{\tilde{V}_k}$, the restriction to $\tilde{V}_k$ of the centralizer of $\tilde{x}_s$ in G. Thus

$$(G^{\tilde{x}_s}|_{\tilde{V}_k},\ Ker(x' + \tilde{x}_s) \cap \tilde{W}_{k,k}) \text{ is isomorphic to } (U_{p,q},\ \mathbb{C}^{p+q}), \tag{10.35}$$

for some p, q.

Let $U \subseteq \mathfrak{g}^{\tilde{x}_s}$ be a completely invariant, open neighborhood of $\tilde{x}_s$, invariant under conjugation by elements of the identity component of $G^{\tilde{x}_s}$, such that (10.33) holds with the $\tilde{x}_s$ replaced by any $x \in U$. Notice that our original element $\tilde{x}$ belongs to U. We choose U small enough so that the adjoint orbits are transversal to U, (see [Va, part I, p. 19]). By [Hö, 8.2.4], the distribution $chc_{x'}$ restricts to U. We describe this restriction in more detail.

Let $\mathfrak{g}^+_{\mathbb{C},x'} = \{z = x+iy;\ \langle y\,,\ \rangle|_{Ker(x'+x)>0},\ x, y \in \mathfrak{g}\} \subseteq \mathfrak{g}_{\mathbb{C}}$. The function $chc_W : sp^+_{\mathbb{C}} \to \mathbb{C}$, defined in (9.3), restricts to a holomorphic function

$$\mathfrak{g}^+_{\mathbb{C},x'} \ni z \to chc_W(x'+z) \in \mathbb{C}.$$

We see from (10.35) that for each $x \in U$ there is an element $y \in \mathfrak{g}$ which preserves the decomposition (10.34) and satisfies the condition $\langle y\,,\ \rangle|_{Ker(x'+x)>0}$. For such x and y, and $z = x + iy$,

$$chc_W(x'+z) = chc_{Ker(x'+\tilde{x}_s)}(x'+z) \cdot chc_{Ker(x'+\tilde{x}_s)^{\perp}}(x'+z),$$

because the function chc_W is the reciprocal of a square root of the determinant, (see (9.3)). By taking limit if $y \to 0$ we obtain the following equation of distributions:

$$chc(x'+x) = chc_{Ker(x'+\tilde{x}_s)}(x'+x) \cdot chc_{Ker(x'+\tilde{x}_s)^{\perp}}(x'+x) \qquad (x \in U), \tag{10.36}$$

where the second factor, on the right hand side, is a real analytic function. Furthermore, by (10.33),

$$chc_{Ker(x'+\tilde{x}_s)}(x'+x) = \prod_{k=1}^{m} chc_{Ker(x'+\tilde{x}_s)\cap \tilde{W}_{k,k}}(x'+x) \qquad (x \in U). \tag{10.37}$$

Since, by [D-P3], the Theorem 10.19 holds for the pair $U_{p,q}, U_1$, the formulas (10.36) and (10.37) imply the equation (19) for $\psi \in C_c^\infty(Ad(G)\,U)$. A partition of unity argument completes the proof. □

Lemma 10.38. *Let* $\phi(v) = (1+|v|)^{-N}$ *or* $\phi(u,v) = (1+|u|)^{-N}(1+|v|)^{-N}$*;* $u, v \in \mathbb{R}$, $N \geq 0$. *Then for* N *large enough, and any* $\epsilon > 0$, *the following integrals are finite:*

(a) $$\int_{\mathbb{R}} \int_{\mathbb{R}} (v^2 + x'^2)^{-1/2} \phi(v)\, dv\, (1+x'^2)^{-\epsilon/2}\, dx',$$

(b) $$\int_{\mathbb{R}^2} \int_{\mathbb{R}} (v^2 + {x'_1}^2)^{-1/2} (v^2 + {x'_2}^2)^{-1/2} \phi(v)\, dv \; (1+{x'_1}^2)^{-\epsilon/2}(1+{x'_2}^2)^{-\epsilon/2}\, dx'_1\, dx'_2,$$

(c) $$\int_{\mathbb{R}} \int_{\mathbb{R}^2} ((u-x')^2 + v^2)^{-1/2} \phi(u,v)\, du\, dv\, (1+x'^2)^{-\epsilon/2}\, dx',$$

(d) $$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} ((u-x'_1)^2 + v^2)^{-1/2} ((u-x'_2)^2 + v^2)^{-1/2} \phi(u,v)\, du\, dv \; (1+{x'_1}^2)^{-\epsilon/2}(1+{x'_2}^2)^{-\epsilon/2}\, dx'_1\, dx'_2.$$

We shall prove this lemma in Appendix B.

Proof of Theorem 10.3. From the formulas (10.19), (10.24–10.29) and (A.3), (A.4), we see that the integral in Theorem 10.3 is a sum of integrals, each of which can be dominated by by one of the integrals (10.38). □

11. A general pair $(\mathfrak{g}, \mathfrak{g}')$ of type I

Here we resume the general case of Sect. 8. Let $H' = T'A' \subseteq G'$ be a Cartan subgroup, as in Sect. 1. Let $V'_c \subseteq V'$ be the subspace on which A' acts trivially. Let $V'_s = V_c'^{\perp} \subseteq V'$ be the orthogonal complement of V'_c. Then V'_s has a complete polarization $V'_s = X' \oplus Y'$, preserved by H'. Let $X' = X'_1 \oplus X'_2 \oplus \ldots$ and $Y' = Y'_1 \oplus Y'_2 \oplus \ldots$ be the decomposition of X', Y' into A'-isotypic components. Altogether we have

(11.1)
$$V' = V'_s \oplus V'_c,\ V'_s = X \oplus Y,\ X' = X'_1 \oplus X'_2 \oplus \ldots,\ Y' = Y'_1 \oplus Y'_2 \oplus \ldots .$$

We assume that the restriction of the form $(\ ,\ )'$ to each space $V'_j = X'_j \oplus Y'_j$, as in (11.1), is non-degenerate. Let $W_s = Hom(V'_s, V)$, $W_c = Hom(V'_c, V)$, $W_j = Hom(V'_j, V)$. Then we have the following direct sum orthogonal decompositions

$$W = W_c \oplus W_s,\ W_s = W_1 \oplus W_2 \oplus \ldots . \tag{11.2}$$

Moreover,

$$W_j = Hom(X'_j, V) \oplus Hom(Y'_j, V) \qquad (j \geq 1). \tag{11.3}$$

The group A'' (= centralizer of A' in Sp) preserves the decompositions (2) and (11.3) and the obvious restrictions yield isomorphisms:

$$\begin{aligned} \mathfrak{a}'' &= sp(W_c) \oplus End_{\mathbb{R}}(Hom(X'_1, V)) \oplus End_{\mathbb{R}}(Hom(X'_2, V)) \oplus \ldots \\ A'' &= Sp(W_c) \times GL_{\mathbb{R}}(Hom(X'_1, V)) \times GL_{\mathbb{R}}(Hom(X'_2, V)) \times \ldots . \end{aligned} \tag{11.4}$$

Let A''' be the centralizer of A'' in Sp. Let A'''_j be the restriction of A''' to X'_j. Then A'''_j is isomorphic to $GL_1(\mathbb{R})$. The restriction of A''' to W_c is isomorphic to O_1, the two element group. Hence, (A'', A''') is a dual pair isomorphic to

$$(Sp_{2n}(\mathbb{R}), O_1) \times (GL_{n_1}(\mathbb{R}), GL_1(\mathbb{R})) \times (GL_{n_2}(\mathbb{R}), GL_1(\mathbb{R})) \times \ldots . \tag{11.5}$$

Let $W_{A'''} = (W_c \setminus \{0\}) \times W_{1\,A_1'''} \times W_{2\,A_2'''} \times \ldots$.

Define an invariant measures $d\dot{w}$ on the quotient manifolds $A'' \setminus W_{A''}$, and on $A_j''' \setminus W_{A_j'''}$, as in (1.3). Then, as a temperate distribution on $\mathfrak{a}'$,

$$(11.6)\quad chc(x) = \int_{A'''\setminus W_{A'''}} \chi_x(w)\, d\dot{w}$$

$$= \int_{W_c} \chi_{x_c}(w)\, dw \otimes \int_{A_1'''\setminus W_{A_1'''}} \chi_{x_1}(w)\, d\dot{w} \otimes \int_{A_1'''\setminus W_{A_2'''}} \chi_{x_2}(w)\, d\dot{w} \; \otimes \dots ,$$

where $x \in \mathfrak{a}''$, $x_c = |_{W_c}$, $x_j = x|_{Hom(X_j',V)}$, and each term is well defined via the results of previous sections. In particular Lemma (1.7) follows.

Proof of Proposition 1.8. Suppose $s_0 \in sp(W_c)$ is of rank one. Then there is a non-zero $w_o \in W_c$ such that

$$s_0(w_0') = \pm\langle w_0, w_0'\rangle w_0 \qquad (w_0' \in W_c).$$

Let $s_j \in End_{\mathbb{R}}(Hom(X_j', V))$, $j \geq 1$, be of rank one. Then there are non-zero elements $x_j \in Hom(X_j', V)$, $y_j \in Hom(Y_j', V)$, such that

$$s_j(u) = tr(u\, y_j^*)x_j \qquad (u \in Hom(X_j', V)).$$

In other words, $s_0 + s_1 + s_2 + \dots = \tau_{\mathfrak{a}''}(w)$, for $w = (w_0, x_1 + y_1, x_2 + y_2, \dots)$. Suppose that $s_0 + s_1 + s_2 + \dots \in \mathfrak{a}''$ is perpendicular to $\mathfrak{g}$. Then $\tau_{\mathfrak{g}}(w) = 0$, i.e.

$$(11.7)\qquad w_0w_0^* + x_1y_1^* + x_2y_2^* + \dots = 0.$$

Let $x \in \mathfrak{g}$ be such that $(x' + x, s_0 + s_1 + s_2 + \dots)$ is in the wave front set of the distribution chc, (11.6). Then by (4.9) and (9.3.d)

$$(11.8)\qquad xw_0 = w_0x_c',\ xx_j = x_jx_j',\ xy_j = y_jx_j' \qquad (j \geq 1).$$

By combining (11.7) and (11.8) we deduce

$$(11.9)\quad 0 = w_0(x')^kw_0^* + x_1(x')^ky_1^* + x_2(x')^ky_2^* + \dots \quad (k = 0, 1, 2, \dots)\ .$$

Since $x' \in \mathfrak{h}'^r$ is regular, the odd powers $(x')^k$ span $\mathfrak{h}'$ over the field of the points in the center of $\mathbb{D}$, fixed by the involution. Hence, by taking linear combinations of both sides of (11.9), with coefficients in in that field, we see that (11.9) holds with the $(x')^k$ replaced by an arbitrary element of $\mathfrak{h}'$. In particular

$$(11.10)\qquad w_0w_0^* = 0,\ x_jy_j^* = 0, \qquad (j \geq 1).$$

The first equation in (11.10) means that the image of w_0^* is an isotropic subspace of V_c'. By (11.8), this image is preserved by x'. Hence, by the classification of Cartan subalgebras in $\mathfrak{g}'$, $w_0 = 0$. Also, as in the proof of this Proposition (1.8), for pairs of type II, we check that $s_j = 0$ for $j \geq 1$. This contradiction completes the proof. □

Suppose $V' = V'_s$. Then the first line of (11.4) reduces to

$$\mathfrak{a}'' = End_{\mathbb{R}}(Hom(X'_1, V)) \oplus End_{\mathbb{R}}(Hom(X'_2, V)) \oplus \dots ,$$

and (11.6) reads

$$chc(x) = \delta \circ det(x_1) \otimes \delta \circ det(x_2) \otimes \dots,$$

where $x \in \mathfrak{a}''$, x_j is the restriction of x to $Hom(X'_j, V)$, and $\delta \circ det$ is as in (4.6). In particular the distribution $chc(x' + x)$, $x \in \mathfrak{g}$, is a positive invariant measure on $\mathfrak{g}$. The following lemma can verified using the same argument as in the proof of (7.10). We leave the details to the reader.

Lemma 11.11. *Suppose $V' = V'_s$.*
(a) If the distribution $chc_{x'}(x) = chc(x' + x)$, $x \in \mathfrak{g}$, is non-zero, then the space V contains an isotropic subspace of the same dimension as the dimension of the isotropic space $X' \subseteq V'$.
(b) The distribution $chc_{x'}$ is regular, in the sense that for any $\psi \in S(\mathfrak{g})$,

$$\begin{aligned}&\int_{\mathfrak{g}} \psi(x) chc(x' + x)\, dx \\ &= \sum \frac{1}{|W(H)|} \int_{\mathfrak{h}^r} chc(x' + x)\, |\pi_{\mathfrak{h}}(x)|^2 \int_{G/H} \psi(ghg^{-1})\, d\dot{g}\, dx,\end{aligned}$$

where the integrals are absolutely convergent, and the summation is over a maximal family of mutually non-conjugate Cartan subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$.

From now on we assume that $chc_{x'} \neq 0$. If $V'_c \neq 0$, then, in terms of (11.2),

$$chc_W(x' + x) = chc_{W_c}(x' + x) \cdot chc_{W_s}(x' + x) \qquad (x \in \mathfrak{g}), \tag{11.12}$$

where the product of distributions is well defined, by (1.8). In particular we see that (11.11.a) holds in general. Hence, we may assume that V'_s is a subspace of V, such that the restriction of the form $(\ ,\ )$ to V'_s is non-degenerate and such that $V'_s = X' \oplus Y'$ is a complete polarization with respect to the form $(\ ,\ )$.

Let $U = V'^{\perp}_s \subseteq V$. Then

$$V = V'_s \oplus U = X' \oplus Y' \oplus U. \tag{11.13}$$

As in (7.20), this gives embeddings, $End(X') \to \mathfrak{g}$, $Hom(U, X') \to \mathfrak{g}$, and $\mathfrak{g}(U) \to \mathfrak{g}$, where $\mathfrak{g}(U)$ is the Lie algebra of the group $G(U)$ of isometries of the restriction of the form $(\ ,\ )$ to U. Notice that $Hom(U, X')$ is contained in the unipotent radical $\mathfrak{n}$ of the parabolic subalgebra of $\mathfrak{g}$ preserving X'. The Levi factor of this parabolic subalgebra coincides with $End(X') + \mathfrak{g}(U)$. Let

$\mathfrak{n}' \subseteq \mathfrak{g}'$ be the unipotent radical of the parabolic subalgebra preserving X. Recall the maximal compact subgroup $K \subseteq G$. For $\psi \in S(\mathfrak{g})$ define

$$\psi_{\mathfrak{n}}^K(x) = \int_{\mathfrak{n}} \int_K \psi(k(x+y)k^{-1})\, dk\, dy \qquad (x \in \mathfrak{g}).$$

By restriction the above formula gives a continuous map

$$S(\mathfrak{g}) \ni \psi \to \psi_{\mathfrak{n}}^K \in S(End(X') + \mathfrak{g}(U)).$$

Proposition 11.14. *For any $x' \in \mathfrak{h}'^r$ and any $\psi \in S(\mathfrak{g})$,*

$$\int_{\mathfrak{g}} chc(x'+x)\psi(x)\, dx$$
$$= \frac{1}{|det(ad(x')|_{\mathfrak{n}'})|} \int_{GL(X')/H_s'} \int_{\mathfrak{g}(U)} chc_{W_c}(x_c'+x)\psi_{\mathfrak{n}}^K(gx_s'g^{-1}+x)\, dx d\dot{g},$$

where $H_s' = H'|_{X'}$, is the restriction of H' to X', $x_c' = x'|_{V_c'}$, and $x_s' = x'|_{X'}$.

Proof. If $V' = V_s'$, i.e., if $U = 0$, then, by (11.11.b), the left hand side can be expressed in terms of integrals over the regular parts of various Cartan subalgebras. The same can be done with the right hand side, and the two appear equal.

The general case follows from the previous one and from the formula (11.12), via a straightforward calculation. □

The pairs of groups

12. The pair $Sp_{2n}(\mathbb{R})), O_1$

Let $W_{\mathbb{C}} = W \otimes \mathbb{C}$ be the complexification of W. The symplectic form $\langle\ ,\ \rangle$ extends uniquely to a complex valued form on $W_{\mathbb{C}}$. Let $Sp(W_{\mathbb{C}})$ and $sp(W_{\mathbb{C}}) = sp(W) \oplus i\, sp(W)$ denote the corresponding complex symplectic group and complex symplectic Lie algebra. It is easy to see that the elements of the subset $sp_{\mathbb{C}}^{++} = \{x+iy;\ x, y \in sp,\ \langle y\ ,\ \rangle > 0\} \subseteq sp(W_{\mathbb{C}})$ don't have 1 as an eigenvalue. Let $Sp^{++}(W_{\mathbb{C}}) = c(sp_{\mathbb{C}}^{++})$. As shown in [H2, (12.4b)], $Sp^{++}(W_{\mathbb{C}})$ is an open sub-semi-group of $Sp(W_{\mathbb{C}})$. Furthermore, see [H2, (23.7.2)], every element $g \in Sp^{++}(W_{\mathbb{C}})$ has a unique factorization $g = u \cdot p$, where $u \in Sp(W)$, $\langle Im(c(p))\ ,\ \rangle > 0$, $c(p) \in i\, sp(W)$ and p (as an endomorphism of $W_{\mathbb{C}}$) has positive eigenvalues $\neq 1$. Set

(12.1)
$$\widetilde{Sp}^{++}(W_{\mathbb{C}}) = \{\tilde{g} = (g, \xi);\ g \in Sp^{++}(W_{\mathbb{C}}),\ \xi^2 = det(i(g-1))^{-1}\},$$
$$\Theta : \widetilde{Sp}^{++}(W_{\mathbb{C}}) \ni \tilde{g} \to \xi \in \mathbb{C}.$$

The distribution Θ, defined in (2.7), coincides with the function equal to the limit of the holomorphic function defined in (12.1), as p tends to the identity.

Lemma 12.2. *The distribution* Θ*, has the wave front set given by the following formula:* $WF(\Theta) = \{(\tilde{g}, \tau_{sp}(w));\ g(w) = w,\ \tilde{g} \in \widetilde{Sp}(W),\ w \in W\}$.

Proof. Let $\Psi \in C_c^\infty(\widetilde{Sp}^c)$ be supported on one sheet of the covering map $\widetilde{Sp} \to Sp$. Let $\tilde{g}_0 \in \widetilde{Sp}^c$. Then

$$\begin{aligned}\int_{\widetilde{Sp}^c} \Psi(\tilde{g})\Theta(\tilde{g}_0\tilde{g})\, d\tilde{g} &= \int_{\widetilde{Sp}^c} \Psi(\tilde{g})T(\tilde{g}_0)\natural T(\tilde{g})(0)\, d\tilde{g} \\ &= \int_W \int_{\widetilde{Sp}^c} \Psi(\tilde{g})T(\tilde{g}_0)(w)T(\tilde{g})(w)\, d\tilde{g}\, dw.\end{aligned} \tag{12.3}$$

Fix a smooth lifting $\tilde{c} : sp^c \to \widetilde{Sp}^c$ of the Cayley transform $c : sp^c \to Sp^c$, so that $supp\,\Psi$ is in the image of $\tilde{c}$. Let $\tilde{g} = \tilde{c}(x)$, $\psi(x) = const\,\Psi(\tilde{c}(x))$ $det(1-x)^{2n+1}$, and let $x_0 = c(g_0)$. Then (12.3) may be rewritten as

$$\int_{\widetilde{Sp}^c} \Psi(\tilde{g})\Theta(\tilde{g}_0\tilde{g})\, d\tilde{g} = \int_W \int_{sp^c} \psi(x)\Theta(\tilde{c}(x))\Theta(\tilde{g}_0)\chi_{x_0+x}(w)\, dx\, dw. \tag{12.4}$$

Define maps

$$\Lambda_{\tilde{g}_0} : \widetilde{Sp} \ni \tilde{g} \to \tilde{g}_0\tilde{g} \in \widetilde{Sp}, \quad \lambda_{\tilde{x}_0} : sp \ni x \to x_0 + x \in sp. \tag{12.5}$$

Then (12.4) shows that, in terms of pullbacks of distributions [Hö, 9.2], we have

$$\tilde{c}^*((\Lambda_{\tilde{g}_0}^*\Theta)|_{\widetilde{Sp}^c}) = \Theta(\tilde{g}_0) \cdot \tilde{c}^*\Theta \cdot \lambda_{x_0}^*\hat{\mu}, \tag{12.6}$$

where $\Theta(\tilde{g}_0)$ is a constant, $c^*\Theta(x) = \Theta(\tilde{c}(x))$ is a smooth function, and $\hat{\mu}$ is defined just before (9.7). Hence, by [Hö, 8.2.4],

$$WF((\Lambda_{\tilde{g}_0}^*\Theta)|_{\widetilde{Sp}^c}) = \tilde{c}^* \circ \lambda_{x_0}^*(WF(\hat{\mu})). \tag{12.7}$$

Recall (9.7) that, under the identification $sp = sp^*$,

$$WF(\hat{\mu}) = \{(x, s) \in sp \times \tau_{sp}(W);\ xs = 0\}.$$

Hence,

$$\lambda_{x_0}^*(WF(\hat{\mu})) = \{(x, s) \in sp \times \tau_{sp}(W);\ (x_0 + x)s = 0\}.$$

It is easy to check that

$$d\tilde{c}(g)(x) = -2(g-1)^{-1}x(g-1)^{-1} \qquad (x \in sp).$$

Hence,

$$\begin{aligned}&(\tilde{g}, s) \in \tilde{c}^* \circ \lambda_{x_0}^*(WF(\hat{\mu})) \\ &\text{iff } \left(x, -\frac{1}{2}(g-1)s(g-1)\right) \in \lambda_{x_0}^*(WF(\hat{\mu}) \\ &\text{iff } (x_0 + x)(g-1)s(g-1) = 0 \\ &\text{iff } (x_0 + x)(g-1)s = 0.\end{aligned} \tag{12.8}$$

Recall [H2] that on a dense subset of $Sp^c \times Sp^c$

$$c(c(x_0)c(x)) = (x-1)(x_0+x)^{-1}(x_0+1)+1.$$

Hence,

$$x_0 + x = (x_0-1)(c(g_0g)-1)^{-1}(x-1) = 2(g_0-1)^{-1}(g_0g-1)(g-1)^{-1}.$$

Therefore, on the whole set $Sp^c \times Sp^c$,

$$x_0 + x = 2(g_0-1)^{-1}(g_0g-1)(g-1)^{-1}.$$

Hence, the last condition in (12.8) is equivalent to

$$(g_0g-1)s = 0.$$

Since $Sp = Sp^c \cdot Sp^c$, we are done. □

Lemma 12.9. *For any $\Psi \in C_c^\infty(\widetilde{Sp})$, the distribution $T(\Psi)$, (see (2.7)), is a function and belongs to $S(W)$. Moreover the map*

$$C_c^\infty(\widetilde{Sp}) \ni \Psi \to T(\Psi) \in S(W)$$

is continuous.

Proof. By the method of stationary phase, the lemma is clear if $\widetilde{Sp}$ is replaced by $\widetilde{Sp}^c$. For the general case we notice that there are $g_1, g_2, \ldots, g_m \in \widetilde{Sp}^c$ such that

$$\widetilde{Sp} = \bigcup_{j=1}^m g_j \widetilde{Sp}^c.$$

Hence there are functions $\Psi_1, \Psi_2, \ldots, \Psi_m \in C_c^\infty(\widetilde{Sp}^c)$, such that

$$\sum_{j=1}^m \Psi_j(g_j^{-1}g) = 1 \qquad (g \in \widetilde{Sp}).$$

Therefore for any $\Psi \in C_c^\infty(\widetilde{Sp})$

$$T(\Psi) = \int_{\widetilde{Sp}} \Psi(g)T(g)\,dg = \sum_{j=1}^m \int_{\widetilde{Sp}} \Psi_j(g_j^{-1}g)\Psi(g)T(g)\,dg \tag{12.10}$$

$$= \sum_{j=1}^m \int_{\widetilde{Sp}} \Psi_j(g)\Psi(g_jg)T(g_jg)\,dg = \sum_{j=1}^m T(g_j)\natural \int_{\widetilde{Sp}^c} \Psi_j(g)\Psi(g_jg)T(g)\,dg.$$

Since for any $g \in \widetilde{Sp}$ the map

$$S(W) \ni \phi \to T(g)\natural\phi \in S(W)$$

is well defined and continuous, the lemma follows from the formula (12.10). □

Lemma 12.9 implies, in particular, that the following formula defines a distribution on $\widetilde{Sp}$

$$Chc(\Psi) = \int_W T(w)\,dw(\Psi) = \int_W \int_{\widetilde{Sp}} \Psi(g)T(w)\,dg\,dw \qquad (\Psi \in C_c^\infty(\widetilde{Sp})). \tag{12.11}$$

Corollary 12.12. *With the above definitions we have*

(a) $$Chc = \int_W T(w)\,dw = \Theta(\widetilde{(-1)})^{-1}\Lambda^*_{\widetilde{(-1)}}\Theta,$$

(b) $$WF(Chc) = \{(\tilde{g}, \tau_{sp}(w));\ g(w) = -w,\ \tilde{g} \in \widetilde{Sp},\ w \in W\}.$$

Proof. For part (a) we notice that

$$\Lambda^*_{\widetilde{(-1)}}\Theta(\Psi) = \int_{\widetilde{Sp}} \Psi(g)\Theta(\widetilde{(-1)}g)\,dg = \int_{\widetilde{Sp}} \Psi(g)T(\widetilde{(-1)}g)(0)\,dg$$
$$= T(\widetilde{(-1)})\natural \int_{\widetilde{Sp}} \Psi(g)T(g)(0)\,dg = \int_W \int_{\widetilde{Sp}} \Theta(\widetilde{(-1)})\Psi(g)T(g)(w)\,dg\,dw.$$

Part (b) follows from (a) and (12.2). □

Let J be a (not necessarily positive definite) compatible complex structure on W and let $U = Sp(W)^J$ be the centralizer of J in the symplectic group. Set

$$H_J(w, w') = \langle Jw, w'\rangle + i\langle w, w'\rangle.$$

This is a positive definite hermitian form on W, viewed as a complex vector space where multiplication by $i \in \mathbb{C}$ is identified with J. For $g \in GL_{\mathbb{C}}(W)$, let $g^* \in GL_{\mathbb{C}}(W)$ be the adjoint element, defined by the equation $H_J(gw, w') = H_J(w, g^*w')$. In these terms, the group $U = \{g \in GL_{\mathbb{C}}(W),\ g^* = g^{-1}\}$.

Let $U_{\mathbb{C}}^{++} = Sp^{++}(W_{\mathbb{C}})^J$ be the centralizer of J in $Sp^{++}(W_{\mathbb{C}})$, and let $\tilde{U}_{\mathbb{C}}^{++}$ be the preimage of $U_{\mathbb{C}}^{++}$ in $\widetilde{Sp}^{++}(W_{\mathbb{C}})$. It is easy to check that $U_{\mathbb{C}}^{++} = \{g \in GL_{\mathbb{C}}(W);\ g^*g < 1\}$, and that $\tilde{U}_{\mathbb{C}}^{++} = \{\tilde{g} = (g, \xi);\ g \in U_{\mathbb{C}}^{++},\ \xi^2 = \frac{det(g)}{det(1-g)^2}\}$. (Here "$g^*g < 1$" means "$H_J((1 - g^*g)w, w) > 0$ for all $w \in W \setminus \{0\}$.") In particular

$$\Theta(\tilde{g}) = \frac{det^{1/2}(g)}{det(1-g)}, \tag{12.13}$$

where $\tilde{g} \in \tilde{U}_{\mathbb{C}}^{++}$ is in the preimage of $g \in U_{\mathbb{C}}^{++}$, and $((det^{1/2}(g))^2 = det(g)$.

The group $GL_{\mathbb{C}}(W)$ is a complexification of U. Let $\widetilde{GL}_{\mathbb{C}}(W) = \{(g, h);\ g \in GL_{\mathbb{C}}(W),\ h^2 = det(g)\}$, and let $\widetilde{GL}_{\mathbb{C}}^{++}(W) = \{(g, h);\ g \in GL_{\mathbb{C}}(W),\ g^* g < 1,\ h^2 = det(g)\}$. A straightforward calculation shows that the map

$$GL_{\mathbb{C}}^{++}(W) \ni (g, h) \to \left(g, \frac{h}{det(1-g)}\right) \in \tilde{U}_{\mathbb{C}}$$

preserves multiplication. In particular we see that the function Θ, described in (12.13), extends to a rational function on $\widetilde{GL}_{\mathbb{C}}(W)$.

13. The pair $GL_n(\mathbb{R}), GL_1(\mathbb{R})$

Here we use the notation developed in Sect. 4.

Lemma 13.1. *For any $\Psi \in C_c^\infty(\widetilde{G})$, $T(\Psi)$ is a function on W^{max} such that*

(a) $$\int_{G' \backslash W^{max}} |T(\Psi)(w)| d\dot{w} < \infty.$$

Let $\Psi \in C_c^\infty(\widetilde{G})$ be supported on one sheet of the covering map. Then, with $\delta \circ det$ understood as in (4.6),

(b) $$\begin{aligned} Chc(\Psi) &= \int_{G' \backslash W^{max}} T(\Psi)(w)\, d\dot{w} \\ &= \int_G \Psi(\tilde{g})\Theta(\tilde{g})|det(g-1)|\delta(det(g+1))\, dg. \end{aligned}$$

Moreover,

(c) $$(\Theta(\tilde{g}) det(g-1))^2 = det\, g \qquad (g \in G).$$

Thus,

(d) $$Chc(\tilde{g}) = det^{1/2}(g)\delta(det(g+1)) \quad (g \in G),$$

where the sign of the square root depends on $\tilde{g}$ in the preimage of g.

Proof. Notice that for $g \in G^c$ and $x = c(g)$,

(13.2) $$\begin{aligned} det(x-1) &= det(2(g-1)^{-1}) \neq 0 \text{ and} \\ det(x+1) &= det(2g(g-1)^{-1}) \neq 0. \end{aligned}$$

Let $\Psi \in C_c^\infty(\widetilde{G}^c)$ be supported on one sheet of the covering map. Then

(13.3) $$T(\Psi)(w) = \int_{\widetilde{G}} \Psi(\tilde{g})\Theta(\tilde{g})\chi_{c(g)}(w)\, dg = \int_{\mathfrak{g}'} \psi(x)\chi_x(w)\, dx,$$

where the function

$$\psi(x) = 2^{n^2}\,\Psi(\tilde{c}(x))\Theta(\tilde{c}(x))|det(x-1)(x+1)|^{-n} \quad (x \in \mathfrak{g}), \tag{13.4}$$

is in $C_c^\infty(\mathfrak{g}^c)$. Hence, for $\Psi \in C_c^\infty(\tilde{G}^c)$, the method of stationary phase implies (a). Moreover, (4.10) shows that

$$\begin{aligned}\int_{G\backslash W^{max}} T(\Psi)(w)\,d\dot{w} &= \int_{G'\backslash W^{max}} \int_{\mathfrak{g}} \psi(x)\chi_x(w)\,dx\,d\dot{w}\\ &= \int_{\mathfrak{g}} \psi(x)\delta(det(x))\,dx = \int_G \Psi(\tilde{g})\Theta(\tilde{g})\delta(c(g))\,dg\\ &= \int_G \Psi(\tilde{g})\Theta(\tilde{g})|det(g-1)|\delta(det(g+1))\,dg.\end{aligned}$$

Thus (b) follows for $\Psi \in C_c^\infty(\widetilde{G}^c)$.

Let $g_0 \in G^c$ and let $x_0 = c(g_0)$. Then for $\Psi \in C_c^\infty(\tilde{G}^c)$,

$$\begin{aligned}&T(\tilde{g}_0)\natural T(\Psi)(w') \\ &= \Theta(\tilde{g}_0)\chi_{x_0}(w')\int_W \chi_{x_0}(w)\int_{\mathfrak{g}} \psi(x)\chi_x(w)\,dx\,\chi\left(\frac{1}{2}\langle(1-x_0)w', w\rangle\right)dw\end{aligned} \tag{13.5}$$

as a distribution on W. As shown in (4.11), this distribution coincides with a function on W^{max}, which is absolutely integrable over $G \setminus W^{max}$. Thus, with the convergence question out of the way, we are free to calculate the following oscillatory integral (with $z_0,\ z \in \mathfrak{g}^c$):

$$\begin{aligned}&\int_{G'\backslash W^{max}} \chi_{z_0}\natural\chi_z(w')\,d\dot{w}'\\ &= \int_{G'\backslash W^{max}}\int_W \chi_{z_0}(w')\chi\left(\frac{1}{2}\langle(1-z_0)w', w\rangle\right)\chi_{z_0+z}(w)\,dw\\ &= \int_{G'\backslash X^{max}}\int_X\int_Y\int_Y \chi\Big(\frac{1}{2}((x'z_0 + x(1-z_0))y'\\ &\qquad + (-x'(1+z_0) + x(z_0+z))y)\Big)\,dy\,dy'\,dx\,d\dot{x}'\\ &= const\int_{G'\backslash X^{max}}\int_X \delta\Big(\frac{1}{2}(x'z_0 + x(1-z_0))\Big)\\ &\qquad \delta\Big(\frac{1}{2}(x'(1+z_0) - x(z_0+z))\Big)\,dx\,d\dot{x}'\end{aligned} \tag{13.6}$$

$$= const\,|det(1-z_0)|^{-1}\int_{G'\backslash X^{max}}\int_X \delta(x'z_0+z)$$

$$\delta(x'(1+z_0)-x(1-z_0)^{-1}(z_0+z))\,dx\,d\dot{x}'$$

$$= const\,|det(1-z_0)|^{-1}\int_{G'\backslash X^{max}} \delta(x'(1+z_0+z_0(1-z_0)^{-1}(z_0+z)))\,d\dot{x}'$$

$$= const\,|det(1-z_0)|^{-1}|det(1-z)|^{-1}$$

$$\int_{G'\backslash X^{max}} \delta(x'(1+z_0+z_0(1-z_0)^{-1}(z_0+z))(1-z)^{-1})\,d\dot{x}'$$

$$= const\,|det(1-z_0)|^{-1}|det(1-z)|^{-1}$$

$$\int_{G'\backslash X^{max}} \delta(x'(1+(1-z_0)^{-1}(z_0+z)(1-z)^{-1}))\,d\dot{x}'$$

$$= const\,|det(g_0-1)det(g-1)|\int_{G'\backslash X^{max}} \delta\left(x'\frac{1}{2}(g_0g+1)\right)\,d\dot{x}'$$

$$= const\,|det(g_0-1)det(g-1)|\int_{G'\backslash X^{max}} \delta(x'(g_0g+1))\,d\dot{x}'$$

Thus for $g_0, g \in G^c$,

(13.7) $$\int_{G'\backslash W^{max}} T(\tilde{g}_0)\natural T(\tilde{g})(w')\,d\dot{w}'$$

$$= const\,\Theta(\tilde{g}_0)|det(g_0-1)|\Theta(\tilde{g})|det(g-1)|\int_{G'\backslash X^{max}} \delta(x'(g_0g+1))\,d\dot{x}'$$

$$= const\,det^{1/2}(\tilde{g}_0)det^{1/2}(\tilde{g})\delta(det(g_0g+1)).$$

By taking the limit if g_0 goes to 1, we see that the constant is equal to 1, ($const = 1$). Since there are $g_0, g_1, ..., g_m \in G^c$ such that

$$G = \bigcup_{j=0}^{m} g_j G^c$$

a partition of unity argument completes the proof of (b). Parts (c) is easy. □

An argument analogous to the one used to prove (12.11) verifies the following statement:

(13.8) $$WF(Chc) = \{(\tilde{g}, \tau_{\mathfrak{g}}(w));\ g(w) = -w,\ \tilde{g} \in \widetilde{G},\ w \in W\}.$$

14. A general pair G, G'

The proof of Proposition (2.10) is based on (12.12.b) and (13.8), and is entirely analogous to the proof of Proposition (1.8). Thus we leave it for the reader.

Fix a Cartan subgroup $H' \subseteq G'$, as in Sect. 2, an element $h' \in \tilde{H}'^r$. Assume that $Chc_{h'} \neq 0$.

Suppose the pair G, G' is of type II. Then we have the decomposition $V = V' \oplus U$, (see (7.20)). Let $P \subseteq \tilde{G}$ be the parabolic subgroup preserving V'. Then the unipotent radical of P, $N = 1 + \mathfrak{n}$, where $\mathfrak{n} = Hom(U, V')$. The Levi factor of P, M, coincides with the double cover of $GL(V') \cdot GL(U)$, via (7.20). Let δ_P be the modular function for P. Recall the Harish-Chandra transform

$$\Psi^P(m) = \delta_P^{1/2}(m) \int_N \int_K \Psi(kmnk^{-1})\, dk\, dn \qquad (\Psi \in C_c^\infty(\tilde{G}),\ m \in M), \tag{14.1}$$

see [W2, 7.2.1]. It is easy to see (as in (7.21)) that (13.1) implies the following proposition.

Proposition 14.2. *For any $\Psi \in C_c^\infty(\tilde{G})$ and any $h' \in \tilde{H}'^r$,*

$$\int_{\tilde{G}} Chc(h'g)\Psi(g)\, dg = \int_{G'/H'} \int_{\tilde{GL}(U)} \Psi^P(gh'g^{-1}h)\, dh\, d\dot{g}.$$

Let G, G' be a pair of type I. Then $V = X' \oplus U \oplus Y'$, as in (11.13). Let $P \subseteq \tilde{G}$ be the parabolic subgroup preserving X', and let $P' \subseteq \tilde{G}'$ be the parabolic subgroup preserving X'. Let $N \subseteq P$ and $N' \subseteq P'$ be the unipotent radicals. The Levi factor of P, M, coincides with the double cover of $GL(X') \cdot G(U)$, where $G(U)$ is the restriction of G to U. Let Ψ^P denote the Harish-Chandra transform of Ψ, as in (14.1). Then an argument analogous to the one verifying (11.14) proves the following proposition.

Proposition 14.3. *For any $\Psi \in C_c^\infty(\tilde{G})$ and any $h' \in \tilde{H}'^r$,*

$$\int_{\tilde{G}} Chc(h'g)\Psi(g)\, dg = \delta_{P'}^{-1/2}(h') \int_{GL(X')/H'_s} \int_{\tilde{G}(U)} Chc_{W_c}(h'_c h)\Psi^P(gh'_s g^{-1}h)\, dh\, d\dot{g},$$

where $H'_s = H'|_{X'}$, is the restriction of H' to X', $h'_c = h'|_{V'_c}$, $h'_s = h'|_{X'}$, and $W_c = Hom(V'_c, U)$.

From now on we assume that the pair G, G' is of type I, and that the Cartan subgroup $H' \subseteq G'$ is compact.

Let $H \subseteq G$ be a compact Cartan subgroup. The function Θ, (12.13), and hence the function Chc, (12.12.a), uniquely extends to a rational function on $\tilde{H}'_{\mathbb{C}}\tilde{H}_{\mathbb{C}}$. In terms of the decomposition (10.4) we have

$$Chc_W(h'h) = \prod_{j \in \mathcal{J}',\ k \in \mathcal{J}} Chc_{W_{j,k}}(h'h) \qquad (h' \in \tilde{H}'_{\mathbb{C}},\ h \in \tilde{H}_{\mathbb{C}}), \tag{14.4}$$

where the subscript (W or $W_{j,k}$) indicates the symplectic space with respect to which the corresponding function is defined. As in (10.7) we have

$$\begin{aligned} &\Delta(h')Chc_W(h'h)\Delta(h) = \sum_L \tilde{F}_{L,h'}(h), \\ &\tilde{F}_{L,h'}(h) = \tilde{P}_L(h')Chc_L(h'h),\ Chc_L(h'h) = \prod_{j \in \mathcal{J}' \setminus \{0\}} Chc_{W_{j,L(j)}}(h'h), \end{aligned} \tag{14.5}$$

where the summation is over all injections $L : \mathcal{J} \setminus \{0\} \to \mathcal{J} \setminus \{0\}$, each $\tilde{P}_L$ is a regular function on $\tilde{H}_{\mathbb{C}}$, $h' \in \tilde{H}'_{\mathbb{C}}$, and $h \in \tilde{H}_{\mathbb{C}}$.

For a strongly orthogonal set $S \subseteq \Phi^n(\mathfrak{h})$, the Cayley transform $c_S : \mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{S,\mathbb{C}}$ lifts to an isomorphism $C_S : \tilde{H}_{\mathbb{C}} \to \tilde{H}_{S,\mathbb{C}}$. Thus, as in (10.11),

$$\begin{aligned} \Delta(h')\, Chc_W(h'h)\, \Delta \circ C_S^{-1}(h) &= \Delta(h')\, Chc_W(h'C_S^{-1}(h))\, \Delta \circ C_S^{-1}(h) \\ &= \sum_L \tilde{F}_{L,h'} \circ C_S(h) \qquad (h' \in \tilde{H}'_{\mathbb{C}},\ h \in \tilde{H}_{S,\mathbb{C}}). \end{aligned} \tag{14.6}$$

In terms of (10.13) let

$$h|_{V_l} = exp(x)|_{V_l} = exp(x|_{V_l}) = e^{ix_l} = h_l \qquad (h \in \tilde{H},\ x \in \mathfrak{h}). \tag{14.7}$$

Thus, each h_l is identified with a complex number of absolute value 1. The formula (14.7) extends to the complexification $\tilde{H}_{\mathbb{C}}$. Then each h_l is a non-zero complex number. In these terms, define the following differential forms on $\tilde{H}_{\mathbb{C}}$ and on $\tilde{H}_{S,\mathbb{C}}$:

$$\tilde{\mu} = \frac{dh_1}{ih_1}\frac{dh_2}{ih_2}\ldots\frac{dh_n}{ih_n},\ \tilde{\mu}_S = i^{|S|}C_S^*\tilde{\mu}. \tag{14.8}$$

Recall the Harish-Chandra integral, defined with respect to negative roots (see [W1, 7.4.8]):

$$\begin{aligned} &\Psi_S(h) = \overline{\Delta} \circ C_S^{-1}(h)\, \tilde{\epsilon}_S(h) \int_{G/H_S} \Psi(ghg^{-1})\, d\dot{g}, \\ &\tilde{\epsilon}_S(h) = \prod_{\alpha \circ c_S^{-1} \text{ real}} sgn(1 - h^{-\alpha \circ c_S^{-1}}) \qquad (\Psi \in C_c^\infty(\tilde{G}),\ h \in \tilde{H}_S^r). \end{aligned} \tag{14.9}$$

As in Sect. 10, we have the Weyl integration formula:

$$\int_{\tilde{G}} \Psi(g)\, dg = \sum n_S \int_{\tilde{H}_S^+} \Delta \circ C_S^{-1}(h)\, \Psi_S(h)\, \tilde{\mu}_S.$$

Recall the notation introduced between (10.18) and (10.19). Let

$$\tilde{\mathcal{C}}_S^L(t,h) = h\, exp(ity_S^L) \qquad (h \in \tilde{H}_S^+,\ 0 \le t \le 1). \tag{14.10}$$

For $\alpha \in \Phi_S^n$, let $\tilde{\mathcal{C}}_S^L|_\alpha$ denote the restriction of $\tilde{\mathcal{C}}_S^L$ to the set of $h \in \tilde{H}_{S,\mathbb{C}}$, where $h^{\alpha\circ c_S^{-1}} = 1$. Let $\tilde{\mathcal{C}}_S^L(1)$ denote the restriction of $\tilde{\mathcal{C}}_S^L$ to $t = 1$. We give orientations to $\tilde{\mathcal{C}}_S^L$, $\tilde{\mathcal{C}}_S^L(1)$, and $\tilde{\mathcal{C}}_S^L|_\alpha$, compatible with those of $\mathcal{C}_S^L$, $\mathcal{C}_S^L(1)$ and $\mathcal{C}_S^L|_\alpha$ via the exponential map, respectively. Let $\tilde{H}_S^{ir} = \{h \in \tilde{H}_S;$ $h^{\alpha\circ c_S^{-1}} \neq 1$ for all $\alpha \in \Phi_S^n\}$. By a theorem of Harish-Chandra, [Va, part II, p. 219], each function Ψ_S, defined in (14.9), extends to a smooth function on $\tilde{H}_S^{ir}$, which shall be denoted by the same symbol Ψ_S. For an integer $N \ge 0$, let $\Psi_{S,N}$ denote the extension of degree N of Ψ_S, to the complexification $\tilde{H}_{S,\mathbb{C}}^{ir}$, (see (A.6)).

Theorem 14.11. *Fix an element $h' \in \tilde{H}'^r$. Then for any integer $N \ge 0$, large enough, and for all $\Psi \in C_c^\infty(\tilde{G}_1)$,*

$$\begin{aligned} &\Delta(h') \int_{\tilde{G}} Chc(h'g)\Psi(g)\, dg \\ &= \sum n_S \left(\int_{\tilde{\mathcal{C}}_S^L(1)} \tilde{\nu}_{L,x',S,N} - \int_{\tilde{\mathcal{C}}_S^L} d(\tilde{\nu}_{L,x',S,N}) - \sum_{\alpha\in\Phi_S^n} \int_{\tilde{\mathcal{C}}_S^L|_\alpha} \tilde{\nu}_{L,x',S,N} \right), \end{aligned}$$

where $\tilde{\nu}_{L,x',S,N} = \tilde{F}_{L,x'} \circ C_S^{-1} \cdot \Psi_{S,N} \cdot \tilde{\mu}_S$, the unmarked summation is over a maximal family of mutually non-conjugate Cartan subgroups H_S, and over all injections $L : \mathcal{J}' \setminus \{0\} \to \mathcal{J} \setminus \{0\}$. Moreover, each integral on the right hand side of the equation is absolutely convergent, (see (A.7)). Furthermore, the integrals over $\tilde{\mathcal{C}}_S^L|_\alpha$ are equal to each other.

Proof. By Harish-Chandra's Method of Descent, [Va, part II], the problem is reduced to the case $G = U_{p,q}$, $G' = U_1$, as in the proof of (10.19). □

Theorem 14.12. *For any $\Psi \in C_c^\infty(\tilde{G}_1)$ the following integral*

$$\int_{\tilde{H}'^r} \left| \Delta(h') \int_{\tilde{G}} Chc(h'g)\, \Psi(g)\, dg \right| dh'$$

is convergent and defines a continuous seminorm on $C_c^\infty(\tilde{G}_1)$.

Proof. Since, by definition (14.10), $\tilde{\mathcal{C}}_S^L(t, exp(x)) = \mathcal{C}_S^L(t,x)$, this theorem follows from (12.12) and (10.38) via the argument used in the proof of (10.19). □

Appendix A

Here we recall the notion of a boundary value of an analytic function in the sense of distribution theory, see [Hö, 3.1.15].

Let V be a finite dimensional space over $\mathbb{R}$. For any $v \in V$, let $\partial(v)$ denote the derivative in the direction v:

$$\partial(v)\psi(x) = \frac{d}{dt}\psi(u+tv)|_{t=0} \qquad (u \in V,\ \psi \in C^\infty(V)).$$

Let $Sym(V_{\mathbb{C}})$ denote the symmetric algebra of $V_{\mathbb{C}}$. The map ∂ extends to an isomorphism from $Sym(V_{\mathbb{C}})$ onto the algebra of constant coefficient differential operators on V. Let

$$e_N(s) = \sum_{p=0}^{N} \frac{1}{p!} s^p \qquad (s \in Sym(V_{\mathbb{C}}),\ N = 0, 1, 2, ...).$$

For an open set $U \subseteq V$ and a function $\psi \in C^\infty(U)$ define an extension ψ_N $(N = 0, 1, 2, ...)$ of degree N of ψ to the complexification $U + iV$ by

$$\text{(A.1)} \qquad \psi_N(u+iv) = \partial(e_N(iv))\psi(u) \qquad (u \in U,\ v \in V).$$

By fixing a basis, we obtain real valued coordinates $x = (x_1, x_2, ..., x_n)$ on V. Let $z = (z_1, z_2, ..., z_n)$ be the corresponding complex coordinates on $V_{\mathbb{C}}$. Let $dx = dx_1 dx_2 ... dx_n$ and let $dz = dz_1 dz_2 ... dz_n$.

Let $U \subseteq V$ be an open set, and let $\Gamma \subseteq V$ be an open convex cone. Fix a norm $|\ |$ on V. For some $\gamma > 0$, set $Z = \{u + iv \in V_{\mathbb{C}};\ u \in U,\ v \in \Gamma, |v| < \gamma\}$. Let f be an analytic function on Z such that for some $N \geq 0$,

$$\text{(A.2)} \qquad |f(u+iv)| \leq const\,|v|^{-N}, \qquad (u + iv \in Z).$$

Let $v_0 \in \Gamma$, with $|v_0| < \gamma$. Then for any $\psi \in S(V)$, the following limit exists and defines a temperate distribution on V

$$\begin{aligned} \int_V \psi f\, dx &:= \lim_{\Gamma \ni v \to 0} \int_V \psi(u)\, f(u+iv)\, dx(u) \\ \text{(A.3)} \qquad &= \int_V \psi_N(u+iv_0)\, f(u+iv_0)\, dx(u) \\ &+ \int_V \int_0^1 \partial((iv_0)^{N+1}/N!)\psi(u)\, f(u+itv_0) t^N\, dt dx(u). \end{aligned}$$

The formula (A.3) is a direct consequence of Stokes Theorem, and can be written in more intrinsic terms as follows.

Let $\mathcal{C}$ be a $(n+1)$ chain in Z defined by

$$\mathcal{C} : [0,1] \times U \ni (t, u) \to u + itv_0 \in Z,$$

Let $\mathcal{C}(0)$, ($\mathcal{C}(1)$) denote the restriction of $\mathcal{C}$ to $t = 0$, ($t = 1$). We declare the following charts to be positive:

$$V = \mathcal{C}(0) \ni u \to x(u) \in \mathbb{R}^n, \quad V = \mathcal{C}(1) \ni u + iv_0 \to x(u) \in \mathbb{R}^n,$$
$$\mathcal{C} \ni u + itv_0 \to (t, x(u)) \in \mathbb{R}^{1+n}.$$

Then

$$\text{(A.4)} \qquad \int_{\mathcal{C}(0)} \psi_N\, f\, dz = \int_{\mathcal{C}(1)} \psi_N\, f\, dz - \int_{\mathcal{C}} d(\psi_N\, f\, dz),$$

where

$$\int_{\mathcal{C}(0)} \psi_N\, f\, dz = \int_V \psi\, f\, dx$$
$$\text{(A.5)} \int_{\mathcal{C}(1)} \psi_N\, f\, dz = \int_V \psi_N(u + iv_0)\, f(u + iv_0)\, dx(u)$$
$$-\int_{\partial\mathcal{C}} d(\psi_N\, f\, dz) = \int_V \int_0^1 \partial((iv_0)^{N+1}/N!)\psi(u)\, f(u + itv_0)t^N\, dtdx(u).$$

More generally, let H be a commutative Lie group of dimension n, and let $H_{\mathbb{C}}$ be the complexification of H. For $y \in \mathfrak{h}$, the Lie algebra of H, let

$$\tilde{\partial}(y)\, \Psi(h) = \frac{d}{dt}\Psi(h \cdot exp(ty))|_{t=0} \qquad (h \in H,\ \Psi \in C^\infty(H)).$$

As is well known, $\tilde{\partial}$ extends to an injective homomorphism from $Sym(\mathfrak{h}_{\mathbb{C}})$ to the algebra of differential operators on H.

For an open subset $X \subseteq H$ and and a function $\Psi \in C^\infty(X)$ define an extension Ψ_N ($N = 0, 1, 2, \ldots$) of degree N of Ψ to $Z = X \cdot exp(i\mathfrak{h}) \subseteq H_{\mathbb{C}}$ by

$$\text{(A.6)} \qquad \Psi_N(h \cdot exp(iy)) = \tilde{\partial}(e_N(iy))\Psi(h) \qquad (h \in X,\ y \in \mathfrak{h}).$$

In particular, if the group H is connected, we have

$$\Psi_N(exp(x + iy)) = (\Psi \circ exp)_N(x + iy) \qquad (x \in exp^{-1}(X),\ y \in \mathfrak{h}),$$

where the right hand side was defined in (A.1).

Let $\Gamma \subseteq \mathfrak{h}$ be an open convex cone. Fix a norm $|\ |$ on $\mathfrak{h}$. For $\gamma > 0$ set $Z_\gamma = \{h \cdot exp(iy);\ h \in X,\ y \in \Gamma,\ |y| < \gamma\}$. Let f be a holomorphic function on $Z_\gamma \setminus X$. Assume that the function f satisfies the following growth condition

$$|\, f(h \cdot exp(iy))| \le const\, |y|^{-N}, \qquad (h \in X,\ y \in \Gamma,\ |y| < \gamma).$$

Suppose $\mathcal{C}$ is a $(n + 1)$ chain in Z_γ, with the boundary $\partial\mathcal{C} = \mathcal{C}_0 - \mathcal{C}_1$, where $\mathcal{C}_0 = X$ and $\mathcal{C}_1 \subseteq Z_\gamma \setminus X$. Let dz be an invariant holomorphic n-form on $H_{\mathbb{C}}$. Then the corresponding limit distribution is given by the following formula,

$$\int_X \Psi(x)\ f(x)\,dx := \int_{\mathcal{C}_1} \Psi_N(z)\ f(z)\,dz - \int_{\mathcal{C}} d\left(\Psi_N(z)\ f(z)\,dz\right), \tag{A.7}$$

where, for N large enough, the integrals on the right hand side are absolutely convergent.

Appendix B

Here we prove Lemma (10.38). Notice that

$$\int_{|v|\geq 1} (v^2+a^2)^{-1/2}\phi(v)\,dv \leq \int_{\mathbb{R}} \phi(v)\,dv,$$
$$\int_{|v|\leq 1} (v^2+a^2)^{-1/2}\phi(v)\,dv \leq 2 \parallel \phi \parallel_\infty \int_0^1 (v^2+a^2)^{-1/2}\phi(v)\,dv,$$
$$\int_0^1 (v^2+a^2)^{-1/2}\phi(v)\,dv = log(v+\sqrt{v^2+a^2})|_0^1 = log\left(\frac{1}{|a|}+\sqrt{\frac{1}{|a|^2}+1}\right).$$

Moreover,

$$\lim_{x\to\infty} \frac{log\left(x+\sqrt{x^2+1}\right)}{log(x+1)} = 1.$$

hence

$$log\left(\frac{1}{|a|}+\sqrt{\frac{1}{|a|^2}+1}\right) \leq const\,log\left(\frac{1}{|a|}+1\right).$$

Furthermore, for $\epsilon > 0$,

$$\int_0^\infty log\left(\frac{1}{a}+1\right)(a^2+1)^{-\epsilon/2}\,da$$
$$= \int_0^1 log\left(\frac{1}{a}+1\right)(a^2+1)^{-\epsilon/2}\,da + \int_1^\infty log\left(\frac{1}{a}+1\right)(a^2+1)^{-\epsilon/2}\,da$$
$$\leq \int_0^1 log\left(\frac{1}{a}+1\right)da + \int_1^\infty a^{-1-\epsilon}\,da$$
$$= \int_1^\infty log\,(a+1)\,a^{-2}\,da + \int_1^\infty a^{-1-\epsilon}\,da < \infty.$$

This verifies (10.38.a).

Notice that, for $\epsilon > 0$,

$$\int_{\mathbb{R}^2}\int_{|v|\geq 1}(v^2+a^2)^{-1/2}(v^2+b^2)^{-1/2}\phi(v)\,dv(1+a^2)^{-\epsilon/2}(1+b^2)^{-\epsilon/2}\,da\,db$$
$$\leq \|\ \phi\ \|_1 \int_{\mathbb{R}^2}(1+a^2)^{-1/2-\epsilon/2}(1+b^2)^{-1/2-\epsilon/2}\,da\,db < \infty.$$

Furthermore,

$$\int_{\mathbb{R}^2}\int_0^1 (v^2+a^2)^{-1/2}(v^2+b^2)^{-1/2}\,dv(1+a^2)^{-\epsilon/2}(1+b^2)^{-\epsilon/2}\,da\,db$$
(B.1)
$$= \int_0^1 \left(\int_{\mathbb{R}}(v^2+a^2)^{-1/2}(1+a^2)^{-\epsilon/2}\,da\right)^2\,dv,$$

and

$$\frac{1}{2}(v+a)^2 \leq v^2+a^2.$$

Moreover, for $v > 0$ and $a > 0$,

$$\int_0^\infty (v+a)^{-1}(1+a)^{-\epsilon}\,da \leq \int_0^1 (v+a)^{-1}\,da + \int_1^\infty a^{-1-\epsilon}\,da$$
$$= log(v^{-1}+1)+\epsilon^{-1}.$$

Hence the right hand side of (B.1) can be dominated by

$$\int_0^1 \left(\int_{\mathbb{R}}(|v|+|a|)^{-1}(1+|a|)^{-\epsilon}\,da\right)^2\,dv$$
$$\leq 4\int_0^1 \left(\int_{\mathbb{R}}(v^2+a^2)^{-1/2}(1+a^2)^{-\epsilon/2}\,da\right)^2\,dv$$
$$\leq 4\int_0^1 \left(log(v^{-1}+1)+\epsilon^{-1}\right)^2\,dv < \infty.$$

This verifies (10.38.b).

Let $z = u + iv$, and let $\psi(z) = \phi(u, v)$. We would like to show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2}|z-a|^{-1}\psi(z)\,du\,dv(1+a^2)^{-\epsilon/2}\,da < \infty. \tag{B.2}$$

It is easy to check that for $|z-a| \geq 1$,

$$\frac{1}{|z-a|} \leq \frac{2}{1+|z-a|} \leq 2\frac{1+|z|}{1+|a|} \leq 2\frac{1+|z|}{(1+a^2)^{1/2}}.$$

Hence,

$$\int_{\mathbb{R}} \int_{|z-a|\geq 1} |z-a|^{-1}\psi(z)\,du\,dv(1+a^2)^{-\epsilon/2}\,da$$
$$\leq 2\int_{\mathbb{R}}\int_{\mathbb{R}^2}(1+|z|)\psi(z)\,du\,dv(1+a^2)^{-1/2-\epsilon/2} < \infty.$$

Moreover,

$$\int_{|z-a|\leq 1} |z-a|^{-1}\psi(z)\,du\,dv \leq \int_{|z|\leq 1} \max_{|z-a|\leq 1} \phi(z)$$
$$\leq const\,(1+a^2)^{-1/2} < \infty.$$

Hence,

$$\int_{\mathbb{R}} \int_{|z-a|\leq 1} |z-a|^{-1}\psi(z)\,du\,dv(1+a^2)^{-\epsilon/2}\,da < \infty.$$

This verifies (10.38.c).

Since,

$$((u-a)^2+v^2)^{-1/2}(1+a^2)^{-\epsilon/2} \leq 4(|u-a|+|v|)^{-1}(1+|a|)^{-\epsilon},$$

we consider the following integral

(B.3) $$\int_0^\infty (|u-a|+|v|)^{-1}(1+|a|)^{-\epsilon}\,da,$$

for an arbitrary $u\in\mathbb{R}$ and $v\geq 0$. If $u\leq 0$, then the integral (B.3) is equal to

$$\int_0^\infty (a-u+v)^{-1}(1+a)^{-\epsilon}\,da \leq log((v-u)^{-1}+1)+\epsilon^{-1}.$$

If $u>0$, then the integral (B.3) is equal to

$$\int_0^u (u-a+v)^{-1}(1+a)^{-\epsilon}\,da + \int_u^\infty (a-u+v)^{-1}(1+a)^{-\epsilon}\,da.$$

Notice that

$$\int_0^u (u-a+v)^{-1}(1+a)^{-\epsilon}\,da \leq \int_0^u (u-a+v)^{-1}\,da$$
$$= log(u+v) - log(u),$$

and that

$$\int_u^\infty (a-u+v)^{-1}(1+a)^{-\epsilon}\,da \leq \int_0^\infty (a-u+v)^{-1}(1+a)^{-\epsilon}\,da$$
$$\leq log((v-u)^{-1}+1)+\epsilon^{-1}.$$

Hence, the square of the integral (B.3) is integrable against the rapidly decreasing function $\phi(u,v)$. This verifies (10.38,d).

References

[A-S] M. Abramowitz, I. Stegun, Handbook of mathematical functions. National Bureau of Standards, Applied Mathematics Series 1964

[B] A. Bouaziz, Sur les caractéres des groupes de Lie réductifs non connexes. J. Func. Anal. **70** (1987), 1–79

[D-P1] A. Daszkiewicz, T. Przebinda, The Oscillator Correspondence of Orbital Integrals, for pairs of type one in the stable range. Duke Math. J. **82** (1996) 1–20

[D-P2] A. Daszkiewicz, T. Przebinda, The Oscillator Character Formula, for isometry groups of split forms in deep stable range. Invent. math. **123** (1996) 349–376

[D-P3] A. Daszkiewicz, T. Przebinda, A Cauchy Harish-Chandra Integral for the pair $\mathfrak{u}_{p,q}, \mathfrak{u}_1$. In preparation

[He] S. Helgason, Groups and Geometric Analysis. Academic Press, INC, 1984

[H1] R. Howe, Transcending Classical Invariant Theory. J. Amer. Math. Soc. **2** (1989) 535–552

[H2] R. Howe, The oscillator semigroup. Proc. Symp. Pure Math., Amer. Math. Soc.: Providence **48** (1988) 61–132

[H3] R. Howe, Wave Front Sets of Representations of Lie Groups In: Automorphic forms, Representation Theory and Arithmetic, pp. 117–140. Tata Institute of Fundamental Research, Bombay 1981

[H4] R. Howe, L^2 Duality in the Stable Range. Preprint

[H5] R. Howe, A century of Lie theory. American Mathematical Society centennial publications, Vol. II (Providence, RI, 1988), 101–320, Amer. Math. Soc., Providence, RI, 1992

[Hö] L. Hörmander, The Analysis of Linear Partial Differential Operators, I. Springer Verlag 1983

[Li] Jian-Shu Li, Singular unitary representations of classical groups. Invent. math. **97** (1989) 237–255

[M] L. G. Macdonald, Symmetric Functions and Hall Polynomials. Clarendon Press, Oxford 1979

[P1] T. Przebinda, Characters, Dual Pairs and Unipotent Representations. J. Func. Anal. **98** (1991) 59–96

[P2] T. Przebinda, Characters, Dual Pairs and Unitary Representations. Duke Math. J. **69** (1993) 547–592

[R1] W. Rossmann, Limit Characters of Reductive Lie Groups. Invent. math. **61** (1980) 53–66

[R2] W. Rossmann, Picard-Lefschetz theory and characters of a semisimple Lie group. Invent. math. **121** (1995) 579–611

[R3] W. Rossmann, Lie Groups, An Introduction Through Linear Groups. An unpublished book

[RR] R. Ranga Rao, Orbital integrals in reductive groups. Ann. Math. **96** (1972) 505–510

[Sch] W. Schmid, On the Characters of the Discrete Series, The Hermitian Symmetric Case. Invent. math. **30** (1975) 47–144

[S] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton University Press 1993

[Va] V. S. Varadarajan, Harmonic Analysis on Real Reductive Groups. Springer Verlag, Lecture Notes in Mathematics **576**, 1977

[W1] N. Wallach, Real Reductive Groups, I. Academic Press, INC, 1988

[W2] N. Wallach, Invariant differential operators on a reductive Lie algebra and Weyl group representations. J. Amer. Math. Soc. **6** (1993) 779–816

[Wy] H. Weyl, The Classical Groups, their invariants and representations. Princeton University Press 1939