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# Local geometry of orbits for an ordinary classical Lie supergroup

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**Abstract:** In this paper we identify a real reductive dual pair of Roger Howe with an Ordinary Classical Lie supergroup. In these terms we describe the semisimple orbits of the dual pair in the symplectic space, a slice through a semisimple element of the symplectic space, an analog of a Cartan subalgebra, the corresponding Weyl group and the corresponding Weyl integration formula.


*Keywords: Dual pairs, Lie supergroups, orbits, integration formulas*
*MSC (2000): 17B05, 17B75, 22E15*

## Introduction

The purpose of this article is to present a few elementary facts about the local structure of orbits in the symplectic space under the action of a real reductive dual pair, see [6] and [7]. We shall use this material later to study the characters of the representations which occur in Howe's correspondence. The corresponding facts for the adjoint action of a real reductive group on its Lie algebra is essentaily contained in section one (eleven pages) of part one of [11].

The main results are presented as quickly as possible, with the proofs deffered to further sections. These proofs, based on elementary linear algebra, are rather non-interesting, but had to be included.

Some of the material included here is contained in an unpublished work of Howe, [5]. However our approach through the Lie superalgebras seems more akin to the standard theory, [11].

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# 1 A slice through a point

Let $M$ be a manifold and let $G$ be a Lie group acting on $M$. Let $x \in M$ and let $G^x$ be the stabilizer of $x$ in $G$. Assume that the orbit $Gx \subseteq M$ is a regularly embedded submanifold.

A connected submanifold $U \subseteq M$ is called an *admissible* slice through $x$ if and only if

$$x \in U, \tag{1.1}$$
$$U \text{ is } G^x - \text{stable}, \tag{1.2}$$
$$\text{the tangent space } T_x(M) = T_x(U) \oplus T_x(Gx), \tag{1.3}$$
$$\text{if } g \in G \text{ and } u, u' \in U \text{ are such that } gu = u' \text{ then } g \in G^x, \tag{1.4}$$
$$\text{the map } G \times U \ni (g,u) \to gu \in M \text{ is a submersion.} \tag{1.5}$$

The condition (1.3) implies that the map

$$\mu : GU \ni gu \to gx \in Gx \tag{1.6}$$

is well defined. As shown in [11, part I, pages 15, 16], $\mu$ is a locally trivial fibration with the fiber $U$. In other words, for every point $gx \in Gx$ there is an open neighborhood $W \subseteq Gx$, and a diffeomorphism $\phi$ such that the following diagram commutes:

$$\begin{array}{ccc} W \times U & \xrightarrow{\phi} & \mu^{-1}(W) \\ \downarrow & & \mu\downarrow \\ W & \xrightarrow{=} & W, \end{array} \tag{1.7}$$

where the left vertical arrow is the projection on the first component.

Let $N \subseteq M$ be a complete metric subspace. Suppose $N$ is the union of a finite set of $G$-orbits. Then, as shown in [12, 8.A.4.5], we can label the orbits $\mathcal{O}_1, \mathcal{O}_2, ...\mathcal{O}_k$ so that for $1 \le j \le k$ the set

$$N_j = \bigcup_{l=j}^{k} \mathcal{O}_l \tag{1.8}$$

is closed in $N$. Suppose $x \in \mathcal{O}_j$ for some $1 \le j \le k$. A connected manifold $U \subseteq M$ is called a *weakly admissible* slice through $x$ if and only if the conditions (1.0), (1.2), (1.5) hold and

$$\begin{array}{l}\text{the intersection of the image of the map (1.5) with } N_j \\ \text{is equal to } \mathcal{O}_j,\end{array} \tag{1.9}$$

and

$$U \cap \mathcal{O}_j = \{x\}. \tag{1.10}$$

## 2 Ordinary classical Lie supergroups and dual pairs

Let $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $V_0$, $V_1$ be two finite dimensional left vector spaces over $\mathbb{D}$. Set

$$V = V_0 \oplus V_1 \tag{2.1}$$

and define an element $S \in End(V)$ by

$$S(v_0 + v_1) = v_0 - v_1 \qquad (v_o \in V_0,\ v_1 \in V_1). \tag{2.2}$$

Set

$$\begin{aligned} End(V)_0 &= \{x \in End(V);\ Sx = xS\}, \\ End(V)_1 &= \{x \in End(V);\ Sx = -xS\}, \\ GL(V)_0 &= GL(V) \cap End(V)_0. \end{aligned} \tag{2.3}$$

The real vector space $End(V)_0$ is a Lie algebra, with the usual commutator $[x, y] = xy - yx$. The adjoint action of $GL(V)_0$ on $End(V)$

$$Ad(g)x = gxg^{-1} \qquad (g \in GL(V)_0,\ x \in End(V))$$

preserves both $End(V)_0$ and $End(V)_1$. Furthermore the anticommutator

$$End(V)_1 \times End(V)_1 \ni (x, y) \to \{x, y\} = xy + yx \in End(V)_0 \tag{2.4}$$

is $\mathbb{R}$-bi-linear and $GL(V)_0$-equivariant. Set

$$\langle x, y\rangle = tr_{\mathbb{D}/\mathbb{R}}\{Sx, y\} \qquad (x, y \in End(V)). \tag{2.4'}$$

(Here $tr_{\mathbb{D}/\mathbb{R}}(y)$ is the trace of $y \in End(V)$ viewed as an endomorphism of $V$ over $\mathbb{R}$.) It is easy to see that the form $\langle\ ,\ \rangle$ is preserved under the action of $GL(V)_0$.

**Lemma 2.1.** *The restriction of the bilinear form $\langle\ ,\ \rangle$ to $End(V)_1$ is symplectic and non-degenerate. Moreover, the group homomorphism*

$$Ad : G \to Sp(End(V)_1, \langle\ ,\ \rangle)$$

*maps the groups*

$$G_0 = \{g \in GL(V)_0;\ g|_{V_1} = 1\},\ \textit{and}\ G_1 = \{g \in GL(V)_0;\ g|_{V_0} = 1\}$$

*injectively onto an irreducible dual pair of type II in the symplectic group $Sp(End(V)_1, \langle\ ,\ \rangle)$.*

**Proof.** The following map

$$\begin{aligned} &Hom(V_0, V_1) \oplus Hom(V_1, V_0) \ni (A, B) \to x_{A,B} \in End(V)_1 \\ &x_{A,B}(v_0 + v_1) = Bv_1 + Av_0 \qquad (v_0 \in V_0,\ v_1 \in V_1), \end{aligned} \tag{2.5}$$

is an $\mathbb{R}$-linear bijection. Furthermore, for $v_0 \in V_0$ and $v_1 \in V_1$, and for any $A, A' \in Hom(V_0, V_1)$ and $B, B' \in Hom(V_1, V_0)$ we have

$$x_{A,B}\, x_{A',B'}(v_0 + v_1) = BA'v_0 + AB'v_1,$$

and therefore

$$S(x_{A,B}\, x_{A',B'} - x_{A',B'}\, x_{A,B})(v_0 + v_1) = (BA' - B'A)v_0 + (A'B - AB')v_1.$$

Hence,

$$\begin{aligned} \langle x_{A,B}, x_{A',B'} \rangle &= tr_{\mathbb{D}/\mathbb{R}}(Sx_{A,B}x_{A',B'} + x_{A',B'}Sx_{A,B}) \\ &= tr_{\mathbb{D}/\mathbb{R}}(S(x_{A,B}x_{A',B'} - x_{A',B'}x_{A,B})) \\ &= tr_{\mathbb{D}/\mathbb{R}}(BA' - B'A) + tr_{\mathbb{D}/\mathbb{R}}(A'B - AB') \\ &= 2tr_{\mathbb{D}/\mathbb{R}}(BA' - B'A). \end{aligned} \tag{2.6}$$

Thus the form $\langle\ ,\ \rangle$ is symplectic. It is easy to check that if $tr_{\mathbb{D}/\mathbb{R}}(BA' - B'A) = 0$ for all $A'$ and $B'$ then $A = 0$ and $B = 0$. Thus the form $\langle\ ,\ \rangle$ is non-degenerate.

The groups $G_0$ and $G_1$ are isomorphic to $GL(V_0)$ and $GL(V_1)$ by restriction. Further, the action of the groups $G_0$ and $G_1$ on $Hom(V_0, V_1)$ induced by the isomorphism (2.5), embeds these groups into $GL(Hom(V_0, V_1))$. It is not hard to check that $G_0$ and $G_1$ are mutual centralizers in $GL(Hom(V_0, V_1))$, and hence form a dual pair of type II in the symplectic group. $\square$

Let $\iota$ be a possibly trivial involution on $\mathbb{D}$. Let $\tau_0$ be a non-degenerate $\iota$-hermitian form on $V_0$, and let $\tau_1$ be a non-degenerate $\iota$-skew-hermitian form on $V_1$. Set $\tau = \tau_0 \oplus \tau_1$. Then

$$\tau(u, v) = \iota(\tau(v, Su)) \qquad (u, v \in V). \tag{2.7}$$

Define

$$\begin{aligned} \mathfrak{g}(V, \tau)_0 &= \{x \in End(V)_0;\ \tau(xu, v) = \tau(u, -xv),\ u, v \in V\}, \\ \mathfrak{g}(V, \tau)_1 &= \{x \in End(V)_1;\ \tau(xu, v) = \tau(u, Sxv),\ u, v \in V\}, \\ G(V, \tau)_0 &= \{g \in GL(V)_0;\ \tau(gu, gv) = \tau(u, v),\ u, v \in V\}. \end{aligned} \tag{2.8}$$

Clearly, $G(V, \tau)_0$ is a Lie subgroup of $GL(V)_0$, with the Lie algebra $\mathfrak{g}(V, \tau)_0$. Moreover, it is easy to check that the anticommutator (2.4) maps $\mathfrak{g}(V, \tau)_1 \times \mathfrak{g}(V, \tau)_1$ into $\mathfrak{g}(V, \tau)_0$. Furthermore, the adjoint action of $G(V, \tau)_0$ preserves $\mathfrak{g}(V, \tau)_0$, $\mathfrak{g}(V, \tau)_1$, and the form $\langle\ ,\ \rangle$.

**Lemma 2.2.** *The restriction of the bilinear form $\langle\ ,\ \rangle$ to $\mathfrak{g}(V, \tau)_1$ is symplectic and non-degenerate. Moreover,*

$$Ad : G(V, \tau)_0 \to Sp(\mathfrak{g}(V, \tau)_1, \langle\ ,\ \rangle)$$

*maps the groups*

$$G_0 = \{g \in G(V, \tau)_0;\ g|_{V_1} = 1\}, \text{ and } G_1 = \{g \in G(V, \tau)_0;\ g|_{V_0} = 1\}$$

*injectively onto an irreducible dual pair of type I in the symplectic group $Sp(\mathfrak{g}(V, \tau)_1, \langle\ ,\ \rangle)$.*

**Proof.** Recall the map

$$\begin{aligned}&End(V_0)\ni A\to A^\sharp\in End(V_0),\\ &\tau_0(Au_0,v_0)=\tau_0(u_0,A^\sharp v_0)\qquad (A\in End(V_0)).\end{aligned}$$

Let $v_1, v_2, v_3, ...$ be a basis of $V_0$, and let $v_1', v_2', v_3', ...$ be the dual basis, in the sense that $\tau_0(v_i, v_j') = \delta_{ij}$. (Here $\delta_{ij}$ is the Kronecker delta: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.) Then for $A \in End(V)_0$

$$\sum_i \tau_0(Av_i, v_i') = \sum_i \tau_0(v_i, A^\sharp v_i') = \sum_i \iota(\tau_0(A^\sharp v_i', v_i)).$$

Thus

$$tr_{\mathbb{D}/\mathbb{R}}(A) = tr_{\mathbb{D}/\mathbb{R}}(A^\sharp)\qquad (A\in End(V_0)). \tag{2.9}$$

Define the following maps

$$\begin{aligned}&Hom(V_0,V_1)\ni w\to w^*\in Hom(V_1,V_0),\\ &\tau_1(wv_0,v_1)=\tau_0(v_0,w^*v_1)\qquad (v_0\in V_0,\ v_1\in V_1),\\ &Hom(V_1,V_0)\ni w\to w^{*\prime}\in Hom(V_0,V_1),\\ &\tau_0(wv_1,v_0)=\tau_1(v_1,w^{*\prime}v_0)\qquad (v_0\in V_0,\ v_1\in V_1).\end{aligned} \tag{2.10}$$

Then

$$\begin{aligned}\tau_1(wv_0,v_1)&=\tau_0(v_0,w^*v_1)=\iota(\tau_0(w^*v_1,v_0))\\ &=\iota(\tau_1(v_1,w^{**\prime}v_0))=\tau_1(-w^{**\prime}v_0,v_1).\end{aligned}$$

Thus (as is well known, [5])

$$w^{**\prime}=-w\qquad (w\in Hom(V_0,V_1)). \tag{2.11}$$

For $x \in \mathfrak{g}(V,\tau)_1$ let $w_x \in Hom(V_0, V_1)$ be the restriction of $x$ to $V_0$. Then

$$x(v_0+v_1)=w_x^*v_1+w_xv_0\qquad (v_0\in V_0,\ v_1\in V_1). \tag{2.12}$$

Since for $x, y \in \mathfrak{g}(V,\tau)_1$

$$\langle x,y\rangle = tr_{\mathbb{D}/\mathbb{R}}(S(xy-yx)),$$

the form $\langle\ ,\ \rangle$ is symplectic. Furthermore, by (2.12),

$$Sxy(v_0+v_1)=w_x^*w_yv_0-w_xw_y^*v_1\qquad (v_0\in V_0,\ v_1\in V_1).$$

Thus

$$S(xy-yx)(v_0+v_1)=(w_x^*w_y-w_y^*w_x)v_0+(w_yw_x^*-w_xw_y^*)v_1.$$

Hence,

$$\begin{aligned}\langle x,y\rangle&=tr_{\mathbb{D}/\mathbb{R}}(w_x^*w_y-w_y^*w_x)+tr_{\mathbb{D}/\mathbb{R}}(w_yw_x^*-w_xw_y^*)\\ &=2tr_{\mathbb{D}/\mathbb{R}}(w_x^*w_y)-2tr_{\mathbb{D}/\mathbb{R}}(w_xw_y^*).\end{aligned}$$

But, by (2.9), (2.10) and (2.11),

$$\begin{aligned} tr_{\mathbb{D}/\mathbb{R}}(w_x w_y^*) &= tr_{\mathbb{D}/\mathbb{R}}(w_y^* w_x) = tr_{\mathbb{D}/\mathbb{R}}((w_y^* w_x)^\sharp) \\ &= tr_{\mathbb{D}/\mathbb{R}}(w_x^* w_y^{**\prime}) = -tr_{\mathbb{D}/\mathbb{R}}(w_x^* w_y). \end{aligned}$$

Thus

$$\langle x, y\rangle = 4tr_{\mathbb{D}/\mathbb{R}}(w_x^* w_y) \qquad (x, y \in \mathfrak{g}(V,\tau)_1). \tag{2.13}$$

As shown in [5], the right hand side of (2.13) defines a non- degenerate symplectic form on $Hom(V_0, V_1)$. Since (2.12) defines an $\mathbb{R}$-linear bijection

$$\mathfrak{g}(V,\tau)_1 \ni x \to w_x \in Hom(V_0, V_1), \tag{2.13}$$

the first part of the Lemma follows.

The groups $G_0$, $G_1$ defined in (b), are isomorphic to the isometry groups $G(V_0, \tau_0)$, $G(V_1, \tau_1)$, by restriction. As is well known, [5], these isometry groups form an irreducible dual pair of type I in the symplectic group on $Hom(V_0, V_1)$, equipped with the symplectic form defined by the right hand side of (2.13). □

**Definition 2.3.** An irreducible ordinary classical Lie supergroup is a pair $(G, \mathfrak{g})$ with $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where either

$$G = GL(V)_0,\ \mathfrak{g}_0 = End(V)_0,\ \mathfrak{g}_1 = End(V)_1,\ \text{as in (2.3)}, \tag{II}$$

or

$$G = G(V,\tau)_0,\ \mathfrak{g}_0 = \mathfrak{g}(V,\tau)_0,\ \mathfrak{g}_1 = \mathfrak{g}(V,\tau)_1,\ \text{as in (2.9)}. \tag{I}$$

The pair $(G, \mathfrak{g})$ is a supergroup of type II in the case (II) and of type I in the case (I). The space $V$ shall be called the defining module or the defining space for $(G, \mathfrak{g})$. If needed, we shall indicate this by writing $G = G(V)$ and $\mathfrak{g} = \mathfrak{g}(V)$.

For the general theory of Lie superalgebras and Lie supergroup see, [8] and [9].

The following Proposition is easy to verify.

**Proposition 2.4.** *The restriction of the form $\langle\ ,\ \rangle$, (see (2.4')), to $\mathfrak{g}_0$ is symmetric, non-degenerate and $G$-invariant. Furthermore, the spaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are orthogonal. If we identify $\mathfrak{g}$ with the dual $\mathfrak{g}^*$ by*

$$y(x) = \langle y, x\rangle \qquad (x, y \in \mathfrak{g}) \tag{a}$$

*then, for $x \in \mathfrak{g}_1$, the map*

$$\mathfrak{g}_1 \ni z \to \{x, z\} \in \mathfrak{g}_0 \tag{b}$$

*is adjoint to the map*

$$\mathfrak{g}_0 \ni y \to [x, y] \in \mathfrak{g}_1. \tag{c}$$

*In other words,*

$$\langle \{x, z\}, y\rangle = \langle z, [x, y]\rangle \qquad (y \in \mathfrak{g}_0,\ x, z \in \mathfrak{g}_1). \tag{d}$$

**Theorem 2.5.** *Let $(G, \mathfrak{g})$ be an irreducible ordinary classical Lie supergroup. Up to conjugation by $G$ there is exactly one automorphism $\theta$ of $\mathfrak{g}$ such that $\theta|_{\mathfrak{g}_0}$ is a Cartan involution on $\mathfrak{g}_0$ and $\theta|_{\mathfrak{g}_1}$ is a positive compatible complex structure on $\mathfrak{g}_1$.*

*The automorphism $\theta$ may be realized as follows. Let $V = V_0 \oplus V_1$ be the defining module for $(G, \mathfrak{g})$. Then there is a positive definite hermitian form $\eta$ on $V$ such that $V_0$ is orthogonal to $V_1$ with respect to $\eta$, and if $End(V) \ni x \to x^\dagger \in End(V)$ is the adjoint with respect to $\eta$, $(\eta(xu, v) = \eta(u, x^\dagger v))$, then*

$$\theta(x) = \begin{cases} -x^\dagger & \text{if } x \in \mathfrak{g}_0, \\ Sx^\dagger & \text{if } x \in \mathfrak{g}_1. \end{cases}$$

*Moreover, if the Lie supergroup $(G, \mathfrak{g})$ is of type I and $(\mathbb{D}, \iota) \neq (\mathbb{C}, 1)$, then there is an element $T \in G$, unique up to conjugation, such that $\eta(\ ,\ ) = \tau(T\ ,\ )$.*

**Proof.** The existence of $\theta$ is known (see, for example [3, 8.1, 10.2]). We shall verify the uniqueness.

Since $\theta$ is an automorphism of $\mathfrak{g}$, the restriction $\theta|_{\mathfrak{g}_1}$ is invertible in $End(\mathfrak{g}_1)$ and

$$[\theta y, x] = \theta[y, \theta^{-1}x] \qquad (y \in \mathfrak{g}_0,\ x \in \mathfrak{g}_1).$$

In other words,

$$ad(\theta y)|_{\mathfrak{g}_1} = (\theta|_{\mathfrak{g}_1})(ad(y)|_{\mathfrak{g}_1})(\theta|_{\mathfrak{g}_1})^{-1} \qquad (y \in \mathfrak{g}_0).$$

Suppose $\theta_1$ is another automorphism of $\mathfrak{g}$ such that $\theta_1|_{\mathfrak{g}_0}$ is a Cartan involution on $\mathfrak{g}_0$ and $\theta_1|_{\mathfrak{g}_1}$ is a positive compatible complex structure on $\mathfrak{g}_1$. Since the Cartan involution on $\mathfrak{g}_0$ is unique up to conjugation by a element of $G$, [12, 2.3.2], we may assume that $\theta_1|_{\mathfrak{g}_0} = \theta|_{\mathfrak{g}_0}$. Then for $y \in \mathfrak{g}_0$,

$$(\theta|_{\mathfrak{g}_1})(ad(y)|_{\mathfrak{g}_1})(\theta|_{\mathfrak{g}_1})^{-1} = ad(\theta y)|_{\mathfrak{g}_1} = ad(\theta_1 y)|_{\mathfrak{g}_1} = (\theta_1|_{\mathfrak{g}_1})(ad(y)|_{\mathfrak{g}_1})(\theta_1|_{\mathfrak{g}_1})^{-1}.$$

Hence,

$$(\theta|_{\mathfrak{g}_1})^{-1}(\theta_1|_{\mathfrak{g}_1}) \in Sp(\mathfrak{g}_1, \langle\ ,\ \rangle)^{ad(\mathfrak{g}_0)}.$$

(Here, $X^Y$ is the centralizer of $Y$ in $X$.) But we know from the structure of dual pairs, [7], that this last set is contained in (the centralizer of the identity component of $Ad(G)$ in) $Ad(G)$. □

For $i = 1, 2, 3, ..., n$, let $(G^{(i)}, \mathfrak{g}^{(i)})$ be irreducible ordinary classical Lie supergroups, not necessarily of the same type, with the defining modules $V^{(i)}$. The group

$$G = G^{(1)} \times G^{(2)} \times G^{(3)} \times ... \times G^{(n)}$$

and the Lie superalgebra

$$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \oplus ... \oplus \mathfrak{g}^{(n)}$$

act on the vector space

$$V = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus \ldots \oplus V^{(n)}$$

componentwise. The resulting pair $(G, \mathfrak{g})$ shall be called the direct product of the ordinary classical Lie supergroups $(G^{(i)}, \mathfrak{g}^{(i)})$, with the defining module $V$.

An ordinary classical Lie supergroup $(G, \mathfrak{g})$ is a finite direct product of irreducible ordinary classical Lie supergroups, as defined above. Notice that the group $G$ corresponds to the unique reductive dual pair obtained via the action of $G$ on the symplectic space $\mathfrak{g}_1$. This correspondence is bijective.

## 3 The tangent space to the $G$-orbit through a point $x \in \mathfrak{g}_1$

Let $(G, \mathfrak{g})$ be an irreducible ordinary classical Lie supergroup and let $x \in \mathfrak{g}$. Since the derivative of the adjoint action of the group $G$ on $\mathfrak{g}$ coincides with the adjoint action of the Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}$, we may identify the tangent space to $Gx$ at $x$ with

$$[\mathfrak{g}_0, x] = \{[y, x];\ y \in \mathfrak{g}_0\}. \tag{3.1}$$

For $x \in \mathfrak{g}_1$ let

$${}^x\mathfrak{g}_1 = \{z \in \mathfrak{g}_1;\ \{x, z\} = 0\}. \tag{3.2}$$

This is the anticommutant of $x$ in $\mathfrak{g}_1$. For any subset $W \subseteq \mathfrak{g}_1$ let

$$W^{\perp} = \{y \in \mathfrak{g}_1;\ \langle y, z\rangle = 0 \text{ for all } z \in W\}. \tag{3.3}$$

Since our symplectic form $\langle\ ,\ \rangle$ is non-degenerate, we have

$$W^{\perp\perp} = W, \tag{3.4}$$

if $W$ is a vector subspace of $\mathfrak{g}_1$.

**Lemma 3.1.** *For any $x \in \mathfrak{g}_1$ we have $[\mathfrak{g}_0, x] = ({}^x\mathfrak{g}_1)^{\perp}$.*

**Proof.** Let $z \in \mathfrak{g}_1$. Then by (2.4.d), $z \in [\mathfrak{g}_0, x]^{\perp}$ if and only if $\{x, z\} = 0$. Thus $[\mathfrak{g}_0, x]^{\perp} = {}^x\mathfrak{g}_1$, and our claim follows from (3.4). □

Let $\theta$ be as in the Theorem 2.5. Then the formula

$$(x, y) = -\langle x, \theta y\rangle \qquad (x, y \in \mathfrak{g}) \tag{3.5}$$

defines a symmetric positive definite form on $\mathfrak{g}$. In particular, for any $x \in \mathfrak{g}_1$, the orthogonal complement to $[\mathfrak{g}_0, x]$ in $\mathfrak{g}_1$, with respect to the form (3.5), is equal to $\theta({}^x\mathfrak{g}_1)$.

The form (3.5) restricts to any subspace of $\mathfrak{g}$, and induces a positive definite form on the quotient of $\mathfrak{g}$ by any subspace.

Let $\mathcal{U}$, $\mathcal{V}$ be two vector spaces, over the reals, of the same dimension. Suppose $\mathcal{U}$, $\mathcal{V}$ are subspaces or quotients of $\mathfrak{g}$ (one could be a subspace and the other one the quotient).

Let $L : \mathcal{U} \to \mathcal{V}$ be a linear map. We define the absolute value of the determinant of $L$, $|det(L)|$, to be the absolute value of the determinant of the matrix of $L$ with respect to any orthonormal basis of $\mathcal{U}$, $\mathcal{V}$, with respect to the form induced by (3.5).

Fix $x \in \mathfrak{g}_1$. The derivative of the map

$$G/G^x \ni gG^x \to gx \in \mathfrak{g}_1 \tag{3.6}$$

at $x$ may be identified with the following linear map

$$\mathfrak{g}_0/\mathfrak{g}_0^x \ni y + \mathfrak{g}_0^x \to [y, x] \in [\mathfrak{g}_0, x]. \tag{3.7}$$

Denote by

$$J(x) \qquad (x \in \mathfrak{g}_1)$$

the absolute value of the determinant of the map (3.7). We shall give a formula for the function $J(x)$, for $x \in \mathfrak{g}_1$ semisimple, in Corollary 6.9.

## 4 A $G$-equivariant localization in $\mathfrak{g}_1$

In this section we state several theorems which shall be verified later.

As in the previous section, let $(G, \mathfrak{g})$ be an irreducible ordinary classical Lie supergroup and let $x \in \mathfrak{g}_1$. The element $x \in \mathfrak{g}_1$ is called semi-simple (or nilpotent) if and only if $x$ is semi-simple (or nilpotent) as an endomorphism of $V$.

**Theorem 4.1.** *Let $x \in \mathfrak{g}_1$ and let $x = x_s + x_n$ be the Jordan decomposition of $x$, as an element of $End(V)$. (Here $x_s$ stands for the semisimple part of $x$, and $x_n$ for the nilpotent part of $x$.) Then $x_s$ and $x_n$ belong to $\mathfrak{g}_1$. Furthermore, an element $y \in \mathfrak{g}_1$ anti-commutes with $x$ if and only if $y$ anti-commutes with $x_s$ and $x_n$.*

**Theorem 4.2.** *For any $x \in \mathfrak{g}_1$ the semisimple part $x_s$ belongs to $Cl(Gx)$, the closure of the orbit $Gx$.*

**Theorem 4.3.** *For any $x \in \mathfrak{g}_1$, $x$ is semisimple if and only if the orbit $Gx$ is closed.*

Notice that if $x \in \mathfrak{g}_1$, then $x^2 = \frac{1}{2}\{x, x\} \in \mathfrak{g}_0$. Let $G^{x^2} \subseteq G$ denote the centralizer of $x^2$, and let $\mathfrak{g}^{x^2}$, $\mathfrak{g}_0^{x^2}$, $\mathfrak{g}_1^{x^2}$ denote the centralizer of $x^2$ in $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{g}_1$ respectively.

**Theorem 4.4.** *Let $x \in \mathfrak{g}_1$ be semisimple.*
*(a) Suppose $ker(x) = 0$. Then $(G^{x^2}, \mathfrak{g}^{x^2})$ is the direct product of the irreducible ordinary classical Lie supergroups, with the corresponding dual pairs isomorphic either to $(U_n, U_n)$ or to $(GL_n(\mathbb{D}), GL_n(\mathbb{D}))$, where the division algebra $\mathbb{D}$ may be different than the division algebra over which the defining module $V$ for the supergroup $(G, \mathfrak{g})$ was defined.*

*The restriction of the symplectic form $\langle\ ,\ \rangle$ to $\mathfrak{g}_1^{x^2}$ is non-degenerate and*

$$\mathfrak{g}_1^{x^2} = {}^x\mathfrak{g}_1 \oplus \mathfrak{g}_1^x$$

*is a complete polarization. (Here* $\mathfrak{g}_1^x = \{y \in \mathfrak{g}_1;\ xy = yx\}$*.)*
*(b) Let* $V^0 = ker(x)$ *and let* $V^+ = xV$*. Then*

$$V = V^0 \oplus V^+$$

*is a direct sum (orthogonal in the type I case) decomposition into graded non-zero subspaces preserved by* $x$*. Moreover,*

$$G^{x^2} = G(V^0) \times G(V^+)^{x^2}$$

*and*

$$\mathfrak{g}_1{}^{x^2} = \mathfrak{g}_1(V^0) \oplus \mathfrak{g}_1(V^+)^{x^2},$$

*where the sum is orthogonal, with respect to the symplectic form* $\langle\ ,\ \rangle$*, and* $\mathfrak{g}_1(V^0) = 0$ *unless* $V^0 \cap V_0 \neq 0$ *and* $V^0 \cap V_1 \neq 0$*. Moreover, the double anticommutant of* $x$ *in* $\mathfrak{g}_1$ *coincides with the double commutant of* $x$ *in* $\mathfrak{g}_1(V^+)$*:*

$$^{({}^x\mathfrak{g}_1)}\mathfrak{g}_1 = \mathfrak{g}_1(V^+)^{(\mathfrak{g}_1(V^+)^x)}.$$

*(c) The maximal possible dimension of the real vector space* $^{({}^x\mathfrak{g}_1)}\mathfrak{g}_1$ *is equal to the minimum of the rank of* $G(V_0)$ *and the rank of* $G(V_1)$*, viewed as real reductive Lie groups. For the* $x$ *such that the dimension of* $^{({}^x\mathfrak{g}_1)}\mathfrak{g}_1$ *is maximal, we have* $V^0 \subseteq V_0$ *or* $V^0 \subseteq V_1$*, so that* $\mathfrak{g}_1(V^0) = 0$*, and*

$$^{({}^x\mathfrak{g}_1)}\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^x)}.$$

An explicit description of the double anticommutant $^{({}^x\mathfrak{g}_1)}\mathfrak{g}_1$ will be given in the proof of the theorem (see (13.13.1), (13.22.1), (13.31.1), (13.42.1), (13.47.1), (13.47.2), and (13.53.1)).

**Theorem 4.5.** *Let* $x \in \mathfrak{g}_1$ *be semisimple. Then* $\mathfrak{g}_1^x$ *has a basis for the* $G^x$*-invariant neighborhoods of* $x$ *consisting of admissible slices* $U_x$ *through* $x$*, such that for* $i = 0, 1$*, the map*

$$U_x \ni y \to y^2|_{V_i} \in \mathfrak{g}_0(V_i)^{x^2}$$

*is an (injective) immersion, (see [10] Vol 1, for the definition of an immersion.)*

**Theorem 4.6.** *[3] The set of nilpotent* $G$*-orbits in* $\mathfrak{g}_1$ *is finite.*

**Theorem 4.7.** *Let* $x \in \mathfrak{g}_1$ *be nilpotent and let* $W \subseteq \mathfrak{g}_1$ *be a subspace such that*

$$\mathfrak{g}_1 = [\mathfrak{g}_0, x] \oplus W.$$

*Then the affine space* $x + W \subseteq \mathfrak{g}_1$ *has a basis for the neighborhoods of* $x$ *consisting of weakly admissible slices through* $x$*.*

The following theorem has a substantial overlap with the Proposition 8.2 in [5]

**Theorem 4.8.** *The map*

$$\mathfrak{g}_1 \supseteq Gx \to Gx^2 \subseteq \mathfrak{g}_0$$

*is injective on the set of semisimple orbits. (Here, in order to simplify the notation we write $Gx$ rather than $Ad(G)x$, and similarly for $Gx^2$.)*

## 5 The $G$-orbits in $\mathfrak{g}_1$

We retain the notation of the previous section. An element $x \in \mathfrak{g}_1$, or the pair $(x, V)$, is called *decomposable* if and only if there are two non-zero $\mathbb{Z}/2\mathbb{Z}$-graded subspaces $V', V'' \subseteq V$, (which are orthogonal if $(G, \mathfrak{g})$ is of type I), preserved by $x$ and such that $V = V' \oplus V''$. In this case we say that $(x, V)$ is the direct sum of the elements $(x|_{V'}, V')$ and $(x|_{V''}, V'')$. The element $(x, V)$ is called *indecomposable* if and only if $(x, V)$ is not decomposable.

**Theorem 5.1.** *For any $x \in \mathfrak{g}_1$, $(x, V)$ is the direct sum of indecomposable elements.*

**Proof (a reduction to the case when $x$ is semisimple).** Let $x = x_s + x_n$ be the Jordan decomposition of $x$. Then, as we know from Theorem 4.1, $x_n \in \mathfrak{g}_1$. Suppose $x_n \neq 0$. There is a decomposition

$$V = V^1 \oplus V^2 \oplus V^3 \oplus ...$$

into indecomposables with respect to $x_n$, [3, sections 5 and 6]. Since $x$ commutes with $x_n$, each $V^j$ is $x$-invariant, and indecomposable with respect to $x$ (because $x_n$ is a polynomial of $x$). Thus we may assume that $x$ is semisimple. We shall consider this case and complete the argument in sections 8 and 9. □

Let $(G, \mathfrak{g})$, $(G', \mathfrak{g}')$ be two irreducible ordinary classical Lie supergroups with the defining spaces $V$, $V'$ respectively. We'll say that two elements $x \in \mathfrak{g}_1$, $x' \in \mathfrak{g}'_1$ are *similar* if and only if the supergroups $(G, \mathfrak{g})$, $(G', \mathfrak{g}')$ are of the same type and there is a $\mathbb{Z}/2\mathbb{Z}$-graded linear bijection $\phi : V \to V'$ (an isometry in the type I case) such that $x = \phi^{-1} x' \phi$. In particular if $V = V'$ and $(G, \mathfrak{g}) = (G', \mathfrak{g}')$ then $x$ is similar to $x'$ if and only if $x$ and $x'$ are in the same $G$-orbit. In that case we shall write $x \approx x'$.

**Theorem 5.2.** *Let $(G, \mathfrak{g})$ be of type I. The following is a complete list of all non-zero semisimple indecomposable elements $(x, V)$, $x \in \mathfrak{g}_1$, up to similarity. In each case we indicate which elements of the list are similar, describe an element $g \in G$ which provides the similarity, and list the eigenvalues of $x$.*

$$
\begin{aligned}
&V_0 = \mathbb{D}v_0 \oplus \mathbb{D}v_0',\ V_1 = \mathbb{D}v_1 \oplus \mathbb{D}v_1';\\
&\tau(v_0, v_0) = \tau(v_0', v_0') = \tau(v_1, v_1) = \tau(v_1', v_1') = 0,\ \tau(v_0, v_0') = \tau(v_1, v_1') = 1;\\
&\textit{if } \iota = 1 \textit{ then let } \xi \in \mathbb{D} \setminus 0,\\
&\textit{if } \iota \neq 1 \textit{ then let } \xi \in \mathbb{C} \subseteq \mathbb{D} \textit{ and } \xi^2 \notin i\mathbb{R};\\
&x = x(\xi) : v_0 \to \xi v_1,\ v_1 \to \xi v_0,\ v_0' \to -\iota(\xi) v_1',\ v_1' \to \iota(\xi) v_0';\\
&\textit{if } \mathbb{D} = \mathbb{R} \textit{ then } x(\xi) \approx x(-\xi) \textit{ has eigenvalues } \xi, i\xi, -\xi, -i\xi;\\
&\textit{if } \mathbb{D} = \mathbb{C} \textit{ and } \iota = 1 \textit{ then } x(\xi) \approx x(i\xi) \approx x(-\xi) \approx x(-i\xi) \textit{ has eigenvalues}\\
&\xi, i\xi, -\xi, -i\xi;\\
&\textit{if } \mathbb{D} = \mathbb{C} \textit{ and } \iota \neq 1 \textit{ then } x(\xi) \approx x(-\xi) \textit{ has eigenvalues}\\
&\xi, i\iota(\xi), -\xi, -i\iota(\xi);\\
&\textit{if } \mathbb{D} = \mathbb{H} \textit{ then}\\
&x(\xi) \approx x(-\xi) \approx x(\iota(\xi)) \approx x(-\iota(\xi))\\
&\textit{has eigenvalues } \xi, \iota(\xi), -\xi, -\iota(\xi), i\xi, i\iota(\xi), -i\xi, -i\iota(\xi);\\
&\textit{for all } \mathbb{D},\ gx(\xi)g^{-1} = x(-\xi) \textit{ if } g : v_0 \to -v_0, v_1 \to v_1, v_0' \to -v_0', v_1' \to v_1';\\
&\textit{for } \mathbb{D} = \mathbb{C} \textit{ and } \iota = 1,\ gx(\xi)g^{-1} = x(i\xi) \textit{ if}\\
&g : v_0 \to -iv_0', v_1 \to -v_1', v_0' \to iv_0, v_1' \to v_1;\\
&\textit{for } \mathbb{D} = \mathbb{H},\ gx(\xi)g^{-1} = x(\iota(\xi)) \textit{ if}\\
&g : v_0 \to jv_0', v_1 \to jv_1', v_0' \to jv_0, v_1' \to jv_1;
\end{aligned}
\tag{a}
$$

$$
\begin{aligned}
&V_0 = \mathbb{D}v_0, V_1 = \mathbb{D}v_1,\ \mathbb{C} \subseteq \mathbb{D},\ \iota \neq 1;\\
&\tau(v_0, v_0) = \epsilon = \pm 1, \tau(v_1, v_1) = \delta i = \pm i;\\
&\xi^2 \in i\mathbb{R} \setminus 0, sgn(im(\xi^2)) = -\epsilon\delta;\\
&x = x(\xi) : v_0 \to \xi v_1, v_1 \to \xi v_0;\\
&x(\xi) \approx x(-\xi) \textit{ has eigenvalues } \xi, -\xi;\\
&gx(\xi)g^{-1} = x(-\xi) \textit{ if } g : v_0 \to -v_0, v_1 \to v_1;
\end{aligned}
\tag{b}
$$

$$
\begin{aligned}
&V_0 = \mathbb{R}v_0 \oplus \mathbb{R}v_0',\ V_1 = \mathbb{R}v_1 \oplus \mathbb{R}v_1',\ \mathbb{D} = \mathbb{R};\\
&\tau(v_0, v_0) = \tau(v_0', v_0') = \epsilon = \pm 1, \tau(v_1, v_1) = \tau(v_1', v_1') = 0,\\
&\tau(v_0, v_0') = 0, \tau(v_1, v_1') = 1;\\
&\xi \in \mathbb{R} \setminus 0;\\
&x = x(\xi) : v_0 \to \xi(v_1 - \epsilon v_1'),\ v_1 \to \xi(v_0 - v_0'),\\
&v_0' \to \xi(v_1 + \epsilon v_1'), v_1' \to \epsilon\xi(v_0 + v_0');\\
&x(\xi) \approx x(-\xi) \textit{ has eigenvalues } \xi(1-i), \xi(1+i), -\xi(1-i), -\xi(1+i);\\
&gx(\xi)g^{-1} = x(-\xi) \textit{ if } g : v_0 \to -v_0, v_1 \to v_1, v_0' \to -v_0', v_1' \to v_1';
\end{aligned}
\tag{c}
$$

$$
\begin{aligned}
&V_0 = (\mathbb{R}u_0 \oplus \mathbb{R}v_0) \oplus (\mathbb{R}u_0' \oplus \mathbb{R}v_0'),\\
&V_1 = (\mathbb{R}u_1 \oplus \mathbb{R}v_1) \oplus (\mathbb{R}u_1' \oplus \mathbb{R}v_1'),\ \mathbb{D} = \mathbb{R},\\
&\textit{the spaces in parenthesis are isotropic, and}\\
&\tau(u_0, u_0') = \tau(v_0, v_0') = \tau(u_1, u_1') = \tau(v_1, v_1') = 1;\\
&\xi, \eta \in \mathbb{R},\ \xi^2 \neq \eta^2, \xi\eta \neq 0;\\
&x = x(\xi, \eta) :\\
&u_0 \to \xi u_1 + \eta v_1, u_1 \to \xi u_0 + \eta v_0,\\
&v_0 \to -\eta u_1 + \xi v_1, v_1 \to -\eta u_0 + \xi v_0,\\
&u_0' \to -\xi u_1' + \eta v_1', u_1' \to \xi u_0' - \eta v_0',\\
&v_0' \to -\eta u_1' - \xi v_1', v_1' \to \eta u_0' + \xi v_0';\\
&x(\xi, \eta) \approx x(-\xi, \eta) \approx x(\xi, -\eta) \approx x(-\xi, -\eta)\\
&\ \approx x(\eta, \xi) \approx x(-\eta, \xi) \approx x(\eta, -\xi) \approx x(-\eta, -\xi)\\
&\textit{has eigenvalues } \xi + i\eta, \xi - i\eta, -\xi + i\eta, -\xi - i\eta,\\
&\eta + i\xi, -\eta + i\xi, \eta - i\xi, -\eta - i\xi;\\
&gx(\xi, \eta)g^{-1} = x(-\xi, \eta) \textit{ if } g : u_0 \to -u_0, v_0 \to v_0, u_0' \to -u_0', v_0' \to v_0',\\
&u_1 \to u_1, v_1 \to -v_1, u_1' \to u_1', v_1' \to -v_1';\\
&gx(\xi, \eta)g^{-1} = x(\eta, \xi) \textit{ if } g : u_0 \to u_0', v_0 \to v_0', u_0' \to u_0, v_0' \to v_0,\\
&u_1 \to v_1', v_1 \to -u_1', u_1' \to -v_1, v_1' \to u_1;
\end{aligned}
\tag{d}
$$

**Theorem 5.3.** *Let $(G, \mathfrak{g})$ be of type II. The following is a complete list of all non-zero semisimple indecomposable elements $(x, V)$, $x \in \mathfrak{g}_1$, up to similarity. In each case we indicate which elements of the list are similar, describe an element $g \in G$ which provides the similarity, and list the eigenvalues of $x$.*

$$
\begin{aligned}
&V_0 = \mathbb{D}v_0,\ V_1 = \mathbb{D}v_1;\\
&\xi \in \mathbb{D} \setminus 0;\\
&x = x(\xi) : v_0 \to \xi v_1, v_1 \to \xi v_0;\\
&\textit{if } \mathbb{D} \neq \mathbb{H} \textit{ then } x(\xi) \approx x(-\xi) \textit{ has eigenvalues } \xi, -\xi;\\
&\textit{if } \mathbb{D} = \mathbb{H} \textit{ then } x(\xi) \approx x(-\xi) \approx x(\iota(\xi)) \approx x(-\iota(\xi))\\
&\textit{has eigenvalues } \xi, -\xi, \iota(\xi), -\iota(\xi);\\
&gx(\xi)g^{-1} = x(-\xi) \textit{ if } g : v_0 \to -v_0, v_1 \to v_1;\\
&gx(\xi)g^{-1} = x(\iota(\xi)) \textit{ if } \mathbb{D} = \mathbb{H} \textit{ and } g : v_0 \to jv_0, v_1 \to jv_1;
\end{aligned}
\tag{a}
$$

$$
\begin{aligned}
&V_0 = \mathbb{R}v_0,\ V_1 = \mathbb{R}v_1;\\
&\xi \in \mathbb{R} \setminus 0;\\
&x = x(\xi) : v_0 \to \xi v_1, v_1 \to -\xi v_0;\\
&x(\xi) \approx x(-\xi) \quad \textit{has eigenvalues } \pm i\xi;\\
&gx(\xi)g^{-1} = x(-\xi) \textit{ if } g : v_0 \to -v_0, v_1 \to v_1;
\end{aligned}
\tag{a'}
$$

$$
\begin{aligned}
&V_0 = \mathbb{R}u_0 \oplus \mathbb{R}v_0,\ \ V_1 = \mathbb{R}u_1 \oplus \mathbb{R}v_1, \mathbb{D} = \mathbb{R};\\
&\xi, \eta \in \mathbb{R} \setminus 0;\\
&x = x(\xi, \eta) : u_0 \to \xi u_1 + \eta v_1, u_1 \to \xi u_0 + \eta v_0,\\
&v_0 \to -\eta u_1 + \xi v_1, v_1 \to -\eta u_0 + \xi v_0;\\
&x(\xi, \eta) \approx x(-\xi, \eta) \approx x(\xi, -\eta) \approx x(-\xi, -\eta)\\
&\textit{has eigenvalues } \xi + i\eta, \xi - i\eta, -\xi + i\eta, -\xi - i\eta;\\
&gx(\xi, \eta)g^{-1} = x(-\xi, \eta) \textit{ if } g : u_0 \to -u_0, v_0 \to v_0, u_1 \to u_1, v_1 \to -v_1;\\
&gx(\xi, \eta)g^{-1} = x(\xi, -\eta) \textit{ if } g : u_0 \to u_0, v_0 \to -v_0, u_1 \to u_1, v_1 \to -v_1.
\end{aligned} \tag{b}
$$

**Proof (of Theorem 4.1).** Let $x \in \mathfrak{g}_1$ and let $x = x_s + x_n$ be the Jordan decomposition of $x$, as an element $End(V)$. Suppose $(G, \mathfrak{g})$ is of type II. Notice that $Sx_sS^{-1}$ is semisimple and $Sx_nS^{-1}$ is nilpotent, and that these elements commute. Moreover,

$$Sx_sS^{-1} + Sx_nS^{-1} = SxS^{-1} = -x = -x_s - x_n.$$

Thus the uniqueness of the Jordan decomposition in $End(V)$ implies that $Sx_sS^{-1} = -x_s$ and $Sx_nS^{-1} = -x_n$. In other words, $x_s, x_n \in \mathfrak{g}_1$.

Suppose $(G, \mathfrak{g})$ is of type I. Then, as shown above, $x_s, x_n \in End(V)_1$. Consider the map

$$
\begin{aligned}
&End(V)_1 \ni y \to y^\sharp \in End(V)_1,\\
&\tau(yu, v) = \tau(u, y^\sharp v) \qquad (u, v \in V).
\end{aligned}
$$

Then $y$ is semisimple if and only if $y^\sharp$ is semisimple, and $y$ is nilpotent if and only if $y^\sharp$ is nilpotent. In particular, $x_s^\sharp$ is semisimple and $x_n^\sharp$ is nilpotent. The Theorem 5.1 (for semisimple elements) and Theorem 5.2 imply that $Sx_s^\sharp$ is semisimple. It is easy to check that $Sx_n^\sharp$ is nilpotent and that $Sx_s^\sharp Sx_n^\sharp = Sx_n^\sharp Sx_s^\sharp$. Since $x \in \mathfrak{g}_1$, we have, by the definition (2.8),

$$x = Sx^\sharp = Sx_s^\sharp + Sx_n^\sharp.$$

Thus the uniqueness of the Jordan decomposition in $End(V)$ implies

$$x_s = Sx_s^\sharp \text{ and } x_n = Sx_n^\sharp.$$

Hence, $x_s, x_n \in \mathfrak{g}_1$.

Let $y \in \mathfrak{g}_1$. If $y$ anticommutes with $x_s$ and $x_n$ then clearly $y$ anticommutes with $x = x_s + x_n$.

Conversely, suppose $yx = -xy$. Then, with $S$ as in (2.2), the following computation holds in $End(V)$:

$$x(Sy) = -Sxy = (Sy)x.$$

Thus $x$ commutes with $Sy$. Therefore $x_s$ and $x_n$ commute with $Sy$. Hence, for $z = x_s$ or $x_n$,

$$zy = zS^2y = -Sz(Sy) = -S(Sy)z = -yz.$$

In other words, $x_s$ and $x_n$ anti-commute with $y$. □

Let

$$\delta(k) = (-1)^{k(k-1)/2} = \begin{cases} 1 \text{ if } k \in 4\mathbb{Z} \text{ or } 4\mathbb{Z}+1, \\ -1 \text{ if } k \in 4\mathbb{Z}+2 \text{ or } 4\mathbb{Z}+3. \end{cases} \tag{5.1'}$$

**Theorem 5.4.** *[3] Let $(G, \mathfrak{g})$ be of type I. The following is a complete list of all non-zero nilpotent indecomposable elements $(x, V)$, $x \in \mathfrak{g}_1$, up to similarity.*

$$\begin{aligned}
&m \in 4\mathbb{Z};\\
&V = \sum_{k=0}^{m} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_k = x^k v_0 \neq 0, 0 \leq k \leq m, xv_m = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m-k, \tau(v_k, v_{m-k}) = \delta(k)\delta(\frac{m}{2})sgn(\tau_0),\\
&\textit{where}\ sgn(\tau_0) = 1\ \textit{if}\ \mathbb{D} = \mathbb{C}\ \textit{and}\ \iota = 1;
\end{aligned} \tag{a}$$

$$\begin{aligned}
&m \in 4\mathbb{Z}, \mathbb{D} \neq \mathbb{R}, \iota \neq 1;\\
&V = \sum_{k=1}^{m+1} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_{k+1} = x^k v_1 \neq 0, 0 \leq k \leq m, xv_{m+1} = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m+2-k, \tau(v_k, v_{m+2-k}) = \delta(k)\tau(v_1, v_{m+1}),\\
&\tau(v_1, v_{m+1}) = i\, sgn(-i\tau_1)\delta(1+\frac{m}{2}) \ \textit{if}\ \mathbb{D} = \mathbb{C};\\
&\tau(v_1, v_{m+1}) = j \ \textit{if}\ \mathbb{D} = \mathbb{H};
\end{aligned} \tag{b}$$

$$\begin{aligned}
&m \in 4\mathbb{Z}, \mathbb{D} \neq \mathbb{H}, \iota = 1;\\
&V = \sum_{k=1}^{m+1} (\mathbb{D}v_k \oplus \mathbb{D}v_k'),\ \ v_{even}, v_{even}' \in V_0,\ \ v_{odd}, v_{odd}' \in V_1;\\
&v_{k+1} = x^k v_1 \neq 0, v_{k+1}' = x^k v_1' \neq 0, 0 \leq k \leq m, xv_{m+1} = 0, xv_{m+1}' = 0;\\
&\tau(v_k, v_l) = \tau(v_k', v_l') = 0,\ \ 1 \leq k, l \leq m+1,\\
&\tau(v_k, v_l') = \tau(v_k', v_l) = 0,\ \ l \neq m+2-k,\\
&\tau(v_k, v_{m+2-k}') = -\tau(v_k', v_{m+2-k}) = \delta(k),\ \ 1 \leq k \leq m+1;
\end{aligned} \tag{c}$$

$$\begin{aligned}
&m \in 2\mathbb{Z} \setminus 4\mathbb{Z}, \mathbb{D} \neq \mathbb{R}, \iota \neq 1;\\
&V = \sum_{k=0}^{m} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_k = x^k v_0 \neq 0, 0 \leq k \leq m, xv_m = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m-k, \tau(v_k, v_{m-k}) = \delta(k) i sgn(-i\tau_1),\\
&\textit{(here}\ -i\tau_1\ \textit{is hermitian)};
\end{aligned} \tag{d}$$

$$\begin{aligned}
&m \in 2\mathbb{Z} \setminus 4\mathbb{Z};\\
&V = \sum_{k=1}^{m+1} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_{k+1} = x^k v_1 \neq 0, 0 \leq k \leq m, xv_{m+1} = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m+2-k, \tau(v_k, v_{m+2-k}) = \delta(k)\tau(v_1, v_{m+1}),\\
&\tau(v_1, v_{m+1}) = \delta(1+\frac{m}{2}) sgn(\tau_0);
\end{aligned} \tag{e}$$

$$\begin{aligned}
&m \in 2\mathbb{Z} \setminus 4\mathbb{Z}, \mathbb{D} \neq \mathbb{H}, \iota = 1;\\
&V = \sum_{k=0}^{m} (\mathbb{D}v_k \oplus \mathbb{D}v'_k),\ \ v_{even}, v'_{even} \in V_0,\ \ v_{odd}, v'_{odd} \in V_1;\\
&v_k = x^k v_0 \neq 0, v'_k = x^k v'_0 \neq 0, 0 \leq k \leq m, xv_m = 0, xv'_m = 0;\\
&\tau(v_k, v_l) = \tau(v'_k, v'_l) = 0,\ \ 0 \leq k, l \leq m,\\
&\tau(v_k, v'_l) = \tau(v'_k, v_l) = 0,\ \ l \neq m-k,\\
&\tau(v_k, v'_{m-k}) = -\tau(v'_k, v_{m-k}) = \delta(k),\ \ 0 \leq k \leq m;
\end{aligned} \tag{f}$$

$$\begin{aligned}
&m \in 2\mathbb{Z}+1;\\
&V = \sum_{k=0}^{m} (\mathbb{D}v_k \oplus \mathbb{D}v'_{k+1}),\ \ v_{even}, v'_{even} \in V_0,\ \ v_{odd}, v'_{odd} \in V_1;\\
&v_k = x^k v_0 \neq 0, v'_{k+1} = x^k v'_1 \neq 0, 0 \leq k \leq m, xv_m = 0, xv'_{m+1} = 0;\\
&\tau(v_k, v_l) = \tau(v'_{k+1}, v'_{l+1}) = 0,\ \ 0 \leq k, l \leq m,\\
&\tau(v_k, v'_{l+1}) = \tau(v'_{k+1}, v_l) = 0,\ \ l \neq m-k,\\
&\tau(v_k, v'_{m+1-k}) = \delta(k)(-1)^k, \tau(v'_{k+1}, v_{m-k}) = \delta(k)\delta(m),\ \ 0 \leq k \leq m;
\end{aligned} \tag{g}$$

The following theorem is well known, and goes back to Jordan. See [3] for details.

**Theorem 5.5.** *Let $(G, \mathfrak{g})$ be of type II. The following is a complete list of all non-zero nilpotent indecomposable elements $(x, V)$, $x \in \mathfrak{g}_1$, up to similarity.*

$$\begin{aligned}
&V = \sum_{k=0}^{m} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_k = x^k v_0 \neq 0, 0 \leq k \leq m, xv_m = 0;
\end{aligned} \tag{a}$$

$$\begin{aligned}
&V = \sum_{k=1}^{m+1} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_{k+1} = x^k v_1 \neq 0, 0 \leq k \leq m, xv_{m+1} = 0;
\end{aligned} \tag{b}$$

For a nilpotent element $x \in \mathfrak{g}_1$ the *height* of $x$, or the height of $(x, V)$, is the integer $m \geq 0$ such that $x^m \neq 0$ and $x^{m+1} = 0$. In particular $x = 0$ if and only if the height of $x$ is 0. The pair $(x, V)$ is called *uniform* if $ker(x^m) = im(x)\ (= x(V))$. These notions are adopted from [1], and have been used in [3].

Let $x \in \mathfrak{g}_1$ and let $x = x_s + x_n$ be the Jordan decomposition of $x$. Let

$$V = V^{(1)} \oplus V^{(2)} \oplus ... \tag{5.2}$$

be the decomposition of $V$ into a direct (and orthogonal in the type I case) sum, such that each $(x_n, V^{(j)})$ is uniform (see [3]). Then each $V^{(j)}$ is preserved by $x_s$. As one can see from the proof of Theorems 5.12 and 6.1 in [3], there is a graded $x_s$- invariant subspace $F^{(j)}$ such that

$$V^{(j)} = F^{(j)} \oplus x_n F^{(j)} \oplus x_n^2 F^{(j)} \oplus ...,$$

where, in the type I case, $x_n^k F^{(j)} \perp x_n^l F^{(j)}$, for $k + l \leq m_j - 1$, and $m_j$ is the height of $(x_n, V^{(j)})$.

Since the $x_n$ and $x_s$ commute, the action of $x_s$ on $V^{(j)}$ is determined by the action on the $F^{(j)}$. This space is equipped with the form

$$\tau_{m_j,j}(u, v) = \tau(u, x_n^{m_j} v) \qquad (u, v \in F^{(j)}), \tag{5.3}$$

in the type I case. As an endomorphism of $F^{(j)}$, $x_s$ is of degree one (i.e. the restriction of $x_s$ to $F^{(j)}$ is in $End(F^{(j)})_1$) and

$$\tau_{m_j,j}(x_s u, v) = \tau_{m_j,j}(u, S x_s v)(-1)^{m_j} \qquad (u, v \in F^{(j)}), \tag{5.4}$$

If $m_j \in 4\mathbb{Z}$, then

$$\tau_{m_j,j} = \tau_{m_j,j}|_{F_0^{(j)}} \oplus \tau_{m_j,j}|_{F_1^{(j)}} \tag{5.5}$$

with $\tau_{m_j,j}|_{F_0^{(j)}}$ hermitian and $\tau_{m_j,j}|_{F_1^{(j)}}$ skew-hermitian.

If $m_j \in 2\mathbb{Z}\backslash 4\mathbb{Z}$, then (5.5) holds with $\tau_{m_j,j}|_{F_0^{(j)}}$ skew-hermitian and $\tau_{m_j,j}|_{F_1^{(j)}}$ hermitian. Thus for $m_j$ even we know how to decompose $(x_s, F^{(j)})$ into indecomposables.

Recall the function $\delta$, (5.1'). For $m_j \in 2\mathbb{Z} + 1$

$$\begin{aligned} &\tau_{m_j,j}(u, v) = -\delta(m_j)\iota(\tau_{m_j,j}(v, u)) \qquad (u, v \in F^{(j)}), \\ &\tau_{m_j,j}|_{F_0^{(j)}} = 0, \text{ and } \tau_{m_j,j}|_{F_1^{(j)}} = 0. \end{aligned} \tag{5.6}$$

Let us write $m = m_j$, $\tau_m = \tau_{m_j,j}$ and $F = F^{(j)}$ in the last case. We may rewrite (5.6) and (5.4) as

$$\begin{aligned} &\tau_m(u, v) = -\delta(m)\iota(\tau_m(v, u)), \\ &F_0,\ F_1 \text{ are isotropic subspaces of } F, \\ &\tau_m(x_s u, v) = -\tau_m(u, S x_s v) \qquad (u, v \in F). \end{aligned} \tag{5.7}$$

As an easy consequence of Theorem 5.5 we deduce the following fact.

**Theorem 5.6.** *Suppose $(x_s, F)$, described in (5.7), is indecomposable and non-zero. Then $\iota \neq 1$ and, up to similarity,*

$$\begin{aligned} &F = \mathbb{D}v_0 \oplus \mathbb{D}v_1 \qquad (v_0 \in F_0,\ v_1 \in F_1), \\ &x_s : v_0 \to a_1 v_1, \quad v_1 \to a_0 v_0, \end{aligned}$$

*where $a_0 = \delta(m)\iota(a_0)$ and $a_1 = -\delta(m)\iota(a_1)$ if $\mathbb{D} \neq \mathbb{R}$.*

# 6 A Cartan subspace, the Weyl group and an integration formula

**Definition 6.1.** An element $x \in \mathfrak{g}_1$ is regular if and only if the $G$-orbit through $x$ is of maximal possible dimension. A Cartan subspace $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ is the double anticommutant $\mathfrak{h}_1 = {}^{({}^{x}\mathfrak{g}_1)}\mathfrak{g}_1$ of a regular semisimple element of $x \in \mathfrak{g}_1$. The Weyl group $W(G, \mathfrak{h}_1)$ is the quotient of the stabilizer of $\mathfrak{h}_1$ in $G$ by the subgroup which acts trivially on $\mathfrak{h}_1$. (We shall identify the Weyl group $W(G, \mathfrak{h}_1)$ with its image in $GL(\mathfrak{h}_1)$.)

**Proposition 6.2.** *The following is a complete list of the Cartan subspaces $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ and the Weyl groups $W(G, \mathfrak{h}_1)$, up to conjugation by an element of $G$, such that $\mathfrak{h}_1$ contains a non-zero regular semisimple indecomposable element:*

**Type I**

$$\begin{aligned}
&V_0 = \mathbb{D}v_0 \oplus \mathbb{D}v_0',\ V_1 = \mathbb{D}v_1 \oplus \mathbb{D}v_1';\\
&\tau(v_0, v_0) = \tau(v_0', v_0') = \tau(v_1, v_1) = \tau(v_1', v_1') = 0,\ \tau(v_0, v_0') = \tau(v_1, v_1') = 1;\\
&x(a) : v_0 \to av_1,\ v_1 \to av_0,\ v_0' \to -\iota(a)v_1',\ v_1' \to \iota(a)v_0',\ a \in \mathbb{D};\\
&\text{if } \mathbb{D} = \mathbb{R} \text{ then } \mathfrak{h}_1 = \{x(a);\ a \in \mathbb{R}\},\ |W(G, \mathfrak{h}_1)| = 2,\\
&\text{the non-trivial element of } W(G, \mathfrak{h}_1) \text{ maps } x(a) \text{ to } x(-a);\\
&\text{if } \mathbb{D} = \mathbb{C} \text{ and } \iota = 1 \text{ then } \mathfrak{h}_1 = \{x(a);\ a \in \mathbb{C}\},\ |W(G, \mathfrak{h}_1)| = 4,\\
&\text{the non-trivial elements of } W(G, \mathfrak{h}_1) \text{ map } x(a) \text{ to } x(-a),\ x(ia),\ x(-ia);\\
&\text{if } \mathbb{D} = \mathbb{C} \text{ and } \iota \neq 1 \text{ then } \mathfrak{h}_1 = \{x(a);\ a \in \mathbb{C}\},\ |W(G, \mathfrak{h}_1)| = 2,\\
&\text{the non-trivial element of } W(G, \mathfrak{h}_1) \text{ maps } x(a) \text{ to } x(-a);\\
&\text{if } \mathbb{D} = \mathbb{H} \text{ then } \mathfrak{h}_1 = \{x(a);\ a \in \mathbb{C}\},\ |W(G, \mathfrak{h}_1)| = 4,\\
&\text{the non-trivial elements of } W(G, \mathfrak{h}_1) \text{ map } x(a) \text{ to } x(-a),\ x(\iota(a)),\ x(-\iota(a));
\end{aligned} \tag{a}$$

$$\begin{aligned}
&V_0 = \mathbb{D}v_0, V_1 = \mathbb{D}v_1, \mathbb{C} \subseteq \mathbb{D}, \iota \neq 1;\\
&\tau(v_0, v_0) = \epsilon = \pm 1, \tau(v_1, v_1) = \delta i = \pm i;\\
&x(a) : v_0 \to av_1, v_1 \to av_0,\ a \in \mathbb{C};\\
&\mathfrak{h}_1 = \{x(a);\ a = -\epsilon\delta i\iota(a) \in \mathbb{C}\},\ |W(G, \mathfrak{h}_1)| = 2;\\
&\text{the non-trivial element of } W(G, \mathfrak{h}_1) \text{ maps } x(a) \text{ to } x(-a);
\end{aligned} \tag{b}$$

$$\begin{aligned}
&V_0 = \mathbb{R}v_0 \oplus \mathbb{R}v_0',\ V_1 = \mathbb{R}v_1 \oplus \mathbb{R}v_1', \mathbb{D} = \mathbb{R};\\
&\tau(v_0, v_0) = \tau(v_0', v_0') = \epsilon = \pm 1, \tau(v_1, v_1) = \tau(v_1', v_1') = 0,\\
&\tau(v_0, v_0') = 0, \tau(v_1, v_1') = 1;\\
&x = x(a) : v_0 \to a(v_1 - \epsilon v_1'),\ v_1 \to a(v_0 - v_0'),\\
&v_0' \to a(v_1 + \epsilon v_1'), v_1' \to \epsilon a(v_0 + v_0'),\ a \in \mathbb{R};\\
&\mathfrak{h}_1 = \{x(a);\ a \in \mathbb{R}\},\ |W(G, \mathfrak{h}_1)| = 2;\\
&\text{the non-trivial element of } W(G, \mathfrak{h}_1) \text{ maps } x(a) \text{ to } x(-a);
\end{aligned} \tag{c}$$

$$\begin{aligned}
&V_0 = (\mathbb{R}u_0 \oplus \mathbb{R}v_0) \oplus (\mathbb{R}u_0' \oplus \mathbb{R}v_0'),\\
&V_1 = (\mathbb{R}u_1 \oplus \mathbb{R}v_1) \oplus (\mathbb{R}u_1' \oplus \mathbb{R}v_1'),\ \mathbb{D} = \mathbb{R},\\
&\text{the spaces in parenthesis are isotropic, and}\\
&\tau(u_0, u_0') = \tau(v_0, v_0') = \tau(u_1, u_1') = \tau(v_1, v_1') = 1;\\
&x = x(a, b) :\\
&u_0 \to au_1 + bv_1, u_1 \to au_0 + bv_0,\\
&v_0 \to -bu_1 + av_1, v_1 \to -bu_0 + av_0,\\
&u_0' \to -au_1' + bv_1', u_1' \to au_0' - bv_0',\\
&v_0' \to -bu_1' - av_1', v_1' \to bu_0' + av_0', \quad a, b \in \mathbb{R};\\
&\mathfrak{h}_1 = \{x(a, b);\ a, b \in \mathbb{R}\},\ |W(G, \mathfrak{h}_1)| = 8;\\
&\text{the non-trivial elements of } W(G, \mathfrak{h}_1) \text{ map } x(a, b) \text{ to}\\
&x(-a, b), x(a, -b), x(-a, -b), x(b, a), x(-b, a), x(b, -a), x(-b, -a);
\end{aligned} \tag{d}$$

**Type II**

$$\begin{aligned}
&V_0 = \mathbb{D}v_0,\ V_1 = \mathbb{D}v_1;\\
&x(a) : v_0 \to av_1, v_1 \to av_0,\ a \in \mathbb{D};\\
&\text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C},\ \text{then } \mathfrak{h}_1 = \{x(a);\ a \in \mathbb{D}\},\ |W(G, \mathfrak{h}_1)| = 2;\\
&\text{the non-trivial element of } W(G, \mathfrak{h}_1) \text{ maps } x(a) \text{ to } x(-a);\\
&\text{if } \mathbb{D} = \mathbb{H},\ \text{then } \mathfrak{h}_1 = \{x(a);\ a \in \mathbb{C}\},\ |W(G, \mathfrak{h}_1)| = 4;\\
&\text{the non-trivial elements of } W(G, \mathfrak{h}_1) \text{ map } x(a) \text{ to } x(-a),\ x(\iota(a)),\ x(-\iota(a));
\end{aligned} \tag{e}$$

$$\begin{aligned}
&V_0 = \mathbb{R}v_0,\ V_1 = \mathbb{R}v_1;\\
&x(a) : v_0 \to av_1, v_1 \to -av_0;\\
&\mathfrak{h}_1 = \{x(a);\ a \in \mathbb{R}\},\ |W(G, \mathfrak{h}_1)| = 2;\\
&\text{the non-trivial element of } W(G, \mathfrak{h}_1) \text{ maps } x(a) \text{ to } x(-a);
\end{aligned} \tag{f}$$

$$\begin{aligned}
&V_0 = \mathbb{R}u_0 \oplus \mathbb{R}v_0,\ V_1 = \mathbb{R}u_1 \oplus \mathbb{R}v_1, \mathbb{D} = \mathbb{R};\\
&x = x(a, b) : u_0 \to au_1 + bv_1, u_1 \to au_0 + bv_0,\\
&v_0 \to -bu_1 + av_1, v_1 \to -bu_0 + av_0;\\
&\mathfrak{h}_1 = \{x(a, b);\ a, b \in \mathbb{R}\},\ |W(G, \mathfrak{h}_1)| = 4;\\
&\text{the non-trivial elements of } W(G, \mathfrak{h}_1) \text{ map } x(a, b) \text{ to}\\
&x(-a, b), x(a, -b), x(-a, -b);
\end{aligned} \tag{g}$$

**Proof.** This Proposition is a straightforward consequence of Theorems 5.2, 5.3 and the proof of Theorem 4.4, (see the section 13). □

In general, a Cartan subspace $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ induces a direct sum decomposition

$$\begin{aligned} V = V^0 &\oplus V^1 \oplus V^2 \oplus \ldots \oplus V^{i_1} \\ &\oplus V^{i_1+1} \oplus V^{i_1+2} \oplus \ldots \oplus V^{i_2} \\ &\ \ldots \\ &\oplus V^{i_{k-1}+1} \oplus V^{i_{k-1}+2} \oplus \ldots \oplus V^{i_k}, \end{aligned} \tag{6.1}$$

orthogonal in the type I case, into graded subspaces preserved by $\mathfrak{h}_1$, such that

$$\begin{aligned} &(a)\ \mathfrak{h}_1(V^0) = 0, \\ &(b)\ \text{for each } 0 \leq i \leq i_k \text{ there is } x \in \mathfrak{h}_1 \text{ such that } (x, V^i) \text{ is indecomposable,} \\ &(c)\ \text{there is } x \in \mathfrak{h}_1 \text{ such that the elements } (x, V^j),\ (x, V^k) \\ &\quad \text{are indecomposable and similar if and only if there is } 1 \leq l \leq k-1, \\ &\quad \text{with } i_l < j \leq i_{l+1} \text{ and } i_l < k \leq i_{l+1}. \end{aligned} \tag{6.2}$$

Then the Weyl group

$$\begin{aligned} &W(G, \mathfrak{h}_1) = \\ &(S_{i_1} \ltimes (W(G(V^1), \mathfrak{h}_1(V^1)) \times W(G(V^2), \mathfrak{h}_1(V^2)) \times \ldots \\ &\qquad \times W(G(V^{i_1}), \mathfrak{h}_1(V^{i_1})))) \\ &\times (S_{i_2 - i_1} \ltimes (W(G(V^{i_1+1}), \mathfrak{h}_1(V^{i_1+1})) \times W(G(V^{i_1+2}), \mathfrak{h}_1(V^{i_1+2})) \times \ldots \\ &\qquad \times W(G(V^{i_2}), \mathfrak{h}_1(V^{i_2})))) \\ &\ldots \\ &\times (S_{i_k - i_{k-1}} \ltimes (W(G(V^{i_{k-1}+1}), \mathfrak{h}_1(V^{i_{k-1}+1})) \times W(G(V^{i_{k-1}+2}), \mathfrak{h}_1(V^{i_{k-1}+2})) \times \ldots \\ &\qquad \times W(G(V^{i_k}), \mathfrak{h}_1(V^{i_k})))), \end{aligned} \tag{6.3}$$

with the action on $\mathfrak{h}_1$ compatible with the decomposition (6.3). (Here $S_m$ stands for the group of all permutations of $m$ objects.)

Proposition 6.2, together with (6.1), imply that there are finitely many conjugacy classes of Cartan subspaces in $\mathfrak{g}_1$ and that any Cartan subspace consists of elements which commute in $End(V)$. Also, it is easy to see from Definition 6.1 that each semisimple element of $\mathfrak{g}_1$ belongs to a Cartan subspace of $\mathfrak{g}_1$. Also, the set or regular elements coincides with the set where certain determinants don't vanish, hence it is open and dense. We record these facts in the following proposition.

**Proposition 6.3.** *There are finitely many $G$-conjugacy classes of Cartan subspaces in $\mathfrak{g}_1$. Every semisimple element of $\mathfrak{g}_1$ belongs to the $G$-orbit through an element of a Cartan subspace. The set of regular semisimple elements is dense in $\mathfrak{g}_1$. Any two elements of a Cartan subspace $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ commute as endomorphisms of $V$.*

The following lemma shall be verified at the end of section 13.

**Lemma 6.4.** *For any two commuting regular semisimple elements $x, y \in \mathfrak{g}_1$ we have*

$$(a)\ \mathfrak{g}_1^{x^2} = \mathfrak{g}_1^{y^2};\ (b)\ \mathfrak{g}_1^{x} = \mathfrak{g}_1^{y};\ (c)\ {}^{x}\mathfrak{g}_1 = {}^{y}\mathfrak{g}_1;\ (d)\ \mathfrak{g}_0^{x^2} = \mathfrak{g}_0^{y^2};\ (e)\ \mathfrak{g}_0^{x} = \mathfrak{g}_0^{y}.$$

Let $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ be a Cartan subspace and let $\mathfrak{h}_1^{reg} \subseteq \mathfrak{h}_1$ be the subset of regular elements. Denote by $\mathfrak{h}_1^2 = \{\mathfrak{h}_1, \mathfrak{h}_1\}$ the linear span of the elements $\{x, y\}$, where $x, y \in \mathfrak{h}_1$. Define

$$^{\mathfrak{h}_1}\mathfrak{g}_1 = \bigcap_{y \in \mathfrak{h}_1} {}^{y}\mathfrak{g}_1, \quad \mathfrak{g}_1^{\mathfrak{h}_1} = \bigcap_{y \in \mathfrak{h}_1} \mathfrak{g}_1^{y}, \quad \mathfrak{g}_i^{\mathfrak{h}_1^2} = \bigcap_{y \in \mathfrak{h}_1^2} \mathfrak{g}_i^{y}, \quad (i = 0, 1). \tag{6.4}$$

**Lemma 6.5.** *For any $x \in \mathfrak{h}_1^{reg}$,*

$$(a)\ {}^{\mathfrak{h}_1}\mathfrak{g}_1 = {}^{x}\mathfrak{g}_1, \quad (b)\ \mathfrak{g}_1^{\mathfrak{h}_1} = \mathfrak{g}_1^{x} = \mathfrak{h}_1, \quad (c)\ \mathfrak{g}_i^{\mathfrak{h}_1^2} = \mathfrak{g}_i^{x^2} \quad (i = 0, 1).$$

**Proof.** In the definition (6.4), it suffices to take the finite intersection over the $y$'s which form a basis of the corresponding linear space. For the equations (a), (b) we may choose a basis of $\mathfrak{h}_1$ consisting of regular elements. Then the equalities follow from Lemma 6.4.

Since the elements of $\mathfrak{h}_1$ commute, the space $\mathfrak{h}_1^2$ is spanned by the squares $y^2$, $y \in \mathfrak{h}_1$. Thus we may choose a basis $y_1^2$, $y_2^2$,..., of $\mathfrak{h}_1^2$ such that each $y_i \in \mathfrak{h}_1^{reg}$. Then the equality (c) also follows from Lemma 6.4. □

**Proposition 6.6.** *Let $x \in \mathfrak{h}_1^{reg}$. Set $V^0 = ker(x)$ and $V^+ = xV$, as in Theorem 4.4(b). Then*

$$\mathfrak{h}_1 = \mathfrak{g}_1(V^+)^{\mathfrak{h}_1} = \mathfrak{g}_1(V^+)^{x}; \tag{a}$$
$$^{\mathfrak{h}_1}\mathfrak{g}_1 = {}^{\mathfrak{h}_1}\mathfrak{g}_1(V^+); \tag{b}$$
$$\mathfrak{g}_1^{\mathfrak{h}_1^2} = \mathfrak{g}_1(V^+)^{\mathfrak{h}_1^2} = \mathfrak{g}_1(V^+)^{x^2}; \tag{c}$$
$$\mathfrak{g}_0^{\mathfrak{h}_1^2} = \mathfrak{g}_0(V^0) \oplus \mathfrak{h}_0, \tag{d}$$

*where $\mathfrak{h}_0 = \mathfrak{g}_0(V^+)^{\mathfrak{h}_1^2}$ is a Cartan subalgebra of $\mathfrak{g}_0(V^+)$, and the sum is orthogonal;*

$$\mathfrak{g}_0^{\mathfrak{h}_1} = \mathfrak{g}_0(V^0) \oplus \mathfrak{h}_1^2; \tag{e}$$

*the restriction of the form $\langle\ ,\ \rangle$ to $\mathfrak{h}_0$ is non-degenerate and* (f)
*$\mathfrak{h}_0 = S\mathfrak{h}_1^2 \oplus \mathfrak{h}_1^2$ is a complete polarization.*

**Proof.** Parts (a), (b), (c) are immediate from Theorem 4.4 and Lemma 6.5. Similarly we have the orthogonal decomposition in (d). A straightforward computation based on Theorem 4.4(a) shows that

$$dim\, \mathfrak{g}_0(V^+)^{x^2} = 2\, dim\, \mathfrak{g}_1(V^+)^{x}.$$

By Theorem 4.4(a) and 4.4(c),

$$dim\, \mathfrak{g}_1(V^+)^{x} = min\{rank\, \mathfrak{g}_0(V_0^+), rank\, \mathfrak{g}_0(V_1^+)\}.$$

Since

$$\mathfrak{g}_0(V^+) = \mathfrak{g}_0(V_0^+) \oplus \mathfrak{g}_0(V_1^+),$$

we see that the restriction of $x^2$ to $V^+$ is a regular element of $\mathfrak{g}_0(V^+)$. Hence, $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0(V^+)$ and (d) follows.

For (e) we may assume that $V^0 = 0$ and that $(x, V)$ is indecomposable. Then the equality follows from Proposition 6.2 via a case by case analysis. Similarly we check (f). $\square$

**Lemma 6.7.** *For any $x \in \mathfrak{h}_1^{reg}$ the following map is a linear bijection*

$$(S\mathfrak{g}_0^{\mathfrak{h}_1})^\perp \ni y \to [y, x] \in ({}^{\mathfrak{h}_1}\mathfrak{g}_1)^\perp. \tag{a}$$

*The map (a) intertwines the adjoint action of $\mathfrak{h}_1^2$ on both spaces. The following are direct sum decompositions into the trivial and the non-trivial $\mathfrak{h}_1^2$-components:*

$$(S\mathfrak{g}_0^{\mathfrak{h}_1})^\perp = S\mathfrak{h}_1^2 \oplus \mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp, \tag{b}$$

$$({}^{\mathfrak{h}_1}\mathfrak{g}_1)^\perp = {}^{\mathfrak{h}_1}\mathfrak{g}_1 \oplus (\mathfrak{g}_1^{\mathfrak{h}_1^2})^\perp. \tag{c}$$

*The map (a) restricts to bijections*

$$S\mathfrak{h}_1^2 \ni y \to [y, x] \in {}^{\mathfrak{h}_1}\mathfrak{g}_1, \tag{d}$$

$$\mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp \ni y \to [y, x] \in (\mathfrak{g}_1^{\mathfrak{h}_1^2})^\perp. \tag{e}$$

**Proof.** We see from Proposition 6.6(e) that

$$(S\mathfrak{g}_0^{\mathfrak{h}_1})^\perp = \mathfrak{g}_0(V^0)^\perp \cap (S\mathfrak{h}_1^2)^\perp.$$

The $\mathfrak{h}_1^2$-trivial component of this space is

$$\mathfrak{g}_0(V^0)^\perp \cap (S\mathfrak{h}_1^2)^\perp \cap \mathfrak{h}_0 = S\mathfrak{h}_1^2.$$

This verifies (b).

We see from (b), and from Proposition 6.6(e), that

$$\mathfrak{g}_0 = (S\mathfrak{g}_0^{\mathfrak{h}_1})^\perp \oplus \mathfrak{g}_0^{\mathfrak{h}_1}.$$

Since ${}^{\mathfrak{h}_1}\mathfrak{g}_1 = {}^{x}\mathfrak{g}_1$, Lemma 3.1 implies that the map (a) is well defined and surjective. Let $z \in \mathfrak{h}_1$ and let $x, y$ be as in (a). Then

$$[z^2, [y, x]] = [[z^2, y], x] + [y, [z^2, x]] = [[z^2, y], x],$$

because $[z^2, x] = 0$. Hence, the map (a) is $\mathfrak{h}_1^2$-intertwining, and the proof of (a) is complete.

Part (c) is clear from Theorem 4.4. Parts (d) and (e) follow from the intertwining property of the map (a). $\square$

The derivative of the map

$$\mathfrak{h}_1 \ni x \to x^2 \in \mathfrak{h}_1^2 \tag{6.5}$$

at $x \in \mathfrak{h}_1$, coincides with the following linear map

$$\mathfrak{h}_1 \ni y \to \{y, x\} \in \mathfrak{h}_1^2, \tag{6.6}$$

which, by Proposition 2.4, is adjoint to the map

$$S\mathfrak{h}_1^2 \ni y \to [x, y] \in S\mathfrak{h}_1. \tag{6.7}$$

(Notice that the range of the map (6.7) is contained in $S\mathfrak{h}_1$. Indeed, let $x, z \in \mathfrak{h}_1^{reg}$. Then $[Sz^2, x] \in [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1 = S\mathfrak{g}_1$ and $[Sz^2, x] = S(z^2x+xz^2) = S2z^2x$. Clearly $z^2x$ commutes with $x$. Thus $[Sz^2, x] \in S\mathfrak{g}_1^x$. Furthermore $z^2x|_{V^0} = 0$. Therefore $[Sz^2, x] \in S\mathfrak{h}_1$.)

Hence,

$$|det(\mathfrak{h}_1 \ni y \to \{y, x\} \in \mathfrak{h}_1^2)| = |det(S\mathfrak{h}_1^2 \ni y \to [y, x] \in S\mathfrak{h}_1)|. \tag{6.8}$$

Define polynomials $D_j(x)$, $x \in \mathfrak{g}_1$, by

$$det(tI - ad(x^2)|_{\mathfrak{g}_1}) = \sum_{j=r}^{R} t^j D_j(x) \qquad (x \in \mathfrak{g}_1), \tag{6.9}$$

where $R = dim(\mathfrak{g}_1)$, $D_R = 1$, and $r \geq 0$ is the smallest integer such that $D_r$ is not identically equal zero.

**Lemma 6.8.** *For $x \in \mathfrak{h}_1$ we have*

$$\begin{aligned} |D_r(x)| &= |det(ad(x^2)|_{(\mathfrak{g}_1^{h_1^2})^\perp})| \\ &= |det(\mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp \ni y \to [y, x] \in (\mathfrak{g}_1^{h_1^2})^\perp)|^2 \\ &= |det(ad(x^2)|_{g_0(V^0)^\perp \cap \mathfrak{h}_0^\perp})|. \end{aligned}$$

**Proof.** The first equality is clear from (6.9). The map

$$ad(x^2)|_{(\mathfrak{g}_1^{h_1^2})^\perp} : (\mathfrak{g}_1^{h_1^2})^\perp \to (\mathfrak{g}_1^{h_1^2})^\perp$$

coincides with ($-1$ times) the composition of the following two maps:

$$(\mathfrak{g}_1^{h_1^2})^\perp \ni y \to \{y, x\} \in g_0(V^0)^\perp \cap \mathfrak{h}_0^\perp, \tag{6.10}$$

and

$$g_0(V^0)^\perp \cap \mathfrak{h}_0^\perp \ni y \to [y, x] \in (\mathfrak{g}_1^{h_1^2})^\perp, \tag{6.11}$$

which, by Proposition 2.4, are adjoint to each other. Hence the second equality follows. Similarly the map

$$ad(x^2) : \mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp \to \mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp$$

is ($-1$ times) the composition of the maps (6.10) and (6.11), and the third equality follows. □

Recall the function $J(x)$, equal to the absolute value of the determinant of the map (3.7):

$$J(x) = |det(\mathfrak{g}_0/\mathfrak{g}_0^x \ni y + \mathfrak{g}_0^x \to [y,x] \in [\mathfrak{g}_0, x])|, \qquad (x \in \mathfrak{g}_1). \tag{6.12}$$

**Corollary 6.9.** *For $x \in \mathfrak{h}_1$ we have*

$$J(x) = |det(\mathfrak{h}_1 \ni y \to \{y,x\} \in \mathfrak{h}_1^2)| \cdot |D_r(x)|^{1/2}.$$

**Proof.** Let $x \in \mathfrak{h}_1^{reg}$. By Proposition 6.6(e),

$$\mathfrak{g}_0^x = \mathfrak{g}_0(V^0) \oplus \mathfrak{h}_1^2.$$

Therefore Proposition 6.6(e) and Proposition 6.6(f)) imply

$$\begin{aligned}(S\mathfrak{g}_0^x)^\perp &= (\mathfrak{g}_0(V^0) \oplus S\mathfrak{h}_1^2)^\perp = \mathfrak{g}_0(V^0)^\perp \cap (S\mathfrak{h}_1^2)^\perp \\ &= (\mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp) \oplus S\mathfrak{h}_1^2.\end{aligned}$$

But we see from Proposition 6.6(d) and Proposition 6.6(f) that

$$\begin{aligned}\mathfrak{g}_0 &= \mathfrak{g}_0(V^0) \oplus (\mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp) \oplus S\mathfrak{h}_1^2 \oplus \mathfrak{h}_1^2 \\ &= (\mathfrak{g}_0(V^0) \oplus \mathfrak{h}_1^2) \oplus (\mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp \oplus S\mathfrak{h}_1^2),\end{aligned}$$

where the middle direct sum is orthogonal. Thus,

$$\mathfrak{g}_0 = \mathfrak{g}_0^x \oplus (\mathfrak{g}_0^x)^\perp = (S\mathfrak{h}_1^2 \oplus \mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp) \oplus \mathfrak{g}_0^x. \tag{6.13}$$

Furthermore, by (3.1), (4.4) and Proposition 6.6(b),

$$[\mathfrak{g}_0, x] = ({}^x\mathfrak{g}_1)^\perp = {}^x\mathfrak{g}_1 \oplus \mathfrak{g}_1(V^0) = {}^x\mathfrak{g}_1(V^+) \oplus \mathfrak{g}_1(V^0) = {}^{\mathfrak{h}_1}\mathfrak{g}_1 \oplus (\mathfrak{g}_1^{\mathfrak{h}_1^2})^\perp.$$

Hence, by Lemma 6.7, $J(x)$ is the absolute value of the determinant of the map Lemma 6.7(d) times the absolute value of the determinant of the map Lemma 6.7(e):

$$J(x) = |det(S\mathfrak{h}_1^2 \ni y \to \{y,x\} \in S\mathfrak{h}_1)| \cdot |det(\mathfrak{g}_0(V^0)^\perp \cap \mathfrak{h}_0^\perp \ni y \to [y,x] \in (\mathfrak{g}_1^{\mathfrak{h}_1^2})^\perp)|.$$

Hence our formula for $J(x)$ follows from (6.8) and Lemma 6.8. □

**Example 6.10.** The dual pair $O_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C})$

For $i = 0, 1$ choose a basis

$$v_{i1}, v_{i2}, ..., v_{in}, v'_{i1}, v'_{i2}, ..., v'_{in}$$

of the vector space $V_i$ such that

$$\tau(v_{ik}, v'_{ik}) = 1 \qquad (k = 1, 2, 3, ..., n)$$

and all the other pairings are zero. Let $\mathfrak{h}_1$ be the Cartan subspace consisting of elements $x(a)$, $a \in \mathbb{C}^n$, such that

$$\begin{aligned}x(a) :& v_{0k} \to a_k v_{1k}, \quad v_{1k} \to a_k v_{0k}, \\ & v'_{0k} \to -a_k v'_{1k}, \quad v'_{1k} \to a_k v'_{0k}, \qquad (k = 1, 2, 3, ..., n).\end{aligned}$$

Then

$$det(\mathfrak{h}_1 \ni y \to \{y,x\} \in \mathfrak{h}_1^2) = \prod_{k=1}^{n}(2a_k),$$

$$D_r(x(a)) = \left(\prod_{j\neq k}(a_j^2 - a_k^2)(a_j^2 + a_k^2)\cdot\prod_{k=1}^{n}(\sqrt{2}a_k)\right)^2, \quad \text{and} \quad r = 4n.$$

**Example 6.11.** The dual pair $GL_n(\mathbb{C}), GL_n(\mathbb{C})$.

For $i = 0, 1$ choose a basis

$$v_{i1}, v_{i2}, ..., v_{in}$$

of the vector space $V_i$. Let $\mathfrak{h}_1$ be the Cartan subspace consisting of elements $x(a)$, $a \in \mathbb{C}^n$, such that

$$x(a) : v_{0k} \to a_k v_{1k}, \quad v_{1k} \to a_k v_{0k}, \qquad (k = 1, 2, 3, ..., n).$$

Then

$$det(\mathfrak{h}_1 \ni y \to \{y,x\} \in \mathfrak{h}_1^2) = \prod_{k=1}^{n}(2a_k),$$

$$D_r(x(a)) = \left(\prod_{j\neq k}(a_j^2 - a_k^2)\right)^2, \quad \text{and} \quad r = 2n.$$

**Corollary 6.12.** *For a Cartan subspace $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ let $\mathfrak{h}_1^+ \subseteq \mathfrak{h}_1^{reg}$ be a (measurable) fundamental domain for the action of the Weyl group $W(G, \mathfrak{h}_1)$. Let $Q : \mathfrak{g}_1 \ni x \to x^2 \in \mathfrak{g}_0$. Let $\mathfrak{g}_1^{reg,ss} \subset \mathfrak{g}_1$ denote the subset of regular semisimple elements. Then for $f \in C_c(\mathfrak{g}_1^{reg,ss})$*

$$\begin{aligned}
\int_{\mathfrak{g}_1} f(x)\,dx &= \sum_{\mathfrak{h}_1} \frac{1}{|W(G,\mathfrak{h}_1)|}\int_{\mathfrak{h}_1^{reg}} J(x)\int_{G/G^{\mathfrak{h}_1}} f(gx)\,d\dot{g}\,dx \\
&= \sum_{\mathfrak{h}_1}\int_{\mathfrak{h}_1^+} J(x)\int_{G/G^{\mathfrak{h}_1}} f(gx)\,d\dot{g}\,dx \\
&= \sum_{Q\mathfrak{h}_1^+}\int_{Q\mathfrak{h}_1^+} |D_r(Q^{-1}(x))|^{1/2}\int_{G/G^{\mathfrak{h}_1}} f(gQ^{-1}(x))\,d\dot{g}\,dx,
\end{aligned}$$

*where the summation is over a maximal family of mutually non-conjugate Cartan subspaces $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$.*

**Proof.** The first equality follows from the fact that the absolute value of Jacobian of the map

$$G/G^{\mathfrak{h}_1} \times \mathfrak{h}_1^{reg} \ni (gG^{\mathfrak{h}_1}, x) \to gx \in \mathfrak{g}_1^{reg,ss}$$

at $(gG^{\mathfrak{h}_1}, x)$ is equal to $J(x)$. The second equality is obvious and the third one follows from Corollary 6.21. $\square$

# 7 A canonical complementary subspace to the tangent space of an orbit

Here we adopt the view point of Harish-Chandra, [4, section 14], that the complementary subspace in Theorem 4.7 should be the orthogonal with respect to a natural positive definite form, (see (7.13) below).

Recall the symmetric positive definite form $(\ ,\ ) = -\langle\ ,\theta\ \rangle$ on $\mathfrak{g}$ defined in (3.5), and the adjoint map:

$$End(\mathfrak{g}) \ni A \to A^{\dagger} \in End(\mathfrak{g}), \quad (A(x), y) = (x, A^{\dagger}(y)), \quad x, y \in \mathfrak{g}. \tag{7.1}$$

**Lemma 7.1.** *For the adjoint representation $ad : \mathfrak{g} \to End(\mathfrak{g})$, we have*

$$ad(x)^{\dagger} = \begin{cases} -ad(\theta(x)) & if \;\; x \in \mathfrak{g}_0, \\ \;\; ad(\theta(x)) & if \;\; x \in \mathfrak{g}_1. \end{cases}$$

**Proof.** Let $x, y, z \in \mathfrak{g}_0$. Then

$$(ad(x)y, z) = -\langle [x, y], \theta z\rangle = -\langle y, -[x, \theta z]\rangle = -\langle y, -\theta[\theta x, z]\rangle = (y, -ad(\theta x)z).$$

Let $x \in \mathfrak{g}_0$, and let $y, z \in \mathfrak{g}_1$. Then the above computation applies without any change. Hence the formula follows for $x \in \mathfrak{g}_0$ (as is well known [4, Lemma 27]).

If $x \in \mathfrak{g}_1$, and either $y, z \in \mathfrak{g}_0$ or $y, z \in \mathfrak{g}_1$, then all the pairings in question are zero. Let $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_1$ and $z \in \mathfrak{g}_0$. Then, by Proposition 2.4,

$$(ad(x)y, z) = -\langle \{x, y\}, \theta z\rangle = -\langle y, [x, \theta z]\rangle = -\langle y, \theta[\theta x, z]\rangle = (y, ad(\theta x)z).$$

Let $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_0$ and $z \in \mathfrak{g}_1$. Then, by Proposition 2.4,

$$\begin{aligned}(ad(x)y, z) &= -\langle [x, y], \theta z\rangle = \langle \theta z, [x, y]\rangle = \langle \{x, \theta z\}, y\rangle = -\langle y,\ \{x, \theta z\}\rangle \\ &= -\langle y, \theta\{\theta x, z\}\rangle = (y, ad(\theta x)z).\end{aligned}$$

This verifies the second formula. □

Let $x \in \mathfrak{g}_0$. Then the $(\ ,\ )$-orthogonal complement to $[\mathfrak{g}_0, x]$ in $\mathfrak{g}_0$ is equal to $\theta(\mathfrak{g}_0^x) = \mathfrak{g}_0^{\theta(x)}$. Thus

$$\mathfrak{g}_0 = [\mathfrak{g}_0, x] \oplus \theta(\mathfrak{g}_0^x). \tag{7.2}$$

In particular, if $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ is a Cartan subalgebra, then

$$\mathfrak{g}_0 = [\mathfrak{g}_0, x] \oplus \theta(\mathfrak{h}_0) \qquad (x \in \mathfrak{h}_0^{reg}), \tag{7.3}$$

which provides a geometric interpretation for the notion of a $\theta$-stable Cartan subalgebra.

If $x \in \mathfrak{g}_0$ is nilpotent and such that $\{x, y = -\theta(x), h = [x, y]\}$ is a Cayley triple, [2], then (7.2) may be rewritten as

$$\mathfrak{g}_0 = [\mathfrak{g}_0, x] \oplus \mathfrak{g}_0^y. \tag{7.4}$$

Moreover, if $\mathfrak{l} = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}h$, then by Lemma 7.1, $ad(\mathfrak{l}) \subseteq End(\mathfrak{g}_0)$ is a self adjoint family of operators. Hence $\mathfrak{g}_0$ decomposes into a direct, ( , )- orthogonal sum of irreducible components, which have managable structure because the Lie algebra $\mathfrak{l}$ is isomorphic to $sl(2,\mathbb{R})$. Furthermore,

$$\mathfrak{g}_0 = [\mathfrak{g}_0, x] \oplus \mathfrak{g}_0^x \tag{7.5}$$

is another ( , )-orthogonal decomposition, and the map

$$[\mathfrak{g}_0, x] \times \mathfrak{g}_0^y \ni (v, w) \to [v, x] + w \in \mathfrak{g}_0 \tag{7.6}$$

is a linear bijection.

Consider an element $x \in \mathfrak{g}_1$. Then $\theta({}^x\mathfrak{g}_1) = {}^{\theta(x)}\mathfrak{g}_1$ is the ( , )-orthogonal complement of $[\mathfrak{g}_0, x]$ in $\mathfrak{g}_1$. Thus

$$\mathfrak{g}_1 = [\mathfrak{g}_0, x] \oplus \theta({}^x\mathfrak{g}_1). \tag{7.7}$$

In particular, if $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ is a Cartan subspace, then

$$\mathfrak{g}_1 = [\mathfrak{g}_0, x] \oplus \theta({}^{\mathfrak{h}_1}\mathfrak{g}_1) \qquad (x \in \mathfrak{h}_1^{reg}). \tag{7.8}$$

Moreover, if $(\ )^\perp$ denotes the orthogonal complement with respect to the form $\langle\ ,\ \rangle$, then $\theta((\mathfrak{g}_0^x)^\perp) = (\mathfrak{g}_0^{\theta(x)})^\perp$ is the ( , )-orthogonal complement of $\mathfrak{g}_0^x$, so that

$$\mathfrak{g}_0 = \theta((\mathfrak{g}_0^x)^\perp) \oplus \mathfrak{g}_0^x, \tag{7.9}$$

and the following map is a linear bijection

$$\theta((\mathfrak{g}_0^x)^\perp) \times \theta({}^x\mathfrak{g}_1) \ni (v, w) \to [v, x] + w \in \mathfrak{g}_1. \tag{7.10}$$

As shown by Harish-Chandra, [4, sections 13 and 14], the Jacobson-Morozov theorem and a theorem of Mostow imply that for a nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}$, there is an element $x \in \mathcal{O}$ such that the Lie algebra generated by $x$ and $\theta(x)$ is isomorphic to $sl(2,\mathbb{R})$. This may be deduced directly from the classification Theorems 5.4 and 5.5, and motivates the following problem.

**Problem 7.2.** Let $\mathcal{O} \subseteq \mathfrak{g}_\infty$ be a non-zero nilpotent orbit. For an element $x \in \mathcal{O}$ let $\mathfrak{s}(x) \subseteq \mathfrak{g}$ be the Lie sub- superalgebra generated by $x$ and $\theta(x)$. Let $n_\mathcal{O} = min\{dim\, \mathfrak{s}(x);\ x \in \mathcal{O}\}$. Describe all the $\mathfrak{s}(x)$ with $dim\, \mathfrak{s}(x) = n_\mathcal{O}$.

**Remark 7.3.** With the notation of Problem 7.2, suppose $(x, V)$ is indecomposable. Then one can show, using the classification Theorems 4.4 and 4.5, that for $x \neq 0$, the height of $x$ is even if and only if there is (a possibly different) $x \in \mathcal{O}$ such that

$$[\{x, \theta(x)\}, x] = x.$$

Consequently $\mathfrak{s}(x)$ is isomorphic to $(\mathfrak{o}_1, sp_2(\mathbb{R}))$ as a dual pair, and $(x^2, -\theta(x)^2, [x^2, -\theta(x)^2])$ is a Cayley triple.

Notice that the adjoint representation maps $\mathfrak{g}$ into the ortho- symplectic Lie sub-superalgebra $osp(\mathfrak{g}) \subseteq End(\mathfrak{g})_0 \oplus End(\mathfrak{g})_1$, defined as in (2.8) with the $\tau$ replaced by $\langle\ ,\ \rangle$. Let $x \in \mathfrak{g}_1$ be nilpotent. Then $ad(x) \in osp(\mathfrak{g})_1$ is nilpotent. Hence the classification Theorems 4.4 and 4.5, applied to $osp(\mathfrak{g})$ provide a decomposition of $\mathfrak{g}$ into a direct orthogonal sum of $ad(x)$-indecomposables.

Moreover

$$\begin{aligned}\theta((\mathfrak{g}_0^x)^\perp) &= \theta((\mathfrak{g}_0 \cap ker(ad(x))^\perp) = \theta((\mathfrak{g}_0 \cap ad(x)(\mathfrak{g})) = ad(\theta(x))(\mathfrak{g}_1)\\ &= \{\theta(x), \mathfrak{g}_1\}.\end{aligned} \tag{7.11}$$

Therefore (7.11) may be rewritten as

$$\{\theta(x), \mathfrak{g}_1\} \times \theta({}^x\mathfrak{g}_1) \ni (v, w) \to [v, x] + w \in \mathfrak{g}_1. \tag{7.12}$$

The space $W$ of Theorem 4.7 may be taken to be $\theta({}^x\mathfrak{g}_1)$ and the local coordinates around $x$ are provided by the map

$$\{\theta(x), \mathfrak{g}_1\} \times \theta({}^x\mathfrak{g}_1) \ni (v, w) \to [v, x + u] + w \in \mathfrak{g}_1 \qquad (u \in \theta({}^x\mathfrak{g}_1)). \tag{7.13}$$

## 8 A proof of Theorem 5.1 for $x$ semisimple and $(G, \mathfrak{g})$ of type II, and a proof of Theorem 5.3

Let

$$U = \begin{cases} V^{\mathbb{C}}, \text{ the complexification of } V, \text{ if } \mathbb{D} = \mathbb{R}, \\ V, \ \text{ if } \mathbb{D} = \mathbb{C}, \\ V, \text{ viewed as a vector space over } \mathbb{C}, \text{ if } \mathbb{D} = \mathbb{H}. \end{cases}$$

Since $x$ is semisimple we have a direct sum decomposition into eigenspaces:

$$U = \sum_\lambda U^\lambda, \ xu = \lambda u, \ u \in U^\lambda. \tag{8.1}$$

Let $L$ be a set of eigenvalues of $x$ such that $L \cap (-L) = \emptyset$ and $L \cup (-L)$ is the set of all non-zero eigenvalues of $x$. Since $SU^\lambda = U^{-\lambda}$, (see (2.2) for $S$),

$$U = U^0 \oplus \sum_{\lambda \in L} (U^\lambda \oplus U^{-\lambda}) \tag{8.2}$$

is a direct sum decomposition into $\mathbb{Z}/2\mathbb{Z}$-graded subspaces preserved by $x$. The space $U^0$ is either zero or decomposes into a direct sum of one dimensional graded subspaces

$$U = \sum_k U^{0,k}. \tag{8.3}$$

For each $\lambda \in L$ let

$$U^\lambda = \sum_l U^{\lambda,l}$$

be a direct sum decomposition into one-dimensional subspaces. Then

$$U^\lambda \oplus SU^\lambda = \sum_l (U^{\lambda,l} \oplus SU^{\lambda,l}) \tag{8.4}$$

is a direct sum decomposition into graded $x$-invariant subspaces, which does not admit any finer decomposition of this type. Thus each term $U^{\lambda,l} \oplus SU^{\lambda,l}$ is indecomposable under the action of $x$ and $S$. This verifies Theorem 5.1 for $(G, \mathfrak{g})$ of type II and $\mathbb{D} = \mathbb{C}$.

Fix $\lambda$ and $l$ as in (8.4), and let $u^\lambda \in U^{\lambda,l}$ be a non-zero vector. Set $v_0 = u^\lambda + Su^\lambda$ and $v_1 = u^\lambda - Su^\lambda$. Then

$$Sv_0 = v_0,\ Sv_1 = -v_1,\ xv_0 = \lambda v_1,\ xv_1 = \lambda v_0. \tag{8.5}$$

Thus Theorem 5.3, for $\mathbb{D} = \mathbb{C}$, follows.

Let $\mathbb{D} = \mathbb{R}$. Let $U \ni u \to \overline{u} \in U$ be the complex conjugation with respect to the real form $V \subseteq U$. Then for each eigenvalue $\lambda$ of $x$ we have

$$\overline{U^\lambda} = U^{\overline{\lambda}}. \tag{8.6}$$

We may split the set of eigenvalues of $x$ into a disjoint union

$$\{0\} \cup L_{\mathbb{R}} \cup (-L_{\mathbb{R}}) \cup L_{\mathbb{C}} \cup (-L_{\mathbb{C}}),$$

where the elements $\lambda \in L_{\mathbb{R}}$ are such that $\lambda^2 \in \mathbb{R}$, and the elements $\lambda \in L_{\mathbb{C}}$ are such that $\lambda^2 \in \mathbb{C} \setminus \mathbb{R}$, so that the four complex numbers $\lambda$, $-\lambda$, $\overline{\lambda}$, $-\overline{\lambda}$ are distinct. Thus

$$U = U^0 \oplus \sum_{\lambda \in L_{\mathbb{R}}} (U^\lambda \oplus U^{-\lambda}) \oplus \sum_{\lambda \in L_{\mathbb{C}}} (U^\lambda \oplus U^{-\lambda} \oplus U^{\overline{\lambda}} \oplus U^{-\overline{\lambda}}). \tag{8.7}$$

Each summand in (8.7) is invariant under $x$, $S$, and the complex conjugation. The terms $U^0$ and $U^\lambda \oplus U^{-\lambda}$, with $\lambda \in \mathbb{R} \setminus 0$, may be treated as in the case $\mathbb{D} = \mathbb{C}$. The indecomposable summands of $U^\lambda \oplus U^{-\lambda}$ are described in part (a) of Theorem 5.3.

Suppose $\lambda = i\xi \in i\mathbb{R} \setminus 0$. The, with the notation (8.5),

$$x : v_0 + \overline{v}_0 \to \xi i(v_1 - \overline{v}_1) \to -\xi(v_0 + \overline{v}_0).$$

Hence, in this case, the indecomposable summands of $U^\lambda \oplus U^{-\lambda}$ are described in part (a') of Theorem 5.3.

Consider $\lambda \in L_{\mathbb{C}}$. Let

$$U^\lambda = \sum_l U^{\lambda,l}$$

be a direct sum decomposition into one dimensional subspaces. Then

$$U^\lambda \oplus U^{-\lambda} \oplus U^{\overline{\lambda}} \oplus U^{-\overline{\lambda}} = \sum_l (U^{\lambda,l} \oplus U^{-\lambda,l} \oplus U^{\overline{\lambda},l} \oplus U^{-\overline{\lambda},l}) \tag{8.8}$$

is a direct sum decomposition into ($x$, $S$, $u \to \overline{u}$ invariant) subspaces of minimal possible dimension (equal 4). This verifies Theorem 5.1 for for $(G, \mathfrak{g})$ of type II and $\mathbb{D} = \mathbb{R}$.

Fix $\lambda \in L_{\mathbb{C}}$ and $l$ as in (8.8). Let $u^\lambda \in U^{\lambda,l}$ be a non-zero vector. Set $u^{\overline{\lambda}} = \overline{u^\lambda}$, $u^{-\lambda} = Su^\lambda$, and $u^{-\overline{\lambda}} = \overline{u^{-\lambda}}$. Then

$$xu^\lambda = \lambda u^\lambda,\ xu^{-\lambda} = -\lambda u^{-\lambda},\ xu^{\overline{\lambda}} = \overline{\lambda} u^{\overline{\lambda}},\ xu^{-\overline{\lambda}} = -\overline{\lambda} u^{-\overline{\lambda}}. \tag{8.9}$$

Let $\lambda = \xi + i\eta$, with $\xi, \eta \in \mathbb{R}$. Then $\xi\eta \neq 0$. We see from (8.9) that

$$\begin{aligned} x : & u^\lambda + u^{-\lambda} + u^{\overline{\lambda}} + u^{-\overline{\lambda}} \to \lambda(u^\lambda - u^{-\lambda}) + \overline{\lambda}(u^{\overline{\lambda}} - u^{-\overline{\lambda}}),\\ & u^\lambda - u^{-\lambda} + u^{\overline{\lambda}} - u^{-\overline{\lambda}} \to \lambda(u^\lambda + u^{-\lambda}) + \overline{\lambda}(u^{\overline{\lambda}} + u^{-\overline{\lambda}}),\\ & u^\lambda + u^{-\lambda} - u^{\overline{\lambda}} - u^{-\overline{\lambda}} \to \lambda(u^\lambda - u^{-\lambda}) - \overline{\lambda}(u^{\overline{\lambda}} - u^{-\overline{\lambda}}),\\ & u^\lambda - u^{-\lambda} - u^{\overline{\lambda}} + u^{-\overline{\lambda}} \to \lambda(u^\lambda + u^{-\lambda}) - \overline{\lambda}(u^{\overline{\lambda}} + u^{-\overline{\lambda}}). \end{aligned} \tag{8.10}$$

Set

$$\begin{aligned} u_0 &= u^\lambda + u^{-\lambda} + u^{\overline{\lambda}} + u^{-\overline{\lambda}},\ v_0 = i(u^\lambda + u^{-\lambda} - u^{\overline{\lambda}} - u^{-\overline{\lambda}}),\\ u_1 &= u^\lambda - u^{-\lambda} + u^{\overline{\lambda}} - u^{-\overline{\lambda}},\ v_1 = i(u^\lambda - u^{-\lambda} - u^{\overline{\lambda}} + u^{-\overline{\lambda}}). \end{aligned}$$

We see from (8.10) that

$$\begin{aligned} x : & u_0 \to \xi u_1 + \eta v_1,\ u_1 \to \xi u_0 + \eta v_0,\\ & v_0 \to -\eta u_1 + \xi v_1,\ v_1 \to -\eta u_0 + \xi v_0. \end{aligned} \tag{8.11}$$

The formulas (8.11) are consistent with the formulas of part (b) of Theorem 5.3. This verifies Theorem 5.3 for $\mathbb{D} = \mathbb{R}$.

Let $\mathbb{D} = \mathbb{H}$. Since $jU^\lambda = U^{\overline{\lambda}}$, each summand in the decomposition (8.7) is a vector space over $\mathbb{H}$. Let $\lambda \in L_{\mathbb{R}} \cup L_{\mathbb{C}}$. Pick a non-zero vector $u^\lambda \in U^\lambda$. Let

$$v_0 = u^\lambda + Su^\lambda,\ v_1 = u^\lambda - Su^\lambda.$$

Then

$$xv_0 = \lambda v_1,\ xv_1 = \lambda v_0, \tag{8.12}$$

and

$$\mathbb{H}v_0 + \mathbb{H}v_1 \subseteq U \tag{8.13}$$

is a graded $x$-invariant subspace over $\mathbb{H}$. The non-zero part of the right hand side of (8.7) may be grouped into a direct sum of spaces of the form (8.13). This verifies Theorems 5.1 and 5.3 for $\mathbb{D} = \mathbb{H}$.

## 9 A proof of Theorem 5.1 for $x$ semisimple and $(G, \mathfrak{g})$ of type I, and a proof of Theorem 5.2

We consider four cases: $(\mathbb{D} = \mathbb{C}, \iota = 1)$, $(\mathbb{D} = \mathbb{C}, \iota \neq 1)$, $(\mathbb{D} = \mathbb{H}, \iota \neq 1)$ and $(\mathbb{D} = \mathbb{R}, \iota = 1)$.

**Case** $(\mathbb{D} = \mathbb{C}, \iota = 1)$. Let

$$V = \sum_\lambda V^\lambda \tag{9.1}$$

be the decomposition of $V$ into the eigenspaces for $x$ ($xv = \lambda v$ for $v \in V^\lambda$). For two eigenvalues $\lambda$, $\mu$ and the corresponding eigenvectors $v^\lambda$, $v^\mu$, we have

$$\mu^2\tau(v^\mu, v^\lambda) = \tau(x^2v^\mu, v^\lambda) = \tau(v^\mu, -x^2v^\lambda) = \tau(v^\mu, v^\lambda)(-\lambda^2).$$

Thus

$$V^\lambda \perp V^\mu \ \text{ if } \ \mu^2 + \lambda^2 \neq 0. \tag{9.2}$$

Let us decompose the set of non-zero eigenvalues of $x$ into a disjoint union

$$L \cup (-L) \cup iL \cup (-iL).$$

Then, by (9.2),

$$V = V^0 \oplus \sum_{\lambda \in L}(V^\lambda \oplus V^{-\lambda} \oplus V^{i\lambda} \oplus V^{-i\lambda}) \tag{9.3}$$

is a direct sum orthogonal decomposition into graded subspaces preserved by $x$.

For $\lambda \in L$, pick a non-zero vector $v^\lambda \in V^\lambda$, and a non-zero vector $v^{i\lambda} \in V^{i\lambda}$. Let

$$\begin{aligned} v_0 &= v^\lambda + Sv^\lambda, \quad v_1 = v^\lambda - Sv^\lambda, \\ v_0' &= i(v^{i\lambda} + Sv^{i\lambda}), \quad v_1' = v^{i\lambda} - Sv^{i\lambda}. \end{aligned}$$

Then

$$x : v_0 \to \lambda v_1,\ v_1 \to \lambda v_0,\ v_0' \to -\lambda v_1',\ v_1' \to \lambda v_0'. \tag{9.4}$$

Hence,

$$\lambda\tau(v_1, v_1') = \tau(xv_0, v_1') = \tau(v_0, Sxv_1') = \tau(v_0, S\lambda v_0') = \tau(v_0, v_0')\lambda.$$

Thus

$$\tau(v_1, v_1') = \tau(v_0, v_0'). \tag{9.5}$$

We may multiply $v_0$ and $v_1$ by the same complex number to ensure

$$\tau(v_1, v_1') = \tau(v_0, v_0') = 1. \tag{9.6}$$

Notice that

$$\tau(v_0, v_0)\lambda = \tau(v_0, xv_1) = \tau(-Sxv_0, v_1) = \tau(-S\lambda v_1, v_1) = \lambda\tau(v_1, v_1)$$

and

$$\lambda\tau(v_0, v_0) = \tau(xv_1, v_0) = \tau(v_1, Sxv_0) = \tau(v_1, S\lambda v_1) = \tau(v_1, v_1)(-\lambda).$$

Hence

$$\tau(v_0, v_0) = \tau(v_1, v_1) = 0. \tag{9.7}$$

Similarly

$$\tau(v_0', v_0')\lambda = \tau(v_0', xv_1') = \tau(-Sxv_0', v_1') = \tau(S\lambda v_1', v_1') = -\lambda\tau(v_1', v_1')$$

and

$$\lambda\tau(v_0', v_0') = \tau(xv_1', v_0') = \tau(v_1', Sxv_0') = \tau(v_1', -S\lambda v_1') = \tau(v_1', v_1')\lambda.$$

Hence

$$\tau(v_0', v_0') = \tau(v_1', v_1') = 0. \tag{9.8}$$

The subspace

$$\mathbb{C}v_0 \oplus \mathbb{C}v_0' \oplus \mathbb{C}v_1 \oplus \mathbb{C}v_1' \subseteq V^{\lambda} \oplus V^{-\lambda} \oplus V^{i\lambda} \oplus V^{-i\lambda} \tag{9.9}$$

is graded and $x$-invariant and the action of $x$ on this subspace is consistent with the formulas of part (a) of Theorem 5.2. Moreover it is clear that we may decompose the right hand side of (9.9) into direct sum of such subspaces. This verifies Theorems 5.1 and 5.2.

**Case** $(\mathbb{C}, \iota \neq 1)$. Here, instead of (9.2) we have

$$V^{\lambda} \perp V^{\mu} \ \text{ if } \ \mu^2 + \overline{\lambda}^2 \neq 0, \tag{9.10}$$

where $\overline{\lambda} = \iota(\lambda)$. We decompose the set of non-zero eigenvalues of $x$ into a disjoint union

$$L_{i\mathbb{R}} \cup (-L_{i\mathbb{R}}) \cup L_{\mathbb{C}} \cup (-L_{\mathbb{C}}) \cup iL_{\mathbb{C}} \cup (-iL_{\mathbb{C}}),$$

where $\lambda^2 \in i\mathbb{R} \setminus 0$ for $\lambda \in L_{i\mathbb{R}}$, and $\lambda^2 \in \mathbb{C} \setminus i\mathbb{R}$ for $\lambda \in L_{\mathbb{C}}$. Then by (9.10),

$$V = V^0 \oplus \sum_{\lambda \in L_{i\mathbb{R}}} (V^{\lambda} \oplus V^{-\lambda}) \oplus \sum_{\lambda \in L_{\mathbb{C}}} (V^{\lambda} \oplus V^{-\lambda} \oplus V^{i\overline{\lambda}} \oplus V^{-i\overline{\lambda}}) \tag{9.11}$$

is a direct sum orthogonal decomposition into graded $x$-invariant subspaces.

For $\lambda \in L_{i\mathbb{R}}$, pick a non-zero vector $v^{\lambda} \in V^{\lambda}$. Let

$$v_0 = v^{\lambda} + Sv^{\lambda}, \quad v_1 = v^{\lambda} - Sv^{\lambda}.$$

Then

$$x : v_0 \longrightarrow \lambda v_1, \ v_1 \longrightarrow \lambda v_0. \tag{9.12}$$

Moreover,

$$\lambda\tau(v_0, v_0) = \tau(xv_1, v_0) = \tau(v_1, Sxv_0) = \tau(v_1, S\lambda v_1) = \tau(v_1, v_1)(-\overline{\lambda}).$$

Thus

$$\tau(v_1, v_1) = -\frac{\lambda}{\overline{\lambda}}\tau(v_0, v_0), \tag{9.13}$$

where

$$-\frac{\lambda}{\overline{\lambda}} = -\frac{\lambda^2}{|\lambda|^2} = -i\, sgn(im(\lambda^2)). \tag{9.14}$$

Since $\tau_0$ is hermitian, $\tau(v_0, v_0) \in \mathbb{R} \setminus 0$. Thus we may multiply $v_0$ and $v_1$ by the same positive real number so that (by (9.13) and (9.14))

$$\tau(v_0, v_0) = \epsilon = \pm 1, \ \tau(v_1, v_1) = \delta i = \pm i, \ \text{with } -\epsilon\delta = sgn(im(\lambda^2)). \tag{9.15}$$

Clearly

$$\mathbb{C}v_0 \oplus \mathbb{C}v_1 \subseteq V^{\lambda} \oplus V^{-\lambda} \tag{9.16}$$

is a graded, $x$-invariant subspace described in part (b) of Theorem 5.2. This subspace is indecomposable and the right hand side of (9.16) decomposes into an orthogonal direct sum of such subspaces.

Let $\lambda \in L_{\mathbb{C}}$ and let $v^\lambda \in V^\lambda$ be a non-zero vector. Set

$$v_0 = v^\lambda + Sv^\lambda, \quad v_1 = v^\lambda - Sv^\lambda,$$
$$v_0' = i(v^{i\overline{\lambda}} + Sv^{i\overline{\lambda}}), \quad v_1' = v^{i\overline{\lambda}} - Sv^{i\overline{\lambda}}.$$

Then

$$x : v_0 \to \lambda v_1, \ v_1 \to \lambda v_0, \ v_0' \to -\overline{\lambda} v_1', \ v_1' \to \overline{\lambda} v_0'. \tag{9.17}$$

Furthermore

$$\lambda\tau(v_1, v_1') = \tau(xv_0, v_1') = \tau(v_0, Sxv_1') = \tau(v_0, v_0')\lambda$$

so that

$$\tau(v_1, v_1') = \tau(v_0, v_0').$$

As before we may scale the vectors $v_0$ and $v_1$ by the same number so that

$$\tau(v_1, v_1') = \tau(v_0, v_0') = 1. \tag{9.18}$$

Moreover,

$$\lambda^2\tau(v_0, v_0) = \tau(x^2v_0, v_0) = \tau(v_0, -x^2v_0) = \tau(v_0, v_0)(-\overline{\lambda}^2)$$

and

$$\lambda^2\tau(v_1, v_1) = \tau(x^2v_1, v_1) = \tau(v_1, -x^2v_1) = \tau(v_1, v_1)(-\overline{\lambda}^2),$$

so that

$$(\lambda^2 + \overline{\lambda}^2)\tau(v_0, v_0) = (\lambda^2 + \overline{\lambda}^2)\tau(v_1, v_1) = 0,$$

which implies

$$\tau(v_0, v_0) = \tau(v_1, v_1) = 0. \tag{9.19}$$

Similarly,

$$\tau(v_0', v_0') = \tau(v_1', v_1') = 0. \tag{9.20}$$

Clearly,

$$\mathbb{C}v_0 \oplus \mathbb{C}v_0' \oplus \mathbb{C}v_1 \oplus \mathbb{C}v_1' \subseteq V^\lambda \oplus V^{-\lambda} \oplus V^{i\overline{\lambda}} \oplus V^{-i\overline{\lambda}} \tag{9.21}$$

is a graded, $x$-invariant subspace, as in part (a) of Theorem 5.2. This subspace is indecomposable and the right hand side of (9.21) decomposes into an orthogonal direct sum of such subspaces.

**Case** ($\mathbb{D} = \mathbb{H}, \iota \neq 1$). Here we view $V$ as a vector space over $\mathbb{C} \subseteq \mathbb{H}$. Then the decomposition (9.1) holds and

$$jV^\lambda = V^{\overline{\lambda}}, \tag{9.22}$$

where $\overline{\lambda} = \iota(\lambda)$. Furthermore,

$$V^\lambda \perp V^\mu \ \text{ if } \ \mu^2 + \lambda^2 \neq 0 \text{ and } \mu^2 + \overline{\lambda}^2 \neq 0. \tag{9.23}$$

Indeed, let $v^\mu \in V^\mu$, $v^\lambda \in V^\lambda$ and let $\tau(v^\mu, v^\lambda) = \alpha + j\beta$, with $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}\mu^2(\alpha + j\beta) &= \mu^2\tau(v^\mu, v^\lambda) = \tau(x^2 v^\mu, v^\lambda) = \tau(v^\mu, -x^2 v^\lambda) = \tau(v^\mu, -\lambda^2 v^\lambda)\\ &= \tau(v^\mu, v^\lambda)(-\overline{\lambda}^2) = (\alpha + j\beta)(-\overline{\lambda}^2).\end{aligned}$$

Hence,

$$(\mu^2 + \overline{\lambda}^2)\alpha = 0 \quad\text{and}\quad (\mu^2 + \lambda^2)j\beta = 0,$$

and (9.23) follows.

We decompose the set of non-zero eigenvalues of $x$ into a disjoint union

$$L_{i\mathbb{R}} \cup (-L_{i\mathbb{R}}) \cup L_{\mathbb{R}} \cup (-L_{\mathbb{R}}) \cup L_{\mathbb{C}} \cup (-L_{\mathbb{C}}), \tag{9.24}$$

where $\lambda^2 \in i\mathbb{R}\setminus 0$ for $\lambda \in L_{i\mathbb{R}}$, $\lambda^2 \in \mathbb{R}\setminus 0$ for $\lambda \in L_{\mathbb{R}}$, and $\lambda^2 \in \mathbb{C}\setminus(i\mathbb{R}\cup\mathbb{R})$ for $\lambda \in L_{\mathbb{C}}$. Then by (9.22) and (9.23),

$$\begin{aligned} V = V^0 &\oplus \sum_{\lambda\in L_{i\mathbb{R}}} (V^\lambda \oplus V^{-\lambda} \oplus V^{\overline{\lambda}} \oplus V^{-\overline{\lambda}})\\ &\oplus \sum_{\lambda\in L_{\mathbb{R}}} (V^\lambda \oplus V^{-\lambda} \oplus V^{i\lambda} \oplus V^{-i\lambda})\\ &\oplus \sum_{\lambda\in L_{\mathbb{C}}} (V^\lambda \oplus V^{-\lambda} \oplus V^{i\lambda} \oplus V^{-i\lambda} \oplus V^{\overline{\lambda}} \oplus V^{-\overline{\lambda}} \oplus V^{-i\overline{\lambda}} \oplus V^{i\overline{\lambda}})\end{aligned} \tag{9.25}$$

is a direct sum orthogonal decomposition into graded $x$-invariant subspaces over $\mathbb{H}$.

Let $\lambda \in L_{i\mathbb{R}}$. Then (9.12) holds, and since $\tau(v_0, v_0) \in \mathbb{R}$, (9.13) holds too. Therefore (9.15) holds and, instead of (9.16), we see that

$$\mathbb{H}v_0 \oplus \mathbb{H}v_1 \subseteq V^\lambda \oplus V^{-\lambda} \oplus V^{\overline{\lambda}} \oplus V^{\overline{-\lambda}} \tag{9.26}$$

is a graded $x$-invariant subspace, as in part (c) of Theorem 5.2. This subspace is indecomposable and the right hand side of (9.26) decomposes into an orthogonal direct sum of such subspaces.

For $\lambda \in L_{\mathbb{R}} \cup L_{\mathbb{C}}$ the argument (9.17)-(9.21) carries over. This completes the proof of both Theorems 5.1 and 5.2 in the case $\mathbb{D} = \mathbb{H}$.

**Case** ($\mathbb{D} = \mathbb{R}, \iota = 1$). Let

$$V^{\mathbb{C}} = \sum_\lambda V^{\mathbb{C},\lambda} \tag{9.27}$$

be the decomposition of $V^{\mathbb{C}}$, the complexification of $V$, into the eigenspaces for $x$. Let $V^{\mathbb{C}} \ni v \to \overline{v} \in V^{\mathbb{C}}$ be the complex conjugation with respect to the real form $V \subseteq V^{\mathbb{C}}$. The form $\tau$ extends uniquely to a complex linear form on $V^{\mathbb{C}}$. As before we check that

$$V^{\mathbb{C},\mu} \perp V^{\mathbb{C},\lambda} \quad\text{if}\quad \mu^2 + \lambda^2 \neq 0. \tag{9.28}$$

Let $L_{i\mathbb{R}}$, $L_{\mathbb{R}}$, $L_{\mathbb{C}}$ be as in (9.24). Then

$$\begin{aligned} V^{\mathbb{C}} = V^{\mathbb{C},0} &\oplus \sum_{\lambda\in L_{i\mathbb{R}}} (V^{\mathbb{C},\lambda} \oplus V^{\mathbb{C},-\lambda} \oplus V^{\mathbb{C},\overline{\lambda}} \oplus V^{\mathbb{C},-\overline{\lambda}})\\ &\oplus \sum_{\lambda\in L_{\mathbb{R}}} (V^{\mathbb{C},\lambda} \oplus V^{\mathbb{C},-\lambda} \oplus V^{\mathbb{C},i\lambda} \oplus V^{\mathbb{C},-i\lambda})\\ &\oplus \sum_{\lambda\in L_{\mathbb{C}}} (V^{\mathbb{C},\lambda} \oplus V^{\mathbb{C},-\lambda} \oplus V^{\mathbb{C},i\lambda} \oplus V^{\mathbb{C},-i\lambda} \oplus V^{\mathbb{C},\overline{\lambda}} \oplus V^{\mathbb{C},-\overline{\lambda}} \oplus V^{\mathbb{C},-i\overline{\lambda}} \oplus V^{\mathbb{C},i\overline{\lambda}})\end{aligned} \tag{9.29}$$

is an orthogonal direct sum decomposition into graded $x$-invariant subspaces invariant under the complex conjugation, $V^{\mathbb{C}} \ni u \to \overline{u} \in V^{\mathbb{C}}$ with respect to the real form $V$.

Let $\lambda \in L_{i\mathbb{R}}$ and let $v^\lambda \in V^{\mathbb{C},\lambda}$ be a non-zero vector. Let

$$u = v^\lambda + Sv^\lambda \text{ and } v = v^\lambda - Sv^\lambda.$$

Then

$$x : u \to \lambda v,\ \ v \to \lambda u,\ \ \overline{u} \to \overline{\lambda}\overline{v},\ \ \overline{v} \to \overline{\lambda}\overline{u}. \tag{9.30}$$

Hence

$$\begin{aligned}\lambda\tau(u,u) &= \tau(xv,u) = \tau(v,Sxu) = \tau(v,-\lambda v) = \tau(v,v)(-\lambda),\\ \lambda\tau(v,v) &= \tau(xu,v) = \tau(u,Sxv) = \tau(u,\lambda u) = \tau(u,u)\lambda.\end{aligned}$$

Therefore

$$\tau(u,u) = \tau(v,v) = \tau(\overline{u},\overline{u}) = \tau(\overline{v},\overline{v}) = 0. \tag{9.31}$$

Furthermore

$$\lambda\tau(v,\overline{v}) = \tau(xu,\overline{v}) = \tau(u,Sx\overline{v}) = \tau(u,\overline{u})\overline{\lambda},$$

so that

$$\tau(v,\overline{v}) = \tau(u,\overline{u})(-i)sgn(im(\lambda^2)). \tag{9.32}$$

Multiplying $u$ and $v$ by the same non-zero real number does not change (9.30), (9.31), (9.32). Thus we may assume

$$\tau(u,\overline{u}) = \epsilon/2, \tag{9.33}$$

where $\epsilon = \pm 1$. Set

$$v_0 = u + \overline{u},\ \ v_0' = i(u-\overline{u}), v_1 = v + \overline{v},\ \ v_1' = i(v - \overline{v}). \tag{9.34}$$

Then, by (9.31) and (9.33),

$$\begin{aligned}\tau(v_0,v_0) &= \tau(u,\overline{u}) + \tau(\overline{u},u) = \epsilon,\\ \tau(v_0',v_0') &= -\tau(u,-\overline{u}) - \tau(-\overline{u},u) = \epsilon,\\ \tau(v_0,v_0') &= \tau(u,-i\overline{u}) + \tau(\overline{u},iu) = 0,\\ \tau(v_1,v_1) &= \tau(v,\overline{v}) + \tau(\overline{v},v) = 0,\\ \tau(v_1',v_1') &= \tau(iv,-\overline{v}) + \tau(-\overline{v},iv) = 0,\\ \tau(v_1,v_1') &= \tau(v,-i\overline{v}) + \tau(\overline{v},iv) = -2i\tau(v,\overline{v})\\ &= -2\tau(u,\overline{u})sgn(im(\lambda^2)) = -\tau(v_0,\overline{v}_0)sgn(im(\lambda^2)).\end{aligned} \tag{9.35}$$

We see from (9.30) and (9.34) that, with $\lambda = \xi + i\eta$, $\xi,\eta \in \mathbb{R}$,

$$\begin{aligned}x : &v_0 \to \xi v_1 + \eta v_1',\ \ v_1 \to \xi v_0 + \eta v_0',\\ &v_0' \to -\eta v_1 + \xi v_1',\ \ v_1' \to -\eta v_0 + \xi v_0'.\end{aligned} \tag{9.36}$$

Notice that

$$im(\lambda^2) = 2\xi\eta. \tag{9.37}$$

Thus the last formula in (9.35) may be rewritten as

$$\tau(v_1, v_1') = \epsilon\delta, \ -\delta = sgn(\xi\eta). \tag{9.38}$$

Thus $\lambda = \xi(1 - i\delta)$ and if we replace $v_1'$ by $\epsilon\delta v_1'$ then (9.36) coincides with

$$\begin{aligned} x : & v_0 \to \xi(v_1 - \epsilon v_1'), \ v_1 \to \xi(v_0 - \delta v_0'), \\ & v_0' \to \delta\xi(v_1 + \epsilon v_1'), \ v_1' \to \epsilon\xi(v_0 + \delta v_0'). \end{aligned} \tag{9.39}$$

Let

$$g : v_0 \to v_0, \ v_0' \to \delta v_0', \ v_1 \to v_1, \ v_1' \to v_1'.$$

Then $g \in G$ and $gxg^{-1}$ acts according to the formula (c) of Theorem 5.2.

The subspace

$$\mathbb{R}v_0 \oplus \mathbb{R}v_0' \oplus \mathbb{R}v_1 \oplus \mathbb{R}v_1' \subseteq V \tag{9.40}$$

is graded, $x$-invariant and indecomposable.

Let $\lambda \in L_{\mathbb{R}}$. Then either $\lambda \in \mathbb{R} \setminus 0$ or $\lambda \in i\mathbb{R} \setminus 0$. Suppose $\lambda \in \mathbb{R} \setminus 0$. Then the eigenspace $V^{\mathbb{C},\lambda}$ is closed under the complex conjugation. Hence we may chose a non-zero vector $v^{\lambda} \in V \cap V^{\mathbb{C},\lambda}$. Let $v^{i\lambda} \in V \cap V^{\mathbb{C},i\lambda}$ be a non-zero vector such that $\overline{v^{i\lambda}} = Sv^{i\lambda}$. Set

$$v_0 = v^{\lambda} + Sv^{\lambda}, \ v_1 = v^{\lambda} - Sv^{\lambda}, \ v_0' = v^{i\lambda} + Sv^{i\lambda}, \ v_1' = -i(v^{i\lambda} - Sv^{i\lambda}). \tag{9.41}$$

Then

$$\overline{v_0} = v_0, \ \overline{v_1} = v_1, \ \overline{v_0'} = v_0', \ \overline{v_1'} = v_1', \tag{9.42}$$

and

$$x : v_0 \to \lambda v_1, \ v_1 \to \lambda v_0, \ v_0' \to -\lambda v_1', \ v_1' \to \lambda v_0'. \tag{9.43}$$

Furthermore,

$$\lambda\tau(v_1, v_1') = \tau(xv_0, v_1') = \tau(v_0, Sxv_1') = \tau(v_0, v_0')\lambda,$$

so that

$$\tau(v_1, v_1') = \tau(v_0, v_0'). \tag{9.44}$$

Similarly we check that the vectors $v_0$, $v_0'$, $v_1$, $v_1'$ are isotropic. Multiplying $v_0$ and $v_1$ by the same number we may assume that

$$\tau(v_1, v_1') = \tau(v_0, v_0') = 1. \tag{9.45}$$

The subspace

$$\mathbb{R}v_0 \oplus \mathbb{R}v_0' \oplus \mathbb{R}v_1 \oplus \mathbb{R}v_1' \subseteq V \tag{9.46}$$

is graded, $x$-invariant and indecomposable. The formulas (9.43) are compatible with the formulas of part (a) of Theorem 5.2.

Finally, let $\lambda \in L_{\mathbb{C}}$. Choose non-zero vectors $v^{\lambda} \in V^{\mathbb{C},\lambda}$, $v^{i\lambda} \in V^{\mathbb{C},i\lambda}$ and let

$$u = v^{\lambda} + Sv^{\lambda}, \ v = v^{\lambda} - Sv^{\lambda}, \ u' = v^{i\lambda} + Sv^{i\lambda}, \ v' = v^{i\lambda} - Sv^{i\lambda}. \tag{9.47}$$

Then

$$\begin{aligned} x :& u \to \lambda v,\ v \to \lambda u,\ \overline{u} \to \overline{\lambda}\overline{v},\ \overline{v} \to \overline{\lambda}\overline{u}, \\ & u' \to i\lambda v',\ v' \to i\lambda u',\ \overline{u'} \to \overline{i\lambda}\,\overline{v'},\ \overline{v'} \to \overline{i\lambda}\,\overline{u'}. \end{aligned} \tag{9.48}$$

We see from (9.28) and (9.24) that

$$\begin{aligned} &\left(V^{\mathbb{C},\lambda} + V^{\mathbb{C},-\lambda}\right) \perp \left(V^{\mathbb{C},-i\overline{\lambda}} + V^{\mathbb{C},i\overline{\lambda}}\right), \text{ and} \\ &\left(V^{\mathbb{C},\lambda} + V^{\mathbb{C},-\lambda}\right) \perp \left(V^{\mathbb{C},\overline{\lambda}} + V^{\mathbb{C},-\overline{\lambda}}\right). \end{aligned} \tag{9.49}$$

Thus by (9.29), the restriction of the form $\tau$ to

$$V^{\mathbb{C},\lambda} + V^{\mathbb{C},-\lambda} + V^{\mathbb{C},i\lambda} + V^{\mathbb{C},-i\lambda}$$

is non-degenerate. Hence we may choose the vectors $v^{\lambda}$, $v^{i\lambda}$ so that

$$\tau(u,u') \neq 0. \tag{9.50}$$

The following calculation

$$\lambda\tau(v,v') = \tau(xu,v') = \tau(u,xv') = \tau(u,i\lambda u') = \tau(u,u')i\lambda$$

shows that

$$\tau(v,v') = \tau(u,u').$$

As before we may assume that

$$\tau(v,v') = \tau(u,u') = \frac{1}{2}. \tag{9.51}$$

Since $\tau$ is the complexification of a real form the usual calculation using (9.24), (9.28) and (9.51) implies

$$\begin{aligned} &\tau(u,u') = \tau(\overline{u},\overline{u'}) = \frac{1}{2}, \\ &\tau(u,u) = \tau(u,\overline{u}) = \tau(\overline{u},u') = \tau(u',u') = 0, \\ &\tau(\overline{u},\overline{u}) = \tau(\overline{u}',\overline{u'}) = 0, \\ &\tau(v,-iv') = \tau(\overline{v},i\overline{v'}) = \frac{1}{2}, \\ &\tau(\overline{v},\overline{v}) = \tau(v,v) = \tau(v,\overline{v'}) = \tau(\overline{v},v') \\ &= \tau(v',v') = \tau(\overline{v}',\overline{v'}) = 0. \end{aligned} \tag{9.52}$$

Set

$$\begin{aligned} &u_0 = u + \overline{u},\ v_0 = i(u - \overline{u}),\ u_0' = u' + \overline{u'},\ v_0' = -i(u' - \overline{u'}), \\ &u_1 = v + \overline{v},\ v_1 = i(v - \overline{v}),\ u_1' = -i(v' - \overline{v'}),\ v_1' = -v' - \overline{v'}. \end{aligned} \tag{9.53}$$

Let $\lambda = \xi + i\eta$, $\xi, \eta \in \mathbb{R}$, Then the formulas of part (d) of Theorem 5.2 hold. Furthermore the subspace

$$\begin{aligned} &\mathbb{R}u_0 \oplus \mathbb{R}u_0' \oplus \mathbb{R}v_0 \oplus \mathbb{R}v_0' \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_1' \oplus \mathbb{R}v_1 \oplus \mathbb{R}v_1' \\ &\subseteq V^{\mathbb{C},\lambda} \oplus V^{\mathbb{C},-\lambda} \oplus V^{\mathbb{C},i\lambda} \oplus V^{\mathbb{C},-i\lambda} \oplus V^{\mathbb{C},\overline{\lambda}} \oplus V^{\mathbb{C},-\overline{\lambda}} \oplus V^{\mathbb{C},-i\overline{\lambda}} \oplus V^{\mathbb{C},i\overline{\lambda}} \end{aligned}$$

is graded, $x$-invariant and indecomposable, and the space on the right hand side of the inclusion decomposes into a direct sum of such subspaces. This completes our proof.

## 10 A proof of Theorem 4.2

We may assume that $(x_n, V)$ is uniform and that

$$V = F \oplus x_n F \oplus x_n^2 F \oplus ... x_n^m F,$$

where the subspace $F$ is graded and $x_s$-invariant. For each set of positive numbers $a_0$, $a_1$, ..., $a_m$ (such that $a_k a_{m-k} = 1$, $0 \le k \le m$, in the type I case) the formula

$$\begin{aligned} & g(u_0 + x_n u_1 + ... + x_n^m u_m) \\ & = a_0 u_0 + a_1 x_n u_1 + ... + a_m x_n^m u_m \qquad (u_0, u_1, ..., u_m \in F) \end{aligned} \tag{10.1}$$

defines an element $g \in G^{x_s}$. Furthermore,

$$g x_n g^{-1} : x_n^k u_i \to \frac{a_{k+1}}{a_k} x_k^{k+1} u_i \qquad (u_i \in F,\ 0 \le i, k \le m) \tag{10.2}$$

An elementary argument shows that there is a sequence $a_k^{(l)}$, $l = 1, 2, 3, ...$ of positive numbers such that for all $0 \le k \le m$,

$$\underset{l \to \infty}{\to} \lim \frac{a_{k+1}^{(l)}}{a_k^{(l)}} = 0. \tag{10.3}$$

Let $g^{(l)} \in G^{x_s}$ be as in (10.1), for the sequence $a_k^{(l)}$. Then, by (10.2) and (10.3),

$$g^{(l)} x g^{(l)-1} = x_s + g^{(l)} x_n (g^{(l)})^{-1} \to x_s \text{ as } l \to \infty.$$

## 11 A proof of Theorem 4.8

Let $x_1 \in \mathfrak{g}_1$ be semisimple and let

$$V = V^{(0)} \oplus V^{(1)} \oplus V^{(2)} \oplus ...$$

be such that each $x_1|_{V^{(0)}} = 0$ and for $j \ge 1$, $(x_1, V^{(j)})$ is indecomposable, with $x_1|_{V^{(j)}} \ne 0$. We assume that the sum is orthogonal in the type I case. Then, by Theorems 5.2 and 5.3, $(x_1^2, V_0^{(j)})$ and $(x_1^2, V_1^{(j)})$, $j \ge 1$, are indecomposable.

Let $x_2 \in \mathfrak{g}_1$ be another semisimple elements such that $Gx_1^2 = Gx_2^2$. Then, by the above argument, we may assume that $x_2|_{V^{(0)}} = 0$ and that for $j \ge 1$ $(x_2, V^{(j)})$ is indecomposable with $x_2|_{V^{(j)}} \ne 0$. Hence we may assume that $(x_1, V^{(j)})$ and $(x_2, V^{(j)})$, $j \ge 1$, are indecomposable. But then the Theorem 4.8 follows from Theorems 5.2 and 5.3 by inspection.

## 12 A proof of Theorem 4.3

Suppose $X \in \mathfrak{G}_1$ is not semisimple. Then in the Jordan decomposition $X = X_S + X_N$, $X_N \ne 0$. By Theorem 4.2, $X_S \in Cl(Gx)$. But $X_S \notin Gx$. Thus the orbit $Gx$ is not closed.

Suppose $X \in \mathfrak{G}_1$ is semisimple. Since the $Gl(V)$-orbit through $X$ in $End(V)$ is closed, we see that $Cl(Gx)$ is a union of semisimple orbits. Since $Gx^2 \subseteq \mathfrak{G}_0$ is closed, Theorem 4.8 implies that $Cl(Gx)$ is a single orbit, hence is equal to $Gx$.

## 13 A proof of Theorem 4.4

Let $x \in \mathfrak{g}_1$ be semisimple. We'll say that $V$ is isotypic for $x$, or that $(x, V)$ is isotypic, if $(x, V)$ decomposes into mutually similar indecomposable pieces. Two isotypic elements $(x, V)$ and $(x', V')$ are of different types if the indecomposable pieces of $(x, V)$ are not similar to the indecomposable pieces of $(x', V')$.

**Lemma 13.1.** *Let $(x, V)$ and $(x', V')$ be two semisimple isotypic elements of different types. Then the only $y \in Hom(V', V)$ such that $xy + yx' = 0$ is $y = 0$.*

**Proof.** We may assume that the elements $(x, V)$, $(x', V')$ are indecomposable. We need to check that the map

$$Hom(V', V) \ni y \to xy + yx' \in Hom(V', V)$$

is injective. The eigenvalues of this map are sums of the eigenvalues of $x$ and $x'$. By Theorems 5.2 and 5.3 these sums are not zero. $\square$

**Lemma 13.2.** *The anticommutant of $\mathfrak{g}_1$ in $\mathfrak{g}_1$ is zero: ${}^{\mathfrak{g}_1}\mathfrak{g}_1 = 0$.*

**Proof.** Suppose the Lie superalgebra $\mathfrak{g}$ is of type II. Then $\mathfrak{g}_1 = S\mathfrak{g}_1$. Let $x \in {}^{\mathfrak{g}_1}\mathfrak{g}_1$ and let $y \in \mathfrak{g}_1$. Then $Sy \in \mathfrak{g}_1$ and therefore $\{Sy, x\} = 0$. In particular, $0 = tr\{Sy, x\} = \langle y, x\rangle$. Thus $x$ is orthogonal to $\mathfrak{g}_1$, and since the form $\langle\ ,\ \rangle$ is non-degenerate, we see that $x = 0$.

Suppose the Lie superalgebra $\mathfrak{g}$ is of type I. In this case $\mathfrak{g}_1 \cap S\mathfrak{g}_1 = 0$, so we are forced to use a different argument. We may assume that $\mathfrak{g}$ is complex (and of type I). Then $dim\, V_0 \geq 2$ or $dim\, V_1 \geq 2$. Consider the case $dim\, V_0 \geq 2$. The second one is analogous. Let $x \in {}^{\mathfrak{g}_1}\mathfrak{g}_1$, and let $y \in \mathfrak{g}_1$. Then, in terms of (2.12), we have

$$(xy + yx)(v_0 + v_1) = (w_x^* w_y + w_y^* w_x)v_0 + (w_x w_y^* + w_y w_x^*)v_1.$$

Thus the condition $xy+yx = 0$ translates to $w_x^* w_y + w_y^* w_x = 0$. Hence, for any $v_0, v_0' \in V_0$,

$$\tau_0(v_0, w_x^* w_y v_0') + \tau_0(v_0, w_y^* w_x v_0') = 0,$$

or equivalently,

$$\tau_1(w_x v_0, w_y v_0') + \tau_1(w_y v_0, w_x v_0') = 0.$$

Now fix $v_0 \in V_0 \setminus 0$ and let $v_0' \in V_0 \setminus 0$ be such that the vectors $v_0, v_0'$ are linearly independent. Then

$$\{w(v_0');\ w(v_0) = 0,\ w \in Hom(V_0, V_1)\} = V_1.$$

Hence, $w_x(v_0)$ is orthogonal to $V_1$ with respect to the form $\tau_1$. Since this form is non-degenerate, $w_x = 0$, and therefore $x = 0$. $\square$

Let

$$V = V^0 \oplus V^1 \oplus V^2 \oplus ...$$

be the isotypic decomposition of $V$, with respect to $x$, with $V^0 = ker(x)$. Then

$$xV = V^1 \oplus V^2 \oplus ...$$

and the first part of (b) follows. Let $(G(V^i), \mathfrak{g}(V^i))$ be the restriction of $(G, \mathfrak{g})$ to $V^i$. Then

$$G^{x^2} = G(V^0)^{x^2} \times G(V^1)^{x^2} \times G(V^2)^{x^2} \times ...,$$
$$\mathfrak{g}^{x^2} = \mathfrak{g}(V^0)^{x^2} \oplus \mathfrak{g}(V^1)^{x^2} \oplus \mathfrak{g}(V^2)^{x^2} \oplus ...,$$

where the summands are orthogonal with respect to the symplectic form $\langle\ ,\ \rangle$ on $\mathfrak{g}_1$.

Moreover, $G(V^0)^{x^2} = G(V^0)$ and $\mathfrak{g}(V^0)^{x^2} = \mathfrak{g}(V^0)$. Hence,

$$G^{x^2} = G(V^0) \times G(V^1)^{x^2} \times G(V^2)^{x^2} \times ...,$$
$$\mathfrak{g}^{x^2} = \mathfrak{g}(V^0) \oplus \mathfrak{g}(V^1)^{x^2} \oplus \mathfrak{g}(V^2)^{x^2} \oplus ... .$$

This verifies the second part of (b). We shall see in section 13 () that there is $y \in {}^x\mathfrak{g}_1$ such that for all $i \neq j$ greater than or equal to 1, the restrictions $(y, V^i)$, $(y, V^j)$ are isotropic of different types. Hence the lemmas 13.1 and 13.2 imply

$${}^{({}^x\mathfrak{g}_1)}\mathfrak{g}_1 = {}^{({}^x\mathfrak{g}_1(V^1))}\mathfrak{g}_1(V^1) \oplus {}^{({}^x\mathfrak{g}_1(V^2))}\mathfrak{g}_1(V^2) \oplus ... .$$

Hence the proof of the last formula in (b) is reduced to the case when $(x, V)$ is isotypic and non-zero. Also, the above formula reduces the proof of (c), which we leave to the reader, to the case when $(x, V)$ is isotypic and non-zero.

From now on, we assume that $(x, V)$ is isotypic and $x \neq 0$. We proceed via a case by case analysis according to Theorems 5.2 and 5.3.

**Case 5.2.a.** Here $\mathbb{D}$ is arbitrary and the vector spaces $V_i$ $(i = 0, 1)$ have basis

$$\begin{aligned} &v_{i,1}, v_{i,2}, v_{i,3}, ..., v_{i,n}, \text{ such that} \\ &\tau(v_{0,k}, v'_{0,k}) = \tau(v_{1,k}, v'_{1,k}) = 1 \qquad (k = 1, 2, 3, ..., n) \end{aligned} \tag{13.1}$$

and all the other pairings are zero. Further,

$$x = x(\xi) : v_{0,k} \to \xi v_{1,k},\ v_{1,k} \to \xi v_{0,k},\ v'_{0,k} \to -\iota(\xi) v'_{1,k},\ v'_{1,k} \to \iota(\xi) v'_{0,k}. \tag{13.2}$$

Thus

$$x^2 : v_{0,k} \to \xi^2 v_{0,k},\ v_{1,k} \to \xi^2 v_{1,k},\ v'_{0,k} \to -\iota(\xi)^2 v'_{0,k},\ v'_{1,k} \to -\iota(\xi)^2 v'_{1,k}. \tag{13.3}$$

Since $\xi^2 \neq -\iota(\xi)^2$, the group $G^{x^2}$ preserves each of the isotropic subspaces

$$\sum_{k=1}^{n} \mathbb{D} v_{i,k} \subseteq V_i,\ \sum_{k=1}^{n} \mathbb{D} v'_{i,k} \subseteq V_i \quad (i = 0, 1). \tag{13.4}$$

By Witt's Theorem the restriction

$$G^{x^2}|_{\sum_{k=1}^{n} \mathbb{D} v_{i,k}} = GL(\sum_{k=1}^{n} \mathbb{D} v_{i,k})^{x^2} \quad (i = 0, 1). \tag{13.5}$$

Hence,

$$G^{x^2} = G^{x^2}|_{V_0} \times G^{x^2}|_{V_1} = \begin{cases} GL_n(\mathbb{D}) \times GL_n(\mathbb{D}) & \text{if } \mathbb{D} \neq \mathbb{H} \text{ or } \mathbb{D} = \mathbb{H} \text{ and } \xi^2 \in \mathbb{R}, \\ GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) & \text{if } \mathbb{D} = \mathbb{H} \text{ and } \xi^2 \notin \mathbb{R}. \end{cases} \tag{13.6}$$

Suppose $w \in Hom(V_0, V_1)$ commutes with $x^2$. Then by (13.3), there are elements $w_{kl}, w'_{kl}, w^*_{kl}, {w^*_{kl}}' \in \mathbb{D}$ commuting with $\xi^2$ and such that

$$\begin{aligned} w(v_{0,k}) &= \sum_{l=1}^n w_{kl} v_{1,l}, \quad w(v'_{0,k}) = \sum_{l=1}^n w'_{kl} v'_{1,l}, \\ w^*(v_{1,k}) &= \sum_{l=1}^n w^*_{kl} v_{0,l}, \quad w^*(v'_{1,k}) = \sum_{l=1}^n {w^*_{kl}}' v'_{0,l}. \end{aligned} \tag{13.7}$$

By (2.10) and (13.1) we have

$$\begin{aligned} \iota({w^*_{pk}}') &= \tau_0(v_{0,k}, \sum_{l=1}^n {w^*_{pl}}' v'_{0,l}) = \tau_0(v_{0,k}, w^*(v'_{1,p})) \\ &= \tau_1(w(v_{0,k}), v'_{1,p}) = \tau_1(\sum_{l=1}^n w_{kl} v_{1,l}, v'_{1,p}) = w_{kp}, \end{aligned}$$

and

$$\begin{aligned} \iota(w^*_{pk}) &= \tau_0(v'_{0,k}, v_{0,k}) \iota(w^*_{pk}) = \tau_0(v'_{0,k}, \sum_{l=1}^n w^*_{pl} v_{0,l}) = \tau_0(v'_{0,k}, w^*(v_{1,p})) \\ &= \tau_1(w(v'_{0,k}), v_{1,p}) = {w_{kp}}' \tau_1(v'_{1,p}, v_{1,p}) = -{w_{kp}}'. \end{aligned}$$

Hence,

$$w^*_{pk} = -\iota({w_{kp}}'), \quad {w^*_{pk}}' = \iota(w_{kp}). \tag{13.8}$$

For $y, z \in \mathfrak{g}_1^{x^2}$ let $w = y|_{V_0}$ and let $u = z|_{V_0}$. Then $w, u \in Hom(V_0, V_1)$ commute with $x^2$ and, by (2.13), (13.7) and (13.8),

$$\begin{aligned} \frac{1}{4}\langle y, z\rangle &= tr_{\mathbb{D}/\mathbb{R}}(w^* u) \\ &= \sum_{k,l=1}^n tr_{\mathbb{D}/\mathbb{R}}(u_{kl} w^*_{lk} + u'_{kl} {w^*_{lk}}') = \sum_{k,l=1}^n tr_{\mathbb{D}/\mathbb{R}}(u'_{kl} \iota(w_{kl}) - u_{kl} \iota({w_{kl}}')). \end{aligned} \tag{13.9}$$

The formula (2.12), (13.2), (13.7) and (13.8) imply

$$\begin{aligned} yx &: v_{0,k} \to -\sum_{l=1}^n \xi \iota({w_{lk}}') v_{0,l}, \quad v'_{0,k} \to -\sum_{l=1}^n \iota(\xi) \iota(w_{lk}) v'_{0,l}, \\ xy &: v_{0,k} \to \sum_{l=1}^n w_{kl} \xi v_{0,l}, \quad v'_{0,k} \to \sum_{l=1}^n {w_{kl}}' \iota(\xi) v'_{0,l}. \end{aligned} \tag{13.10}$$

Hence,

$$\begin{aligned} y \in \mathfrak{g}_1^x &\text{ if and only if } {w_{kl}}' = -\iota(\xi)\iota(w_{lk})\iota(\xi)^{-1}, \\ y \in {}^x\mathfrak{g}_1 &\text{ if and only if } {w_{kl}}' = \iota(\xi)\iota(w_{lk})\iota(\xi)^{-1}. \end{aligned} \tag{13.11}$$

Suppose $y, z \in \mathfrak{g}_1^x$. Then by (13.9) and (13.11),

$$\begin{aligned}\frac{1}{4}\langle y, z\rangle &= \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(-\iota(\xi)\iota(u_{lk})\iota(\xi)^{-1}\iota(w_{kl}) + u_{kl}\xi^{-1}w_{lk}\xi)\\ &= -\sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(w_{kl}\xi^{-1}u_{lk}\xi) + \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(u_{lk}\xi^{-1}w_{kl}\xi) \qquad (13.12)\\ &= -\sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(w_{kl}\xi u_{lk}\xi^{-1}) + \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(u_{lk}\xi^{-1}w_{kl}\xi) = 0,\end{aligned}$$

where the equation $\xi^{-1}u_{lk}\xi = \xi u_{lk}\xi^{-1}$ follows from the fact that $u_{lk}$ commutes with $\xi^2$. The computation (13.12) shows that $\mathfrak{g}_1^x$ is an isotropic subspace of $\mathfrak{g}_1$.

Suppose $y \in \mathfrak{g}_1^x$ and $z \in {}^x\mathfrak{g}_1$. Then $w_{kl}' = \iota(\xi^{-1}w_{lk}\xi)$ and, as in (13.12), we show that

$$\frac{1}{4}\langle y, z\rangle = -2\sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(u_{lk}\xi w_{kl}\xi^{-1}), \qquad (13.13)$$

which implies that the symplectic form $\langle\ ,\ \rangle$ provides a non-degenerate pairing between $\mathfrak{g}_1^x$ and ${}^x\mathfrak{g}_1$. The supergroup $(G^{x^2}, \mathfrak{g}^{x^2})$ is irreducible, of type II, and the ranks of $G^{x^2}|_{V_0}$ and $G^{x^2}|_{V_1}$ are equal.

Furthermore a straightforward computation shows that

$$^{(x(\xi)}\mathfrak{g}_1)\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^{x(\xi)})} = \{x(\zeta);\ \zeta^2 \in \mathbb{D}^{(\mathbb{D}^{\xi^2})}\}. \qquad (13.13.1)$$

**Case 5.2.b** Here $\mathbb{D} \supseteq \mathbb{C}$ and the vector spaces $V_i$ $(i = 0, 1)$ have basis

$$\begin{aligned}&v_{i,1}, v_{i,2}, v_{i,3}, ..., v_{i,n}, \text{ such that}\\ &\tau(v_{0,k}, v_{0,k}) = \epsilon = \pm 1, \quad \tau(v_{1,k}, v_{1,k}) = \delta i = \pm i \qquad (k = 1, 2, 3, ..., n)\end{aligned} \qquad (13.14)$$

and all the other pairings are zero. Furthermore,

$$x = x(\xi) : v_{0,k} \to \xi v_{1,k},\ v_{1,k} \to \xi v_{0,k}. \qquad (13.15)$$

Thus

$$x^2 : v_{0,k} \to \xi^2 v_{0,k},\ v_{1,k} \to \xi^2 v_{1,k}. \qquad (13.16)$$

Since $\xi^2 \in i\mathbb{R} \setminus 0$,

$$G^{x^2} = G^{x^2}|_{V_0} \times G^{x^2}|_{V_1} = U_n(\mathbb{C}) \times U_n(\mathbb{C}). \qquad (13.17)$$

Suppose $w \in Hom(V_0, V_1)$ commutes with $x^2$. Then, by (13.16), there are $w_{kl}, w_{kl}^* \in \mathbb{D}$ commuting with $\xi^2$ and such that

$$w(v_{0,k}) = \sum_{l=1}^{n} w_{kl}v_{1,l}, \quad w^*(v_{1,k}) = \sum_{l=1}^{n} w_{kl}^* v_{0,l}. \qquad (13.18)$$

By (2.10) and (13.14) we have

$$\begin{aligned}\iota(w^*_{pk}) &= \tau_0(v_{0,k}, \sum_{l=1}^{n} w^*_{pl} v_{0,l})\epsilon = \tau_0(v_{0,k}, w^*(v_{1,p}))\epsilon \\ &= \tau_1(w(v_{0,k}), v_{1,p})\epsilon = w_{kp}\tau_1(v_{1,p}, v_{1,p})\epsilon = w_{kp}\epsilon\delta i.\end{aligned}$$

Thus,

$$w^*_{pk} = -\epsilon\delta i\iota(w_{kp}). \tag{13.19}$$

For $y, z \in \mathfrak{g}_1^{x^2}$ let $w = y|_{V_0}$ and let $u = z|_{V_0}$. Then $w, u \in Hom(V_0, V_1)$ commute with $x^2$ and, by (2.13),

$$\begin{aligned}\frac{1}{4}\langle y, z\rangle &= tr_{\mathbb{D}/\mathbb{R}}(w^* u) \\ &= \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(u_{kl} w^*_{lk}) = \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(-\epsilon\delta i\iota(w_{kl})u_{kl}).\end{aligned} \tag{13.20}$$

The formulas (2.12) and (13.15) imply

$$\begin{aligned} yx &: v_{0,k} \to \sum_{l=1}^{n} \xi w^*_{lk} v_{0,l}, \\ xy &: v_{0,k} \to \sum_{l=1}^{n} w_{kl}\xi v_{0,l}.\end{aligned} \tag{13.20'}$$

Furthermore, since $\xi^2 \in \mathbb{R}\setminus 0$, the centralizer of $\xi^2$ in $\mathbb{D}$ coincides with the the centralizer of $\xi$ in $\mathbb{D}$. In particular, $w_{kl}\xi = \xi w_{kl}$. By combining this with (13.19) and (13.20') we see that

$$\begin{aligned} y \in \mathfrak{g}_1^x \ &\text{ if and only if } \ w_{kl} = -\epsilon\delta i\iota(w_{lk}), \\ y \in {}^x\mathfrak{g}_1 \ &\text{ if and only if } \ w_{kl} = \epsilon\delta i\iota(w_{lk}).\end{aligned} \tag{13.21}$$

Suppose $y, z \in {}^x\mathfrak{g}_1$. Then (13.20) and (13.21) imply

$$\begin{aligned}\frac{1}{4}\langle y, z\rangle &= \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(-\xi^{-1} w_{lk}\xi u_{kl}) \\ &= \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(-\epsilon\delta i\iota(u_{lk})w_{kl}) = \frac{1}{4}\langle z, y\rangle.\end{aligned} \tag{13.22}$$

Thus $\langle y, z\rangle = 0$. Hence ${}^x\mathfrak{g}_1$ is an isotropic subspace of $\mathfrak{g}_1$. Similarly we check that $\mathfrak{g}_1^x$ is an isotropic subspace of $\mathfrak{g}_1$, and that the symplectic form provides a non-degenerate pairing between ${}^x\mathfrak{g}_1$ and $\mathfrak{g}_1^x$. The dual pair corresponding to the supergroup $(G^{x^2}, \mathfrak{g}^{x^2})$ is isomorphic to $(U_n, U_n)$.

Furthermore a straightforward computation shows that

$${}^{({}^{x(\xi)}\mathfrak{g}_1)}\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^{x(\xi)})} = \{x(\zeta);\ \zeta^2 \in i\mathbb{R},\ sgn(im(\zeta^2)) = sgn(im(\xi^2))\}. \tag{13.22.1}$$

**Case 5.2.c.** Here $\mathbb{D} = \mathbb{R}$ and the vector spaces $V_i$ $(i = 0, 1)$ have basis

$$\begin{aligned} &v_{i,1}, v_{i,2}, v_{i,3}, ..., v_{i,n},\ \ v'_{i,1}, v'_{i,2}, v'_{i,3}, ..., v'_{i,n}, \\ &\text{such that with } \epsilon = \pm 1 \\ &\tau(v_{0,k}, v_{0,k}) = \tau(v'_{0,k}, v'_{0,k}) = \epsilon,\quad \tau(v_{1,k}, v'_{1,k}) = 1 \qquad (k = 1, 2, 3, ..., n) \end{aligned} \tag{13.23}$$

and all the other pairings are zero. Further, with $\xi \in \mathbb{R} \setminus 0$,

$$\begin{aligned} x = x(\xi, \epsilon) : &v_{0,k} \to \xi(v_{1,k} - \epsilon v'_{1,k}),\ \ v_{1,k} \to \xi(v_{0,k} - v'_{0,k}), \\ &v'_{0,k} \to \xi(v_{1,k} + \epsilon v'_{1,k}),\ \ v'_{1,k} \to \epsilon\xi(v_{0,k} + v'_{0,k}). \end{aligned} \tag{13.24}$$

Therefore,

$$\begin{aligned} x^2 : &v_{0,k} \to -2\xi^2 v'_{0,k},\ \ v_{1,k} \to -2\epsilon\xi^2 v'_{1,k}, \\ &v'_{0,k} \to 2\xi^2 v_{0,k},\ \ v'_{1,k} \to 2\epsilon\xi^2 v_{1,k}. \end{aligned} \tag{13.25}$$

Since $\xi \neq 0$, (13.25) implies

$$G^{x^2} = G^{x^2}|_{V_0} \times G^{x^2}|_{V_1} = U_n \times U_n. \tag{13.26}$$

Suppose $w \in Hom(V_0, V_1)$ commutes with $x^2$. Then, by (13.25), there are elements $w_{kl}, w'_{kl}, w^*_{kl}, {w^*_{kl}}' \in \mathbb{R}$ such that

$$\begin{aligned} w(v_{0,k}) &= \sum_{l=1}^{n} (w_{kl} v_{1,l} - w'_{kl} v'_{1,l}), \\ w(v'_{0,k}) &= \sum_{l=1}^{n} \epsilon(w'_{kl} v_{1,l} + w_{kl} v'_{1,l}), \\ w^*(v_{1,k}) &= \sum_{l=1}^{n} (w^*_{kl} v_{0,l} - {w^*_{kl}}' v'_{0,l}), \\ w^*(v'_{1,k}) &= \sum_{l=1}^{n} \epsilon({w^*_{kl}}' v_{0,l} + w^*_{kl} v'_{0,l}). \end{aligned} \tag{13.27}$$

From (13.23) and (13.27) we see that

$$\begin{aligned} {w^*_{pk}}' &= \epsilon\tau_0(v_{0,k}, {w^*_{pk}}' v_{0,k}) = \epsilon\tau_0(v_{0,k}, \epsilon w^*(v'_{1,p})) \\ &= \tau_1(w(v_{0,k}), v'_{1,p}) = \tau_1(w_{k,p} v_{1,p}, v'_{1,p}) = w_{kp}, \end{aligned}$$

and

$$\begin{aligned} w^*_{pk} &= \epsilon\tau_0(v_{0,k}, w^*_{pk} v_{0,k}) = \epsilon\tau_0(v_{0,k}, w^*(v_{1,p})) \\ &= \epsilon\tau_1(w(v_{0,k}), v_{1,p}) = \epsilon\tau_1(-{w_{kp}}' v'_{1,p}, v_{1,p}) = \epsilon {w_{kp}}'. \end{aligned}$$

Thus,

$$w^*_{pk} = \epsilon {w_{kp}}',\quad {w^*_{pk}}' = w_{kp}. \tag{13.28}$$

For $y, z \in \mathfrak{g}_1^{x^2}$ let $w = y|_{V_0}$ and let $u = z|_{V_0}$. Then $w, u \in Hom(V_0, V_1)$ commute with $x^2$ and, by (2.13), (13.27) and (13.28),

$$\begin{aligned} &\frac{1}{4}\langle y, z\rangle = tr(w^* u) \\ &= 2\sum_{k,l=1}^{n} (u_{kl} w^*_{lk} - \epsilon u'_{kl} {w^*_{lk}}') = 2\epsilon \sum_{k,l=1}^{n} (u_{kl} {w_{kl}}' - {u_{kl}}' w_{kl}). \end{aligned} \tag{13.29}$$

We calculate using (13.24), (13.27) and (13.28):

$$\begin{aligned}
\frac{1}{\xi}yx :v_{0,k} &\to \sum_{l=1}^{n}(\epsilon w_{lk}{}' - w_{lk})v_{0,l} - \sum_{l=1}^{n}(w_{lk} + \epsilon w_{lk}{}')v'_{0,l},\\
v'_{0,k} &\to \sum_{l=1}^{n}(\epsilon w_{lk}{}' + w_{lk})v_{0,l} - \sum_{l=1}^{n}(w_{lk} - \epsilon w_{lk}{}')v'_{0,l},\\
\frac{1}{\xi}xy :v_{0,k} &\to \sum_{l=1}^{n}(w_{kl} - \epsilon w_{kl}{}')v_{0,l} - \sum_{l=1}^{n}(w_{kl} + \epsilon w_{kl}{}')v'_{0,l},\\
v'_{0,k} &\to \sum_{l=1}^{n}(\epsilon w_{kl}{}' + w_{kl})v_{0,l} - \sum_{l=1}^{n}(\epsilon w_{kl}{}' - w_{kl})v'_{0,l}
\end{aligned} \tag{13.30}$$

Hence,

$$\begin{aligned}
&y \in \mathfrak{g}_1^x \ \text{ if and only if } \ w_{kl}{}' = \epsilon w_{lk},\\
&y \in {}^x\mathfrak{g}_1 \ \text{ if and only if } \ w_{kl}{}' = -\epsilon w_{lk}.
\end{aligned} \tag{13.31}$$

It is clear from (13.29) and (13.31) that the spaces ${}^x\mathfrak{g}_1$, $\mathfrak{g}_1^x$ are isotropic, and that the symplectic form provides a non-degenerate pairing between them. The supergroup $(G^{x^2}, \mathfrak{g}^{x^2})$ is irreducible, of type I, and the corresponding dual pair is isomorphic to $(U_n, U_n)$.

Furthermore a straightforward computation shows that

$$^{({}^{x(\xi,\epsilon)}\mathfrak{g}_1)}\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^{x(\xi,\epsilon)})} = \{x(\zeta,\epsilon);\ \zeta \in \mathbb{R}\}. \tag{13.31.1}$$

**Case 5.2.d.** Here $\mathbb{D} = \mathbb{R}$ and the vector spaces $V_i$ $(i = 0, 1)$ have basis

$$\begin{aligned}
&u_{i,1}, u_{i,2}, u_{i,3}, ..., u_{i,n},\ \ u'_{i,1}, u'_{i,2}, u'_{i,3}, ..., u'_{i,n},\\
&v_{i,1}, v_{i,2}, v_{i,3}, ..., v_{i,n},\ \ v'_{i,1}, v'_{i,2}, v'_{i,3}, ..., v'_{i,n},\\
&\text{such that}\\
&\tau(u_{i,k}, u'_{i,k}) = \tau(v_{i,k}, v'_{i,k}) = 1, \qquad (i = 0, 1;\ k = 1, 2, 3, ..., n)
\end{aligned} \tag{13.32}$$

and all the other pairings are zero. Moreover,

$$\begin{aligned}
x = x(\xi,\eta) :u_{0,k} &\to \xi u_{1,k} + \eta v_{1,k},\ \ u_{1,k} \to \xi u_{0,k} + \eta v_{0,k},\\
v_{0,k} &\to -\eta u_{1,k} + \xi v_{1,k},\ \ v_{1,k} \to -\eta u_{0,k} + \xi v_{0,k},\\
u'_{0,k} &\to -\xi u'_{1,k} + \eta v'_{1,k},\ \ u'_{1,k} \to \xi u'_{0,k} - \eta v'_{0,k},\\
v'_{0,k} &\to -\eta u'_{1,k} - \xi v'_{1,k},\ \ v'_{1,k} \to \eta u'_{0,k} + \xi v'_{0,k},
\end{aligned} \tag{13.33}$$

Therefore, with $\alpha = \xi^2 - \eta^2$ and $\beta = 2\xi\eta$,

$$\begin{aligned}
x^2 :u_{0,k} &\to \alpha u_{0,k} + \beta v_{0,k},\ \ u_{1,k} \to \alpha u_{1,k} + \beta v_{1,k},\\
v_{0,k} &\to -\beta u_{0,k} + \alpha v_{0,k},\ \ v_{1,k} \to -\beta u_{1,k} + \alpha v_{1,k},\\
u'_{0,k} &\to -\alpha u'_{0,k} + \beta v'_{0,k},\ \ u'_{1,k} \to -\alpha u'_{1,k} + \beta v'_{1,k},\\
v'_{0,k} &\to -\beta u'_{0,k} - \alpha v'_{0,k},\ \ v'_{1,k} \to -\beta u'_{1,k} - \alpha v'_{1,k}.
\end{aligned} \tag{13.34}$$

Since $\alpha, \beta \neq 0$, (13.34) and (13.32) imply

$$G^{x^2} = G^{x^2}|_{V_0} \times G^{x^2}|_{V_1} = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}). \tag{13.35}$$

Suppose $w \in Hom(V_0, V_1)$ commutes with $x^2$. Then, by (13.34), $w$ maps the span of the $u_{0,k}$, $v_{0,k}$ to the span of the $u_{1,k}$, $v_{1,k}$ and the span of the $u'_{0,k}$, $v'_{0,k}$ to the span of the $u'_{1,k}$, $v'_{1,k}$. More precisely, there are numbers $w_{kl}, \tilde{w}_{kl}, w'_{kl}, \tilde{w}'_{kl}$, $w^*_{kl}, \tilde{w}^*_{kl}, {w^*_{kl}}', {\tilde{w}^*_{kl}}' \in \mathbb{R}$ such that

$$\begin{aligned} w(u_{0,k}) &= \sum_{l=1}^{n} (w_{kl} u_{1,l} + \tilde{w}_{kl} v_{1,l}), \\ w(v_{0,k}) &= \sum_{l=1}^{n} (-\tilde{w}_{kl} u_{1,l} + w_{kl} v_{1,l}), \\ w(u'_{0,k}) &= \sum_{l=1}^{n} (w'_{kl} u'_{1,l} + \tilde{w}'_{kl} v'_{1,l}), \\ w(v'_{0,k}) &= \sum_{l=1}^{n} (-\tilde{w}'_{kl} u'_{1,l} + w'_{kl} v'_{1,l}). \end{aligned} \tag{13.36}$$

and

$$\begin{aligned} w^*(u_{1,k}) &= \sum_{l=1}^{n} (w^*_{kl} u_{0,l} + \tilde{w}^*_{kl} v_{0,l}), \\ w^*(v_{1,k}) &= \sum_{l=1}^{n} (-\tilde{w}^*_{kl} u_{0,l} + w^*_{kl} v_{0,l}), \\ w^*(u'_{1,k}) &= \sum_{l=1}^{n} ({w^*_{kl}}' u'_{0,l} + {\tilde{w}^*_{kl}}' v'_{0,l}), \\ w^*(v'_{1,k}) &= \sum_{l=1}^{n} (-{\tilde{w}^*_{kl}}' u'_{0,l} + {w^*_{kl}}' v'_{0,l}). \end{aligned} \tag{13.37}$$

Using (13.32), (13.36) and (13.37) we show that

$$w^*_{kl} = -w'_{lk}, \quad {w^*_{kl}}' = w_{lk}, \quad \tilde{w}^*_{kl} = \tilde{w}'_{lk}, \quad {\tilde{w}^*_{kl}}' = -\tilde{w}_{lk}. \tag{13.38}$$

For $y, z \in \mathfrak{g}_1^{x^2}$ let $w = y|_{V_0}$ and let $u = z|_{V_0}$. Then

$$\begin{aligned} \frac{1}{4}\langle y, z\rangle &= tr(w^* u) \\ &= 2\sum_{k,l=1}^{n} (u_{kl} w^*_{lk} - \tilde{u}_{kl} \tilde{w}^*_{lk} + u'_{kl} {w^*_{lk}}' - \tilde{u}'_{kl} {\tilde{w}^*_{lk}}') \\ &= 2\sum_{k,l=1}^{n} (w_{kl} u'_{kl} + \tilde{w}_{kl} \tilde{u}'_{kl} - w'_{kl} u_{kl} - \tilde{w}'_{kl} \tilde{u}_{kl}) \end{aligned} \tag{13.39}$$

Furthermore,

$$\begin{aligned} yx: u_{0,k} &\to \sum_{l=1}^{n}((-\xi w'_{lk} - \eta \tilde{w}'_{lk})u_{0,l} + (\xi \tilde{w}'_{lk} - \eta w'_{lk})v_{0,l}), \\ v_{0,k} &\to \sum_{l=1}^{n}((\eta w'_{lk} - \xi \tilde{w}'_{lk})u_{0,l} + (-\eta \tilde{w}'_{lk} - \xi w'_{lk})v_{0,l}), \\ u'_{0,k} &\to \sum_{l=1}^{n}((-\xi w_{lk} + \eta \tilde{w}_{lk})u'_{0,l} + (\xi \tilde{w}_{lk} + \eta w_{lk})v'_{0,l}), \\ v'_{0,k} &\to \sum_{l=1}^{n}((-\eta w_{lk} - \xi \tilde{w}_{lk})u'_{0,l} + (\eta \tilde{w}_{lk} - \xi w_{lk})v'_{0,l}), \end{aligned} \tag{13.40}$$

and

$$\begin{aligned} xy: u_{0,k} &\to \sum_{l=1}^{n}((\xi w_{kl} - \eta \tilde{w}_{kl})u_{0,l} + (\eta w_{kl} + \xi \tilde{w}_{kl})v_{0,l}), \\ v_{0,k} &\to \sum_{l=1}^{n}((-\xi \tilde{w}_{kl} - \eta w_{kl})u_{0,l} + (-\eta \tilde{w}_{kl} + \xi w_{kl})v_{0,l}), \\ u'_{0,k} &\to \sum_{l=1}^{n}((\xi w'_{kl} + \eta \tilde{w}'_{kl})u'_{0,l} + (-\eta w'_{kl} + \xi \tilde{w}'_{kl})v'_{0,l}), \\ v'_{0,k} &\to \sum_{l=1}^{n}((-\xi \tilde{w}'_{kl} + \eta w'_{kl})u'_{0,l} + (\eta \tilde{w}'_{kl} + \xi w'_{kl})v'_{0,l}). \end{aligned} \tag{13.41}$$

Hence,

$$\begin{aligned} y \in \mathfrak{g}_1^x \quad \text{if and only if} \quad & \xi w_{kl} - \eta \tilde{w}_{kl} + \xi w'_{lk} + \eta \tilde{w}'_{lk} = 0, \\ & \eta w_{kl} + \xi \tilde{w}_{kl} + \eta w'_{lk} - \xi \tilde{w}'_{lk} = 0; \\ y \in {}^x\mathfrak{g}_1 \quad \text{if and only if} \quad & \xi w_{kl} - \eta \tilde{w}_{kl} - \xi w'_{lk} - \eta \tilde{w}'_{lk} = 0, \\ & \eta w_{kl} + \xi \tilde{w}_{kl} - \eta w'_{lk} + \xi \tilde{w}'_{lk} = 0. \end{aligned}$$

Thus

$$\begin{aligned} y \in \mathfrak{g}_1^x \quad &\text{if and only if} \quad w'_{kl} = -w_{lk}, \quad \tilde{w}'_{kl} = \tilde{w}_{lk}; \\ y \in {}^x\mathfrak{g}_1 \quad &\text{if and only if} \quad w'_{kl} = w_{lk}, \quad \tilde{w}'_{kl} = -\tilde{w}_{lk}. \end{aligned} \tag{13.42}$$

It is easy to see from (13.39) and (13.42) that the spaces ${}^x\mathfrak{g}_1$, $\mathfrak{g}_1^x$ are isotropic, and that the symplectic form provides a non-degenerate pairing between them. The supergroup $(G^{x^2}, \mathfrak{g}^{x^2})$ is irreducible, of type II, and as a dual pair is isomorphic to $(GL_n(\mathbb{C}), GL_n(\mathbb{C}))$.

Furthermore a straightforward computation shows that

$$^{({}^{x(\xi,\eta)}\mathfrak{g}_1)}\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^{x(\xi,\eta)})} = \{x(\zeta,\gamma);\ \zeta,\gamma \in \mathbb{R}\}. \tag{13.42.1}$$

**Case 5.3.a** The spaces $V_0$, $V_1$ have basis

$$v_{i,1}, v_{i,2}, v_{i,3}, ..., v_{i,n} \qquad (i = 0, 1)$$

such that

$$x(\xi): v_{0,k} \to \xi v_{1,k}, \quad v_{1,k} \to \xi v_{0,k}. \tag{13.43}$$

Hence,

$$x^2 : v_{0,k} \to \xi^2 v_{0,k}, \quad v_{1,k} \to \xi^2 v_{1,k}. \tag{13.44}$$

Therefore

$$G^{x^2} = G^{x^2}|_{V_0} \times G^{x^2}|_{V_1} = \begin{cases} GL_n(\mathbb{D}) \times GL_n(\mathbb{D}) & \text{if } \xi^2 \in \mathbb{R}, \\ GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) & \text{if } \xi^2 \notin \mathbb{R}. \end{cases} \tag{13.45}$$

Suppose $A \in Hom(V_0, V_1)$ and $B \in Hom(V_1, V_0)$ commute with $x^2$. Then there are elements $a_{k,l}, b_{k,l} \in \mathbb{D}$ commuting with $\xi^2$ such that

$$A : v_{0,k} \to \sum_{l=1}^{n} a_{k,l} v_{1,l}, \quad B : v_{1,k} \to \sum_{l=1}^{n} b_{k,l} v_{0,l}.$$

Furthermore the formula

$$y(v_0 + v_1) = Bv_1 + Av_0 \qquad (v_0 \in V_0,\ v_1 \in V_1)$$

defines an element $y \in \mathfrak{g}_1^{x^2}$ and all elements of $\mathfrak{g}_1^{x^2}$ may be described as above. Suppose $y' \in \mathfrak{g}_1^{x^2}$. Let $A'$, $B'$ be the corresponding elements of $Hom(V_0, V_1)$, $Hom(V_1, V_0)$ respectively. Then, by (2.6),

$$\frac{1}{2}\langle y, y' \rangle = tr_{\mathbb{D}/\mathbb{R}}(BA' - B'A) = \sum_{k,l=1}^{n} tr_{\mathbb{D}/\mathbb{R}}(b_{k,l}a'_{l,k} - b'_{k,l}a_{l,k}). \tag{13.46}$$

We see from (13.43) that

$$\begin{aligned} y \in \mathfrak{g}_1^x &\text{ if and only if } b_{k,l} = \xi a_{k,l} \xi^{-1}, \\ y \in {}^x\mathfrak{g}_1 &\text{ if and only if } b_{k,l} = -\xi a_{k,l} \xi^{-1}. \end{aligned} \tag{13.47}$$

It is clear from (13.46) and (13.47) that ${}^x\mathfrak{g}_1$ and $\mathfrak{g}_1^x$ are isotropic subspaces of $\mathfrak{g}_1$ and that the symplectic form $\langle\ ,\ \rangle$ provides a non-degenerate pairing between them. The Lie supergroup $(G^{x^2}, \mathfrak{g}^{x^2})$ is of type II, is irreducible and the corresponding dual pair is isomorphic to $(GL_n(\mathbb{C}), GL_n(\mathbb{C}))$ or $(GL_n(\mathbb{D}), GL_n(\mathbb{D}))$, as indicated in (13.45).

Furthermore a straightforward computation shows that

$$^{({}^{x(\xi)}\mathfrak{g}_1)}\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^{x(\xi)})} = \{x(\zeta);\ \zeta^2 \in \mathbb{D}^{(\mathbb{D}^{\xi^2})}\}. \tag{13.47.1}$$

**Case 5.3.a'** Here the spaces $V_0$, $V_1$ have basis

$$v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,n} \qquad (i = 0, 1),$$

$$x = x(\xi) : v_{0,k} \to \xi v_{1,k},\ v_{1,k} \to -\xi v_{0,k},\ (\xi \in \mathbb{R} \setminus 0),$$

$$^{({}^{x(\xi)}\mathfrak{g}_1)}\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^{x(\xi)})} = \{x(\zeta);\ \zeta \in \mathbb{R}\}. \tag{13.47.2}$$

and the proof is as in the previous case.

**Case 5.3.b** Here the spaces $V_0$, $V_1$ have basis

$$u_{i,1}, u_{i,2}, u_{i,3}, ..., u_{i,n};\ v_{i,1}, v_{i,2}, v_{i,3}, ..., v_{i,n} \qquad (i = 0, 1)$$

such that

$$\begin{aligned} x = x(\xi, \eta) : & u_{0,k} \to \xi u_{1,k} + \eta v_{1,k}, \quad u_{1,k} \to \xi u_{0,k} + \eta v_{0,k}, \\ & v_{0,k} \to -\eta u_{1,k} + \xi v_{1,k}, \quad v_{1,k} \to -\eta u_{0,k} + \xi v_{0,k} \end{aligned} \tag{13.48}$$

Therefore, with $\alpha = \xi^2 - \eta^2$ and $\beta = 2\xi\eta$,

$$\begin{aligned} x^2 : & u_{0,k} \to \alpha u_{0,k} + \beta v_{0,k}, \quad u_{1,k} \to \alpha u_{1,k} + \beta v_{1,k}, \\ & v_{0,k} \to -\beta u_{0,k} + \alpha v_{0,k}, \quad v_{1,k} \to -\beta u_{1,k} + \alpha v_{1,k} \end{aligned} \tag{13.49}$$

Hence,

$$G^{x^2} = G^{x^2}|_{V_0} \times G^{x^2}|_{V_1} = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}). \tag{13.50}$$

Suppose $y \in \mathfrak{g}_1^{x^2}$. Then there are numbers $a_{k,l}$, $\tilde{a}_{k,l}$, $b_{k,l}$, $\tilde{b}_{k,l}$ in $\mathbb{R}$ such that

$$\begin{aligned} y : & u_{0,k} \to \sum_{l=1}^{n} (a_{k,l} u_{1,l} + \tilde{a}_{k,l} v_{1,l}), \quad u_{1,k} \to \sum_{l=1}^{n} (b_{k,l} u_{0,l} + \tilde{b}_{k,l} v_{0,l}), \\ & v_{0,k} \to \sum_{l=1}^{n} (-\tilde{a}_{k,l} u_{1,l} + a_{k,l} v_{1,l}), \quad v_{1,k} \to \sum_{l=1}^{n} (-\tilde{b}_{k,l} u_{0,l} + b_{k,l} v_{0,l}). \end{aligned} \tag{13.51}$$

If $y' \in \mathfrak{g}_1^{x^2}$, then (with the notation (13.51)),

$$\frac{1}{2}\langle y, y' \rangle = \sum_{k,l=1}^{n} (a'_{k,l} b_{l,k} - \tilde{a}'_{k,l} \tilde{b}_{l,k} - a_{k,l} b'_{l,k} + \tilde{a}_{k,l} \tilde{b}'_{l,k}). \tag{13.52}$$

Furthermore

$$\begin{aligned} yx : & u_{0,k} \to \sum_{l=1}^{n} ((\xi b_{k,l} - \eta \tilde{b}_{k,l}) u_{0,l} + (\xi \tilde{b}_{k,l} + \eta b_{k,l}) v_{0,l}), \\ & v_{0,k} \to \sum_{l=1}^{n} ((-\eta b_{k,l} - \xi \tilde{b}_{k,l}) u_{0,l} + (-\eta \tilde{b}_{k,l} + \xi b_{k,l}) v_{0,l}) \end{aligned}$$

and

$$\begin{aligned} xy : & u_{0,k} \to \sum_{l=1}^{n} ((\xi a_{k,l} - \eta \tilde{a}_{k,l}) u_{0,l} + (\xi \tilde{a}_{k,l} + \eta a_{k,l}) v_{0,l}), \\ & v_{0,k} \to \sum_{l=1}^{n} ((-\xi \tilde{a}_{k,l} - \eta a_{k,l}) u_{0,l} + (\xi a_{k,l} - \eta \tilde{a}_{k,l}) v_{0,l}) \end{aligned}$$

Therefore,

$$\begin{aligned} & y \in \mathfrak{g}_1^x \ \text{ if } \ a_{k,l} = b_{k,l}, \quad \tilde{a}_{k,l} = \tilde{b}_{k,l}; \\ & y \in {}^x\mathfrak{g}_1 \ \text{ if } \ a_{k,l} = -b_{k,l}, \quad \tilde{a}_{k,l} = -\tilde{b}_{k,l}. \end{aligned} \tag{13.53}$$

It is clear from (13.52) and (13.53) that ${}^x\mathfrak{g}_1$ and $\mathfrak{g}_1^x$ are isotropic subspaces of $\mathfrak{g}_1$ and that the form $\langle\ ,\ \rangle$ provides a non-degenerate pairing between them. The Lie supergroup

$(G^{x^2}, \mathfrak{g}^{x^2})$ is irreducible, of type II and the corresponding dual pair is isomorphic to $(GL_n(\mathbb{C}), GL_n(\mathbb{C}))$.

Furthermore a straightforward computation shows that

$$({}^{x(\xi,\eta)}\mathfrak{g}_1)\mathfrak{g}_1 = \mathfrak{g}_1^{(\mathfrak{g}_1^{x(\xi,\eta)})} = \{x(\zeta,\gamma);\ \zeta,\gamma \in \mathbb{R}\}. \tag{13.53.1}$$

This completes the proof of Theorem 4.4.

**Proof (of Lemma 6.4).** Since the elements $x$ and $y$ commute, they preserve the same isotypic decomposition of $V$. For a fixed isotypic component, all the sets which occur in Lemma 6.4 (a), (b) and (c) are described in (13.11), (13.21), (13.31), (13.42), (13.47) and (13.53). One checks the equalities (a), (b), (c) via a case by case analysis. Similarly one verifies (d) and (e). □

## 14 A proof of Theorem 4.5

Consider the map

$$G \times \mathfrak{g}_1^x \ni (g,y) \to gy \in \mathfrak{g}_1. \tag{14.1}$$

The derivative of (14.1) at $(g,y)$ coincides with the following linear map

$$\mathfrak{g}_0 \times \mathfrak{g}_1^x \ni (A,B) \to [A,gy] + gB \in \mathfrak{g}_1. \tag{14.2}$$

The range of the map (14.2) is equal to

$$[\mathfrak{g}_0, gy] + g(\mathfrak{g}_1^x) = g([\mathfrak{g}_0, y] + \mathfrak{g}_1^x). \tag{14.3}$$

We see from Lemma 3.1 and Theorem 4.4 that

$$[\mathfrak{g}_0, x] + \mathfrak{g}_1^x = ({}^x\mathfrak{g}_1)^\perp + \mathfrak{g}_1^x = \mathfrak{g}_1. \tag{14.4}$$

Hence, the set

$$U = \{y \in \mathfrak{g}_1^x;\ [\mathfrak{g}_0, y] + \mathfrak{g}_1^x = \mathfrak{g}_1\} \tag{14.5}$$

is non-empty. The set $U$ is open and $G^x$-invariant. Furthermore, (14.3) shows that the map (14.1) restricted to $G \times U$ is a submersion. The set $U$ satisfies the conditions (1.0), (1.1), (1.2) and (1.5).

Suppose we have a non-empty open subset $\tilde{U}_x \subseteq \mathfrak{g}_1^x$ such that

$$\begin{aligned} &\text{(1.4) holds for the supergroup } (G^{x^2}, \mathfrak{g}^{x^2}): \\ &\text{if } g \in G^{x^2} \text{ and } y, y' \in \tilde{U}_x \text{ are such that } gy = y', \text{ then } g \in G^x. \end{aligned} \tag{14.6}$$

Since $x^2$ is semisimple, there is an admissible slice $U_{x^2}$ through $x^2$ in $\mathfrak{g}_0^{x^2}$, with respect to the group $G$. We may assume that $\tilde{U}_x$ is contained in the preimage of $U_{x^2}$ under the map

$$\mathfrak{g}_1^x \ni y \to y^2 \in \mathfrak{g}_0^{x^2}. \tag{14.7}$$

Then for $y, y' \in \tilde{U}_x$ and $g \in G$ such that $gy = y'$ we have $gy^2 = y'^2$. Hence $g \in G^{x^2}$. But this together with (14.6) implies the $g \in G^x$. Thus $\tilde{U}_x$ satisfies (1.3). Hence, if we set $U_x = \tilde{U}_x \cap U$, where $U$ is the set defined in (14.5), then $U_x$ satisfies all the conditions (1.0)-(1.5).

Next we shall verify the statement (14.6). The Theorem 4.4 implies that we may assume that either $x = 0$ or $x \neq 0$ and that the supergroup $(G^{x^2}, \mathfrak{g}^{x^2})$ corresponds either to the dual pair $(U_n, U_n)$ or to $(GL_n(\mathbb{D}), GL_n(\mathbb{D}))$.

Since any $G$-invariant open neighborhood of 0 in $\mathfrak{g}_1$ is an admissible slice through 0, we may assume that $x \neq 0$.

We proceed via a case by case analysis performing the computations in terms of matrices. It will be clear from what follows that the sets constructed may be made arbitrarily small and thus form the desired basis for the neighborhoods of $x$ in $\mathfrak{g}_1^x$.

**Case** $(U_n, U_n)$.

Set $W = M_n(\mathbb{C})$, $V_0 = V_1 = \mathbb{C}^n$ and

$$\tau_0(u_0, v_0) = \overline{v}_0^T u_0, \quad \tau_1(u_1, v_1) = \overline{v}_1^T i u_1 \qquad (u_0, v_0 \in V_0,\ u_1, v_1 \in V_1).$$

The space $W$ is identified with $Hom(V_0, V_1)$ by $w(v) = wv$, $w \in W$, $v \in V_0$. Then

$$w^* = -i\overline{w}^T \qquad (w \in W).$$

The restriction of $x$ to $Hom(V_0, V_1)$ coincides with $\xi I \in W$, where $\xi \in i\mathbb{R} \setminus 0$, $im(\xi^2) < 0$. Thus $\xi = re^{-\frac{\pi}{4}i}$, where $r \in \mathbb{R} \setminus 0$. A straightforward calculation using the formula (2.12) shows that under the identification

$$\mathfrak{g}_1 = \mathfrak{g}_1|_{V_0} = Hom(V_0, V_1) = W$$

we have

$$^x\mathfrak{g}_1 = \{\xi A;\ A = -\overline{A}^T \in W\} \quad \text{and} \quad \mathfrak{g}_1^x = \{\xi H;\ H = \overline{H}^T \in W\}. \tag{14.8}$$

For $0 < \epsilon \leq \frac{1}{2}$ let $\tilde{U}_{x,\epsilon}$ be the set of all points $w \in \mathfrak{g}_1^x$ such that

$$\lambda_1 + \lambda_2 \neq 0 \ \text{ and } \ |\lambda - \xi| < \epsilon|\xi| \ \text{ for all eigenvalues } \ \lambda, \lambda_1, \lambda_2 \ \text{ of } \ w. \tag{14.9}$$

Let $\| \ \|$ be the operator norm on $W$, viewed as the space of operators on the Hilbert space $(V_0, \tau_0)$. Suppose $g, h \in U_n$ and $\xi H \in \tilde{U}_{x,\epsilon}$ are such that $\xi g H h^{-1} \in \tilde{U}_{x,\epsilon}$. Then, by (14.9),

$$\| H - I \| < \epsilon \ \text{ and } \ \| gHh^{-1} - I \| < \epsilon.$$

Hence $\| H - g^{-1}h \| < \epsilon$, and by the triangle inequality

$$\| g^{-1}h - I \| < 2\epsilon.$$

Since $2\epsilon \leq 1$, we have $A = log(g^{-1}h) \in \mathfrak{u}_n$. Thus

$$g^{-1}h = \exp(A).$$

Since $\xi gHh^{-1} \in \tilde{U}_{x,\epsilon}$, we have

$$\overline{gHh^{-1}}^T = gHh^{-1}.$$

Thus $hHg^{-1} = gHh^{-1}$, or equivalently $Hg^{-1}h = h^{-1}gH$. Therefore

$$H\exp(A) = \exp(-A)H.$$

Since, by (14.9), $H$ is invertible and since *log* is injective, this last condition may be expressed as $HAH^{-1} = -A$, or equivalently as

$$HA + AH = 0. \tag{14.10}$$

Conjugating both sides of (14.10) by an appropriate element of $U_n$ we may assume that $H$ is diagonal. Then (14.9) shows that $A = 0$. Hence $g^{-1}h = I$, i.e. $g = h$. Since the diagonal subgroup $\{(g,g) \in U_n \times U_n\}$ coincides with $G^x$, we see that the set $\tilde{U}_{x,\epsilon}$ satisfies (1.3). This also shows that the derivative of the map $H \to H^2$ at $H$ is an injective linear map. Furthermore, since $H$ is close to the identity, it is positive definite. Thus $H$ is the unique positive definite square root of $H^2$. Hence the map $H \to H^2$ is injective.

**Case** $(GL_n(\mathbb{D}), GL_n(\mathbb{D}))$.

For $\alpha > 0$ let

$$M_n(\mathbb{C})[\alpha] = \{A \in M_n(\mathbb{C});\ |Im(a)| < \alpha \text{ for all eigenvalues } a \text{ of } A\},$$

and let

$$GL_n(\mathbb{C})[\alpha] = \exp(M_n(\mathbb{C})[\alpha]).$$

Then, as is well known [11][part. II, p. 17],

$$\exp : M_n(\mathbb{C})[\pi] \to GL_n(\mathbb{C})[\pi] \tag{14.11}$$

is a bijective analytic diffeomorphism. Moreover, the closure,

$$Cl(GL_n(\mathbb{C})[\alpha]) \subseteq GL_n(\mathbb{C})[\beta] \qquad (0 < \alpha < \beta \leq \pi). \tag{14.12}$$

Let $'M_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ be the set of all matrices $A$ such that the map

$$M_n(\mathbb{C}) \ni B \to AB + BA \in M_n(\mathbb{C}) \tag{14.13}$$

is surjective. Clearly $'M_n(\mathbb{C})$ is a Zariski open, $Ad(GL_n(\mathbb{C}))$-invariant neighborhood of the identity $I \in {}'M_n(\mathbb{C})$.

**Lemma 14.1.** *For any $0 < \alpha < \beta < \pi$ there is an open $Ad(GL_n(\mathbb{C}))$-invariant neighborhood of the identity*

$$V_{\alpha,\beta} = V_{\alpha,\beta}^{-1} \subseteq GL_n(\mathbb{C})[\alpha] \cap {}'M_n(\mathbb{C})a$$

*such that*

$$GL_n(\mathbb{C})[\alpha]\, V_{\alpha,\beta} \subseteq GL_n(\mathbb{C})[\beta].b$$

**Proof.** The set of eigenvalues of a matrix $A \in M_n(\mathbb{C})$ may be viewed as an orbit in $\mathbb{C}^n$ under the action of the permutation group. The family of all such orbits has a natural topology. In these terms the set of eigenvalues of a matrix $A \in M_n(\mathbb{C})$, is a continuous function.

Let $S^1 \subseteq M_n(\mathbb{C})$ denote the unit sphere with respect to the operator norm $\| \ \|$. Since by Jordan's Theorem

$$GL_n(\mathbb{C})[\alpha] = \{A \in M_n(\mathbb{C});\ a \neq 0,\ |Arg(a)| < \alpha \text{ for all eigenvalues } a \text{ of } A\} \quad (14.14)$$

the previous paragraph shows that there is an open neighborhood $V^{(\alpha,\beta)} \subseteq GL_n(\mathbb{C})$ of the identity $I$, such that

$$(S^1 \cap GL_n(\mathbb{C})[\alpha])\, V^{(\alpha,\beta)} \subseteq GL_n(\mathbb{C})[\beta]. \quad (14.15)$$

Notice that the set (14.14) is closed under the dilations $A \to tA$, $t > 0$. Hence, (14.15) implies

$$GL_n(\mathbb{C})[\alpha]V^{(\alpha,\beta)} \subseteq GL_n(\mathbb{C})[\beta]. \quad (14.16)$$

Let

$$V^{\alpha,\beta} = Ad(GL_n(\mathbb{C}))V^{(\alpha,\beta)}.$$

As the union of open sets, $V^{\alpha,\beta}$ is open. Clearly, $V^{\alpha,\beta}$ is $Ad(GL_n(\mathbb{C}))$-invariant and contains the identity. The inclusion (14.16) together with the $Ad(GL_n(\mathbb{C}))$-invariance of the sets $GL_n(\mathbb{C})[\gamma]$, $\gamma = \alpha, \beta$, implies (b). Hence the Lemma holds for $V_{\alpha,\beta} = V^{\alpha,\beta} \cap \left(V^{\alpha,\beta}\right)^{-1}$. □

**Corollary 14.2.** *Let $V_{\alpha,\beta}$ be as in the Lemma 14.1. Suppose $A \in V_{\alpha,\beta}$ and $g, h \in GL_n(\mathbb{C})$ are such that $hAg^{-1} = gAh^{-1} \in V_{\alpha,\beta}$. Then $g = h$. In particular, if $A, B \in V_{\alpha,\beta}$ and $A^2 = B^2$, then $A = B$. Furthermore, the derivative of the map $A \to A^2$ at $A \in V_{\alpha,\beta}$ is injective.*

**Proof.** Set $u = g^{-1}h$. Then $uA \in V_{\alpha,\beta}$. Since $A^{-1} \in V_{\alpha,\beta}$, Lemma 14.1 implies

$$\begin{aligned} u = (uA)A^{-1} &\in V_{\alpha,\beta}V_{\alpha,\beta} \subseteq GL_n(\mathbb{C})[\alpha]\, V_{\alpha,\beta} \subseteq GL_n(\mathbb{C})[\beta] \\ &\subseteq GL_n(\mathbb{C})[\pi]. \end{aligned}$$

Hence there is a unique $B \in M_n(\mathbb{C})[\pi]$ such that

$$u = \exp(B).$$

Furthermore,

$$uA = g^{-1}hA = Ah^{-1}g = Au^{-1}.$$

Hence

$$\exp(B)A = Aexp(-B),$$

or equivalently

$$\exp(A^{-1}BA) = exp(-B). \quad (14.17)$$

Since $A^{-1}BA$ and $-B$ belong to $M_n(\mathbb{C})[\pi]$, (14.17) implies $A^{-1}BA = -B$, or equivalently

$$AB + BA = 0. \tag{14.18}$$

Since $A \in {}'M_n(\mathbb{C})$, (14.18) implies $B = 0$, which means that $u = 1$. Thus $g = h$. Let $A, B \in V_{\alpha,\beta}$. Then $A^2 = B^2$ is equivalent to $AAB^{-1} = BAA^{-1}$, which implies $A = B$. The injectivity of the derivative follows from the fact that $A \subseteq {}'M_n(\mathbb{C})$. $\square$

Let $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and let $\xi \in \mathbb{D} \setminus 0$ be such that $\xi^2$ is in the center of $\mathbb{D}$. Set

$$x = \begin{pmatrix} 0 & \xi I \\ \xi I & 0 \end{pmatrix}. \tag{14.19}$$

Under the usual identifications we have

$$\begin{aligned} \mathfrak{g}_1 = End(\mathbb{D}^n \oplus \mathbb{D}^n)_1 &= \{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} ;\ A, B \in M_n(\mathbb{D})\}, \\ G = GL(\mathbb{D}^n \oplus \mathbb{D}^n)_0 &= \{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} ;\ g, h \in GL(\mathbb{D}^n)\}. \end{aligned} \tag{14.20}$$

A straightforward calculation shows that

$$\begin{aligned} \mathfrak{g}_1^x &= \{ \begin{pmatrix} 0 & \xi B \xi^{-1} \\ B & 0 \end{pmatrix} ;\ B \in M_n(\mathbb{D})\}, \\ G^x &= \{ \begin{pmatrix} g & 0 \\ 0 & \xi g \xi^{-1} \end{pmatrix} ;\ g \in GL(\mathbb{D}^n)\}. \end{aligned} \tag{14.21}$$

Let $V_{\alpha,\beta}$ be the set constructed in Lemma 14.1 for the group $GL_n(\mathbb{D})$ if $\mathbb{D} = \mathbb{C}$, and for the complexification of $GL_n(\mathbb{D})$ if $\mathbb{D} \neq \mathbb{C}$. Set

$$V_{\alpha,\beta}(\mathbb{D}) = V_{\alpha,\beta} \cap GL_n(\mathbb{D}). \tag{14.22}$$

Then $V_{\alpha,\beta}(\mathbb{D})$ is an open $Ad(GL_n(\mathbb{D}))$-invariant neighborhood of the identity $I \in GL_n(\mathbb{D})$. Let

$$\begin{aligned} U_{\alpha,\beta,\xi}(\mathbb{D}) &= \{ \begin{pmatrix} 0 & \xi B \xi^{-1} \\ B & 0 \end{pmatrix} ;\ B \in \xi V_{\alpha,\beta}(\mathbb{D})\} \\ &= \{ \begin{pmatrix} 0 & A\xi \\ \xi A & 0 \end{pmatrix} ;\ A \in V_{\alpha,\beta}(\mathbb{D})\}. \end{aligned} \tag{14.23}$$

The set (14.23) is an open $G^x$-invariant neighborhood of $x$ in $\mathfrak{g}_1^x$.

**Lemma 14.3.** *Suppose $y, z \in U_{\alpha,\beta,\xi}(\mathbb{D})$ and $s \in G$ are such that $sys^{-1} = z$. Then $s \in G^x$.*

**Proof.** Let

$$s = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & A\xi \\ \xi A & 0 \end{pmatrix}.$$

Then

$$sys^{-1} = \begin{pmatrix} 0 & gA\xi h^{-1} \\ h\xi A g^{-1} & 0 \end{pmatrix}$$

Since $sys^{-1} \in U_{\alpha,\beta,\xi}(\mathbb{D})$, we have

$$gA\xi h^{-1} = \xi(h\xi A g^{-1})\xi^{-1} = \xi h \xi^{-1} A g^{-1} \xi.$$

Thus

$$(\xi h \xi^{-1}) A g^{-1} = gA(\xi h^{-1} \xi^{-1}). \tag{14.24}$$

Furthermore

$$h\xi A g^{-1} \in \xi U_{\alpha,\beta,\xi}(\mathbb{D}).$$

Thus

$$(\xi h \xi^{-1}) A g^{-1} = \xi^{-1} h \xi A g^{-1} \in U_{\alpha,\beta,\xi}(\mathbb{D}). \tag{14.25}$$

By combining (14.24), (14.25) and Corollary 14.2 we see that $\xi h \xi^{-1} = g$, so that $h = \xi^{-1} g \xi = \xi g \xi^{-1}$. $\square$

The Lemma 14.3 shows that the set (14.23) satisfies the condition (1.3). The rest is also clear from Corollary 14.2.

## 15 A proof of Theorem 4.7

The ideas presented below originate in [4, sec. 12]. Recall the following Lemma.

**Lemma 15.1.** *[12, 8.A.4.5] Let $N$ be a complete metric space and let $G$ be a $\sigma$-compact topological group acting on $N$. Suppose $N$ is the union of a finite number of $G$-orbits. The we can label the orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$, so that for each $1 \le j \le k$, the set*

$$N_j = \bigcup_{l=j}^{k} \mathcal{O}_l \quad \text{is closed in} \quad N.$$

We apply the above Lemma to our ordinary classical Lie supergroup $(G, \mathfrak{g})$, with $N \subseteq \mathfrak{g}_1$ equal to the set of nilpotent elements. As is well known, [3], $N$ is the union of a finite number of $G$-orbits.

In particular, for each $1 \le j \le k$ there is an open $G$-invariant set $W_j \subseteq \mathfrak{g}_1$ such that

$$W_j \cap N_j = \mathcal{O}_j. \tag{15.1}$$

Fix $1 \le j \le k$ and an element $x \in \mathcal{O}_j$. Let $U \subseteq \mathfrak{g}_1$ be a subspace complementary to the tangent space to the orbit through $x$. Thus

$$\mathfrak{g}_1 = [\mathfrak{g}_0, x] \oplus U. \tag{15.2}$$

For each $z \in U$ we have a linear map

$$T_z : \mathfrak{g}_0 \oplus U \ni (y, z') \to [y, x+z] + z' \in \mathfrak{g}_1. \tag{15.3}$$

Notice that, by (15.2), $T_0$ is surjective. Further, the map

$$U \ni z \to T_z \in Hom(\mathfrak{g}_0 \oplus U, \mathfrak{g}_1)$$

is affine and hence continuous. Therefore the set of all $z \in U$ such that

$$rank(T_z) \ge rank(T_0) \ (= dim\, \mathfrak{g}_1) \tag{15.4}$$

is an open neighborhood of 0 in $U$. Let us denote this neighborhood by $U_1$. Let

$$\Phi : G \times U_1 \ni (g, z) \to g(x+z)g^{-1} \in \mathfrak{g}_1. \tag{15.5}$$

The derivative of $\Phi$ at $(g, z)$ coincides with the following linear map

$$\mathfrak{g}_0 \oplus U \ni (y, z') \to g([y, x+z] + z')g^{-1} \in \mathfrak{g}_1. \tag{15.6}$$

By (15.4), the map (15.6) is surjective. Thus $\Phi$ is a submersion.

Recall the set $W_j$, (15.1). Let

$$U_2 = \{z \in U_1;\ x + z \in W_j\}. \tag{15.7}$$

Then

$$\Phi(G \times U_2) \subseteq W_j. \tag{15.8}$$

Let $W \subseteq \mathfrak{g}_0$ be a subspace complementary to the kernel of the map

$$ad(x) : \mathfrak{g}_0 \ni y \to [y, x] \in \mathfrak{g}_1 \tag{15.9}$$

so that the map

$$W \ni y \to [y, x] \in [\mathfrak{g}_0, x] \tag{15.10}$$

is a linear bijection. By (15.6), the derivative of the map

$$W \times U_2 \ni (y, z) \to \Phi(\exp(y), z) = \exp(y)(x+z)\exp(-y) \in \mathfrak{g}_1 \tag{15.11}$$

at $(0, 0)$ coincides with the following linear map

$$W \oplus U \ni (y, z') \to [y, x] + z' \in \mathfrak{g}_1, \tag{15.12}$$

which, by (15.2) and (15.10), is a linear bijection. Hence, there is an open neighborhood $W_1$ of 0 in $W$ and an open neighborhood $U_3$ of 0 in $U_2$, such that the map

$$W_1 \times U_3 \ni (y, z) \to \exp(y)(x + z)\exp(-y) \in \mathfrak{g}_1 \tag{15.13}$$

is an diffeomorphism onto an open neighborhood of $x$ in $\mathfrak{g}_1$.

Let $W_2 \subseteq W_1$ be an open neighborhood of 0 such that

$$\exp(ad(W_2))x \subseteq N_j$$

is an open neighborhood of $x$ in $N_j$. This is compatible with (15.8). Choose an open neighborhood $W_0 \subseteq W_2$ of 0 and an open neighborhood $U_0 \subseteq U_3$ of 0 such that

$$\Phi(\exp(W_0), U_0) \cap N_j \subseteq \exp(ad(W_2))x \cap N_j. \tag{15.14}$$

Suppose $z \in U_0$ is such that

$$x + z \in \mathcal{O}_j. \tag{15.15}$$

Since $x + z = \Phi(\exp(0), z)$, (15.14) implies that there is $y \in W_2$ such that

$$x + z = \exp(y)\, x\, \exp(-y). \tag{15.16}$$

Thus

$$\Phi(\exp(0), z) = \Phi(\exp(y), 0). \tag{15.17}$$

But then (15.13) implies that $y = 0$ and $z = 0$. Thus

$$(x + U_0) \cap \mathcal{O}_j = \{x\}. \tag{15.18}$$

## References


[1] N. Burgoyne and R. Cushman: "Conjugacy Classes in Linear Groups", *J. Algebra*, Vol. 44, (1975), pp. 339–362.

[2] D. Collingwood and W. McGovern: *Nilpotent orbits in complex semisimple Lie algebras*, Reinhold, Van Nostrand, New York, 1993.

[3] A. Daszkiewicz, W. Kraśkiewicz and T. Przebinda: "Dual Pairs and Kostant-Sekiguchi Correspondence. II. Classification of Nilpotent Elements", *Centr. Eur. J. Math.*, Vol. 3, (2005), pp. 430-464.

[4] Harish-Chandra: "Invariant Distributions on Lie algebras", *Amer. J. of Math.*, Vol. 86, (1964), pp. 271–309.

[5] R. Howe: Analytic Preliminaries, preprint.

[6] R. Howe: "Remarks on classical invariant theory", *Trans. Amer. Math. Soc.*, Vol. 313, (1989), pp. 539–570,

[7] R. Howe: "Transcending Classical Invariant Theory", *J. Amer. Math. Soc.*, Vol. 2, (1989), pp. 535–552.

[8] V. Kac: "Lie superalgebras", *Adv. Math.*, Vol. 26, (1977), pp. 8–96.

[9] B. Kostant: *Graded manifolds, graded Lie theory, and prequantization*, Lecture Notes in Math., Vol. 570, Springer-Verlag, Berlin-New York, 1977, pp. 177–306.

[10] M. Spivak: *A comprehensive introduction to differential geometry*, Brandeis University, Waltham, Massachusetts, 1970.

[11] V.S. Varadarajan: *Harmonic Analysis on Real Reductive Groups I and II*,Lecture Notes in Math., Vol. 576, Springer Verlag, 1977.

[12] N. Wallach: *Real Reductive Groups I*, Academic Press, INC, 1988.