

# The Cauchy Harish-Chandra Integral and the Invariant Eigendistributions

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In this paper, we prove that the Cauchy Harish-Chandra integral maps invariant eigendistributions to invariant eigendistributions with the correct infinitesimal character and that it maps the orbital integrals on the larger group to the orbital integrals on the smaller one. This is the last paper of a series of three.

## 1 The Main Results

Let  $W$  be a finite-dimensional vector space over the reals, with a nondegenerate symplectic form  $\langle \cdot, \cdot \rangle$ . Let  $\mathcal{J}$  be a positive compatible complex structure on  $W$ ,  $\mathrm{Sp}(W)$  (resp.  $\mathfrak{sp}(W)$ ) the symplectic group (resp. the symplectic Lie algebra) associated to  $\langle \cdot, \cdot \rangle$  and let  $\chi(r) = \exp(2\pi ir)$ ,  $r \in \mathbb{R}$ . Fix a Lebesgue measure  $dw$  on  $W$  so that

$$\int_W \chi\left(\frac{i}{2}\langle \mathcal{J}w, w \rangle\right) dw = 1.$$

The conjugation by  $\mathcal{J}$  is a Cartan involution  $\theta$  on  $\mathfrak{sp}(W)$ . The formula

$$\tilde{\kappa}(x, y) = -\mathrm{tr}(\theta x, y) \quad (x, y \in \mathfrak{sp}(W))$$

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defines a positive definite symmetric form on  $\mathfrak{sp}(W)$ . We normalize the Lebesgue measure  $\mu$  on any subspace of  $\mathfrak{sp}(W)$  by requiring that the volume of the unit cube is 1. For any Lie subgroups  $F \subseteq E \subseteq \mathrm{Sp}(W)$ , the measure  $\mu$  induces the left invariant Haar measure on  $E$  and invariant measures on the quotients  $E/F$  and  $F \backslash E$ . We shall denote these induced measures also by  $\mu$ . It should be clear from the context, (such as  $d\mu(a)$  or  $d\mu(A_s''' w)$  below), which version of  $\mu$  we are using.

Let  $(G, G')$  be a reductive dual pair of  $\mathrm{Sp}(W)$  (see [17] for the definition), with the rank of  $G'$  less or equal to the rank of  $G$ . Denote by  $\mathfrak{g}, \mathfrak{g}'$  the Lie algebras of  $G, G'$ , respectively. We lift the Cartan involution  $\theta$  to the group and assume (as we may) that  $G$  and  $G'$  are preserved by  $\theta$ . Let  $H'$  be a Cartan subgroup of  $G'$  preserved by  $\theta$ . Consider the Cartan decomposition of  $H' : H' = T' A'$ , where  $T'$  (resp.  $A'$ ) is the compact (resp. split) part of  $H'$ . Consider the commutant  $A''$  (resp.  $A'''$ ) of  $A'$  (resp.  $A''$ ) in  $\mathrm{Sp}(W)$ . Then  $(A'', A''')$  is a reductive dual pair of  $\mathrm{Sp}(W)$ ; see [18]. Denote by  $V'$  the defining module for  $G'$ . This is a finite-dimensional vector space  $V'$  over a division algebra  $\mathbb{D} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ; see [15, p. 278]. Let  $V'_c = \{v \in V' \mid av = v \ \forall a \in A'\}$ . There exists a unique complement  $V'_s$  of  $V'_c$  in  $V'$  such that the decomposition

$$V' = V'_c \oplus V'_s$$

is preserved by  $H'$ . As  $A''' \subset H'$ , we may consider

$$A_s''' = \{v \in A''' \mid v|_{V'_c} = \mathrm{id}\}.$$

Then  $A''' = A_s'''$  if and only if  $V'_c = 0$  and  $A''' = A_s''' \times \{\pm \mathrm{id}|_{V'_c}\}$  otherwise. There exists a dense open subset  $W_{A_s''}$  of  $W$  such that the quotient  $A_s'' \backslash W_{A_s''}$  is a smooth manifold. Define the measure  $\mu$  on the quotient  $A_s'' \backslash W_{A_s''}$  by

$$\int_W f(w) \, dw = \int_{A_s'' \backslash W_{A_s''}} \int_{A_s''} f(aw) \, d\mu(a) \, d\mu(A_s'' w).$$

Let

$$\chi_x(w) = \chi \left( \frac{1}{4} \langle xw, w \rangle \right), \quad (x \in \mathfrak{sp}(W), w \in W).$$

Recall [18, p. 302] the Cauchy Harish-Chandra integral on the Lie algebra:

$$\widetilde{\mathrm{chc}}(x' + x) = \int_{A_s'' \backslash W_{A_s''}} \chi_{x'+x}(w) \, d\mu(A_s'' w) \quad (x' \in \mathfrak{h}^{\mathrm{reg}}, x \in \mathfrak{g}).$$

Define the normalized Cauchy Harish-Chandra integral by

$$\text{chc}(x' + x) = \frac{1}{\mu(A_s''' \setminus H')} \widetilde{\text{chc}}(x' + x) \quad (x' \in \mathfrak{h}^{\text{reg}}, x \in \mathfrak{g}).$$

Let  $\widetilde{\text{Sp}}(W)$  be the connected two-fold cover of  $\text{Sp}(W)$ . This is the metaplectic group with the canonical surjection

$$\widetilde{\text{Sp}}(W) \longrightarrow \text{Sp}(W). \tag{1}$$

For a subset  $E$  of  $\text{Sp}(W)$ , denote by  $\tilde{E}$  the preimage of  $E$  in  $\widetilde{\text{Sp}}(W)$ .

Let  $S'(W)$  be the space of temperate distributions on  $W$ . Recall [18, Theorem 2.8] Howe’s embedding of the metaplectic group into the space of temperate distributions on the symplectic space:

$$T : \widetilde{\text{Sp}}(W) \longrightarrow S'(W).$$

Recall the definition of the Cauchy Harish-Chandra integral on the group  $\tilde{G}'$ , [18, Definition 2.11]:

$$\widetilde{\text{Chc}}(x'x) = \int_{A_s''' \setminus W_{A_s'''}} T(x'x)(w) \, d\mu(A_s''' w) \quad (x' \in \tilde{H}^{\text{reg}}, x \in \tilde{G}).$$

Define the normalized integral by

$$\text{Chc}(x'x) = \frac{1}{\mu(A_s''' \setminus H')} \widetilde{\text{Chc}}(x'x) \quad (x' \in \tilde{H}^{\text{reg}}, x \in \tilde{G}).$$

### 1.1 Statement on the Lie algebra level

For  $\phi \in \mathcal{D}(\mathfrak{g})$  (the space of the compactly supported smooth functions on  $\mathfrak{g}$ ) let

$$\text{chc}(\phi)(x') = \int_{\mathfrak{g}} \text{chc}(x' + x)\phi(x) \, d\mu(x) \quad (x' \in \mathfrak{h}^{\text{reg}}).$$

The above formula defines a  $G'$ -invariant function,  $\text{chc}(\phi)$ , on  $\mathfrak{g}^{\text{reg}}$ . Let us denote by  $\tilde{\mathcal{I}}(\mathfrak{g}')$  the space of unnormalized orbital integrals on  $\mathfrak{g}'$  with no condition on the support. This means that  $\tilde{\mathcal{I}}(\mathfrak{g}')$  is the set of all  $\psi \in C^\infty(\mathfrak{g}^{\text{reg}})$  such that the functions

$$|\det(\text{ad}(x))_{\mathfrak{g}'/\mathfrak{g}'x}|^{1/2} \psi(x) \quad (x \in \mathfrak{g}^{\text{reg}})$$

satisfy the conditions  $I_1, I_2,$  and  $I_3$  of [5, Section 3.2] and we do not assume that the condition  $I_4$  is satisfied. We denote also by  $\mathcal{I}(\mathfrak{g})$  the space of unnormalized orbital integrals

on  $\mathfrak{g}$  assuming this time that all the conditions  $I_1, I_2, I_3$ , and  $I_4$  are satisfied. In other words, by a result of Bouaziz [5],  $\mathcal{I}(\mathfrak{g})$  is the space of all the functions of the form

$$\int_{G/G^x} \psi(g.x) d\mu(gG^x) \quad (x \in \mathfrak{g}^{\text{reg}}, \psi \in \mathcal{D}(\mathfrak{g})).$$

The first result of this paper is:

**Theorem 1.1.** Let  $\phi \in \mathcal{D}(\mathfrak{g})$ . Then  $\text{chc}(\phi) \in \tilde{\mathcal{I}}(\mathfrak{g}')$ . □

Concerning the support, the functions in the range of  $\text{chc}$  do not necessarily satisfy the condition  $I_4$ . The functions restricted to a given Cartan subalgebra  $\mathfrak{h}'$  have a compact support modulo the elliptic part of  $\mathfrak{h}'$ .

### 1.2 Statements on the Lie group level

Let  $G_1$  denote the Zariski identity component of  $G$  multiplied by the center of  $G$ . Then  $G_1 = G$  unless  $G$  is a real, even orthogonal group  $O_{2p,2q}$ . Similarly, we define  $G'_1$ . Let  $\phi \in \mathcal{D}(\tilde{G}_1)$  (the space of the compactly supported smooth functions on  $\tilde{G}_1$ ). Define  $\text{Chc}(\phi)$  to be a  $\tilde{G}$ -invariant function on  $\tilde{G}_1^{\text{reg}}$  by the formula

$$\text{Chc}(\phi)(x') = \int_{\tilde{G}_1} \text{Chc}(x'x)\phi(x) d\mu(x) \quad (x' \in \tilde{H}^{\text{reg}}).$$

Let us denote by  $\mathcal{I}(\tilde{G}_1)$  the space of unnormalized orbital integrals on  $\tilde{G}_1$ . This means that  $\mathcal{I}(\tilde{G}_1)$  is the set of function of the form

$$\tilde{G}_1^{\text{reg}} \ni x \longrightarrow \int_{\tilde{G}_1/\tilde{G}_1^x} \psi(g.x) d\mu(g\tilde{G}_1^x) \in \mathbb{C},$$

where  $\psi \in \mathcal{D}(\tilde{G}_1)$ . As explained in the remark following Theorem 6.7,  $\mathcal{I}(\tilde{G}_1)$  is isomorphic as a vector space to the space of the Harish-Chandra orbital integrals, which is endowed with a topology, as explained on [6, p. 580]. Hence,  $\mathcal{I}(\tilde{G}_1)$  is a linear topological vector space.

We denote by  $I_{\tilde{G}}$  the map from  $\mathcal{D}(\tilde{G}_1)$  onto  $\mathcal{I}(\tilde{G}_1)$ , defined above, and similarly for  $\tilde{G}$ . One of the main results of this paper is

**Theorem 1.2.** Let  $\phi \in \mathcal{D}(\tilde{G}_1)$ . Then  $\text{Chc}(\phi) \in \mathcal{I}(\tilde{G}_1)$ . Moreover, the resulting map

$$\text{Chc} : \mathcal{D}(\tilde{G}_1) \rightarrow \mathcal{I}(\tilde{G}_1)$$

is continuous and, if  $I_{\tilde{G}}(\phi)$  is 0, then  $\text{Chc}(\phi)$  is also 0. □

This theorem proves that  $\text{Chc}$  induces a map

$$\text{Chc} : \mathcal{I}(\tilde{G}_1) \rightarrow \mathcal{I}(\tilde{G}_1). \tag{2}$$

Concerning the support of the functions in the range of  $\text{Chc}$ , their restrictions to a given Cartan subgroup  $\tilde{H}'$  of  $\tilde{G}$  have a compact support modulo the compact part of  $\tilde{H}'$ . Therefore, they are compactly supported. Let  $\mathcal{D}'(\tilde{G}_1)$  be the space of distributions on  $\tilde{G}_1$  and let  $\mathcal{D}'^{\tilde{G}}(\tilde{G}_1)$  be the space of  $\tilde{G}$ -invariant distributions on  $\tilde{G}_1$ . Recall, [5] that the transpose of the map  $I_{\tilde{G}}$  induces an isomorphism between  $\mathcal{I}(\tilde{G}_1)'$  (the dual of  $\mathcal{I}(\tilde{G}_1)$ ) and  $\mathcal{D}'^{\tilde{G}}(\tilde{G}_1)$ . Hence, we have

$$\begin{aligned} {}^t I_{\tilde{G}} : \mathcal{I}(\tilde{G}_1)' &\xrightarrow{\cong} \mathcal{D}'^{\tilde{G}}(\tilde{G}_1), \\ {}^t I_{\tilde{G}} : \mathcal{I}(\tilde{G}_1)' &\xrightarrow{\cong} \mathcal{D}'^{\tilde{G}}(\tilde{G}_1). \end{aligned} \tag{3}$$

We denote also by  $\text{Chc}$  the resulting map

$$\text{Chc} : \mathcal{D}'^{\tilde{G}}(\tilde{G}_1) \mapsto \mathcal{D}'^{\tilde{G}}(\tilde{G}_1) \tag{4}$$

induced by the transpose of  $\text{Chc}$  and the isomorphisms (3). Thus, for  $u \in \mathcal{D}'^{\tilde{G}}(\tilde{G}_1)$ ,

$$\text{Chc}(u) = ({}^t I_{\tilde{G}})^{-1}(u) \circ \text{Chc}. \tag{5}$$

In particular, if  $u$  is given in terms of a locally integrable function, then, for  $\phi \in \mathcal{D}(\tilde{G}_1)$ ,

$$\text{Chc}(u)(\phi) = \sum \frac{1}{|W(H')|} \int_{H'^{\text{reg}}} |\det(1 - \text{Ad}(h^{-1}))_{\mathfrak{g}'/\mathfrak{h}'}| u(h) \text{Chc}(\phi)(h) d\mu(h), \tag{6}$$

where the summation is over a maximal family of mutually nonconjugate Cartan subgroups  $H' \subseteq \tilde{G}$  and  $|W(H')|$  stands for the cardinality of the Weyl group of  $H'$  in  $\tilde{G}$ .

We denote by  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  (resp.  $\mathcal{U}(\mathfrak{g}'_{\mathbb{C}})$ ) the enveloping algebra of  $\mathfrak{g}_{\mathbb{C}}$  (resp.  $\mathfrak{g}'_{\mathbb{C}}$ ). Consider the Capelli Harish-Chandra homomorphism (cf. equality (62)):

$$\mathcal{C}_{\mathfrak{g},\mathfrak{g}'} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G \longrightarrow \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}.$$

Let  $L$  be the left regular representation of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$  on  $\mathcal{D}(\tilde{G})$  (cf. equality (56)) and similarly for  $\mathfrak{g}'$ .

**Theorem 1.3.** Let  $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ . Then for  $\phi \in \mathcal{D}(\tilde{G}_1)$

$$\text{Chc}(L(\check{z})\phi) = L(C_{\mathfrak{g}, \mathfrak{g}'}(z)) \text{Chc}(\phi),$$

where  $z \rightarrow \check{z}$  is the involution on the universal enveloping algebra, extending the map  $\mathfrak{g}_{\mathbb{C}} \ni z \rightarrow -z \in \mathfrak{g}_{\mathbb{C}}$ . □

The action of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$  on  $\mathcal{D}(\tilde{G})$  induces an action on  $\mathcal{D}'(\tilde{G})$  denoted also by  $L$ . By definition, we have

$$L(z)u(\phi) = u(L(\check{z})\phi)$$

for  $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ ,  $u \in \mathcal{D}'(\tilde{G})$ , and  $\phi \in \mathcal{D}(\tilde{G})$ . Moreover,  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$  stabilizes  $\mathcal{D}'^{\tilde{G}}(\tilde{G})$ . The same holds for  $\tilde{G}$ . The following theorem explains the title of the paper.

**Theorem 1.4.** Let  $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$  and  $u \in \mathcal{D}'^{\tilde{G}}(\tilde{G}_1)$ . Then

$$\text{Chc}(L(C_{\mathfrak{g}, \mathfrak{g}'}(z))u) = L(\check{z}) \text{Chc}(u). \quad \square$$

This result was the main aim of our project. One might deduce it directly from [3, Theorem 7.4]; however, it is conceptually easier to see that it follows from Theorems 1.2 and 1.3, because the action of the centers of the universal enveloping algebras intertwine the maps (3).

### 1.3 Relation with Howe’s correspondence

Let  $\mathfrak{v} \in \widetilde{\text{Sp}}(\mathbb{W})$  be any element in the preimage of  $-1 \in \text{Sp}(\mathbb{W})$ . Let  $\Pi'$  be an irreducible admissible representation of  $\tilde{G}$  and let  $\Theta_{\Pi'}$  denote the distribution character of  $\Pi'$ . Denote by  $\chi_{\Pi'}(\mathfrak{v})$  the scalar by which  $\Pi'(\mathfrak{v})$  acts on the Hilbert space of  $\Pi'$ . Theorem 1.2 implies that for a test function  $\phi \in \mathcal{D}(\tilde{G})$  there is a test function  $\phi' \in \mathcal{D}(\tilde{G}_1)$  whose orbital integrals coincide with  $\text{Chc}(\phi)$ . Let

$$\Theta'_{\Pi'}(\phi) = \chi_{\Pi'}(\mathfrak{v})\Theta(\mathfrak{v}) \int_{\tilde{G}_1} \Theta_{\Pi'}(g^{-1})\phi'(g) \, d\mu(g). \tag{7}$$

Though the function  $\phi'$  is not uniquely determined, its orbital integrals are. Hence, formula (7) defines an invariant distribution on  $\tilde{G}_1$ . (Recall that  $\Theta_{\Pi'}(g^{-1}) = \bar{\Theta}_{\Pi'}(g)$  if  $\Pi'$  is unitary.)

Let  $\mathcal{H}^\infty$  denote the space of the smooth vectors in the Hilbert space of the oscillator representation  $\omega$  corresponding to the character  $\chi$  of the additive group of the real numbers, as in [16]. Suppose the representation  $\Pi'$  occurs as a quotient of  $\mathcal{H}^\infty$  by a

closed invariant subspace  $\mathcal{H}_1^\infty$ . Let  $\mathcal{H}_{\Pi'}^\infty$  be the intersection of all such subspaces  $\mathcal{H}_1^\infty$ . As shown in [17],

$$\mathcal{H}^\infty/\mathcal{H}_{\Pi'}^\infty \text{ is isomorphic to } \Pi'_1 \otimes \Pi'$$

as a  $\tilde{G} \times \tilde{G}$ -module, where  $\Pi'_1$  is a finitely generated admissible quasisimple representation of  $\tilde{G}$ , which has a unique quotient  $\Pi$  (the “theta lift” of  $\Pi'$ ). (The notation  $\Pi'_1$  is consistent with that of Howe [17, (1.1)] except that the members of the dual pair are reversed. Therefore  $\Pi'_1$  is a representation of  $\tilde{G}$  not of  $\tilde{G}$ .)

Here, we offer a more precise version of a conjecture formulated in [18, Conjecture 2.18].

**Conjecture 1.5.** Suppose  $\Pi'$  occurs as a quotient of  $\mathcal{H}^\infty$  and the function  $\Theta_{\Pi'}$  is supported in  $\tilde{G}_1$ , that is, the restriction to the complement is zero:  $\Theta_{\Pi'}|_{\tilde{G} \setminus \tilde{G}_1} = 0$ . Then,

$$\Theta'_{\Pi'} = \Theta_{\Pi'_1}|_{\tilde{G}_1}. \quad \square$$

It is clear from Theorem 1.4 that if  $\gamma' : \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'} \rightarrow \mathbb{C}$  is the infinitesimal character of  $\Pi'$ , then  $\Theta'_{\Pi'}$  is an  $\gamma' \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}$ -eigendistribution, as are  $\Theta_{\Pi'_1}$  and  $\Theta_{\Pi}$ . Thus,  $\Theta'_{\Pi'}$  is an invariant eigendistribution with the correct infinitesimal character!

If our dual pair is in the “deep stable range” with  $G'$  the smaller member (see [8]), and  $\Pi'$  is genuine and unitary, then  $\Pi'_1 = \Pi$  and  $\Theta'_{\Pi'} = \Theta_{\Pi}$ . (In fact [8] was the main motivation for [18].) The same holds if the group  $G'$  is compact. If the dual pair is of type II, then the explicit formulas for the Chc in [4, Theorem 6] show that the conjecture holds. Furthermore, these formulas combined with the equation

$$\frac{1}{x - i0} - \frac{1}{x + i0} = 2\pi i \delta(x) \quad (x \in \mathbb{R}),$$

imply and generalize the results of Adams [1] and Renard [20], concerning stable orbital integrals and the theta lift, see Section 3.

The plan of this paper is as follows. In Section 2, we provide explicit formulas for chc and for Chc. In Section 3, we consider sums of the integrals corresponding to various real forms of a dual pair. In Section 4, we prove Theorem 1.1. In [3, 4], we proved the boundedness of chc. (Specifically, the explicit formulas for chc in [4, Corollary 4 and Corollary 8] reduce the problem to the case of an elliptic Cartan subalgebra of  $\mathfrak{g}'$  and in that case [3, Theorem 1] proves the result.)

In [4, Theorem 10], we proved that the jump relations are satisfied for the dual pairs  $(\mathrm{Sp}_{2n}(\mathbb{R}), O_{1,2})$ ,  $(O_{p,q}, \mathrm{Sp}_2(\mathbb{R}))$ , and  $(U_{p,q}, U_{1,1})$ . Here, we prove a result of reduction and deduce that the jump relations are satisfied for all dual pairs. The proof of

Theorem 1.2 is similar to that of Theorem 1.1, and we provide a sketch in Section 5. In Section 6, we define a certain open subset of  $\tilde{G}$  on which Chc is an explicitly known smooth function. We prove that on this open subset Chc is compatible with the Capelli Harish-Chandra homomorphism (see Theorem 6.3). This implies that the compatibility is satisfied for any regular semisimple element belonging to a fundamental Cartan subgroup. Then, after recalling some classical result of Harish-Chandra (see Section 6.2), we prove Theorem 1.3 using the induction method.

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## 2 Explicit Formulas

Theorem 9 in [4] shows how to compute the Cauchy Harish-Chandra integral on every Cartan subgroup of  $\tilde{G}$ , assuming that we know how to do it for a compact Cartan subgroup. Thus, let  $H' \subseteq G'$  be a compact Cartan subgroup. As a generalized function, the unnormalized Cauchy Harish-Chandra integral satisfies the following identity:

$$\widetilde{\text{Chc}}(h'g) = \frac{1}{\Theta(\mathfrak{d})} \Theta(\mathfrak{d}h'g) \quad (h' \in \tilde{H}^{\text{reg}}, g \in \tilde{G}), \tag{8}$$

where  $\Theta$  is the character of the oscillator representation  $\omega$  and  $H^{\text{reg}} \subseteq H'$  stands for the subset of regular elements. In Theorem 2.2, we shall give an explicit formula for

$$\int_{\tilde{G}_1} \Theta(h'g) f(g) \, d\mu(g) \quad (h' \in \tilde{H}^{\text{reg}}, f \in \mathcal{D}(\tilde{G}_1)).$$

Since  $G'$  has a compact Cartan subgroup, there is a division algebra  $\mathbb{D} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , with an involution  $\iota$ , and a finite-dimensional space  $V'$  over  $\mathbb{D}$  with a nondegenerate form  $(\cdot, \cdot)'$  which is either  $\iota$ -hermitian or  $\iota$ -skew-hermitian, so that  $G'$  may be identified with the isometry group of that form. Let

$$V' = V'_0 \oplus V'_1 \oplus \cdots \oplus V'_n \tag{9}$$

be the decomposition into  $H'$ -irreducibles over  $\mathbb{D}$ . Here  $V'_0 = 0$  unless  $\mathbb{D} = \mathbb{R}$ , the form  $(\cdot, \cdot)'$  is symmetric and  $\dim(V')$  is odd. In that case  $h'$  acts trivially on  $V'_0$  and  $\dim(V'_0) = 1$ . There is an element  $J' \in \mathfrak{h}'$  such that  $J'^2 = -1$  on  $V'_1 \oplus \cdots \oplus V'_n$ . Let  $J'_j$  denote the restriction of  $J'$  to  $V'_j$ ,  $1 \leq j \leq n$ .



The group  $G$  coincides with the isometry group of a form  $(\cdot, \cdot)$  of opposite type to  $(\cdot, \cdot)'$  on a finite-dimensional vector space  $V$  over  $\mathbb{D}$ . The symplectic space  $W = \text{Hom}(V, V')$ , with the symplectic form given by the formula

$$\langle w', w \rangle = \text{tr}_{\mathbb{D}/\mathbb{R}}(w^* w') \quad (w, w' \in W),$$

where  $w^* \in \text{Hom}(V', V)$  is defined by

$$(wv, v')' = (v, w^* v') \quad (v \in V, v' \in V').$$

We have the obvious inclusions:  $G, G' \subseteq \text{Sp}(W)$  and  $\mathfrak{g}, \mathfrak{g}' \subseteq \mathfrak{sp}(W)$ .

If  $V'_0 \neq 0$ , we choose an element  $J \in \mathfrak{g}$  such that  $J^2 = -1$  in  $\text{End}(V)$  and the restriction of the symmetric form  $\langle J, \cdot \rangle$  to the subspace  $\text{Hom}(V, V'_0) \subseteq W$  is positive definite:  $\langle J, \cdot \rangle_{\text{Hom}(V, V'_0)} > 0$ . We shall view  $W$  as a complex vector space by

$$iw = \begin{cases} J'(w) & \text{if } w \in \text{Hom}(V, V'_1 + V'_2 + \dots + V'_n), \\ J(w) & \text{if } w \in \text{Hom}(V, V'_0). \end{cases} \tag{10}$$

Let  $\det : \text{End}_{\mathbb{C}}(W) \rightarrow \mathbb{C}$  denote the corresponding determinant and let

$$\widetilde{\text{GL}}_{\mathbb{C}}(W) = \{\tilde{g} = (g, \xi); \xi^2 = \det(g), g \in \text{GL}_{\mathbb{C}}(W)\}.$$

Note that this is a linear algebraic group. Define

$$\det^{1/2}(\tilde{g}) = \xi, \quad (\tilde{g} = (g, \xi) \in \widetilde{\text{GL}}_{\mathbb{C}}(W)).$$

Let  $H'_{\mathbb{C}}, G^i_{\mathbb{C}} \subseteq \text{GL}_{\mathbb{C}}(W)$  be the complexifications of  $H'$  and  $G^i$ , the centralizer of  $i$  in  $G$ , respectively. The character  $\Theta$  extends to a rational function

$$\Theta(\tilde{h}'\tilde{g}) = (-1)^{p-} \frac{\det^{1/2}(\tilde{h}'\tilde{g})}{\det(1 - h'g)} \quad (\tilde{h}' \in \tilde{H}'_{\mathbb{C}}, \tilde{g} \in \tilde{G}^i_{\mathbb{C}}), \tag{11}$$

where  $\tilde{H}'_{\mathbb{C}}, \tilde{G}^i_{\mathbb{C}} \subseteq \widetilde{\text{GL}}_{\mathbb{C}}(W)$  are the preimages of the complexifications of  $H', G^i$ , and  $2p-$  is the maximal dimension of real subspace of  $W$  on which the symmetric form  $\langle J', \cdot \rangle$ , (or equivalently  $\langle i, \cdot \rangle$ ), is negative definite. This follows from Theorem 2.13 and formula (10.10) in [18]. (Indeed, it suffices to consider the pair  $G = U_{p,q}, G' = U_1$ . This is because  $\text{Sp}(W)^i$ , the centralizer of  $i$  in  $\text{Sp}(W)$ , coincides with the subgroup preserving the

following nondegenerate hermitian form

$$\langle iw, w' \rangle + i \langle w, w' \rangle \quad (w, w' \in W)$$

and both  $H'$  and  $G^i$  are contained in  $\text{Sp}(W)^i$ . Then  $\dim_{\mathbb{R}}(W) = 2(p + q)$ . Let  $n = p + q$ . Theorem 2.13 in [18] shows that

$$\lim_{t \rightarrow 0} t^n \Theta(\tilde{c}_-(tx)) = 2^{-\dim_{\mathbb{R}}(W)} \widetilde{\text{chc}}(x) \quad (x \in \mathfrak{g}),$$

where  $\tilde{c}_-$  is the lift of the Cayley transform  $c_-(x) = (1 + x)(1 - x)^{-1}$  such that  $\tilde{c}_-(0)$  is the identity. On the other hand,

$$\lim_{t \rightarrow 0} t^n \frac{\det^{1/2}(\tilde{c}_-(tx))}{\det(1 - c_-(tx))} = \frac{1}{\det(-2)} \frac{1}{\det(x)}.$$

Since by Przebinda [18, (10.10)],

$$\widetilde{\text{chc}}(x) = (-1)^p 2^n \frac{1}{\det(x)},$$

we see that

$$\Theta(\tilde{g}) \frac{\det(1 - g)}{\det^{1/2}(\tilde{g})} = 2^{-\dim_{\mathbb{R}}(W)} \det(-2) (-1)^p 2^n = (-1)^{n+p} = (-1)^q,$$

and the claim follows.)

Let  $H \subseteq G$  be a fundamental Cartan subgroup and let

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_n \tag{12}$$

be the decomposition into  $H$ -irreducibles over  $\mathbb{D}$ . Here  $V_0 = 0$  unless  $\mathbb{D} = \mathbb{R}$ , the form  $(\ , \ )$  is symmetric and  $\dim(V)$  is odd. In that case  $\mathfrak{h}$  acts trivially on  $V_0$  and  $\dim(V_0) = 1$ . The group  $H$  is compact unless  $\mathbb{D} = \mathbb{R}$ , the form  $(\ , \ )$  is symmetric,  $\dim(V)$  is even but the Witt index of  $(\ , \ )$  is odd (i.e.,  $G$  is isomorphic to  $O_{2p+1, 2q+1}$ ). If  $H$  is compact, then there is an element  $J \in \mathfrak{h}$  (consistent with (10)) whose square equals minus identity on  $V_1 + V_2 + \cdots + V_n$ . Let  $J_j = J|_{V_j}$ ,  $1 \leq j \leq n$ . Suppose  $H$  is not compact. Then we may assume that  $H|_{V_j}$  is compact for each  $2 \leq j \leq n$  and that there is  $J_1 \in \text{End}(V_1)$  with  $J_1^2 = -1$ , such that  $\mathfrak{h}|_{V_1}$  is conjugate over  $\mathbb{C}$  to  $\mathbb{R}iJ_1$ . As before, let  $J = J_1 + J_2 + \cdots + J_n$ . In any case,  $J_1, J_2, \dots, J_n$  is a basis of the complex vector space  $\mathfrak{h}_{\mathbb{C}}$ . Let  $J_1^*, J_2^*, \dots, J_n^*$  be the dual basis, and let

$e_j = iJ_j^*$ ,  $1 \leq j \leq n$ . Also, we shall identify

$$V'_j = V_j, \quad J'_j = J_j \quad (1 \leq j \leq n'). \tag{13}$$

In particular,  $\mathfrak{h}'_{\mathbb{C}} \subseteq \mathfrak{h}_{\mathbb{C}}$ . Let

$$W^{\mathfrak{h}'} = \sum_{j=1}^{n'} \text{Hom}(V_j, V_j)^{J_j}.$$

This is a complex subspace of  $W$  consisting of all elements that commute with  $J$ .

Let  $H_{\mathbb{C}} \subseteq G_{\mathbb{C}}$  denote the complexification of  $H$  and let  $H_{\mathbb{C},1}$  denote the identity component of  $H_{\mathbb{C}}$ . Then  $H_{\mathbb{C},1}$  is isomorphic to

$$\mathfrak{h}_{\mathbb{C}} / \left\{ \sum_{j=1}^n 2\pi x_j J_j \mid x_j \in \mathbb{Z} \right\}.$$

Let  $\check{H}_{\mathbb{C},1}$  denote the connected two-fold cover of  $H_{\mathbb{C},1}$  isomorphic to

$$\check{H}_{\mathbb{C},1} / \left\{ \sum_{j=1}^n 2\pi x_j J_j \mid \sum_{j=1}^n x_j J_j \in 2\mathbb{Z}, \quad x_j \in \mathbb{Z}, \quad 1 \leq j \leq n \right\}.$$

Then we have the following covering maps:

$$\check{p}: \check{H}_{\mathbb{C},1} \rightarrow \check{H}_{\mathbb{C},1}, \quad \tilde{p}: \check{H}_{\mathbb{C},1} \rightarrow H_{\mathbb{C},1}, \quad p = \tilde{p} \circ \check{p}: \check{H}_{\mathbb{C},1} \rightarrow H_{\mathbb{C},1}. \tag{14}$$

Here,  $\check{p}$  is either an isomorphism or  $\check{H}_{\mathbb{C},1}$  coincides with the direct product  $H_{\mathbb{C},1} \times \{\pm 1\}$  and  $\check{p} = p$ . Similarly, we have

$$\check{p}: \check{H}'_{\mathbb{C},1} \rightarrow \check{H}'_{\mathbb{C},1}, \quad \tilde{p}: \check{H}'_{\mathbb{C},1} \rightarrow H'_{\mathbb{C},1}, \quad p = \tilde{p} \circ \check{p}: \check{H}'_{\mathbb{C},1} \rightarrow H'_{\mathbb{C},1}. \tag{15}$$

Fix a system  $\Psi'$  of positive roots of  $\mathfrak{h}'_{\mathbb{C}}$  in  $\mathfrak{g}'_{\mathbb{C}}$  and a system  $\Psi$  of positive roots of  $\mathfrak{h}_{\mathbb{C}}$  in  $\mathfrak{g}_{\mathbb{C}}$ . Let  $\Phi = -\Psi$ . Let  $Z$  denote the centralizer of  $\mathfrak{h}'_{\mathbb{C}}$  in  $G$  and let  $W(H_{\mathbb{C}}, Z_{\mathbb{C}})$  be the corresponding Weyl group. Let  $\Phi(Z) = \Phi \cap \mathfrak{h}'^{\perp}$ . This is a system of positive roots of  $\mathfrak{h}_{\mathbb{C}}$  in the Lie algebra of  $Z_{\mathbb{C}}$ . For  $h' \in \check{H}'_{\mathbb{C},1}$  and  $h \in \check{H}_{\mathbb{C},1}$  define

$$\Delta_{\Psi'}(h') = h'^{\frac{1}{2} \sum_{\alpha \in \Psi'} \alpha} \prod_{\alpha \in \Psi'} (1 - h'^{-\alpha}),$$

$$\Delta_{\Phi}(h) = h^{\frac{1}{2} \sum_{\alpha \in \Phi} \alpha} \prod_{\alpha \in \Phi} (1 - h^{-\alpha}),$$

$$\Delta_{\Phi(Z)}(h) = h^{\frac{1}{2} \sum_{\alpha \in \Phi(Z)} \alpha} \prod_{\alpha \in \Phi(Z)} (1 - h^{-\alpha}), \tag{16}$$

$$\det^{1/2}(h)_{W^{b'}} = h^{\frac{1}{2} \sum_{j=1}^{n'} e_j}.$$

The choice of the covering is such that the above definitions make sense. Furthermore,

$$(\det^{1/2}(h)_{W^{b'}})^2 = \det(p(h))_{W^{b'}},$$

where  $\det(p(h))_{W^{b'}}$  is the usual determinant of  $p(h) \in \text{End}_{\mathbb{C}}(W^{b'})$ .

Define a number  $k = -1, 0$  or  $1$  as follows:

$$k = \begin{cases} -1 & \text{if } (G_{\mathbb{C}}, G'_{\mathbb{C}}) = (GL_n(\mathbb{C}), GL_{n'}(\mathbb{C})) \text{ and } n - n' \in 2\mathbb{Z}, \\ & \text{or } (G_{\mathbb{C}}, G'_{\mathbb{C}}) = (O_{2n+1}(\mathbb{C}), Sp_{2n'}(\mathbb{C})), \\ 1 & \text{if } (G_{\mathbb{C}}, G'_{\mathbb{C}}) = (Sp_{2n}(\mathbb{C}), O_{2n'+1}(\mathbb{C})), \\ 0 & \text{otherwise.} \end{cases}$$

Define a character sign of the Weyl group  $W(H_{\mathbb{C}}, G_{\mathbb{C}}) = W(H_{\mathbb{C}})$  by

$$\Delta_{\Phi}(s.h) \frac{\det^{1/2}(s.h)_{\text{Hom}(V, V_0)}}{\det(1 - p(s.h))_{\text{Hom}(V, V_0)}} = \text{sign}(s) \Delta_{\Phi}(h) \frac{\det^{1/2}(h)_{\text{Hom}(V, V_0)}}{\det(1 - p(h))_{\text{Hom}(V, V_0)}} \tag{17}$$

$(s \in W(H_{\mathbb{C}}), h \in \check{H}_{\mathbb{C},1}).$

The group  $W(H_{\mathbb{C}})$  acts on the real span of the  $J_1, \dots, J_n$  (which is the same as  $\mathfrak{h}$  if  $H$  is compact) and is realized as conjugations by elements of  $GL_{\mathbb{C}}(W)$  as in [3, Section 3].

**Proposition 2.1.** There is a constant  $v = \pm 1$ , which depends only on the choice of the positive root systems  $\Psi', \Phi$ , such that for  $h' \in \check{H}'_{\mathbb{C},1}$  and  $h \in \check{H}_{\mathbb{C},1}$ ,

$$\begin{aligned} & \det^{k/2}(h')_{W^{b'}} \Delta_{\Psi'}(h') \Theta(\check{p}(h') \check{p}(h)) \Delta_{\Phi}(h) \\ &= \sum_{s \in W(H_{\mathbb{C}})} (-1)^{p-} \frac{v \text{sign}(s)}{|W(H_{\mathbb{C}}, Z_{\mathbb{C}})|} \frac{\det^{k/2}(s^{-1}.h)_{W^{b'}} \Delta_{\Phi(Z)}(s^{-1}.h)}{\det(1 - p(h') p(s^{-1}.h))_{W^{b'}}} \frac{\det^{1/2}(s^{-1}.h)_{W_0}}{\det(1 - p(s^{-1}.h))_{W_0}}, \end{aligned}$$

where  $W_0 = \text{Hom}(\sum_{j=n'+1}^n V_j, V_0)$ . □

This is verified by the argument used in [3, Appendix B] to prove the corresponding statement for the Lie algebra. The factor  $\det^{k/2}$  is necessary for the partial fraction

decomposition to work. Unfortunately, it was overlooked in [18, (14.5)]. Let

$$U = \begin{cases} V_{\mathbb{C}} & \text{if } \mathbb{D} \neq \mathbb{C}, \\ V & \text{if } \mathbb{D} = \mathbb{C}. \end{cases}$$

Let  $\Psi_{st}^n$  denote the family of strongly orthogonal noncompact imaginary roots in  $\Psi$ , as in [3, (1.1)–(1.2)]. For each  $S \in \Psi_{st}^n$ , there is an element  $C(S) \in GL_{\mathbb{C}}(U)$  such that the map

$$\text{End}(U) \ni x \rightarrow c(S)(x) = C(S)xC(S)^{-1} \in \text{End}(U),$$

when restricted to  $\mathfrak{g}_{\mathbb{C}}$ , coincides with the Cayley transform. Let  $H(S) \subseteq G$  be the corresponding Cartan subgroup and let

$$H_S = c(S)^{-1}(H(S)) \subseteq H_{\mathbb{C}}.$$

The map  $c(S)$  lifts to the covering, and we shall use the same symbol  $c(S)$  to denote these lifts. Let  $H_{S,1} = H_S \cap H_{\mathbb{C},1}$  and let  $\check{H}_{S,1} \subseteq \check{H}_{\mathbb{C},1}$ ,  $\check{H}_{S,1} \subseteq \check{H}_{\mathbb{C},1}$  be the corresponding preimages under the covering maps  $\check{\rho}$ ,  $\check{\rho}$ . Let  $\Psi_{S,\mathbb{R}} \subseteq \Psi$  denote the set of the real roots for  $H_S$ . Define

$$\epsilon_{\Psi_{S,\mathbb{R}}}(h) = \text{sign} \left( \prod_{\alpha \in \Psi_{S,\mathbb{R}}} (1 - h^{-\alpha}) \right) \quad (h \in \check{H}_{S,1}^{\text{reg}}),$$

where  $\check{H}_{S,1}^{\text{reg}}$  stands for the set of regular elements in  $\check{H}_{S,1}$ . Recall the Harish-Chandra orbital integral of a function  $f \in \mathcal{D}(\check{G}_1)$ :

$$\mathcal{H}_S f(h) = \epsilon_{\Psi_{S,\mathbb{R}}}(h) \Delta_{\psi}(h) \int_{G/H(S)} f(g.c(S)(\check{\rho}(h))) d\mu(gH(S)) \quad (h \in \check{H}_{S,1}^{\text{reg}}).$$

Note that the function  $\Delta_{\phi}(h)\mathcal{H}_S f(h)$  is constant on the fibers of the covering map  $\check{\rho}$ . Hence, the Weyl integration formula for  $\check{G}_1$  looks as follows:

$$\int_{\check{G}_1} f(g) d\mu(g) = \sum_{S \in \Psi_{st}^n} m_S \int_{\check{H}_S} \epsilon_{\Psi_{S,\mathbb{R}}}(h) \Delta_{\phi}(h) \mathcal{H}_S f(h) d\mu(h),$$

where the  $m_S$  are appropriate, uniquely determined, constants.

For a subset  $A \subseteq \Psi$ , let  $\underline{A} = \{j \mid \text{there is } \alpha \in A \text{ such that } \alpha(J_j) \neq 0\}$ . For  $s \in W(H_{\mathbb{C}})$  and  $S \in \Psi_{st}^n$  define

$$\Gamma_{s,S} = \{Y \in \mathfrak{h} \mid \langle Y, \cdot \rangle_{sW^{b'} \cap \sum_{j \in \underline{S}} \text{Hom}(V_j, V')} > 0\},$$

as in [3, Lemma 7.1], and let  $E_{s,S} = \exp(\Gamma_{s,S}) \subseteq \tilde{H}_{\mathbb{C},1}$ . Furthermore, let

$$M_S(s) = (-1)^{p-} \frac{vm_S \operatorname{sign}(s)}{|W(H_{\mathbb{C}}, Z_{\mathbb{C}})|}$$

**Theorem 2.2.** Let  $h' \in \check{H}'_1 (= \check{H}'_{\emptyset,1})$  and let  $f \in \mathcal{D}(\tilde{G}_1)$ . Then

$$\begin{aligned} & \det^{k/2}(h')_{W_{\mathfrak{h}'}} \Delta_{\psi'}(h') \int_{\tilde{G}_1} \Theta(\check{p}(h')g) f(g) \, d\mu(g) \\ &= \sum_{s \in W(H_{\mathbb{C}})} \sum_{S \in \Psi_{st}^n} M_S(s) \lim_{r \in E_{s,S}, r \rightarrow 1} \\ & \int_{\check{H}_{S,1}^{\text{reg}}} \frac{\det^{k/2}(s^{-1} \cdot h)_{W_{\mathfrak{h}'}} \Delta_{\Phi(Z)}(s^{-1} \cdot h)}{\det(1 - p(h)rp(h))_{SW_{\mathfrak{h}'}}} \frac{\det^{1/2}(s^{-1} \cdot h)_{W_0}}{\det(1 - p(s^{-1} \cdot h))_{W_0}} \epsilon_{\Phi_{S,\mathbb{R}}}(h) \mathcal{H}_S f(h) \, d\mu(h). \end{aligned}$$

Here the factor  $\frac{\Delta_{\Phi(Z)}(s^{-1} \cdot h)}{\det(1 - p(s^{-1} \cdot h))_{W_0}}$  is a smooth function. □

This follows from Proposition 2.1 by localization, as in the proof of [3, Theorem 7.3]. Note that if  $G'$  is an odd real orthogonal group (i.e.,  $\mathbb{D} = \mathbb{R}$ , the form  $(\ , \ )'$  is symmetric and  $\dim(V')$  is odd), then  $\mathfrak{d} \notin \check{H}'_{\emptyset,1}$ . However, the element  $\mathfrak{d}$  is in the center of the metaplectic group, and therefore, we may view it as an element of  $\tilde{G}_1$ .

### 3 A Relation with Stable Orbital Integrals

The theory of Stable Orbital Integrals or Stable Invariant Eigendistributions leads to certain identities among some averages of irreducible characters, see [1] and Renard [20]. Our approach attempts to produce some irreducible characters. In this section, we check that our construction leads to the same identities and thus provide some evidence for Conjecture 1.5.

Let us define the following two distributions on the unit circle in the complex plane:

$$\frac{1}{1 - 1_{+1}z} = \lim_{r>1, r \rightarrow 1} \frac{1}{1 - rz}, \quad \frac{1}{1 - 1_{-1}z} = \lim_{r<1, r \rightarrow 1} \frac{1}{1 - rz} \quad (z \in \mathbb{C}, |z| = 1).$$

Then, as is well known [14, Example 3.1.13],

$$\frac{1}{2\pi} \left( \frac{1}{1 - 1_{-1}z} - \frac{1}{1 - 1_{+1}z} \right) = \delta(z), \tag{18}$$

where  $\delta$  stands for the Dirac delta at the identity.

If  $\mathbb{D} = \mathbb{C}$ , then the Weyl group  $W(H_{\mathbb{C}}, G_{\mathbb{C}})$  may be identified with the permutation group  $\Sigma_n$  so that

$$\sigma.J_j = J_{\sigma(j)}, \quad (\sigma \in W(H_{\mathbb{C}}, G_{\mathbb{C}}), 1 \leq j \leq n).$$

If  $\mathbb{D} \neq \mathbb{C}$ , then  $W(H_{\mathbb{C}}, G_{\mathbb{C}})$ , is identified with the semidirect product of  $\Sigma_n$  and  $(\mathbb{Z}/2\mathbb{Z})^n$ , where  $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$  with the addition modulo 2, so that

$$s.J_j = \hat{\epsilon}_j J_{\sigma(j)}, \quad (s = \sigma \epsilon \in W(H_{\mathbb{C}}, G_{\mathbb{C}}), \hat{\epsilon}_j = (-1)^{\epsilon_j}, 1 \leq j \leq n).$$

In any case ( $\mathbb{D} = \mathbb{C}$  or  $\mathbb{D} \neq \mathbb{C}$ ) we shall think of elements  $s \in W(H_{\mathbb{C}}, G_{\mathbb{C}})$  as  $s = \sigma \epsilon$ , where  $\epsilon = 0$  if  $\mathbb{D} = \mathbb{C}$ .

Let  $\text{Hom}(V_l, V_j)^0 = \text{Hom}(V_l, V_j)^J$  be the subspace of elements that commute with  $J$  and let  $\text{Hom}(V_l, V_j)^1$  be the subspace of elements that anti-commute with  $J$ . (The last space is zero if  $\mathbb{D} = \mathbb{C}$ .) Thus, for  $\epsilon_j = 0$  or  $1$ , we have the subspace  $\text{Hom}(V_{\sigma(j)}, V_j)^{\epsilon_j} \subseteq \text{Hom}(V_{\sigma(j)}, V_j)$ . Let

$$Y_s = \sum_{j=1}^n \text{sign}\langle J, \cdot \rangle_{\text{Hom}(V_{\sigma(j)}, V_j)^{\epsilon_j}} J_{\sigma(j)} \in \mathfrak{h} \quad (s = \sigma \epsilon \in W(H_{\mathbb{C}}, G_{\mathbb{C}})). \tag{19}$$

Here, by definition, the  $\text{sign}\langle J, \cdot \rangle_{\text{Hom}(V_{\sigma(j)}, V_j)^{\epsilon_j}}$  is equal 1 if this form is positive definite and  $-1$  if it is negative definite.

Let  $H' \subseteq G'$  be the compact Cartan subgroup considered in section 2. For  $h \in H$  and  $1 \leq j \leq n$  let  $h_j = h^{\epsilon_j} \in \mathbb{C}$ , and similarly for  $H'$ . Then, in terms of Theorem 2.2,

$$\lim_{r \in E_{s, \theta}, r \rightarrow 1} \frac{1}{\det(1 - h'rh)_{sW^{h'}}} = \prod_{j=1}^{n'} \frac{1}{1 - h'_j (1_{J_{\sigma(j)}^*(Y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \quad (h' \in H'^{\text{reg}}, h \in H).$$

We note that (18) implies the following identity:

$$\begin{aligned} & \sum_{\lambda \in \{\pm 1\}^{n'}} \prod_{j=1}^{n'} \frac{\lambda_j}{1 - h'_j (1_{\lambda_j J_{\sigma(j)}^*(Y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \\ &= \prod_{j=1}^{n'} (-\hat{\epsilon}_j) \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(Y_s) \cdot \prod_{j=1}^{n'} \left( \frac{1}{1 - h'_j 1_{+1} h_{\sigma(j)}^{-\hat{\epsilon}_j}} - \frac{1}{1 - h'_j 1_{-1} h_{\sigma(j)}^{-\hat{\epsilon}_j}} \right) \\ &= (2\pi)^{n'} \cdot \prod_{j=1}^{n'} \hat{\epsilon}_j \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(Y_s) \cdot \delta(h'(s^{-1}.h)^{-1}), \end{aligned} \tag{20}$$

where  $\delta$  stands for the Dirac delta at the identity of  $H'$ . Furthermore,

$$\lim_{r \in E_{s,S}, r \rightarrow 1} \frac{1}{\det(1 - h'rh)_{sW^{b'}}} = \prod_{j=1, j \notin \underline{S}}^{n'} \frac{1}{1 - h'_j(1_{J_{\sigma(j)}^*(Y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \prod_{j=1, j \in \underline{S}}^{n'} \frac{1}{1 - h'_j h_{\sigma(j)}^{-\hat{\epsilon}_j}}, \tag{21}$$

and if  $S \neq \emptyset$ , then

$$\sum_{\lambda \in \{\pm 1\}^{n'}} \prod_{j=1, j \notin \underline{S}}^{n'} \frac{\lambda_j}{1 - h'_j(1_{J_{\sigma(j)}^*(Y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \prod_{j=1, j \in \underline{S}}^{n'} \frac{\lambda_j}{1 - h'_j h_{\sigma(j)}^{-\hat{\epsilon}_j}} = 0. \tag{22}$$

The generalized function  $\Theta(h'g)$ , (8), and the sets  $E_{s,S}$  of Theorem 2.2 depend on the form  $(, )'$  (and the form  $(, )$ ). Let  $\Theta^\lambda(h'g)$  and  $E_{s,S}^\lambda$  denote the corresponding objects for the form

$$\lambda_1(, )'|_{V_1} + \lambda_2(, )'|_{V_2} + \dots + \lambda_{n'}(, )'|_{V_{n'}}, \tag{23}$$

where  $\lambda = (\lambda_1, \dots, \lambda_{n'}) \in \{\pm 1\}^{n'}$ .

Assume from now on that  $n' = n$ . Then  $H' = H$  via identification (13) and  $\check{H}'_1 = \check{H}'_1$ . Formulas (20) and (22) imply that, in terms of Theorem 2.2,

$$\begin{aligned} & \sum_{\lambda \in \{\pm 1\}^{n'}} \lambda_1 \lambda_2 \dots \lambda_n \det^{k/2}(h')_{W^{b'}} \Delta_{\Psi'}(h') \int_{\check{G}_1} \Theta^\lambda(\check{p}(h')g) f(g) d\mu(g) \\ &= \sum_{s \in W(H_{\mathbb{C}}, G_{\mathbb{C}})} M_{\emptyset}(s) \sum_{\lambda \in \{\pm 1\}^{n'}} \lambda_1 \lambda_2 \dots \lambda_n, \end{aligned} \tag{24}$$

$$\begin{aligned} & \lim_{r \in E_{s,\emptyset}^\lambda, r \rightarrow 1} \int_{\check{H}_1} \frac{1}{\det(1 - p(h')rp(h))_{sW^{b'}}} \det^{k/2}(s^{-1} \cdot h)_{W^{b'}} \mathcal{H}_{\emptyset} f(h) d\mu(h) \\ &= \sum_{s \in W(H_{\mathbb{C}}, G_{\mathbb{C}})} M_{\emptyset}(s) (2\pi)^{n'} \cdot \prod_{j=1}^{n'} \hat{\epsilon}_j \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(Y_s), \end{aligned} \tag{25}$$

$$\begin{aligned} & \int_{\check{H}_1} \delta(p(h'(s^{-1} \cdot h)^{-1})) \det^{k/2}(s^{-1} \cdot h)_{W^{b'}} \mathcal{H}_{\emptyset} f(h) d\mu(h) \\ &= \det^{k/2}(h')_{W^{b'}} \mu(H') (-1)^{p-n+n'} \frac{v}{|W(H, G)|} \sum_{s \in W(H_{\mathbb{C}}, G_{\mathbb{C}})} \text{sign}(s) \prod_{j=1}^n \hat{\epsilon}_j \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(Y_s) \cdot \mathcal{H}_{\emptyset} f(s \cdot h'). \end{aligned}$$

Suppose, from now on, that the form  $(, )'$  is hermitian. Then the element  $y_{\sigma\epsilon}$  (19) does not depend on  $\epsilon$ . Moreover,

$$\prod_{j=1}^{n'} J_{\sigma(j)}^*(Y_s) = (-1)^{-n+p-(W^{b'})},$$



where  $2p_-(W^{b'})$  is the maximal dimension of a subspace of  $W^{b'}$  on which the symmetric form  $\langle J', \cdot \rangle$  is negative definite.

Under our assumptions, we have the identification  $W(H'_\mathbb{C}, G'_\mathbb{C}) = W(H_\mathbb{C}, G_\mathbb{C})$ . Let us denote both groups by  $W(H_\mathbb{C})$ . Furthermore, for  $s = \sigma \epsilon \in W(H_\mathbb{C})$ ,

$$\text{sign}(s) \prod_{j=1}^n \hat{\epsilon}_j = \text{sign}'(s),$$

where the  $\text{sign}'$  is defined by  $\Delta_{\psi'}(s.h') = \text{sign}'(s) \Delta_{\psi'}(h')$ . For  $\lambda \in \{\pm 1\}^{n'}$ , define

$$q(\lambda) = (-1)^{p_- + p_-(W^{b'})} \frac{(-1)^{n'v}}{|W(H_\mathbb{C})|} \lambda_1 \lambda_2 \cdots \lambda_{n'}.$$

Then (24) shows that

$$\begin{aligned} & \sum_{\lambda \in \{\pm 1\}^{n'}} q(\lambda) \frac{1}{\mu(H')} \int_{\tilde{G}} \Theta^\lambda(\check{p}(h)g) f(g) \, d\mu(g) \\ &= \frac{1}{|W(H, G)||W(H_\mathbb{C})|} \sum_{s \in W(H_\mathbb{C})} \frac{1}{\Delta_{\psi'}(s.h')} \mathcal{H}_\emptyset f(s.h'). \end{aligned} \tag{26}$$

This function is clearly  $W(H_\mathbb{C})$ -invariant.

Let  $\gamma \in \mathfrak{h}^* = \mathfrak{h}'^*$  be a regular element. For  $h \in \check{H}^{\text{reg}}$  define

$$F(h) = \frac{\sum_{s \in W(H_\mathbb{C})} \text{sign}'(s) h^{s\gamma}}{\Delta_{-\psi'}(h)}. \tag{27}$$

Then (26) shows that

$$\begin{aligned} & \int_{\check{H}^{\text{reg}}} |\Delta_{\psi'}(h)|^2 F(h) \sum_{\lambda \in \{\pm 1\}^n} q(\lambda) \frac{1}{\mu(H')} \int_{\tilde{G}} \Theta^\lambda(\check{p}(h)g) f(g) \, d\mu(g) \, d\mu(h) \\ &= \frac{1}{|W(H, G)|} \int_{\check{H}^{\text{reg}}} |\Delta_\psi(h)|^2 \frac{1}{|W(H_\mathbb{C})|} \sum_{s \in W(H_\mathbb{C})} \left( F(s.h) \frac{\Delta_{-\psi'}(s.h)}{\Delta_{-\psi}(s.h)} \right) \\ & \quad \times \int_{G/H} f(g.\check{p}(s.h)) \, d\mu(gH) \, d\mu(h). \end{aligned} \tag{28}$$

The group  $\Sigma_n$  acts on the set  $\{\pm 1\}^n$  so that

$$\Theta^{\sigma\lambda}(hg) = \Theta^\lambda((\sigma^{-1}.h)g) \quad (\sigma \in \Sigma_n, \lambda \in \{\pm 1\}^n, h \in \check{H}^{\text{reg}}, g \in \tilde{G}).$$

Hence, the left-hand side of (28) may be rewritten as

$$\begin{aligned}
 & \sum_{\lambda \in \{\pm 1\}^n} q(\lambda) \int_{\check{H}^{\text{reg}}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(H')} \int_{\check{G}} \Theta^\lambda(\check{p}(h)g) f(g) \, d\mu(g) \, d\mu(h) \\
 &= \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} \sum_{[\sigma] \in \Sigma_n / \text{Stab}_{\Sigma_n}(\lambda)} q(\sigma\lambda) \int_{\check{H}^{\text{reg}}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(H')} \int_{\check{G}} \Theta^{\sigma\lambda}(\check{p}(h)g) f(g) \, d\mu(g) \, d\mu(h) \\
 &= \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} \sum_{[\sigma] \in \Sigma_n / \text{Stab}_{\Sigma_n}(\lambda)} q(\lambda) \int_{\check{H}^{\text{reg}}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(H')} \int_{\check{G}} \Theta^\lambda(\check{p}(h)g) f(g) \, d\mu(g) \, d\mu(h).
 \end{aligned} \tag{29}$$

Let  $G'_\lambda$  be the isometry group of the form (23). Then

$$|\Sigma_n / \text{Stab}_{\Sigma_n}(\lambda)| = \frac{|W(H_{\mathbb{C}})|}{|W(H', G'_\lambda)|}.$$

Hence, (29) is equal to

$$\sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} q(\lambda) \frac{|W(H_{\mathbb{C}})|}{|W(H', G'_\lambda)|} \int_{\check{H}^{\text{reg}}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(H')} \int_{\check{G}} \Theta^\lambda(\check{p}(h)g) f(g) \, d\mu(g) \, d\mu(h). \tag{30}$$

We know from Theorem 1.2 that there are functions  $f_\lambda \in \mathcal{D}(\check{G}_\lambda)$  such that

$$\frac{1}{\mu(H')} \int_{\check{G}} \Theta^\lambda(\check{p}(h)g) f(g) \, d\mu(g) = \int_{\check{G}_\lambda / \check{H}'} f_\lambda(g \cdot \check{p}(h)) \, d\mu(gH'). \tag{31}$$

Furthermore,

$$\begin{aligned}
 F(h) &= \sum_{[s] \in W(H', G'_\lambda) \setminus W(H_{\mathbb{C}})} F_{\lambda, [s]}(h), \quad \text{where} \\
 F_{\lambda, [s]}(h) &= \frac{\sum_{s' \in W(H', G'_\lambda)} \text{sign}'(s's) h^{s's\gamma}}{\Delta_{-\Psi'}(h)}.
 \end{aligned} \tag{32}$$

Hence, (30) is equal to

$$\begin{aligned}
 & \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} q(\lambda) |W(H_{\mathbb{C}})| \sum_{[s] \in W(H', G'_\lambda) \setminus W(H_{\mathbb{C}})} \frac{1}{|W(H', G'_\lambda)|} \\
 & \times \int_{\check{H}^{\text{reg}}} |\Delta_{\Psi'}(h)|^2 F_{\lambda, [s]}(h) \int_{\check{G}_\lambda / \check{H}'} f_\lambda(g \cdot \check{p}(h)) \, d\mu(gH').
 \end{aligned} \tag{33}$$

We may assume that the forms  $(\cdot, \cdot)'$  and  $\langle J, \cdot \rangle$  are positive definite. Then  $(-1)^{p_- + p_-(W^{b'})} = 1$ . Thus, we see from (33) that the equality (28) may be rewritten as

$$\begin{aligned} & \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} (-1)^{n'} v_{\lambda_1 \lambda_2 \cdots \lambda_n} \sum_{[s] \in W(H', G'_\lambda) \setminus W(H_C)} \frac{1}{|W(H', G'_\lambda)|} \\ & \quad \times \int_{\check{H}^{reg}} |\Delta_{\psi'}(h)|^2 F_{\lambda, [s]}(h) \int_{\check{G}_\lambda / \check{H}'} f_\lambda(g \cdot \check{p}(h)) \, d\mu(gH') \\ & = \frac{1}{|W(H, G)|} \int_{H^{reg}} |\Delta_\psi(h)|^2 \frac{1}{|W(H_C)|} \\ & \quad \times \sum_{s \in W(H_C)} \left( F(s \cdot h) \frac{\Delta_{-\psi'}(s \cdot h)}{\Delta_{-\psi}(s \cdot h)} \right) \int_{G/H} f(g \cdot \check{p}(s \cdot h)) \, d\mu(gH) \, d\mu(h). \end{aligned} \tag{34}$$

If  $\mathbb{D} = \mathbb{R}$  and  $\dim(V) = 2n + 1$ , then

$$\lambda_1 \lambda_2 \cdots \lambda_n = (-1)^{\frac{1}{2} \dim(G'_\lambda / K'_\lambda)},$$

where  $K'_\lambda \subseteq G'_\lambda$  is a maximal compact subgroup. In this case (34) simplifies further to

$$\begin{aligned} & (-1)^{n'} v (-1)^{\frac{1}{2} \dim(G'_\lambda / K'_\lambda)} \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} \sum_{[s] \in W(H', G'_\lambda) \setminus W(H_C)} \frac{1}{|W(H', G'_\lambda)|} \\ & \quad \times \int_{\check{H}^{reg}} |\Delta_{\psi'}(h)|^2 F_{\lambda, [s]}(h) \int_{\check{G}_\lambda / \check{H}'} f_\lambda(g \cdot \check{p}(h)) \, d\mu(gH') \\ & = \frac{1}{|W(H, G)|} \int_{\check{H}^{reg}} |\Delta_\psi(h)|^2 \left( F(h) \frac{\Delta_{-\psi'}(h)}{\Delta_{-\psi}(h)} \right) \sum_{s \in W(H_C)} \int_{G/H} f(g \cdot \check{p}(s \cdot h)) \, d\mu(gH) \, d\mu(h). \end{aligned} \tag{35}$$

Thus, in this case, the weighted sum of the Cauchy Harish-Chandra integrals (26) coincides with the transfer map studied by Adams [1, Definition 4.5] and Renard [20]. We formulate this conclusion in the following proposition.

**Proposition 3.1.** Suppose  $\mathbb{D} = \mathbb{R}$ ,  $\dim(V) = 2n' + 1$ , the forms  $(\cdot, \cdot)'$  and  $\langle J, \cdot \rangle$  are positive definite. For  $\lambda \in \{\pm 1\}^{n'}$  let  $(\cdot, \cdot)'_\lambda$  denote the symmetric form on  $V$  defined by the condition (23) and let  $G'_\lambda$  be the corresponding isometry group. Assume  $n' = n$ , so that  $(G, G'_\lambda)$  is an orthosymplectic dual pair consisting of groups of equal rank. Let us identify  $H' = H$  as

in (13). Let  $F$  denote the function (27). Then for any test function  $f \in \mathcal{D}(\tilde{G})$ ,

$$\begin{aligned} & \int_{\check{H}^{\text{reg}}} |\Delta_{\psi'}(h)|^2 F(h) \frac{v'}{|W(H_C)|} \sum_{\lambda \in \{\pm 1\}^{r'}} \frac{(-1)^{\frac{1}{2} \dim(G'_\lambda/K'_\lambda)}}{\mu(H')} \int_{\tilde{G}} \Theta^\lambda(\check{p}(h')g) f(g) \, d\mu(g) \, d\mu(h) \\ &= \frac{1}{|W(H, G)|} \int_{\check{H}^{\text{reg}}} |\Delta_{\psi}(h)|^2 \left( F(h) \frac{\Delta_{-\psi'}(h)}{\Delta_{-\psi}(h)} \right) \sum_{s \in W(H_C)} \int_{G/H} f(g.\check{p}(s.h)) \, d\mu(gH) \, d\mu(h), \end{aligned}$$

where  $v' = (-1)^{r'} v = \pm 1$  depends only on the choice of the positive root systems. □

### 4 Proof of Theorem 1.1

#### 4.1 Properties of Cauchy Harish-Chandra integral on the Lie algebra

Let  $\mathfrak{h}' \subseteq \mathfrak{g}'$  be a  $\theta$ -stable Cartan subalgebra. Denote by  $\Delta_I(\mathfrak{g}'_C, \mathfrak{h}'_C)$  the set of imaginary roots of  $\mathfrak{h}'$  in  $\mathfrak{g}'_C$ . Fix a positive system  $\Psi \subseteq \Delta_I(\mathfrak{g}'_C, \mathfrak{h}'_C)$ . For  $\psi \in \mathcal{S}(\mathfrak{g})$  and for  $x' \in \mathfrak{h}'^{\text{reg}}$ , define

$$\text{chc}_{W, \psi, \mathfrak{h}'}(\psi)(x') = \prod_{\alpha \in \Psi} \frac{\alpha(x')}{|\alpha(x')|} |\det(\text{ad}(x'))_{\mathfrak{g}'/\mathfrak{h}'}|^{1/2} \int_{\mathfrak{g}} \text{chc}_W(x' + x)\psi(x) \, d\mu(x). \tag{36}$$

Let  $\mathfrak{h}'_{I\text{-reg}} = \{x' \in \mathfrak{h}' \mid \alpha(x') \neq 0 \text{ for all } \alpha \in \Psi\}$ .

The explicit formulas for chc in [4, Corollaries 4 and 8] together with [3, Theorem 1] show that that the function  $\text{chc}_{W, \psi, \mathfrak{h}'}(\psi)$  is smooth on  $\mathfrak{h}'^{\text{reg}}$  and for any  $w \in \text{Sym}(\mathfrak{h}'_C)$  the derivative  $\partial(w) \text{chc}_{W, \psi, \mathfrak{h}'}(\psi)$  is locally bounded. The remaining property to be proved is the jump relation (see (37)).

Fix a single noncompact imaginary root  $\alpha \in \Psi$ . Let  $x' \in \mathfrak{h}'$  be such that  $\alpha(x') = 0$  and the derived Lie algebra  $[\mathfrak{g}'^{x'}, \mathfrak{g}'^{x'}]$  is isomorphic to  $\mathfrak{so}_2(\mathbb{R})$ . This means that  $x'$  is a subregular element attached to the noncompact imaginary root  $\alpha$ . Let  $H_\alpha \in i\mathfrak{h}'$ ,  $X_{\pm\alpha} \in \mathfrak{g}'_{C, \pm\alpha}$  be such that

$$[X_\alpha, X_{-\alpha}] = H_\alpha, \quad [H_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}, \quad \bar{X}_\alpha = X_{-\alpha},$$

where  $X \rightarrow \bar{X}$  stands for the conjugation with respect to the real form  $\mathfrak{g}' \subseteq \mathfrak{g}'_C$ . In particular,  $(X_\alpha, H_\alpha, X_{-\alpha})$  is a  $\mathfrak{so}_2$ -triple. Then, we have the decomposition

$$\mathfrak{h}' = \mathbb{R}iH_\alpha \oplus \ker(\alpha).$$

Let

$$c(\alpha) = \exp\left(-i\frac{\pi}{4} \text{ad}(X_\alpha + X_{-\alpha})\right) \in \text{End}(\mathfrak{g}'_C).$$

Then

$$c(\alpha)(\mathfrak{h}'_{\mathbb{C}}) \cap \mathfrak{g}' = \mathbb{R}s_{\alpha}H_{\alpha} \oplus \ker \alpha$$

is another Cartan subalgebra of  $\mathfrak{g}'$  denoted by  $\mathfrak{h}'_{\alpha}$ . Let  $\Psi \cap \alpha^{\perp} = \{\beta \in \Psi \mid \beta(H_{\alpha}) = 0\}$ . Then

$$\Psi \cap \alpha^{\perp} \circ c(\alpha)^{-1}$$

is a system of positive imaginary roots of  $\mathfrak{h}'_{\alpha}$  in  $\mathfrak{g}'_{\mathbb{C}}$  denoted  $\Psi_{\alpha}$ . Set

$$d(\alpha) = \begin{cases} 2 & \text{if the reflection with respect to } \alpha \text{ is realized by an element of } \tilde{G}^x, \\ 1 & \text{otherwise.} \end{cases}$$

Let  $w \in \text{Sym}(\mathfrak{h}'_{\mathbb{C}})$ . Put

$$\langle \partial(w) \text{chc}_{W, \psi, \mathfrak{h}'(\psi)} \rangle(x') = \lim_{t \rightarrow 0_+} \partial(w) \text{chc}_{W, \psi, \mathfrak{h}'(\psi)}(x' + tiH_{\alpha}) - \lim_{t \rightarrow 0_+} \partial(w) \text{chc}_{W, \psi, \mathfrak{h}'(\psi)}(x' - tiH_{\alpha}).$$

We need to prove the following jump relation

$$\langle \partial(w) \text{chc}_{W, \psi, \mathfrak{h}'(\psi)} \rangle(x') = id(x') \partial(s_{\alpha} w) \text{chc}_{W, \psi_{\alpha}, \mathfrak{h}'_{\alpha}(\psi)}(x'). \quad (37)$$

#### 4.2 Some useful facts

**Lemma 4.1.** Suppose that  $H$  is a compact Cartan subgroup of  $G$ . Then  $x \in \mathfrak{h}$  is not annihilated by any noncompact imaginary roots if and only if  $G^x$  is compact.  $\square$

**Proof.** Recall the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}} = \bigoplus_{\alpha} \mathfrak{k}_{\mathbb{C}, \alpha} \oplus \bigoplus_{\beta} \mathfrak{p}_{\mathbb{C}, \beta},$$

where the  $\alpha$ 's are the compact roots and the  $\beta$ 's are the noncompact roots. Thus,

$$\mathfrak{g}_{\mathbb{C}}^x = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha(x)=0} \mathfrak{k}_{\mathbb{C}, \alpha} \oplus \bigoplus_{\beta(x)=0} \mathfrak{p}_{\mathbb{C}, \beta},$$

and the lemma follows.  $\blacksquare$

**Lemma 4.2.** Let  $\mathfrak{h} \subseteq \mathfrak{k}$  be a Cartan subalgebra of  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Let

$$V = \bigoplus_j V_j$$

be the decomposition into  $\mathfrak{h}$ -isotypic components over  $\mathbb{D}$ . Let  $x \in \mathfrak{h}$ . Then  $x$  is annihilated by precisely one noncompact imaginary root of  $\mathfrak{h}$  in  $\mathfrak{g}_{\mathbb{C}}$  if and only if  $G^x$  is not compact and either

- (1) there is exactly one pair  $j < k$  such that  $\text{eig}(x|_{V_j}) = \text{eig}(x|_{V_k})$ , and  $x|_{V_l} \neq 0$  for all  $l$  if  $\mathbb{D} = \mathbb{R}$  and the form  $(\cdot, \cdot)$  is skew-symmetric, or
- (2)  $\mathbb{D} = \mathbb{R}$ , the form  $(\cdot, \cdot)$  is skew symmetric, the sets  $\text{eig}(x|_{V_j})$  are distinct, and there is exactly one  $l$  such that  $x|_{V_l} = 0$ . □

Here  $\text{eig}(x)$  stands for the set of the eigenvalues of  $x$ .

**Proof.** There is  $t \in G$  such that  $\theta = \text{Ad}(t)$ . Since  $x = \theta(x)$ ,  $t$  preserves the decomposition of  $V$ . More precisely, if the form  $(\cdot, \cdot)$  is hermitian, then  $t|_{V_j} = \epsilon_j I_{V_j}$ , where  $\epsilon_j = 1$  if  $(\cdot, \cdot)_{V_j} > 0$ , and  $\epsilon_j = -1$  if  $(\cdot, \cdot)_{V_j} < 0$ . If the form  $(\cdot, \cdot)$  is skew hermitian, then  $t$  is a positive compatible complex structure on  $V$  and there are real numbers  $r_j$  such that  $x|_{V_j} = r_j t|_{V_j}$  for all  $j$ . As an  $\mathfrak{h}$ -module,

$$\mathfrak{p} = {}^t\mathfrak{g} = \bigoplus_j {}^t\mathfrak{g}(V_j) \oplus \bigoplus_{j < k} {}^t\text{Hom}(V_j, V_k).$$

Here  ${}^tE \subseteq F$  stands for the anticommutant of  $E$  in  $F$ , that is,  ${}^tE = \{Y \in F; xY + Yx = 0 \text{ for all } x \in E\}$ . Suppose the form  $(\cdot, \cdot)$  is hermitian. Then  ${}^t\mathfrak{g}(V_j) = 0$  for all  $j$ . Moreover,  ${}^t\text{Hom}(V_j, V_k) \neq 0$  if and only if  $\epsilon_j + \epsilon_k = 0$ , and in this case,  ${}^t\text{Hom}(V_j, V_k) = \text{Hom}(V_j, V_k)$ . Thus,

$$\mathfrak{p} = \bigoplus_{j < k, \epsilon_j + \epsilon_k = 0} \text{Hom}(V_j, V_k).$$

Let  $\alpha$  be a noncompact imaginary root. Then there is exactly one pair  $j < k$ , with  $\epsilon_j + \epsilon_k = 0$ , such that  $\alpha$  is an eigen-character of  $\mathfrak{h}$  in  $\text{Hom}(V_j, V_k)_{\mathbb{C}}$ , that is,  $\text{Hom}(V_j, V_k)_{\mathbb{C}, \alpha} \neq 0$ . Then  $\alpha(x) = 0$  is equivalent to

$$\det(\text{ad}(x))_{\text{Hom}(V_j, V_k)} = 0,$$

which, by a case-by-case verification, is equivalent to

$$\text{eig}(x|_{V_j}) = \text{eig}(x|_{V_k}).$$

Suppose the form  $(\ , \ )$  is skew-hermitian. We may assume that  $\mathbb{D} \neq \mathbb{C}$ . If  $\alpha$  is a noncompact imaginary root, then either

$${}^t\mathrm{Hom}(V_j, V_k)_{\mathbb{C}, \alpha} \neq 0 \quad \text{for some } j < k,$$

or

$${}^t\mathfrak{g}(V_l)_{\mathbb{C}, \alpha} \neq 0 \quad \text{for some } l.$$

Note that

$$\mathrm{ad}(x)_{{}^t\mathrm{Hom}(V_j, V_k)} = (r_j + r_k) \mathrm{ad}\left(\frac{1}{2}t\right)_{{}^t\mathrm{Hom}(V_j, V_k)}$$

and

$$\mathrm{ad}(x)_{{}^t\mathfrak{g}(V_l)} = r_l \mathrm{ad}(t)_{{}^t\mathfrak{g}(V_l)}.$$

This implies our lemma. ■

**Corollary 4.3.** Let  $\mathfrak{h} \subseteq \mathfrak{k}$  be a Cartan subalgebra of  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Let  $x \in \mathfrak{h}$  be such that  $x$  is annihilated by precisely one noncompact imaginary root of  $\mathfrak{h}$  in  $\mathfrak{g}_{\mathbb{C}}$ . Then the space  $V$  has a direct sum decomposition

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_m,$$

such that  $[\mathfrak{g}^x|_{V_0}, \mathfrak{g}^x|_{V_0}]$  is isomorphic to  $\mathfrak{sl}_2(\mathbb{R})$ , and for each  $j \geq 1$ , the element  $(x, V_j)$  is indecomposable and  $\mathfrak{g}^x|_{V_j} = \mathfrak{h}|_{V_j}$ .

Moreover, the sets

$$\mathrm{eig}(x|_{V_j}) \quad (j = 0, 1, 2, \dots, m)$$

are disjoint.

If  $(x, V_0)$  is indecomposable, then  $\mathbb{D} = \mathbb{R}$ , the form  $(\ , \ )$  is skew-symmetric and  $x|_{V_0} = 0$ . In this case,  $\mathfrak{g}^x|_{V_0} = \mathfrak{g}(V_0)$  is isomorphic to  $\mathfrak{sp}_2(\mathbb{R})$ .

Suppose the element  $(x, V_0)$  is decomposable. Then  $(x, V_0)$  is the sum of two distinct indecomposables. If  $x|_{V_0} \neq 0$ , then both components are nonzero and  $\mathfrak{g}^x|_{V_0} = \mathfrak{u}_{1,1}$ . If  $x|_{V_0} = 0$ , then  $\mathbb{D} = \mathbb{R}$ , the form  $(\ , \ )$  is symmetric,  $(\ , \ )_{V_0}$  has signature  $(2, 1)$  or  $(1, 2)$  and  $\mathfrak{g}^x|_{V_0} = \mathfrak{g}(V_0)$  is isomorphic to  $\mathfrak{so}(1, 2)$ . □

4.3 The reduction

**Proposition 4.4.** Suppose  $H'$  is a compact Cartan subgroup of  $G'$ . Let  $\psi \in \mathcal{D}(\mathfrak{g})$ . Then the function

$$\mathfrak{h}'_{In-reg} \ni x' \rightarrow \text{chc}(\psi)(x') = \int_{\mathfrak{g}} \text{chc}(x' + x)\psi(x) d\mu(x) \in \mathbb{C}$$

is smooth. □

**Proof.** This may be done via a wave front set computation as in Lemma 15.6, [18], or explicitly as follows.

Fix  $x' \in \mathfrak{h}'_{In-reg}$ . It will suffice to consider the function  $\text{chc}(\psi)(x')$  for  $x'$  in some small neighborhood of  $x'$ . Furthermore, we may assume that  $\psi$  is supported in an arbitrarily small completely invariant neighborhood of a semisimple point  $x \in \mathfrak{g}$ , which belongs to the singular support of  $\text{chc}(x'+)$ . Let

$$V' = \bigoplus_j V'_j \quad \text{and} \quad V = \bigoplus_k V_k$$

be the isotypic decompositions with respect to  $x'$  and  $x$ , respectively. It is not difficult to check that the sets  $\text{eig}(x'|_{V'_j})$  are disjoint. For each  $j$ , let  $\tilde{V}_j$  be the sum of all the  $V_k$  such that  $\text{eig}(x'|_{V'_j}) = \text{eig}(x|_{V_k})$ . We arrange the indices so that  $\tilde{V}_j \neq 0$  if and only if  $j = 1, 2, 3, \dots, m$ . Then

$$W = \bigoplus_{j=1}^m \ker(x' + x) \cap \text{Hom}(V'_j, \tilde{V}_j) \oplus \ker(x' + x)^\perp.$$

Let  $\mathcal{U}' \subseteq \mathfrak{g}^{x'}$  and  $\mathcal{U} \subseteq \mathfrak{g}^x$  be slices through  $x'$  and  $x$ , respectively. Then for  $x' \in \mathcal{U}'$  and for  $x \in \mathcal{U}$ ,

$$\text{chc}_W(x' + y) = \prod_{j=1}^m \text{chc}_{\ker(x'+x) \cap \text{Hom}(V'_j, \tilde{V}_j)}(x' + y) \text{chc}_{\ker(x'+x)^\perp}(x' + y),$$

where the last factor is a smooth function. Since, by Lemma 4.1,  $G'^{x'}$  is compact, the above decomposition of  $\text{chc}_W$  reduces the proof to the case when  $G'$  is compact. In this case,

$$\int_{\mathfrak{g}} \widetilde{\text{chc}}(x' + x)\psi(x) d\mu(x) = \int_{\mathfrak{g}} \chi \left( \frac{1}{4} \langle xw, w \rangle \right) \phi(w) dw,$$

where  $\phi \in \mathcal{S}(W)$  is the pullback via the moment map  $W \rightarrow \mathfrak{g}^*$  of a Fourier transform of  $\psi$ . (Here the fact that  $\phi$  is rapidly decreasing follows from the compactness of  $G'$ .) In particular, it is clear that the function in question is smooth. ■



**Proof of Theorem 1.1.** Theorem 1.1 holds for pairs of type II, as was observed by Bernon [2]. This is a consequence of an explicit formula [18, Proposition 7.21]. Thus, we may assume that the pair  $(G, G')$  is of type I. Then  $G$  and  $G'$  are the isometry groups  $G = G(V, (\cdot, \cdot))$  and let  $G' = G(V', (\cdot, \cdot)')$ , as in Section 2. Recall the  $\theta$ -stable Cartan subgroup  $H' = T'A' \subseteq G'$ . Let  $V'_c \subseteq V'$  be the subspace on which  $A'$  acts trivially, and let  $V'_s = V'_c^\perp$ . Then  $V'_s$  has a complete polarization

$$V'_s = X' \oplus Y'$$

preserved by  $H'$ . We may and do assume that  $V'_s \subseteq V$  and that the above is also a complete polarization with respect to the form  $(\cdot, \cdot)$ . Let  $U = V'_s^\perp \subseteq V$ . Then

$$V = V'_s \oplus U.$$

The above decompositions induce embeddings:

$$\begin{aligned} \text{GL}(X') \times G(U) &\subseteq G, \\ \mathfrak{n}' = \text{Hom}(X', V'_c) + \text{Hom}(X', Y') \cap \mathfrak{g}' &\subseteq \mathfrak{g}', \\ \mathfrak{n} = \text{Hom}(X', U) + \text{Hom}(X', Y') \cap \mathfrak{g} &\subseteq \mathfrak{g}. \end{aligned} \tag{38}$$

We assume that the subgroup  $\text{GL}(X') \times G(U) \subseteq G$  is preserved by  $\theta$ . Let  $W_c = \text{Hom}(V'_c, U)$ . Then, by Bernon and Przebinda [4, Corollary 8], there is a nonzero constant

$$c = \frac{\gamma(V, V', X')}{\eta(V, V', X')},$$

such that, for  $\psi \in \mathcal{S}(\mathfrak{g})$ , we have

$$\begin{aligned} &\text{chc}_{W, \psi, \mathfrak{h}'}(\psi)(X') \\ &= c \prod_{\alpha \in \Psi(\mathfrak{gl}(X'))} \frac{\alpha(X')}{|\alpha(X')|} |\det(\text{ad}(X'))_{\mathfrak{gl}(X')/\mathfrak{h}'|_{X'}}|^{1/2} \\ &\cdot \int_{\text{GL}(X')/H'|_{X'}} \int_{\mathfrak{g}(U)} \prod_{\alpha \in \Psi(\mathfrak{g}(V'_c))} \frac{\alpha(X')}{|\alpha(X')|} \text{chc}_{W_c}(X'|_{V'_c} + y) \psi_{\mathfrak{n}}^K(g \cdot X'|_{X'} + y) d\mu(y) d\mu(gH'|_{X'}), \end{aligned} \tag{39}$$

where, for the purposes of normalization of the measure  $\mu$ ,  $H'|_{X'} \subseteq G' \subseteq \text{Sp}(W)$ . Moreover,  $\Psi$  is the disjoint union of  $\Psi(\mathfrak{gl}(X')) \subseteq \Delta_I(\mathfrak{gl}(X')_{\mathbb{C}}, \mathfrak{h}'_{X', \mathbb{C}})$  and  $\Psi(\mathfrak{g}(V'_c)) \subseteq \Delta_I(\mathfrak{g}(V'_c)_{\mathbb{C}}, \mathfrak{h}'_{V'_c, \mathbb{C}})$ , and

the integral over  $\mathfrak{g}(\mathbb{U})$  is not there if  $W_c = 0$ . If  $W_c \neq 0$  and  $\mathfrak{h}'$  acts trivially on  $W_c$  (or equivalently on  $V'_c$ ), then  $\text{chc}_{W_c}(x'|_{V'_c} + y)$  is replaced by  $\text{chc}_{W_c}(y)$ .

If  $\text{chc}_{W_c}(y) = 0$  or  $\text{chc}_{W_c}(y) \neq 0$  but  $\mathfrak{h}'|_{V'_c} = 0$ , then we see from (39) that  $\text{chc}_{W, \psi, \mathfrak{h}'}(\psi)$  is an orbital integral, so there is nothing to prove.

If  $\mathfrak{h}'|_{V'_c} \neq 0$ , then  $\text{chc}_{W, \psi, \mathfrak{h}'}(\psi)$  is essentially the tensor product of an orbital integral for  $\mathfrak{gl}(X')$  and  $\text{chc}_{W, \psi(\mathfrak{g}(V'_c)), \mathfrak{h}'|_{V'_c}}$ . Thus, formula (39) shows that, if the jump relations are satisfied under the assumption that  $H'$  is compact, then they are satisfied in general. Therefore, we can assume that  $H'$  is compact.

Fix an element  $x' \in \mathfrak{h}'$  that is annihilated by precisely one noncompact imaginary root of  $\mathfrak{h}'$  in  $\mathfrak{g}'_{\mathbb{C}}$ . Then, by Corollary 4.3, the space  $V'$  has a direct sum decomposition

$$V' = V'_0 \oplus V'_1 \oplus V'_2 \oplus \cdots \oplus V'_m,$$

such that  $[\mathfrak{g}^{x'}|_{V'_0}, \mathfrak{g}^{x'}|_{V'_0}]$  is isomorphic to  $\mathfrak{sl}_2(\mathbb{R})$ , and  $\mathfrak{g}^{x'}|_{V'_j} = \mathfrak{h}'|_{V'_j}$  for each  $j \geq 1$ . Let  $x \in \mathfrak{g}$  be a semisimple element in the singular support of the distribution  $\text{chc}(x' + \cdot)$ . Let

$$V = \bigoplus_k V_k$$

be the  $x$ -isotypic decomposition of  $V$ . Since  $\det(x' + x) = 0$ , there are  $j$  and  $k$  such that  $\text{eig}(x'|_{V'_j}) \cap \text{eig}(x|_{V_k}) \neq \emptyset$ . (Then  $\text{eig}(x'|_{V'_j}) = \text{eig}(x|_{V_k}$ .) For each  $j$ , let  $\tilde{V}_j$  be the sum of all the  $V_k$  such that  $\text{eig}(x'|_{V'_j}) = \text{eig}(x|_{V_k})$ . Let  $\mathcal{J} \subseteq \{0, 1, 2, \dots, m\}$  be the set such that  $\tilde{V}_j \neq 0$ . Then

$$W = \left( \bigoplus_{j \in \mathcal{J}} \ker(x' + x) \cap \text{Hom}(V'_j, \tilde{V}_j) \right) \oplus \ker(x' + x)^\perp.$$

Let  $\mathcal{U}' \subseteq \mathfrak{g}^{x'}$  and  $\mathcal{U} \subseteq \mathfrak{g}^x$  be slices through  $x'$  and  $x$ , respectively. Then for  $x' \in \mathcal{U}'$  and for  $x \in \mathcal{U}$ ,

$$\text{chc}_W(x' + y) = \prod_{j \in \mathcal{J}} \text{chc}_{\ker(x'+x) \cap \text{Hom}(V'_j, \tilde{V}_j)}(x' + y) \text{chc}_{\ker(x'+x)^\perp}(x' + y),$$

where the last factor is a smooth function. Also, by Proposition 4.4, for each  $j \in \mathcal{J} \setminus \{0\}$ , the function

$$\mathcal{U}' \ni x' \rightarrow \text{chc}_{\text{Hom}(V'_j, \tilde{V}_j) \cap \ker(x'+x)}(x' + \cdot) \in \mathcal{D}'(\mathcal{U})$$

is smooth. Hence, the jump may occur if and only if  $0 \in \mathcal{J}$ . In that case we are led to consider the pair

$$G^{x'}|_{V'_0}, G^x|_{\tilde{V}_0} \subseteq \text{Sp}(\ker(x' + x) \cap \text{Hom}(V'_0, \tilde{V}_0)). \tag{40}$$

Now we use Corollary 4.3. If  $x'|_{V_0} = 0$ , then  $x|_{\tilde{V}_0} = 0$  and the pair (40) coincides either with  $(O_{p,q}, \text{Sp}_2(\mathbb{R}))$  or with  $(\text{Sp}_{2n}(\mathbb{R}), O_{1,2})$ .

If  $x'|_{V_0} \neq 0$ , then  $x|_{\tilde{V}_0} \neq 0$  and there are complex structures  $\mathcal{J}' \in \mathfrak{g}'$  and  $\mathcal{J} \in \mathfrak{g}$  such that  $x' = r'\mathcal{J}'$  and  $x = r\mathcal{J}$ , for some  $r, r' \in \mathbb{R} \setminus \{0\}$ . Furthermore, if we view  $V'$  and  $V$  as complex vector spaces via the actions of  $\mathcal{J}'$  and  $\mathcal{J}$ , then

$$G^{x'}|_{V_0} = U_{1,1}, \quad G^x|_{\tilde{V}_0} = U_{p,q},$$

$$\ker(x' + x) \cap \text{Hom}(V'_0, \tilde{V}_0) = \text{Hom}_{\mathbb{C}}(V'_0, \tilde{V}_0).$$

By Bernon and Przebinda [4, Theorem 10], we know that for the dual pairs  $(O_{p,q}, \text{Sp}_2(\mathbb{R}))$ ,  $(\text{Sp}_{2n}(\mathbb{R}), O_{1,2})$ , and  $(U_{p,q}, U_{1,1})$ , the jump relations are satisfied. We deduce the result using a partition of unity. ■

### 5 Proof of Theorem 1.2

As in the case of Theorem 1.1, we need to check the boundedness and the jump relations. The boundedness shall be verified in Proposition 5.1, where we use the notation of Section 2 without comments. The verification of the jump relations is done via the reduction to smaller cases (as in the proof of Theorem 1.1), lifting to the Lie algebra via the Cayley transform and localization, as in the proof of Proposition 5.1. We leave the details to the reader.

**Proposition 5.1.** Let  $f \in \mathcal{D}(\tilde{G}_1)$ . Then for any  $u \in \text{Sym}(\mathfrak{h}'_{\mathbb{C}})$ ,

$$\sup_{h' \in \tilde{H}'^{\text{reg}}} \left| L(u) \left( \Delta_{\psi'}(h') \int_{\tilde{G}} \Theta(\check{p}(h')g) f(g) d\mu(g) \right) \right| < \infty. \quad \square$$

**Proof.** Let us write  $\Theta = \Theta_{V,V'}$  in order to indicate the dependence of the character  $\Theta$  on the underlying spaces,  $V$  and  $V'$ . (For different vector spaces we get different symplectic spaces and thus different characters.)

Consider first the case when  $-1$  is not an eigenvalue of any element of  $\text{supp}(f)$ . Let  $P = (P^+, P^-)$  be a partition of the set  $\{1, 2, \dots, n\}$ :

$$\{1, 2, \dots, n\} = P^+ \cup P^-. \tag{41}$$

For  $\epsilon > 0$  let

$$H'(P, \epsilon) = \{h' \in H' \mid |h'_j + 1| > \epsilon \text{ if } j \in P^+, \text{ and } |h'_j + 1| < 2\epsilon \text{ if } j \in P^-\}. \tag{42}$$

Also, let

$$V^+ = \sum_{j \in P^+} V'_j, \quad V^- = \sum_{j \in P^-} V'_j. \tag{43}$$

Then, for  $\epsilon > 0$  small enough, the sets (42) are nonempty and form an open covering of  $H'$ , and the family of functions

$$\Theta_{V, V^-}(h'g) f(g) \quad (h' \in \tilde{H}'(P, \epsilon))$$

is bounded in  $\mathcal{D}(\tilde{G})$ . From now on we fix such an  $\epsilon$ .

Let  $\Psi'_+ = \{\alpha \in \Psi' \mid \underline{\alpha} \subseteq P^+\}$ . Then for any  $v \in \text{Sym}(\mathfrak{h}'_C)$ , the function

$$L(v) \frac{\Delta_{\Psi'}(h')}{\Delta_{\Psi'_+}(h')} \quad (h' \in H' \tilde{p}) \tag{44}$$

is bounded. Furthermore,

$$\begin{aligned} &\Delta_{\Psi'}(h') \int_{\tilde{G}} \Theta(\check{p}(h')g) f(g) \, d\mu(g) \\ &= \frac{\Delta_{\Psi'}(h')}{\Delta_{\Psi'_+}(h')} \Delta_{\Psi'_+}(h') \int_{\tilde{G}} \Theta_{V, V'_+}(\check{p}(h')g) (\Theta_{V, V^-}(\check{p}(h')g) f(g)) \, d\mu(g). \end{aligned} \tag{45}$$

Thus, we may assume that  $V_0 = 0$  (i.e.,  $P^- = \emptyset$ ) and consider only the  $h' \in \tilde{H}'$  with  $|\check{p}(h')'_j + 1| > \epsilon$  for all  $1 \leq j \leq n'$ . Let

$$\check{p}(h') = \tilde{c}(x')\mathfrak{v}, \quad g = \tilde{c}(x)\mathfrak{v}^{-1}.$$

Then

$$\Theta(\check{p}(h')g) = \Theta(\tilde{c}(x')\tilde{c}(x)) = \Theta(h'\mathfrak{v}^{-1})\widetilde{\text{chc}}(x' + x)\Theta(g\mathfrak{v}). \tag{46}$$

Thus, (45) coincides with

$$\frac{\Delta_{\Psi'}(h')}{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')} \Theta(\check{p}(h')\mathfrak{v}^{-1}) \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} \widetilde{\text{chc}}(x' + x) (\Theta(\tilde{c}(x)) f(\tilde{c}(x)\mathfrak{v}^{-1})) j_{\mathfrak{g}}(x) \, d\mu(x),$$

where  $\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') = \prod_{\alpha \in \Psi'} \alpha(x')$  and  $j_{\mathfrak{g}}$  is the Jacobian of the Cayley transform  $c: \mathfrak{g} \rightarrow G$ . Since any derivative of the function  $\mathbb{R} \ni t \rightarrow \frac{\sin(t)}{t} \in \mathbb{R}$  is bounded, it is easy to check that

for any  $v \in S(\mathfrak{h}'_{\mathbb{C}})$ , the function

$$L(v) \frac{\Delta_{\psi'}(\mathfrak{h}')}{\pi_{\mathfrak{g}'/\mathfrak{h}'(\mathfrak{X}')}} \quad (\mathfrak{h}' \in \check{H}') \tag{47}$$

is bounded. Also,

$$|\Theta(\check{p}(\mathfrak{h}')\mathfrak{d}^{-1})| \leq \left| \frac{1}{\det(1 + p(\mathfrak{h}'))_{\mathbb{W}}} \right|.$$

Hence, the proposition follows from [3, Theorem 1].

Suppose 1 is not an eigenvalue of any element of  $\text{supp}(f)$ . Then the elements of the support of the function  $f(\mathfrak{d}g)$  do not have  $-1$  as an eigenvalue. Furthermore,  $\mathfrak{d} \in \tilde{G} \cap \tilde{G}$  and  $\Delta_{\psi'}(\mathfrak{h}') = \Delta_{\psi'}(\mathfrak{h}'\mathfrak{d})$ . Hence, by the left invariance of the measure  $\mu$ , the left-hand side of (45) is equal to

$$\Delta_{\psi'}(\mathfrak{h}'\mathfrak{d}) \int_{\tilde{G}} \Theta(\check{p}(\mathfrak{h}')\mathfrak{d}g) f(\mathfrak{d}g) \, d\mu(g), \tag{48}$$

and we are in the case considered previously, with  $\check{p}(\mathfrak{h}')$  replaced by  $\check{p}(\mathfrak{h}')\mathfrak{d}$ , so the result follows. Thus, we need to consider functions  $f$  supported in a completely invariant open neighborhood  $\mathcal{V}$  of a semisimple element  $g_0$  which has both 1 and  $-1$  as eigenvalues.

If  $\mathbb{D} = \mathbb{C}$ , then  $\tilde{G} \cap \tilde{G}$  is a double cover of the unitary group  $U_1$ . Hence, by choosing  $\mathcal{V}$  small enough, we may translate, as in (48), by an element of  $\tilde{G} \cap \tilde{G}$  to reduce to the case considered in (45). In order to resolve the general case, we proceed as follows.

For a partition  $P$ , as in (41), let

$$H'(P) = \{\mathfrak{h}' \in H'; |h'_j + 1| > \frac{11}{10}|h'_j - 1| \text{ if } j \in P^+, \text{ and } |h'_j - 1| > \frac{11}{10}|h'_j + 1| \text{ if } j \in P^-\}.$$

The sets  $H'(P)$  are not empty and form an open covering of  $H'$ . Let

$$V = V^+ \oplus V^-$$

be a direct sum decomposition preserved by  $g_0$  and such that  $-1$  is not an eigenvalue of  $g_0|_{V^+}$ , and 1 is not an eigenvalue of  $g_0|_{V^-}$ . We may assume that the neighborhood  $\mathcal{V}$  is small enough so that the families of functions

$$\mathcal{V} \ni g \rightarrow \Theta_{V^-, V^+}(\mathfrak{h}'g) \quad (\mathfrak{h}' \in \check{H}'(P)),$$

$$\mathcal{V} \ni g \rightarrow \Theta_{V^+, V^-}(\mathfrak{h}'g) \quad (\mathfrak{h}' \in \check{H}'(P))$$

are bounded in  $C^\infty(\mathcal{V})$ . Then the families of functions

$$\begin{aligned} \mathcal{V} \ni g &\rightarrow \frac{\Theta_{V^-,V^+}(h'g)}{\Theta_{V^-,V^+}(\partial h'g)} \quad (h' \in \tilde{H}'(P)), \\ \mathcal{V} \ni g &\rightarrow \frac{\Theta_{V^+,V^-}(h'g)}{\Theta_{V^+,V^-}(\partial h'g)} \quad (h' \in \tilde{H}'(P)) \end{aligned}$$

are also bounded in  $C^\infty(\mathcal{V})$ . Furthermore, for  $h' \in \tilde{H}'(P)$  and  $g \in \mathcal{V}$ ,

$$\begin{aligned} \Theta(h'g) &= \Theta_{V^+,V^+}(h'g)\Theta_{V^+,V^-}(\partial h'g) \frac{\Theta_{V^+,V^-}(h'g)}{\Theta_{V^+,V^-}(\partial h'g)} \\ &\quad \cdot \Theta_{V^-,V^-}(h'g)\Theta_{V^-,V^+}(\partial h'g) \frac{\Theta_{V^-,V^+}(h'g)}{\Theta_{V^-,V^+}(\partial h'g)}. \end{aligned} \tag{49}$$

Let

$$h' = (\partial_+ \tilde{c}(x^+))\tilde{c}(x^-), \quad \partial_+ \in \tilde{G}(V^+), \quad x^\pm \in \mathfrak{h}'|_{V^\pm},$$

and let

$$g = (\partial_+ \tilde{c}(x^+))\tilde{c}(x^-), \quad \partial_+ \in \tilde{G}(V^+),$$

$x^\pm \in \mathfrak{g}$  are conjugate to elements of  $\mathfrak{g}(V^\pm)$ , respectively.

Then, as in (46),

$$\begin{aligned} &\Theta_{V^+,V^+}(h'g)\Theta_{V^+,V^-}(\partial h'g) \\ &= \Theta_{V^+,V^+}(h'\partial_+^{-1})\widetilde{\text{chc}}_{V^+,V^+}(x^+ + x^+)\Theta_{V^+,V^+}(g\partial_+), \\ &\Theta_{V^+,V^-}(h')\widetilde{\text{chc}}_{V^+,V^-}(x^- + x^+)\Theta_{V^+,V^-}(g) \\ &= \Theta_{V^+,V^+}(h'\partial_+^{-1})\Theta_{V^+,V^-}(h')\widetilde{\text{chc}}_{V^+,V^-}(x' + x^+)\Theta_{V^+,V^+}(g\partial_+)\Theta_{V^+,V^-}(g), \end{aligned} \tag{50}$$

where  $x' = x^- + x^+$ . Similarly,

$$\begin{aligned} \Theta_{V^-,V^-}(h'g)\Theta_{V^-,V^+}(\partial h'g) &= \Theta_{V^-,V^-}(h')\Theta_{V^-,V^+}(h'\partial_+^{-1})\widetilde{\text{chc}}_{V^-,V^-}(x' + x^-) \\ &\quad \cdot \Theta_{V^-,V^-}(g)\Theta_{V^-,V^+}(g\partial). \end{aligned} \tag{51}$$

Hence, (49) is equal to

$$\begin{aligned} &\Theta_{V^+,V^+}(h'\partial_+^{-1})\Theta_{V^+,V^-}(h')\Theta_{V^-,V^-}(h')\Theta_{V^-,V^+}(h'\partial_+^{-1})\widetilde{\text{chc}}_{V^+,V^-}(x' + x) \\ &\quad \cdot \left( \Theta_{V^+,V^+}(g\partial_+)\Theta_{V^+,V^-}(g)\Theta_{V^-,V^-}(g)\Theta_{V^-,V^+}(g\partial^{-1}) \frac{\Theta_{V^+,V^-}(h'g)}{\Theta_{V^+,V^-}(\partial h'g)} \frac{\Theta_{V^-,V^+}(h'g)}{\Theta_{V^-,V^+}(\partial h'g)} \right). \end{aligned} \tag{52}$$

The term in parenthesis is a smooth function and the product of the first four terms on the right-hand side of (52) is dominated by

$$\left| \frac{1}{\det(1 + \tilde{p}(h'))_{\text{Hom}(V^+, V^+)}} \frac{1}{\det(1 - \tilde{p}(h'))_{\text{Hom}(V^+, V^-)}} \frac{1}{\det(1 - \tilde{p}(h'))_{\text{Hom}(V^-, V^+)}} \frac{1}{\det(1 + \tilde{p}(h'))_{\text{Hom}(V^-, V^+)}} \right|.$$

Thus, since the  $\widetilde{\text{chc}}_{V, V}$  in (52) corresponds to the situation when the rank of  $\mathfrak{g}'$  is less or equal to the rank of  $\mathfrak{g}$ , the theorem follows from [3, Theorem 1]. ■

### 6 Proof of Theorem 1.3

As before, let  $\Theta$  be the distribution character of the Shale Weil (oscillator) representation (of  $\widetilde{\text{Sp}}(W)$ ). In the first subsection, we prove that there exists a certain subset on which we can restrict  $\Theta$ . This proves that  $\text{Chc}$  has a nice restriction on a dense open subset of  $\tilde{G}$  denoted  $\tilde{G}_{\text{npb}}$ . Proposition 6.3 says that the compatibility of  $\text{Chc}$  with the Capelli Harish-Chandra homomorphism is satisfied on  $\tilde{G}_{\text{npb}}$ . The crucial point is that any regular element belonging to a fundamental Cartan subgroup belongs to  $\tilde{G}_{\text{npb}}$ . The extension to the whole  $\tilde{G}$  is immediate from Theorem 1.2. In Section 1.2, we introduce some notations and recall some classical results due to Harish-Chandra. In Section 1.3, we prove Theorem 1.3.

#### 6.1 The restriction of $\Theta$ to a dense subset of $\tilde{G}$

For  $g \in \text{Sp}(W)$  the tangent space  $T_g \text{Sp}(W)$  may be identified with  $g \mathfrak{sp}(W) \subseteq \text{End}(W)$ . Then the dual space  $T_g^* \text{Sp}(W)$  is identified with  $\{g\} \times \mathfrak{sp}^*(W)$  by

$$\mathfrak{sp}^*(W) \ni \xi \rightarrow (T_g \text{Sp}(W) \ni gx \rightarrow \xi(x) \in \mathbb{R}) \in T_g^* \text{Sp}(W)$$

and the cotangent bundle  $T^* \mathfrak{sp}(W)$  with  $\text{Sp}(W) \times \mathfrak{sp}^*(W)$ . Denote by  $\rho$  the canonical projection

$$\rho : T^* \text{Sp}(W) \ni (g, \xi) \rightarrow g \in \text{Sp}(W).$$

Similarly,  $T^* \widetilde{\text{Sp}}(W)$  is identified with  $\widetilde{\text{Sp}}(W) \times \mathfrak{sp}^*(W)$ .

Let  $(G, G')$  be a dual pair in  $\text{Sp}(W)$ . The conormal bundle to the embedding  $\tilde{G} \tilde{G} \rightarrow \widetilde{\text{Sp}}(W)$  is given by

$$N_{\tilde{G} \tilde{G}} = \{(gg', \xi) \mid g \in \tilde{G}, g' \in \tilde{G}, \xi \in \mathfrak{sp}^*(W), \xi|_{\mathfrak{g}+\mathfrak{g}'} = 0\}.$$

Recall, [18, Lemma 12.2], that the wave front set of the distribution character  $\Theta \in \mathcal{D}'(\widetilde{\text{Sp}}(\mathbb{W}))$  of the oscillator representation is

$$\text{WF}(\Theta) = \{(g, \tau_{\mathfrak{sp}}(w)) \mid g \in \widetilde{\text{Sp}}(\mathbb{W}) \times \mathfrak{sp}^*(\mathbb{W}), w \in \mathbb{W}, w \neq 0, \underline{g}(w) = w\}, \tag{53}$$

where  $\tau_{\mathfrak{g}} : \mathbb{W} \rightarrow \mathfrak{g}^*$ ,  $\tau(w)(x) = \langle xw, w \rangle$ . Also, the elements of  $\widetilde{\text{Sp}}(\mathbb{W})$  act on  $\mathbb{W}$  via the map (1). Clearly,

$$\begin{aligned} &\text{WF}(\Theta) \cap N_{\tilde{G}\tilde{G}} \\ &= \{(p, \tau_{\mathfrak{sp}}(w)) \in \tilde{G}\tilde{G} \times \mathfrak{sp}(\mathbb{W}) \mid w \in \mathbb{W} \setminus \{0\}, \tau_{\mathfrak{g}}(w) = 0, \tau_{\mathfrak{g}'}(w) = 0, pw = w\}, \end{aligned} \tag{54}$$

and therefore,

$$\rho(\text{WF}(\Theta) \cap N_{\tilde{G}\tilde{G}}) = \{p \in \tilde{G}\tilde{G}' \mid w \in \mathbb{W} \setminus \{0\} \text{ with } \tau_{\mathfrak{g}}(w) = 0, \tau_{\mathfrak{g}'}(w) = 0, pw = w\}.$$

Thus,

$$\tilde{G}\tilde{G} \setminus (\rho(\text{WF}(\Theta) \cap N_{\tilde{G}\tilde{G}})) = \{p \in \tilde{G}\tilde{G} \mid \tau_{\mathfrak{g}}^{-1}(0) \cap \tau_{\mathfrak{g}'}^{-1}(0) \cap \ker(p - \text{id}_{\mathbb{W}}) = \{0\}\}.$$

Let  $\tilde{G}_{\leq n}$  the preimage of the set of all elements of  $G'$  which do not preserve any nonzero (isotropic in the type I case) subspaces of the defining module for  $g'$ , of dimension  $\leq n$ .

**Lemma 6.1.** Let  $V$  denote the defining module for  $G$ . If  $G$  is the isometry group of a form  $(\ , \ )$  on  $V$ , denote by  $n$  the Witt index of the form  $(\ , \ )$ . If  $G = \text{GL}(V)$ , let  $n = \dim(V)$ . Then the set of the elements  $g' \in \tilde{G}$ , such that  $gg' \in \tilde{G}\tilde{G} \setminus \rho(\text{WF}(\Theta) \cap N_{\tilde{G}\tilde{G}})$  for all  $g \in \tilde{G}$ , is equal to  $\tilde{G}_{\leq n}$ . □

**Proof.** Suppose the pair  $(G, G')$  is of type I. Then  $G'$  is the Lie group of the isometries of a form  $(\ , \ )'$  on  $V'$ , and the symplectic space  $\mathbb{W}$  can be realized as  $\mathbb{W} = \text{Hom}(V, V')$ . For  $w \in \mathbb{W}$  define  $w^* \in \text{Hom}(V', V)$  by

$$(w(v), v')' = (v, w^*(v')) \quad (v \in V, v' \in V').$$

Let  $g \in \tilde{G}$ ,  $g' \in \tilde{G}$ , and  $w \in \mathbb{W} \setminus \{0\}$  be such that  $\tau_{\mathfrak{g}}(w) = 0$ ,  $\tau_{\mathfrak{g}'}(w) = 0$ , and  $gg'(w) = w$ . Then  $w^*w = 0$  and  $ww^* = 0$ . Hence,  $\text{im}(w) \subseteq \ker(w^*) = \text{im}(w)^\perp$ , and  $\ker(w) \supseteq \text{im}(w^*) = \ker(w)^\perp$ . Thus,  $\text{im}(w) \subseteq V'$  is an isotropic subspace, and  $\ker(w) \subseteq V$  is a co-isotropic subspace. In particular, the dimension of the image of  $w$  is not greater than the Witt index of the



form  $(\cdot, \cdot)$ . The equation  $gg'(w) = w$  translates to  $g'w = wg$  and implies that  $g'$  preserves  $\text{im}(w)$ .

Conversely, suppose  $g' \in G'$  preserves an isotropic subspace  $X' \subseteq V'$  of dimension not greater than the Witt index of  $(\cdot, \cdot)$ . Then there is an isotropic subspace  $X \subseteq V$  and a linear bijection  $w : X \rightarrow X'$ . Define an element  $g \in \text{GL}(X)$  by

$$g'w = wg.$$

Let  $Y \subseteq V$  be a subspace complementary to  $X^\perp$ , and let  $U = (X + Y)^\perp$ . Then  $Y$  is isotropic,  $V = X \oplus U \oplus Y$ , and  $g$  extends to an element  $g \in G$  that preserves  $X, Y$ , and  $U$ . Let us extend  $w$  to an element of  $W$  so that  $\ker(w) = U \oplus Y (= X^\perp)$ . Then  $w$  is a nonzero element of  $W$  with  $w^*w = 0, ww^* = 0$ , and  $g'w = wg$ .

Suppose, from now on, that the pair  $(G, G')$  is of type II. Let us realize the symplectic space  $W$  as  $W = \text{Hom}(V, V') \oplus \text{Hom}(V', V)$ . Let  $g \in G, g' \in G'$ , and  $w \in W \setminus \{0\}$  be such that  $\tau_g(w) = 0, \tau_{g'}(w) = 0$ , and  $gg'(w) = w$ . Then there are  $S \in \text{Hom}(V, V')$  and  $T \in \text{Hom}(V', V)$ , with  $S$  or  $T$  nonzero, such  $w = (S, T)$ . Hence, our condition translates to  $ST = 0, TS = 0, g'Sg^{-1} = S$ , and  $gTg'^{-1} = T$ . Clearly,  $g'$  preserves the image of  $S$  and the kernel of  $T$ . Furthermore,  $\dim(\text{im}(S)) \leq \dim(V)$ ,  $\text{co-dim}(\ker(T)) \leq \dim(V)$  and at least one of these spaces is not zero.

Conversely, suppose  $g' \in G'$  preserves a nonzero subspace  $X' \subseteq V'$ , with  $\dim(X') \leq \dim(V)$ . Let  $X \subseteq V$  be a subspace of the same dimension. Let  $Y \subseteq V$  be a complementary subspace. Let  $S \in \text{Hom}(V, V')$  be such that  $\ker(S) = Y$  and  $S|_X : X \rightarrow X'$  is a bijection. There is  $g \in G$  with  $g'Sg^{-1} = S$ . Let  $T = 0$  and let  $w = (S, T)$ . Then  $w \neq 0$  and  $ST = 0, TS = 0, g'Sg^{-1} = S$ , and  $gTg'^{-1} = T$ . ■

Let  $m : \tilde{G} \times \tilde{G} \ni (g, g') \rightarrow gg' \in \tilde{\text{Sp}}(W)$  denote the multiplication. Lemma 6.1 implies that the pull-back  $m^*(\Theta) \in \mathcal{D}'(\tilde{G} \times \tilde{G}_{\leq n})$  is well defined. Let  $K_\Theta$  denote the corresponding integral kernel operator:

$$K_\Theta : \mathcal{D}(\tilde{G}) \rightarrow \mathcal{D}'(\tilde{G}_{\leq n}),$$

$$K_\Theta(\psi)(\psi') = m^*(\Theta)(\psi \otimes \psi') \quad (\psi \in \mathcal{D}(\tilde{G}), \psi' \in \mathcal{D}(\tilde{G}_{\leq n})).$$

Let  $G'_{\text{npb}} \subseteq G'$  be the set of all the elements of  $G'$  that do not belong to any proper parabolic subgroup of  $G'$ . In other words  $G'_{\text{npb}} = G'_{\leq \infty}$  is the set of all the elements of the group that do not preserve any nonzero (isotropic in the type I case) subspaces of  $V'$ . In particular,  $G'_{\text{npb}} \subseteq G'_{\leq n}$  for any  $n = 1, 2, \dots$ . Clearly it might happen that the set  $G'_{\text{npb}}$

is empty. In fact  $G'_{\text{npb}}$  coincides with the set of the orbits passing through the regular elements of a compact Cartan subgroup of  $G'$ . In the following lemma, we shall focus on the cases when  $G'_{\text{npb}} \neq \emptyset$ .

**Lemma 6.2.** For any  $\psi \in \mathcal{D}(\tilde{G})$ , the restriction of the distribution  $K_\Theta(\psi)$  to  $\tilde{G}_{\text{npb}}$  is a smooth function. We shall denote this function by

$$\int_{\tilde{G}} \Theta(g'g)\psi(g) \, d\mu(g) \quad (g' \in \tilde{G}_{\text{npb}}). \quad \square$$

**Proof.** We know from [14, Theorem 8.2.12] and (53) that

$$\begin{aligned} \text{WF}(K_\Theta(\psi)|_{\tilde{G}_{\text{npb}}}) &\subseteq \{(g', \tau_{g'}(w)) \mid g' \in \tilde{G}_{\text{npb}} \text{ and there is } g \in \text{supp}(\psi) \\ &\text{and } w \in \mathbb{W} \setminus \{0\}, \text{ such that } \tau_g(w) = 0 \text{ and } g'g(w) = w\}. \end{aligned}$$

Let  $g, g'$ , and  $w$  be as above. Then

$$\tau_{g'}(w) = \tau_{g'}(gg'(w)) = g' \tau_{g'}(w) g'^{-1}. \tag{55}$$

On the other hand,  $\tau_g(w) = 0$  implies that  $\tau_{g'}(w)$  is nilpotent. But since  $g' \in \tilde{G}_{\text{npb}}$ , (55) implies that this nilpotent element has to be zero (see [7, Lemma 3.8.4]). Therefore,  $\text{WF}(K_\Theta(\psi)|_{\tilde{G}_{\text{npb}}}) = \emptyset$ , and the lemma follows. ■

Recall the left regular representation of  $\mathfrak{g}$ :

$$L(x)\psi(g) = \frac{d}{dt} \psi(\exp(-tx)g)|_{t=0} \quad (x \in \mathfrak{g}, g \in \tilde{G}, \psi \in C^\infty(\tilde{G})), \tag{56}$$

and similarly for  $g'$ . Then  $L$  extends to an injective homomorphism from the universal enveloping algebra  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  to the algebra of the analytic differential operators on  $G$ . Recall the involution  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \ni z \rightarrow \check{z} \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  defined by  $\check{z} = -z, z \in \mathfrak{g}$ . The action of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  on the space  $\mathcal{D}'(\tilde{G})$ , of the distributions, is defined by

$$L(z)u(\psi) = u(L(\check{z})\psi) \quad (u \in \mathcal{D}'(\tilde{G}), z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})).$$

(Note that this definition is consistent with the embedding of  $C^\infty(\tilde{G})$  into  $\mathcal{D}'(\tilde{G})$  via the Haar measure.)

Assume from now on that the rank of  $g'$  is less than or equal to the rank of  $g$ . Recall the Capelli Harish-Chandra homomorphism  $\mathcal{C}_{\mathfrak{g},g'} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G \rightarrow \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}$ , [19, equality (5.5)].

**Theorem 6.3.** For any  $\psi \in \mathcal{D}(\tilde{G})$  the function

$$\tilde{G}_{\text{npb}} \ni g' \rightarrow \int_{\tilde{G}} \Theta(g'g)\psi(g) \, d\mu(g) \in \mathbb{C}$$

is smooth and invariant under the conjugation by  $\tilde{G}$ . Moreover, for  $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ ,

$$\int_{\tilde{G}} \Theta(g'g)L(z)\psi(g) \, d\mu(g) = L(\mathcal{C}_{\mathfrak{g},g'}(\check{z})) \int_{\tilde{G}} \Theta(g'g)\psi(g) \, d\mu(g) \quad (g' \in \tilde{G}_{\text{npb}}).$$

The above statements hold with  $\Theta$  replaced by  $\text{Chc}$  (see (8)). □

**Proof.** The first part is clear from Lemma 6.2. Furthermore, in terms of the Weyl calculus as in [18],

$$\begin{aligned} \int_{\tilde{G}} \Theta(g'g)L(z)\psi(g) \, d\mu(g) &= T(g') \natural T(L(z)\psi)(0) = T(g') \natural T(z) \natural T(\psi)(0) \\ &= T(g') \natural T(\mathcal{C}_{\mathfrak{g},g'}(z)) \natural T(\psi)(0) = T(\mathcal{C}_{\mathfrak{g},g'}(z)) \natural T(g') \natural T(\psi)(0) \\ &= L(\mathcal{C}_{\mathfrak{g},g'}(\check{z}))(T(g') \natural T(\psi))(0) = L(\mathcal{C}_{\mathfrak{g},g'}(\check{z})) \int_{\tilde{G}} \Theta(g'g)\psi(g) \, d\mu(g), \end{aligned}$$

where the third equality follows from [19, 6.12]. ■

### 6.2 Recapitulation of some results of Harish-Chandra

Let  $H$  be a Cartan subgroup of a real reductive group  $G$ . Let  $\mathfrak{h}$  and  $\mathfrak{g}$  denote the Lie algebras of  $H$  and  $G$ , respectively. Fix a positive root system for the pair  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ .

For a root  $\alpha$ , let  $\mathfrak{g}_{\alpha} \subseteq \mathfrak{g}_{\mathbb{C}}$  denote the corresponding  $\mathfrak{h}_{\mathbb{C}}$ -eigenspace. Let  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \subseteq \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  denote the center of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ .

**Theorem 6.4** ([9, Lemma 36], [10, Lemma 18]). For each element  $z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ , there is a unique element  $\gamma'(z) \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$  such that  $z - \gamma'(z) \in \sum_{\alpha > 0} \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{g}_{\alpha}$ . Moreover, the map

$$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \ni z \rightarrow \gamma'(z) \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$$

is an injective algebra homomorphism. □

Let  $W$  be the Weyl group for the pair  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Let  $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ . Let  $\lambda$  denote the following automorphism of  $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$

$$\lambda(x) = x - \rho(x) \quad (x \in \mathfrak{h}_{\mathbb{C}}).$$

Define

$$\gamma = \lambda \circ \gamma'.$$

**Theorem 6.5** ([10, Lemma 19, Lemma 20]). The map  $\gamma : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathcal{U}(\mathfrak{h}_{\mathbb{C}})^W$  is a surjective algebra isomorphism. This map does not depend on the choice of the positive root system. If  $\sigma$  is an automorphism of  $\mathfrak{g}_{\mathbb{C}}$  which preserves  $\mathfrak{h}_{\mathbb{C}}$ , then

$$\gamma(\sigma(z)) = \sigma(\gamma(z)) \quad (z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})).$$

If  $z \rightarrow \check{z}$  denote the anti-automorphism of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  defined by  $\check{z} = -z$  if  $z \in \mathfrak{g}_{\mathbb{C}}$ , then

$$\gamma(\check{z}) = \check{\gamma(z)} \quad (z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})). \quad \square$$

For a root  $\alpha$ , let  $\eta_{\alpha}$  denote the corresponding character of  $H$ :

$$(\exp(x))^{\alpha} = \exp(\alpha(x)) \quad (x \in \mathfrak{h}).$$

(We assume that these characters exist.) Set

$$\Delta'(h) = \prod_{\alpha > 0} (1 - h^{-\alpha}) \quad (h \in H).$$

Define  $S_c(H^{\text{reg}})$  to be the space of all smooth functions  $f$  on  $H^{\text{reg}}$ , such that the support of  $f$  is bounded and for every  $z \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ ,

$$\sup_{h \in H^{\text{reg}}} |L(z) f(h)| < \infty. \quad (57)$$

Define a topology on  $S_c(H^{\text{reg}})$  by taking all the quantities (57) as seminorms. Let

$$\tilde{I}_G(f)(h) = \Delta'(h) \int_{G/H} f(ghg^{-1}) d\mu(gH) \quad (h \in H^{\text{reg}}, f \in \mathcal{D}(G)). \quad (58)$$

**Theorem 6.6** ([11, Theorems 2 and 3]). For a function  $f \in \mathcal{D}(G)$ ,  $\tilde{I}_G(f) \in S_c(H^{\text{reg}})$ . Moreover, the map

$$\mathcal{D}(G) \ni f \rightarrow \tilde{I}^G(f) \in S_c(H^{\text{reg}})$$

is continuous. Furthermore,

$$\tilde{I}^G(z.f) = \gamma'(z).\tilde{I}^G(f) \quad (z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}), f \in \mathcal{D}(G)).$$

(Here we identify the  $z$  with  $L(z)$ ) □

For  $h \in H$  let

$$\begin{aligned} \Delta(h) &= h^\rho \Delta'(h), \\ \Delta'_\mathbb{R}(h) &= \prod_{\alpha > 0, \alpha \text{ real}} (1 - \eta_\alpha(h^{-1})), \\ \epsilon_\mathbb{R}(h) &= \text{sign}(\Delta'_\mathbb{R}(h)) \end{aligned} \tag{59}$$

**Theorem 6.7** ([12, Section 22]). For a function  $f \in \mathcal{D}(G)$  and for  $h \in H^{\text{reg}}$  define

$$\check{I}^G(f)(h) = \epsilon_\mathbb{R}(h) \Delta(h) \int_{G/H} f(g.h) \, d\dot{g}.$$

Then

$$\check{I}^G(z.f) = \gamma(z). \check{I}^G(f) \quad (z \in \mathcal{Z}, f \in \mathcal{D}(G)). \quad \square$$

**Remark.** In Section 1, we defined an unnormalized orbital integral denoted  $I^G$  which is not bounded. The normalized orbital integrals  $\tilde{I}^G$  and  $\check{I}^G$  are bounded. The last one is the original Harish-Chandra orbital integral.  $\square$

If needed, we shall write  $\epsilon_\mathbb{R}^G$  and  $\Delta^G$ , in order to indicate the group with respect to which these functions are defined. Consider a parabolic subalgebra  $\mathfrak{q} \subset \mathfrak{g}$ , with the Langlands decomposition

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

From now on we assume that the Cartan subalgebra  $\mathfrak{h}$  is  $\theta$ -stable and that  $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{a}$ . Let  $\mathcal{Z}(\mathfrak{m}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C})$  denote the center of the universal enveloping algebra of the complexification of  $\mathfrak{m} \oplus \mathfrak{a}$ . As in [12, Section 12] set

$$\nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} = \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}} : \mathcal{Z}(\mathfrak{g}_\mathbb{C}) \rightarrow \mathcal{Z}(\mathfrak{m}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C}). \tag{60}$$

Let  $Q$  be the parabolic subgroup of  $G$  such that  $\text{Lie}(Q) = \mathfrak{m} \oplus \mathfrak{a}$  and  $L$  the Levi factor of  $Q$ . We have  $\text{Lie}(L) = \mathfrak{m} \oplus \mathfrak{a}$ . For  $f \in \mathcal{D}(G)$ , denote by  $f^Q$  the Harish-Chandra transform of  $f$  (see [4, (0.4)]).

**Theorem 6.8** ([13, Corollary 2, p. 94, and Corollary of Lemma 14, p. 96]). For any function  $f \in \mathcal{D}(G)$  and any  $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ ,

$$(z.f)^L = \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}(z) f^L.$$

Moreover,

$$\check{I}^G(f) = \check{I}^L(f^L) \quad (f \in \mathcal{D}(G)). \quad \square$$

Recall, [9, p. 117], the Harish-Chandra radial component map  $\delta'_{G/H}$  from the algebra of analytic differential operators on  $G$  to the algebra of analytic differential operators on  $H^{\text{reg}}$ . For any analytic  $\text{Ad}(G)$ -invariant differential operator  $D$  on  $G$ ,  $\delta'_{G/H}(D)$  is the unique analytic differential operator on  $H^{\text{reg}}$  such that

$$D\psi|_{H^{\text{reg}}} = \delta'_{G/H}(D)(\psi|_{H^{\text{reg}}}) \quad (\psi \in C^\infty(G.[H^{\text{reg}}])^G),$$

(see [21, Proposition 6, p. 225]). Let  $\pi_{G/H}$  denote any analytic square root of the determinant

$$H^{\text{reg}} \ni h \rightarrow \det(\text{Ad}(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}} \in \mathbb{C}.$$

Then

$$\delta'_{G/H}(L(z)) = \pi_{G/H}^{-1} L(\gamma_{\mathfrak{g}/\mathfrak{h}}(z)) \pi_{G/H} \quad (z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G) \quad (61)$$

(see [10, Theorem 2, p. 125; 12, Lemma 13, p.466]).

### 6.3 Chc and the Capelli Harish-Chandra homomorphism

Let  $(G, G') \subseteq \text{Sp}(W)$  be a dual pair with the rank of  $G'$  less than or equal than to the rank of  $G$ . Recall the Cartan subgroup  $H' \subseteq G'$  and the parabolic subgroups  $Q' \subseteq G'$ ,  $Q \subseteq G$ . We identify  $\mathfrak{h}'$  with a subspace of  $\mathfrak{g}$ , as in [19, Proposition 1.14]. Let  $\mathfrak{z} \subseteq \mathfrak{g}$  be the centralizer of  $\mathfrak{h}'$  and let  $Z \subseteq G$  be the normalizer of  $\mathfrak{z}$  in  $G$ . If the pair  $(G, G')$  is of type I, then there is a nondegenerate subspace  $U^0 \subseteq U$  such that  $\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{z}''$ , with  $\mathfrak{z}'' = \mathfrak{g}(U^0)$ . If the pair  $(G, G')$  is of type II, then  $\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{z}''$ , with  $\mathfrak{z}'' = \mathfrak{gl}(U)$ . Let  $Z'' = G(U^0)$  in the first case, and let  $Z'' = \text{GL}(U)$  in the second case. Then

$$\mathcal{U}(\mathfrak{z}_{\mathbb{C}})^Z = \mathcal{U}(\mathfrak{h}'_{\mathbb{C}})^{W'} \otimes \mathcal{U}(\mathfrak{z}''_{\mathbb{C}})^{Z''},$$

where  $W'$  is the appropriate Weyl group for  $G'$ . Let

$$\epsilon_{\mathfrak{z}''} : \mathcal{U}(\mathfrak{z}''_{\mathbb{C}}) \rightarrow \mathbb{C}$$

be the algebra homomorphism by which  $\mathcal{U}(\mathfrak{z}''_{\mathbb{C}})$  acts on the trivial representation of  $\mathfrak{z}''$ , unless  $G'$  is an orthogonal group of type  $B$  (i.e., the defining module of  $G'$  has odd dimension). In this case,  $Z''$  is a symplectic group and  $\epsilon_{\mathfrak{z}''}$  is the infinitesimal character of the

oscillator representation. Choose a Cartan subalgebra  $\mathfrak{h}'' \subseteq \mathfrak{z}''$ . Then  $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$  is a Cartan subalgebra of  $\mathfrak{g}$ . Recall the Capelli Harish-Chandra homomorphism, [19, equality (5.5)],

$$C_{\mathfrak{g}, \mathfrak{g}'} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G \longrightarrow \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}, \quad C_{\mathfrak{g}, \mathfrak{g}'} = \gamma_{\mathfrak{g}'/\mathfrak{h}'}^{-1} \circ (1 \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}}. \tag{62}$$

Similarly, in the type I case we have,

$$C_{\mathfrak{g}(\mathbf{U}), \mathfrak{g}(V_c)} : \mathcal{U}(\mathfrak{g}(\mathbf{U})_{\mathbb{C}})^{G(\mathbf{U})} \rightarrow \mathcal{U}(\mathfrak{g}(V_c)_{\mathbb{C}})^{G(V_c)}.$$

Also, let  $\mathfrak{h}'_s = \mathfrak{h}'|_{X'} \subseteq \mathfrak{gl}(X')$  and let  $\mathfrak{h}'_c = \mathfrak{h}'|_{V_c} \subseteq \mathfrak{g}(V_c)$ . Then

$$\mathcal{U}(\mathfrak{h}'_{\mathbb{C}})^{W'} = \mathcal{U}(\mathfrak{h}'_{s, \mathbb{C}})^{W'_s} \otimes \mathcal{U}(\mathfrak{h}'_{c, \mathbb{C}})^{W'_c}, \tag{63}$$

where  $W'_s$  (resp.  $W'_c$ ) is the Weyl group of  $\mathfrak{h}'_{s, \mathbb{C}}$  (resp.  $\mathfrak{h}'_{c, \mathbb{C}}$ ).

**Lemma 6.9.** Suppose the pair  $(G, G')$  is of type I. Then

$$\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g}, \mathfrak{g}'} = (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{g}(V_c)/\mathfrak{h}'_c} \circ C_{\mathfrak{g}(\mathbf{U}), \mathfrak{g}(V_c)}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}.$$

If the pair  $(G, G')$  is of type II, then

$$\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g}, \mathfrak{g}'} = (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \epsilon_{\mathfrak{z}''}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}. \quad \square$$

**Proof.** From (62) we see that

$$\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g}, \mathfrak{g}'} = (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}},$$

and similarly,

$$\gamma_{\mathfrak{g}(V_c)/\mathfrak{h}'_c} \circ C_{\mathfrak{g}(\mathbf{U}), \mathfrak{g}(V_c)} = (1_c \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}(\mathbf{U})/\mathfrak{h}(\mathbf{U})}^{-1} \circ \gamma_{\mathfrak{g}(\mathbf{U})/\mathfrak{h}(\mathbf{U})},$$

where  $\mathfrak{z}(\mathbf{U}) = \mathfrak{g}(\mathbf{U}) \cap \mathfrak{z} = \mathfrak{h}'_c \oplus \mathfrak{z}''$  and  $1_c$  stands for the identity on  $\mathcal{U}(\mathfrak{h}'_{c, \mathbb{C}})^{W'_c}$ .

Let us write  $1 = 1_s \otimes 1_c$  for the identity, according to the decomposition (63). Then,

$$\begin{aligned}
 \gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g},\mathfrak{g}'} &= (1 \otimes \epsilon_3'') \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}} \\
 &= (1_s \otimes (1_c \otimes \epsilon_3'')) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\
 &= (1_s \otimes (1_c \otimes \epsilon_3'')) \circ \left( 1_s \otimes \gamma_{\mathfrak{z}(\mathbb{U})/\mathfrak{h}(\mathbb{U})}^{-1} \right) \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\
 &= \left( 1_s \otimes (1_c \otimes \epsilon_3'') \circ \gamma_{\mathfrak{z}(\mathbb{U})/\mathfrak{h}(\mathbb{U})}^{-1} \right) \circ \left( \gamma_{\mathfrak{gl}(\mathbb{X}')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{g}(\mathbb{U})/\mathfrak{h}(\mathbb{U})} \right) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\
 &= \left( \gamma_{\mathfrak{gl}(\mathbb{X}')/\mathfrak{h}'_s} \otimes (1_c \otimes \epsilon_3'') \circ \gamma_{\mathfrak{z}(\mathbb{U})/\mathfrak{h}(\mathbb{U})}^{-1} \circ \gamma_{\mathfrak{g}(\mathbb{U})/\mathfrak{h}(\mathbb{U})} \right) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\
 &= \left( \gamma_{\mathfrak{gl}(\mathbb{X}')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{g}(\mathbb{V}_c)/\mathfrak{h}'_c} \circ \mathcal{C}_{\mathfrak{g}(\mathbb{U}),\mathfrak{g}(\mathbb{V}_c)} \right) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}.
 \end{aligned}$$

This verifies the first equality. The second one is simpler:

$$\begin{aligned}
 \gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g},\mathfrak{g}'} &= (1 \otimes \epsilon_3'') \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\
 &= (1 \otimes \epsilon_3'') \circ (1 \otimes \gamma_{\mathfrak{z}''/\mathfrak{h}''})^{-1} \circ (\gamma_{\mathfrak{gl}(\mathbb{X}')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{z}''/\mathfrak{h}''}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\
 &= (1 \otimes \epsilon_3'') \circ (\gamma_{\mathfrak{gl}(\mathbb{X}')/\mathfrak{h}'_s} \otimes 1) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\
 &= (\gamma_{\mathfrak{gl}(\mathbb{X}')/\mathfrak{h}'_s} \otimes \epsilon_3'') \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}. \quad \blacksquare
 \end{aligned}$$

**Proof of Theorem 1.3.** Consider first a dual pair  $(G, G')$  of type I. Let  $\psi \in \mathcal{D}(\tilde{G})$  be supported in a connected set of a single sheet of the covering map  $\tilde{G} \rightarrow G$ . Our task is to show that for all  $x' \in H^{\text{reg}}$ ,

$$\int_{\tilde{G}} \text{Chc}(x'g)L(z)\psi(g) \, d\mu(g) = \delta'_{G'/H'}(L(\mathcal{C}_{\mathfrak{g},\mathfrak{g}'}(z))) \int_{\tilde{G}} \text{Chc}(x'g)\psi(g) \, d\mu(g). \tag{64}$$

Let  $A'$  (resp  $T'$ ) be the split (resp. compact) part of  $H'$ . Put, as before,

$$V'_c = \{v \in V' \mid a.v = 0 \, \forall a \in A'\}.$$

Then, there exists a unique complement  $V'_s$  of  $V'_c$  in  $V'$  such that the decomposition

$$V' = V'_c \oplus V'_s$$



is preserved by  $H'$ . The space  $V'_c$  has a complete polarization

$$V'_s = X' \oplus Y'.$$

Let  $x' \in \tilde{H}^{\text{reg}}$ . We put  $x' = x'_c x'_s$  according to the equality  $\tilde{H} = \tilde{T}\tilde{A}$ . We can assume that  $V'_s$  is contained in  $V$  (otherwise  $\text{Chc}$  is trivial) and we have a decomposition

$$V = V'_s \perp \oplus U.$$

Recall that  $W_c = \text{Hom}(V'_c, U)$ . In [4, Theorem 9], three cases are considered. The first two are simpler than the last one; therefore, we assume that  $\mathfrak{h}'$  does not act trivially on  $W_c$ . We have

$$\begin{aligned} & \epsilon_{\mathbb{R}}^{\text{GL}(X')} (x'_s) \Delta^{\text{GL}(X')} (x'_s) |\det(\text{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det((\text{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'})| \int_{\tilde{G}} \text{Chc}_{W_c}(x'g) \psi(g) \, d\mu(g) \\ &= C_0 \epsilon(x'_s) \int_{\text{GL}(X')/H'_{X'}} \int_{\tilde{G}(U)} \epsilon_{\mathbb{R}}^{\text{GL}(X')} (x'_s) \Delta^{\text{GL}(X')} (x'_s) \epsilon(\partial x'_s Y_s) \\ & \cdot \text{Chc}_{W_c}(x'_c Y_c) \psi^{\tilde{L}}(h.(\partial x'_s) Y) \, d\mu(Y) \, d\mu(hH'_{X'}), \end{aligned} \tag{65}$$

where the constant  $C_0$  is explicitly known. Since  $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{gl}(X') \oplus \mathfrak{g}(U)$ , there are finitely many elements  $s_i \in \mathcal{U}(\mathfrak{gl}(X')_{\mathbb{C}})^{\text{GL}(X')}$ ,  $c_i \in \mathcal{U}(\mathfrak{g}(U)_{\mathbb{C}})^{G(U)}$ , such that

$$v_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}(z) = \sum_i s_i \otimes c_i.$$

Then, by Theorem 6.8,

$$(L(z)\psi)^{\tilde{L}} = \sum_i (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}.$$

Hence, if we replace  $\psi$  by  $L(z)\psi$  on the right-hand side of (65), we obtain

$$\begin{aligned} & C_0 \epsilon(x'_s) \sum_i \int_{\text{GL}(X')/H'_{X'}} \int_{\tilde{G}(U)} \epsilon_{\mathbb{R}}^{\text{GL}(X')} (x'_s) \Delta^{\text{GL}(X')} (x'_s) \epsilon(\partial x'_s Y_s) \text{Chc}_{W_c}(x'_c Y_c) \\ & \cdot (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(h.(\partial x'_s) Y) \, d\mu(Y) \, d\mu(hH'_{X'}). \end{aligned} \tag{66}$$

We may rewrite (66) as follows

$$\sum_i \int_{\tilde{\text{GL}}(X')} \int_{\tilde{G}(U)} \epsilon(\partial x'_s Y_s) \check{I}_{\partial x'_s}^{\tilde{\text{GL}}(X')} (x) \text{Chc}_{W_c}(x'_c Y_c) (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(x' Y) \, d\mu(Y) \, d\mu(x). \tag{67}$$

By Theorems 6.3 and 6.7, (67) coincides with

$$\begin{aligned}
 C_0 \in (X'_s) \sum_i \int_{\widetilde{GL}(X')} \int_{\widetilde{G}(U)} L(\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_{X'}}(\check{s}_i)) (\in \mathfrak{d}X'_s Y_s) \check{I}_{\mathfrak{d}X'_s}^{\widetilde{GL}(X')}(x) \\
 \cdot \delta'_{\widetilde{G}(V_c)/\widetilde{H}'_c}(L(C_{\mathfrak{g}(U), \mathfrak{g}(V_c)}(\check{c}_i)))(\text{Chc}_{W_c}(x'_c Y_c)) \psi^{\check{L}}(x' y) d\mu(y) d\mu(x).
 \end{aligned} \tag{68}$$

The reason why we have  $\check{s}_i$  in (68) rather than  $s_i$ , as in (67), is that  $s_i \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  in (67) and  $s_i \in \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})$  in (68). More precisely, on the one hand,  $GL(X') \subseteq G'$  acts on the symplectic space  $W = \text{Hom}(V, V')$  via the post-multiplication by the inverse, and on the other,  $GL(X')$  is identified with a subgroup of  $G$  via the embedding (38). This inverse forces the transition  $s_i \rightarrow \check{s}_i$ .

The formula (61) implies that (68) is equal to

$$\begin{aligned}
 C_0 \left( 1 \otimes \frac{1}{\pi_{\widetilde{G}(V_c)/\widetilde{H}'_c}} \right) L \left( \sum_i (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_{X'}}(\check{s}_i)) \otimes \gamma_{\mathfrak{g}(V_c)/\mathfrak{h}'_c} \circ C_{\mathfrak{g}(U), \mathfrak{g}(V_c)}(\check{c}_i) \right) (1 \otimes \pi_{\widetilde{G}(V_c)/\widetilde{H}'_c}) \\
 \cdot \int_{\widetilde{GL}(X')} \int_{\widetilde{G}(U)} \check{I}_{\mathfrak{d}X'_s}^{\widetilde{GL}(X')}(x) \text{Chc}_{W_c}(x'_c Y_c) \psi^{\check{L}}(xy) d\mu(y) d\mu(x).
 \end{aligned} \tag{69}$$

Then, Lemma 6.9 shows that (69) coincides with

$$\begin{aligned}
 C_0 \left( 1 \otimes \frac{1}{\pi_{\widetilde{G}(V_c)/\widetilde{H}'_c}} \right) L(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g}, \mathfrak{g}'}(\check{z})) (1 \otimes \pi_{\widetilde{G}(V_c)/\widetilde{H}'_c}) \\
 \cdot \int_{\widetilde{GL}(X')} \int_{\widetilde{G}(U)} \check{I}_{\mathfrak{d}X'_s}^{\widetilde{GL}(X')}(x') \text{Chc}_{W_c}(x'_c Y_c) \psi^{\check{L}}(xy) d\mu(y) d\mu(x).
 \end{aligned} \tag{70}$$

Hence, by (65),

$$\begin{aligned}
 \left( 1 \otimes \frac{1}{\pi_{\widetilde{G}(V_c)/\widetilde{H}'_c}} \right) L(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g}, \mathfrak{g}'}(\check{z})) (1 \otimes \pi_{\widetilde{G}(V_c)/\widetilde{H}'_c}) \\
 \cdot \epsilon_{\mathbb{R}}^{\text{GL}(X')}(X'_s) \Delta^{\text{GL}(X')}(X'_s) |\det(\text{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det((\text{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'})| \int_{\widetilde{G}} \text{Chc}_W(x'g) \psi(g) d\mu(g) \\
 = \epsilon_{\mathbb{R}}^{\text{GL}(X')}(X'_s) \Delta^{\text{GL}(X')}(X'_s) |\det(\text{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det((\text{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'})| \\
 \cdot \int_{\widetilde{G}} \text{Chc}_W(x'g) (L(z) \psi(g)) d\mu(g)
 \end{aligned} \tag{71}$$

Note that

$$\Delta^{\text{GL}(X')}(X'_s) |\det(\text{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det((\text{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'})| (1 \otimes \pi_{\widetilde{G}(V_c)/\widetilde{H}'_c})(x') = \Delta^{G'}(x').$$

Hence, (71) may be rewritten as

$$\begin{aligned} & \frac{1}{\epsilon_{\mathbb{R}}^{\text{GL}(X')} (x'_s) \Delta^{G'}(x')} L(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g}, \mathfrak{g}'}(\check{z})) \epsilon_{\mathbb{R}}^{\text{GL}(X')} (x'_s) \Delta^{G'}(x') \int_{\tilde{G}} \text{Chc}_W(x'g) \psi(g) \, d\mu(g) \\ &= \int_{\tilde{G}} \text{Chc}_W(x'g) (L(z)\psi(g)) \, d\mu(g). \end{aligned} \tag{72}$$

Since

$$\frac{1}{\epsilon_{\mathbb{R}}^{\text{GL}(X')} (x'_s) \Delta^{G'}(x')} (\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g}, \mathfrak{g}'}(\check{z})) \epsilon_{\mathbb{R}}^{\text{GL}(X')} (x'_s) \Delta^{G'}(x') = \delta'_{\tilde{G}/\tilde{H}'}(C_{\mathfrak{g}, \mathfrak{g}'}(\check{z})),$$

we are done.

We consider now a dual pair  $(G, G')$  of type II. Again, we need to verify the equality (64). From [4, Theorem 6.5], we see that

$$\begin{aligned} & \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}} \text{Chc}_W(x'g) \psi(g) \, d\mu(g) \\ &= C_0 \int_{G'/H'} \int_{\tilde{\text{GL}}(\mathbb{U})} \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \psi_{\tilde{\alpha}}(x.(\eta x')u) \, d\mu(u) \, d\mu(xH'), \end{aligned} \tag{73}$$

where  $C_0$  is an explicitly known constant. Since  $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}' \oplus \mathfrak{gl}(\mathbb{U})$ , there are finitely many elements  $s_i \in \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}$ ,  $c_i \in \mathcal{U}(\mathfrak{gl}(\mathbb{U})_{\mathbb{C}})^{\text{GL}(\mathbb{U})}$ , such that

$$\nu_{\mathfrak{g}'/\mathfrak{m} \oplus \mathfrak{a}}(z) = \sum_i s_i \otimes c_i.$$

Then, by Theorem 6.8,

$$(L(z)\psi)^{\tilde{L}} = \sum_i (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}.$$

Hence, if we replace  $\psi$  by  $L(z)\psi$  on the right-hand side of (73), we obtain

$$C_0 \sum_i \int_{G'/H'} \int_{\tilde{\text{GL}}(\mathbb{U})} \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(x.(\partial x'_s)u) \, d\mu(u) \, d\mu(xH'). \tag{74}$$

We may rewrite (62) as follows

$$\sum_i \int_{\tilde{G}} \int_{\tilde{\text{GL}}(\mathbb{U})} \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \epsilon(\partial x'_s I_{\partial x'_s})(x) (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(xu) \, d\mu(u) \, d\mu(x). \tag{75}$$

By Theorem 6.7, (75) is equal to

$$\sum_i \int_{\tilde{G}} \int_{\tilde{GL}(U)} L(\gamma_{\mathfrak{g}'/\mathfrak{h}'}(\check{s}_i)) (\epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') I_{\mathfrak{d}x'}(x)) \epsilon_{\check{s}_i}(\check{C}_i) \psi^{\tilde{L}}(xu) \, d\mu(u) \, d\mu(x). \tag{76}$$

Lemma 6.9(ii) shows that (76) coincides with

$$L(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g},\mathfrak{g}'}(\check{Z})) \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}} \int_{\tilde{GL}(U)} I_{\mathfrak{d}x'}(x) \psi^{\tilde{L}}(xu) \, d\mu(u) \, d\mu(x). \tag{77}$$

Hence, by (73),

$$\begin{aligned} & L(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g},\mathfrak{g}'}(\check{Z})) \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}} \text{Chc}_W(x'g) \psi(g) \, d\mu(g) \\ &= \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}} \text{Chc}_W(x'g) (L(z) \psi(g)) \, d\mu(g). \end{aligned}$$

Since

$$\frac{1}{\epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x')} (\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ C_{\mathfrak{g},\mathfrak{g}'}(\check{Z})) \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') = \delta'_{\tilde{G}/\tilde{H}'}(C_{\mathfrak{g},\mathfrak{g}'}(\check{Z})),$$

we are done. ■

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### References

- [1] Adams, J. "Lifting of characters on orthogonal and metaplectic groups." *Duke Mathematical Journal* 92, no. 1 (1998): 129–78.
- [2] Bernon, F. "Propriétés de l'intégrale de Cauchy Harish-Chandra pour les paires duales d'algèbres de Lie réductives." Thèse, Université de Poitiers, 2001.
- [3] Bernon, F. and T. Przebinda. "Boundedness of the Cauchy Harish-Chandra integral." *Journal of Lie Theory* 11, no. 3 (2011): 499–613.
- [4] Bernon, F. and T. Przebinda. "Normalization of the Cauchy Harish-Chandra integral." *Journal of Lie Theory* 11, no. 3 (2011): 615–702.
- [5] Bouaziz, A. "Intégrales orbitales sur les algèbres de Lie réductives." *Inventiones Mathematicae* 115, no. 1 (1994): 163–207.
- [6] Bouaziz, A. "Intégrales orbitales sur les groupes de Lie réductifs." *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série* 27, no. 5 (1994): 573–609.
- [7] Collingwood, D. and W. McGovern. *Nilpotent Orbits in Complex Semisimple Lie Algebras*. New York: Reinhold, Van Nostrand, 1993.

- [8] Daszkiewicz, A. and T. Przebinda. "The oscillator character formula, for isometry groups of split forms in deep stable range." *Inventiones Mathematicae* 123, no. 2 (1996): 349–76.
- [9] Harish-Chandra. "On some applications of the universal enveloping algebra of a semisimple Lie algebra." *Transactions of the American Mathematical Society* 70 (1951): 28–96.
- [10] Harish-Chandra. "The characters of semisimple Lie groups." *Transactions of the American Mathematical Society* 83 (1956): 98–163.
- [11] Harish-Chandra. "A formula for semisimple Lie groups." *American Journal of Mathematics* 79 (1957): 733–60.
- [12] Harish-Chandra. "Invariant eigendistributions on a semisimple Lie groups." *Transactions of the American Mathematical Society* 119 (1965): 457–508.
- [13] Harish-Chandra. "Two theorems on semisimple Lie groups." *Annals of Mathematics* 83 (1966): 74–128.
- [14] Hörmander, L. *The Analysis of Linear Partial Differential Operators I*. Berlin: Springer, 1983.
- [15] Howe, R. " $\theta$ -series and Invariant Theory." In *Automorphic Forms, Representations and L-Functions (Proceedings of Symposia in Pure Mathematics, Oregon State University, Corvallis, Oregon, 1977), Part 1*, 275–85. Proceedings of Symposia in Pure Mathematics XXXIII. Providence, RI: American Mathematical Society, 1979.
- [16] Howe, R. "The oscillator semigroup." *Proceedings of Symposia in Pure Mathematics, American Mathematical Society* 48 (1988): 61–132.
- [17] Howe, R. "Transcending classical invariant theory." *Journal of the American Mathematical Society* (2) 2 (1989): 535–52.
- [18] Przebinda, T. "A Cauchy Harish-Chandra integral, for a real reductive dual pair." *Inventiones Mathematicae* 141, no. 2 (2000): 299–363.
- [19] Przebinda, T. "A Capelli Harish-Chandra homomorphism." *Transactions of the American Mathematical Society* 356, no. 3 (2004): 1121–54.
- [20] Renard, D. "Transfer of orbital integrals between  $\mathrm{Mp}(2n, \mathbb{R})$  and  $\mathrm{SO}(n, n+1)$ ." *Duke Mathematical Journal* 95, no. 2 (1998): 125–450.
- [21] Varadarajan, V. S. *Harmonic Analysis on Real Reductive Groups*. Lecture Notes in Mathematics 576. Berlin: Springer, 1977.