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The Cauchy Harish-Chandra Integral, for the pair $\mathfrak{u}_{p,q}, \mathfrak{u}_1$

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Abstract: For the dual pair considered, the Cauchy Harish-Chandra Integral, as a distribution on the Lie algebra, is the limit of the holomorphic extension of the reciprocal of the determinant. We compute that limit explicitly in terms of the Harish-Chandra orbital integrals.

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1 Introduction

This article was completed in summer 1998, while the work [5] was still in progress. We publish the original version without any essential changes.

One of the main problems in the theory of dual pairs is the description of the correspondence of characters of representations in Howe duality, [3]. In [2] a formula describing this correspondence was obtained under some very strong assumptions. In [5] the second author has developed a notion of a Cauchy Harish–Chandra integral for any real reductive pair, in order to describe this correspondence of characters. In this paper a special case of this integral will be studied. The results obtained here are crucial for the estimates needed in [5]. (See the proof of Theorem 10.19, page 343, in [5].)

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In this paper we consider the Lie algebra $\mathfrak{g} = \mathfrak{u}_{p,q} = \{z \in \mathfrak{gl}_{p+q}(\mathbb{C});\ z\, I_{p,q} + I_{p,q}\, \overline{z}^t = 0\}$. We assume, for convenience, that $p \le q$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be the diagonal Cartan subalgebra. Here $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, as usual. Let $H_j = E_{jj} \in \mathfrak{h}_{\mathbb{C}}$, $1 \le j \le p+q$, be the diagonal matrix with 1 in the jth row and jth column and zeros elsewhere. Then $H_1, H_2, ..., H_{p+q}$ is a basis of the vector space $\mathfrak{h}_{\mathbb{C}}$. Let $e_1, e_2, ..., e_{p+q} \in \mathfrak{h}_{\mathbb{C}}^*$ denote the dual basis. We fix the following system of positive roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, $\Phi(\mathfrak{h}) = \{e_j - e_k;\ 1 \le j < k \le p+q\}$. Let $\mathfrak{k} \subseteq \mathfrak{g}$ be the centralizer of $I_{p,q}$. Then $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of $G = U_{p,q} = \{g \in GL(\mathbb{C});\ g\, I_{p,q}\, \overline{g}^t = I_{p,q}\}$, and $\mathfrak{h} \subseteq \mathfrak{k}$. The set of non-compact roots in $\Phi(\mathfrak{h})$ is $\Phi^n(\mathfrak{h}) = \{e_j - e_k;\ 1 \le j \le p < k \le p+q\}$. Let π denote the product of all the roots in $\Phi(\mathfrak{h})$:

$$\pi = \prod_{1 \le j < k \le p+q} (e_j - e_k)\,.$$

For a root $\alpha \in \Phi^n(\mathfrak{h})$, let $c_\alpha \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ be the Cayley transform and let $H_\alpha \in i\,\mathfrak{h}$ be the corresponding element, as in [1, 3.1], ($H_\alpha = H_j - H_k$, if $\alpha = e_j - e_k$). For a strongly orthogonal set $S \subseteq \Phi^n(\mathfrak{h})$ let $c_S = \prod_{\alpha \in S} c_\alpha$, and let $\mathfrak{h}_S = \mathfrak{g} \cap c_S(\mathfrak{h}_{\mathbb{C}})$ be the corresponding Cartan subalgebra, as in [6, sec. 2]. Denote by $H_S \subseteq G$ the corresponding Cartan subgroup. For any $\alpha \in S$, the root $-\alpha \circ c_S^{-1} = \overline{\alpha} \circ c_S^{-1}$ of $\mathfrak{h}_{S,\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, is real. For $x \in \mathfrak{h}_S^r$, the set of regular elements in $\mathfrak{h}_S$, set $\epsilon_S(x) = \prod_{\alpha \in S} \mathrm{sgn}(\overline{\alpha} \circ c_S^{-1}(x))$. The formula,

$$P(x, y) = \mathrm{tr}(x\overline{y}^t) \qquad (x, y \in \mathfrak{g}) \tag{1.0}$$

defines a real valued, positive definite scalar product on $\mathfrak{g}$, viewed as a real vector space. This scalar product determines a Lebesgue measure dx on $\mathfrak{g}$, such that for any basis e_1, e_2, ..., e_n of $\mathfrak{g}$ the volume of the parallelepiped $Ie_1 + Ie_2 + ... + Ie_n$, where $I = (0, 1)$ is the unit interval, is equal to $\det(P(e_i, e_j))^{1/2}$. Similarly P determines an Lebesgue measure on each subspace of $\mathfrak{g}$, an invariant measure on the group G, on each closed unimodular Lie subgroup and on each quotient of two such subgroups.

Recall the Harish–Chandra integral defined with respect to the negative roots:

$$\psi_S(x) = \overline{\pi} \circ c_S^{-1}(x)\, \epsilon_S(x) \int_{G/H_S} \psi(gxg^{-1})\, d\dot{g} \qquad (x \in \mathfrak{h}_S^r,\ \psi \in S(\mathfrak{g})). \tag{1.1}$$

Here $\mathfrak{h}^r \subseteq \mathfrak{h}$ is the subset of regular elements, (see [8, 0.2.1]). Set $\alpha_j = e_j - e_{p+j}$, $1 \le j \le p$, and for $m = 1, 2, 3, ..., p$, define $S_m = \{\alpha_1, \alpha_2, ..., \alpha_m\}$. Let $S_0 = \emptyset$. Then each S_m, $0 \le m \le p$, is a strongly orthogonal set and, in terms of the Cartan subalgebras corresponding to these sets, the Weyl integration formula may be written as follows

$$\int_{\mathfrak{g}} \psi(x)\, dx = \sum_{m=0}^{p} \frac{1}{(p-m)!(q-m)!} \int_{\mathfrak{h}_{S_m}^+} \pi \circ c_{S_m}^{-1}(x)\, \psi_{S_m}(x)\, dx, \tag{1.2}$$

where $\psi \in S(\mathfrak{g})$, $\mathfrak{h}_{S_0}^+ = \mathfrak{h}^r$, and for $m \ge 1$, $\mathfrak{h}_{S_m}^+ = \{x \in \mathfrak{h}_S;\ \tilde{\alpha}_1(x) > ... > \tilde{\alpha}_m(x) > 0\}$, and $\tilde{\alpha}_j = -\frac{1}{2}\alpha_j \circ c_{S_m}^{-1}$. Let

$$Y = \sum_{j=1}^{p} i\, H_j - \sum_{j=p+1}^{p+q} i\, H_j, \quad Y_{S_m} = \sum_{e_j \perp S_m} e_j(Y)\, H_j \qquad (0 \le m \le p). \tag{1.3}$$

Here, "$e_j \perp S_m$" means "$\alpha(H_j) = 0$, for all $\alpha \in S_m$". If this condition is empty then $Y_{S_m} = 0$. Let $\mathfrak{g}^+ = \{y \in \mathfrak{g};\ -iyI_{p,q} > 0\}$. This is an open convex cone in $\mathfrak{g}$. In terms of limits of holomorphic functions, [4, 3.1.15], define the following temperate distribution:

$$\frac{1}{\det(x + i\,0)} = \lim_{\mathfrak{g}^+ \ni y \to 0} \frac{1}{\det(x + i\,y)} \qquad (x \in \mathfrak{g}). \tag{1.4}$$

Clearly, this distribution is $Ad(G)$-invariant. The goal of this paper is to prove the following, seemingly obvious theorem, which expresses the distribution (1.4) in terms of integrals over various Cartan subgroups.

Theorem 1.5. *For any* $\psi \in S(\mathfrak{g})$,

$$\begin{aligned}&\int_{\mathfrak{g}} \frac{1}{\det(x + i\,0)} \psi(x)\,dx \\ &= \lim_{\epsilon \to 0+} \sum_{m=0}^{p} \frac{1}{(p-m)!(q-m)!} \int_{\mathfrak{h}^+_{S_m,\epsilon}} \frac{\pi \circ c_{S_m}^{-1}(x + i\,\epsilon\, Y_{S_m})}{\det \circ c_{S_m}^{-1}(x + i\,\epsilon\, Y_{S_m})} \psi_{S_m}(x)\,dx\end{aligned}$$

where $\mathfrak{h}^+_{S_0,\epsilon} = \mathfrak{h}^r$, $\mathfrak{h}^+_{S_m,\epsilon} = \{x \in \mathfrak{h}_S;\ \tilde{\alpha}_1(x) > ... > \tilde{\alpha}_m(x) > \epsilon\}$ for $m \geq 1$, *and the integrals on the right hand side are absolutely convergent.*

Let

$$G^+_{\mathbb{C}} = \{g \in GL_n(\mathbb{C});\ \text{the hermitian matrix } \left(I_{p,q} - \overline{g}^t I_{p,q} g\right) \text{ is positive definite}\}.$$

Clearly, $G^+_{\mathbb{C}}$ is a sub-semigroup of $GL_n(\mathbb{C})$, and $G \cdot G^+_{\mathbb{C}} \subseteq G^+_{\mathbb{C}}$. In terms of limits of holomorphic functions, define the following distribution on G:

$$\frac{1}{\det(1 - g \cdot 1)} = \lim_{\{p \to 1,\ p \in G^+_{\mathbb{C}}\}} \frac{1}{det(1 - g \cdot p)} \qquad (g \in G). \tag{1.6}$$

For a strongly orthogonal set $S \subseteq \Phi^n(\mathfrak{h})$, let $H_S = \exp(\mathfrak{h}_S) \subseteq G$ be the corresponding Cartan subgroup, and let $C_S : H_{\mathbb{C}} \to H_{S,\mathbb{C}}$, be the Cayley transform. Let

$$\Delta(h) = \prod_{j<k} (h^{(e_j - e_k)/2} - h^{(e_k - e_j)/2}) \qquad (h \in H_{\mathbb{C}}).$$

Recall the Harish–Chandra integral, defined with respect to the negative roots:

$$\begin{aligned}\Psi_S(h) &= \overline{\Delta} \circ C_S^{-1}(h) \tilde{\epsilon}_S(h) \int_{G/H_S} \Psi(ghg^{-1})d\dot{g}, \\ \tilde{\epsilon}(h) &= \prod_{\alpha \circ c_S^{-1} \text{ real}} \operatorname{sgn}(1 - h^{-\alpha \circ c_S^{-1}}) \qquad (h \in H_S^r),\end{aligned} \tag{1.7}$$

where $H_S^r \subseteq H_S$ is the subset of regular elements. Set $H^+_{S_m} = \exp(\mathfrak{h}^+_{S_m})$. Then the Weyl integration formula for G says

$$\int_G \Psi(g)\,dg = \sum_{m=0}^{p} \frac{1}{(p-m)!(q-m)!} \int_{H^+_{S_m}} \Delta \circ C_{S_m}^{-1}(h) \Psi_{S_m}(h)\,dh \tag{1.8}$$

where $\Psi \in C_c(G)$. With the $\mathfrak{h}_{S_m,\epsilon}$ as in (1.5), set $H^+_{S_m,\epsilon} = \exp(\mathfrak{h}_{S_m,\epsilon})$.

Theorem 1.9. *For $\Psi \in C_c^\infty(G)$,*

$$\int_G \frac{1}{\det(1 - g \cdot 1)} \Psi(g)\, dg$$
$$= \lim_{\epsilon \to 0} \sum_{m=0}^{p} \frac{1}{(p-m)!(q-m)!} \int_{H^+_{S_m,\epsilon}} \frac{\Delta \circ C_{S_m}^{-1}(h \exp(i\epsilon Y_{S_m}))}{\det(1 - C_{S_m}^{-1}(h \exp(i\epsilon Y_{S_m})))} \Psi_{S_m}(h)\, dh,$$

where the integrals on the right hand side are absolutely convergent.

2 Integration by parts

Let V be a finite dimensional space over the reals. Let V^* denote the linear dual to V. Fix elements $e \in V$ and $e^* \in V^*$ such that $e^*(e) = 1$. Let f be a smooth function on V and let ϕ be a bounded, smooth function on $V \setminus \ker e^*$, the complement of $\ker e^*$ in V. Recall the directional derivative:

$$\partial(e) f(x) = \frac{d}{dt} f(x + t\, e)|_{t=0} \qquad (x \in V). \tag{2.1}$$

Assume that f and all derivatives of f are of at most polynomial growth at infinity, and that ϕ and all derivatives of ϕ are rapidly decreasing at infinity. Suppose we have an Euclidean norm on V. Assume that e has norm 1 and that it is orthogonal to $\ker(e^*)$. Then every subspace of V is equipped with a Lebesgue measure dx, normalized so that the volume of the unit cube is 1. Integration by parts verifies the following formula

$$\begin{aligned} &\int_{e^*(x)>\epsilon} (f(x)(\partial(e^n)^t \phi(x)) - (\partial(e^n) f(x))\phi(x))dx \\ &= \int_{\ker e^*} \sum_{k=0}^{n-1} \partial(e^{n-1-k}) f(x + \epsilon e) \partial(e^k)^t \phi(x + \epsilon e)\, dx \qquad (\epsilon > 0,\ n = 1, 2, ...). \end{aligned} \tag{2.2}$$

Here $\partial(e^n)^t = (-1)^n \partial(e^n)$ stands for the adjoint of the differential operator $\partial(e^n)$. Let us assume that the following limits exist

$$\partial(e^k)\phi(x \pm 0\, e) = \lim_{t \to 0+} \partial(e^k)\phi(x \pm t\, e) \qquad (x \in \ker e^*;\ k = 0, 1, 2, ...). \tag{2.3}$$

Then (2.2) implies

$$\begin{aligned} &\int_{e^*(x)\neq 0} (f(x)(\partial(e^n)^t \phi(x)) - (\partial(e^n) f(x))\phi(x))dx \\ &= \int_{\ker e^*} \sum_{k=0}^{n-1} \partial(e^{n-1-k}) f(x) (\partial(e^k)^t \phi(x + 0\, e) - \partial(e^k)^t \phi(x - 0\, e))\, dx, \end{aligned} \tag{2.4}$$

where $n = 1, 2, ...$.

Recall that the map ∂, defined in (2.1), extends to an isomorphism of the symmetric algebra $Sym(V_{\mathbb{C}})$ and the algebra of constant coefficient differential operators on V. For any $w \in Sym(V_{\mathbb{C}})$ there are uniquely determined elements $w_n \in Sym(\ker e^*)$ such that

$$w = \sum_{n\geq 0} w_n e^n. \tag{2.5}$$

By combining (2.2), (2.4) and (2.5) we deduce that for any $w \in Sym(V_{\mathbb{C}})$ and any $\epsilon > 0$, the following two formulas hold

$$\begin{aligned}&\int_{e^*(x)>\epsilon} (f(x)(\partial(w)^t\phi(x)) - (\partial(w)f(x))\phi(x))dx \\ &= \sum_{n\geq 1}\sum_{k=0}^{n-1}\int_{\ker\, e^*} \partial(e^{n-1-k})\partial(w_n)f(x+\epsilon\, e)\partial(e^k)^t\phi(x+\epsilon\, e)\, dx,\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\int_{e^*(x)\neq 0} (f(x)(\partial(w)^t\phi(x)) - (\partial(w)f(x))\phi(x))dx \\ &= \sum_{n\geq 1}\sum_{k=0}^{n-1}\int_{\ker\, e^*} \partial(e^{n-1-k})\partial(w_n)f(x)(\partial(e^k)^t\phi(x+0\, e) - \partial(e^k)^t\phi(x-0\, e))\, dx,\end{aligned} \tag{2.7}$$

3 Proof of the Theorem 1.5

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the bilinear form

$$B(x,y) = \mathrm{tr}(xy) \qquad (x,y\in\mathfrak{g}).$$

(Notice that B takes only real values.) Given a polynomial function P on $\mathfrak{g}_{\mathbb{C}}$, let $P^{\#}$ be the corresponding element of the symmetric algebra $Sym(\mathfrak{g}_{\mathbb{C}})$.

Lemma 3.1. *In terms of germs of holomorphic functions, the following formula holds:*

$$\partial(\det{}^{\#})\log(\det(z)) = \frac{(n-1)!}{\det(z)} \qquad (z\in\mathfrak{g}_{\mathbb{C}},\ \det(z)\neq 0),$$

where $n = p+q$, *and* log *is the natural logarithm.*

Proof. Notice that, by the definition (2.1), the map ∂ depends on the real form $\mathfrak{g}\subseteq\mathfrak{g}_{\mathbb{C}}$. We shall write $\partial_{\mathfrak{g}}$ in order to indicate this dependence. For a function f defined on $\mathfrak{g}$ and for an element $g\in GL_n(\mathbb{C})$, let $\lambda(g)f(x) = f(g^{-1}x)$ be a function defined on the set $g\mathfrak{g}$. Then for a polynomial P on $\mathfrak{g}_{\mathbb{C}}$,

$$\lambda(g)\partial_{\mathfrak{g}}(P^{\#})f = \partial_{g\mathfrak{g}}((\rho(g)P)^{\#})\lambda(g)f, \tag{3.2}$$

where $\rho(g)P(x) = P(x\,g)$. Let $g = I_{p,q}$. Then $g\mathfrak{g} = \mathfrak{u}_n$. Hence, (3.2) implies that, with $f(z) = \log(\det(z))$,

$$\begin{aligned}\partial_{\mathfrak{u}_{p,q}}(\det{}^{\#})f &= \lambda(g^{-1})\partial_{\mathfrak{u}_n}(\det(g)\cdot\det{}^{\#})\lambda(g)f \\ &= \det(g)\lambda(g^{-1})\partial_{\mathfrak{u}_n}(\det{}^{\#})\lambda(g)f = \partial_{\mathfrak{u}_n}(\det{}^{\#})f,\end{aligned}$$

where the last equation holds because (locally) f is a holomorphic function on $\mathfrak{u}_{n,\mathbb{C}} = \mathfrak{u}_n + i\,\mathfrak{u}_n = gl_n(\mathbb{C})$. Thus, in order to prove the lemma, we may assume that $\mathfrak{g} = \mathfrak{u}_n$.

Harish–Chandra's theorem on the radial component of an invariant differential operator, [8, 7.A.2.9], implies that our lemma will follow as soon as we show that

$$\frac{1}{\pi(x)}\partial((\det|_{\mathfrak{h}})^{\#})\pi(x)\,\log(\det(x)) = \frac{(n-1)!}{\det(x)} \qquad (x \in \mathfrak{h}^r). \tag{3.3}$$

The equation (3.3) is equivalent to

$$\det(z)\,\partial_{z_1}\partial_{z_2}\dots\partial_{z_n}(\pi(z)\,\log(\det(z))) = (n-1)!\pi(z) \qquad (z \in \mathfrak{h}_{\mathbb{C}}), \tag{3.4}$$

where $z_j = e_j(z),\ 1 \le j \le n$. Let $\mathfrak{S}_n$ denote the group of permutations of elements of the set $\{0, 1, 2, \dots, (n-1)\}$. Recall (Vandermonde), that

$$\pi(z) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) z_1^{\sigma(n-1)} z_2^{\sigma(n-2)} \dots z_n^{\sigma(0)} \qquad (z \in \mathfrak{h}_{\mathbb{C}}).$$

A straightforward calculation shows that for $\gamma_k = 0, 1, 2, \dots,$

$$\begin{aligned}
&z_1 z_2 \dots z_n\, \partial_{z_1}\partial_{z_2}\dots\partial_{z_n}(z_1^{\gamma_1} z_2^{\gamma_2}\dots z_n^{\gamma_n}\log(z_1 z_2 \dots z_n))\\
&= \gamma_1\gamma_2\dots\gamma_n z_1^{\gamma_1} z_2^{\gamma_2}\dots z_n^{\gamma_n}\log(z_1 z_2\dots z_n) + \sum_{k=1}^{n}\gamma_1\gamma_2\dots\widehat{\gamma}_k\dots\gamma_n\, z_1^{\gamma_1} z_2^{\gamma_2}\dots z_n^{\gamma_n},
\end{aligned}$$

where the hat, $\widehat{\gamma}_k$, indicates that γ_k is missing in the product. Hence, the left hand side of (3.4) coincides with

$$\begin{aligned}
&\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\Big(\sigma(n-1)\sigma(n-2)\dots\sigma(0) z_1^{\sigma(n-1)} z_2^{\sigma(n-2)}\dots z_n^{\sigma(0)}\log(z_1 z_2 \dots z_n)\\
&+\sum_{k=1}^{n}\sigma(n-1)\sigma(n-2)\dots\widehat{\sigma}(n-k)\dots\sigma(0) z_1^{\sigma(n-1)} z_2^{\sigma(n-2)}\dots z_n^{\sigma(0)}\Big)\\
&= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)(n-1)!\, z_1^{\sigma(n-1)} z_2^{\sigma(n-2)}\dots z_n^{\sigma(0)} = (n-1)!\pi(z),
\end{aligned}$$

which coincides with the right hand side of (3.4). □

Lemma 3.5. *Let* $u = (\det_{\mathfrak{h}})^{\#} \in Sym(\mathfrak{h}_{\mathbb{C}})$, *and let* $F(z) = \log\circ\det(z)\cdot\pi(z),\ z \in \mathfrak{h}_{\mathbb{C}}$. *Then for* $\psi \in S(\mathfrak{g})$

$$\begin{aligned}
&\int_{\mathfrak{g}} \frac{(p+q-1)!}{\det(x+i0)}\psi(x)\,dx\\
&= \lim_{\epsilon\to 0+}\sum_{m=0}^{p}\frac{1}{(p-m)!(q-m)!}\int_{\mathfrak{h}^+_{S_m,\epsilon}} F(c_{S_m}^{-1}(x+i\epsilon Y_{S_m}))\partial(c_{S_m}u)^t\psi_{S_m}(x)\,dx.
\end{aligned}$$

Proof. The limit

$$\log(\det(x+i\,0)) = \lim_{\{y\to 0,\ y\in\mathfrak{g}^+\}}\log(\det(x+i\,y)) \qquad (x \in \mathfrak{g})$$

exists in the sense of distributions, [4, 3.1.15], and coincides with the indicated locally integrable function. Also, by (3.1),

$$\int_{\mathfrak{g}} \frac{(p+q-1)!}{\det(x+i0)}\psi(x)\,dx = \int_{\mathfrak{g}} \log(\det(x+i\,0))\partial(\det{}^{\#})^t\psi(x)\,dx.$$

Thus the lemma follows from the Weyl integration formula and Harish-Chandra's theorem on the radial component of an invariant differential operator, [8, 7.A.2.9]. □

Set

$$\begin{aligned} I_{S_m,\epsilon} = \frac{1}{(p-m)!(q-m)!}\int_{\mathfrak{h}^+_{S_m,\epsilon}} &(F\circ c_{S_m}^{-1}(x+i\epsilon Y_{S_m})\partial(c_{S_m}u)^t\psi_{S_m}(x)- \\ &\partial(c_{S_m}u)(F\circ c_{S_m}^{-1})(x+i\epsilon Y_{S_m})\psi_{S_m}(x))\,dx, \end{aligned} \tag{3.6}$$

where $u=(\det|_{\mathfrak{h}})^{\#}$, $\psi\in S(\mathfrak{g}_{\mathbb{C}})$, and $\mathfrak{h}^+_{S_m,\epsilon}$ is as in (1.5). In order to prove Theorem 1.5 it will suffice to show that

$$\lim_{\epsilon\to 0}\sum_{m=0}^{p} I_{S_m,\epsilon}=0. \tag{3.7}$$

For a root $\alpha\in\Phi^n(\mathfrak{h})$ let (as in (2.5))

$$u=\sum_{n\geq 0} u_{n,\alpha}(iH_\alpha)^n \qquad (u_{n,\alpha}\in Sym(\ker\,\alpha)). \tag{3.8}$$

Lemma 3.9. *Let us multiply the form (1.0) by a positive constant such that the norm of each iH_α is 1. Then, with the above notation we have*

$$\begin{aligned} \sum_{m=0}^{p} I_{S_m,\epsilon} = \sum_{m=1}^{p}\frac{1}{(p-m)!(q-m)!}\sum_{n\geq 1}\sum_{k=0}^{n-1}\int_{\mathfrak{h}^+_{S_m,\epsilon,0}} &\partial((iH_{\alpha_m})^{n-1-k})\partial(u_{n,\alpha_m}) \\ F(c_{S_m}^{-1}(x+i\epsilon Y_{S_{m-1}}))i\,\big(&\partial((c_{\alpha_m}iH_{\alpha_m})^k)^t\psi_{S_m}(x) \\ -&\partial((c_{\alpha_m}iH_{\alpha_m})^k)^t\psi_{S_m}(x-\epsilon c_{\alpha_m}H_{\alpha_m})\big)\,dx, \end{aligned}$$

where $\mathfrak{h}^+_{S_m,\epsilon,0}=\{x\in\mathfrak{h}_{S_m};\ \tilde\alpha_1(x)>...>\tilde\alpha_{m-1}(x)>\epsilon,\ \tilde\alpha_m(x)=0\}$.

Proof. Consider the integral (3.6). Suppose first that

$$\operatorname{supp}\psi_{S_m}\cap\bigcup_{\beta\in\Phi^n(\mathfrak{h})\cap S_m^{\perp}}\ker(\beta\circ c_{S_m}^{-1})=\emptyset.$$

Notice that

$$\begin{aligned} c_{S_m}u &= \sum_{n\geq 0} c_{S_m}u_{n,\alpha_m}(c_{S_m}iH_{\alpha_m})^n = \sum_{n\geq 0} c_{\alpha_m}u_{n,\alpha_m}(c_{\alpha_m}iH_{\alpha_m})^n \\ &= \sum_{n\geq 0}(-i)^n c_{\alpha_m}u_{n,\alpha_m}(-c_{\alpha_m}H_{\alpha_m})^n. \end{aligned}$$

By applying (2.6) with $V = \mathfrak{h}_S$, $e = -c_{\alpha_m} H_{\alpha_m}$ and the w replaced by $c_{S_m} u$, we see that

$$\begin{aligned} & I_{S_m,\epsilon} \\ & = \frac{1}{(p-m)!(q-m)!} \sum_{n\geq 1} \sum_{k=0}^{n-1} \int_{\mathfrak{h}^+_{S_m,\epsilon,0}} \partial((-c_{\alpha_m} H_{\alpha_m})^{n-1-k})(-i)^n \partial(c_{S_m} u_{n,\alpha_m}) \\ & (F \circ c_{S_m}^{-1})(x - \epsilon c_{\alpha_m} H_{\alpha_m} + i\epsilon Y_{S_m}) \partial((-c_{\alpha_m} H_{\alpha_m})^k)^t \psi_{S_m}(x - \epsilon c_{\alpha_m} H_{\alpha_m})\, dx \qquad (3.10) \\ & = -\frac{1}{(p-m)!(q-m)!} \sum_{n\geq 1} \sum_{k=0}^{n-1} \int_{\mathfrak{h}^+_{S_m,\epsilon,0}} \partial((iH_{\alpha_m})^{n-1-k}) \partial(u_{n,\alpha_m}) \\ & F(c_{S_m}^{-1}(x - \epsilon c_{\alpha_m} H_{\alpha_m} + i\epsilon Y_{S_m})) i\partial((c_{\alpha_m} iH_{\alpha_m})^k)^t \psi_{S_m}(x - \epsilon c_{\alpha_m} H_{\alpha_m})\, dx \end{aligned}$$

Recall that, with $S = S_m$, for $\beta \in \Phi^n(\mathfrak{h}) \cap S^\perp$ we have the Harish-Chandra's matching condition, [1, 3.1],

$$\partial(v)\psi_S(x + 0iH_\beta) - \partial(v)\psi_S(x - 0iH_\beta) = i\partial(c_\beta v)\psi_{S\cup\{\beta\}}(x), \qquad (3.11)$$

where $\beta \circ c_S^{-1}(x) = 0$, and x is not annihilated by any other non-compact imaginary root of $\mathfrak{h}_S$.

Suppose now that the support of ψ_S is disjoint with the set where $\tilde{\alpha}(x) = \epsilon$ for all $\alpha \in S$. The we apply (2.7) with $V = \mathfrak{h}_S$ and $e = iH_\beta$ for $\beta \in \Phi^n(\mathfrak{h}) \cap S^\perp$, and the matching condition (3.11), to see that

$$\begin{aligned} & I_{S_m,\epsilon} \\ & = \frac{1}{(p-m)!(q-m)!} \sum_{\beta} \sum_{n\geq 1} \sum_{k=0}^{n-1} \int_{\mathfrak{h}^+_{S_m,\epsilon},\ \beta\circ c_{S_m}^{-1}(x)=0} \partial((iH_\beta)^{n-1-k}) \partial(c_{S_m} u_{n,\beta}) \qquad (3.12) \\ & (F \circ c_{S_m}^{-1})(x + i\epsilon Y_{S_m}) i\partial((c_{S_m\cup\{\beta\}} iH_\beta)^k)^t \psi_{S_m\cup\{\beta\}}(x)\, dx. \end{aligned}$$

Notice also that, by definition (1.3),

$$-H_{\alpha_{m+1}} + iY_{S_{m+1}} = i(iH_{\alpha_{m+1}} + Y_{S_{m+1}}) = iY_{S_m}. \qquad (3.13)$$

From the definition (3.6) we deduce that the summands corresponding to various β in (3.12) are all equal to each other. Since the set $\Phi^n(\mathfrak{h}) \cap S_m^\perp$ has $(p-m)(q-m)$ elements, (3.12) is equal to

$$\begin{aligned} & I_{S_m,\epsilon} \\ & = \frac{1}{(p-m-1)!(q-m-1)!} \sum_{n\geq 1} \sum_{k=0}^{n-1} \int_{\mathfrak{h}^+_{S_{m+1},\epsilon,0}} \partial((iH_{\alpha_{m+1}})^{n-1-k}) \partial(u_{n,\alpha_{m+1}}) \qquad (3.14) \\ & F(c_{S_{m+1}}^{-1}(x - \epsilon c_{\alpha_{m+1}} H_{\alpha_{m+1}} + i\epsilon Y_{S_{m+1}})) i\partial((c_{S_{m+1}} iH_{\alpha_{m+1}})^k)^t \psi_{S_{m+1}}(x)\, dx. \end{aligned}$$

The integral (3.14) is non-zero only if $m < p$. (Otherwise there are no non-compact roots β.)

Hence, the lemma follows (via partition of unity) by adding (3.10) and (3.14) and grouping the terms with the same ψ_{S_m}. □

Proof of Theorem 1.5 Notice that

$$(\det|_{\mathfrak{h}})^{\#} = \prod_{j=1}^{p+q} H_j. \tag{3.15}$$

Also, for $1 \leq m \leq p$, $H_m H_{p+m} = \frac{1}{4}((H_m + H_{p+m})^2 - (H_m - H_{p+m})^2)$. Since $H_m - H_{p+m} = H_{\alpha_m}$, the decomposition (3.8) can be rewritten as

$$(\det|_{\mathfrak{h}})^{\#} = \frac{1}{4}(H_m + H_{p+m})^2 \left(\prod_{j\neq m, p+m} H_j \right) + \frac{1}{4} \left(\prod_{j\neq m, p+m} H_j \right) (iH_{\alpha_m})^2. \tag{3.16}$$

Therefore (3.9) shows that

$$\begin{aligned} &\sum_{m=0}^{p} I_{S_m,\epsilon} \\ &= \sum_{m=1}^{p} \frac{1}{4(p-m)!(q-m)!} \sum_{k=0}^{1} \int_{\mathfrak{h}^+_{S_m,\epsilon,0}} \partial((iH_{\alpha_m})^{1-k}) \partial(\prod_{j\neq m,p+m} H_j) \\ &F(c_{S_m}^{-1}(x + i\epsilon Y_{S_{m-1}})) i (\partial((c_{\alpha_m} iH_{\alpha_m})^k)^t \psi_{S_m}(x) \\ &- \partial((c_{\alpha_m} iH_{\alpha_m})^k)^t \psi_{S_m}(x - \epsilon c_{\alpha_m} H_{\alpha_m}))\, dx. \end{aligned} \tag{3.17}$$

In the formula (3.17), $c_{S_m}^{-1}(x + i\epsilon Y_{S_{m-1}}) = c_{S_{m-1}}^{-1}(x) + i\epsilon Y_{S_{m-1}}$. Hence,

$$c_{S_m}^{-1}(x + i\epsilon Y_{S_{m-1}}) = i \left(\sum_{j=1}^{m-1} (z_j H_j + \overline{z}_j H_{p+j}) + \sum_{j=m}^{p} z_j H_j + \sum_{j=p+m}^{p+q} z_j H_j \right), \tag{3.18}$$

where

$$\begin{aligned} \mathrm{Im}\, z_j &> \epsilon \quad \text{for } 1 \leq j \leq m-1 \\ \mathrm{Im}\, z_j &= \epsilon \quad \text{for } m \leq j \leq p \\ \mathrm{Im}\, z_j &= -\epsilon \quad \text{for } p+m \leq j \leq p+q. \end{aligned} \tag{3.19}$$

and

$$Re\, z_m = Re\, z_{p+m}. \tag{3.20}$$

Consider the functions

$$\partial(\prod_{j\neq m,p+m} H_j) F(z) = \partial_{z_1} \partial_{z_2} \dots \widehat{\partial}_{z_m} \dots \widehat{\partial}_{z_{m+p}} \dots \partial_{z_{p+q}} \log(z_1 z_2 \dots z_{p+q}), \tag{3.21}$$

and

$$\begin{aligned} &\partial(H_m - H_{p+m}) \partial(\prod_{j\neq m,p+m} H_j) F(z) \\ &= (\partial_{z_m} - \partial_{z_{p+m}}) \partial_{z_1} \partial_{z_2} \dots \widehat{\partial}_{z_m} \dots \widehat{\partial}_{z_{m+p}} \dots \partial_{z_{p+q}} \log(z_1 z_2 \dots z_{p+q}). \end{aligned} \tag{3.22}$$

The function (3.21) is a linear combination of terms

$$\begin{aligned} &\log(z_1 z_2 \dots z_{p+q}) \cdot \text{polynomial}(z_1, z_2, \dots, z_{p+q}), \\ &\text{polynomial}(z_1, z_2, \dots, z_{p+q}), \\ &\frac{z_m z_{p+m}}{z_l} \cdot \prod_{1\leq j<k\leq p+q;\ j\neq l,\ k\neq l} (z_j - z_k), \quad (l \neq m,\ l \neq p+m) \end{aligned} \tag{3.23}$$

The function (3.22) is a linear combination of terms

$$\begin{aligned}
&\log(z_1 z_2 \dots z_{p+q}) \cdot \text{polynomial}(z_1, z_2, \dots, z_{p+q}),\\
&\text{polynomial}(z_1, z_2, \dots, z_{p+q}),\\
&\frac{z_{p+m}}{z_m} \cdot \prod_{1 \le j < k \le p+q;\ j \ne m,\ k \ne m} (z_j - z_k),\\
&\frac{z_m}{z_{p+m}} \cdot \prod_{1 \le j < k \le p+q;\ j \ne p+m,\ k \ne p+m} (z_j - z_k),\\
&\frac{z_a}{z_l} \cdot \prod_{1 \le j < k \le p+q;\ j \ne l,\ k \ne l} (z_j - z_k),
\end{aligned} \tag{3.24}$$

where $a = m$, or $a = p + m$; $l \neq m$, and $l \neq p + m$. By combining (3.18) - (3.24) we see that each of the functions (3.21), (3.22) when evaluated at $c_{S_m}^{-1}(x + i\epsilon Y_{S_{m-1}})$, can be dominated by a constant multiple of $|\log(|x|)| + |x|^M$, for $M \geq 0$ large enough, independently of $0 < \epsilon < 1$. Hence, by dominated convergence, (3.7) holds, and we are done. □

4 A sketch of a proof of the Theorem 1.9

If the support of the test function Ψ (see the statement of Theorem 1.9) is disjoint with the singular support of the distribution (1.6), then the limit formula (1.9) holds, for trivial reasons.

Consider a semisimple point h in the singular support of the distribution (1.6). We may, and shall, assume that h belongs to one of the Cartan subgroups H_S, (see (1.7)). Let $Z = G^h$ denote the centralizer of h in G. Let $U \subseteq Z$ be a connected, completely invariant open neighborhood of h, contained in the set of regular elements of Z, (see [7]). Since the sets of the form $G \cdot U = \{gug^{-1};\ g \in G,\ u \in U\}$ cover the singular support of the distribution (1.6), we may assume that $\Psi \in C_c^\infty(G \cdot U)$.

The group G acts on the space $V = \mathbb{C}^{p+q}$ as the group of isometries of the hermitian form $(u, v) = \overline{v}^t I_{p,q} u$, $(u, v \in V)$. Let $V = V_1 \oplus V_2 \oplus \dots$ be a decomposition of V into the direct sum of eigenspaces for h. It is easy to see that the restriction of the form (,) to a V_j is either non-degenerate or zero. Hence, the group Z is isomorphic to a Cartesian product $GL_{n_1}(\mathbb{C}) \times \dots GL_{n_s}(\mathbb{C}) \times U_{p_1,q_1} \times \dots U_{p_c,q_c}$, by restriction. Moreover, there is only one factor (necessarily U_{p_j,q_j}) in this product, such that the restriction of the distribution (1.6) to U is singular on it. Thus, by descent, we may assume that $h = 1$.

Let $U_0 \subseteq \mathfrak{g}$ be a completely invariant open neighborhood of $0 \in \mathfrak{g}$, such that $x \to \exp(x)$ is an analytic diffeomorphism of U_0 onto $U = \exp(U_0)$. Then, with the standard normalization of the Haar measure on G, we have

$$\int_G \Psi(g)\, dg = \int_{\mathfrak{g}} \Psi(\exp(x)) \left| \det\left(\frac{\exp(-ad(x) - 1}{-ad(x)} \right) \right| dx.$$

The function

$$j(x) = \det\left(\frac{\exp(-ad(x) - 1}{-ad(x)} \right) \qquad (x \in U_0)$$

is invariant, analytic and positive. Hence the positive square root $j^{1/2}(x)$, $x \in U_0$, is well defined and extends to a holomorphic function in a neighborhood of U_0 in $\mathfrak{g}_{\mathbb{C}}$. Thus

$$\int_G \frac{1}{\det(1-g\cdot 1)}\Psi(g)\,dg = \int_{\mathfrak{g}} \frac{1}{\det(1-\exp(x)\cdot 1)} j^{1/2}(x)\cdot \overline{j^{1/2}(x)}\Psi(\exp(x))\,dx$$
$$= \int_{\mathfrak{g}} \frac{1}{\det(x+i0)}\left[\det\left(\frac{x}{1-\exp(x)}\right) j^{1/2}(x)\right]\overline{j^{1/2}(x)}\Psi(\exp(x))\,dx,$$

where the function in brackets is invariant and holomorphic in a neighborhood of the support of the test function $\Psi(\exp(x))$.

A slight modification of the proof of Theorem 1.5 shows that this theorem holds with the $\frac{1}{\det(x+i0)}$ replaced by $\frac{f(x)}{\det(x+i0)}$, where f is any invariant, holomorphic function in a neighborhood of the support of the test function ψ. A straightforward application of the limit formula 1.5 completes the proof.

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