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The Oscillator Character Formula, for isometry groups of split forms in deep stable range

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Abstract. Let $\widetilde{G}$ and $\widetilde{G}'$ be a reductive dual pair of the type mentioned in the title, with $\widetilde{G}$ the smaller member. Let Π and Π' be unitary representations of $\widetilde{G}$, $\widetilde{G}'$ which occur in Howe's correspondence. We express the distribution character of Π' in terms of the character of Π via an explicit integral kernel operator.

1. Introduction

Let W be a finite dimensional vector space over $\mathbf{R}$ with a non-degenerate symplectic form $\langle\, ,\rangle$. Let $Sp = Sp(W) \subseteq \mathrm{End}(W)$ denote the corresponding symplectic group with the Lie algebra $sp = sp(W) \subseteq \mathrm{End}(W)$. Let

$$c(z) = (z+1)(z-1)^{-1} \quad (z \in \mathrm{End}(W) \text{ with } z-1 \text{ invertible}) \tag{1.1}$$

denote the Cayley transform. This is a birational isomorphism form sp to Sp and vice versa. Set $c_- = -c$, so that $c_-(0) = 1$ is the identity.

Let $\chi(x) = \exp(2\pi i x)$, $x \in \mathbf{R}$. Normalize the Lebesgue measure dw on W so that for any positive definite compatible complex structure J on W

$$\int_W \chi\left(\frac{i}{2}\langle J(w), w\rangle\right) dw = 1\ . \tag{1.2}$$

Let $\widetilde{Sp}$ be the metaplectic group and let Θ denote the distribution character of the oscillator representation ω of $\widetilde{Sp}$ attached to χ, as in [H2]. We view Θ as a function on $\widetilde{Sp}$, see (2.18). Set

$$\chi_z(w) = \chi\left(\frac{1}{4}\langle zw, w\rangle\right) \quad (z \in sp,\ w \in W)\ . \tag{1.3}$$

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Let $G, G' \subseteq Sp$ be an irreducible dual pair, see [H1]. Denote the corresponding Lie algebras by $\mathfrak{g}, \mathfrak{g}'$ respectively. Recall the unnormalized moment maps $\tau_{\mathfrak{g}} : W \to \mathfrak{g}^*$, $\tau_{\mathfrak{g}'} : W \to \mathfrak{g}'^*$:

$$\tau_{\mathfrak{h}}(w)(z) = \tfrac{1}{4}\langle z w, w\rangle \quad (w \in W,\ z \in \mathfrak{h},\ \mathfrak{h} = \mathfrak{g} \text{ or } \mathfrak{g}')\,. \tag{1.4}$$

By the Howe–Witt theorem [H3] there is a $G \cdot G'$-invariant open dense subset $W^{\max} \subseteq W$ such that for any G orbit $\mathcal{O} \subseteq \tau_{\mathfrak{g}}(W^{\max})$ the set

$$\mathcal{O}' = \tau_{\mathfrak{g}'}(W^{\max} \cap \tau_{\mathfrak{g}}^{-1}(\mathcal{O})) \tag{1.5}$$

is a single G' orbit. Let $W^{\max} \ni w \to \dot{w} \in G\backslash W^{\max}$ denote the quotient map and let $d\dot{w}$ denote the measure on $G\backslash W^{\max}$ defined by

$$\int_W \phi(w)\, dw = \int_{G\backslash W^{\max}} \int_G \phi(gw)\, dg\, d\dot{w} \quad (\phi \in C_c(W^{\max}))\,. \tag{1.6}$$

Let ω^∞ be the smooth representation of $\widetilde{Sp}$ associated to ω. Let ω be realized on a Hilbert space H_ω, and let the subspace of smooth vectors, on which ω^∞ is defined, be written H_ω^∞.

Let $\widetilde{G}, \widetilde{G}'$ be the preimages of G, G' in $\widetilde{Sp}$ under the covering map $\widetilde{Sp} \ni \widetilde{g} \to g \in Sp$.

For $E = G, G'$ or $G \cdot G'$ let $\mathscr{R}(E, \omega)$ denote the set of infinitesimal equivalence classes of continuous irreducible admissible representations of $\widetilde{E}$ on locally convex spaces which are realized as quotients by $\omega^\infty(\widetilde{E})$ – invariant closed subspaces on H_ω^∞.

Theorem [H1, Theorem 1]. *The set $\mathscr{R}(G \cdot G', \omega)$ is the graph of bijection between all of $\mathscr{R}(G, \omega)$ and all of $\mathscr{R}(G', \omega)$. Moreover an element $\Pi \otimes \Pi'$ occurs as a quotient of ω^∞ in a unique way.*

The bijection $\mathscr{R}(G, \omega) \ni \Pi \leftrightarrow \Pi' \in \mathscr{R}(G', \omega)$ is called Howe's correspondence.

We assume from now on that G, G' is an irreducible dual pair of type I in the stable range with G the smaller member. (This notion was introduced by Howe and is explained in Sect. 3).

Fix an irreducible unitary representation Π of $\widetilde{G}$, whose restriction to the kernel of the covering map $\widetilde{G} \to G$ is a multiple of the unique nontrivial character of this kernel. Then, as shown by Li, [Li], Π occurs in Howe's correspondence and the irreducible admissible representation Π' of $\widetilde{G}'$ associated to Π is unitary. Let Θ_Π denote the character of Π and let $\Theta_{\Pi'}$ denote the character of Π'. Recall that G is the isometry group of a non-degenerate form $(\,,)$ on a finite dimensional vector space V over a division $\mathbf{R}$-algebra $\mathbf{D}$. Similarly G' is the isometry group of a non-degenerate form $(\,,)'$ on a finite dimensional vector space V' over $\mathbf{D}$. We prove the following theorem:

Theorem 1.7. *Suppose that*

(a) *the form $(\,,)'$ is split*;

(b) *if* G *is a symplectic group over* $\mathbf{D} = \mathbf{R}$ *or* $\mathbf{C}$ *then* $\dim_{\mathbf{D}} V <$ *the Witt index of* $(\,,\,)'$;

(c) *if* G' *is an orthogonal group over* $\mathbf{D} = \mathbf{R}$ *or* $\mathbf{C}$ *then* $\dim_{\mathbf{D}} V'$ *is a multiple of* 4;

(d) *if* G *is an orthogonal group over* $\mathbf{D} = \mathbf{R}$ *or* $\mathbf{C}$ *and* G_1 *is the Zariski identity component of* G *then the restriction of* Π *to* $\widetilde{G}_1$ *is reducible*;

(e) $d' \geqq (d+2)^2$, *where* $d' = \dim_{\mathbf{D}} V'$ *and* $d = \dim_{\mathbf{D}} V$.

Then there is a non-empty Zariski open subset $G''' \subseteq G'$ *and a measurable section* $G\backslash W^{\max} \ni \dot{w} \to w \in W^{\max}$ *such that for any* $\Psi \in C_c^\infty(\widetilde{G'''})$

$$\Theta_{\Pi'}(\Psi) = \int_{\widetilde{G}} \int_{G\backslash W^{\max}} \int_{\widetilde{G'}} \Psi(\widetilde{g}')\,\Theta(\widetilde{g}')\chi_{c(g')+c(g)}(w)\Theta(\widetilde{g})\overline{\Theta}_{\Pi}(\widetilde{g})\,d\widetilde{g}'d\dot{w}\,d\widetilde{g}\,,$$

where each consecutive integral is absolutely convergent. In other words, the function of $\widetilde{g}$ *and* w *obtained after integrating over* $\widetilde{G'}$ *is absolutely integrable over* $G\backslash W^{\max}$, *and the resulting function of* $\widetilde{g}$ *is absolutely integrable over* $\widetilde{G}$.

Remarks. (a) The title of this paper refers to the character formula of Theorem 1.7, and the "deep stable range" to the assumption (e). Since

$$\chi_{c(g')+c(g)}(w) = \chi_{c(g')}(w)\chi_{c(g)}(w) = \chi(\tau_{\mathfrak{g}'}(w)(c(g')))\chi(\tau_{\mathfrak{g}}(w)(c(g)))\,,$$

the above character formula may be understood as *a microlocalization of the orbit correspondence* (1.5) *provided by the moment maps.*

(b) Changing the order of integration in the character formula of Theorem 1.7 is impossible. We shall explain it in some detail. Consider the integral

$$(*) \qquad \int_{G\backslash W^{\max}} \int_{\widetilde{G}} \int_{\widetilde{G'}} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')+c(g)}(w)\Theta(\widetilde{g})\overline{\Theta}_{\Pi}(\widetilde{g})d\widetilde{g}'d\widetilde{g}\,d\dot{w}\,.$$

By our choice of G''', the support of the function Ψ does not touch the singularities of Θ. In other words, $\Psi \cdot \Theta$ is a compactly supported bounded function, hence the integral over $\widetilde{G'}$ is absolutely convergent. Let

$$(**) \qquad \phi(w) = \int_{\widetilde{G'}} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')}(w)\,d\widetilde{g}'\,.$$

Then

$$\int_{\widetilde{G}} \left| \int_{\widetilde{G'}} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')+c(g)}(w)\Theta(\widetilde{g})\overline{\Theta}_{\Pi}(\widetilde{g})d\widetilde{g}' \right| d\widetilde{g}$$
$$= \int_{\widetilde{G}} |\phi(w)\chi_{c(g)}(w)\Theta(\widetilde{g})\Theta_{\Pi}(\widetilde{g})|\,d\widetilde{g} = |\phi(w)| \int_{\widetilde{G}} |\Theta(\widetilde{g})\Theta_{\Pi}(\widetilde{g})|\,d\widetilde{g}\,.$$

Thus "every consecutive integral in $(*)$ is absolutely convergent" if and only if

$$\int_{\widetilde{G'}} |\Theta(\widetilde{g})\Theta_{\Pi}(\widetilde{g})|\,d\widetilde{g} < \infty\,.$$

But this never happens! If G is not compact then obviously the above integral is infinite. Suppose G is compact. Then Θ_{Π} is a continuous positive definite function, so it is enough to see that Θ is not locally absolutely integrable in any neighborhood of the identity. This is clear from the Weyl integral formula on G and from the formula (2.18) for Θ. For example, let $G = O(2)$ and let $G' = Sp_{2n}(\mathbb{R})$. Then

$$\int_{\widetilde{G}} |\Theta(\widetilde{g})|\, d\widetilde{g} = \text{const} \int_G |\det(g-1)|^{-n}\, dg$$
$$\geqq \text{const} \int_{SO(2)} |\det(g-1)|^{-n} dg = \text{const} \int_0^{2\pi} |1-\cos(t)|^{-n}\, dt = +\infty\,.$$

Thus the integral $(*)$ does not make sense.

Now few words explaining why the original integral in (1.7) works in the case when G is compact. Then

$$\int_{\widetilde{G}} \overline{\Theta}_\Pi(\widetilde{g})\omega(\widetilde{g})\, d\widetilde{g}$$

makes sense as *an operator in the Hilbert space of* ω. By definition (2.14) $\omega(\widetilde{g}) = \rho(T(\widetilde{g}))$, (see (2.7) and (2.8) for T). The above statement follows from the fact that the following integral

$$\int_{\widetilde{G}} \overline{\Theta}_\Pi(\widetilde{g})T(\widetilde{g})\, d\widetilde{g}$$

makes sense as a tempered distribution on W. But this means, in particular, that for any test function $\phi \in S(W)$

$$\int_{\widetilde{G}} \left| \int_W \overline{\Theta}_\Pi(\widetilde{g})T(\widetilde{g})(w)\phi(w)dw \right| d\widetilde{g} < \infty$$

If ϕ is defined by $(**)$ then in the following formula

$$\Theta_{\Pi'}(\Psi) = \int_{\widetilde{G}} \int_W \int_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')+c(g)}(w)\Theta(\widetilde{g})\overline{\Theta}_\Pi(\widetilde{g})d\widetilde{g}'\, dw\, d\widetilde{g}$$
$$= \int_{\widetilde{G}} \int_{G\backslash W^{\max}} \int_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')+c(g)}(w)\Theta(\widetilde{g})\overline{\Theta}_\Pi(\widetilde{g})d\widetilde{g}'\, d\dot{w}\, d\widetilde{g}$$

every consecutive integral is absolutely convergent. It gives the character $\Theta_{\Pi'}$ as shown in [P1], and the last equality is clear because G is compact.

The analytic difficulty in the case of a non-compact G is to *kill as many oscillations as possible before we get to the integral over* $\widetilde{G}$. This is accomplished by a correct choice of the model of ω and by lemma 4.3.

In the rest of this section we work under the assumptions of (1.7).

For $s \in \mathrm{End}_{\mathbf{D}}(V)$ let $\det_{\mathbf{R}}(s)$ denote the determinant of s viewed as an element of $\mathrm{End}_{\mathbf{R}}(V)$. Define

(1.8) $\qquad r = 2\dim_{\mathbf{R}}(\mathfrak{g})/\dim_{\mathbf{R}}(V), \qquad ch_{\mathfrak{g}}(z) = |\det_{\mathbf{R}}(z-1)|^{1/2} \quad (z \in \mathfrak{g})\,.$

Let us fix a real analytic lifting $\widetilde{c} : sp \to \widetilde{Sp}$ of the Cayley transform $c : sp \to Sp$. Set $\widetilde{c}_-(z) = \widetilde{c}(z)\widetilde{c}(0)^{-1}$, $z \in sp$. Then $\widetilde{c}_-$ is a lifting of the

Cayley Transform c_- and $\widetilde{c}_-(0) = 1$, the identity element of $\widetilde{Sp}$. By restriction $\widetilde{c} : \mathfrak{g} \to \widetilde{G}$, and similarly for G'. One can normalize the Lebesgue measure on $\mathfrak{g}$ so that

$$(1.9) \qquad \int_{G_1} \Psi(g)\, dg = \int_{\mathfrak{g}} \Psi(c(z))\, ch_{\mathfrak{g}}(z)^{-2r}\, dz \quad (\Psi \in C_c(G_1))\,,$$

see [P1, (3.11)]. The same formula (1.9) holds with c replaced by c_-, and similarly for G'.

Let

$$(1.10) \qquad \theta_{\Pi}(z) = \Theta(\widetilde{c}(z))\overline{\Theta}_{\Pi}(\widetilde{c}(z))\, ch_{\mathfrak{g}}(z)^{-2r} \quad (z \in \mathfrak{g})\,.$$

This function does not depend on the choice of $\widetilde{c}$. It follows from the explicit formula (2.18) for Θ that for a specific choice of $\widetilde{c}$, $\Theta(\widetilde{c}(z)) = \mathrm{const} \cdot ch_{\mathfrak{g}}^{d'}(z)$, $z \in \mathfrak{g}$, so that the product $\Theta(\widetilde{c}(z))ch_{\mathfrak{g}}(z)^{-2r}$ is often *a polynomial.*

Corollary 1.11. *Let* $\mathfrak{g}''' = c(G''')$. *Then for* $\psi \in C_c^{\infty}(\mathfrak{g}''')$

$$\frac{\widetilde{c}_-^{*}\Theta_{\Pi'}}{\widetilde{c}^{*}\Theta}(\psi) = \mathrm{const} \int_{\mathfrak{g}} \int_{G\backslash W^{\max}} \int_{\mathfrak{g}'} \psi(z')\chi_{z'+z}(w)\, \theta_{\Pi}(z)\, dz'\, d\dot{w}\, dz\,,$$

where each consecutive integral is absolutely convergent, and "const" *is such that* $\Pi'(\widetilde{c}(0)^{-1}) = \mathrm{const} \cdot \mathrm{identity}$.

For a function $\psi \in S(\mathfrak{g})$ define the Fourier transform $\mathscr{F}_{\mathfrak{g}}\psi = \hat{\psi} \in S(\mathfrak{g}^*)$ by

$$(1.12) \qquad \mathscr{F}_{\mathfrak{g}}\psi(\xi) = \hat{\psi}(\xi) = \int_{\mathfrak{g}} \chi(\xi(x))\psi(x)\, dx \quad (\xi \in \mathfrak{g}^*)\,.$$

This extends to a map $\mathscr{F}_{\mathfrak{g}} : S^*(\mathfrak{g}) \to S^*(\mathfrak{g}^*)$ by taking the adjoint of the inverse. Similarly we have the Fourier transform $\mathscr{F}_{\mathfrak{g}'}$ on $\mathfrak{g}'$. Here is our main theorem, which should be viewed as a natural contribution to Howe's theory of reductive dual pairs.

Theorem 1.13. *Suppose the function* θ_{Π} (1.10) *is a finite sum of homogeneous functions. Then* $\widetilde{c}_-^{*}\Theta_{\Pi'}/\widetilde{c}^{*}\Theta$ *is a finite sum of homogeneous functions. If* $\mathbf{D} = \mathbf{H}$, *the quaternions, assume* (*in addition to* (1.7.a–e)) *that* $\dim_{\mathbf{D}} V' > \frac{4}{3}(\dim_{\mathbf{D}} V)^2$. *Then both* θ_{Π} *and* $\widetilde{c}_-^{*}\Theta_{\Pi'}/\widetilde{c}^{*}\Theta$ *define tempered distributions via integration against Lebesgue measures on* $\mathfrak{g}, \mathfrak{g}'$ *respectively and*

$$\begin{aligned} WF(\Pi') = \mathrm{supp}\left(\mathscr{F}_{\mathfrak{g}'}\left(\frac{\widetilde{c}_-^{*}\Theta_{\Pi'}}{\widetilde{c}^{*}\Theta}\right)\right) &= \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(\mathrm{supp}\, \mathscr{F}_{\mathfrak{g}'}(\theta_{\Pi}))) \\ &= \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(WF(\Pi)))\,. \end{aligned}$$

In particular if $WF(\Pi)$ *is the closure of a single orbit, then so is* $WF(\Pi')$.

The point of Theorem 1.13 is that it provides *an algorithm for constructing irreducible unitary representations of isometry groups of split forms, whose characters are supported on single nilpotent coadjoint orbits.* Indeed, if the wave front set of Π is the closure of a single nilpotent orbit, then so is $\mathrm{supp}(\mathscr{F}_{\mathfrak{g}'}(\widetilde{c}^*_-\Theta_{\Pi'}/\widetilde{c}^*\Theta))$. Then we can take another dual pair G', G'' with G' the smaller member, and under the assumptions of (1.13) construct an irreducible unitary representation of $\widetilde{G}''$, whose character has the Fourier transform supported on a single orbit … .

We would like to thank W. Rossmann for keeping us informed on the recent progress in his character theory [R]. In fact [R] had a catalytic influence on our project.

2. The oscillator representation

Here we recall Howe's construction of the oscillator representation ω of the metaplectic group $\widetilde{Sp}$ [H2], in a way suitable for our applications.

Let W be a real vector space of dimension $2n$, with a non-degenerate symplectic form $\langle\, ,\, \rangle$. The symplectic group Sp and the symplectic Lie algebra sp are defined as follows

$$Sp = Sp(W) = \{g \in \mathrm{End}(W);\ \langle g(w), g(w')\rangle = \langle w, w'\rangle \text{ for all } w, w' \in W\}\ ,$$

$$sp = sp(W) = \{z \in \mathrm{End}(W);\ \langle z(w), w'\rangle = -\langle w, z(w')\rangle \text{ for all } w, w' \in W\}\ .$$

We shall use the superscript c to indicate the domain of the Cayley transform c. Thus Sp^c is the domain of c in Sp and sp^c is the domain of c in sp. Define the following set

$$\widetilde{Sp^c} = \{(g, \xi); g \in Sp^c,\ \xi^2 = \det(i(g-1))^{-1}\}\ .$$

This is a real analytic manifold, and a two fold covering of Sp^c via the map

$$\widetilde{Sp^c} \ni (g, \xi) \to g \in Sp^c\ . \tag{2.1}$$

For $z \in sp$ the formula $\langle z(w), w'\rangle$ defines a symmetric bilinear form $\langle z,\ \rangle$ on W. The signature of this form, $\mathrm{sgn}\langle z,\ \rangle$, is the difference between the dimension of the maximal subspace on which the form is positive definite, and the dimension of the maximal subspace on which the form is negative definite. Let

$$\gamma(z) = |\det(z)|^{1/2} \exp\left(-\frac{\pi}{4} i\, \mathrm{sgn}\langle z,\ \rangle\right) \quad (z \in sp,\ \det(z) \neq 0)\ . \tag{2.2}$$

This is a Fourier transform of one of the two non-zero minimal nilpotent adjoint orbits in $sp(W)$. For $(g_1, \xi_1),\ (g_2, \xi_2) \in \widetilde{Sp^c}$ with $c(g_1) + c(g_2)$ invertible in $\mathrm{End}(W)$ set

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1 g_2, 2^n \xi_1 \xi_2 \gamma(c(g_1) + c(g_2))^{-1})\ . \tag{2.3}$$

Theorem 2.4 [H2]. (a) *Up to a group isomorphism there is a unique connected group* $\widetilde{Sp}$ *containing* $\widetilde{Sp^c}$ *with the multiplication given by* (2.3) *on the indicated subset of* $\widetilde{Sp^c} \times \widetilde{Sp^c}$.

(b) *The group* $\widetilde{Sp}$ *is a connected Lie group which contains* $\widetilde{Sp^c}$ *as an open submanifold.*

(c) *The map* (2.1) *extends to a double covering homomorphism of Lie groups*: $\widetilde{Sp} \to Sp$.

The metaplectic group $\widetilde{Sp}$ may be realized as a subset of $S^*(W)$, the space of tempered distributions on W, as follows.

For $\phi, \phi' \in S(W)$, the Schwartz space of W, define the twisted convolution $\phi \natural \phi'$ and ϕ^* by

(2.5)
$$\phi \natural \phi'(w') = \int_W \phi(w)\phi'(w' - w)\chi(\tfrac{1}{2}\langle w, w'\rangle)dw \, ,$$
$$\phi^*(w) = \overline{\phi(-w)} \quad (w, w' \in W) \, .$$

For a tempered distribution $f \in S^*(W)$ define $f^* \in S^*(W)$ by $f^*(\phi) = f(\phi^*)$. The functions χ_z (1.3) do not belong to $S(W)$, but we may convolve them in the sense analogous to the formula of (2.5). Indeed, let $\phi \in S(W)$. Then for $w' \in W$ and $y \in sp$

$$\int_W \chi_y(w)\phi(w' - w)\chi(\tfrac{1}{2}\langle w, w'\rangle)\, dw$$
$$= \chi_y(w') \int_W \chi(\tfrac{1}{2}\langle (1 - y)(w'), w\rangle)\chi_y(w)\phi(w)\, dw \, .$$

Thus, for $1 - y$ invertible, the above is a Schwartz function of w'. Denote this function by $\chi_y \natural \phi(w')$. Suppose $x \in sp^c$. Then by the same argument $\chi_x \natural (\chi_y \natural \phi) \in S(W)$. Suppose moreover that $x + y$ is invertible in $\mathrm{End}(W)$. Let $z = (y - 1)(x + y)^{-1}(x - 1) + 1$. Then $z \in sp^c$ and, by [Hö, 3.4], $\chi_x \natural (\chi_y \natural \phi) = 2^n \gamma(x + y)^{-1}\chi_z \natural \phi$. Thus,

(2.6)
$$\chi_x \natural \chi_y = 2^n \gamma(x + y)^{-1}\chi_z \, .$$

The formula (2.6) is the key to Howe's construction.

Define the following functions

(2.7)
$$\Theta : \widetilde{Sp^c} \ni \tilde{g} = (g, \xi) \to \xi \in \mathbf{C} \, ,$$
$$T : \widetilde{Sp^c} \ni \tilde{g} = (g, \xi) \to \Theta(\tilde{g})\chi_{c(g)} \in S^*(W) \, .$$

(We shall see shortly that Θ coincides with the distribution character of ω).

Theorem 2.8 [H2]. *The map T extends to a unique injective continuous map $T : \widetilde{Sp} \to S^*(W)$, and the following formulas hold*

(a) $\quad T(\tilde{g}_1)\natural T(\tilde{g}_2) = T(\tilde{g}_1 \cdot \tilde{g}_2) \quad (\tilde{g}_1, \tilde{g}_2 \in \widetilde{Sp^c}, \ \det(c(g_1) + c(g_2)) \neq 0)$

(b) $$T(\tilde{g})^* = T(\tilde{g}^{-1}) \quad (\tilde{g} \in \widetilde{Sp})$$

(c) $$T(1) = \delta \, .$$

Here $\delta \in S^*(W)$ is the Dirac delta at the origin.

In order to realize $\widetilde{Sp}$ in the group of unitary operators on a Hilbert space, choose a complete polarization $W = X + Y$ and Lebesgue measures dx, dy on

X, Y respectively, so that

$$\int_W \phi(w)\,dw = \int_X \int_Y \phi(x+y)\,dy\,dx \quad (\phi \in S(W))\,. \tag{2.9}$$

For $f \in S(W)$ define $\mathcal{K}(f) \in S(X \times X)$ by

$$\mathcal{K}(f)(x,x') = \int_Y f(x - x' + y)\chi(\tfrac{1}{2}\langle y, x + x'\rangle)dy \quad (x, x' \in X)\,. \tag{2.10}$$

Conversely, for $K \in S(X \times X)$ define $f \in S(W)$ by

(2.11)

$$f(x+y) = 2^{-n}\int_X K\left(\frac{x'+x}{2}, \frac{x'-x}{2}\right)\chi\left(\frac{1}{2}\langle x', y\rangle\right)dx' \quad (x \in X, y \in Y)\,.$$

Furthermore, given $f \in S(W)$ define a bounded operator $\rho(f)$ on the Hilbert space $L^2(X)$:

$$\rho(f)v(x) = \int_X \mathcal{K}(f)(x,x')v(x')\,dx' \quad (x \in X,\ v \in L^2(X))\,. \tag{2.12}$$

Here is the classical Stone–von Neumann, Segal, Schwartz, ….

Theorem 2.13 [H4]. *The maps* (2.10) *and* (2.11) *are mutual inverses, and the map* ρ (2.12) *has the following properties*

(a) $$\rho(f_1 \natural f_2) = \rho(f_1)\rho(f_2) \quad (f_1, f_2 \in S(W))\,,$$

(b) $$\rho(f)^* = \rho(f^*) \quad (f \in S(W))\,.$$

Moreover the map (2.10) *extends to a linear bijection*

$$\mathcal{K} : S^*(W) \ni f \to \mathcal{K}(f) \in S^*(X \times X)$$

with the inverse given by the corresponding extension of (2.11) (*we view functions as distributions via multiplication by the Lebesgue measure dw*). *The map* ρ *extends to a linear bijection*

$$\rho : S^*(W) \to \mathrm{Hom}(S(X), S^*(X))\,,$$

with $\rho(\delta) =$ *the identity. The above* ρ *restricts to a bijective isometry*

$$\rho : L^2(W) \to H.S.(L^2(X))\,,$$

the space of Hilbert–Schmidt operators on $L^2(X)$.

Definition 2.14. The Schrödinger model of the oscillator representation of $\widetilde{Sp}$ attached to the polarization $W = X + Y$ is the unitary representation ω on the Hilbert space $L^2(X)$ defined by

$$\omega(\widetilde{g}) = \rho(T(\widetilde{g})) \quad (\widetilde{g} \in \widetilde{Sp})\,.$$

Thus $\omega(\widetilde{g})$ is an integral kernel operator with the integral kernel $\mathcal{K}(T(\widetilde{g}))$, $\widetilde{g} \in \widetilde{Sp}$.

It is important for us to realize that for g in a Zariski open subset of Sp this integral kernel can be explicitly calculated.

Define the following map

$$\mathrm{End}(X) \ni A \to A^* \in \mathrm{End}(Y), \qquad \langle A(x), y\rangle = \langle x, A^*(y)\rangle, \quad (x \in X, \ y \in Y)$$

and let

$$\mathrm{Hom}(X,Y)_- = \{B \in \mathrm{Hom}(X,Y); \langle B(x), x'\rangle = -\langle x, B(x')\rangle, \ x, x' \in X\}$$

$$\mathrm{Hom}(Y,X)_- = \{C \in \mathrm{Hom}(Y,X); \langle C(y), y'\rangle = -\langle y, C(y')\rangle, \ y, y' \in Y\} .$$

We embed $\mathrm{End}(X)$, $\mathrm{Hom}(X,Y)_-$ and $\mathrm{Hom}(Y,X)_-$ into $sp(W)$ as follows

$$A(x+y) = A(x) - A^*(y) \quad (A \in \mathrm{End}(X))$$

$$B(x+y) = B(x), \quad C(x+y) = C(y) \quad (B \in \mathrm{Hom}(X,Y)_-, C \in \mathrm{Hom}(Y,X)_-) .$$

This gives a direct sum decomposition

$$sp(W) = \mathrm{End}(X) \oplus \mathrm{Hom}(X,Y)_- \oplus \mathrm{Hom}(Y,X)_- . \tag{2.15}$$

Notice that for an invertible element $C \in \mathrm{Hom}(Y,X)_-$ the inverse $C^{-1} \in \mathrm{Hom}(X,Y)_-$. Moreover the formula $\langle C(y), y'\rangle$, $y, y' \in Y$ defines a symmetric form on Y. Denote the signature of this form by $\mathrm{sgn}\langle C, \ \rangle$, as before.

Lemma 2.16. *Let $A \in \mathrm{End}(X)$, $B \in \mathrm{Hom}(X,Y)_-$, $C \in \mathrm{Hom}(Y,X)_-$ and let $z = A + B + C \in sp$. Assume that $z \in sp^c$ and that C is invertible. Let $\widetilde{c}(z) \in \widetilde{Sp}$ be in the preimage of $c(z) \in Sp$ under the covering map* (2.1). *Then the distribution $\mathscr{K}(T(\widetilde{c}(z))) \in S^*(X \times X)$ coincides with the following function*

$$\mathscr{K}(T(\widetilde{c}(z)))(x,x') = \mathrm{const}\, \Theta(\widetilde{c}(z))|\det(C)|^{-1/2} \exp\left(\frac{\pi}{4} i \,\mathrm{sgn}\langle C, \ \rangle\right)$$
$$\times \chi_B(x - x')\chi_{-C^{-1}}(x + x' - A(x - x')) \quad (x, x' \in X)$$

(*multiplied by the Lebesgue measure $dx\, dx'$*).

Proof. This is a straightforward calculation:

$$\begin{aligned}
\mathscr{K}(T(\widetilde{c}(z)))(x,x') &= \int_Y \Theta(\widetilde{c}(z))\chi_z(x - x' + y)\chi(\tfrac{1}{2}\langle y, x + x'\rangle)dy \\
&= \Theta(\widetilde{c}(z))\chi_z(x - x')\int_Y \chi_z(y)\chi(\tfrac{1}{2}\langle y, x + x' - z(x - x')\rangle)dy \\
&= \Theta(\widetilde{c}(z))\chi_B(x - x')\int_Y \chi(\tfrac{1}{4}\langle C(y), y\rangle) \\
&\quad \times \chi(\tfrac{1}{2}\langle y, x + x' - A(x - x')\rangle)dy \\
&= \Theta(\widetilde{c}(z))\chi_B(x - x')\mathrm{const}_\chi |\det(C)|^{-1/2} \exp\left(\frac{\pi}{4} i \ \mathrm{sgn}\langle C, \ \rangle\right) \\
&\quad \times \chi_{-C^{-1}}(x + x' - A(x - x'))
\end{aligned}$$

where the last equality follows from [Hö, 7.6.1]. □

Similarly we verify the following lemma:

Lemma 2.17. *Let* $A \in \mathrm{End}(X)$. *Assume* $\det(A \pm 1) \neq 0$. *Consider* A *as an element of* sp, *as in* (2.15). *Let* $\widetilde{c}(A) \in \widetilde{Sp}$ *be in the preimage of* $c(A) \in Sp$. *Then*

$$\mathscr{K}(T(\widetilde{c}(A)))(x,x') = 2^n \Theta(\widetilde{c}(A)) |\det(A+1)_X|^{-1} \delta(c(A)^{-1}x - x') \, .$$

Moreover

$$2^n |\Theta(\widetilde{c}(A))| |\det(A+1)_X|^{-1} = |\det(c(A)_X)|^{-1/2} \, .$$

Since formally $\mathrm{tr}\,\omega(\widetilde{g}) = \mathrm{tr}\,\rho(T(\widetilde{g})) = T(\widetilde{g})(0) = \Theta(\widetilde{g})$, $\widetilde{g} \in \widetilde{Sp}$ (see (2.7) and (2.10)), the distribution character of ω can be identified with the function Θ. Since this function figures prominently throughout our paper we recall it again in a concise form:

$$\Theta(\widetilde{g}) = \xi, \quad \widetilde{g} = (g, \xi), \quad g \in Sp^c, \quad \xi^2 = \det(i(g-1))^{-1} \, . \tag{2.18}$$

3. Explicit realization of representations in the stable range

Let G, G' be a reductive dual pair of type I in the stable range with G the smaller member. Let $\widetilde{G}, \widetilde{G}'$ denote the preimages of G, G' in $\widetilde{Sp}$ respectively. Let $\Pi \otimes \Pi' \in \mathscr{R}(G \cdot G', \omega)$ be unitary [H1]. Suppose Π is realized on a Hilbert space H_Π. Following [Li] we describe in this section an explicit Hilbert space realization of the representation Π'. We shall do it under the following additional assumption on the pair G, G'.

Let $\mathbf{D} = (\mathbf{R}, \mathbf{C}, \mathbf{H})$ be a finite-dimensional division algebra, with an involution, over $\mathbf{R}$. Let V and V' be finite dimensional vector spaces over $\mathbf{D}$ equipped with non-degenerate forms $(\,,\,)$ and $(\,,\,)'$, respectively – one hermitian and the other skew-hermitian. The groups G, G' are the isometry groups of the forms $(\,,\,)$, $(\,,\,)'$ respectively.

We assume that the form $(\,,\,)'$ *is split*, (*see* (1.7a)).

Define a map $\mathrm{Hom}(V', V) \ni w \to w^* \in \mathrm{Hom}(V, V')$ by

$$(w(v'), v) = (v', w^*(v)) \quad (w \in W, v \in V, v' \in V') \, . \tag{3.1}$$

Let $W = \mathrm{Hom}(V', V)$. The formula

$$\langle w, w' \rangle = \mathrm{tr}_{\mathbf{D}/\mathbf{R}}(w w'^*) \quad (w, w' \in W) \tag{3.2}$$

defines a symplectic form on W. The groups G and G' act on W via post-multiplication and pre-multiplication by the inverse, respectively. These actions embed G and G' into the symplectic group $Sp(W)$.

Let $V' = X' \oplus Y'$ be a complete polarization of V'. The assumption that the pair G, G' is in the stable range with G the smaller member, means that $\dim V \leqq \dim X'$.

Set $X = \mathrm{Hom}(X', V)$ and $Y = \mathrm{Hom}(Y', V)$. Then $W = X \oplus Y$ is a complete polarization of W. Let $\mathscr{B}(X')$ denote the space of forms on X' of the same type as (,). Recall the pullback of forms

$$\beta : X \ni x \to (,) \circ x \in \mathscr{B}(X') .$$

Let $X^{\max} = \{x \in X;$ rank of $\beta(x)$ is equal to the dimension of $V\}$. This is a G-stable subset of X. Let $X^{\max} \ni x \to \dot{x} \in G\backslash X^{\max}$ denote the quotient map. Define a Borel measure $d\dot{x}$ on $G\backslash X^{\max}$ by the formula

$$\int_{X^{\max}} v(x)\,dx = \int_{G\backslash X^{\max}} \int_G v(gx)\,dg\,d\dot{x} \quad (v \in C_c(X^{\max})) . \tag{3.3}$$

Fix a positive definite scalar product on the real vector space $X : x \cdot x'$, $x, x' \in X$. Let $|x| = (x \cdot x)^{1/2}$ denote the corresponding norm. For $g \in \mathrm{End}_{\mathbf{R}}(X)$ let $g^t \in \mathrm{End}_{\mathbf{R}}(X)$ denote the adjoint, defined by $gx \cdot x' = x \cdot g^t x'$ and let $|g|^2 = \mathrm{tr}(g^t g)$ be the Hilbert–Schmidt norm of g. Notice that since G is an isometry group we may choose the above scalar product on X so that $|g| = |g^{-1}|$ for $g \in G$.

Suppose $x \in X^{\max}$. Then $x : X' \to V$ is surjective, so there is a constant const, depending on x, such that $|gx| \geqq \mathrm{const}|g|$ for all $g \in \mathrm{End}_{\mathbf{R}}(X)$. Recall [W, 2.A.2.4] that for $N > 0$ sufficiently large $\int_G |g|^{-N} dg < \infty$. Hence for $N > 0$ large enough

$$\int_G (1 + |gx|^2)^{-N} dg < \infty \quad (x \in X^{\max}) \tag{3.4}$$

and *under the assumption* (1.7.b), *which from now on is in effect*,

$$\int_{G\backslash X^{\max}} \left(\int_G (1 + |gx|^2)^{-N} dg \right)^2 d\dot{x} < \infty . \tag{3.5}$$

It follows from (2.17) that

$$\omega(\widetilde{g}) v(x) = 2^n \Theta(\widetilde{g}) |\det(c(g) + 1)_X|^{-1} v(g^{-1}x) \quad \text{and} \tag{3.6}$$

$$|2^n \Theta(\widetilde{g})| |\det\ (c(g) + 1)_X|^{-1} = 1 \quad (\widetilde{g} \in \widetilde{G^c}, v \in S(X)) .$$

In fact the function $2^n \Theta(\widetilde{g}) |\det(c(g) + 1)_X|^{-1}$ extends to a character $\zeta(\widetilde{g})$ of the group $\widetilde{G}$. Let $\overline{\Pi}$ denote the representation conjugate to Π, realized on a Hilbert space $H_{\overline{\Pi}}$. (The character of $\overline{\Pi}$ is $\Theta_{\overline{\Pi}} = \overline{\Theta}_{\Pi}$). Since Π occurs as a non-zero quotient of the oscillator representation, the restriction of Π to the (two element) kernel of the covering map $\widetilde{G} \to G$ has to be a multiple of the unique non-trivial character of this subgroup. The same holds for $\overline{\Pi}$. Hence the function $\widetilde{g} \to \zeta(\widetilde{g}) \overline{\Pi}(\widetilde{g})$ is constant on the fibers of this covering map. Let

$$\pi(g) = \zeta(\widetilde{g}) \overline{\Pi}(\widetilde{g}) . \tag{3.7}$$

This is an irreducible unitary representation of G on $H_{\overline{\Pi}}$.

The Haar measure on $\widetilde{G}$ is normalized in such a way that the measure of a small open set in G is equal to the measure of its inverse image in $\widetilde{G}$.

By (3.4) and (3.6) the integral

(3.8)
$$\int_{\widetilde{G}} \omega(\widetilde{g})v(x)\overline{\Pi}(\widetilde{g})\eta\, d\widetilde{g} = \int_{G} v(g^{-1}x)\pi(g)\eta\, dg \quad (v \in S(X),\ x \in X^{\max},\ \eta \in H_{\overline{\Pi}})$$

is absolutely convergent and defines a continuous, $H_{\overline{\Pi}}$-valued, function on $X^{\max}$. Notice that for $h \in G$

$$\int_{G} v(g^{-1}hx)\pi(g)\eta\, dg = \pi(h)\int_{G} v(g^{-1}x)\pi(g)\eta\, dg\ . \tag{3.9}$$

Thus the norm of the function (3.8) is G-invariant, and by (3.5) square integrable with respect to the measure $d\dot{x}$ on $G\backslash X^{\max}$. Let H be the space of $H_{\overline{\Pi}}$-valued functions on $X^{\max}$ which satisfy the equivariance condition (3.9) and are square integrable with respect to the measure $d\dot{x}$. Let $S(X)\otimes H_{\overline{\Pi}}$ denote the algebraic tensor product of the vector spaces $S(X)$ and $H_{\overline{\Pi}}$. Define a linear map $Q : S(X)\otimes H_{\overline{\Pi}} \to H$ by

$$\begin{aligned} Q(v\otimes\eta)(x) &= \int_{\widetilde{G}} \omega(\widetilde{g})v(x)\overline{\Pi}(\widetilde{g})\eta\, d\widetilde{g} \\ &= \int_{G} v(g^{-1}x)\pi(g)\eta\, dg \quad (x \in X^{\max}, v \in S(X), \eta \in H_{\overline{\Pi}})\ . \end{aligned} \tag{3.10}$$

Lemma 3.11. *The range of Q is dense in H.*

Proof. Indeed, notice that $G\backslash X^{\max}$ is a single orbit under the obvious action of the group $GL(X')$ and that one can choose a smooth densely defined section $\sigma : G\backslash X^{\max} \to X^{\max}$. The pullback by σ gives an isomorphism of H and $L^2(G\backslash X^{\max}, H_{\overline{\Pi}})$. Fix $\eta \in H_{\overline{\Pi}}$ and $u \in C_c^{\infty}(G\backslash X^{\max})$ supported in the set of regular points of σ. Let $\lambda \in C_c^{\infty}(G)$. Define a function v on $X^{\max}$ by

$$v(g^{-1}\sigma(\dot{x})) = \lambda(g)u(\dot{x}) \quad (g \in G,\ \dot{x} \in G\backslash X^{\max})\ .$$

Clearly $v \in C_c^{\infty}(X^{\max})$. Moreover

$$Q(v\otimes\eta)(x) = u(\dot{x})\int_{G} \lambda(g)\pi(g)\eta\, dg\ .$$

Thus $u(\dot{x})\eta$ is in the closure of the range of Q. Since the set of such functions is dense in $L^2(G\backslash X^{\max}, \mu, H_{\overline{\Pi}})$, we are done. □

Let $(\ ,\ )_{\Pi}$ denote the pullback of the inner product of H to $S(X)\otimes H_{\overline{\Pi}}$ via Q. A straightforward calculation analogous to the one verifying (3.5) shows that

$$(v\otimes\eta, v'\otimes\eta')_{\Pi} = \int_{\widetilde{G}}\int_{X} \omega(\widetilde{g})v(x)\overline{v'(x)}\, dx\, (\overline{\Pi}(g)\eta, \eta')\, d\widetilde{g}\ . \tag{3.12}$$

The group $\widetilde{G}'$ acts on $S(X)\otimes H_{\overline{\Pi}}$ via ω on $S(X)$ and identity on $H_{\overline{\Pi}}$. Since the integral over X is a matrix coefficient of ω it is clear from (3.12) that the form $(\ ,\ )_{\Pi}$ is $\widetilde{G}'$-invariant. Hence the radical of $(\ ,\ )_{\Pi}$ is $\widetilde{G}'$-invariant. But this radical coincides with the kernel of Q.

Theorem 3.13 [Li, 6.1]. *The representation Π' associated to Π via Howe's correspondence coincides with the completion of $(S(X) \otimes H_{\overline{\Pi}})/\ker(Q)$ with respect to the inner product defined by the form $(\ ,\ )_{\Pi}$.*

This gives a realization of Π' on the Hilbert space H, which we shall denote from now on by $H_{\Pi'}$. It remains to shed some light at the action of $\widetilde{G}'$ on $H_{\Pi'}$. We shall do it in the next section.

4. $\widetilde{G}'$ acts on $H_{\Pi'}$ via integral kernel operators

We retain the notation and assumptions of the previous section. Let $\Psi \in C_c^\infty(\widetilde{G}')$. Then $T(\Psi) = \int_{\widetilde{G}'} \Psi(\widetilde{g})T(\widetilde{g})\,d\widetilde{g}$ is a well defined tempered distribution on W, (see (2.8)). Further, $\mathscr{K}(T(\Psi))$ is a tempered distribution on $X \times X$, (see (2.13)). *Under an additional assumption on the support of Ψ this last distribution is a function.*

Indeed, by our choice of the complete polarization $W = X + Y$ the direct sum decomposition (2.15) of *sp* induces, by restriction, an analogous decomposition of $\mathfrak{g}'$. Thus, as in (2.16) we may write each element $z \in \mathfrak{g}'$ as $z = A + B + C$. Let $\mathfrak{g}''$ denote the Zariski open subset of $\mathfrak{g}'^c$ consisting of these z for which the $C \in \operatorname{Hom}(Y,X)_-$ is invertible. For this set to be non-empty we must *exclude the pairs Sp_{2n}, $O_{m,m}$ with m odd* (*this is the assumption* (1.7.c)). Let $G'' = c(\mathfrak{g}'')$.

Suppose, in addition, that $\operatorname{supp}\Psi \subseteq \widetilde{G}''$. Then by (2.16) $\mathscr{K}(T(\Psi))$ is a function given by

$$\begin{aligned}(4.1)\qquad &\mathscr{K}(T(\Psi))(x,x') \\ &= \int_{\widetilde{G}'} \Psi(\widetilde{g})\operatorname{const}\Theta(\widetilde{g})|\det(C)|^{-1/2}\exp\left(\frac{\pi}{4} i \operatorname{sgn}\langle C,\ \rangle\right) \\ &\quad \times \chi_B(x-x')\chi_{-C^{-1}}(x+x'-A(x-x'))\,d\widetilde{g} \quad (x,x' \in X)\,,\end{aligned}$$

where under the integral $c(g) = z = A + B + C$ is the decomposition explained above. The integral (4.1) is absolutely convergent because the support of Ψ does not touch neither the singularities of $\Theta(\widetilde{g})$ nor the singularities of $|\det(C)|^{-1/2}$.

Recall the map $\operatorname{Hom}(V',V) \ni w \to w^* \in \operatorname{Hom}(V,V')$ defined by

$$(4.2)\qquad (w(v'),v) = (v',w^*(v)) \quad (w \in W, v \in V, v' \in V')\,.$$

By abuse of notation we have for $x \in X = \operatorname{Hom}(X',V)$ the corresponding element $x^* \in \operatorname{Hom}(V,Y')$. Thus for $x,\ x' \in X$ the composition $x^*x' \in \operatorname{Hom}(X',Y')$. Fix a norm $|\,|$ on the real vector space $\operatorname{Hom}(X',Y')$. Here is the main technical lemma of this paper.

Lemma 4.3. *There are: a non-empty Zariski open subset $G''' \subseteq G''$ and continuous seminorms $q_N : C_c^\infty(\widetilde{G}''') \to \mathbf{R}$, $N = 0,1,2,3,\ldots,$ such that the* *following estimate holds*

$$|\mathscr{K}(T(\Psi))(x,x')| \leqq q_N(\Psi)(1+\tfrac{1}{2}|x^*x+x'^*x'|+|x^*x'|)^{-N}$$
$$(\Psi \in C_c^\infty(\widetilde{G}'''); \; x,x' \in X; \; N=0,1,2,\ldots)\,.$$

Proof. Consider the integral (4.1) with $x,x' \in X$ fixed. This is an oscillatory integral, [Hö, 7.8.1], with the phase function

$$\widetilde{g} \to \langle B(x-x'),x-x'\rangle + \langle -C^{-1}(x+x'-A(x-x')),\; x+x'-A(x-x')\rangle\,.$$

In the coordinates provided by the Cayley transform this function is linear with respect to B. Hence its derivative, which is a linear functional on $\mathfrak{g}'$, can be estimated from below by $|(x-x')^*(x-x')|$. Hence, by the method of stationary phase [Hö, (7.7.1)′], there are continuous seminorms $q'_N : C_c^\infty(\widetilde{G}'') \to \mathbf{R}$ such that

$$(4.4)\quad |\mathscr{K}(T(\Psi))(x,x')| \leqq q'_N(\Psi)(1+|(x-x')^*(x-x')|)^{-N}$$
$$(\Psi \in C_c^\infty(\widetilde{G}''); \; x,x' \in X; \; N=0,1,2,\ldots)\,.$$

Fix an element $h \in G'^c$ which preserves X and Y. Then by (2.17)

$$(4.5)\qquad \mathscr{K}(T(\Psi))(x,x') = \mathscr{K}(T(\Psi_{\tilde{h}}))(h(x),x')$$
$$\text{where } \Psi_{\tilde{h}}(\widetilde{g}) = 2^{-n}\Psi(\widetilde{h}\widetilde{g})\Theta(\widetilde{h}^{-1})^{-1}|\det((c(h)^{-1}+1)_X)|\,.$$

Choose a *finite* number of elements $h_1,h_2,h_3,\ldots \in G'^c$ which preserve X and Y. Let

$$(4.6)\qquad G''' = G'' \cap h_1^{-1}G'' \cap h_2^{-1}G'' \cap \ldots\,.$$

Suppose $\Psi \in C_c^\infty(\widetilde{G}''')$. Then (4.4) and (4.5) imply that for all j

$$|\mathscr{K}(T(\Psi))(x,x')| \leqq q'_N(\Psi_{h_j})(1+|(h_j(x)-x')^*(h_j(x)-x')|)^{-N}$$
$$(x,x' \in X; \; N=0,1,2,\ldots)\,.$$

Let $q''_N(\Psi) = q'_N(\Psi_{h_1}) + q'_N(\Psi_{h_2}) + q'_N(\Psi_{h_3}) + \ldots$. We may replace the $q'_N(\Psi_{h_j})$ in the above estimate by $q''_N(\Psi)$, and then take minimum over j. This gives the following estimate

$$|\mathscr{K}(T(\Psi))(x,x')|$$
$$\leqq q''_N(\Psi)(1+\max\{|(h_j(x)-x')^*(h_j(x)-x')|; j=1,2,3,\ldots\})^{-N}\,.$$

Since the average of a finite number of non-negative numbers does not exceed their maximum, we obtain

$$(4.7)\qquad |\mathscr{K}(T(\Psi))(x,x')| \leqq q'''_N(\Psi)(1+|(h_1(x)-x')^*(h_1(x)-x')|$$
$$+\,|(h_2(x)-x')^*(h_2(x)-x')|+\ldots)^{-N}\,,$$

where $q_N''' = \mathrm{const}\, q_N''(\Psi)$ and the finite constant const depends on the number of elements in (4.6). Define a map $\mathrm{Hom}(X', Y') \ni z \to z^\star \in \mathrm{Hom}(X', Y')$ by

$$(z(v'), v)' = (v', z^\star(v))' \quad (z \in \mathrm{Hom}(X', Y');\ v, v' \in X') .$$

Notice that for h, x, x' as in (4.5)

$$\begin{aligned} &|(h(x) - x')^*(h(x) - x')| + |(h(x) + x')^*(h(x) + x')| \\ &\quad \geqq |h(x)^* x' + x'^\star h(x)| = |hx^* x' + x'^* x h^{-1}| = |hx^* x' + (hx^* x')^\star| . \end{aligned}$$

Some elementary linear algebra shows that the only $z \in \mathrm{Hom}(X', Y')$ such that $hz + (hz)^\star = 0$ for all h is $z = 0$. Thus there are finitely many elements $h_1, h_2, h_3, \ldots$ as in (4.6) and a nonzero constant const such that

$$|h_1 z + (h_1 z)^\star| + |h_2 z + (h_2 z)^\star| + \ldots \geqq \mathrm{const}\, |z| \quad (z \in \mathrm{Hom}(X', Y')) .$$

Since we may assume that $\pm h_1,\ \pm h_2,\ \pm h_3, \ldots$ belong to the sequence $h_1, h_2, h_3, \ldots$, the estimate (4.7) implies

$$\begin{aligned} |\mathscr{K}(T(\Psi))(x, x')| &\leqq q_N'''(\Psi)(1 + |h_1 x^* x' + (h_1 x^* x')^\star| \\ &\qquad + |h_2 x^* x' + (h_2 x^* x')^\star| + \ldots)^{-N} \\ &\leqq q_N(\Psi)(1 + |x^* x'|)^{-N} , \end{aligned}$$

where the q_N is a constant multiple of q_N'''. This combined with (4.4) completes the proof of the lemma. □

Proposition 4.8. *Suppose* $\Psi \in C_c^\infty(\widetilde{G}''')$. *Then* $\Pi'(\Psi) = \int_{\widetilde{G}'} \Psi(\widetilde{g})\Pi'(\widetilde{g})\, d\widetilde{g}$ *is an integral kernel operator on* $H_{\Pi'} \cong L^2(G\backslash X^{\max}, H_{\overline{\Pi}})$. *Explicitly*

$$\Pi'(\Psi)u(x) = \int_{G\backslash X^{\max}} K_{\Pi'}(\Psi)(x, x')u(x')\, d\dot{x}' \quad (u \in H_{\Pi'}, x \in X^{\max}) ,$$

where

$$K_{\Pi'}(\Psi)(x, x') = \int_G \mathscr{K}(T(\Psi))(g^{-1}x, x')\pi(g)\, dg \quad (x, x' \in X^{\max})$$

is a continuous, operator valued, $G \times G$*-invariant function on* $X^{\max} \times X^{\max}$. *Moreover*

$$\mathrm{tr}(K_{\Pi'}(\Psi)(x, x')) = \int_{\widetilde{G}} \mathscr{K}(T(\Psi))(g^{-1}x, x')\zeta(\widetilde{g})\Theta(\widetilde{g})\overline{\Theta}_\Pi(\widetilde{g})\, d\widetilde{g} \quad (x, x' \in X^{\max}) .$$

All the above integrals are absolutely convergent.

Proof. Recall that for any $x, x' \in X^{\max}$ there is a non-zero constant const such that $|x^* g x'| \geqq \mathrm{const}\, |g|$, for all $g \in G$. Thus for $v \in C_c^\infty(X^{\max})$ and $N > 0$ sufficiently large

$$\begin{aligned} &\int_G (1 + |x^* g x'|)^{-N}\, dg \leqq \mathrm{const} \int_G (1 + |g|)^{-N}\, dg < \infty \text{ and} \\ &\int_G \int_X (1 + |x^* g x'|)^{-N} |v(x')| dx'\, dg < \infty \quad (x, x' \in X^{\max}) . \end{aligned} \tag{4.9}$$

Let $\Psi \in C_c^\infty(\widetilde{G}''')$ and let $\eta \in H_{\overline{\Pi}}$. Recall the quotient map Q (3.10). It is clear from (4.9) and (4.3) that for a fixed $x \in X^{\max}$ the following integrals are absolutely convergent:

$$\begin{aligned}
&Q(\omega(\Psi)v \otimes \eta)(x)\\
&\quad = \int_G \omega(\Psi)v(g^{-1}x)\pi(g)\eta\, dg\\
&\quad = \int_G \int_X \mathscr{K}(T(\Psi))(g^{-1}x,x')v(x')\pi(g)\eta\, dx'\, dg\\
&\quad = \int_X \int_G \mathscr{K}(T(\Psi))(g^{-1}x,x')v(x')\pi(g)\eta\, dg\, dx'\\
&\quad = \int_{G\backslash X^{\max}} \int_G \int_G \mathscr{K}(T(\Psi))(g^{-1}x,h^{-1}x')v(h^{-1}x')\pi(g)\eta\, dh\, dg\, d\dot{x}'\\
&\quad = \int_{G\backslash X^{\max}} \int_G \int_G \mathscr{K}(T(\Psi))(g^{-1}x,x')v(h^{-1}x')\pi(gh)\eta\, dh\, dg\, d\dot{x}'\\
&\quad = \int_{G\backslash X^{\max}} \left(\int_G \mathscr{K}(T(\Psi))(g^{-1}x,x')\pi(g)\eta\, dg\right)\left(\int_G v(h^{-1}x')\pi(h)\eta\, dh\right) d\dot{x}'\ ,
\end{aligned}$$

where the fifth equality holds because G commutes with G'. This verifies the first part of our proposition. It remains to calculate the trace.

Fix $x,x' \in X^{\max}$. By the argument given at the beginning of this proof, (4.3) implies that the function

$$G \ni g \to \mathscr{K}(T(\Psi))(g^{-1}x,x') \in \mathbf{C} \tag{4.10}$$

is rapidly decreasing in the sense of [W, 2.A.2.1, 7.1.1]. The usual argument involving (2.17) shows that this function is differentiable and all the derivatives are rapidly decreasing too. Hence by [W, 8.1.2] the integral

$$\int_G \mathscr{K}(T(\Psi))(g^{-1}x,x')\pi(g)\, dg$$

is a trace class operator with the trace equal to Θ_π, the character of π, applied to the function (4.10). Thus

$$\operatorname{tr}\left(\int_G \mathscr{K}(T(\Psi))(g^{-1}x,x')\pi(g)\, dg\right) = \int_G \mathscr{K}(T(\Psi))(g^{-1}x,x')\Theta_\pi(g)\, dg\ ,$$

where the integral is absolutely convergent.

Indeed, let us first work under the assumption (1.7.d). Then the character Θ_π can be majorized by some power Ξ^α, $\alpha \leqq 0$, of the Harish–Chandra Ξ-function divided by the square root of the Weyl denominator (see [M, Theorem 1, page 69]):

$$|\Theta_\pi(g)| \leqq \operatorname{const} \Xi(g)^\alpha |d_G(g)|^{-1/2} \quad (g \in G)\ . \tag{4.11}$$

Moreover $\int_G |d_G(g)|^{-1/2}\Xi(g)\, dg < \infty$ [Wa, 8.5.7.3]. But there is $\beta > 0$ such that for any $g \in G$, $|g|^{-1} \leqq \operatorname{const} \Xi^\beta(g)$, [W, 2.A.2.3, 4.5.3]. Hence the claim follows.

The argument in the remaining case (without the assumption (1.7.d)) is the same but uses results of [B].

Since by the definition of π (3.6–7) $\Theta_\pi(g) = \zeta(\widetilde{g})\Theta(\widetilde{g})\overline{\Theta_\Pi(\widetilde{g})}$, we are done. □

5. The trace

We retain the notation and assumptions of the previous two sections.

Lemma 5.1. *For all $N \geqq 0$ there are continuous seminorms $q_N : C_c^\infty(\widetilde{G}''') \to$ **R** such that for $\Psi \in C_c^\infty(\widetilde{G}''')$*

$$\int_{G\backslash X^{\max}} |\mathrm{tr}\, K_{\Pi'}(\Psi)(x,x)| d\dot{x}$$

$$\leqq q_N(\Psi) \int_{G\backslash X^{\max}} \int_G (1 + |x^*x| + |x^*gx|)^{-N} |\Theta_\pi(g)|\, dg\, d\dot{x}\ .$$

The integral on the right hand side is finite for N large enough if $\dim_{\mathbf{D}} V' \geqq (\dim_{\mathbf{D}} V + 2)^2$, (*this is the* "*deep stable range*" *assumption* (1.7.e)).

Proof. The inequality follows directly from (4.3) and (4.8). We defer the proof of finiteness to the Appendix. □

If the form (,) is hermitian and positive definite then $|x^*x| \geqq \mathrm{const}\,|x|^2$ for some non-zero constant const and $x \in X$. Hence for $N > 0$ sufficiently large the function under the integral (5.1) is bounded. Since, in this case, G is compact, the integral is finite without the additional assumption. However in this case the Theorems 1.7 and 1.13 are already known [P1]. *Therefore in this and in the next section we shall work under the additional assumption* (1.7.e).

Let H be a Cartan subgroup of G. As in (3.3), we define a measure $d\dot{x}$ on the quotient space $H\backslash X^{\max}$ by the formula

$$\int_{X^{\max}} v(x)\, dx = \int_{H\backslash X^{\max}} \int_H v(hx)\, dh\, d\dot{x} \quad (v \in C_c(X^{\max}))\ , \tag{5.2}$$

where $X^{\max} \ni x \to \dot{x} \in H\backslash X^{\max}$ denotes the quotient map. Recall the Weyl denominator $d_G(h)$, $h \in H$, [W, 2.4.4].

Theorem 5.3. *The character of the representation Π', when applied to a test function $\Psi \in C_c^\infty(\widetilde{G}''')$, is equal to*

(a)
$$\Theta_{\Pi'}(\Psi) = \int_{G\backslash X^{\max}} \mathrm{tr}\, K_{\Pi'}(\Psi)(x,x)\, d\dot{x}$$

$$= \int_{G\backslash X^{\max}} \int_G \mathscr{K}(T(\Psi))(g^{-1}x,x)\Theta_\pi(g)\, dg\, d\dot{x}\ .$$

Under the additional assumption (1.7.d)

(b)
$$\Theta_{\Pi'}(\Psi) = \sum_H \frac{1}{\iota} \int_{H\backslash X^{\max}} \int_H \mathscr{K}(T(\Psi))(h^{-1}x,x)\Theta_\pi(h)|d_G(h)|\, dh\, d\dot{x}\ ,$$

where ι is the index of G_1 in G (equal to 1 *unless G is an orthogonal group, in which case it is equal to* 2) *and the summation is over a set of representatives of conjugacy classes of Cartan subgroups $H \subseteq G$.*

All the integrals are absolutely convergent and may be estimated by $q(\Psi)$, for some continuous seminorm $q\colon C_c^\infty(\widetilde{G}''') \to \mathbf{R}$.

Proof. Everything except the formula (b) is a straightforward consequence of (5.1), (3.11) and a standard fact about integral kernel operators [Ki, 13.6]. Since, by (1.7.d), $\Theta_\pi = 0$ on the complement of G_1 in G (see [P2, 6.5]), (b) follows by a direct calculation. □

From now on we work under the additional assumption (1.7.d).

Theorem 5.4. *The character $\Theta_{\Pi'}$ of the representation Π', when applied to the test function $\Psi \in C_c^\infty(\widetilde{G}''')$, is equal to*

$$\Theta_{\Pi'}(\Psi) = \int\limits_{G\backslash X^{\max}} \int\limits_{G} \int\limits_{Y} \int\limits_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')+c(g)}(x+y)\Theta(\widetilde{g})\overline{\Theta}_\Pi(\widetilde{g})\, d\widetilde{g}'\, dy\, dg\, d\dot{x}\ ,$$

where the consecutive integrals over $\widetilde{G}'$, Y and $G\backslash X^{\max} \times G$ are absolutely convergent.

In the proof of Theorem 5.4 we'll need the following lemma the proof of which is left to the reader.

Lemma 5.5. *Suppose H is a unimodular subgroup of G, μ is the canonical measure on the quotient space $H\backslash X^{\max}$ defined in* (5.2), *v is an H-invariant function on $X^{\max}$-integrable over $H\backslash X^{\max}$, and $s \in GL(X)$ commutes with H. Then*

$$\int\limits_{H\backslash X^{\max}} v(sx)\, d\mu(\dot{x}) = |\det(s)|^{-1} \int\limits_{H\backslash X^{\max}} v(x)\, d\mu(\dot{x})\ .$$

Proof of Theorem 5.4. We begin by establishing convergence of the integrals involved. Since the function $\Psi(\widetilde{g})\Theta(\widetilde{g})$ is smooth and compactly supported, the integral

$$\int\limits_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')}(x+y)\, d\widetilde{g}' \tag{5.6}$$

is a rapidly decreasing function of $(x+y)^*(x+y)$. But $(x+y)^*(x+y) = x^*x + y^*y + x^*y + y^*x$, and (with obvious identifications) $x^*x \in \mathrm{Hom}(X', X')$, $y^*y \in \mathrm{Hom}(Y', Y')$, $x^*y \in \mathrm{Hom}(Y', X')$, $y^*x \in \mathrm{Hom}(X', Y')$. Hence the norm of $(x+y)^*(x+y)$ can be estimated from below by the norm of x^*y. But $x \in X^{\max}$ implies that x is surjective, so x^* is injective and therefore the norm of x^*y can be estimated from below by the norm of y. Hence the integral (5.6) is a rapidly decreasing function of y. Therefore

$$\int\limits_{Y} \left| \int\limits_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')}(x+y)\, d\widetilde{g}' \right| dy < \infty \quad (x \in X^{\max})\ . \tag{5.7}$$

By Theorem 5.3 the integral over $G\backslash X^{\max} \times G$ is absolutely convergent.

Now we prove the character formula. By Theorem 5.3.b,

(5.8)

$$\begin{aligned}\Theta_{\Pi'}(\Psi) &= \sum_H \frac{1}{\iota} \int_{H\backslash X^{\max}} \int_H \mathscr{K}(T(\Psi))(h^{-1}x, x)\Theta_\pi(h)|d_G(h)|\, dh\, d\dot{x} \\ &= \sum_H \frac{1}{\iota} \int_{H\backslash X^{\max}} \int_H \mathscr{K}(T(\Psi))(-\tfrac{1}{2}h^{-1}(c(h)+1)x, -\tfrac{1}{2}(c(h)+1)x) \\ &\quad \times 2^{-n}|\det(c(h)+1)_X|\Theta_\pi(h)|d_G(h)|\, dh\, d\dot{x}\,,\end{aligned}$$

where the summation is over a set of representatives of conjugacy classes of Cartan subgroups, the last equation follows from (5.5) and all integrals are absolutely convergent. Then by [W, 2.4.4] (5.8) is equal to

$$(5.9) \qquad \begin{aligned}&\int_{G\backslash X^{\max}} \int_G \mathscr{K}(T(\Psi))(-\tfrac{1}{2}g^{-1}(c(g)+1)x, -\tfrac{1}{2}(c(g)+1)x) \\ &\qquad \times 2^{-n}|\det(c(g)+1)_X|\Theta_\pi(g)\, dg\, d\dot{x}\,.\end{aligned}$$

By the definition (2.7), (5.9) is equal to

(5.10)

$$\begin{aligned}&\int_{G\backslash X^{\max}} \int_G \int_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\mathscr{K}(T(\chi_{c(g')}))(-\tfrac{1}{2}g^{-1}(c(g)+1)x, -\tfrac{1}{2}(c(g)+1)x) \\ &\qquad \times 2^{-n}|\det(c(g)+1)_X|\Theta_\pi(g)\, d\widetilde{g}'\, dg\, d\dot{x} \\ &= \int_{G\backslash X^{\max}} \int_G \int_Y \int_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')}(x+y)\chi(\tfrac{1}{2}\langle c(g)x, y\rangle) \\ &\qquad \times 2^{-n}|\det(c(g)+1)_X|\Theta_\pi(g)\, d\widetilde{g}'\, dy\, dg\, d\dot{x} \\ &= \int_{G\backslash X^{\max}} \int_G \int_Y \int_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')}(x+y)\chi_{c(g)}(x+y) \\ &\qquad \times 2^{-n}|\det(c(g)+1)_X|\Theta_\pi(g)\, d\widetilde{g}'\, dy\, dg\, d\dot{x}\,,\end{aligned}$$

where the consecutive integrals over $\widetilde{G}'$, Y and $G\backslash X^{\max}$ are absolutely convergent. Since by the definition (3.7) $2^{-n}|\det(c(g)+1)_X|\Theta_\pi(g) = \Theta(\widetilde{g})\overline{\Theta_\Pi(\widetilde{g})}$, the formula follows. □

Let $\widetilde{c}^*\Theta_{\Pi'}(z) = \Theta_{\Pi'}(\widetilde{c}(z))$, $z \in \mathfrak{g}'^c$. By (1.9) we have

$$(5.11) \qquad \widetilde{c}^*\Theta_{\Pi'}(\psi) = \Theta_{\Pi'}(\Psi) \quad \text{if } \psi(z) = \Psi(\widetilde{c}(z))\, ch_{\mathfrak{g}'}(z)^{-2r'}(z), \quad (z \in \mathfrak{g}',\ \Psi \in C_c(\widetilde{G}'^c))\,.$$

This is consistent with the notion of pullback of distributions, [Hö, 6.1.2].

Similarly we define $\widetilde{c}^*_-\Theta_{\Pi'}$. Notice that $\widetilde{c}(0)^{-1}$ is in the center of $\widetilde{Sp}$, and hence in the center of $\widetilde{G}'$. Therefore, since Π' is irreducible, $\Pi'(\widetilde{c}(0)^{-1}) =$ const · identity. Hence,

$$(5.12) \qquad \widetilde{c}^*_-\Theta_{\Pi'} = \text{const} \cdot \widetilde{c}^*\Theta_{\Pi'}\,.$$

Corollary 5.13. *With the above notation*

$$\frac{\widetilde{c}^*_- \Theta_{\Pi'}}{\widetilde{c}^* \Theta}(\psi) = \text{const} \int\limits_{G\backslash X^{\max}} \int\limits_{\mathfrak{g}} \int\limits_{Y} \int\limits_{\mathfrak{g}'} \psi(z')\chi_{z'+z}(x+y)\theta_\Pi(z)\, dz'\, dy\, dz\, d\dot{x}\,,$$

where $\theta_\Pi(z) = \Theta(\widetilde{c}(z))\overline{\Theta}_\Pi(\widetilde{c}(z))\, ch_{\mathfrak{g}}(z)^{-2r}$ *does not depend on the choice of* $\widetilde{c}$ *and the consecutive integrals over* $\mathfrak{g}'$, Y *and* $(G\backslash X^{\max}) \times \mathfrak{g}$ *are absolutely convergent.*

Recall that the symplectic space $W = \mathrm{Hom}(V', V)$. Let

$$W^{\max} = \{w \in W; (\,,\,)\circ w \text{ and } (\,,\,)'\circ w^* \text{ are forms of maximal rank}\}\,.$$

Then (1.5) holds. Let $W^{\max} \ni w \to \dot{w} \in G\backslash W^{\max}$ denote the quotient map. Recall the measure $d\dot{w}$ (1.6). Notice that $(X^{\max} + Y) \cap W^{\max}$ is a dense subset of W and that, by (2.9), for ϕ as in (1.6)

$$\int\limits_{W^{\max}} \phi(w)\, dw = \int\limits_{G\backslash X^{\max}} \int\limits_{Y} \int\limits_{G} \phi(g(x+y))\, dg\, dy\, d\dot{x}\,.$$

Hence we may think of the double integral over $(G\backslash X^{\max}) \times Y$ as of an integral over $G\backslash W^{\max}$.

Moreover, by fixing any measurable section $G\backslash X^{\max} \ni \dot{x} \to x \in X^{\max}$, we may reverse the integrals over $G\backslash X^{\max}$ and G in (5.4) and the integrals over $G\backslash X^{\max}$ and $\mathfrak{g}$ in (5.13). Therefore (5.4) and (5.13) may be rewritten as

Theorem 5.14. *There is a measurable section* $G\backslash W^{\max} \ni \dot{w} \to w \in W^{\max}$ *such that for* $\Psi \in C_c^\infty(\widetilde{G''})$

(a) $$\Theta_{\Pi'}(\Psi) = \int\limits_{\widetilde{G}} \int\limits_{G\backslash W^{\max}} \int\limits_{\widetilde{G}'} \Psi(\widetilde{g}')\Theta(\widetilde{g}')\chi_{c(g')+c(g)}(w)\Theta(\widetilde{g})\overline{\Theta}_\Pi(\widetilde{g})\, d\widetilde{g}'\, d\dot{w}\, d\widetilde{g}\,,$$

and (*with the notation of* (5.11) *and* (5.13))

(b) $$\frac{\widetilde{c}^*_- \Theta_{\Pi'}}{\widetilde{c}^* \Theta}(\psi) = \text{const} \int\limits_{\mathfrak{g}} \int\limits_{G\backslash W^{\max}} \int\limits_{\mathfrak{g}'} \psi(z')\chi_{z'+z}(w)\theta_\Pi(z)\, dz'\, d\dot{w}\, dz\,,$$

where each consecutive integral is absolutely convergent.

Proof. We only need to check that the integral over $G\backslash W^{\max}$ (i.e. over $G\backslash X^{\max} \times Y$) is absolutely convergent. For that, it suffices to see that if $\psi \in S(\mathfrak{g})$ then

$$\int\limits_{G\backslash W^{\max}} |\hat{\psi} \circ \tau_{\mathfrak{g}'}(w)|\, d\dot{w} < \infty\,. \tag{5.15}$$

The integral (5.15) does not depend on the choice of the section $\dot{w} \to w$ because the function $\hat{\psi} \circ \tau_{\mathfrak{g}'}$ is G-invariant. For a specific choice of this section the finiteness (5.15) is obvious, see [D-P. Theorem 1.6]. □

Theorem 1.7 coincides with part (a) of Theorem 5.14.

6. Representations with characters supported on nilpotent orbits

Recall the moment maps $\tau_{\mathfrak{g}} : W \to \mathfrak{g}^*$, $\tau_{\mathfrak{g}'} : W \to \mathfrak{g}'^*$ (1.4) and the notion of the wave front set of a representation [H5].

Theorem 6.1. *Suppose the function* θ_Π (1.10) *is a finite sum of homogeneous functions*

$$\text{(a)} \qquad \theta_\Pi(z) = \sum_{j=1}^{m} \theta_{\Pi,j}(z) \quad (z \in \mathfrak{g}) .$$

Then $\frac{\widetilde{c}^*_- \Theta_{\Pi'}}{\widetilde{c}^* \Theta}$ *is a finite sum of homogeneous functions*

$$\text{(b)} \qquad \frac{\widetilde{c}^*_- \Theta_{\Pi'}}{\widetilde{c}^* \Theta}(z') = \sum_{j=1}^{m} \theta_{\Pi',j}(z') \quad (z' \in \mathfrak{g}') ,$$

and (*with the notation of* (5.14)) *for* $\psi \in C_c^\infty(c(G'''))$

$$\text{(c)} \qquad \theta_{\Pi',j}(\psi) = \int_{\mathfrak{g}'} \theta_{\Pi',j}(z')\psi(z')\,dz'$$
$$= \text{const} \int_{\mathfrak{g}} \int_{G\backslash W^{\max}} \int_{\mathfrak{g}'} \psi(z')\chi_{z'+z}(w)\theta_{\Pi,j}(z)\,dz'\,d\dot{w}\,dz ,$$

where each single integral is absolutely convergent.

If $\theta_{\Pi,j}$ *is homogeneous of degree* d_j, *i.e.* $\theta_{\Pi,j}(tz) = t^{d_j}\theta_{\Pi,j}(z)$, $z \in \mathfrak{g}$, *then* $\theta_{\Pi',j}$ *is homogeneous of degree* $d'_j = d_j - n + \dim_{\mathbf{R}}\mathfrak{g}$, *where* $n = \frac{1}{2}\dim W$. *Moreover*

$$\text{(d)} \qquad WF(\Pi') = \tau_{\mathfrak{g}'}(\tau_{\mathfrak{g}}^{-1}(WF(\Pi))) .$$

Proof. For any $\varepsilon > 0$ there are continuous functions $p_j : [1, 1+\varepsilon] \to \mathbf{R}$, $j = 1, 2, \ldots, m$, such that

$$\int_1^{1+\varepsilon} p_j(t)t^{d'_{j'}}\,dt = \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j' . \end{cases}$$

Here the finiteness (a) is crucial of course. Define

$$\text{(6.2)} \qquad \psi_t(z) = t^{-p}\psi(t^{-1}z) \quad (\psi \in S(\mathfrak{g}),\ z \in \mathfrak{g},\ p = \dim_{\mathbf{R}}\mathfrak{g})$$

and similarly for $\mathfrak{g}'$. Fix ψ as in (c). We can choose $\varepsilon > 0$ so small that the dilates ψ_t for $1 \leqq t \leqq 1+\varepsilon$ are in $C_c^\infty(c(G'''))$. Define the functions Ψ_t on $\widetilde{G}'''$ by $\psi_t(z) = \Psi_t(\widetilde{c}(z))\,ch_{\mathfrak{g}'}(z)^{-2r'}$, $z \in \mathfrak{g}^c$. Then clearly the map $[1, 1+\varepsilon] \ni t \to \Psi_t \in C_c^\infty(\widetilde{G}''')$ is continuous. Therefore a straightforward calculation using (5.5) and (5.14) shows that

$$\int_1^{1+\varepsilon} p_j(t)\frac{\widetilde{c}^*_- \Theta_{\Pi'}}{\widetilde{c}^* \Theta}(\psi_t)\,dt = \int_{\mathfrak{g}} \int_{G\backslash W^{\max}} \int_{\mathfrak{g}'} \psi(z')\chi_{z'+z}(w)\theta_{\Pi,j}(z)\,dz'\,d\dot{w}\,dz .$$

Hence the last integral is well defined and

$$\text{(6.3)} \qquad \frac{\widetilde{c}^*_- \Theta_{\Pi'}}{\widetilde{c}^* \Theta}(\psi) = \text{const} \sum_{j=1}^{m} \int_{\mathfrak{g}} \int_{G\backslash W^{\max}} \int_{\mathfrak{g}'} \psi(z')\chi_{z'+z}(w)\theta_{\Pi,j}(z)\,dz'\,d\dot{w}\,dz .$$

An easy and well known consequence of Harish–Chandra's regularity theorem [W, 8.4.1] (and (5.11)) is that $\widetilde{c}_-^* \Theta_{\Pi'}/\widetilde{c}^* \Theta$, when restricted to some Zariski open subset of $\mathfrak{g}'$, is a possibly infinite sum of real analytic homogeneous functions, and that this sum converges in the sense of distributions on $\mathfrak{g}'^c$. Since our test functions are arbitrary in the Zariski open subset $c(G''') \subseteq \mathfrak{g}'^c$, we see that the formula (6.3) verifies (b) and (c).

We are left with (d). Suppose $d_1 < d_2 < \cdots < d_m$. Then $\theta_{\Pi,1}$ is the lowest term in the asymptotic expansion of Θ_Π (see [P2, (5.11)]), and $\theta_{\Pi',1}$ is the lowest term in the asymptotic expansion of $\Theta_{\Pi'}$. By [R, Theorem C]

$$WF(\Pi') = \text{ the wave front set of } \frac{\widetilde{c}_-^* \Theta_{\Pi'}}{\widetilde{c}^* \Theta} \text{ at } 0 = \operatorname{supp} \hat{\theta}_{\Pi',1} ,$$

and similarly for Π. It remains to show the following equation

$$\operatorname{supp} \hat{\theta}_{\Pi',1} = \tau_{\mathfrak{g}'} \circ \tau_{\mathfrak{g}}^{-1}(\operatorname{supp} \hat{\theta}_{\Pi,1}) . \tag{6.4}$$

Since $\theta_{\Pi,1}$ and $\theta_{\Pi',1}$ are finite sums of Fourier transforms of nilpotent orbital integrals, the equation (6.4) follows from Proposition 7.3 in the next section. □

Proof of Theorem 1.13. In view of (6.3) and the fact that the fiber of the wave front set at the origin of a finite sum of homogeneous distributions coincides with the support of its Fourier transform, it will suffice to check that $\widetilde{c}_-^* \Theta_{\Pi'}/\widetilde{c}^* \Theta$ is locally integrable and defines a tempered distribution on $\mathfrak{g}'$. Thus we want to find $N \geqq 0$ such that

$$\int_{\mathfrak{g}'} ch_{\mathfrak{g}'}^{-d}(z) |\Theta_{\Pi'}(\widetilde{c}_-(z))| (1 + |z|)^{-N} dz < \infty , \tag{6.5}$$

because $|\widetilde{c}^* \Theta(z)| = \text{const} \cdot ch_{\mathfrak{g}'}^{-d}(z)$, $z \in \mathfrak{g}$. We know from [P2, 4.11] that $\Theta_{\Pi'}$ has the rate of growth

$$\gamma' = 1 - \left(1 - \frac{1+\gamma}{d'}(r-1)\right) \frac{d}{r'-1} , \tag{6.6}$$

where γ is the rate of growth of Θ_Π. Let $\iota = 1$ if $\mathbf{D} \neq \mathbf{H}$ and $\iota = 1/2$ if $\mathbf{D} = \mathbf{H}$. Then [P2, 5.12, (5.12*')] implies that for (6.5) it suffices to have the following inequality

$$-d > \gamma'(r'-1) - r' - \iota ,$$

which is equivalent to

$$d' > \frac{1+\gamma}{1+\iota}(r-1)d . \tag{6.7}$$

Since Π is unitary, $\gamma \leqq 1$. Also, in all cases $r - 1 \leqq d$. Hence (6.7) holds if $d' > \frac{2}{1+\iota} d^2$.

The fact that θ_Π defines a tempered distribution was verified in [P2, 6.5]. □

7. The correspondence of orbital integrals

We identify the real vector space $\mathfrak{g}$ with the dual space $\mathfrak{g}^*$ via the bilinear form

$$\mathfrak{g} \times \mathfrak{g} \ni (z,s) \to \tfrac{1}{4}\mathrm{tr}(zs) \in \mathbf{R}\,, \tag{7.1}$$

where $\mathrm{tr} = \mathrm{tr}_{\mathbf{D}/\mathbf{R}}$. Then the moment map $\tau_{\mathfrak{g}}$ coincides with τ, where

$$\tau : W \ni w \to ww^* \in \mathfrak{g}\,. \tag{7.2}$$

Similarly we identify $\mathfrak{g}'$ with $\mathfrak{g}'^*$ and define $\tau' : W \ni w \to w^*w \in \mathfrak{g}'$.

Let $\mathcal{O} \subseteq \mathfrak{g}$ be a G-orbit, and let $\mu_{\mathcal{O}}$ be the canonical G-invariant measure on $\mathcal{O}$. Then by Harish–Chandra the Fourier transform $\hat{\mu}_{\mathcal{O}}$ coincides with a function $\hat{\mu}_{\mathcal{O}}(z)$, $z \in \mathfrak{g}$, which is absolutely integrable against any Schwartz function on $\mathfrak{g}$, [W, 8.4.1]. Let $\mathcal{O}' = \tau'(\tau^{-1}(\mathcal{O}) \cap W^{\max})$, as in (1.5). This is a single G'-orbit, which is nilpotent if $\mathcal{O}$ is. Let $\mu_{\mathcal{O}'}$ denote the canonical invariant measure on $\mathcal{O}'$. The goal of this section is to verify the following

Proposition 7.3. *There is a constant* const > 0 *such that for any* $\psi \in S(\mathfrak{g}')$

$$\int_{G\backslash X^{\max}} \int_{\mathfrak{g}} \hat{\mu}_{\mathcal{O}}(z) \int_Y \int_{\mathfrak{g}'} \psi(z')\chi_{z'+z}(x+y)\, dz'\, dy\, dz\, d\dot{x} = \mathrm{const} \int_{\mathfrak{g}'} \psi(z')\, d\hat{\mu}_{\mathcal{O}'}(z')\,,$$

where each consecutive integral is absolutely convergent.

We need some preparation. Since we work in the stable range we may assume that

$$V' = V \oplus V'' \oplus V\,, \tag{7.4}$$

with

$$((u_1,u_2,u_3),(v_1,v_2,v_3))' = (u_1,v_3) + (u_2,v_2)' - (u_3,v_1)\,,$$

where $u_1, v_1, u_3, v_3 \in V$ and $u_2, v_2 \in V''$. Then

$$W = \mathrm{Hom}(V',V) = \mathrm{End}(V) \oplus \mathrm{Hom}(V'',V) \oplus \mathrm{End}(V)$$

and we may write $w \in W$ as

$$w = (w_1,w_2,w_3) \quad (w_1,w_3 \in \mathrm{End}(V),\ w_2 \in \mathrm{Hom}(V'',V))\,. \tag{7.5}$$

Define a map $\mathrm{End}(V) \ni s \to s^\# \in \mathrm{End}(V)$ by

$$(su,v) = (u,s^\# v) \quad (u,v \in V)\,. \tag{7.6}$$

Then for w as in (7.5) the element $w^* \in \mathrm{Hom}(V,V')$, defined in (4.2), may be identified with $w^* = (-w_3^\#, w_2^*, w_1^\#)$ in the sense that for $v \in V$

$$w^*(v) = (-w_3^\#(v), w_2^*(v), w_1^\#(v))\,. \tag{7.7}$$

In terms of (7.5) define

$$\sigma : \mathfrak{g} \ni z \to (I, 0, \tfrac{1}{2}z) \in W\,, \tag{7.8}$$

where $I \in \mathrm{End}(V)$ is the identity. Then (7.7) implies that σ *is a section of the map* τ (7.2).

The orthogonal complement of $\mathfrak{g}$ in $\mathrm{End}(V)$, $\mathfrak{g}^{\perp} = \{s \in \mathrm{End}(V); s = s^{\#}\}$. Define a map

$$(7.9)\qquad \begin{aligned} &e : \mathfrak{g}^{\perp} \times \mathrm{Hom}(V'', V) \times GL(V) \to \mathrm{End}(V'),\ \text{by}\\ &e(S,T,g) = e(S)e(T)e(g),\ \text{where}\\ &e(S)(v_1,v_2,v_3) = (v_1 + Sv_3, v_2, v_3)\\ &e(T)(v_1,v_2,v_3) = (v_1 + Tv_2 - \tfrac{1}{2}TT^{*}v_3, v_2 - T^{*}v_3, v_3)\\ &e(g)(v_1,v_2,v_3) = (gv_1, v_2, (g^{\#})^{-1}v_3)\ . \end{aligned}$$

and $v_1, v_3 \in \mathrm{End}(V), v_2 \in \mathrm{Hom}(V'', V)$. Then the image of e coincides with the parabolic subgroup $P' \subseteq G'$ preserving the isotropic subspace $0 \oplus 0 \oplus V \subseteq V'$. Notice that in terms of (7.5), (7.8) and (7.9)

$$(7.10)\qquad \sigma(z)e(S,T,g) = (g, T, (\tfrac{1}{2}z + S - \tfrac{1}{2}TT^{*})(g^{\#})^{-1})\ .$$

Since we assume that the form $(\,,)'$ is split the restriction of it to V'' is split. Let $V'' = X'' \oplus Y''$ be a complete polarization. Let $X' = V \oplus X'', Y' = Y'' \oplus V$, (see (7.4)). Then $V' = X' \oplus Y'$ is a complete polarization, and we may assume that the proposition (7.3) is written in terms of

$$(7.11)\qquad \begin{aligned} X &= \mathrm{Hom}(X', V) = \mathrm{End}(V) \oplus \mathrm{Hom}(X'', V)\ ,\\ Y &= \mathrm{Hom}(Y', V) = \mathrm{Hom}(Y'', V) \oplus \mathrm{End}(V)\ . \end{aligned}$$

Let $x = (g, x')$, $g \in GL(V)$, $x' \in \mathrm{Hom}(X'', V)$. Then a straightforward argument shows that for a test function ϕ on $W^{\max}$

(7.12)

$$\int_Y \phi(x+y)\,dy = \mathrm{const}|\det_{\mathbf{R}}(g)|^{-d} \int_{\mathrm{Hom}(Y'',V)} \int_{\mathfrak{g}^{\perp}} \int_{\mathfrak{g}} \phi(\sigma(z)e(S, x', y', g))\,dz\,dS\,dy'\ ,$$

where $d = \dim_{\mathbf{D}} V$. Moreover, if ϕ is G-invariant then

$$(7.13)\qquad \begin{aligned} &\int_{G\backslash X^{\max}} \int_Y \phi(x+y)\,dy\,d\dot{x}\\ &= \mathrm{const} \int_{G\backslash GL(V)} \int_{\mathrm{Hom}(V'',V)} \int_{\mathfrak{g}^{\perp}} \int_{\mathfrak{g}} \phi(\sigma(z)e(S,T,g))\,dz\,dS\,dT\,d\dot{g}\ . \end{aligned}$$

where $d\dot{g}$ stands for the invariant measure on the quotient space $G\backslash GL(V)$.

Proof of Proposition 7.3. The argument (5.6)–(5.7) shows that for a fixed $x \in X^{\max}$ and any $\psi \in S(\mathfrak{g}')$ the following function belongs to $S(Y)$:

$$Y \ni y \to \int_{\mathfrak{g}'} \psi(z')\chi_{z'}(x+y)\,dz' = \hat{\psi} \circ \tau'(x+y) \in \mathbf{C}\ .$$

Hence the function

$$\mathfrak{g} \ni z_1 \to \int_Y \int_{\mathfrak{g}'} \psi(z')\chi_{z_1+z'}(x+y)\,dz'\,dy \in \mathbf{C}$$

is in $S(\mathfrak{g})$. Therefore, by Harish–Chandra [W, 8.4.1], the integral

$$\int_{\mathfrak{g}} \hat{\mu}_{\mathcal{O}}(z_1) \int_Y \int_{\mathfrak{g}'} \psi(z')\chi_{z_1+z'}(x+y)\,dz'\,dy\,dz_1 \quad (x \in X^{\max}) \tag{7.14}$$

is absolutely convergent and defines a G-invariant function on $X^{\max}$. In terms of the coordinates described in (7.9) and (7.12), the integral (7.14) may be rewritten as a constant multiple of

$$\begin{aligned} (7.15)\quad & |\det_{\mathbf{R}}(g)|^{-d} \int_{\mathfrak{g}} \hat{\mu}_{\mathcal{O}}(z_1) \int_{\mathrm{Hom}(Y'',V)} \int_{\mathfrak{g}^\perp} \int_{\mathfrak{g}} \int_{\mathfrak{g}'} \psi(z') \\ & \times \chi_{z_1+z'}(\sigma(z)e(S,x',y',g))\,dz'\,dz\,dS\,dy'\,dz_1 \\ & = |\det_{\mathbf{R}}(g)|^{-d} \int_{\mathrm{Hom}(Y'',V)} \int_{\mathfrak{g}^\perp} \int_{\mathfrak{g}} \hat{\psi} \circ \tau'(\sigma(z)e(S,x',y',g))\,d\mu_{\mathcal{O}}(z)\,dS\,dy' \,. \end{aligned}$$

There are rapidly decreasing functions (not necessarily smooth) $\psi_+ \geqq 0$ and $\psi_- \leqq 0$ such that $\hat{\psi} = \psi_+ + \psi_-$. Hence, by (7.13) the integral of (7.15) over $G\backslash X^{\max}$ coincides with the sum of the following two integrals:

$$\int_{G\backslash GL(V)} \int_{\mathrm{Hom}(V'',V)} \int_{\mathfrak{g}^\perp} \int_{\mathfrak{g}} \psi_\pm \circ \tau'(\sigma(z)e(S,T,g))\,d\mu_{\mathcal{O}}(z)\,dS\,dT\,d\dot{g} \,. \tag{7.16}$$

Fix $z_0 \in \mathcal{O}$. Then (7.16) may be rewritten as

$$\int_{GL(V)_{z_0}\backslash GL(V)} \int_{\mathrm{Hom}(V'',V)} \int_{\mathfrak{g}^\perp} \psi_\pm \circ \tau'(\sigma(z_0)e(S,T,g))\,d\mu_{\mathcal{O}}(z)\,dS\,dT\,d\dot{g} \,, \tag{7.17}$$

where $GL(V)_{z_0}$ is the stabilizer of z_0 in $GL(V)$, and $d\dot{g}$ stands for the invariant measure on the indicated homogeneous space. Notice that the stabilizer of $\tau'(\sigma(z_0))$ in P' coincides with $P'_{z_0} = GL(V)_{z_0} \subseteq P'$ (see (7.9)). Hence (7.17) may be rewritten as

$$\int_{P'_{z_0}\backslash P'} \psi_\pm(g^{-1}\tau'(\sigma(z_0))g)\,d\dot{g} \,. \tag{7.18}$$

where $d\dot{g}$ is the (right) invariant measure on $P'_{z_0}\backslash P'$. Since the P'-orbit of $\tau'(\sigma(z_0))$ is dense in $\mathcal{O}'$, (7.18) coincides with a constant multiple of

$$\int_{\mathcal{O}'} \psi_\pm(z')\,d\mu_{\mathcal{O}'}(z') \,,$$

which by [RR] is absolutely convergent. □

Appendix

Here we complete the proof of Lemma 5.1. We are going to use the obvious fact that $G\backslash GL(V)$ is a symmetric space [F-J] corresponding to the involution

$\mathrm{End}(V) \ni x \to -x^{\#} \in \mathrm{End}(V)$ where

$$(1) \qquad (xu, v) = (u, x^{\#} v) \quad (u, v \in V)\,.$$

Let $(\,,)_+$ and $(\,,)'_+$ be positive hermitian (or symmetric) forms on V and V', chosen so that the subgroups of G and G' preserving these forms are maximal compact. We may assume that $V' = X' \oplus Y'$ is orthogonal for $(\,,)'_+$. Let $U' \subseteq G'$ be the subgroup preserving X', Y' and the form $(\,,)'_+$. Then U' is isomorphic to a maximal compact subgroup of $GL(X')$, by restriction. We shall identify U' with that subgroup.

Let $U \subseteq GL(V)$ be the isometry group of $(\,,)_+$. Let $e_1, e_2, \ldots, e_d$ be an $(\,,)_+$-orthogonal basis of V and let $B = \{b \in GL(V); be_i = b_i e_i,\ b_i \in \mathbf{R}, b_i > 0\}$.

Suppose that either $\mathbf{D} = \mathbf{R}$ and the form $(\,,)$ is symmetric; or $\mathbf{D} \neq \mathbf{R}$, the involution on $\mathbf{D}$ is non-trivial and the form $(\,,)$ hermitian. Let p, q, be the signature of the form $(\,,)$, with $p \leqq q$ and $p + q = d = \dim_{\mathbf{D}} V$. Let

$$(2) \quad B^+ = \{b \in B;\ b_1 > b_2 > \cdots > b_p > 0,\ b_{p+1} > b_{p+2} > \cdots > b_d > 0\}\,,$$

$$\delta(b) = (b_1^{d-1} b_2^{d-3} \cdots b_d^{-d+1})^{\dim_{\mathbf{R}} \mathbf{D}} \quad (b \in B)\,.$$

If $\mathbf{D} = \mathbf{C}$ is equipped with the trivial involution and the form $(\,,)$ is symmetric we use the notation (2) with $p = d$ and $q = 0$.

Suppose $\mathbf{D} \neq \mathbf{H}$ and the form $(\,,)$ is skew-symmetric or $\mathbf{D} = \mathbf{H}$ and the form $(\,,)$ is skew-hermitian. Then $d = \dim_{\mathbf{D}} V = 2p$ is an even integer and we may assume that $b_i = b_{p+i}$, for $b \in B$ and $1 \leqq i \leqq p$. Let

$$(3) \qquad B^+ = \{b \in B;\ b_1 > b_2 > \cdots > b_p > 0\}\,,$$

$$\delta(b) = (b_1^{d-2} b_2^{d-6} \cdots b_d^{-d+2})^{\dim_{\mathbf{R}} \mathbf{D}} \quad (b \in B)\,.$$

In all cases [F-J, (2.13–2.14)] implies that with an appropriate normalization of the Haar measure dg on $GL(V)$ the following inequality holds:

$$(4) \qquad \int_{GL(V)} |v(g)|\, dg \leqq \int_G \int_{B^+} \int_U |v(g\, b\, k)| \delta(b)\, dk\, db\, dg \quad (v \in C_c(GL(V))\,.$$

Since, by the stable range assumption, $\dim V \leqq \dim X'$, we may assume that $V \subseteq X'$ and that the restriction of the form $(\,,)'_+$ to V coincides with $(\,,)'_+$. Let $e_1, e_2, \ldots, e_d, e_{d+1}, \ldots, e_n$ be an $(\,,)'_+$-orthonormal basis of X'. Let $I' \in X = \mathrm{Hom}(X', V)$ denote the map defined by

$$I'(e_i) = \begin{cases} e_i & \text{if } 1 \leqq i \leqq d \\ 0 & \text{if } d < i \leqq n\,. \end{cases}$$

By Witt's theorem $X^{\max} = GL(V) \cdot I' \cdot U'$ is a single $GL(V) \times U'$-orbit. Hence

$$(5) \qquad \int_{X^{\max}} v(x)\, dx = \int_{GL(V)} \int_{U'} v(gI'k) |\det_{\mathbf{R}} g|^n\, dk\, dg \quad (v \in C_c(X^{\max}))\,.$$

Indeed, both sides are U'-invariant and transform by $|\det_{\mathbf{R}}|^n$ under the action of $GL(V)$.

By combining (4) and (5) we obtain the following integral inequality

$$(6)\quad \int_{X^{\max}} |v(x)|\,dx \leqq \int_G \int_{B^+} \int_{U'} |v(gbI'k)||\det_{\mathbf{R}} b|^n \delta(b)\,dk\,db\,dg \quad (v \in C_c(X^{\max}))\,.$$

Proof of Lemma 5.4. Define a norm $|\ |$ on (the real vector space) $\mathrm{End}(V)$ by $|x|^2 = \sum_{i,j=1}^d |(e_i, xe_j)|^2$. We may choose a norm $|\ |$ on $\mathrm{Hom}(X', Y')$, invariant under multiplication by U', so that

$$(7)\qquad |(xI')^*(xI')| = |x^\sharp x| \quad (x \in \mathrm{End}(V))\,.$$

Recall the character Θ_π of the representation π (3.7). It follows from (6) and (7) that for $N > 0$ we have

$$(8)\qquad \int_{G\backslash X^{\max}} \int_G |\Theta_\pi(g)|(1 + |x^*x|^2 + |x^*gx|^2)^{-N}\,dg\,d\dot{x} \leqq \int_G |\Theta_\pi(g)|\,L(g)\,dg\,,$$

where

$$L(g) = \int_{B^+} (1 + |(bI')^*(bI')|^2 + |(bI')^*g(bI')|^2)^{-N} |\det_{\mathbf{R}} b|^n \delta(b)\,db$$

$$= \int_{B^+} (1 + |b^\sharp b|^2 + |b^\sharp gb|^2)^{-N} |\det_{\mathbf{R}} b|^n \delta(b)\,db\,.$$

Notice that $|b^\sharp b|^2 = |b^2|^2 = \sum_{i=1}^d b_i^4$ and

$$|b^\sharp gb|^2 = \sum_{i,j=1}^d b_i^2 b_j^2 |(e_i, ge_j)|^2 \geqq b_k^4 |g|^2\,,$$

where $b_k^4 = \min\{b_1, b_2, \ldots\}$. Hence,

$$L(g) \leqq \int_{B^+} \left(1 + \sum_{i \neq k}^d b_i^4 + b_k^4(1 + |g|^2)\right)^{-N} |\det_{\mathbf{R}} b|^n \delta(b)\,db\,.$$

We replace B^+ in the above integral by B, use invariance of the Haar measure on B, the assumption $n \geqq d$ and the formulas (2) and (3) for $\delta(b)$, to see that for N large enough there is a finite constant const such that for all $g \in G$

$$(9)\qquad L(g) \leqq \mathrm{const}\,|g|^{-m} \quad (g \in G)\,,$$

where $m = \dim_{\mathbf{R}} \mathbf{D}(n - d + 1)/2 = \dim_{\mathbf{R}} \mathbf{D}(d'/2 - d + 1)/2$ if the form $(\,,)$ is hermitian (or symmetric), and $m = \dim_{\mathbf{R}} \mathbf{D}(n - p + 1) = \dim_{\mathbf{R}} \mathbf{D}(d'/2 - d/2 + 1)$ otherwise.

A straightforward calculation (see the proof of 4.11 in [P2]), using a fundamental estimate of Harish–Chandra for his Ξ-function, shows that $|g|^{-m} \leqq \mathrm{const}\,\Xi^{2+2\varepsilon}$, for all $g \in G$, if $m > \dim_{\mathbf{R}} \mathbf{D}(r - p)p$, where r is as in (1.8). Since the representation π is irreducible and unitary it has the rate of growth at most 1, in the sense of Miličić, [M]. Thus for any $\varepsilon > 0$ there is a constant const $< \infty$ such that

$$|\Theta_\pi(g)| \leqq \text{const} \cdot |d_G(g)|^{-1/2} \Xi(g)^{-1-\varepsilon} \quad (g \in G)\,,$$

where d_G is the Weyl denominator. Moreover, for any $\varepsilon > 0$

$$\int_G |d_G(g)|^{-1/2} \Xi(g)^{1+\varepsilon}\, dg < \infty\,,$$

see [Wa, 8.3.7.6]. Hence, for $m > \dim_{\mathbf{R}} \mathbf{D}(r-p)p$,

$$\int_G |\Theta_\pi(g)| L(g)\, dg \leqq \text{const} \int_G |\Theta_\pi(g)| |g|^{-m}\, dg \leqq \text{const} \int_G |\Theta_\pi(g)| \Xi(g)^{2+2\varepsilon}\, dg$$
$$\leqq \int_G |d_G(g)|^{-1/2} \Xi(g)^{1+\varepsilon}\, dg < \infty\,.$$

Since in all cases $r \leqq d+1$ and $(r-p)p \leqq (r/2)^2$, the assumption $d' \geqq (d+2)^2$ implies that $m > \dim_{\mathbf{R}} \mathbf{D}(r-p)p$. Thus the integral (8) is convergent, and the lemma follows. □

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