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On the occurrence of admissible representations in the real Howe correspondence in stable range

Abstract. Let $(G, G') \subset \mathrm{Sp}(W)$ be an irreducible real reductive dual pair of type I in stable range, with G the smaller member. In this note, we prove that all irreducible genuine representations of $\tilde{G}$ occur in the Howe correspondence. The proof uses structural information about the groups forming a reductive dual pair and estimates of matrix coefficients.

1. Introduction and the main result

One of the most important constructions in representation theory of reductive groups over local fields and the theory of automorphic representations is the reductive dual pair correspondence (or theta-correspondence) defined by R. Howe, see [3,6]. In his pioneering works [4,5] R. Howe demonstrated that a certain singular part of the unitary dual of $G' = Sp_{2n}(\mathbb{R})$ (or its two-fold cover) can be obtained from the unitary duals of various orthogonal groups $G = O_{p,q}(\mathbb{R})$, where G is "much smaller" than G'. This theory was completed by J.-S. Li [7] by incorporating all irreducible reductive dual pairs (G, G') of type I in stable range with G the smaller member over an arbitrary local field F. In this setting, J.-S. Li proved that any irreducible admissible genuine *unitary* representation π of $\tilde{G}$ occurs in the correspondence.

The purpose of this note is to give a uniform proof of an analogous result, Theorem 1, without the unitarity assumption on π. This statement seems to be known to the experts in the area; a sketch of a proof for a dual pair $(Sp_{2n}(\mathbb{R}), O_{2m,2n}(\mathbb{R}))$, $m \geq n$, appears in [9]. Our argument is similar to both [9, III.5] and [7, Theorem A]. We should note that the Langlands parameters of the representations occurring in the reductive dual pair correspondence have been worked out explicitly in many cases in [1,8,10]. However, this involves a fairly detailed analysis of representations, in particular, their K-types, and case-by-case considerations. By contrast, our proof uses only the standard results from the theory of reductive dual pairs (which we recall below) and some elementary estimates on the matrix coefficients from [12].

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Let W be a finite dimensional vector space over the reals, with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Let $\mathrm{Sp}(W)$ be the corresponding symplectic group and let $\widetilde{\mathrm{Sp}}(W)$ denote the metaplectic group with the covering map

$$\widetilde{\mathrm{Sp}}(W) \ni \tilde{g} \to g \in \mathrm{Sp}(W). \tag{1}$$

Let $(G, G') \subset \mathrm{Sp}(W)$ be an irreducible reductive dual pair of type I. The Classification Theorem due to R. Howe describes G and G′ as isometry groups of certain non-degenerate bilinear or sesquilinear forms on their defining modules V and V'. One says that the pair (G, G') is *in the stable range with* G *the smaller memeber* if the defining module for G′ containes an isotropic subspace of the same dimension as the defining module for G. For example, the reductive dual pair $(O_{p,q}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$ is in the stable range with $O_{p,q}(\mathbb{R})$ the smaller member if and only if $n \geq p + q$.

Every irreducible admissibe representation of a real reductive Lie group can be realized on some Hilbert space $\mathcal{H}_\pi$ so that the maximal compact subgroup acts by unitary operators. Let $\tilde{G}, \tilde{G}' \subset \widetilde{\mathrm{Sp}}(W)$ be the preimages of G, G′ in $\widetilde{Sp}(W)$. Let Z_2 be the kernel of the covering map (1). This is a two-element subgroup of $\widetilde{\mathrm{Sp}}(W)$, contained in both $\tilde{G}$ and $\tilde{G}'$. A representation π of $\tilde{G}$ on a Hilbert space $\mathcal{H}_\pi$ is called *genuine* if and only if the restriction of π to Z_2 is a multiple of the unique non-trivial character of Z_2.

Fix a character of the additive group of the real numbers and let ω be the corresponding oscillator representation of $\widetilde{\mathrm{Sp}}(W)$ realized on a Hilbert space $\mathcal{H}_\omega$, as in [3]. Let $\mathcal{H}_\omega^\infty$ be the space of smooth vectors and let $\mathcal{H}_\omega^{\infty*}$ be the linear topological dual, with the natural action of $\widetilde{\mathrm{Sp}}(W)$ also denoted by ω.

Theorem 1. *Let* $(G, G') \subset \mathrm{Sp}(W)$ *be an irreducible real reductive dual pair of type I in the stable range, with* G *the smaller memeber. Let* π *be an irreducible admissible genuine representation of* $\tilde{G}$ *on a Hilbert space* $\mathcal{H}_\pi$. *Then there is a continuous injective intertwining map*

$$Q : \mathcal{H}_\pi \to \mathcal{H}_\omega^{\infty*}. \tag{2}$$

In fact, there exists a non-zero linear functional $v^* \in \mathcal{H}_\omega^{\infty*}$ *such that for any non-zero vector* $v_0 \in \mathcal{H}_\pi$ *the following integral is absolutely convergent and defines a map Q with the required properties*:

$$Q(v)(v) = \int_{\tilde{G}} (\pi(\tilde{g})v, v_0)_{\mathcal{H}_\pi} v^*(\omega(\tilde{g})v) \, d\tilde{g} \qquad (v \in \mathcal{H}_\pi,\ v \in \mathcal{H}_\omega^\infty). \tag{3}$$

Remark. In the language of the theta correspondence (or Howe correspondence), the theorem says that each irreducible genuine representation of $\tilde{G}$ *occurs* in the correspondence. That only genuine representations occur in the correspondence is immediate. On the other hand, we do not assume that the representation π is unitary.

The intertwining property of the map (3) follows by formal manipulations, provided that the integral is absolutely convergent. We will verify the absolute convergence and the injectivity below. This is not difficult but requires some care.

The argument proceeds similar to the proof of Theorem A in [7] (where the representation π was assumed to be unitary), but instead of considering all smooth matrix coefficients of the oscillator representation ω as in [7], we concentrate on certain specially constructed ones that are sufficient for our purpose and use some estimates on the matrix coefficients which may be found in [12]. Very briefly, for the absolute convergence we show that the first factor in the integrand in (3) is of moderate growth, while the second one is rapidly decreasing, with respect to some norm on $\tilde{G}$. In order to establish the injectivity, we construct an equivariant embedding of G into an isotropic subspace X of W and analyze the action of $\tilde{G}$ in the mixed model of ω (cf. [9]).

2. Mixed model of ω

Let $J \in \text{End}(W)$ be a positive definite complex structure. Specifically, $J \in \mathfrak{sp}(W)$, $J^2 = -I$, and the symmetric bilinear form $\langle J\,,\,\rangle$ is positive definite. Since (G, G') is a type I reductive dual pair, we may assume that the conjugation by J preserves G and thus induces a Cartan involution θ on G. (This is explained in detail in [11]).

Let us view the symplectic vector space W as a real Hilbert space with the scalar product $(u, v) = \langle Ju, v \rangle$. Denote by $|\,|$ the corresponding operator norm on $End(W)$:

$$|T| = \max_{w \in W:\ (w,w)=1} (Tw, Tw)^{1/2}$$

Then the restriction of $|\,|$ to G is a norm on this group, in the sense of Wallach, [12, 2.A.2].

Recall that any irreducible real reductive dual pair of type I can be obtained as follows, [3,6]. There exist (1) a division algebra D over $\mathbb{R}$, with involution #, and (2) left D-vector spaces V and V' with non-degenerate #-sesquilinear forms $(\,,\,)$ and $(\,,\,)'$, one #-hermitian and the other #-skew-hermitian such that G and G' are the isometry groups of $(\,,\,)$ and $(\,,\,)'$, and $W = Hom_D(V', V)$. By the stable range assumption, the formed space V' contains an isotropic subspace X' of the same dimension as V. Since the form $(\,,\,)'$ is non-degenerate, we may choose another isotropic subspace Y' such that the restriction of the form $(\,,\,)'$ to $X' + Y'$ is non-degenerate, and we let V_0' be the orthogonal complement to $X' + Y'$ in V'. Then $V' = X' \oplus Y' \oplus V_0'$. Introduce the following subspaces of W:

$$X = Hom_D(X', V), \quad Y = Hom_D(Y', V), \quad W_0 = Hom_D(V_0', V).$$

Then X, Y are isotropic and W_0 is the orthogonal complement of $X + Y$ in W. Thus

$$W = X \oplus Y \oplus W_0.$$

Let us identify $X' = V$. Then $X = End_D(V)$ and the right multiplication of $w \in W = Hom_D(V', V)$ by $x \in X$ induces an embedding

$$\iota : X \to End(W).$$

We pull back the operator norm $|\ |$ from $End(\mathsf{W})$ to X via ι. The key observation now is that

$$\mathrm{G} \subseteq End_D(V) = X. \tag{4}$$

Recall [12, 2.A.2.4] that there is $d > 0$ such that $\int_{\mathrm{G}} |g|^{-d}\, dg < \infty$. Hence, for any $r \in \mathbb{R}$ there is N such that

$$\int\limits_{\mathrm{G}} |g|^r (1 + |g|)^{-N}\, dg < \infty. \tag{5}$$

For any $\tilde{g} \in \tilde{\mathrm{G}}$, its image under the projection (1) to G will be denoted by g, and we set $|\tilde{g}| = |g|$. We shall realize the oscillator representation ω in a mixed model associated to the decomposition (4) so that

$$\mathcal{H}_\omega^\infty = S(X, \mathcal{H}_{\omega_0}^\infty), \tag{6}$$

where ω_0 is the oscillator representation of $\widetilde{\mathrm{Sp}}(W_0)$ on the Hilbert space $\mathcal{H}_{\omega_0}$. Then $\tilde{\mathrm{G}}$ acts as follows:

$$\omega(\tilde{g})v(x) = \omega_0(\tilde{g})(v(g^{-1}x)) \qquad (\tilde{g} \in \tilde{\mathrm{G}},\ v \in S(X, \mathcal{H}_{\omega_0}^\infty),\ x \in X), \tag{7}$$

see section 4 in [7].

3. Construction of the intertwining map Q

Lemma 1. *For a seminorm q on $\mathcal{H}_{\omega_0}^\infty$ there is a seminorm q' on $\mathcal{H}_{\omega_0}^\infty$ and a constant C such that*

$$q(\omega_0(\tilde{g})v_0) \leq q'(v_0)|\tilde{g}|^C \qquad (\tilde{g} \in \tilde{\mathrm{G}},\ v_0 \in \mathcal{H}_{\omega_0}^\infty).$$

Proof. Our Cartan involution θ induces a Cartan decomposition

$$\tilde{\mathrm{G}} = KAK.$$

Clearly, it will suffice to verify the statement with $\tilde{\mathrm{G}}$ replaced by A. We may realize ω_0 in a Schrödinger model corresponding to a complete polarization $W_0 = X_0 \oplus Y_0$ preserved by A, and such that $\mathcal{H}_{\omega_0}^\infty = S(X_0)$. Then there is a group homomorphism $\delta : A \to \mathbb{R}^\times$ such that

$$\omega_0(\tilde{a})v_0(x_0) = \delta(\tilde{a})v_0(a^{-1}x_0) \qquad (\tilde{a} \in A,\ v_0 \in S(X_0),\ x_0 \in X_0).$$

Since the seminorm q is given in terms of derivatives and multiplication by polynomials on X_0, see [2], the estimate follows. $\square$

Lemma 2. *For a seminorm q on $\mathcal{H}_{\omega_0}^\infty$ and any $N \geq 0$ there is a seminorm q_N on $\mathcal{H}_\omega^\infty$ such that*

$$q(v(x)) \leq q_N(v)(1 + |x|)^{-N} \qquad (v \in \mathcal{H}_\omega^\infty,\ x \in X).$$

Proof. This is clear if we realize ω_0 as in the proof of Lemma 1. $\qquad\square$

Lemma 3. *For a seminorm q on $\mathcal{H}_{\omega_0}^\infty$ and any $N \geq 0$ there is a seminorm q_N on $\mathcal{H}_\omega^\infty$ such that*

$$q(\omega_0(\tilde{g})v(g^{-1})) \leq q_N(v)(1 + |g|)^{-N} \qquad (\tilde{g} \in \tilde{G}, \ v \in \mathcal{H}_\omega^\infty, \ x \in X).$$

Proof. Lemmas 1 and 2 show that for any $M \geq 0$, the left hand side may be dominated by

$$q'(v(g^{-1}))|g|^C \leq q'_M(v)(1 + |g|)^{-M}|g|^C,$$

where C does not depend on M. Since there is M such that

$$(1 + |g|)^{-M}|g|^C \leq (1 + |g|)^{-N},$$

we are done. $\qquad\square$

Fix a non-zero element $v_0^* \in \mathcal{H}_{\omega_0}^{\infty\,*}$ and define a linear functional $v^* \in \mathcal{H}_\omega^{\infty\,*}$ by the formula

$$v^*(v) = v_0^*(v(1)) \qquad (v \in \mathcal{H}_\omega^\infty). \tag{8}$$

Here 1 is the identity of the group G, viewed as an element of X via (4), and $v(1)$ is in $\mathcal{H}_{\omega_0}^\infty$, cf. (6). Lemma 3 shows that there are seminorms q_N on $\mathcal{H}_\omega^\infty$ such that

$$|v^*(\omega(\tilde{g})v)| \leq q_N(v)(1 + |g|)^{-N} \qquad (\tilde{g} \in \tilde{G}, \ v \in \mathcal{H}_\omega^\infty; \ N = 0, 1, 2, \ldots). \tag{9}$$

As shown in [12, 2.A.2.2], there are constants C, r such that the operator norm

$$\| \pi(\tilde{g}) \| \leq C|\tilde{g}|^r \qquad (\tilde{g} \in \tilde{G}). \tag{10}$$

By combining the estimates (9) and (10) of the matrix coefficients and taking into account the convergence statement (5), we see that the integral (3) is absolutely convergent and that the resulting map (2) is continuous. Thus the map Q is well-defined and has the required intertwining property. To complete the proof of Theorem 1, we need to check that this map is injective.

4. Injectivity of the intertwining map Q

Recall the non-zero vectors $v_0 \in \mathcal{H}_\pi$ (3) and $v_0^* \in \mathcal{H}_{\omega_0}^{\infty\,*}$ (8). Fix $v \in \mathcal{H}_\pi, v \neq 0$. We need to find $v \in \mathcal{H}_\omega^\infty$ such that $Q(v)(v) \neq 0$.

Since the representation π is irreducible and $v, v_0 \neq 0$, the function $\tilde{g} \mapsto (\pi(\tilde{g})v, v_0)_{\mathcal{H}_\pi}$ is not identically zero. Hence there is an element $\tilde{g}_v \in \tilde{G}$ and $c \in \mathbb{C}$ such that $c(\pi(\tilde{g}_v)v, v_0)_{\mathcal{H}_\pi} > 0$. Therefore, there is an open neighborhood $\tilde{U}$ of $\tilde{g}_v$ in $\tilde{G}$ such that

$$Re\, c(\pi(\tilde{g})v, v_0)_{\mathcal{H}_\pi} > 0 \qquad (\tilde{g} \in \tilde{U}).$$

Since $\omega_0(\tilde{g}_\nu)$ is invertible, $v_0^* \neq 0$, we see that $v_0^* \circ \omega_0(\tilde{g}_\nu)$ is a non-zero element of $\mathcal{H}_{\omega_0}^{\infty *}$ and there exists an element $v_\nu \in \mathcal{H}_{\omega_0}^\infty$ such that

$$v_0^*(\omega(\tilde{g}_\nu)v_\nu) = 1.$$

We may shrink the neighbourhood $\tilde{U}$ if necessary so that

$$Re\left(c(\pi(\tilde{g})v, v_0)_{\mathcal{H}_\pi} v_0^*(\omega_0(\tilde{g})v_\nu)\right) > 0 \quad (\tilde{g} \in \tilde{U}). \tag{11}$$

Let $U \subseteq G$ be the image of $\tilde{U}$ under the covering map (1). Since π is genuine, the last inequality continues to hold if we replace $\tilde{U}$ by the preimage of U in $\tilde{G}$, which we will again denote by $\tilde{U}$. Choose a function $v_s \in C_c^\infty(X)$ such that

$$v_s \geq 0, \ v_s(g_\nu^{-1}) > 0, \quad \operatorname{supp} v_s \subseteq U^{-1}. \tag{12}$$

Let

$$v(x) = cv_s(x)v_\nu \quad (x \in X).$$

Then $v \in C_c^\infty(X, \mathcal{H}_{\omega_0}^\infty)$ and the definition (8) of v^*, together with the formula (7) for the action of $\tilde{G}$ in the mixed model imply that

$$(\pi(\tilde{g})v, v_0)_{\mathcal{H}_\pi} v^*(\omega(\tilde{g})v) = c(\pi(\tilde{g})v, v_0)_{\mathcal{H}_\pi} v_0^*(\omega_0(\tilde{g})v_\nu)v_s(g^{-1}) \quad (\tilde{g} \in \tilde{G}). \tag{13}$$

The right hand side of the last expression is a product, where the first factor has positive real part for $\tilde{g} \in \tilde{U}$ by (11) and the second factor is positive for g in the interior of $(\operatorname{supp} v_s)^{-1} \subseteq U$ and zero on its complement, by (12). Therefore, the real part of the product is non-negative for all $\tilde{g} \in \tilde{G}$ and strictly positive on an open subset containing $\tilde{g}_\nu$. It follows that $Q(\nu)(v)$, which is the integral of the left hand side of (13) over $\tilde{G}$, is non-zero.

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