A REVERSE ENGINEERING APPROACH TO THE WEIL REPRESENTATION (EXTENDED)

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Abstract. We describe a new approach to the Weil representation attached to a symplectic group over a finite or a local field. We dissect the representation into small pieces, study how they work, and put them back together. This way, we obtain a reversed construction of that of T. Thomas, skipping most of the literature on which the latter is based.

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Date: July 23, 2020.
2000 *Mathematics Subject Classification.* Primary 22E45, Secondary 20C33, 22E46.
Key words and phrases. Weil representation, oscillator representation, metaplectic representation.
The second author was partially supported by the NSA grant H98230-13-1-0205.

1. Introduction

The Weil representation is a magnificent structure which keeps appearing in a variety of places throughout Mathematics and Physics. This is evident from a simple google or mathscinet search for "oscillator representation", "Weil representation", "Howe correspondence" or "local theta correspondence". The last two terms refer to a correspondence of irreducible representation for certain pairs of groups, conjectured to exist in [17], proven to exist over the reals in [21], over p-adic fields (p odd) in [48] and essentially proven not to exist over finite fields in [1]. A concise description of the Weil representation may be found in [45]. Anyone interested in a short and complete presentation should read that paper and stop right there. That work is really hard to improve upon. In this article we take the opposite approach. We dissect the Weil representation into small pieces, study how they work, and put them back together, in effect checking that the formulas of [45, Theorem C] are correct, thus reversing Thomas' proofs and skipping most of the literature on which it is based. Hence the title of this article. The methods we use are elementary, i.e. contained in a graduate curriculum of an average university in the USA. In contrast, a reader well versed in Algebraic Geometry will certainly enjoy [6], [7] or [8]. In the real case one should also mentions some classics, such as [26] or [4].

The Weil representation concerns a symplectic group defined over a field or over the adeles (or, more recently, over a ring [2], [9], [23], or a finite abelian group [37]). The field could be finite or local. We always assume that the characteristic is not 2, skip the case of the complex numbers as not interesting, and the adeles, the rings and the finite abelian groups as very interesting but requiring more energy, which we have just exhausted. Here is a brief description of what we do.

Let $\mathbb{F}$ be a finite field of odd characteristic and let W be a finite dimensional vector space over $\mathbb{F}$ equipped with a non-degenerate symplectic form $\langle\ ,\ \rangle$. The symplectic form induces a *twisted convolution* $\natural$ on the space $\mathrm{L}^2(\mathrm{W})$, making it into an associative algebra with identity over $\mathbb{C}$. One may think of it as of "the essential part" of the group algebra of the Heisenberg group attached to $(\mathrm{W}, \langle\ ,\ \rangle)$. For any subspace $\mathsf{X} \subseteq \mathrm{W}$, define a measure μ_{X} on W by

$$\int_{\mathsf{X}} \psi(x)\, d\mu_{\mathsf{X}}(x) := |\mathsf{X}|^{-1/2} \sum_{x \in \mathsf{X}} \psi(x),$$

where $|\mathsf{X}|$ is the cardinality of X and $\psi\colon \mathsf{X} \to \mathbb{C}$ is a function. Fix a non-trivial character χ of the additive group $\mathbb{F}$. Then the twisted convolution (with respect to χ) of two functions $\phi, \psi\colon \mathrm{W} \to \mathbb{C}$ is defined as

$$\phi\natural\psi(w) := \int_{\mathrm{W}} \phi(u)\psi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{1}$$

The algebra $\mathrm{H.S.}(\mathrm{L}^2(\mathsf{X}))$ of the Hilbert-Schmidt operators on $\mathrm{L}^2(\mathsf{X})$ may be identified with $\mathrm{L}^2(\mathsf{X}\times\mathsf{X})$ by assigning the integral kernel $K \in \mathrm{L}^2(\mathsf{X}\times\mathsf{X})$ to each operator $\mathrm{Op}(K) \in \mathrm{H.S.}(\mathrm{L}^2(\mathsf{X}))$ by setting

$$\mathrm{Op}(K)v(x) := \int_{\mathsf{X}} K(x, x')v(x')\, d\mu_{\mathsf{X}}(x').$$

Suppose that X is a part of a complete polarization $\mathrm{W} = \mathsf{X}\oplus\mathsf{Y}$. Let $\mathcal{K}\colon \mathrm{L}^2(\mathrm{W}) \to \mathrm{L}^2(\mathsf{X}\times\mathsf{X})$ be the corresponding the Weyl transform:

$$\mathcal{K}(\phi)(x, x') = \int_{\mathsf{Y}} \phi(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathsf{Y}}(y)\,.$$

Then we have the following sequence of algebra isomorphisms:

$$\mathrm{L}^2(\mathrm{W}) \overset{\mathcal{K}}{\to} \mathrm{L}^2(\mathsf{X}\times\mathsf{X}) \overset{\mathrm{Op}}{\to} \mathrm{H.S.}(\mathrm{L}^2(\mathsf{X})). \tag{2}$$

Let $\mathrm{Sp}(\mathrm{W})$ denote the symplectic group, that is the isometry group of the form $\langle\ ,\ \rangle$. The main result of [45, Theorem C] gives an explicit formula for a map $T\colon \mathrm{Sp}(\mathrm{W}) \to \mathrm{L}^2(\mathsf{W})$ such that the resulting composition

$$\omega\colon \mathrm{Sp}(\mathrm{W}) \overset{T}{\to} \mathrm{L}^2(\mathrm{W}) \overset{\mathcal{K}}{\to} \mathrm{L}^2(\mathsf{X}\times\mathsf{X}) \overset{\mathrm{Op}}{\to} \mathrm{H.S.}(\mathrm{L}^2(\mathsf{X})), \tag{3}$$

is an injective group homomorphism of the symplectic group into the group $\mathrm{U}(\mathrm{L}^2(\mathsf{X}))$ of the unitary operators on $\mathrm{L}^2(\mathsf{X})$,

$$\omega : \mathrm{Sp}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X})), \tag{4}$$

which has the following "conjugation property"

$$\left(\omega(g)\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(g^{-1})\right)(w)=\phi(g^{-1}w)\qquad (g\in \mathrm{Sp}(\mathsf{W}),\ \phi\in \mathrm{L}^2(\mathrm{W})). \tag{5}$$

A less explicit formula for $T(g)$ occurred already in [16, Theorem 2.9]. The missing ingredient was the description of the trace $\mathrm{tr}(\omega(g))$, which was done in [44] and led to [45, Theorem C]. A proof of the existence of ω satisfying (4) and (5) is also available in [5, Theorem 2.4]. In chapter 3 we check, via a straightforward but non-trivial computation, that the ω given in [45, Theorem C] is indeed a group homomorphism.

Our approach is the following. For any $g\in \mathrm{Sp}(\mathrm{W})$, the left and right radicals of the bilinear form $(w,w')\mapsto \langle (g-1)w,w'\rangle$ coinciding with $\mathrm{Ker}(g-1)$, we get a non-degenerate bilinear form B_g on the quotient $\mathrm{W}/\mathrm{Ker}(g-1)$. Let $\mathrm{dis}(B_g)$ denote its discriminant. We set

$$\Theta(g):=|\mathrm{Ker}(g-1)|^{1/2}\,\gamma(1)^{\dim(g-1)\mathrm{W}}\ \mathrm{dis}(B_g), \tag{6}$$

where

$$\gamma(1)=\int_{\mathbb{F}}\chi(x^tx)d\mu_{\mathbb{F}}(x).$$

Then we define $T(g)$ by

$$T(g):=\Theta(g)\,\chi_{c(g)}\,\mathbb{I}_{(g-1)W},$$

where for $u\in(g-1)\mathrm{W}$

$$\chi_{c(g)}(u)=\chi(\frac{1}{4}\langle c(g)u,u\rangle), \tag{7}$$

$c(g)\colon (g-1)\mathrm{W}\to \mathrm{W}/\mathrm{Ker}(g-1)$ denoting the Cayley transform, and $\mathbb{I}_{(g-1)W}$ is the indicator function of $(g-1)W$.

Our first main result (Theorem 31) asserts that

$$T(g_1)\natural T(g_2)=T(g_1g_2),\qquad \text{for any } g_1,g_2\in \mathrm{Sp}(\mathrm{W}). \tag{8}$$

Let $\omega:=\mathrm{Op}\circ\mathcal{K}\circ T$. Our second main result (Theorem 33) asserts that ω is an injective group homomorphism from $\mathrm{Sp}(\mathrm{W})$ to $\mathrm{U}(\mathrm{L}^2(\mathsf{X}))$, that the function Θ coincides with the character of the resulting representation, and that Eqn. (5) holds true.

In the case $\mathbb{F}=\mathbb{R}$, the reals, one has to deal with the "smog overspreading the infinite field" [16, page 2]. In particular the first two Hilbert spaces which occur in (2) have to be replaced by the spaces of tempered distributions. Hence, the algebra structure breaks down, but enough of it survives to make sense out of the formulas like

$$T(\tilde g_1)\natural T(\tilde g_2)=T(\tilde g_1\tilde g_2), \tag{9}$$

where $\tilde g_1,\tilde g_2\in \widetilde{\mathrm{Sp}}(\mathrm{W})$, a double cover of $\mathrm{Sp}(\mathrm{W})$ (see below). The resulting representation ω of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ appeared first in [43], as a natural development in Quantum Mechanics, [47]. Explicit formulas for $\omega(\tilde g)$, $\tilde g\in\widetilde{\mathrm{Sp}}(\mathrm{W})$, may be found in [39, Theorem 5.3] and for $T(\tilde g)$ in [28]. Furthermore, if one thinks of $\omega(\tilde g)$ as of a pseudo-differential operator, then its Weyl symbol, see [15], is $T(\tilde g)$.

Our approach consists of defining first, for $g\in \mathrm{Sp}(\mathrm{W})$,

$$\Theta^2(g):=\gamma(1)^{2\dim(g-1)\mathrm{W}}\ \left(\det(g-1\colon \mathrm{W}/\mathrm{Ker}(g-1)\to(g-1)\mathrm{W})\right)^{-1}, \tag{10}$$

setting next

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) := \{(g,\xi);\ g \in \mathrm{Sp}(\mathrm{W}),\, \xi \in \mathbb{C}^\times,\, \xi^2 = \Theta^2(g)\},$$

and finally

$$\Theta(\tilde{g}) := \xi, \qquad \text{for } \tilde{g} = (g,\xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W}).$$

Let $\chi(r) = \exp(2\pi i r)$ for $r \in \mathbb{R}$. Define $\chi_{c(g)}$ as in (7). Then we set

$$T(\tilde{g}) := \Theta(\tilde{g})\, \chi_{c(g)}\, \mu_{(g-1)W},$$

where $\mu_{(g-1)W}$ is an appropriately normalized Haar measure on $(g-1)W$, and prove that the formula (9) is satisfied.

Similar difficulties as for the reals occur when $\mathbb{F}$ is a p-adic field, with some new ones, see chapter 6 for details. The representation ω was constructed in [51] and the explicit formulas for $\omega(\tilde{g})$, $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, may be found in [36]. Our construction in the p-adic case occurs to be a mixed version of the finite and the reals cases, as shows the definition of $\Theta(\tilde{g})^2$ (see Definition 122).

Checking the equality (8) (or (9)) requires some effort. First we compute the twisted convolution of the unnormalized Gaussians $\chi_{c(g)}\, \mathbb{I}_{(g-1)W}$ (or $\chi_{c(g)}\, \mu_{(g-1)W}$) and obtain a cocycle $C(g_1, g_2)$. This is straightforward, but not easy in the sense that one has to keep track of various determinants, which are explained in section 2. Then we "guess" the normalization factor $\Theta(g)$ (or $\Theta(\tilde{g})$) and verify (8) (or (9)). This second step is more difficult. "Guessing" the normalizing factor, which happens to be the distribution character of the Weil representation, was done for us by Teruji Thomas in the finite case and others in the remaining two cases. We show that the normalized Gaussians form a group by a direct computation involving the cocycle. The point is that this computation is the same in all three cases (finite, real and p-adic) and avoids the holomorphic continuation to the oscillator semigroup studied in [20], [29] or [34]. In a sense, we replace analytic difficulties by some convoluted linear algebra of section 2. Our methods are equivalent, but not equal, to those used in [26, sec. 1.4-1.7] where the authors describe the cocycle $C(g_1, g_2)/|C(g_1, g_2)|$ and give a formula for the Weil representation acting in some Schrödinger model. Proving that $C(g_1, g_2)/|C(g_1, g_2)|$ is a cocycle relies on Kashiwara's description of Maslov index associated to three maximal isotropic subspaces of W. We deduce this fact from the associativity of the twisted convolution of the Gaussians. Thus our "convoluted linear algebra" replaces the beautiful theory of Maslov index. (Another justification for the title of our article.)

Weil's construction covers the cases of all locally compact non-discrete fields (including the reals) and adeles and gives applications to the theory of automorphic forms. Hence the name "Weil representation", taking away some of the credit from David Shale - a student of Erza Segal. Possibly in an attempt to find a middle ground Roger Howe proposed the name "the oscillator representation", [16, page 1]. The names "Segal-Shale-Weil representation", [24], "metaplectic representation", [35], and "spin representation of the symplectic group", [25] have also been used. Since, as the reader will see, understanding the Fourier transform of a Gaussian is the only prerequisite to follow our reverse engineering process, a name like "Gauss-Fourier-Segal-Shale-Weil representation" is another option. (In fact many researchers have been (and most likely will be) fascinated by the Gaussians and wrote volumes about them, see for example [30].) We chose to use the name "Weil representation", because it is the shortest one.

We would like to thank Angela Pasquale and Allan Merino for their careful reading of this article, some corrections and suggestions. Also, we express our gratitude to the two referees for their time and guidance through some related literature.

2. Linear algebra preliminaries

The first aim of this section is to collect various results, valid for arbitrary commutative fields of characteristic not equal to 2, that we will use in each of the three next sections. It is the object of the subsections 2.1 to 2.4. The two other subsections are devoted to determinants over the reals, and over a p-adic field, respectively; the main result is Lemma 11 (resp. Lemma 23), which will be used in the proof of Lemma 50 (resp. Lemma 123).

2.1. **General results on quadratic forms.** Let $\mathbb{F}$ be a commutative field of characteristic not equal to 2. Let U be a finite dimensional vector space over $\mathbb{F}$. Suppose q is a non-degenerate symmetric bilinear form on U. Then the formula

$$\Phi(u)(v) = q(u, v) \qquad (u, v \in \mathsf{U}) \tag{11}$$

defines a linear isomorphism $\Phi\colon \mathsf{U} \to \mathsf{U}^*$, where U^* is the vector space dual to U. The form q^* dual to q is given by

$$q^*(u^*, v^*) = v^*(\Phi^{-1}(u^*)) \quad (u^*, v^* \in \mathsf{U}^*).$$

Let Q be the matrix obtained from any basis u_1, u_2, …, u_n of U by

$$Q_{i,j} = q(u_i, u_j) \qquad (1 \le i, j \le n). \tag{12}$$

Lemma 1. *If Q is the matrix corresponding to q and a basis u_1, u_2, …, u_n of U, as above, then Q^{-1} corresponds to q^* and the dual basis u_1^*, u_2^*, …, u_n^* of U^*.*

Proof. Suppose $\Phi(u) = u^*$. Then for any $v \in \mathsf{U}$,

$$u^*(v) = q(u, v) = \sum_{i,j=1}^{n} u_i^*(u) q(u_i, u_j) u_j^*(v).$$

Thus

$$u^* = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} u_i^*(u) q(u_i, u_j) \right) u_j^*.$$

Therefore

$$u^*(u_j) = \sum_{i=1}^{n} u_i^*(u) q(u_i, u_j) \qquad (1 \le j \le n).$$

In matrix form the above equations may be written as

$$(u^*(u_1), u^*(u_2), \ldots, u^*(u_n)) = (u_1^*(u), u_2^*(u), \ldots, u_n^*(u))Q.$$

Hence,

$$(u^*(u_1), u^*(u_2), \ldots, u^*(u_n))Q^{-1} = (u_1^*(u), u_2^*(u), \ldots, u_n^*(u)).$$

Thus

$$u = \sum_{j=1}^{n} u_j^*(u) u_j = \sum_{j=1}^{n} \sum_{i=1}^{n} u^*(u_i)(Q^{-1})_{i,j} u_j.$$

Therefore,

$$q^*(u^*,u^*)=\sum_{j=1}^n\sum_{i=1}^n u^*(u_i)(Q^{-1})_{i,j}u^*(u_i).$$

In other words,

$$q^*(u_i^*,u_j^*)=(Q^{-1})_{i,j}.$$

□

2.2. **Symplectic spaces.** Let W be a finite dimensional vector space over $\mathbb{F}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$ and let $\mathsf{U}\subseteq \mathrm{W}$ be a subspace. We shall identify W with the dual W^* by

$$w^*(w)=\langle w,w^*\rangle \qquad (w,w^*\in \mathrm{W}). \tag{13}$$

Then

$$\mathsf{U}^*=\mathrm{W}/\mathsf{U}^\perp \text{ and } (\mathsf{U}/\mathsf{V})^*=\mathsf{V}^\perp/\mathsf{U}^\perp, \tag{14}$$

where the orthogonal complements are taken in W, with respect to the symplectic form $\langle\ ,\ \rangle$.

Lemma 2. *Let $\mathsf{V}_1,\mathsf{V}_2\subseteq \mathrm{W}$ be two subspaces and let $w\in\mathrm{W}$ be such that $\mathsf{V}_1\cap(\mathsf{V}_2+w)\neq\emptyset$. Then for any $v\in\mathsf{V}_1\cap(\mathsf{V}_2+w)$,*

$$\mathsf{V}_1\cap(\mathsf{V}_2+w)=\mathsf{V}_1\cap\mathsf{V}_2+v.$$

Proof. There are vectors $v_1\in\mathsf{V}_1$ and $v_2\in\mathsf{V}_2$ such that

$$v=v_1=v_2+w.$$

Then

$$\mathsf{V}_1\cap(\mathsf{V}_2+w)-v=\mathsf{V}_1\cap(\mathsf{V}_2+w)-v_1\subseteq\mathsf{V}_1-v_1=\mathsf{V}_1$$

and

$$\mathsf{V}_1\cap(\mathsf{V}_2+w)-v=\mathsf{V}_1\cap(\mathsf{V}_2+w)-(v_2+w)\subseteq(\mathsf{V}_2+w)-(v_2+w)=\mathsf{V}_2.$$

Hence,

$$\mathsf{V}_1\cap(\mathsf{V}_2+w)-v\subseteq\mathsf{V}_1\cap\mathsf{V}_2.$$

Conversely, let $\mathsf{V}_1\ni v_1'=v_2'\in\mathsf{V}_2$. Then

$$v_1'+v=v_1'+v_1\in\mathsf{V}_1 \text{ and } v_2'+v=v_2'+v_2+w\in\mathsf{V}_2+w.$$

Therefore

$$\mathsf{V}_1\cap\mathsf{V}_2+v\subseteq\mathsf{V}_1\cap(\mathsf{V}_2+w).$$

□

Let $\mathrm{Sp}(W)$ denote the isometry group of $\langle\ ,\ \rangle$:

$$\mathrm{Sp}(\mathrm{W})=\{g\in\mathrm{GL}(\mathrm{W})\ :\ \langle gw,gw'\rangle=\langle w,w'\rangle\quad\forall w,w'\in\mathrm{W}\}.$$

Let $\dim(\mathrm{W})=2n$. Then there is a group isomorphism

$$\mathrm{Sp}(\mathrm{W})\simeq\mathrm{Sp}_{2n}(\mathbb{F}):=\{A\in\mathrm{GL}_{2n}(\mathbb{F})\ :\ A^tJ'A=J'\}, \tag{15}$$

where A^t means the transpose of A, and

$$J' = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

The Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{F})$ is equal to

$$\mathfrak{sp}_{2n}(\mathbb{F}) = \{X \in \mathfrak{gl}_{2n}(\mathbb{F}) \; : \; X^t J' + J'X = 0\}.$$

Matrices which belong to $\mathrm{Sp}_{2n}(\mathbb{F})$ are called *symplectic matrices.* It clearly follows from (15) that the square of the determinant of any symplectic matrix is 1. In fact, the determinant itself is always 1. Indeed, the determinant of any antisymmetric matrix can be expressed as the square of a polynomial in the entries of the matrix. This polynomial Pf is called the Pfaffian. The following identity holds true: $\mathrm{Pf}(A^t J' A) = \det(A)\,\mathrm{Pf}(J')$. Since $A^t J' A = J'$, we get $\det(A) = 1$.

2.3. **The Cayley transform.** For $g \in \mathrm{Sp}(\mathrm{W})$, we set

$$g^{\pm} := g \pm 1, \tag{16}$$

and define the Cayley transform by

$$c(g)\colon g^{-}\mathrm{W} \ni g^{-}w \to g^{+}w + \mathrm{Ker}(g^{-}) \in \mathrm{W}/\mathrm{Ker}(g^{-}). \tag{17}$$

Then the bilinear form

$$\langle c(g)u', u''\rangle = \langle g^{+}w', g^{-}w''\rangle \qquad (u' = g^{-}w',\; u'' = g^{-}w'',\; w', w'' \in \mathrm{W}) \tag{18}$$

on the space $g^{-}\mathrm{W}$ is well defined and symmetric.

Lemma 3. *For any $g \in \mathrm{Sp}(\mathrm{W})$ the map*

$$g^{+} : \mathrm{Ker}(g^{-}) \to \mathrm{Ker}(g^{-})$$

is bijective.

Also, for any $u = g^{-}w$, with $w \in \mathrm{W}$, the preimage of $c(g)u \in \mathrm{W}/\mathrm{Ker}(g^{-})$ under the quotient map $\mathrm{W} \to \mathrm{W}/\mathrm{Ker}(g^{-})$ is equal to $g^{+}w + \mathrm{Ker}(g^{-})$.

Proof. Since g^{+} commutes with g^{-}, g^{+} preserves $\mathrm{Ker}(g^{-})$. Suppose $w \in \mathrm{Ker}(g^{-})$ and $g^{+}w = 0$. Then

$$g^{-}w = 0 \text{ and } g^{+}w = 0,$$

which implies $w = 0$. The second statement is obvious. □

Notation 4. For $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$, let

$$\mathsf{U}_1 := g_1^{-}\mathrm{W}, \quad \mathsf{U}_2 := g_2^{-}\mathrm{W} \quad \text{and} \quad \mathsf{U}_{12} := (g_1 g_2)^{-}\mathrm{W},$$

$$K_1 := \mathrm{Ker}\, g_1^{-}, \quad K_2 := \mathrm{Ker}\, g_2^{-} \quad \text{and} \quad K_{12} := \mathrm{Ker}(g_1 g_2)^{-}.$$

Lemma 5. *Let* $g_1, g_2 \in \mathrm{Sp(W)}$ *and let* $w, v \in \mathrm{W}$ *be such that*

$$v \in \mathsf{U}_1 \cap (\mathsf{U}_2 + w).$$

Then for any $u' \in \mathsf{U}_1 \cap \mathsf{U}_2$

$$\begin{aligned} & \langle c(g_1)(u'+v), u'+v\rangle + \langle c(g_2)(w-u'-v), w-u'-v\rangle + 2\langle u'+v, w\rangle \\ = & \langle (c(g_1)+c(g_2))u', u'\rangle - 2\langle u', c(g_1)v - c(g_2)(w-v) - w\rangle \\ + & \langle c(g_1)v, v\rangle + \langle c(g_2)(w-v), w-v\rangle + 2\langle v, w\rangle. \end{aligned}$$

Proof. Notice that all the terms in the above expression make sense. Also,

$$\langle c(g_1)(u'+v), u'+v\rangle = \langle c(g_1)u', u'\rangle + 2\langle c(g_1)u', v\rangle + \langle c(g_1)v, v\rangle$$

and

$$\langle c(g_2)(w-u'-v), w-u'-v\rangle = \langle c(g_2)(w-v), w-v\rangle - 2\langle c(g_2)(w-v), u'\rangle + \langle c(g_2)u', u'\rangle.$$

Hence

$$\begin{aligned} & \langle c(g_1)(u'+v), u'+v\rangle + \langle c(g_2)(w-u'-v), w-u'-v\rangle \\ = & \langle (c(g_1)+c(g_2))u', u'\rangle + \langle c(g_1)v, v\rangle + \langle c(g_2)(w-v), w-v\rangle \\ + & 2\langle c(g_1)u', v\rangle - 2\langle c(g_2)(w-v), u'\rangle. \end{aligned}$$

Furthermore

$$\begin{aligned} & \langle c(g_1)u', v\rangle - \langle c(g_2)(w-v), u'\rangle \\ = & -\langle u', c(g_1)v\rangle + \langle u', c(g_2)(w-v)\rangle = -\langle u', c(g_1)v - c(g_2)(w-v)\rangle \end{aligned}$$

and the desired equality follows. □

Notation 6. *For two elements* $g_1, g_2 \in \mathrm{Sp(W)}$, *let* $\mathsf{U} := \mathsf{U}_1 \cap \mathsf{U}_2$, *and let* q_{g_1,g_2} *denote the following symmetric form on* U*:*

$$q_{g_1,g_2}(u', u'') = \frac{1}{2}\left(\langle c(g_1)u', u''\rangle + \langle c(g_2)u', u''\rangle\right) \qquad (u', u'' \in \mathsf{U}). \tag{19}$$

Let $\mathsf{V} \subseteq \mathsf{U}$ *be the radical of* q_{g_1,g_2} *and let* $\tilde{q}_{g_1,g_2}$ *be the corresponding non-degenerate form on the quotient* U/V.

Lemma 7. *Let* g_1, g_2, U *and* V *be as in Notation 6. Then*

(a) $\dim(K_1 \cap K_2) + \dim \mathsf{V} = \dim K_{12}$;
(b) $\dim \mathrm{W} - \dim \mathsf{U} - \dim \mathsf{V} = \dim K_1 + \dim K_2 - \dim K_{12}$;
(c) $\dim \mathsf{U}_1 + \dim \mathsf{U}_2 - \dim \mathsf{U}_{12} = \dim \mathsf{U} + \dim \mathsf{V}$;
(d) $\mathsf{V} = g_2^- K_{12} = (g_1^{-1} - 1)K_{12}$.

Proof. It is easy to check that the kernel of the following map

$$\mathrm{W} \oplus \mathrm{W} \ni (w_1, w_2) \to (a, b, c) \in \mathrm{W} \oplus \mathrm{W} \oplus \mathrm{W}$$

where

$$a = g_1^- w_1 - g_2^- w_2, \quad b = g_1^- w_1 + g_2^- w_2 \ \text{ and } c = g_1^+ w_1 + g_2^+ w_2,$$

is equal to

$$\{(w,-w);\ w\in K_1\cap K_2\} \tag{20}$$

and that the set of the pairs (w_1,w_2) such that $a=0$ and $c=0$ is equal to

$$\{(-g_2w_2,w_2);\ w_2\in K_{12}\}. \tag{21}$$

Let $u\in\mathsf{U}$. Then there are $w_1,w_2\in \mathrm{W}$ such that $u=g_1^-w_1=g_2^-w_2$. In particular the element "a" is zero. The condition that $u\in\mathsf{V}$ means that

$$g_1^+w_1+g_2^+w_2\in\mathsf{U}^\perp. \tag{22}$$

Since $\mathsf{U}^\perp=K_1+K_2$, there are elements $x_1\in K_1$ and $x_2\in K_2$ such that

$$g_1^+w_1+g_2^+w_2=x_1+x_2.$$

Lemma 3 shows that there are unique elements $y_1\in K_1$ and $y_2\in K_2$ such that $g_1^+y_1=-x_1$ and $g_2^+y_1=-x_2$. Let $w_1'=w_1+y_1$ and $w_2'=w_2+y_2$. Then

$$g_1^+w_1'+g_2^+w_2'=0 \ \text{ and } \ u=g_1^-w_1'=g_2^-w_2'.$$

Therefore V is equal to the projection on the "b component" of the set (21).

Hence, $\dim\mathsf{V}$ is equal to the dimension of the set (21) minus the dimension of the kernel (20):

$$\dim\mathsf{V}=\dim K_{12}-\dim(K_1\cap K_2).$$

This verifies (a).

Since

$$\dim\mathsf{U}^\perp=\dim(K_1+K_2)=\dim K_1+\dim K_2-\dim(K_1\cap K_2)$$

and since $\dim\mathsf{U}^\perp=\dim\mathrm{W}-\dim\mathsf{U}$, (b) follows from (a).

We have

$$\begin{aligned}\dim\mathsf{U}_1+\dim\mathsf{U}_2-\dim\mathsf{U}_{12} &= (\dim\mathrm{W}-\dim K_1)+(\dim\mathrm{W}-\dim K_2)-(\dim\mathrm{W}-\dim K_{12})\\ &= \dim\mathrm{W}+\dim K_{12}-\dim K_1-\dim K_2=\dim\mathsf{U}+\dim\mathsf{V},\end{aligned}$$

because of (b). It proves (c).

As we already noticed,

$$\begin{aligned}\mathsf{V} &= \{g_1^-(-g_2w_2)+g_2^-w_2;\ w_2\in K_{12}\}=\{g_1^-(-g_1^{-1}w_2)+g_2^-w_2;\ w_2\in K_{12}\}\\ &= \{(g_1^{-1}-1)w_2+g_2^-w_2;\ w_2\in K_{12}\}=\{2g_2^-w_2;\ w_2\in K_{12}\}\\ &= \{g_2^-w_2;\ w_2\in K_{12}\}=\{(g_1^{-1}-1)w_2;\ w_2\in K_{12}\}.\end{aligned}$$

This verifies (d). □

Lemma 8. *Let $g\in\mathrm{Sp}(\mathrm{W})$. Then there is a direct sum decomposition*

$$\mathrm{W}=\mathsf{X}\oplus\mathrm{W}_0\oplus\mathsf{Y}\oplus\mathrm{W}_1$$

such that the subspaces X and Y are isotropic,

$$(\mathsf{X}+\mathsf{Y})^\perp=\mathrm{W}_0+\mathrm{W}_1,\ \mathsf{X}\oplus\mathrm{W}_0\oplus\mathsf{Y}=\mathrm{W}_1^\perp,$$
$$\mathsf{X}\oplus\mathrm{W}_0=\mathrm{Im}(g^-),\ \mathsf{X}\oplus\mathrm{W}_1=\mathrm{Ker}(g^-),\ \textit{and}\ \mathsf{X}=\mathrm{Ker}(g^-)\cap\mathrm{Ker}(g^-)^\perp,$$

where $\mathrm{Im}(g^-) = g^-\mathrm{W}$. *Furthermore, there are unique elements*

$$g_0 \in \mathrm{Sp}(\mathrm{W}_0),\ T \in \mathrm{Hom}(\mathrm{W}_0, \mathsf{X}),\ S \in \mathrm{Hom}(\mathsf{Y}, \mathsf{X})$$

such that for $x \in \mathsf{X}$, $w_0 \in \mathrm{W}_0$, $y \in \mathsf{Y}$ *and* $w_1 \in \mathrm{W}_1$

$$g(x + w_0 + y + w_1) = (x + Tw_0 + Sy) + (g_0 w_0 - g_0 T^* y) + y + w_1,$$

where $T^* \in \mathrm{Hom}(\mathsf{Y}, \mathrm{W}_0)$ *is the conjugate of* T *with respect to the pairing* $\langle\ ,\ \rangle$, *and the map*

$$\mathrm{W}_0 \oplus \mathsf{Y} \ni w_0 + y \to (Tw_0 + Sy) + ((g_0 - 1)w_0 - g_0 T^* y) \in \mathsf{X} \oplus \mathrm{W}_0$$

is invertible.

In particular if $g_1 \in \mathrm{End}(\mathrm{W})$ *is defined by*

$$g_1(x + w_0 + y + w_1) = -x - g_0^{-1} w_0 - y - w_1,$$

then $g_1 \in \mathrm{Sp}(\mathrm{W})$ *and* $\mathrm{Ker}(g_1 g - 1) = \mathrm{Ker}(g g_1 - 1) = 0$.

Proof. Clearly $\mathsf{X} = \mathrm{Ker}(g^-) \cap \mathrm{Ker}(g^-)^\perp$ is an isotropic subspace. Let $\mathsf{Y} \subseteq \mathrm{W}$ be another isotropic subspace such that the restriction of the symplectic form to the sum $\mathsf{X} \oplus \mathsf{Y}$ is non-degenerate. Define $\mathrm{W}' = (\mathsf{X} + \mathsf{Y})^\perp$. Then we have $\mathrm{W} = \mathsf{X} \oplus \mathrm{W}' \oplus \mathsf{Y}$.

Also,

$$\mathsf{X} \oplus \mathrm{W}' = \mathsf{X}^\perp = \mathrm{Ker}(g^-) + \mathrm{Ker}(g^-)^\perp \supseteq \mathrm{Ker}(g^-).$$

Set $\mathrm{W}_1 = \mathrm{Ker}(g^-) \cap \mathrm{W}'$. Then the above inclusion implies that $\mathrm{Ker}(g^-) = \mathsf{X} \oplus \mathrm{W}_1$. Let $\mathrm{W}_0 = \mathrm{W}_1^\perp \cap \mathrm{W}'$. Then

$$\mathrm{W}' = \mathsf{W}_0 \oplus \mathrm{W}_1 \quad \text{and} \quad \mathrm{Im}(g^-) = \mathrm{Ker}(g^-)^\perp = \mathsf{X} \oplus \mathrm{W}_0.$$

Since g acts as the identity on W_1, g preserves $\mathrm{W}_1^\perp$. Then $g|_{\mathrm{W}_1^\perp}$ acts as the identity on X. Also, the stabilizer of X in $\mathrm{Sp}(\mathrm{W})$ is a parabolic subgroup. Hence the formula for g follows from the well known structure of these subgroups.

Clearly the element g_1 belongs to $\mathrm{Sp}(\mathrm{W})$. Let $w = x + w_0 + y + w_1$ as in the lemma.

Suppose $g_1 g w = w$. Then

$$x = -x - Tw_0 - Sy,\ w_0 = -w_0 + g_0^{-1} T^* y,\ y = -y \text{ and } w_1 = -w_1.$$

Since the characteristic of the field $\mathbb{F}$ is not 2, we see that $w = 0$.

Suppose $g g_1 w = w$. Then

$$x = -x - Tg_0^{-1} w_0 - Sy,\ w_0 = -w_0 + T^* y,\ y = -y \text{ and } w_1 = -w_1.$$

Again, since the characteristic of the field $\mathbb{F}$ is not 2, we see that $w = 0$. $\square$

2.4. **More lemmas.** Assume from now on till the end of this subsection that $K_1 = \mathrm{Ker}\, g_1^- = \{0\}$.

In this case $\mathsf{U} = g_2^- \mathrm{W}$. Then

$$K_2 \cap K_{12} = K_1 \cap K_2 = \{0\}.$$

Hence there is a subspace $\mathrm{W}_2 \subseteq \mathrm{W}$ such that

$$\mathrm{W} = K_{12} \oplus \mathrm{W}_2 \oplus K_2. \tag{23}$$

Pick a subspace $\mathsf{U}' \subseteq \mathrm{W}$ such that

$$\mathrm{W} = \mathsf{U} \oplus \mathsf{U}'.$$

Then $\mathsf{U} = K_2^{\perp}$ and $\dim\, \mathsf{U}' = \dim\, K_2$. Fix a basis w_{b+1}, w_{b+2}, ... of K_2 and let w'_{b+1}, w'_{b+2}, ... be the dual basis of U' in the sense that

$$\langle w_i, w'_j \rangle = \delta_{i,j} \qquad (b < i, j).$$

Define an element $h \in \mathrm{GL}(\mathrm{W})$ by

$$h|_{K_{12}\oplus \mathrm{W}_2} = (g_1^{-1} - 1)^{-1} g_2^-,\ hw_i = (g_1^{-1} - 1)^{-1} w'_i,\ b < i. \tag{24}$$

Let us extend the basis w_i of K_2 to a basis of W so that $w_i \in K_{12}$ if $i \leq a$ and $w_i \in \mathrm{W}_2$ if $a < i \leq b$. Then

$$hw_i = w_i \qquad (i \leq a). \tag{25}$$

Lemma 9. *The following equalities hold:*

$$\begin{aligned} & \det(\langle (g_1g_2)^- w_i, hw_j \rangle_{a<i,j}) & (26)\\ = \ & \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) g_2^- w_i, g_2^- w_j \rangle_{a<i,j\leq b}) \\ = \ & \det(\langle (g_1g_2)^- w_i, w_j \rangle_{a<i,j}) \det(h). \end{aligned}$$

Moreover, we have

$$\det(\langle w_i, (g_1^{-1} - 1) hw_j \rangle_{i,j}) = (-1)^{\dim \mathsf{U}} \det(\langle g_2^- w_i, w_j \rangle_{i,j\leq b}). \tag{27}$$

Proof. Notice that both $c(g_1)$ and $c(g_2)$ are well defined on the space U and

$$g_1^- \frac{1}{2}(c(g_1) + c(g_2)) g_2^- = \frac{1}{2}(g_1^+ g_2^- + g_1^- g_2^+) + g_1^- K_2 = (g_1g_2)^- + g_1^- K_2. \tag{28}$$

Suppose $a < i, j \leq b$. Then (28) shows that

$$\begin{aligned} \langle (g_1g_2)^- w_i, hw_j \rangle &= \langle (g_1g_2)^- w_i, (g_1^{-1} - 1)^{-1} g_2^- w_j \rangle & (29)\\ &= \langle (g_1^-)^{-1} (g_1g_2)^- w_i, g_2^- w_j \rangle \\ &= \langle (g_1^-)^{-1} g_1^- \frac{1}{2}(c(g_1) + c(g_2)) g_2^- w_i, g_2^- w_j \rangle \\ &= \langle \frac{1}{2}(c(g_1) + c(g_2)) g_2^- w_i, g_2^- w_j \rangle. \end{aligned}$$

Suppose $j \leq b < i$. Then $(g_1g_2)^- w_i = g_1^- w_i$. Hence,

$$\begin{aligned} \langle (g_1g_2)^- w_i, hw_j \rangle &= \langle g_1^- w_i, (g_1^{-1} - 1)^{-1} g_2^- w_j \rangle & (30)\\ &= \langle w_i, g_2^- w_j \rangle \\ &= \langle (g_2^{-1} - 1) w_i, w_j \rangle \\ &= \langle -g_2^{-1} g_2^- w_i, w_j \rangle \\ &= \langle 0, w_j \rangle \\ &= 0. \end{aligned}$$

If $b < i, j$, then

$$\langle (g_1 g_2)^- w_i, h w_j \rangle = \langle g_1^- w_i, h w_j \rangle. \tag{31}$$

Notice that

$$\det(\langle g_1^- w_i, h w_j \rangle_{b<i,j}) = \det(\langle w_i, (g_1^{-1} - 1) h w_j \rangle_{b<i,j}) = \det(\langle w_i, w_j' \rangle_{b<i,j}) = 1. \tag{32}$$

The first equality in (26) follows from relations (29), (30), (31) and (32).

Since h preserves the subspace K_{12}, it makes sense to define $\tilde{h} \in \mathrm{GL}(\mathrm{W}/K_{12})$ by

$$\tilde{h}(w + K_{12}) = hw \qquad (w \in \mathrm{W}).$$

Then

$$\det(\langle (g_1 g_2)^- w_i, h w_j \rangle_{a<i,j}) = \det(\langle (g_1 g_2)^- w_i, w_j \rangle_{a<i,j}) \det(\tilde{h}).$$

But (25) implies $\det(\tilde{h}) = \det(h)$. Hence the second equality in (26) follows.

Also, if $j \le b < i$, then

$$\langle w_i, (g_1^{-1} - 1) h w_j \rangle = \langle w_i, g_2^- w_j \rangle = 0$$

because $K_2 \perp \mathsf{U}$. Hence,

$$\begin{aligned}
\det(\langle w_i, (g_1^{-1} - 1) h w_j \rangle_{i,j}) &= \det(\langle w_i, (g_1^{-1} - 1) h w_j \rangle_{i,j \le b}) \det(\langle w_i, (g_1^{-1} - 1) h w_j \rangle_{b<i,j}) \\
&= \det(\langle w_i, (g_1^{-1} - 1) h w_j \rangle_{i,j \le b}) \\
&= \det(\langle w_i, g_2^- w_j \rangle_{i,j \le b}) \\
&= (-1)^{\dim K_{12} + \dim \mathrm{W}_2} \det(\langle g_2^- w_i, w_j \rangle_{i,j \le b}) \\
&= (-1)^{\dim \mathsf{U}} \det(\langle g_2^- w_i, w_j \rangle_{i,j \le b}),
\end{aligned}$$

This verifies (27). □

Corollary 10. *With the above notation we have*

$$\begin{aligned}
&\det(\langle \frac{1}{2}(c(g_1) + c(g_2)) g_2^- w_i, g_2^- w_j \rangle_{a<i,j \le b}) \\
= \quad &(-1)^{\dim \mathsf{U}} \frac{\det(\langle (g_1 g_2)^- w_i, w_j \rangle_{a<i,j})}{\det(\langle g_1^- w_i, w_j \rangle_{i,j}) \det(\langle g_2^- w_i, w_j \rangle_{i,j \le b})^{-1}}.
\end{aligned}$$

2.5. **Determinants over the reals.** Consider two vector spaces U', U'' over $\mathbb{R}$ of the same dimension equipped with positive definite bilinear symmetric forms B', B'' respectively. Let u_1', u_2', …, u_n' be a B'-orthonormal basis of U' and let u_1'', u_2'', …, u_n'' be a B''-orthonormal basis of U''. Suppose $L \colon \mathsf{U}' \to \mathsf{U}''$ is a linear bijection. Denote by M the matrix of L with respect to the two ordered basis:

$$L u_j' = \sum_{i=1}^{n} M_{i,j} u_i'' \qquad (j = 1, 2, \ldots, n).$$

Then $(\det(M))^2$ does not depend on the choice of the orthonormal basis. (Indeed, if we change the orthonormal bases in the two spaces, we get two matrices $P = (P^t)^{-1}$ and $Q = (Q^t)^{-1}$, so that the new matrix is $M' = PMQ$. Thus $\det(M') = \det(P)\det(M)\det(Q)$.

Since $(\det(P))^2 = (\det(Q))^2 = 1$, we see that $(\det(M'))^2 = (\det(M))^2$.) Thus we may define $(\det(L))^2 := (\det(M))^2$.

We shall also need a notion of a determinant for a linear map between two vector spaces (under some additional assumptions of course). For that reason we fix an element $J \in \mathrm{Sp}(\mathrm{W})$ and the corresponding positive definite symmetric bilinear form B, that is,

$$B(w, w') = \langle J(w), w' \rangle \qquad (w, w' \in \mathrm{W}). \tag{33}$$

Then every subspace of W has a B-orthonormal basis.

For a subset $S \subseteq \mathrm{W}$ let $S^{\perp_B} \subseteq \mathrm{W}$ be the B-orthogonal complement of S. It is easy to see that

$$S^{\perp_B} = J^{-1} S^{\perp} = J S^{\perp}. \tag{34}$$

For an element $h \in \mathrm{End}(\mathrm{W})$ define $h^{\#} \in \mathrm{End}(\mathrm{W})$ by

$$\langle hw, w' \rangle = \langle w, h^{\#} w' \rangle \qquad (w, w' \in \mathrm{W}). \tag{35}$$

Then $(\mathrm{Ker}\, h^{\#})^{\perp} = h\mathrm{W}$.

Consider an element $h \in \mathrm{End}(\mathrm{W})$ such that $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$. (In our applications h will be equal to g^-, where $g \in \mathrm{Sp}(\mathrm{W})$. Then $g^{\#} = g^{-1} - 1 = -g^{-1}g^-$ has the same kernel as g^-.) Let $L = J^{-1}h$. Denote by L^* the adjoint to L with respect to B, $(B(Lw, w') = B(w, L^* w'))$. Then $L^* = Jh^{\#}$. Hence $Ker\, L = Ker\, L^*$. Since B is anisotropic, L maps $(Ker\, L)^{\perp_B} = L\mathrm{W}$ bijectively onto itself. Thus it makes sense to talk about $\det(L|_{L\mathrm{W}})$, the determinant of the restriction of L to $L\mathrm{W}$. If w_1, w_2, ..., w_m is a B-orthonormal basis of $(Ker\, L)^{\perp_B}$, then

$$\begin{aligned} \det(L|_{L\mathrm{W}}) &= \frac{\det(B(Lw_i, w_j)_{1 \le i,j \le m})}{\det(B(w_i, w_j)_{1 \le i,j \le m})} = \det(B(Lw_i, w_j)_{1 \le i,j \le m}) \\ &= \det(\langle hw_i, w_j \rangle_{1 \le i,j \le m}). \end{aligned} \tag{36}$$

Under the condition $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$, we define $\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})$ to be the quantity (36).

Suppose $\mathsf{U} \subseteq \mathrm{W}$ is a subspace and $x \in \mathrm{Hom}(\mathsf{U}, \mathrm{W})$ is a linear map such that the formula

$$\langle xu, u' \rangle \qquad (u, u' \in \mathsf{U})$$

defines a symmetric bilinear form on U with the radical $\mathsf{V} \subseteq \mathsf{U}$. The form B induces a positive definite form on the quotient U/V. Pick a B-orthonormal basis $u_1 + \mathsf{V}, \dots, u_k + \mathsf{V} \in \mathsf{U}/\mathsf{V}$ and set

$$\det(\langle x\ ,\ \rangle_{\mathsf{U}/\mathsf{V}}) = \det(\langle xu_i, u_j \rangle_{1 \le i,j \le k}). \tag{37}$$

It is easy to see that the quantity (37) does not depend on the choice of the B-orthonormal basis.

Lemma 11. *Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and assume that $K_1 = \{0\}$. Then*

$$\begin{aligned} &\frac{\det((g_1 g_2)^- : \mathrm{W}/K_{12} \to \mathsf{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \ & (-1)^{\dim \mathsf{U}} \det(\langle \tfrac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}}) \, (\det(g_2^- : K_{12} \to \mathsf{V}))^{-2}. \end{aligned} \tag{38}$$

Proof. Let $\mathrm{W}_2 \subseteq \mathrm{W}$ be the B-orthogonal complement of $K_{12} + K_2$. Then (23) holds, because B is anisotropic. Let w_1, w_2, ... be a basis of W such that w_1, w_2, ..., w_a is a B-orthonormal basis of K_{12}, w_{a+1}, w_{a+2}, ..., w_b is a B-orthonormal basis of W_2 and w_{b+1}, w_{b+2}, ... is a B-orthonormal basis of K_2. Let $Q \in \mathrm{GL}(\mathrm{W})$ be such that

$$\begin{aligned}
&Qw_1, Qw_2, \dots \text{is a } B\text{-orthonormal basis of W},\\
&Qw_i = w_i \text{ if } i \le b,\\
&Qw_i \perp_B K_{12} + \mathrm{W}_2 \text{ if } b < i.
\end{aligned}$$

Define the matrix elements $Q_{j,i}$ by

$$Qw_i = \sum_j Q_{j,i} w_j.$$

Then

$$Q_{j,i} = \delta_{j,i} \text{ if } i \le b.$$

Hence,

$$\det(Q) = \det((Q_{j,i})_{1\le j,i}) = \det((Q_{j,i})_{b<j,i}) = \det((Q_{j,i})_{a<j,i})$$

and

$$\begin{aligned}
1 &= \det(J^{-1}) = \det(B(J^{-1}Qw_i, Qw_j)_{1\le i,j}) = \det(\langle Qw_i, Qw_j\rangle_{1\le i,j})\\
&= (\det(Q))^2 \det(\langle w_i, w_j\rangle_{1\le i,j}).
\end{aligned}$$

Therefore

$$\det((Q_{j,i})_{a<j,i})^2 \det(\langle w_i, w_j\rangle_{1\le i,j}) = 1. \tag{39}$$

Let u_1, u_2, ..., u_b be B-orthogonal basis of U such that u_1, u_2, ..., u_a span V. Define the matrix elements $(g_2^-)_{k,i}$ by

$$g_2^- w_i = \sum_{k=1}^{b} (g_2^-)_{k,i} u_k \qquad (1 \le i \le b).$$

Since $g_2^- K_{12} = \mathsf{V}$, we see that

$$(g_2^-)_{k,i} = 0 \text{ if } i \le a < k.$$

Hence

$$\det(((g_2^-)_{k,i})_{1\le k,i\le b}) = \det(((g_2^-)_{k,i})_{1\le k,i\le a}) \det(((g_2^-)_{k,i})_{a<k,i\le b}). \tag{40}$$

Also,

$$\begin{aligned}
&(\det(g_2^- : K_{12} \to \mathsf{V}))^2 = (\det(((g_2^-)_{k,i})_{1\le k,i\le a}))^2 \text{ and}\\
&(\det(g_2^- : \mathrm{W}_2 \to \mathsf{U}/\mathsf{V}))^2 = (\det(((g_2^-)_{k,i})_{a<k,i\le b}))^2.
\end{aligned} \tag{41}$$

Define $h \in \mathrm{GL}(\mathrm{W})$ as in (24). Then (26) shows that

$$\det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}) \det(h) = \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) g_2^- w_i, g_2^- w_j\rangle_{a<i,j\le b}). \tag{42}$$

Furthermore, by (27),

$$\begin{aligned} &\det(h) = \det((g_1^{-1}-1)^{-1}(g_1^{-1}-1)h) && (43)\\ =\ & \det(g_1^{-1}-1)^{-1}\ \det((g_1^{-1}-1)h)\\ =\ & \det(g_1^{-1}-1)^{-1}\ \det(\langle w_i,(g_1^{-1}-1)hw_j\rangle_{1\le i,j})\ \det(\langle w_i,w_j\rangle_{1\le i,j})^{-1}\\ =\ & \det(g_1^{-1}-1)^{-1}(-1)^{\dim \mathsf{U}}\ \det(\langle g_2^- w_i,w_j\rangle_{i,j\le b})\ \det(\langle w_i,w_j\rangle_{1\le i,j})^{-1} \end{aligned}$$

Also,

$$\begin{aligned} &\det(\langle\frac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\le b}) && (44)\\ =\ & \det(\langle\frac{1}{2}(c(g_1)+c(g_2))u_k,u_l\rangle_{a<k,l\le b})\ \det(((g_2^-)_{k,i})_{a<k,i\le b})^2. \end{aligned}$$

By (36),

$$\begin{aligned} &\det((g_1g_2)^- : \mathrm{W}/K_{12}\to \mathsf{U}_{12}) = \det(\langle (g_1g_2)^- Qw_i, Qw_j\rangle_{a<i,j}) && (45)\\ =\ & \det((Q_{i,j})_{a<i,j})^2\ \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}). \end{aligned}$$

Define an element $q\in \mathrm{GL(W)}$ by

$$\begin{aligned} qw_i &= J^{-1}u_i \text{ if } i\le b,\\ qw_i &= w_i \text{ if } b<i. \end{aligned}$$

Then qw_1, qw_2, …, qw_b is a B-orthonormal basis of $J^{-1}\mathsf{U} = K_2^{\perp_B}$ so that

$$\det(g_2^- : \mathrm{W}/K_2\to\mathsf{U}) = \det(\langle g_2^- qw_i, qw_j\rangle_{i,j\le b}).$$

Define the coefficients $q_{i,j}$ by

$$qw_i = \sum_j q_{j,i}w_j.$$

Then

$$q_{j,i} = \delta_{j,i} \text{ if } b<i$$

so that

$$\det(q) = \det((q_{j,i})_{1\le i,j}) = \det((q_{j,i})_{1\le i,j\le b}).$$

Also,

$$g_2^- qw_i = \sum_j q_{j,i}g_2^- w_j = \sum_{j\le b} q_{j,i}g_2^- w_j \qquad (i\le b).$$

Therefore,

$$\det(\langle g_2^- qw_i, qw_j\rangle_{i,j\le b}) = \det(q)^2\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}).$$

Define the coefficients $q_{i,j}^{-1}$ of the inverse map q^{-1} by

$$w_i = q^{-1}(qw_i) = \sum_j q_{i,j}^{-1}qw_j.$$

Since, the qw_i form an orthonormal basis of W,

$$q_{i,j}^{-1} = B(q^{-1}qw_i, qw_j) = B(w_i, qw_j) = B(qw_j, w_i),$$

so that

$$q^{-1}_{i,j} = \begin{cases} \langle u_j, w_i\rangle \text{ if } j \le b, \\ B(w_j, w_i) \text{ if } j > b, \\ B(w_j, w_i) = \delta_{i,j} \text{ if } i, j > b. \end{cases}$$

In particular, $q^{-1}_{i,j} = 0$ if $j \le b < i$ so that

$$\det(q)^{-1} = \det(q^{-1}) = \det((q^{-1}_{i,j})_{i,j\le b}) = \det(\langle u_j, w_i\rangle_{i,j\le b}).$$

Thus

$$\begin{aligned} & \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U}) = (\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}))^2 \det(q)^2 \qquad (46) \\ = \; & (\det(\langle \sum_{k=1}^{b} (g_2^-)_{k,i} u_k, w_j\rangle_{i,j\le b}))^2 \det(q)^2 = (\det((g_2^-)_{k,i})_{k,i\le b}) \det(\langle u_k, w_j\rangle_{k,j\le b}))^2 \det(q)^2 \\ = \; & (\det((g_2^-)_{k,i})_{k,i\le b}))^2 = (\det(g_2^- : K_{12} \to \mathsf{V}))^2 (\det(g_2^- : \mathrm{W}_2 \to \mathsf{U}/\mathsf{V}))^2, \end{aligned}$$

where the last equality follows from (40) and (41). The formula (38) follows from (39) - (46) via a straightforward computation:

$$\begin{aligned} & \frac{\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathsf{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \; & \frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \; & \frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1) + c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det((g_(2^-)_{k,i})_{a<k,i\le b})^2}{\det(h) \det g_1^- \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \; & \frac{(-1)^{\dim \mathsf{U}} \det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1) + c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2}{\det(g_1^{-1} - 1)^{-1} \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(\langle w_i, w_j\rangle_{1\le i,j})^{-1} \det g_1^- \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \; & \frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2}{\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \; & \frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \;,\; \rangle_{\mathsf{U}/\mathsf{V}}) (\det(g_2^- : \mathrm{W}_2 \to \mathsf{U}/\mathsf{V}))^2}{\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \; & \frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \;,\; \rangle_{\mathsf{U}/\mathsf{V}}) (\det(g_2^- : \mathrm{W}_2 \to \mathsf{U}/\mathsf{V}))^2}{(\det(g_2^- : K_{12} \to \mathsf{V}))^2 (\det(g_2^- : \mathrm{W}_2 \to \mathsf{U}/\mathsf{V}))^2} \\ = \; & \frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \;,\; \rangle_{\mathsf{U}/\mathsf{V}})}{(\det(g_2^- : K_{12} \to \mathsf{V}))^2}. \end{aligned}$$

(Here the second equality follows from (42) and (44), and the third one from (43).) □

2.6. **Determinants over p-fields.** Let $\mathbb{F}$ be a commutative p-field in the terminology of [52, Def 2, page 12], that is, $\mathbb{F}$ is a local non Archimedean field with finite residue field. Hence $\mathbb{F}$ is a finite extension of either the p-adic field $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$ (the fraction field of the ring $\mathbb{F}_p[[t]]$ of formal power series in one indeterminate t with coefficient in $\mathbb{F}_p$).

Denote by $|\ |_{\mathbb{F}}$ the module on $\mathbb{F}$, as in [52, page 4]. Then $\mathfrak{o}_{\mathbb{F}} = \{a \in \mathbb{F} \ : \ |a|_{\mathbb{F}} \le 1\}$ is the ring of integers of $\mathbb{F}$, and we have $\mathfrak{o}_{\mathbb{F}}^{\times} = \{a \in \mathbb{F} \ : \ |a|_{\mathbb{F}} = 1\}$ as in [52, page 12].

Being locally compact, $\mathbb{F}$ has a real-valued Haar measure: the unique translation invariant measure $\mu_{\mathbb{F}}$ with the properties

$$d\mu(ax) = |a|_{\mathbb{F}}\, d\mu(x) \quad (x \in \mathbb{F},\ a \in \mathbb{F}^{\times}),$$

$$\mu_{\mathbb{F}}(\mathfrak{o}_{\mathbb{F}}) = \int_{|x|_{\mathbb{F}} \le 1} d\mu(x) = 1.$$

Let $r \in \mathbb{Z}$. One has

$$\mu_{\mathbb{F}}(\varpi_{\mathbb{F}}^{r}\, \mathfrak{o}_{\mathbb{F}}) = \int_{|x|_{\mathbb{F}} \le q^{r}} d\mu_{\mathbb{F}}(x) = q^{r}. \tag{47}$$

Then Eqn. (47) gives

$$\int_{|x|_{\mathbb{F}} = q^{r}} d\mu_{\mathbb{F}}(x) = \int_{|x|_{\mathbb{F}} \le q^{r}} d\mu_{\mathbb{F}}(x) - \int_{|x|_{\mathbb{F}} \le q^{r-1}} d\mu_{\mathbb{F}}(x) = q^{r}(1 - q^{-1}). \tag{48}$$

More generally, let $r, R \in \mathbb{Z}$ with $r \le R$. One gets

$$\int_{q^{r} \le |x|_{\mathbb{F}} \le q^{R}} d\mu_{\mathbb{F}}(x) = \int_{|x|_{\mathbb{F}} \le q^{R}} d\mu_{\mathbb{F}}(x) - \int_{|x|_{\mathbb{F}} \le q^{r}} d\mu_{\mathbb{F}}(x) = q^{R} - q^{r}. \tag{49}$$

Let $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{Z}^n$ and $\mathbf{R} = (R_1, R_2, \ldots, R_n) \in \mathbb{Z}^n$ where $r_i \le R_i$ for every $i \in \{1, \ldots, n\}$. We set

$$B(\mathbf{r}, \mathbf{R}) := \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n \ : \ q^{r_i} \le |x_i|_{\mathbb{F}} \le q^{R_i} \text{ for } i = 1, \ldots, n \right\}.$$

It follows from (49) that

$$\mu_{\mathbb{F}^n}(B(\mathbf{r}, \mathbf{R})) = \prod_{i=1}^{n} (q^{R_i} - q^{r_i}). \tag{50}$$

The following Lemma relates the volume of the linear image of the set in $\mathbb{F}^n$ to the volume of the set itself.

Lemma 12. *Let $L \colon \mathbb{F}^n \to \mathbb{F}^n$ be an invertible linear transformation then*

$$\mu_{\mathbb{F}^n}(L(B)) = |\det(L)|_{\mathbb{F}}\ \mu_{\mathbb{F}^n}(B), \qquad \textit{for all } B \in \mathfrak{B}(\mathbb{F}^n). \tag{51}$$

Proof. Call $B(\mathbf{r}, \mathbf{R})^t := \{\mathbf{x}^t \ : \ x \in B(\mathbf{r}, \mathbf{R})\}$ a cell in $\mathbb{F}^n$. (Here $\mathbf{x}^t$ means the transpose of $\mathbf{x}$.) We will first check that the relation (51) for every cell $B(\mathbf{r}, \mathbf{R})^t$. The matrix representing L can be written as a product of elementary matrices, and since determinant preserves products, it is sufficient to show that the relation (51) holds for elementary matrices.

Let $i \in \{1, \ldots, n\}$, let $y \in \mathbb{F}^{\times}$ and let $E_i(y)$ be the elementary matrix obtained by multiplying by y the i-th row of the identity $n \times n$ matrix. We have $\det(E_i(y)) = y$ and

$$\begin{aligned} E_i(y) \cdot B(\mathbf{r}, \mathbf{R})^t = \ & \left\{ (x_1, \ldots, x_{i-1}, y x_i, x_{i+1}, \ldots, x_n)^t \ : \ q^{r_k} \le |x_k|_{\mathbb{F}} \le q^{R_k} \text{ for } k = 1, \ldots, n \right\} \\ = \ & \left\{ (x_1, \ldots, x_n) \ : \ q^{r_k} \le |x_k|_{\mathbb{F}} \le q^{R_k} \text{ for } k \ne i,\ \ |y|_{\mathbb{F}_q}^{r_i} \le |x_i|_{\mathbb{F}} \le |y|_{\mathbb{F}_q}^{R_i} \right\}, \end{aligned}$$

since $|yx_i|_{\mathbb{F}} = |y|_{\mathbb{F}} \cdot |x_i|_{\mathbb{F}}$. Hence

$$\mu_{\mathbb{F}^n}(E_i(y) \cdot B(\mathbf{r},\mathbf{R})^t) = |y|_{\mathbb{F}} \cdot \prod_{k=1}^{n} (q^{R_k} - q^{r_k}) = |\det(E_i(y))|_{\mathbb{F}} \cdot \mu_{\mathbb{F}^n}(B(\mathbf{r},\mathbf{R})^t).$$

Let $i, j \in \{1, \dots, n\}$. Let $E_{i,j}$ be the elementary matrix corresponding to the interchange of row i with row j. We have $\det(E_{i,j}) = -1$ and

$$E_{i,j} \cdot B(\mathbf{r},\mathbf{R})^t = \left\{ (x_1, \dots, x_n) \ : \ \begin{cases} q^{r_k} \leq |x_k|_{\mathbb{F}} \leq q^{R_k} & \text{for } k \neq i, j \\ q^{r_i} \leq |x_j|_{\mathbb{F}} \leq q^{R_i} \\ q^{r_j} \leq |x_i|_{\mathbb{F}} \leq q^{R_j} \end{cases} \right\}.$$

Hence $|\det(E_{i,j})|_{\mathbb{F}} = 1$ and $\mu_{\mathbb{F}^n}(E_{i,j} \cdot B(\mathbf{r},\mathbf{R})^t) = \mu_{\mathbb{F}^n}(B(\mathbf{r},\mathbf{R})^t)$.

Let $E_{i\cup j}$ be the elementary matrix obtained by replacing row i by the sum of row i and row j. By multiplying by the matrix $E_{i,1}$ if necessary, we may assume that $i = 1$. We have $E_{1\cup j}(x_1, \dots, x_n)^t = (x_1 + x_j, x_2 \dots, x_n)^t$. Hence $\det(E_{1\cup j}) = 1$. We can view $\mathbb{F}^n$ as the Cartesian product $\mathbb{F} \times \mathbb{F}^{n-1}$. For every $\mathbf{x}' = (x_2, \dots, x_n)^t \in \mathbb{F}^{n-1}$, let

$$B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'} := \left\{ z \in \mathbb{F} \ : \ (z, x_2, \dots, x_n)^t \in B(\mathbf{r},\mathbf{R})^t \right\}$$

and similarly

$$(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'} := \left\{ z + x_j \in \mathbb{F} \ : \ (z + x_j, x_2, \dots, x_n)^t \in E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t \right\}.$$

We have

$$(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'} = \ \{ z \in \mathbb{F} \ : \ (z, x_2, \dots, x_n)^t \in B(\mathbf{r},\mathbf{R})^t \} \ + \ x_j$$

that is,

$$(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'} = B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'} + x_j.$$

Thus, for all $\mathbf{x}' \in \mathbb{F}^{n-1}$, $(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'}$ is a translation of $B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'}$ and since, the measure $\mu_{\mathbb{F}}$ is translation-invariant, we have $\mu_{\mathbb{F}}((E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'}) = \mu_{\mathbb{F}}(B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'})$. On the other hand, by Fubini's Theorem, we get

$$\begin{aligned} \mu_{\mathbb{F}^n}(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t) = & \int_{\mathbb{F}^{n-1}} \mu_{\mathbb{F}}(((E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'}) d\mu_{\mathbb{F}^{n-1}}(\mathbf{x}') \\ = & \int_{\mathbb{F}^{n-1}} \mu_{\mathbb{F}} B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'}) d\mu_{\mathbb{F}^{n-1}}(\mathbf{x}') = \mu_{\mathbb{F}^n}(B(\mathbf{r},\mathbf{R})^t). \end{aligned}$$

Every open set in $\mathbb{F}^n$ can be written as a countable union of cells in $\mathbb{F}^n$ and therefore, by the countable additivity of the Haar measure on $\mathbb{F}$, the measure $\mu_{\mathbb{F}^n}$ satisfies the relation (51) is for any open set. Then the regularity of $\mu_{\mathbb{F}^n}$ implies that (51) holds for any Borel set. □

Lemma 12 shows that Lemma 38 is still valid on the local nonarchimedean field $\mathbb{F}$ with the pullback $L^*(\mu_V)$ defined as in Eqn. (304) up to replacing the absolute value $| \ \ |$ by $| \ \ |_{\mathbb{F}}$, that is, we obtain here:

$$L^*(\mu_V) = |\det(\tilde{L})|^{-1}_{\mathbb{F}} \, \mu_{L^{-1}(V)}.$$

(See (6.1) for the normalization of the Haar measure μ_V.) Let W be a finite dimensional vector space over $\mathbb{F}$ and let $\mathcal{L} \subseteq \mathrm{W}$ be a lattice, [52, page 28]. Let $\mathrm{W}^* = \mathrm{Hom}(\mathrm{W}, \mathbb{F})$ be the dual vector space and let

$$\mathcal{L}_* = \{w^* \in \mathrm{W}^* :\ w^*(w) \in \mathfrak{o}_{\mathbb{F}} \text{ for all } w \in \mathcal{L}\}.$$

This is the lattice dual to $\mathcal{L}$.

Lemma 13. *For any subspace $\mathsf{U} \subseteq \mathrm{W}$, the restriction map*

$$\mathrm{W}^* \ni w^* \to w^*|_{\mathsf{U}} \in \mathsf{U}^*$$

induces the following short exact sequence

$$0 \to \mathcal{L}_* \cap \mathsf{U}^\perp \to \mathcal{L}_* \to (\mathcal{L} \cap \mathsf{U})_* \to 0,$$

where $\mathsf{U}^\perp \subseteq \mathrm{W}^*$ is the annihilator of U. In particular we have the isomorphisms of lattices

$$(\mathcal{L}_* + \mathsf{U}^\perp)/\mathsf{U}^\perp = \mathcal{L}_*/\mathcal{L}_* \cap \mathsf{U}^\perp = (\mathcal{L} \cap \mathsf{U})_*.$$

Proof. By [52, Theorem 1, page 29], there is a basis $w_1, \dots, w_m, \dots$ of W such that $w_1, \dots, w_m$ is a basis of U and $\mathcal{L} = \mathfrak{o}_{\mathbb{F}} w_1 + \mathfrak{o}_{\mathbb{F}} w_2 + \dots$. Hence, $\mathcal{L} \cap \mathsf{U} = \mathfrak{o}_{\mathbb{F}} w_1 + \dots + \mathfrak{o}_{\mathbb{F}} w_m$. Let $w_1^*, \dots, w_m^*, \dots$ be the dual basis of W^* ($w_i^*(w_j) = \delta_{i,j}$). Then $\mathcal{L}_* = \mathfrak{o}_{\mathbb{F}} w_1^* + \dots + \mathfrak{o}_{\mathbb{F}} w_m^* + \dots$ and $(\mathcal{L} \cap \mathsf{U})_* = \mathfrak{o}_{\mathbb{F}} w_1^* + \dots + \mathfrak{o}_{\mathbb{F}} w_m^*$. Hence the restriction map is surjective. The rest is obvious. ☐

Recall the notion of a norm, [52, page 24], and the norm associated to a lattice

$$N_{\mathcal{L}}(w) = \inf\{|x|_{\mathbb{F}}^{-1} :\ x \in \mathbb{F}^\times,\ xw \in \mathcal{L}\} \qquad (w \in \mathrm{W}),$$

[52, page 28]. Then $\mathcal{L} = \{w \in \mathrm{W} :\ N_{\mathcal{L}}(w) \le 1\}$. The following fact is stated in [52, page 29]

Lemma 14. *Let N be a norm on W. The $N = N_{\mathcal{L}}$ if and only if*

$$\mathcal{L} = \{w \in \mathrm{W} :\ N(w) \le 1\} \tag{52}$$

and

$$\{N(w) :\ w \in \mathrm{W}\} = \{|x|_{\mathbb{F}} :\ x \in \mathbb{F}\}. \tag{53}$$

Let N be a norm on W. As in [52, p. 26], we shall say that two subspaces W′, W″ of W are *N-orthogonal* to each other whenever $\mathrm{W} = \mathrm{W}' \oplus \mathrm{W}''$, and $N(w' + w'') = \sup(N(w'), N(w''))$ for all $w' \in \mathrm{W}'$ and all $w'' \in \mathrm{W}''$.

Lemma 15. *Let $\mathsf{V} \subseteq \mathrm{W}$ be a subspace. Then*

$$N_{(\mathcal{L}+\mathsf{V})/\mathsf{V}}(w + \mathsf{V}) = \inf\{N_{\mathcal{L}}(w + v) :\ v \in \mathsf{V}\} \qquad (w \in \mathrm{W}). \tag{54}$$

Proof. [52, Theorem 1, page 29] implies that there is a subspace $\mathsf{V}' \subseteq \mathrm{W}$ which is $N_{\mathcal{L}}$-orthogonal to V and such that

$$\mathrm{W} = \mathsf{V}' \oplus \mathsf{V} \tag{55}$$

and

$$\mathcal{L} = \mathcal{L} \cap \mathsf{V}' \oplus \mathcal{L} \cap \mathsf{V}. \tag{56}$$

Let $N(w+\mathsf{V})$ denote the right hand side of (54). For $w \in \mathrm{W}$ let $w' \in \mathsf{V}'$ denote the V'-component of w, according to the decomposition (55). Then clearly

$$N(w+\mathsf{V}) = N_{\mathcal{L}}(w') \qquad (w \in \mathrm{W}).$$

In particular N is a norm on W/V. Also, the range of N coincides with the range of $N_{\mathcal{L}}$. Hence Lemma 14 implies that $N = N_{\mathcal{L}'}$, where $\mathcal{L}' = \{w+\mathsf{V} \in \mathrm{W}/\mathsf{V} :\ N(w+\mathsf{V}) \leq 1\}$. The condition $N(w+\mathsf{V}) \leq 1$ means that $N_{\mathcal{L}}(w') \leq 1$, which is equivalent to $w' \in \mathcal{L}'$. Thus

$$\mathcal{L}' = \{w+\mathsf{V} \in \mathrm{W}/\mathsf{V} :\ w' \in \mathcal{L}\}.$$

But (56) shows that the condition $w' \in \mathcal{L}$ is equivalent to $w \in \mathcal{L}+\mathsf{V}$. (Indeed, if $w' \in \mathcal{L}$ then $w \in \mathcal{L}+\mathsf{V}$. Conversely, suppose $w \in \mathcal{L}+\mathsf{V}$. Then there is $w_0 \in \mathcal{L}$ and $v \in \mathsf{V}$ such that $w = w_0 + v$. Hence, $w' = w_0'$. But $w_0' \in \mathcal{L} \cap \mathsf{V}'$ by (56). Thus $w' \in \mathcal{L}$.) Therefore

$$\mathcal{L}' = (\mathcal{L}+\mathsf{V})/\mathsf{V}.$$

□

Corollary 16. *Under the identifications of Lemma 13, the following equalities hold for any* $w^* \in \mathrm{W}^*$*:*

$$\begin{aligned} & N_{(\mathcal{L}\cap\mathsf{U})_*}(w^*|_{\mathsf{U}}) = N_{(\mathcal{L}_*+\mathsf{U}^\perp)/\mathsf{U}^\perp}(w^*+\mathsf{U}^\perp) = \inf\{N_{\mathcal{L}_*}(w^*+w_0^*) :\ w_0^* \in \mathsf{U}^\perp\} \\ = \ & \max\{|w^*(u)|_{\mathbb{F}} :\ u \in \mathcal{L}\cap\mathsf{U}\}. \end{aligned}$$

(The second equality means that the norm on the quotient is the usual quotient norm.)

Proof. The first equality amounts to the last identification of Lemma 13. The second equality follows from Lemma 15 with W, $\mathcal{L}$ and V replaced by W^*, $\mathcal{L}_*$ and $\mathsf{U}^\perp$ respectively. The third equality follows from the fact that

$$N_{\mathcal{L}_*}(w^*) = \max\{|w^*(w)|_{\mathbb{F}} :\ w \in \mathcal{L}\}. \tag{57}$$

One may verify the equality (57) as follows. The right hand side of (57) defines a norm on W^* whose range coincides with the range of $|\ |_{\mathbb{F}}$. The set of the w^* such that the right hand side is less or equal than 1 coincides with the set of the w^* such that $w^*(w) \in \mathfrak{o}_{\mathbb{F}}$ for all $w \in \mathcal{L}$. But this is $\mathcal{L}_*$. Hence Lemma 14 implies (57). □

Let $\mathcal{L} \subseteq \mathrm{W}$ be a lattice. We know from [52, Theorem1, page 29], that there is a basis $w_1, w_2, \dots$ of W such that

$$\mathcal{L} = \mathfrak{o}_{\mathbb{F}} w_1 + \mathfrak{o}_{\mathbb{F}} w_2 + \dots. \tag{58}$$

In particular the spaces $\mathbb{F}w_1, \mathbb{F}w_2, \dots$ are $N_{\mathcal{L}}$-orthogonal and $1 = N_{\mathcal{L}}(w_1) = N_{\mathcal{L}}(w_2) = \dots$. Thus we may define a basis of W to be $N_{\mathcal{L}}$-orthonormal if the condition (58) holds.

Let $w_1^*, w_2^*, \dots$ be the dual basis of W^*. Then

$$\mathcal{L}_* = \mathfrak{o}_{\mathbb{F}} w_1^* + \mathfrak{o}_{\mathbb{F}} w_2^* + \dots.$$

Hence the basis $w_1^*, w_2^*, \dots$ is $N_{\mathcal{L}_*}$-orthonormal.

Lemma 17. *For any* $T \in \mathrm{End}(\mathrm{W})$ *any Haar measure* μ *on the additive group* W *and any measurable set* $B \subseteq \mathrm{W}$

$$\mu(T(B)) = |\det(T)|_{\mathbb{F}}\, \mu(B).$$

Proof. This is a direct consequence of Lemma 12. □

Lemma 18. *Suppose w_1, $w_2, \dots$ is an $N_{\mathcal{L}}$-orthonormal basis of* W *and $T \in \mathrm{End}(\mathrm{W})$ is such that Tw_1, $Tw_2, \dots$ is also an $N_{\mathcal{L}}$-orthonormal basis of* W*. Then $|\det(T)|_{\mathbb{F}} = 1$.*

Proof. Since, by the assumption, $T(\mathcal{L}) = \mathcal{L}$, the map T preserves the Haar measure on W. Hence, Lemma 17 shows that $|\det(T)|_{\mathbb{F}} = 1$. □

Recall the $N_{\mathcal{L}}$-orthonormal basis w_1, $w_2, \dots$ of W. Suppose W$'$ is another finite dimensional vector space over $\mathbb{F}$ with a lattice $\mathcal{L}'$ and an $\mathrm{N}_{\mathcal{L}'}$-orthonormal basis $w_1', w_2', \dots$. Given $h \in \mathrm{Hom}(\mathrm{W}, \mathrm{W}')$, there is the corresponding matrix

$$M(h) = [h_{ji}], \quad h(w_i) = \sum_j h_{ji} w_j'.$$

The determinant $\det(M(h))$ does depend on the choice of the bases, but, as we see from Lemma 18, $|\det(M(h))|_{\mathbb{F}}$ does not. Hence we may define

$$|\det(h : \mathrm{W} \to \mathrm{W}')|_{\mathbb{F}} = |\det(M(h))|_{\mathbb{F}} \in \mathbb{R}. \tag{59}$$

Lemma 19. *Let $h \in \mathrm{Hom}(\mathrm{W}, \mathrm{W}')$ and let $h^* \in \mathrm{Hom}(\mathrm{W}'^*, \mathrm{W}^*)$ be the adjoint map. Then*

$$\det(h : \mathrm{W} \to \mathrm{W}') = \det(h^* : \mathrm{W}'^* \to \mathrm{W}^*)\,.$$

Proof. Let w_1^*, $w_2^*, \dots \in \mathrm{W}^*$ be the basis dual to w_1, $w_2, \dots$ and let ${w_1'}^*$, ${w_2'}^*, \dots \in \mathrm{W}'^*$ be the basis dual to w_1', $w_2', \dots$. Then,

$$h(w_i) = \sum_j h_{ji} w_j' \text{ if and only if } h^*({w_j'}^*) = \sum_i h_{ji} w_i^*,$$

because

$$h_{ji} = {w_j'}^*(\sum_j h_{ji} w_j') = {w_j'}^*(h(w_i)) = h^*({w_j'}^*)(w_i).$$

Hence, the matrix $M(h^*)$ is the transpose of the matrix $M(h)$ and the claim follows. □

From now on we assume that the space W is equipped with a non-degenerate symplectic form $\langle\ ,\ \rangle$. We shall identify W with the dual W* by

$$w(u) = \langle u, w \rangle \qquad (u, w \in \mathrm{W}). \tag{60}$$

Then, for a subspace $\mathsf{U} \subseteq \mathrm{W}$ the annihilator $\mathsf{U}^\perp$ in the dual coincides with the $\langle\ ,\ \rangle$-orthogonal complement. We shall say that the lattice $\mathcal{L}$ is self-dual in the sense that $\mathcal{L} = \mathcal{L}_*$. Let us fix a self-dual lattice $\mathcal{L} \subseteq \mathrm{W}$.

For any two subspaces $\mathsf{V} \subseteq \mathsf{U} \subseteq \mathrm{W}$, N shall denote the quotient norm of $N_{\mathcal{L}}$:

$$N(u + \mathsf{V}) = \inf\{N_{\mathcal{L}}(u + v) :\ v \in \mathsf{V}\} \qquad (u \in \mathsf{U}). \tag{61}$$

For an element $h \in \mathrm{End}(\mathrm{W})$ define $h^\# \in \mathrm{End}(\mathrm{W})$ by

$$\langle hw, w' \rangle = \langle w, h^\# w' \rangle \qquad (w, w' \in \mathrm{W}).$$

Then $(\mathrm{Im}\, h)^\perp = \mathrm{Ker} h^\#$. Hence, if $\mathrm{Ker} h = \mathrm{Ker} h^\#$ then we have the following short exact sequence

$$0 \to (\mathrm{Im}\, h)^\perp \to \mathrm{W} \to \mathrm{Im}\, h \to 0. \tag{62}$$

In the next Lemma, we shall consider $\mathrm{Im}\, h$ as the quotient $\mathrm{W}/(\mathrm{Im}\, h)^{\perp}$, and N will be the corresponding quotient norm as defined in (61).

Lemma 20. *Suppose $h \in \mathrm{End}(\mathrm{W})$ is such that $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$. Let $u_1, \ldots, u_k$ be an N-orthonormal basis of $\mathrm{Im}\, h$ and let $w_1 + \mathrm{Ker} h, \ldots, w_k + \mathrm{Ker} h$ be the dual basis of $\mathrm{W}/\mathrm{Ker} h$. Let $M = M(h)$ be the matrix of the induced bijection $h\colon \mathrm{W}/\mathrm{Ker} h \to \mathrm{Im}\, h$ with respect to these two ordered basis,*

$$hw_i = \sum_j M_{j,i}(h)\, u_j.$$

Then

$$\det(M(h)) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le k}).$$

Also, we may choose the elements $w_1, \ldots, w_k$ so that the spaces $\mathbb{F}w_1, \ldots, \mathbb{F}w_k$, $\mathrm{Ker}\, h$ are N-orthogonal.

Proof. Since

$$\langle hw_i, w_j\rangle = \langle \sum_l M_{l,i} u_l, w_j\rangle = M_{j,i},$$

the formula for the determinant follows. The last statement follows from Lemma 15 and Corollary 16. □

Notice that if $u_1', \ldots, u_k'$ is another N-orthonormal basis of $\mathrm{Im}\, h$, with dual basis $w_1' + \mathrm{Ker} h, \ldots, w_k' + \mathrm{Ker} h$, then

$$\det(\langle hw_i', w_j'\rangle_{1\le i,j\le k}) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le k})\, a^2,$$

where $a \in \mathbb{F}^{\times}$ is the determinant of the transition matrix from $u_1, \ldots, u_k$ to $u_1', \ldots, u_k'$ (which is also the determinant of the transition matrix from the corresponding dual basis). We know from Lemma 18 that $|a|_{\mathbb{F}} = 1$. Hence without any ambiguity we may define

$$\det(h\colon \mathrm{W}/\mathrm{Ker} h \to \mathrm{Im}\, h) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le k})\, (\mathfrak{o}_{\mathbb{F}}^{\times})^2 \tag{63}$$

as an element of $\mathbb{F}^{\times}/(\mathfrak{o}_{\mathbb{F}}^{\times})^2$. (Here $(\mathfrak{o}_{\mathbb{F}}^{\times})^2 = \{(a^2;\ a \in \mathfrak{o}_{\mathbb{F}}^{\times}\}$.) Also, without any ambiguity we may define

$$|\det(h\colon \mathrm{W}/\mathrm{Ker} h \to \mathrm{Im}\, h)|_{\mathbb{F}} = |\det(\langle hw_i, w_j\rangle_{1\le i,j\le k})|_{\mathbb{F}} \tag{64}$$

as a positive real number.

Similarly, if $\mathsf{U} \subset \mathrm{W}$ is a subspace and $x \in \mathrm{Hom}(\mathsf{U}, \mathrm{W})$ is such that the bilinear form

$$\langle xu, u'\rangle \qquad (u, u' \in \mathsf{U})$$

is symmetric, with the radical $\mathsf{V} \subseteq \mathsf{U}$, we define

$$\det(\langle x\ ,\ \rangle_{\mathsf{U}/\mathsf{V}}) = \det(\langle xu_i, u_j\rangle_{1\le i,j\le k})\, (\mathfrak{o}_{\mathbb{F}}^{\times})^2 \tag{65}$$

and

$$|\det(\langle x\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})|_{\mathbb{F}} = |\det(\langle xu_i, u_j\rangle_{1\le i,j\le k})|_{\mathbb{F}}, \tag{66}$$

where $u_1 + \mathsf{V}$, $u_2 + \mathsf{V}$, $\ldots$, is an N-orthonormal basis of U/V.

Lemma 21. *If w_1, w_2, … is a $N_{\mathcal{L}}$-orthonormal basis of* W*, then*

$$|\det(\langle w_i, w_j\rangle_{1\leq i,j})|_{\mathbb{F}} = 1.$$

Proof. Since $\langle w_i, w_j\rangle \in \mathfrak{o}_{\mathbb{F}}$,

$$|\det(\langle w_i, w_j\rangle_{1\leq i,j})|_{\mathbb{F}} \leq 1.$$

Since the lattice $\mathcal{L}$ is self-dual the same inequality holds for the dual basis. The product of the two matrices is 1. Hence the equality follows. □

Lemma 19 may be rephrased as

Lemma 22. *Let $h \in \mathrm{End}(\mathrm{W})$ and let $K \subseteq \mathrm{W}$ be a subspace. Assume that $h : K \to hK$ is injective. Then, with the definition* (**??**),

$$h^{\#}((hK)^{\perp}) \subseteq K^{\perp}, \tag{67}$$

and

$$|\det(h : K \to hK)|_{\mathbb{F}} = |\det(h^{\#} : \mathrm{W}/(hK)^{\perp} \to \mathrm{W}/K^{\perp})|_{\mathbb{F}}. \tag{68}$$

Proof. The point is that $\mathrm{W}/K^{\perp} = K^*$, $\mathrm{W}/(hK)^{\perp} = (hK)^*$ and $h^{\#} = h^*$. □

In the next Lemma, we keep the notation defined in Notation 4 and Notation 6, that is, for $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$,

$$\mathsf{U} = \mathsf{U}_1 \cap \mathsf{U}_2 = g_1^-\mathrm{W} \cap g_2^-\mathrm{W} \quad \text{and} \quad \mathsf{U}_{12} = (g_1g_2)^-\mathrm{W},$$
$$K_1 = \mathrm{Ker}\, g_1^-, \quad K_2 = \mathrm{Ker}\, g_2^- \quad \text{and} \quad K_{12} = \mathrm{Ker}(g_1g_2)^-.$$

Lemma 23. *Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and assume that $K_1 = \{0\}$. Then, with the definition* (63),

$$\begin{aligned} &\frac{\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathsf{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \quad &(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}}) \, (\det(g_2^- : K_{12} \to \mathsf{V}))^{-2} \end{aligned} \tag{69}$$

and

$$\begin{aligned} &\frac{|\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathsf{U}_{12})|_{\mathbb{F}}}{|\det(g_1^- : \mathrm{W} \to \mathrm{W})|_{\mathbb{F}} |\det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})|_{\mathbb{F}}} \\ = \quad &|\det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})|_{\mathbb{F}} \, |\det(g_2^- : K_{12} \to \mathsf{V})|_{\mathbb{F}}^{-2} \end{aligned} \tag{70}$$

Proof. Clearly (70) follows from (69). We shall verify (69). Let $\mathrm{W}_2 \subseteq \mathrm{W}$ be the $N_{\mathcal{L}}$-orthogonal complement of $K_{12} + K_2$. Then (23) holds. Let w_1, w_2, … be a basis of W such that w_1, w_2, …, w_a is a $N_{\mathcal{L}}$-orthonormal basis of K_{12}, w_{a+1}, w_{a+2}, …, w_b is a $N_{\mathcal{L}}$-orthonormal basis of W_2 and w_{b+1}, w_{b+2}, … is a $N_{\mathcal{L}}$-orthonormal basis of K_2. Then w_1, w_2, …, w_b is $N_{\mathcal{L}}$-orthonormal basis of $K_{12} + \mathrm{W}_2$. Theorem 1 on page 29 in [52] implies that we may extend it to an $N_{\mathcal{L}}$-orthonormal basis of W:

$$w_1, \dots, w_b, w'_{b+1}, w'_{b+2}, \dots.$$

Define an element $Q \in \mathrm{GL}(\mathrm{W})$ by

$$Q(w_i) = \begin{cases} w_i & \text{if } i \le b, \\ w_i' & \text{if } i > b. \end{cases}$$

Then

$$\begin{array}{l} Qw_1, Qw_2, \dots \text{ is a } N_{\mathcal{L}}\text{-orthonormal basis of W}, \\ Qw_i = w_i \text{ if } i \le b, \\ \mathbb{F}Qw_{b+1} + \mathbb{F}Qw_{b+2} + \dots \text{ is } N_{\mathcal{L}}\text{-orthogonal to } K_{12} + \mathrm{W}_2. \end{array}$$

We see from Lemma 21 that

$$|\det(\langle Qw_i, Qw_j\rangle_{1\le i,j})|_{\mathbb{F}} = 1.$$

Hence, we may replace one of the w_i by a suitable $(\mathfrak{o}_{\mathbb{F}})^{\times}$-multiple of it so that

$$\det(\langle Qw_i, Qw_j\rangle_{1\le i,j}) = 1. \tag{71}$$

Define the matrix elements $Q_{j,i}$ by

$$Qw_i = \sum_j Q_{j,i} w_j.$$

Then

$$Q_{j,i} = \delta_{j,i} \text{ if } i \le b.$$

In particular the matrix $((Q_{j,i})_{1\le j,i})$ looks as follows

$$((Q_{j,i})_{1\le j,i}) = \begin{pmatrix} \mathrm{I} & * \\ 0 & ((Q_{j,i})_{b<j,i}) \end{pmatrix},$$

where I is the identity matrix of size b. Hence,

$$\det(Q) = \det((Q_{j,i})_{1\le j,i}) = \det((Q_{j,i})_{b<j,i}) = \det((Q_{j,i})_{a<j,i}).$$

Therefore (71) implies

$$\det((Q_{j,i})_{a<j,i})^2 \det(\langle w_i, w_j\rangle_{1\le i,j}) = 1. \tag{72}$$

Let $u_1, u_2, \dots, u_b$ be a $N_{\mathcal{L}}$-orthogonal basis of U such that $u_1, u_2, \dots, u_a$ span V. (The existence of such a basis follows from [52, Theorem 1, page 29].) Define the matrix elements $(g_2^-)_{k,i}$ by

$$g_2^- w_i = \sum_{k=1}^{b} (g_2^-)_{k,i} u_k \qquad (1 \le i \le b).$$

Since $g_2^- K_{12} = \mathsf{V}$, we see that

$$(g_2^-)_{k,i} = 0 \text{ if } i \le a < k.$$

Hence

$$\begin{aligned} & \det(((g_2^-)_{k,i})_{1\le k,i\le b}) \\ = \; & \det(((g_2^-)_{k,i})_{1\le k,i\le a}) \det(((g_2^-)_{k,i})_{a<k,i\le b}). \end{aligned} \tag{73}$$

Define $h \in \mathrm{GL}(\mathrm{W})$ as in (24). Then (26) shows that

$$\begin{aligned} & \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}) \det(h) \\ = \ & \det(\langle \frac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\leq b}). \end{aligned} \tag{74}$$

Furthermore, by (27),

$$\begin{aligned} & \det(h) = \det((g_1^{-1}-1)^{-1}(g_1^{-1}-1)h) \\ = \ & \det(g_1^{-1}-1)^{-1} \det((g_1^{-1}-1)h) \\ = \ & \det(g_1^{-1}-1)^{-1} \det(\langle w_i, (g_1^{-1}-1)^{-1}hw_j\rangle_{1\leq i,j}) \det(\langle w_i, w_j\rangle_{1\leq i,j})^{-1} \\ = \ & \det(g_1^{-1}-1)^{-1}(-1)^{\dim \mathsf{U}} \det(\langle g_2^- w_i, w_j\rangle_{i,j\leq b}) \det(\langle w_i, w_j\rangle_{1\leq i,j})^{-1} \end{aligned} \tag{75}$$

Also,

$$\begin{aligned} & \det(\langle \frac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\leq b}) \\ = \ & \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\leq b}) \det(((g_2^-)_{k,i})_{a<k,i\leq b})^2. \end{aligned} \tag{76}$$

By (63),

$$\begin{aligned} & \det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathsf{U}_{12}) = \det(\langle (g_1g_2)^- Qw_i, Qw_j\rangle_{a<i,j}) \, (\mathfrak{o}_{\mathbb{F}}^{\times})^2 \\ = \ & \det((Q_{i,j})_{a<i,j})^2 \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}) \, (\mathfrak{o}_{\mathbb{F}}^{\times})^2. \end{aligned} \tag{77}$$

We see from Lemma 20 that there are elements $qw_i \in \mathrm{W}$, $i \leq b$, such that

$$\langle u_j, qw_i\rangle = \delta_{j,i} \qquad (j, i \leq b) \tag{78}$$

and the spaces $\mathbb{F}qw_1, \ldots, \mathbb{F}qw_b$, K_2 are N-orthogonal. Define an element $q \in \mathrm{GL}(\mathrm{W})$ by

$$\begin{aligned} q(w_i) &= qw_i \text{ if } i \leq b, \\ q(w_i) &= w_i \text{ if } b < i. \end{aligned}$$

Then

$$\det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U}) = \det(\langle g_2^- qw_i, qw_j\rangle_{i,j\leq b}) \, (\mathfrak{o}_{\mathbb{F}}^{\times})^2. \tag{79}$$

Define the coefficients $q_{i,j}$ by

$$qw_i = \sum_j q_{j,i} w_j.$$

Then

$$q_{j,i} = \delta_{j,i} \text{ if } b < i$$

so that

$$\det(q) = \det((q_{j,i})_{1\leq i,j}) = \det((q_{j,i})_{1\leq i,j\leq b}).$$

Also,

$$g_2^- qw_i = \sum_j q_{j,i} g_2^- w_j = \sum_{j\leq b} q_{j,i} g_2^- w_j \qquad (i \leq b).$$

Therefore,

$$\det(\langle g_2^- qw_i, qw_j\rangle_{i,j\leq b}) = \det(q)^2 \det(\langle g_2^- w_i, w_j\rangle_{i,j\leq b}). \tag{80}$$

Define the coefficients $q_{i,j}^{-1}$ of the inverse map q^{-1} by

$$w_i = q^{-1}(qw_i) = \sum_j q_{i,j}^{-1} q w_j.$$

Then, by (78),

$$q_{i,j}^{-1} = \begin{cases} \langle u_j, w_i\rangle \text{ if } j \le b, \\ \delta_{i,j} \text{ if } i > b. \end{cases}$$

Hence,

$$\det(q)^{-1} = \det(q^{-1}) = \det((q_{i,j}^{-1})_{i,j\le b}) = \det(\langle u_j, w_i\rangle_{i,j\le b}).$$

Thus

$$\begin{aligned} & \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U}) \qquad (81) \\ = \ & (\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}))^2 \det(q)^2 \, (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ = \ & (\det(\langle \sum_{k=1}^{b} (g_2^-)_{k,i} u_k, w_j\rangle_{i,j\le b}))^2 \det(q)^2 \, (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ = \ & (\det((g_2^-)_{k,i})_{k,i\le b})^2 \det(\langle u_k, w_j\rangle_{k,j\le b}))^2 \det(q)^2 \, (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ = \ & (\det((g_2^-)_{k,i})_{k,i\le b}))^2 \, (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ = \ & (\det(g_2^- : K_{12} \to \mathsf{V}))^2 \, (\det(g_2^- : \mathrm{W}_2 \to \mathsf{U}/\mathsf{V}))^2 \, (\mathfrak{o}_{\mathbb{F}}^\times)^2, \end{aligned}$$

where the first equality follows from (79) combined with (80), and the last equality follows from (73). Now the formula (69) may be verified via a straightforward computation, where we ignore the factor $(\mathfrak{o}_{\mathbb{F}}^\times)^2$ for convenience:

$$\begin{aligned} & \frac{\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathsf{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \ & \frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \ & \frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2}{\det(h) \det g_1^- \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \ & \frac{(-1)^{\dim \mathsf{U}} \det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2}{\det(g_1^{-1}-1)^{-1} \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(\langle w_i, w_j\rangle_{1\le i,j})^{-1} \det g_1^- \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \ & \frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) (\det((g_2^-)_{k,i})_{a<k,i\le b}))^2}{\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ = \ & \frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2)) \ , \ \rangle_{\mathsf{U}/\mathsf{V}}) (\det((g_2^-)_{k,i})_{a<k,i\le b})^2}{(\det(g_2^- : K_{12} \to \mathsf{V}))^2 (\det(g_2^- : \mathrm{W}_2 \to \mathsf{U}/\mathsf{V}))^2} \\ = \ & \frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2)) \ , \ \rangle_{\mathsf{U}/\mathsf{V}})}{(\det(g_2^- : K_{12} \to \mathsf{V}))^2}. \end{aligned}$$

(Here the first equality follows from (77), the second equality from (74) and (76), the third from (75), the forth from (72) and the fifth from (81).) □

3. The Weil representation over a finite field of odd characteristic

Let $\mathbb{F}$ be a finite field of odd characteristic and let $\chi : \mathbb{F} \to \mathbb{C}^\times$ be a non-trivial character of the additive group $\mathbb{F}$. In this section we provide an elementary construction of the corresponding the Weil representation, [5].

3.1. **The Fourier transform.** Let U be a finite dimensional vector space over $\mathbb{F}$. Define a measure μ_{U} on U by

$$\int_{\mathsf{U}} \phi(u)\, d\mu_{\mathsf{U}}(u) = |\mathsf{U}|^{-1/2} \sum_{u\in\mathsf{U}} \phi(u),$$

where $|\mathsf{U}|$ is the cardinality of U and $\phi : \mathsf{U} \to \mathbb{C}$ is a function. For E a subset of U let denote by $\mathbb{I}_E$ the indicator function of E, that is, the normalized characteristic function of E:

$$\mathbb{I}_E(u) := \begin{cases} |E|^{-1} & \text{if } u \in E; \\ 0 & \text{otherwise.} \end{cases}$$

Define the Fourier transform $\mathcal{F}$ by

$$\mathcal{F}\phi(u^*) = \int_{\mathsf{U}} \phi(u)\chi(-u^*(u))\, d\mu_{\mathsf{U}}(u) \qquad (u^* \in \mathsf{U}^*).$$

Then μ_{U^*} is the measure dual to μ_{U} in the sense that

$$\phi(u) = \int_{\mathsf{U}^*} \mathcal{F}\phi(u^*)\chi(u^*(u))\, d\mu_{\mathsf{U}^*}(u^*) \qquad (u \in \mathsf{U}).$$

We record by the way the following, easy to verify, formula

$$\mathcal{F}\mathbb{I}_{\mathsf{V}} = |\mathsf{V}||\mathsf{U}|^{-1/2}\mathbb{I}_{\mathsf{V}^\perp}, \tag{82}$$

where $\mathsf{V} \subseteq \mathsf{U}$ is a vector subspace with the orthogonal complement $\mathsf{V}^\perp \subseteq \mathsf{U}^*$.

3.2. **Gaussians on $\mathbb{F}^n$.** For a symmetric matrix $A \in \mathrm{GL}(\mathbb{F}^n)$ define the corresponding Gaussian γ_A by

$$\gamma_A(x) = \chi(\frac{1}{2}x^t Ax) \qquad (x \in \mathbb{F}^n),$$

where we view the x as a column vector. Also, let

$$\gamma(A) = \mathcal{F}\gamma_A(0) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^t Ax)\, d\mu_{\mathbb{F}^n}(x).$$

Lemma 24. *If we identify $\mathbb{F}^n$ with the dual $(\mathbb{F}^n)^*$ by*

$$y(x) = x^t y \qquad (x, y \in \mathbb{F}^n),$$

then

$$\mathcal{F}\gamma_A = \gamma(A)\gamma_{-A^{-1}}.$$

Proof. Notice that

$$\frac{1}{2}x^tAx = \frac{1}{2}(x - A^{-1}y)^tA(x - A^{-1}y) - \frac{1}{2}y^tA^{-1}y + x^ty.$$

Hence,

$$\begin{aligned}
\mathcal{F}\gamma_A(y) &= \int_{\mathbb{F}^n} \gamma_A(x)\chi(-x^ty)\, d\mu_{\mathbb{F}^n}(x) \\
&= \int_{\mathbb{F}^n} \chi(\frac{1}{2}(x - A^{-1}y)^tA(x - A^{-1}y))\, d\mu_{\mathbb{F}^n}(x)\chi(-\frac{1}{2}y^tA^{-1}y) \\
&= \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^tAx)\, d\mu_{\mathbb{F}^n}(x)\chi(-\frac{1}{2}y^tA^{-1}y).
\end{aligned}$$

□

Lemma 25. *Suppose $n = 1$. Then*

(a) $\gamma(a) = \gamma(ab^2) \qquad (a, b \in \mathbb{F}^\times)$,
(b) $\gamma(-a) = \overline{\gamma(a)} = \gamma(a)^{-1} \qquad (a \in \mathbb{F}^\times)$,
(c) *the function*

$$a \mapsto s(a) = \gamma(a)\gamma(-1) \qquad (a \in \mathbb{F}^\times)$$

coincides with the unique non-trivial character of the group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$.

Proof. Part (a) and the first equation in (b) are obvious. Let us extend the character s to $\mathbb{F}$ by letting $s(0) = 0$. Then, since $\frac{1}{2}a \neq 0$, we see from (82) that

$$\begin{aligned}
\gamma(a) &= \int_{\mathbb{F}}(1 + s)(y)\chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y) \\
&= \int_{\mathbb{F}} \chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y) + \int_{\mathbb{F}} s(y)\chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y) \\
&= \int_{\mathbb{F}^\times} s(y)\chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y) = \int_{\mathbb{F}^\times} s(a^{-1}y)\chi(\frac{1}{2}y)\, d\mu_{\mathbb{F}}(y) \\
&= s(a^{-1}) \int_{\mathbb{F}^\times} s(y)\chi(\frac{1}{2}y)\, d\mu_{\mathbb{F}}(y) = s(a)\gamma(1).
\end{aligned}$$

Also,

$$\begin{aligned}
\gamma(1)\overline{\gamma(1)} &= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(y)s(z)\chi(\frac{1}{2}(y-z))\,d\mu_{\mathbb{F}}(y)\,d\mu_{\mathbb{F}}(z)\\
&= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(yz)\chi(\frac{1}{2}(y-z))\,d\mu_{\mathbb{F}}(y)\,d\mu_{\mathbb{F}}(z)\\
&= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(y)\chi(\frac{1}{2}(y-1)z)\,d\mu_{\mathbb{F}}(y)\,d\mu_{\mathbb{F}}(z)\\
&= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(y)\chi((y-1)z)\,d\mu_{\mathbb{F}}(z)\,d\mu_{\mathbb{F}}(y)\\
&= \int_{\mathbb{F}^\times} s(y)\left(\int_{\mathbb{F}}\chi((y-1)z)\,d\mu_{\mathbb{F}}(z)-|\mathbb{F}|^{-1/2}\right)\,d\mu_{\mathbb{F}}(y)\\
&= \int_{\mathbb{F}^\times} s(y)|\mathbb{F}|^{1/2}\mathbb{I}_0(y-1)\,d\mu_{\mathbb{F}}(y)-|\mathbb{F}|^{-1/2}\int_{\mathbb{F}^\times} s(y)\,d\mu_{\mathbb{F}}(y)=s(1),
\end{aligned}$$

because the restriction of $\mu_{\mathbb{F}}$ to $\mathbb{F}^\times$ is a Haar measure on $\mathbb{F}^\times$ and s is a non-trivial character of the abelian group $\mathbb{F}^\times$. Since $s(1)=1$, we see that

$$\gamma(1)\overline{\gamma(1)}=1.$$

In particular $|\gamma(1)|=1$. Therefore the first computation in this proof shows that $|\gamma(a)|=1$ for all $a\in\mathbb{F}^\times$. This implies the second equality in (b). Finally

$$s(a)=\gamma(a)\gamma(1)^{-1}=\gamma(a)\gamma(-1),$$

as claimed in (c). □

Corollary 26. *For arbitrary $n\geq 1$ and a symmetric matrix $A\in\mathrm{GL}(\mathbb{F}^n)$,*

$$\gamma(A)=\gamma(1)^n\,s(\det(A)).$$

Proof. There is $g\in\mathrm{GL}(\mathbb{F}^n)$ and a diagonal matrix $D=\mathrm{diag}(a_1,a_2,\dots,a_n)\in\mathrm{GL}(\mathbb{F}^n)$ such that $A=g^tDg$. Hence,

$$\begin{aligned}
\gamma(A) &= \int_{\mathbb{F}^n}\chi(\frac{1}{2}x^tAx)\,d\mu_{\mathbb{F}^n}(x)=\int_{\mathbb{F}^n}\chi(\frac{1}{2}x^tDx)\,d\mu_{\mathbb{F}^n}(x)\\
&= \int_{\mathbb{F}^n}\prod_{j=1}^n\chi(\frac{1}{2}a_jx_j^2)\,d\mu_{\mathbb{F}^n}(x)=\prod_{j=1}^n\gamma(a_j)=\prod_{j=1}^n(\gamma(1)s(a_j))\\
&= \gamma(1)^ns(\prod_{j=1}^n a_j)=\gamma(1)^ns(\det(D))=\gamma(1)^ns(\det(A)).
\end{aligned}$$

□

3.3. **Gaussians on a vector space.** Let $\gamma(q)=\gamma(Q)$, where Q is defined as in Eq. (12).

Lemma 27. *If q is a non-degenerate symmetric bilinear form on U, then*

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u)=\gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)) \qquad (u^*\in\mathsf{U}^*).$$

Proof. Let $x_i=u_i^*(u)$ and let $y_j=u^*(u_j)$. Then

$$\begin{aligned}\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u)&=\int_{\mathbb{F}^n}\chi(\frac{1}{2}x^tQx)\chi(-x^ty)\,d\mu_{\mathbb{F}^n}(x)\\ &= \gamma(Q)\chi(-\frac{1}{2}y^tQ^{-1}y)=\gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)),\end{aligned}$$

where the second equality follows from Lemma 24 and the last one follows from Lemma 1. □

Corollary 28. *Let q be a symmetric form on U with the radical V. Denote by $\tilde{q}$ the induced non-degenerate form on U/V. Then, for any $u^*\in\mathsf{U}^*$,*

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u)=|\mathsf{V}|^{1/2}\gamma(\tilde{q})\mathbb{I}_{\mathsf{V}^\perp}(u^*)\chi(-\frac{1}{2}\tilde{q}^*(u^*,u^*)),$$

where we identify $\mathsf{V}^\perp=(\mathsf{U}/\mathsf{V})^*$.

Proof. The left hand side is equal to

$$\begin{aligned}&\int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}}\chi(\frac{1}{2}q(u+v,u+v))\chi(-u^*(u+v))\,d\mu_{\mathsf{V}}(v)\,d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\\ =&\int_{\mathsf{U}/\mathsf{V}}\chi(\frac{1}{2}\tilde{q}(u+\mathsf{V},u+\mathsf{V}))\left(\int_{\mathsf{V}}\chi(-u^*(u+v))\,d\mu_{\mathsf{V}}(v)\right)d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\\ =&\int_{\mathsf{U}/\mathsf{V}}\chi(\frac{1}{2}\tilde{q}(u+\mathsf{V},u+\mathsf{V}))\left(\chi(-u^*(u))|\mathsf{V}|^{1/2}\mathbb{I}_{\mathsf{V}^\perp}(u^*)\right)d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\\ =&\ |\mathsf{V}|^{1/2}\mathbb{I}_{\mathsf{V}^\perp}(u^*)\int_{\mathsf{U}/\mathsf{V}}\chi(\frac{1}{2}\tilde{q}(u+\mathsf{V},u+\mathsf{V}))\chi(-u^*(u))\,d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\\ =&\ |\mathsf{V}|^{1/2}\mathbb{I}_{\mathsf{V}^\perp}(u^*)\gamma(\tilde{q})\chi(-\frac{1}{2}\tilde{q}^*(u^*,u^*)).\end{aligned}$$

□

3.4. **Gaussians on a symplectic space.**

Lemma 29. *Suppose $x\in\mathrm{Hom}(\mathsf{U},\mathrm{W}/\mathsf{U}^\perp)$ is such that*

$$\langle xu,v\rangle=\langle xv,u\rangle \qquad (u,v\in\mathsf{U}).$$

Set

$$q(u,v)=\frac{1}{2}\langle xu,v\rangle \qquad (u,v\in\mathsf{U}).$$

Let V be the radical of q and let $\tilde{q}$ be the induced non-degenerate form on U/V. Then

(a) $\mathsf{V}=\mathrm{Ker}(x)$;

(b) *for any* $w \in \mathsf{V}^\perp$ *there is* $u \in \mathsf{U}$ *such that* $xu + (w + \mathsf{U}^\perp) = 0$*;*
(c) *for any* $w \in \mathrm{W}$

$$\int_{\mathsf{U}} \chi(\frac{1}{4}\langle xu', u'\rangle)\chi(-\frac{1}{2}\langle u', w\rangle)\, d\mu_{\mathsf{U}}(u') = |\mathsf{V}|^{1/2}\gamma(\tilde{q})\mathbb{I}_{\mathsf{V}^\perp}(w)\chi(-\frac{1}{4}\langle u, w\rangle)$$

where $u \in \mathsf{U}$ *is such that* $xu + (w + \mathsf{U}^\perp) = 0$.

Proof. Part (a) is obvious. Part (b) means that $\mathrm{Ker}(x)^\perp = \mathrm{Im}(x)$, which is true.
We know from Corollary 28 that the left hand side of (c) is equal to

$$|\mathsf{V}|^{1/2}\gamma(\tilde{q})\mathbb{I}_{\mathsf{V}^\perp}(w)\chi(-\frac{1}{2}\tilde{q}^*(\frac{1}{2}w, \frac{1}{2}w)).$$

Hence we may assume that $w \in \mathsf{V}^\perp$. Recall the map $\Phi\colon \mathsf{U}/\mathsf{V} \to (\mathsf{U}/\mathsf{V})^* = \mathsf{V}^\perp/\mathsf{U}^\perp$:

$$\Phi(u + \mathsf{V})(u' + \mathsf{V}) = \tilde{q}(u' + \mathsf{V}, u + \mathsf{V}) = \frac{1}{2}\langle xu', u\rangle.$$

Suppose $u \in \mathsf{U}$ is such that $\Phi(u + \mathsf{V}) = \frac{1}{2}w + \mathsf{U}^\perp$. Then, by the above,

$$\langle u', \frac{1}{2}w\rangle = \frac{1}{2}\langle xu', u\rangle = \langle u', -\frac{1}{2}xu\rangle \qquad (u' \in \mathsf{U}).$$

Therefore, $xu + \frac{1}{2}w \in \mathsf{U}^\perp$. In other words, $xu + (\frac{1}{2}w + \mathsf{U}^\perp) = 0$ and we see that

$$\tilde{q}^*(\frac{1}{2}w + \mathsf{U}^\perp, \frac{1}{2}w + \mathsf{U}^\perp) = \langle u, \frac{1}{2}w\rangle,$$

so that

$$-\frac{1}{2}\tilde{q}^*(\frac{1}{2}w + \mathsf{U}^\perp, \frac{1}{2}w + \mathsf{U}^\perp) = -\frac{1}{4}\langle u, w\rangle.$$

The formula (c) follows. □

3.5. **Twisted convolution of Gaussians.** Recall the twisted convolution of two functions $\phi, \psi : \mathrm{W} \to \mathbb{C}$:

$$\phi\natural\psi(w) = \int_{\mathrm{W}} \phi(u)\psi(w - u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{83}$$

Let

$$\chi_{c(g)}(u) = \chi(\frac{1}{4}\langle c(g)u, u\rangle) \qquad (u \in g^-\mathrm{W}).$$

More generally, for x and U as in Lemma 29, let

$$\chi_x(u) = \chi(\frac{1}{4}\langle xu, u\rangle) \qquad (u \in \mathsf{U}).$$

Denote the un-normalized characteristic function of E by:

$$\mathbf{1}_E(u) := \begin{cases} 1 & \text{if } u \in E; \\ 0 & \text{otherwise.} \end{cases}$$

By a Gaussian we understand the following function,

$$\mathbf{1}_{g^-\mathrm{W}}(w)\chi_{c(g)}(w) \qquad (w \in \mathrm{W}). \tag{84}$$

The goal of this subsection is to verify the following proposition.

Proposition 30. *For any* $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$,

$$\left(\mathbf{1}_{\mathsf{U}_1}\chi_{c(g_1)}\right) \natural \left(\mathbf{1}_{\mathsf{U}_2}\chi_{c(g_2)}\right) = C(g_1,g_2)\,\mathbf{1}_{\mathsf{U}_{12}}\,\chi_{c(g_1g_2)},$$

where

$$C(g_1,g_2) = \frac{|K_{12}|^{1/2}}{|K_1|^{1/2}|K_2|^{1/2}}\,\gamma(\tilde{q}_{g_1,g_2}).$$

Proof. Notice first that, by the definition of the twisted convolution (83),

$$\left(\mathbf{1}_{\mathsf{U}_1}\chi_{c(g_1)}\right) \natural \left(\mathbf{1}_{\mathsf{U}_2}\chi_{c(g_2)}\right)(w) = 0$$

if $(\mathsf{U}_1 \cap (\mathsf{U}_2 + w) = \emptyset$. Therefore we may assume that there is $v \in \mathsf{U}_1$ such that $w - v \in \mathsf{U}_2$. Lemmas 2 and 5 plus a straightforward computation show that

$$\begin{aligned} &\left(\mathbf{1}_{\mathsf{U}_1}\chi_{c(g_1)}\right) \natural \left(\mathbf{1}_{\mathsf{U}_2}\chi_{c(g_2)}\right)(w) \qquad (85)\\ = \;& \frac{|\mathsf{U}|^{1/2}}{|\mathrm{W}|^{1/2}}\int_{\mathsf{U}} \chi_{c(g_1)+c(g_2)}(u')\chi(-\frac{1}{2}\langle u', c(g_1)v + c(g_2)(v-w) - w\rangle)\,d\mu_{\mathsf{U}}(u')\\ \cdot\;& \chi_{c(g_1)}(v)\chi_{c(g_2)}(v-w)\chi(\frac{1}{2}\langle v, w\rangle). \end{aligned}$$

Since $\mathsf{V}^{\perp} = \mathrm{Ker}(c(g_1)+c(g_2))^{\perp}$ is the image of $c(g_1)+c(g_2)$, we see from Lemma 29 that the expression (85) is not zero if and only if there is $u \in \mathsf{U}$ such that

$$(c(g_1)+c(g_2))u + (c(g_1)v + c(g_2)(v-w) - w) \in \mathsf{U}^{\perp}. \qquad (86)$$

Let

$$u = g_1^- v_1 = g_2^- v_2,\;\; v = g_1^- w_1 \text{ and } w - v = g_2^- w_2. \qquad (87)$$

Then,

$$g_1^+ v_1 + g_2^+ v_2 + g_1^+ w_1 - g_2^+ w_2 - w \in \mathsf{U}^{\perp} = K_1 + K_2.$$

Hence, Lemma 3 shows that, without changing v or $w - v$, we may choose w_1 and w_2 in (87) so that

$$g_1^+ v_1 + g_2^+ v_2 + (g_1^+) w_1 - g_2^+ w_2 - w = 0. \qquad (88)$$

Multiplying (88) by g_1^- we get

$$g_1^- g_1^+ v_1 + g_1^- g_2^+ v_2 + g_1^- g_1^+ w_1 - g_1^- g_2^+ w_2 - g_1^- w = 0.$$

Since, $g_1^-(g_1^+)v_1 = (g_1^+)g_1^- v_1 = (g_1^+)g_2^- v_2$, we see that

$$g_1^+ g_2^- v_2 + g_1^- g_2^+ v_2 + g_1^+ g_1^- w_1 - g_1^- g_2^+ w_2 - g_1^- w = 0.$$

But, by (87), $g_1^- w_1 = w - g_2^- w_2$. Hence,

$$g_1^+ g_2^- v_2 + g_1^- g_2^+ v_2 + g_1^+ w - g_1^+ g_2^- w_2 - g_1^- g_2^+ w_2 - g_1^- w = 0.$$

Thus

$$(g_1^+ g_2^- + g_1^- g_2^+)(v_2 - w_2) + 2w = 0.$$

Therefore

$$w = (g_1 g_2)^-(w_2 - v_2). \qquad (89)$$

Hence, $w \in (g_1g_2)^-\mathrm{W}$.

Conversely, suppose $w = (g_1g_2)^- w_0$ for some $w_0 \in \mathrm{W}$. Then

$$w = g_1^- g_2 w_0 + g_2^- w_0.$$

Let $w_1 = g_2 w_0$ and let $w_2 = w_0$, so that

$$v = g_1^- w_1 \text{ and } w - v = g_2^- w_2,$$

as in (87). Then,

$$c(g_1)v + c(g_2)(v - w) - w = g_1^+ w_1 - g_2^+ w_2 = (g_1^+ g_2 - g_2^+ - (g_1g_2)^-)w_0 = 0 \in \mathsf{U}^\perp.$$

Therefore (86) holds with $u = 0$. Thus we have the indicator function $\mathbb{I}_{(g_1g_2)^-\mathrm{W}}$ in the formula of Proposition 30.

Furthermore, with u as in Lemma 29 (b),

$$\begin{aligned} - \quad & \langle u, c(g_1)v + c(g_2)(v - w) - w\rangle + \langle c(g_1)v, v\rangle + \langle c(g_2)(v - w), v - w\rangle + 2\langle v, w\rangle \qquad (90)\\ = \quad & \langle g_2^- v_2, -g_1^+ w_1 + g_2^+ w_2 + w\rangle + \langle g_1^+ w_1, g_1^- w_1\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle g_1^- w_1, w\rangle\\ = \quad & \langle g_2^- v_2, -g_1^+ w_1 + g_2^+ w_2 + w\rangle + \langle g_1^+ w_1, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w - g_2^- w_2, w\rangle\\ = \quad & \langle g_2^- v_2, -g_1^+ w_1 + g_2^+ w_2 + w\rangle + \langle g_1^+ w_1, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle\\ = \quad & \langle g_2^- v_2, g_2^+ w_2 + w\rangle + \langle g_1^+ w_1, g_2^- v_2\rangle + \langle g_1^+ w_1, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle. \end{aligned}$$

Notice that

$$\begin{aligned} & \langle g_1^+ w_1, g_2^- v_2\rangle = \langle g_1^+ w_1, g_1^- v_1\rangle = \langle (g_1^{-1} - 1)g_1^+ w_1, v_1\rangle\\ = \quad & -\langle g_1^{-1} g_1^-(g_1^+) w_1, v_1\rangle = -\langle g_1^{-1} g_1^+ g_1^- w_1, v_1\rangle = -\langle g_1^{-1} g_1^+ (w - g_2^- w_2), v_1\rangle\\ = \quad & -\langle (1 + g_1^{-1})(w - g_2^- w_2), v_1\rangle = -\langle w - g_2^- w_2, g_1^+ v_1\rangle = \langle g_1^+ v_1, w - g_2^- w_2\rangle. \end{aligned}$$

Hence, (90) is equal to

$$\begin{aligned} & \langle g_2^- v_2, g_2^+ w_2 + w\rangle + \langle g_1^+ v_1, w - g_2^- w_2\rangle + \langle g_1^+ w_1, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle\\ & \quad = \langle g_2^- v_2, g_2^+ w_2 + w\rangle + \langle g_1^+ w_1 + g_1^+ v_1, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle. \qquad (91) \end{aligned}$$

Now we compute $g_1^+ w_1 + g_1^+ v_1$ from (88) and substitute in (91) to see that (91) is equal to

$$\begin{aligned} & \langle g_2^- v_2, g_2^+ w_2 + w\rangle + \langle w + g_2^+ w_2 - g_2^+ v_2, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle \qquad (92)\\ = \quad & \langle g_2^- v_2, g_2^+ w_2\rangle + \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle - \langle w, g_2^- w_2\rangle\\ & -\langle g_2^+ w_2, g_2^- w_2\rangle + \langle g_2^+ v_2, g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle\\ = \quad & \langle g_2^- v_2, g_2^+ w_2\rangle + \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle + \langle g_2^+ v_2, g_2^- w_2\rangle + \langle w, g_2^- w_2\rangle\\ = \quad & \langle (g_2 - g_2^{-1})v_2, w_2\rangle + \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle + \langle (g_2^{-1} - g_2)v_2, w_2\rangle + \langle w, g_2^- w_2\rangle\\ = \quad & \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle - \langle g_2^- w_2, w\rangle = \langle 2(w_2 - v_2), w\rangle. \end{aligned}$$

But we know from (89) that $w = (g_1g_2)^-(w_2 - v_2)$. Hence, (92) is equal to

$$\begin{aligned} & \langle 2(w_2 - v_2), (g_1g_2)^-(w_2 - v_2)\rangle = \langle (g_1g_2)^-(w_2 - v_2) + 2(w_2 - v_2), (g_1g_2)^-(w_2 - v_2)\rangle\\ & = \langle (g_1 g_2^+)(w_2 - v_2), (g_1g_2)^-(w_2 - v_2)\rangle = \langle c(g_1g_2)w, w\rangle. \end{aligned}$$

(Notice that the computation (90) - (93) may be simplified as follows. We already know from (89) that $w = (g_1g_2)^- w_0$ for some $w_0 \in \mathrm{W}$. Hence, we may choose $w_1 = g_2 w_0$, $v = g_1^- g_2 w_0$ and $w_2 = w_0$ in (87). Then

$$c(g_1)v + c(g_2)(v - w) - w = 0$$

and therefore it will suffice to show that

$$\langle c(g_1)v, v\rangle + \langle c(g_2)(w - v), w - v\rangle + 2\langle v, w - v\rangle = \langle c(g_1g_2)w, w\rangle. \tag{93}$$

The left hand side of (93) is equal to

$$\begin{aligned} & \langle g_1^+ w_1, g_1^- w_1\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle g_1^- w_1, g_2^- w_2\rangle \\ = \; & \langle (g_1g_2 + g_2)w_0, (g_1g_2 - g_2)w_0\rangle + \langle g_2^+ w_0, g_2^- w_0\rangle + 2\langle (g_1g_2 - g_2)w_0, g_2^- w_0\rangle \\ = \; & 2\langle w_0, g_1g_2w_0\rangle = \langle (g_1g_2^+)w_0, (g_1g_2)^- w_0\rangle, \end{aligned}$$

which coincides with the right hand side.) Therefore Lemma 29 shows that for $w \in \mathsf{V}^\perp$,

$$\begin{aligned} & \int_{\mathsf{U}} \chi_{c(g_1)+c(g_2)}(u')\chi(-\frac{1}{2}\langle u', c(g_1)v + c(g_2)(v - w) - w\rangle)\, d\mu_{\mathsf{U}}(u') \\ \cdot \; & \chi_{c(g_1)}(v)\chi_{c(g_2)}(v - w)\chi(\frac{1}{2}\langle v, w\rangle) \\ = \; & |\mathsf{V}|^{1/2}\gamma(\tilde{q}_{g_1,g_2})\chi_{c(g_1g_2)}(w). \end{aligned}$$

By combining this with (85) we see that

$$\left(\mathbf{1}_{g_1^- \mathrm{W}}\chi_{c(g)}\right) \natural \left(\mathbf{1}_{g_2^- \mathrm{W}}\chi_{c(g)}\right) = \frac{|\mathsf{U}|^{1/2}|\mathsf{V}|^{1/2}}{|\mathrm{W}|^{1/2}}\gamma(\tilde{q}_{g_1,g_2})\mathbf{1}_{(g_1g_2)^- \mathrm{W}}\chi_{c(g_1g_2)}$$

But Lemma 7 implies

$$\frac{|\mathsf{U}|^{1/2}|\mathsf{V}|^{1/2}}{|\mathrm{W}|^{1/2}} = \frac{|K_{12}|^{1/2}}{|K_1|^{1/2}|K_2|^{1/2}}.$$

□

3.6. **Normalization of Gaussians.** Let B be a non-degenerate bilinear form on a finite dimensional vector space over $\mathbb{F}$. Define the discriminant of B as

$$\mathrm{dis}(B) = s(\det(A)), \tag{94}$$

where A is the matrix obtained from a basis u_1, u_2, …, u_n of the space by

$$A_{i,j} = B(u_i, u_j) \qquad (1 \le i, j \le n).$$

Clearly the discriminant does not depend on the choice of the basis.

For any $g \in \mathrm{Sp(W)}$ the formula

$$\langle g^- w, w'\rangle \qquad (w, w' \in \mathrm{W})$$

defines a bilinear form whose left and right radicals coincide with $\mathrm{Ker}(g^-)$. Hence we get a non-degenerate bilinear form B_g on the quotient $\mathrm{W}/\mathrm{Ker}(g^-)$. Then, for $g \neq 1$,

$$\mathrm{dis}(B_g) = s(\det(\langle g^- w_i, w_j\rangle_{1\le i,j\le r})),$$

where $w_1 + \mathrm{Ker}(g^-)$, $w_2 + \mathrm{Ker}(g^-)$, …, $w_r + \mathrm{Ker}(g^-)$ is a basis of $\mathrm{W}/\mathrm{Ker}(g^-)$. For completeness set $\mathrm{dis}(B_1) = 1$.

For $g \in \mathrm{Sp(W)}$ define

$$\begin{aligned} \Theta(g) &= |\mathrm{Ker}(g^-)|^{1/2}\,\gamma(1)^{\dim g^-\mathrm{W}}\ \mathrm{dis}(B_g), \\ T(g) &= \Theta(g)\,\mathbf{1}_{g^-\mathrm{W}}\,\chi_{c(g)}. \end{aligned} \tag{95}$$

Theorem 31. *For any $g_1, g_2 \in \mathrm{Sp(W)}$,*

$$T(g_1) \natural T(g_2) = T(g_1 g_2).$$

Proof. Proposition 30 implies that we'll be done as soon as we show that

$$C(g_1, g_2) = \frac{\Theta(g_1 g_2)}{\Theta(g_1)\Theta(g_2)} \qquad (g_1, g_2 \in \mathrm{Sp(W)}). \tag{96}$$

Also, we see from Proposition 30 that the absolute values of both sides of (96) are equal. Hence, (96) is equivalent to

$$\gamma(\tilde{q}_{g_1,g_2}) = \frac{\theta(g_1 g_2)}{\theta(g_1)\theta(g_2)} \qquad (g_1, g_2 \in \mathrm{Sp(W)}), \tag{97}$$

where

$$\theta(g) = \gamma(1)^{\dim g^-\mathrm{W}}\,\mathrm{dis}(B_g) \qquad (g \in \mathrm{Sp(W)}).$$

Since the twisted convolution is associative, the function $C(g_1, g_2)$ is a cocycle:

$$C(g_1, g_2)C(g_1 g_2, g_3) = C(g_1, g_2 g_3)C(g_2, g_3) \qquad (g_1, g_2, g_3 \in \mathrm{Sp(W)}).$$

Recall the non-degenerate symmetric form $\tilde{q}_{g_1,g_2}$ defined in Notation 6. Hence, by the formula for $C(g_1, g_2)$ in Proposition 30, the function $\gamma(\tilde{q}_{g_1,g_2})$ is also a cocycle:

$$\gamma(\tilde{q}_{g_1,g_2})\gamma(\tilde{q}_{g_1g_2,g_3}) = \gamma(\tilde{q}_{g_1,g_2g_3})\gamma(\tilde{q}_{g_2,g_3}) \qquad (g_1, g_2, g_3 \in \mathrm{Sp(W)}).$$

Let

$$C'(g_1, g_2) = \frac{\theta(g_1 g_2)}{\theta(g_1)\theta(g_2)} \qquad (g_1, g_2 \in \mathrm{Sp(W)}).$$

This is also a cocycle. Fix two elements $g_2, g_3 \in \mathrm{Sp(W)}$. We have seen in Lemma 8 that there is $g_1 \in \mathrm{Sp(W)}$ such that $K_1 = \mathrm{Ker}\, g_1^- = \{0\}$ and $K_{12} = \mathrm{Ker}(g_1 g_2)^- = \{0\}$. Assume that (97) holds when $K_1 = \{0\}$. Then

$$\gamma(\tilde{q}_{g_2,g_3}) = \frac{\gamma(\tilde{q}_{g_1,g_2})\gamma(\tilde{q}_{g_1g_2,g_3})}{\gamma(\tilde{q}_{g_1,g_2g_3})} = \frac{C'(g_1, g_2)C'(g_1g_2, g_3)}{C'(g_1, g_2g_3)} = C'(g_2, g_3).$$

Hence, in order to verify (97) we may assume that $K_1 = \{0\}$. Then Corollary 10 implies

$$\begin{aligned} \mathrm{dis}(\tilde{q}_{g_1,g_2}) &= \mathrm{dis}(B_{g_1g_2})s(-1)^{\dim \mathsf{U}}\,\mathrm{dis}(B_{g_1})\,\mathrm{dis}(B_{g_2}) \\ &= s(-1)^{\dim \mathsf{U}}\frac{\mathrm{dis}(B_{g_1g_2})}{\mathrm{dis}(B_{g_1})\,\mathrm{dis}(B_{g_2})}. \end{aligned} \tag{98}$$

But, it follows from Lemma 7 that

$$\frac{\gamma(1)^{\dim \mathsf{U}_{12}}}{\gamma(1)^{\dim \mathsf{U}_1}\,\gamma(1)^{\dim \mathsf{U}_2}} = \gamma(1)^{(-\dim \mathsf{U}-\dim \mathsf{V})}. \tag{99}$$

On the other hand, we see from Corollary 26 that

$$\gamma(\tilde{q}_{g_1,g_2}) = \gamma(1)^{\dim \mathsf{U}-\dim \mathsf{V}}\operatorname{dis}(\tilde{q}_{g_1,g_2}) = s(-1)^{\dim \mathsf{U}}\gamma(1)^{-\dim \mathsf{U}-\dim \mathsf{V}}\operatorname{dis}(\tilde{q}_{g_1,g_2}),$$

because $\gamma(1)^2 = s(-1)$. Therefore (98) implies (97). □

3.7. **The conjugation property.** Let $\omega_{1,1}$ denote the permutation representation of $\mathrm{Sp}(\mathrm{W})$ on $\mathrm{L}^2(\mathrm{W})$:

$$\omega_{1,1}(g)\phi(w) = \phi(g^{-1}w) \qquad (g\in \mathrm{Sp}(\mathrm{W}),\ \phi\in \mathrm{L}^2(\mathrm{W})).$$

Also, let

$$\phi^*(w) = \overline{\phi(-w)} \qquad (w\in \mathrm{W},\ \phi\in \mathrm{L}^2(\mathrm{W})).$$

Proposition 32. *For any $\phi \in \mathrm{L}^2(\mathrm{W})$ and $g\in \mathrm{Sp}(\mathrm{W})$ we have*

(a) $T(1)\natural\phi = \phi\natural T(1) = \phi$,
(b) $T(g)\natural\phi\natural T(g^{-1}) = \omega_{1,1}(g)\phi$,
(c) $T(g)^* = T(g^{-1})$.

Proof. Since,

$$T(1) = |\mathrm{W}|^{1/2}\mathbf{1}_{\{0\}}$$

part (a) is easy to check. We see from (95) that the equality (c) is equivalent to

$$\gamma(1)^{-\dim g^-\mathrm{W}}\operatorname{dis}(B_g) = \gamma(1)^{\dim g^-\mathrm{W}}\operatorname{dis}(B_{g^{-1}}),$$

which is the same as

$$s(-1)^{\dim g^-\mathrm{W}} = \operatorname{dis}(B_g)\operatorname{dis}(B_{g^{-1}}).$$

But the last equality holds because

$$\operatorname{dis}(B_{g^{-1}}) = \operatorname{dis}(-B_g) = s(-1)^{\dim g^-\mathrm{W}}\operatorname{dis}(B_g).$$

Thus it remains to prove the equality (b), which is equivalent to

$$T(g)\natural\mathbf{1}_{w_0} = \mathbf{1}_{gw_0}\natural T(g). \tag{100}$$

The left hand side of (100) evaluated at w' is equal to

$$|\mathrm{W}|^{-1/2}\Theta(g)\mathbf{1}_{g^-\mathrm{W}}(w'-w_0)\chi(\frac{1}{4}(\langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle w'-w_0, w'\rangle))$$

and the right hand side is equal to

$$|\mathrm{W}|^{-1/2}\Theta(g)\mathbf{1}_{g^-\mathrm{W}}(w'-gw_0)\chi(\frac{1}{4}(\langle c(g)(w'-gw_0), w'-gw_0\rangle + 2\langle gw_0, w'\rangle)).$$

Since,

$$w'-gw_0 = (w'-w_0) - g^-w_0$$

both sides have the same support. Also,

$$\begin{aligned}
& \langle c(g)(w'-gw_0), w'-gw_0\rangle + 2\langle gw_0, w'\rangle - (\langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle w'-w_0, w'\rangle) \\
= \; & \langle c(g)(w'-gw_0), w'-gw_0\rangle - \langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle g^+w_0, w'\rangle \\
= \; & \langle c(g)((w'-w_0) - g^-w_0, (w'-w_0) - g^-w_0\rangle - \langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle g^+w_0, w'\rangle \\
= \; & \langle c(g)g^-w_0, g^-w_0\rangle - 2\langle c(g)g^-w_0, w'-w_0\rangle + 2\langle g^+w_0, w'\rangle \\
= \; & \langle g^+w_0, g^-w_0\rangle - 2\langle g^+w_0, w'-w_0\rangle + 2\langle g^+w_0, w'\rangle \\
= \; & \langle g^+w_0, g^-w_0\rangle + 2\langle g^+w_0, w_0\rangle = \langle (g^{-1}-1)g^+w_0, w_0\rangle + 2\langle g^+w_0, w_0\rangle \\
= \; & \langle (g^{-1}-g)w_0, w_0\rangle + 2\langle gw_0, w_0\rangle = 0.
\end{aligned}$$

Therefore the two sides of (100) are equal. □

3.8. **The Weyl transform and the Weil representation.** Pick a complete polarization

$$\mathsf{W} = \mathsf{X} \oplus \mathsf{Y} \tag{101}$$

and recall that our normalization of measures is such that $d\mu_{\mathsf{W}}(x+y) = d\mu_{\mathsf{X}}(x)d\mu_{\mathsf{Y}}(y)$. Recall the Weyl transform

$$\begin{aligned}
& \mathcal{K}\colon \mathrm{L}^2(\mathsf{W}) \to \mathrm{L}^2(\mathsf{X}\times\mathsf{X}), \\
& \mathcal{K}(\phi)(x,x') = \int_{\mathsf{Y}} \phi(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y).
\end{aligned} \tag{102}$$

Each element $K \in \mathrm{L}^2(\mathsf{X}\times\mathsf{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathrm{L}^2(\mathsf{X}), \mathrm{L}^2(\mathsf{X}))$ by

$$\mathrm{Op}(K)v(x) = \int_{\mathsf{X}} K(x,x')v(x')\, d\mu_{\mathsf{X}}(x'). \tag{103}$$

A straightforward computation shows that $\mathrm{Op}\circ\mathcal{K}$ transforms the twisted convolution of functions into the composition of the corresponding operators. Also,

$$\mathrm{tr}\ \mathrm{Op}\circ\mathcal{K}(\phi) = \int_{\mathsf{X}} \mathcal{K}(\phi)(x,x)\, d\mu_{\mathsf{X}}(x) = \phi(0) \text{ and } (\mathrm{Op}\circ\mathcal{K}(\phi))^* = \mathrm{Op}\circ\mathcal{K}(\phi^*). \tag{104}$$

Hence, the map

$$\mathrm{Op}\circ\mathcal{K}\colon \mathrm{L}^2(\mathsf{W}) \to \mathrm{H.S.}(\mathrm{L}^2(\mathsf{X})) \tag{105}$$

is an isometry. (Here $\mathrm{H.S.}(\mathrm{L}^2(\mathsf{X}))$ stands for the space of the Hilbert-Schmidt operators on $\mathrm{L}^2(\mathsf{X})$.) Let $\mathrm{U}(\mathrm{L}^2(\mathsf{X}))$ denote the group of the unitary operators on the Hilbert space $\mathrm{L}^2(\mathsf{X})$.

By combining (101) - (105) with Theorem 31 and Proposition 32 we deduce the following theorem.

Theorem 33. *Let* $\omega = \mathrm{Op}\circ\mathcal{K}\circ T$. *Then*

$$\omega\colon \mathrm{Sp}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X}))$$

is an injective group homomorphism. The function Θ *coincides with the character of the resulting representation:*

$$\Theta(g) = \mathrm{tr}\ \omega(g) \qquad (g \in \mathrm{Sp}(\mathrm{W})).$$

Moreover,

$$\omega(g)\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(g^{-1}) = \mathrm{Op}\circ\mathcal{K}(\omega_{1,1}(g)\phi) \qquad (g\in \mathrm{Sp(W)},\ \phi\in \mathrm{L}^2(\mathrm{W})).$$

We end this section by recalling some well known formulas for the action of $\omega(g)$ for some special elements $g\in \mathrm{Sp(W)}$.

Proposition 34. *Let* $\mathrm{M}\subseteq \mathrm{Sp(W)}$ *be the subgroup of all the elements that preserve* X *and* Y*. Then the restriction to* X *defines a group isomorphism* $\mathrm{M}\ni g\to g|_{\mathsf{X}}\in \mathrm{GL}(\mathsf{X})$ *and*

$$\omega(g)v(x) = s(\det(g|_{\mathsf{X}}))v(g^{-1}x) \qquad (g\in\mathrm{M},\ v\in \mathrm{L}^2(\mathsf{X}),\ x\in\mathsf{X}). \tag{106}$$

Proof. Fix an element $g\in\mathrm{M}$. Let $x_1, x_2, \ldots, x_k$ be elements of X such that the vectors $x_1+\mathrm{Ker}(g^-)|_{\mathsf{X}},\ x_2+\mathrm{Ker}(g^-)|_{\mathsf{X}},\ \ldots,\ x_k+\mathrm{Ker}(g^-)|_{\mathsf{X}}$ form a basis of the vector space $\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}$. Pick $y_1, y_2, \ldots, y_k$ in Y so that $\langle x_i, y_j\rangle = 1$. Then the vectors $y_1+\mathrm{Ker}(g^-)|_{\mathsf{Y}}$, $y_2+\mathrm{Ker}(g^-)|_{\mathsf{Y}},\ \ldots,\ y_k+\mathrm{Ker}(g^-)|_{\mathsf{Y}}$ form a basis of the vector space $\mathsf{Y}/\mathrm{Ker}(g^-)|_{\mathsf{Y}}$. Let $w_1 := x_1, \ldots, w_{2k} := y_k$. Then $w_1+\mathrm{Ker}(g^-), \ldots, w_{2k}+\mathrm{Ker}(g^-)$ for a basis of $\mathrm{W}/\mathrm{Ker}(g^-)$. Furthermore g defines an endomorphism $g^{-1}|_{\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}}$ of the space $\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}$ and

$$\begin{aligned}
&\det(\langle g^- w_i, w_j\rangle_{1\le i,j\le 2k})\\
=\ &(-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(\langle g^- x_i, y_j\rangle_{1\le i,j\le k})\det(\langle g^- y_i, x_j\rangle_{1\le i,j\le k})\\
=\ &(-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(\langle g^- x_i, y_j\rangle_{1\le i,j\le k})\det(\langle y_i, (g^{-1}-1)x_j\rangle_{1\le i,j\le k})\\
=\ &(-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(\langle g^- x_i, y_j\rangle_{1\le i,j\le k})\det(\langle g^{-1}g^- x_j, y_i\rangle_{1\le i,j\le k})\\
=\ &(-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\left(\det(\langle g^- x_i, y_j\rangle_{1\le i,j\le k})\right)^2\det(g^{-1}|_{\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}}).
\end{aligned}$$

But $\det(g^{-1}|_{\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}}) = \det(g|_{\mathsf{X}}^{-1})$. Therefore

$$\begin{aligned}
\Theta(g) &= |\mathrm{Ker}(g^-)|^{\frac{1}{2}}\cdot\gamma(1)^{\dim g^-\mathrm{W}}\cdot s\left((-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(g|_{\mathsf{X}}^{-1})\right)\\
&= \left(\frac{|\mathrm{W}|}{|g^-\mathrm{W}|}\right)^{\frac{1}{2}}\cdot\gamma(1)^{2\dim g^-\mathsf{X}}\cdot s\left((-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(g|_{\mathsf{X}}^{-1})\right)\\
&= \frac{|\mathsf{Y}|}{|g^-\mathsf{Y}|}\cdot(s(-1))^{\dim g^-\mathsf{X}}\cdot s\left((-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(g|_{\mathsf{X}}^{-1})\right)\\
&= \frac{|\mathsf{Y}|}{|g^-\mathsf{Y}|}\cdot s\left(\det(g|_{\mathsf{X}}^{-1})\right).
\end{aligned}$$

Let $x, x'\in\mathsf{X}$ and let $y\in\mathsf{Y}$ be such that $x-x'+y\in g^-\mathrm{W}$. Then $x-x'\in g^-\mathsf{X}$ and $y\in g^-\mathsf{Y}$. Moreover,

$$\frac{1}{4}\langle c(g)(x-x'+y), x-x'+y\rangle\rangle = \frac{1}{2}\langle c(g)(x-x'), y\rangle.$$

Hence, (82) shows that

$$\begin{aligned} & \int_{g^-\mathsf{Y}} \chi_{c(g)}(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\ = & \left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} \int_{g^-\mathsf{Y}} \chi(\frac{1}{2}\langle y, x+x'-c(g)(x-x')\rangle)\, d\mu_{g^-\mathsf{Y}}(y) \\ = & \left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} \left(|g^-\mathsf{Y}|\right)^{\frac{1}{2}} \mathbb{I}_{\mathrm{Ker}(g^-)|_{\mathsf{X}}} \left(\frac{1}{2}(x+x'-c(g)(x-x'))\right), \end{aligned}$$

because the annihilator of $g^-\mathsf{Y}$ in X coincides with $\mathrm{Ker}(g^-)|_{\mathsf{X}}$. But the condition $x+x'-c(g)(x-x') \in \mathrm{Ker}(g^-)|_{\mathsf{X}}$ means that $x' = g^{-1}x$. Indeed, if $x - x' = g^-\tilde{x}$, then

$$\begin{aligned} 0 &= g^-(x+x'-c(g)(x-x')) = g^-(x+x'-g^+\tilde{x}) \\ &= g^-(x+x') - g^+(x-x') = 2(gx'-x). \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathcal{K}(T(g))(x,x') \\ = & \Theta(g) \left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} \left(|g^-\mathsf{Y}|\right)^{\frac{1}{2}} \delta_0(g^{-1}x - x') \\ = & \frac{|\mathsf{Y}|}{|g^-\mathsf{Y}|}\, s\left(\det(g|_{\mathsf{X}}^{-1})\right) \left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} \left(|g^-\mathsf{Y}|\right)^{\frac{1}{2}} \delta_0(g^{-1}x - x') \\ = & |\mathsf{Y}|^{\frac{1}{2}} s\left(\det(g|_{\mathsf{X}}^{-1})\right) \delta_0(g^{-1}x - x') \end{aligned}$$

and the formula for $\omega(g)$ follows. □

Proposition 35. *Suppose* $g \in \mathrm{Sp}(\mathrm{W})$ *acts trivially on* Y *and on* W/Y. *Then* $\det((-g)-1) \neq 0$ *and*

$$\omega(g)v(x) = \chi_{c(-g)}(2x)v(x) \qquad (v \in \mathrm{L}^2(\mathsf{X}),\ x \in \mathsf{X}).$$

Proof. Since $-g$ acts as minus the identity on Y and on W/Y, $\det((-g)-1) \neq 0$ and $z := c(-g) \in \mathfrak{sp}(\mathrm{W})$ is well defined. Furthermore

$$z\colon \mathsf{X} \to \mathsf{Y} \to 0.$$

Hence,

$$\begin{aligned} & \int_{\mathsf{Y}} \chi_z(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y) = \int_{\mathsf{Y}} \chi_z(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\ = & \chi_z(x-x')|Y|^{\frac{1}{2}}\delta_0(\frac{1}{2}(x+x')) = \chi_z(2x)|Y|^{\frac{1}{2}}\delta_0(x+x') \end{aligned}$$

Moreover,

$$\Theta(-g) = \gamma(1)^{\dim(\mathrm{W})} s(\det(-2)) = s(-1)^{\dim(\mathsf{X})} s((-2)^{\dim(\mathrm{W})}) = s(-1)^{\dim(\mathsf{X})}.$$

Thus

$$\mathcal{K}(T(-g))(x,x') = s(-1)^{\dim(\mathsf{X})}\chi_z(2x)|Y|^{\frac{1}{2}}\delta_0(x+x').$$

Therefore,

$$\omega(-g)v(x) = s(-1)^{\dim(\mathsf{X})}\chi_z(2x)v(-x).$$

Since, by Proposition 34,

$$\omega(-1)v(x) = s(-1)^{\dim(\mathsf{X})}v(-x),$$

the formula for $\omega(g)$ follows. □

Proposition 36. *Suppose $g \in \mathrm{Sp}(\mathrm{W})$ maps X bijectively onto Y and Y onto X and $g^2 = -1$. Then*

$$\omega(g)v(x) = \gamma(1)^{\dim(\mathsf{X})}\int_{\mathsf{X}} \chi(\langle gx, x'\rangle)v(x')\,d\mu_{\mathsf{X}}(x') \qquad (v \in \mathrm{L}^2(\mathsf{X}),\ x \in \mathsf{X}).$$

(Thus $\omega(g)$ is a Fourier transform on $\mathrm{L}^2(\mathsf{X})$.)

Proof. The formula

$$\langle gx, x'\rangle \qquad (x, x' \in \mathsf{X})$$

defines a non-degenerate symmetric bilinear form on X. Hence, there is a basis x_1, x_2, …, x_n of X and scalars $a_j \in \mathbb{F}^\times$ such that

$$\langle gx_i, x_j\rangle = a_j\delta_{i,j} \qquad (1 \le i, j \le n).$$

Set $y_j := -a_j^{-1}gx_j$, $1 \le j \le n$. Then y_1, y_2, …, y_n is a basis of Y and $\langle x_i, y_j\rangle = \delta_{i,j}$ for all $1 \le i, j \le n$. We have

$$g^-x_j = -a_jy_j - x_j \quad \text{and} \quad g^-y_j = a_j^{-1}x_j - y_j.$$

Set $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. Then, with $\mathrm{I} = \mathrm{I}_n$,

$$\begin{aligned}
\det\left(g^-\right) &= \det\begin{pmatrix} -\mathrm{I} & A^{-1} \\ -A & -\mathrm{I} \end{pmatrix} \\
&= \det\begin{pmatrix} \mathrm{I} & 0 \\ -A & \mathrm{I} \end{pmatrix}\begin{pmatrix} -\mathrm{I} & A^{-1} \\ -A & -\mathrm{I} \end{pmatrix} \\
&= \det\begin{pmatrix} -\mathrm{I} & A^{-1} \\ 0 & -2\,\mathrm{I} \end{pmatrix} = 2^n \ne 0.
\end{aligned}$$

Thus $\mathrm{Ker}(g^-) \ne 0$ so that $g^-\mathsf{W} = \mathsf{W}$. Moreover, with $w_i = x_i$ and $w_{n+i} = y_i$ for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\det\left(\langle g^-w_i, w_j\rangle_{1\le i,j\le 2n}\right) &= \det\begin{pmatrix} -\mathrm{I} & A^{-1} \\ -A & -2\,\mathrm{I} \end{pmatrix}^t\begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \\
&= 2^n\det\begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} = 2^n.
\end{aligned}$$

Thus

$$\mathrm{dis}(B_g) = s(2^n).$$

Hence,

$$\Theta(g) = \gamma(1)^{2n}\, s(2^n) = s(-1)^n\, s(2^n) = s(-2)^n.$$

Since $g^+ = g^-(-g)$, we see that $c(g) = -g$. Further,

$$\begin{aligned}\langle (c(g)(x - x' + y), x - x' + y\rangle &= \langle -g(x - x' + y), x - x' + y\rangle \\ &= \langle -g(x - x'), x - x'\rangle + \langle -gy, y\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathsf{Y}} \chi_{c(g)}(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\ =\;& \chi_{-g}(x - x') \int_{\mathsf{Y}} \chi_{-g}(y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\ =\;& \chi_{-g}(x - x')\, \gamma(\tilde{q})\, \chi_g(x + x') = \gamma(\tilde{q})\, \chi(\langle gx, x'\rangle),\end{aligned}$$

where $\tilde{q}$ is the following symmetric bilinear form on Y

$$\tilde{q}(y, y') = \frac{1}{2}\langle -gy, y'\rangle \qquad (y, y' \in \mathsf{Y}).$$

Since,

$$\det(\tilde{q}(y_i, y_j)_{1\leq i,j\leq n}) = \left(-\frac{1}{2}\right)^n,$$

we see that

$$\gamma(\tilde{q}) = \gamma(1)^n s\left(-\frac{1}{2}\right)^n.$$

Therefore,

$$\mathcal{K}(T(g))(x, x') = s(-2)^n\, \gamma(1)^n\, s\left(-\frac{1}{2}\right)^n \chi(\langle gx, x'\rangle) = \gamma(1)^n\, \chi(\langle gx, x'\rangle).$$

□

4. The Weil representation over $\mathbb{R}$

Let $\chi(r) = \exp(2\pi i r)$, $r \in \mathbb{R}$. This is a non-trivial character of the additive group $\mathbb{R}$. In this section we provide a construction of the corresponding Weil representation, [43], [51].

4.1. **The Fourier transform.** Let U be a finite dimensional vector space over $\mathbb{R}$ and let B be a positive definite scalar product on U. We normalize the Lebesgue measure μ_{U} on U so that the volume of the unit cube (with respect to B) is 1. The formula

$$\Phi(u)(v) = B(u, v) \qquad (u, v \in \mathsf{U})$$

defines a linear isomorphism $\Phi : \mathsf{U} \to \mathsf{U}^*$. The form B^* dual to B is given by

$$B^*(u^*, v^*) = v^*(\Phi^{-1}(u^*)) \quad (u^*, v^* \in \mathsf{U}^*).$$

This is a symmetric positive definite bilinear form on U^*. Denote by μ_{U^*} the corresponding Lebesgue measure.

Let $\mathcal{S}(\mathsf{U})$ be the Schwartz space on U, [14, Definition 7.1.2]. For $\phi \in \mathcal{S}(\mathsf{U})$ let

$$\mathcal{F}\phi(u^*) = \int_{\mathsf{U}} \phi(u)\chi(-u^*(u))\, d\mu_{\mathsf{U}}(u) \qquad (u^* \in \mathsf{U}^*)$$

be the Fourier transform of ϕ. Then, as is well known, $\mathcal{F}\phi \in \mathcal{S}(\mathsf{U}^*)$ and

$$\phi(u) = \int_{\mathsf{U}^*} \mathcal{F}\phi(u^*)\chi(u^*(u))\, d\mu_{\mathsf{U}^*}(u^*) \qquad (u^* \in \mathsf{U}^*),$$

see [14, Theorem 7.1.5].

Let $\mathcal{S}^*(\mathsf{U})$ denote the space of the tempered distributions on U, [14, Definition 7.1.7]. When convenient we shall identify any bounded locally integrable function $f : \mathsf{U} \to \mathbb{C}$ with the tempered distribution $f\mu_{\mathsf{U}}$. In particular, $\mathcal{S}(\mathsf{U}) \subseteq \mathcal{S}^*(\mathsf{U})$. Then the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathsf{U}) \to \mathcal{S}(\mathsf{U}^*)$$

extends to

$$\mathcal{F} : \mathcal{S}^*(\mathsf{U}) \to \mathcal{S}^*(\mathsf{U}^*),$$

[14, Definition 7.1.9].

Let $\mathsf{V} \subseteq \mathsf{U}$ be a non-zero subspace. The form B restricts to V and determines the Lebesgue measure μ_{V}. We may view μ_{V} as a tempered distribution on U by

$$\mu_{\mathsf{V}}(\phi) = \int_{\mathsf{V}} \phi(v)\, d\mu_{\mathsf{V}}(v) \qquad (\phi \in \mathcal{S}(\mathsf{U})).$$

In the case when V is zero we define $\mu_{\mathsf{V}} = \mu_0$ to be the unit measure at 0. In other words $\mu_0 = \delta_0$ is the Dirac delta at 0,

$$\mu_0(\phi) = \delta_0(\phi) = \phi(0) \qquad (\phi \in C(\mathsf{U})).$$

Also, for future reference, let $\delta_u \in \mathcal{S}(\mathsf{U})$ be the Dirac delta at $u \in \mathsf{U}$,

$$\delta_u(\phi) = \phi(u) \qquad (\phi \in C(\mathsf{U})).$$

For an arbitrary subspace $\mathsf{V} \subseteq \mathsf{U}$, let $\mathsf{V}^\perp \subseteq \mathsf{U}^*$ be the annihilator of V. Then,

$$\mathcal{F}\mu_{\mathsf{V}} = \mu_{\mathsf{V}^\perp}, \tag{107}$$

see [14, Theorem 7.1.25].

The quotient space U/V may be identified with the B-orthogonal complement of V in U. Hence it inherits the natural scalar product.

Consider two real vector spaces U', U'' of the same dimension equipped with scalar products B', B'' respectively. Let $u_1', u_2', \dots, u_n'$ be a B'-orthonormal basis of U' and let $u_1'', u_2'', \dots, u_n''$ be a B''-orthonormal basis of U''. Suppose $L\colon \mathsf{U}' \to \mathsf{U}''$ is a linear bijection. Denote by M the matrix of L with respect to the two ordered basis:

$$Lu_j' = \sum_{i=1}^n M_{i,j} u_i'' \qquad (j = 1, 2, \dots, n).$$

Then $|\det(M)|$ does not depend on the choice of the orthonormal basis. Thus we may define $|\det(L)| = |\det(M)|$ (see section 2.5).

Lemma 37. *With the above notation we have*

$$\int_{\mathsf{U}'} \phi(L(u'))\, d\mu_{\mathsf{U}'}(u')\, |\det(L)| = \int_{\mathsf{U}''} \phi(u'')\, d\mu_{\mathsf{U}''}(u'') \qquad (\phi \in \mathcal{S}(\mathsf{U}'')). \tag{108}$$

Proof. Since $\int_0^1 \int_0^1 \cdots \int_0^1 dx_n \cdots dx_2\, dx_1 = 1$ and by definition of $\mu_{\mathsf{U}'}$, $\mu_{\mathsf{U}'}([0,1]u_1' + [0,1]u_2' + \cdots + [0,1]u_n') = 1$,

$$\int_{\mathsf{U}'} \phi(u')\, d\mu_{\mathsf{U}'}(u') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(x_1'u_1' + x_2'u_2' + \cdots + x_n'u_n')\, dx_n' \cdots dx_2'\, dx_1',$$

and similarly for U''. Therefore the right hand side of (108) equals

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\sum_{i=1}^{n} x_i''u_i'')\, dx_n'' \cdots dx_2''\, dx_1'' \\ = &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\sum_{i=1}^{n} \sum_{j=1}^{n} M_{i,j} x_j' u_i'')\, dx_n' \cdots dx_2'\, dx_1' |\det(M)| \\ = &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\sum_{j=1}^{n} x_j' L(u_j'))\, dx_n' \cdots dx_2'\, dx_1' |\det(M)| \end{aligned}$$

which coincides with the left hand side. □

Lemma 38. *Suppose* X *is a finite dimensional vector space over* $\mathbb{R}$ *with a positive definite symmetric bilinear form and* $L\colon \mathsf{X} \to \mathsf{U}$ *is a surjective linear map. Let*

$$\tilde{L} : \mathsf{X}/L^{-1}(\mathsf{V}) \to \mathsf{U}/\mathsf{V}$$

be the induced bijection. Then

$$L^*(\mu_{\mathsf{V}}) = |\det(\tilde{L})|^{-1} \mu_{L^{-1}(\mathsf{V})},$$

where the pullback $L^*(\mu_{\mathsf{V}})$ *is defined as in* [14, Theorem 6.1.2].

Proof. Let $\mathsf{X}' \subseteq \mathsf{X}$ be the orthogonal complement of $\mathrm{Ker}(L)$. Denote by L' the restriction of L to X' and by L'' the restriction of L to $\mathsf{X}' \cap L^{-1}(\mathsf{V})$. Then

$$L'\colon \mathsf{X}' \to \mathsf{U} \text{ and } L''\colon \mathsf{X}' \cap L^{-1}(\mathsf{V}) \to \mathsf{V}$$

are bijections.

According to [14, Theorem 6.1.2], for a test function ϕ we have

$$L^*(\mu_{\mathsf{V}})(\phi) = \int_{\mathrm{Ker}(L)} \int_{\mathsf{V}} \phi(x + L'^{-1}(v))\, d\mu_{\mathsf{V}}(v)\, d\mu_{\mathrm{Ker}(L)}(x)\, |\det(L')|^{-1}. \tag{109}$$

Lemma 37 shows that the right hand side of (109) is equal to

$$\begin{aligned} &\int_{\mathrm{Ker}(L)} \int_{L''^{-1}(\mathsf{V})} \phi(x+y)\, d\mu_{L''^{-1}(\mathsf{V})}(y)\, d\mu_{\mathrm{Ker}(L)}(x)\, |\det(L'')|\, |\det(L')|^{-1} \\ = &\int_{L^{-1}(\mathsf{V})} \phi(z)\, d\mu_{L^{-1}(\mathsf{V})}(z)\, |\det(L'')|\, |\det(L')|^{-1}. \end{aligned}$$

Since $|\det(L'')|^{-1}\,|\det(L')| = |\det(\tilde{L})|$, we are done. □

4.2. **Gaussians on $\mathbb{R}^n$.** Let B be the usual dot product on $\mathbb{R}^n$,

$$B(x,y) = x^t y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \qquad (x, y \in \mathbb{R}^n).$$

Then $d\mu_{\mathbb{R}^n}(x) = dx$ is the usual Lebesgue measure on $\mathbb{R}^n$, see [42, Theorem 10.33].

For a symmetric matrix $A \in \mathrm{GL}(\mathbb{R}^n)$ define the corresponding Gaussian γ_A by

$$\gamma_A(x) = \chi(\frac{1}{2}x^t Ax) \qquad (x \in \mathbb{R}^n).$$

Also, let

$$\gamma(A) = \mathcal{F}\gamma_A(0) = \int_{\mathbb{R}^n} \chi(\frac{1}{2}x^t Ax)\, dx.$$

As customary, we shall identify $\mathbb{R}^n$ with the dual $(\mathbb{R}^n)^*$ via the dot product. In these terms we have the following theorem, [14, Theorem 7.6.1].

Theorem 39. *For any symmetric matrix $A \in \mathrm{GL}(\mathbb{R}^n)$,*

$$\mathcal{F}\gamma_A = \frac{e^{\frac{\pi i}{4}\mathrm{sgn}(A)}}{\sqrt{|\det A|}}\gamma_{-A^{-1}},$$

where $\mathrm{sgn}(A)$ *is the number of the positive eigenvalues of A (counted with the multiplicities) minus the number of the negative eigenvalues of A (counted with the multiplicities). In particular,*

$$\gamma(A) = \frac{e^{\frac{\pi i}{4}\mathrm{sgn}(A)}}{\sqrt{|\det A|}}. \tag{110}$$

Remark 1. Eqn.(110) follows also from [51, Chap. I Théorème 2 and Chap. II § 26].

Remark 2. Eqn.(110) implies that

$$\gamma(A) = \pm\gamma(1)^{n-1}\,\gamma(\det A), \tag{111}$$

which can be viewed as the analog on $\mathbb{R}$ of Corollary 26.

Indeed, by applying Eqn.(110) to both 1 and $\det A$, we get

$$\gamma(1) = e^{\frac{\pi i}{4}} \quad \text{and} \quad \gamma(\det A) = \frac{e^{\frac{\pi i}{4}}\,\mathrm{sign}(\det A)}{\sqrt{|\det A|}},$$

where $\mathrm{sign}(\det A)$ is the sign of the determinant of A. Hence we are reduced to compare the congruence modulo 4 of $\mathrm{sgn}(A)$ with those of $n-1+\mathrm{sign}(\det A)$. Let p (resp. q) denote the number of the positive (resp. negative) eigenvalues of A. We have $\mathrm{sgn}(A) = p - q$ and $n = p + q$. It follows that

$$n - 1 + \mathrm{sign}(\det A) - \mathrm{sgn}(A) = 2q - 1 + \mathrm{sign}(\det A) \equiv 0 \pmod 4,$$

since $\mathrm{sign}(\det A) = (-1)^q$.

Remark 3. It is easy to see from (111) that

$$\left(\frac{\gamma(a)}{\gamma(1)}\right)^2 = \frac{1}{a} \qquad (a \in \mathbb{R}^\times). \tag{112}$$

4.3. **Gaussians on a vector space.** Let U be a finite dimensional vector space over $\mathbb{R}$ with a symmetric positive definite bilinear form B. Suppose q is a non-degenerate symmetric bilinear form on U. Let $\gamma(q) = \gamma(Q)$, where Q is the matrix obtained from any B-orthonormal basis u_1, u_2,..., u_n of U by

$$Q_{i,j} = q(u_i, u_j) \qquad (1 \leq i, j \leq n).$$

Also, we define $\gamma(0) = 1$.

Lemma 40. *If q is a non-degenerate symmetric bilinear form on U, then*

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\, d\mu_{\mathsf{U}}(u) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)) \qquad (u^* \in \mathsf{U}^*).$$

Proof. Fix a B-orthonormal basis u_1, u_2, …, u_n of U and let u_1^*, u_2^*, …, u_n^* be the dual basis of U^*. This is a B^*-orthonormal basis. As we have seen in the proof of Lemma 27, if Q is the matrix corresponding to q, as above, then Q^{-1} corresponds to q^*.

Let $x_i = u_i^*(u)$ and let $y_j = u^*(u_j)$. Then

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\, d\mu_{\mathsf{U}}(u) = \int_{\mathbb{R}^n} \chi(\frac{1}{2}x^tQx)\chi(-x^ty)\, dx$$

$$= \gamma(Q)\chi(-\frac{1}{2}y^tQ^{-1}y) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)),$$

where the second equality follows from Theorem 39. □

4.4. **Gaussians on a symplectic space.** Let W be a finite dimensional vector space over $\mathbb{R}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Fix a positive definite compatible complex structure J on W. In other words, $J \in \mathfrak{sp}(\mathrm{W})$, $J^2 = -\mathrm{I}$ and the form

$$B(w,w') = \langle J(w), w'\rangle \qquad (w, w' \in \mathrm{W}) \tag{113}$$

is positive definite. As explained in section 4.1, this leads to a normalization of the Lebesgue measures on any subspace of $\mathsf{U} \subseteq \mathrm{W}$ and on any quotient U/V, where V is a subspace of U.

We shall identify W with the dual W^* by

$$w^*(w) = \langle w, w^*\rangle \qquad (w, w^* \in \mathrm{W}). \tag{114}$$

Then

$$\mathsf{U}^* = \mathrm{W}/\mathsf{U}^\perp \text{ and } (\mathsf{U}/\mathsf{V})^* = \mathsf{V}^\perp/\mathsf{U}^\perp, \tag{115}$$

where the orthogonal complements are taken in W, with respect to the symplectic form $\langle\ ,\ \rangle$.

Lemma 41. *Suppose* $x \in \mathrm{Hom}(\mathsf{U}, \mathrm{W}/\mathsf{U}^\perp)$ *is such that*

$$\langle xu, v\rangle = \langle xv, u\rangle \qquad (u, v \in \mathsf{U}).$$

Set

$$q(u,v) = \frac{1}{2}\langle xu, v\rangle \qquad (u, v \in \mathsf{U}).$$

Let V *be the radical of* q *and let* $\tilde{q}$ *be the induced non-degenerate form on* U/V*. Then*

(a) $\mathsf{V} = \mathrm{Ker}(x)$;
(b) *The element* x *determines a bijection*

$$\underline{x} : \mathsf{U}/\mathsf{V} \to \mathsf{V}^\perp/\mathsf{U}^\perp,$$

with the inverse

$$\underline{x}^{-1} : \mathsf{V}^\perp/\mathsf{U}^\perp \to \mathsf{U}/\mathsf{V};$$

(c) *Let* $x^{-1} : \mathsf{V}^\perp \to \mathsf{U}/\mathsf{V}$ *be the composition of* $\underline{x}^{-1}$ *with the quotient map* $\mathsf{V}^\perp \to \mathsf{V}^\perp/\mathsf{U}^\perp$*. Define*

$$\chi_x(u) = \chi(\frac{1}{4}\langle xu, u\rangle) \qquad (u \in \mathsf{U}), \tag{116}$$

$$\chi_{x^{-1}}(w) = \chi(\frac{1}{4}\langle x^{-1}w, w\rangle) \qquad (w \in \mathsf{V}^\perp). \tag{117}$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned} &\int_{\mathsf{U}}\int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathsf{U}}(u) \qquad (118)\\ = \;& 2^{\dim(\mathsf{V})}\gamma(\tilde{q}) \int_{\mathsf{V}^\perp} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathsf{V}^\perp}(w)\\ = \;& 2^{\dim(\mathsf{V})}\gamma(\tilde{q}) \int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \chi_{x^{-1}}(w+\mathsf{U}^\perp) \int_{\mathsf{U}^\perp} \phi(w+v)\, d\mu_{\mathsf{U}^\perp}(v)\, d\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp}(w+\mathsf{U}^\perp). \end{aligned}$$

Also, for any $\phi \in \mathcal{S}(\mathrm{W}/\mathsf{U}^\perp)$,

$$\begin{aligned} &\int_{\mathsf{U}}\int_{\mathrm{W}/\mathsf{U}^\perp} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{U}}(u) \qquad (119)\\ = \;& 2^{\dim(\mathsf{V})}\gamma(\tilde{q}) \int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \chi_{\underline{x}^{-1}}(w)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp}(w+\mathsf{U}^\perp). \end{aligned}$$

Proof. Part (a) is obvious. Part (b) means that $\mathrm{Ker}(x)^\perp = \mathrm{Im}(x)$, which is true. For $\phi \in \mathcal{S}(\mathrm{W})$ we have,

$$\begin{aligned}
&\int_{\mathsf{U}}\int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathsf{U}}(u)\\
=\ &\int_{\mathrm{W}} \mathcal{F}(\gamma_q\mu_{\mathsf{U}})(\frac{1}{2}w)\phi(w)\, d\mu_{\mathrm{W}}(w)\\
=\ &\int_{\mathrm{W}} \mathcal{F}(\gamma_q\mu_{\mathsf{U}})(w)\phi(2w)\, d\mu_{\mathrm{W}}(w)\, 2^{\dim \mathrm{W}}\\
=\ &\gamma(\tilde{q})\int_{\mathsf{V}^\perp} \gamma_{-\tilde{q}^*}(w)\phi(2w)\, d\mu_{\mathsf{V}^\perp}(w)\, 2^{\dim \mathrm{W}}\\
=\ &\gamma(\tilde{q})\int_{\mathsf{V}^\perp} \gamma_{-\tilde{q}^*}(\frac{1}{2}w)\phi(w)\, d\mu_{\mathsf{V}^\perp}(w)\, 2^{\dim \mathrm{W}-\dim \mathsf{V}^\perp}\\
=\ &\gamma(\tilde{q})\int_{\mathsf{V}^\perp} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathsf{V}^\perp}(w)\, 2^{\dim \mathsf{V}}.
\end{aligned}$$

This verifies (118). For $\phi \in \mathcal{S}(\mathrm{W}/\mathsf{U}^\perp)$ we have,

$$\begin{aligned}
&\int_{\mathsf{U}}\int_{\mathrm{W}/\mathsf{U}^\perp} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{U}}(u)\\
=\ &\int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}}\int_{\mathrm{W}/\mathsf{U}^\perp} \chi_{\underline{x}}(u+\mathsf{V})\chi(\frac{1}{2}\langle u+v, w\rangle)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{V}}(v)\, d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\\
=\ &\int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}}\int_{\mathrm{W}/\mathsf{U}^\perp} \gamma_{\tilde{q}}(u+\mathsf{V})\chi(\langle u+v, w\rangle)\phi(2w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{V}}(v)\, d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\\
&2^{\dim \mathrm{W}/\mathsf{U}^\perp}\\
=\ &\int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \gamma_{\tilde{q}}(u+\mathsf{V})\chi(\langle u, w\rangle)\phi(2w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\, 2^{\dim \mathrm{W}/\mathsf{U}^\perp}\\
=\ &\gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \gamma_{-\tilde{q}^*}(w+\mathsf{U}^\perp)\phi(2w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, 2^{\dim \mathrm{W}/\mathsf{U}^\perp}\\
=\ &\gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \gamma_{-\tilde{q}^*}(\frac{1}{2}w+\mathsf{U}^\perp)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, 2^{\dim \mathrm{W}/\mathsf{U}^\perp-\dim \mathsf{V}^\perp/\mathsf{U}^\perp}\\
=\ &\gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \chi_{\underline{x}^{-1}}(w+\mathsf{U}^\perp)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, 2^{\dim \mathsf{V}}.
\end{aligned}$$

This verifies (119). □

By a Gaussian on the symplectic space W we shall understand any non-zero constant multiple of the tempered distribution

$$\chi_x\mu_{\mathsf{U}} \in \mathcal{S}^*(\mathrm{W}) \tag{120}$$

where the function χ_x is defined in Lemma 41. In these terms Lemma 41 says that the Fourier transform of a Gaussian is another Gaussian.

4.5. **Twisted convolution of Gaussians.** Recall the twisted convolution of two Schwartz functions $\psi, \phi \in \mathcal{S}(\mathrm{W})$:

$$\psi\natural\phi(w) = \int_{\mathrm{W}} \psi(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{121}$$

It is easy to see that the above integral converges and that $\psi\natural\phi \in \mathcal{S}(\mathrm{W})$. Also, the twisted convolutions

$$\delta_{w_0}\natural\phi(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w_0, w\rangle) \text{ and } \phi\natural\delta_{w_0}(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w, w_0\rangle) \tag{122}$$

are well defined for any continuous function ϕ.

Let

$$t(g) = \chi_{c(g)}\mu_{g^-\mathrm{W}} \qquad (g \in \mathrm{Sp}(\mathrm{W}))\,. \tag{123}$$

For any $\phi \in \mathcal{S}(\mathrm{W})$, the twisted convolution $t(g)\natural\phi$ is a continuous function given by the following absolutely convergent integral

$$t(g)\natural\phi(w) = \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{124}$$

Lemma 42. *For any $g \in \mathrm{Sp}(\mathrm{W})$,*

$$t(g)\natural(\delta_{w_0}\natural\phi) = \delta_{gw_0}\natural(t(g)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ w_0 \in \mathrm{W}).$$

Proof. The left hand side evaluated at $w \in \mathrm{W}$ is equal to

$$\begin{aligned}
&\int_{g^-\mathrm{W}} \chi_{c(g)}(u)(\delta_{w_0}\natural\phi)(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
= &\int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u-w_0)\chi(\frac{1}{2}\langle w_0, w-u\rangle)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
= &\int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)u, u\rangle + 2\langle w_0, w-u\rangle + 2\langle u, w\rangle))\, d\mu_{g^-\mathrm{W}}(u)
\end{aligned}$$

and the right hand side is equal to

$$\begin{aligned}
&(t(g)\natural\phi)(w-gw_0)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
= &\int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-gw_0-u)\chi(\frac{1}{2}\langle u, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
= &\int_{g^-\mathrm{W}} \chi_{c(g)}(u-g^-w_0)\phi(w-gw_0-(u-g^-w_0))\\
&\chi(\frac{1}{2}\langle u-g^-w_0, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
= &\int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)(u-g^-w_0), u-g^-w_0\rangle\\
+ &\ 2\langle u-g^-w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle))\, d\mu_{g^-\mathrm{W}}(u).
\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned} & \langle c(g)(u-g^{-}w_0), u-g^{-}w_0\rangle + 2\langle u-g^{-}w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle \\ - \quad & (\langle c(g)u,u\rangle + 2\langle w_0, w-u\rangle + 2\langle u,w\rangle) = 0. \end{aligned}$$

Hence, the two sides are equal. ☐

Let

$$\partial_{w_0} = \lim_{t\to 0}\frac{\delta_{tw_0}-\delta_0}{t}.$$

Then, for any $\phi\in\mathcal{S}(\mathrm{W})$ and $w_0\in\mathrm{W}$,

$$\begin{aligned} \partial_{w_0}\natural\phi(w) &= \pi i\langle w_0,w\rangle\phi(w) + \partial_{w_0}*\phi(w) \\ \phi\natural\partial_{w_0}(w) &= -\pi i\langle w_0,w\rangle\phi(w) + \partial_{w_0}*\phi(w) \end{aligned} \tag{125}$$

where $\partial_{w_0}*\phi(w) = \frac{d}{dt}\phi(w-tw_0)|_{t=0}$ is the directional derivative in the direction of $-w_0$.

Corollary 43. *For any $g\in\mathrm{Sp}(\mathrm{W})$,*

$$t(g)\natural(\partial_{w_0}\natural\phi) = \partial_{gw_0}\natural(t(g)\natural\phi) \qquad (\phi\in\mathcal{S}(\mathrm{W}),\ w_0\in\mathrm{W}).$$

Proposition 44. *For any $g\in\mathrm{Sp}(\mathrm{W})$ and $\phi\in\mathcal{S}(\mathrm{W})$, $t(g)\natural\phi\in\mathcal{S}(\mathrm{W})$. Moreover the map*

$$\mathcal{S}(\mathrm{W})\ni\phi\to t(g)\natural\phi\in\mathcal{S}(\mathrm{W})$$

is continuous.

Proof. We see from Corollary 43 with the formulas (125) that for any $w_0, w\in\mathrm{W}$,

$$\begin{aligned} 2\pi i\langle w_0,w\rangle(t(g)\natural\phi)(w) &= \partial_{w_0}\natural(t(g)\natural\phi)(w) - (t(g)\natural\phi)\natural\partial_{w_0}(w) \\ &= t(g)\natural(\partial_{g^{-1}w_0}\natural\phi - \phi\natural\partial_{w_0})(w) \end{aligned}$$

and similarly

$$2\partial_{w_0}*(t(g)\natural\phi)(w) = t(g)\natural(\partial_{g^{-1}w_0}\natural\phi + \phi\natural\partial_{w_0})(w).$$

Hence, for any polynomial coefficient differential operator P on W there is a polynomial coefficient differential operator Q on W such that

$$P(t(g)\natural\phi) = t(g)\natural Q(\phi) \qquad (\phi\in\mathcal{S}(\mathrm{W})). \tag{126}$$

Notice also that by the definition (124)

$$\| t(g)\natural\phi \|_\infty \le \sup_{w\in\mathrm{W}}\int_{g^{-}\mathrm{W}} |\phi(w-u)|\, d\mu_{g^{-}\mathrm{W}}(u) < \infty \tag{127}$$

and that the right hand side is a continuous seminorm on $\mathcal{S}(\mathrm{W})$. The proposition clearly follows from these two facts. ☐

Since the left and right twisted convolutions commute, Proposition (44) together with Corollary 65 below show that for any two elements $g_1, g_2\in\mathrm{Sp}(\mathrm{W})$ there is a tempered distribution $t(g_1)\natural t(g_2)\in\mathcal{S}^*(\mathrm{W})$ such that

$$(t(g_1)\natural t(g_2))\natural\phi = t(g_1)\natural(t(g_2)\natural\phi) \qquad (\phi\in\mathcal{S}(\mathrm{W})). \tag{128}$$

In order to verify Proposition 46 below, we shall need an explicit formula for the function $t(g)\natural\phi$ of Proposition 44. This is provided by the following Lemma.

Lemma 45. *Fix an element $g \in \mathrm{Sp}(\mathrm{W})$. Let $\mathsf{U} = g^-\mathrm{W}$. The map*

$$\mathsf{U} \ni u \to \langle\ ,(1-c(g))u\rangle \in \mathsf{U}^* = \mathrm{W}/\mathsf{U}^\perp = \mathrm{W}/\mathrm{Ker}(g^-) \tag{129}$$

is bijective.

Fix a complement Z of U in W so that

$$\mathrm{W} = \mathsf{U} \oplus \mathsf{Z}.$$

We shall denote the elements of U by u and elements of Z by z. In particular every $w \in \mathrm{W}$ has a unique decomposition

$$w = u + z.$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$ and any $w' = u' + z' \in \mathrm{W}$,

$$\begin{aligned} &t(g)\natural\phi(w') \qquad (130)\\ = \;& \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle) \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(u+z')\chi(-\frac{1}{2}\langle u, (1-c(g))u' + z'\rangle)\, d\mu_{\mathsf{U}}(u). \end{aligned}$$

In particular, (130) implies that $t(g)\natural\phi \in \mathcal{S}(\mathrm{W})$.

Proof. Suppose $\langle\ ,(1-c(g))u\rangle = 0$. Then $(1-c(g))u \in \mathrm{Ker}\, g^-$. There is $u_0 \in \mathrm{W}$ such that $u = g^-u_0$. Therefore

$$\begin{aligned} 0 &= g^-(1-c(g))u = g^-(1-c(g))g^-u_0 = g^-(g^-)u_0 - g^-g^+u_0\\ &= g^-(g^-)u_0 - g^+g^-u_0 = (g^- - g^-)g^-u_0 = -2g^-u_0 = -2u. \end{aligned}$$

This verifies (129).

The left hand side of (130) is equal to

$$\begin{aligned} t(g)\natural\phi(w') &= \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(w'-u)\chi(\frac{1}{2}\langle u, w'\rangle)\, d\mu_{\mathsf{U}}(u)\\ = & \int_{\mathsf{U}} \chi_{c(g)}(u+u')\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u)\\ = & \int_{\mathsf{U}} \chi_{c(g)}(u')\chi_{c(g)}(u)\chi(\frac{1}{2}\langle c(g)u', u\rangle)\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u)\\ = & \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle) \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(z'-u)\chi(\frac{1}{2}\langle u, w' - c(g)u'\rangle)\, d\mu_{\mathsf{U}}(u), \end{aligned}$$

which coincides with the right hand side. □

In the following proposition we use Notation 4 and Notation 6.

Proposition 46. *Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$. Let $\mathsf{U}_1' \subseteq \mathsf{U}_1$ be the orthogonal complement of U with respect to the positive definite form B, so that*

$$\mathsf{U}_1 = \mathsf{U}_1' \oplus \mathsf{U}.$$

Then the map

$$L\colon \mathsf{U}_1' + \mathsf{U}_2 \ni u_1' + u_2 \to c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 + \mathsf{U}^\perp \in \mathrm{W}/\mathsf{U}^\perp$$

is well defined, surjective and $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) = \mathsf{U}_{12}$*. Denote by*

$$\begin{aligned}&\tilde{L}\colon (\mathsf{U}_1 + \mathsf{U}_2)/\mathsf{U}_{12} \ni u_1 + u_2 + \mathsf{U}_{12} \to c(g_1)u_1 - c(g_2)u_2 - u_1 - u_2 + \mathsf{V}^\perp \in \mathrm{W}/\mathsf{V}^\perp\\ =\ &(\mathrm{W}/\mathsf{U}^\perp)/(\mathsf{V}^\perp/\mathsf{U}^\perp)\end{aligned}$$

the induced bijection and set

$$C(g_1, g_2) = \gamma(\tilde{q}_{g_1,g_2})2^{\dim \mathsf{V}}|\det(\tilde{L})|^{-1}. \tag{131}$$

Then C is a cocycle, with $C(g_1, 1) = C(1, g_2) = 1$, and

$$t(g_1)\natural t(g_2) = C(g_1, g_2)t(g_1g_2). \tag{132}$$

Furthermore, $C(g_1, g_2) = C(g_2, g_1)$*.*

Here, and elsewhere in this paper, the determinant of the zero map on a zero vector space is by definition equal 1.

Proof. Since $\mathsf{V}^\perp/\mathsf{U}^\perp = (c(g_1) + c(g_2))\mathsf{U}$, the map $\tilde{L}$ is well defined. Suppose $u_1' \in \mathsf{U}_1'$ and $u_2 \in \mathsf{U}_2$ are such that $L(u_1' + u_2) \in \mathsf{V}^\perp/\mathsf{U}^\perp$. Then there is $u \in \mathsf{U}$ such that

$$(c(g_1) + c(g_2))u + c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 \in \mathsf{U}^\perp.$$

Let $u = g_1^- v_1 = g_2^- v_2$, $v = u_1' = g_1^- w_1$, and $w - v = u_2 = g_2^- w_2$. Then

$$(c(g_1) + c(g_2))u + c(g_1)v + c(g_2)(v - w) - w \in \mathsf{U}^\perp.$$

Hence, the computation (87) - (89) shows that $w = (g_1g_2)^-(w_2 - v_2) \in \mathsf{U}_{12}$. Therefore $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) \subseteq \mathsf{U}_{12}$.

The map L is surjective. Indeed, for every $w \in W$, set $u_2 = g_2^- w_2$ with $w_2 = -\frac{1}{2}\, g_2^{-1}w$. Then

$$\begin{aligned}L(u_2) &= -c(g_2)u_2 - u_1 + U^\perp = -g_2^+ w_2 - g_2^- w_2 + U^\perp\\ &= -2g_2w_2U^\perp = w + U^\perp\,.\end{aligned}$$

Lemma 7 (b) shows that $\dim((\mathsf{U}_1+\mathsf{U}_2)/\mathsf{U}_{12}) = \dim((\mathrm{W}/\mathsf{U}^\perp)/(\mathsf{V}^\perp/\mathsf{U}^\perp))$. Thus $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) = \mathsf{U}_{12}$.

Here is a direct proof of this last equality. We already know that $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) \subseteq \mathsf{U}_{12}$. Therefore it will suffice to show that $L(U_{12}) \subset V^\perp/U^\perp$. This is true, because one can show (as was done in the first part of the proof), that for $u = (g_1g_2)^- w = u_1 + u_2 = (u_1' + u') + u_2$, with $u_1 = g_1^- g_2 w$ and $u_2 = g_2^- w$, one has:

$$\begin{aligned}L(u) &= c(g_1)g_1^- g_2 w - c(g_2)g_2^- w - (c(g_1) + c(g_2))u' - (u_1 + u_2) + U^\perp\\ &= (g_1^+)g_2 w - g_2^+ w - u - (c(g_1) + c(g_2))u' + U^\perp\\ &= (g_1g_2)^- w - u - (c(g_1) + c(g_2))u' + U^\perp\\ &= u - u - (c(g_1) + c(g_2))u' + U^\perp \in V^\perp/U^\perp.\end{aligned}$$

The computation (89) - (93) shows that, if $u_1'+u_2\in \mathsf{U}_{12}$ then

$$\begin{aligned}&\langle c(g_1)u_1',u_1'\rangle+\langle c(g_2)u_2,u_2\rangle+2\langle u_1',u_2\rangle+\langle (c(g_1)+c(g_2))^{-1}L(u_1'+u_2),L(u_1'+u_2)\rangle\\ =\ &\langle c(g_1g_2)(u_1'+u_2),u_1+u_2\rangle\end{aligned}$$

so that

$$\chi_{c(g_1)}(u_1')\chi_{c(g_2)}(u_2)\chi(\frac{1}{2}\langle u_1',u_2\rangle)\chi_{(c(g_1)+c(g_2))^{-1}}(L(u_1'+u_2))=\chi_{c(g_1g_2)}(u_1'+u_2). \tag{133}$$

Any $u_1\in\mathsf{U}_1$ has a unique decomposition $u_1=u_1'+u$, where $u_1'\in\mathsf{U}_1'$ and $u\in\mathsf{U}$. With this notation, Lemma 45 shows that for any $\phi\in\mathcal{S}(\mathrm{W})$,

$$\begin{aligned}&t(g_1)\natural(t(g_2)\natural\phi)(0) \qquad (134)\\ =\ &\int_{\mathsf{U}_1}\chi_{c(g_1)}(u_1)t(g_2)\natural\phi(u_1)\,d\mu_{\mathsf{U}_1}(u_1)\\ =\ &\int_{\mathsf{U}_1}\int_{\mathsf{U}_2}\chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u,u_1'\rangle)\chi(\frac{1}{2}\langle u_2,(c(g_2)-1)u\rangle)\\ &\chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2,u_1'\rangle)\phi(u_2+u_1')\,d\mu_{\mathsf{U}_2}(u_2)\,d\mu_{\mathsf{U}_1}(u_1)\\ =\ &\int_{\mathsf{U}}\int_{\mathsf{U}_1'}\int_{\mathsf{U}_2}\chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u,u_1'\rangle)\chi(\frac{1}{2}\langle u_2,(c(g_2)-1)u\rangle)\\ &\chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2,u_1'\rangle)\phi(u_2+u_1')\,d\mu_{\mathsf{U}_2}(u_2)\,d\mu_{\mathsf{U}_1'}(u_1')\,d\mu_{\mathsf{U}}(u)\end{aligned}$$

The formula (119) applied with $x=c(g_1)+c(g_2)$ shows that

$$\begin{aligned}&\int_{\mathsf{U}}\chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u,u_1'\rangle)\chi(\frac{1}{2}\langle u_2,(c(g_2)-1)u\rangle)\,d\mu_{\mathsf{U}}(u) \qquad (135)\\ =\ &\chi_{c(g_1)}(u_1')\int_{\mathsf{U}}\chi_{c(g_1)+c(g_2)}(u)\chi(\frac{1}{2}\langle u,c(g_1)u_1'-c(g_2)u_2-u_1'-u_2\rangle)\,d\mu_{\mathsf{U}}(u)\\ =\ &2^{\dim\mathsf{V}}\gamma(\tilde q_{g_1,g_2})\chi_{c(g_1)}(u_1')(\chi_{\underline{(c(g_1)+c(g_2))}^{-1}}\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp})(c(g_1)u_1'-c(g_2)u_2-u_1'-u_2).\end{aligned}$$

Furthermore, Lemma 38 shows that, for $u_1'+u_2\in\mathsf{U}_{12}$,

$$\begin{aligned}&\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp}(c(g_1)u_1'-c(g_2)u_2-u_1'-u_2)=L^*(\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp})(u_1'+u_2) \qquad (136)\\ =\ &|\det(\tilde L)|^{-1}\mu_{\mathsf{U}_{12}}(u_1'+u_2).\end{aligned}$$

The formula (132) follows directly from (133) - (136).

We see from (122) that

$$\begin{aligned}&t(g_1)\natural(t(g_2)\natural\phi)(w)=(t(g_1)\natural(t(g_2)\natural\phi))\natural\delta_{-w}(0)=(t(g_1)\natural(t(g_2)\natural(\phi\natural\delta_{-w})))(0)\\ =\ &((t(g_1)\natural t(g_2))\natural(\phi\natural\delta_{-w}))(0)=((t(g_1)\natural t(g_2))\natural\phi)\natural\delta_{-w}(0)=(t(g_1)\natural t(g_2))\natural\phi(w).\end{aligned}$$

Therefore

$$(t(g_1)\natural t(g_2))\natural\phi=t(g_1)\natural(t(g_2)\natural\phi).$$

Hence, $t(g_1)\natural t(g_2)$ coincides with the composition of $t(g_1)$ and $t(g_2)$ as elements of the associative algebra $\mathrm{End}(\mathcal{S}(\mathrm{W}))$. Therefore the function C is a cocycle.

The last statement is easy to check if $g_1 - 1$, $g_2 - 1$ and $g_1 g_2 - 1$ are invertible. Since the cocycle C is a continuous function, the equality for all group elements follows. □

4.6. **Normalization of Gaussians and the metaplectic group.** For a subset $S \subseteq \mathrm{W}$ let $S^{\perp_B} \subseteq \mathrm{W}$ be the B-orthogonal complement of S and for an element $h \in \mathrm{End}(\mathrm{W})$ let $h^{\#} \in \mathrm{End}(\mathrm{W})$ be as in (35). In particular, $(\mathrm{Ker}\, h^{\#})^{\perp} = h\mathrm{W}$.

Lemma 47. *Let $h \in \mathrm{End}(\mathrm{W})$ and let $K \subseteq \mathrm{W}$ be a subspace. Then*

$$h^{\#}((hK)^{\perp}) \subseteq K^{\perp} \tag{137}$$

and

$$|\det(h : K \to hK)| = |\det(h^{\#} : \mathrm{W}/(hK)^{\perp} \to \mathrm{W}/K^{\perp})|. \tag{138}$$

Proof. The inclusion (137) follows directly from (35).

Let $w_1, \ldots, w_a$ be a B-orthonormal basis of K and let u_1, ..., u_a be a B-orthonormal basis of hK. Since J is a B-isometry, $Jw_1, \ldots, Jw_a \in JK$ and $Ju_1, \ldots, Ju_a \in JhK$ are B-orthonormal basis. Define a matrix $(h_{k,i})_{1 \leq k,i \leq a}$ by

$$hw_i = \sum_{k=1}^{a} h_{k,i} u_k \qquad (1 \leq i \leq a).$$

Then

$$|\det(h : K \to hK)| = |\det((h_{k,i})_{1 \leq k,i \leq a})|. \tag{139}$$

We see from (34) that

$$JhK = (hK)^{\perp \perp_B} \text{ and } JK = K^{\perp \perp_B}.$$

Therefore

$$|\det(h^{\#} : \mathrm{W}/(hK)^{\perp} \to \mathrm{W}/K^{\perp})| = |\det((h^{\#}_{k,i})_{1 \leq k,i \leq a})|, \tag{140}$$

where

$$h^{\#} Ju_i \in \sum_{k=1}^{a} h^{\#}_{k,i} Jw_k + K^{\perp} \qquad (1 \leq i \leq a).$$

But,

$$\begin{aligned} h_{j,i} &= \sum_{k=1}^{a} h_{k,i} B(u_j, u_k) = -\sum_{k=1}^{a} h_{k,i} \langle u_k, Ju_j \rangle = -\langle hw_i, Ju_j \rangle \\ &= -\langle w_i, h^{\#} Ju_j \rangle = -\langle w_i, \sum_{k=1}^{a} h^{\#}_{k,j} Jw_k \rangle = -\sum_{k=1}^{a} h^{\#}_{k,j} \langle w_i, Jw_k \rangle \\ &= \sum_{k=1}^{a} h^{\#}_{k,j} B(w_k, w_i) = h^{\#}_{i,j}. \end{aligned}$$

Hence, (138) follows from (139) and (140). □

Lemma 48. *Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and assume that $K_1 = \mathrm{Ker}\, g_1^- = 0$. Then*

$$2^{-\dim \mathsf{V}} |\det(\tilde{L})| = |\det(g_2^- : K_{12} \to \mathsf{V})|^{-1}.$$

Proof. Since, by Lemma 7 (c), $\mathsf{V} = g_2^- K_{12} = (g_1^{-1} - 1)K_{12}$, the right hand side of the equation we need to prove makes sense. Also,

$$2^{-\dim \mathsf{V}} |\det(\tilde{L})| = |\det(\frac{1}{2}\tilde{L})|$$

and a straightforward computation shows that

$$\frac{1}{2}\tilde{L} \colon \mathrm{W}/\mathsf{U}_{12} \ni w + \mathsf{U}_{12} \to \frac{1}{2}(c(g_1) - 1)w + \mathsf{V}^\perp = (g_1^-)^{-1}w + \mathsf{V}^\perp \in \mathrm{W}/\mathsf{V}^\perp.$$

Hence,

$$|\det(\frac{1}{2}\tilde{L})|^{-1} = |\det(g_1^- \colon \mathrm{W}/\mathsf{V}^\perp \to \mathrm{W}/\mathsf{U}_{12})|.$$

Notice that $g_1^{-1} - 1 = g_1^{\#}$. Since $\mathsf{V} = g_2^- K_{12}$ and $\mathsf{U}_{12} = K_{12}^\perp$, Lemma 47 shows that

$$|\det(g_1^- \colon \mathrm{W}/\mathsf{V}^\perp \to \mathrm{W}/\mathsf{U}_{12})| = |\det(g_1^{-1} - 1 \colon K_{12} \to \mathsf{V})|.$$

Since the restrictions of g_1^{-1} and g_2 to K_{12} are equal, we are done. □

Consider an element $h \in \mathrm{End}(\mathrm{W})$ such that $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$. (In our applications h will be equal to g^-, where $g \in \mathrm{Sp}(\mathrm{W})$. Then $g^{\#} = g^{-1} - 1 = -g^{-1}g^-$ has the same kernel as g^-.) Let $L = J^{-1}h$. Denote by L^* the adjoint to L with respect to B (i.e. $B(Lw, w') = B(w, L^*w')$). Then $L^* = Jh^{\#}$. Hence $\mathrm{Ker}\, L = \mathrm{Ker}\, L^*$. Therefore L maps $(\mathrm{Ker}\, L)^{\perp_B} = L\mathrm{W}$ bijectively onto itself. Thus it makes sense to talk about $\det(L|_{L\mathrm{W}})$, the determinant of the restriction of L to $L\mathrm{W}$. If $w_1, w_2, \ldots, w_m$ is a B-orthonormal bais of $(\mathrm{Ker}\, L)^{\perp_B}$, then

$$\det(L|_{L\mathrm{W}}) = \det(B(Lw_i, w_j)_{1\le i,j\le m}) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le m}). \tag{141}$$

Under the condition $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$, we define $\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})$ to be the quantity (141).

Since

$$B(Jw_i, Jw_j) = \langle JJw_i, Jw_j\rangle = \langle Jw_i, w_j\rangle = B(w_i, w_j),$$

$Jw_1, Jw_2, \ldots, Jw_m$ is a B-orthonormal basis of $h\mathrm{W}$ ($=JL\mathrm{W}$). Further, if the coefficients $h_{j,i}$ are defined by

$$hw_i = \sum_j h_{j,i} Jw_j,$$

then

$$\begin{aligned} &\det(\langle hw_i, w_j\rangle_{1\le i,j\le m}) = \det(\langle \sum_k h_{k,i} Jw_k, w_j\rangle_{1\le i,j\le m}) \\ = \ &\det((h_{k,i})_{1\le k,i\le m}) \det(\langle Jw_k, w_j\rangle_{1\le k,j\le m}) = \det((h_{k,i})_{1\le k,i\le m}) \det(B(w_k, w_j)_{1\le k,j\le m}) \\ = \ &\det((h_{k,i})_{1\le k,i\le m}). \end{aligned}$$

Thus $|\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})| = |\det((h_{k,i})_{1\le k,i\le m})|$ coincides with the absolute value of the determinant defined previously in section 4.1. In particular,

$$\begin{aligned} &\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W}) \\ = \ &\mathrm{sgn}(\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})) |\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})|. \end{aligned} \tag{142}$$

Hence, if we identify $\mathbb{R}^\times/(\mathbb{R}^\times)^2$ with $\{\pm 1\}$ via the sgn, then $\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})$ is equal to the discriminant of the bilinear form induced by $\langle h\ ,\ \rangle$ on the quotient $\mathrm{W}/\mathrm{Ker} h$ times $|\det(h\colon \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})|$.

Also,

$$(hJ^{-1})Jw_i = \sum_j h_{j,i} Jw_j \quad \text{and} \quad (J^{-1}h)w_i = \sum_j h_{j,i} w_j$$

and hJ^{-1} maps $h\mathrm{W}$ into itself bijectively. Hence,

$$\begin{aligned} \det(h\colon \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W}) &= \det((h_{k,i})_{1\le k,i\le m}) \\ &= \det((hJ^{-1})|_{h\mathrm{W}}) = \det((J^{-1}h)|_{Jh\mathrm{W}})\,. \end{aligned} \tag{143}$$

Definition 49. *For $g \in \mathrm{Sp}(\mathrm{W})$ define*

$$\begin{aligned} \Theta^2(g) &:= \gamma(1)^{2\dim g^-\mathrm{W}} \det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})^{-1} \\ &= \gamma(1)^{2\dim g^-\mathrm{W}-2} (\gamma(\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W}))^2. \end{aligned}$$

(Here the second equality follows from (112).)

Lemma 50. *We have*

$$\frac{\Theta^2(g_1g_2)}{\Theta^2(g_1)\Theta^2(g_2)} = C(g_1, g_2)^2 \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})). \tag{144}$$

Proof. Both sides of the equality (144) are cocycles. Hence, Lemma 8 shows that we may assume that $K_1 = \{0\}$. In terms of the notation of Lemma 11 we have

$$-\dim \mathsf{U}_{12} + \dim \mathrm{W} + \dim \mathsf{U} = \dim K_{12} + \dim \mathsf{U} = \dim \mathsf{V} + \dim \mathsf{U} = -\dim(\mathsf{U}/\mathsf{V}) + 2\dim \mathsf{U}.$$

Hence,

$$\gamma(1)^{2(-\dim \mathsf{U}_{12}+\dim \mathrm{W}+\dim \mathsf{U})} = \gamma(1)^{4\dim U}\gamma(1)^{-2\dim(U/V)} = (-1)^{\dim \mathsf{U}}\gamma(1)^{-2\dim(\mathsf{U}/\mathsf{V})}. \tag{145}$$

Therefore the equality (38) is equivalent to

$$\begin{aligned} &\frac{\gamma(1)^{-2\dim \mathsf{U}_{12}} \det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathsf{U}_{12})}{\gamma(1)^{-2\dim \mathrm{W}} \det(g_1^- : \mathrm{W} \to \mathrm{W})\gamma(1)^{-2\dim \mathsf{U}} \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ &= \gamma(1)^{-2\dim(\mathsf{U}/\mathsf{V})} \det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}}) \det(g_2^- : K_{12} \to \mathsf{V})^{-2}. \end{aligned} \tag{146}$$

By Remark 2, we get

$$\gamma(\tilde{q}_{g_1,g_2}) = e^{\frac{\pi i}{4}\mathrm{sgn}(\tilde{q}_{g_1,g_2})} |\det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})|^{-1/2}$$

and

$$\mathrm{sgn}(\tilde{q}_{g_1,g_2}) = p - q,$$

where p is the dimension of the maximal subspace of U/V on which the form $\langle c(g_1) + c(g_2)\ ,\ \rangle$ is positive definite and q is the dimension of the maximal subspace of U/V on which the form $\langle c(g_1)+c(g_2)\ ,\ \rangle$ is negative definite. Hence,

$$\begin{aligned}
\gamma(\tilde{q}_{g_1,g_2})^2 &= i^{p-q}|\det(\langle\frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})|^{-1}\\
&= i^{p-q}(-1)^q\det(\langle\frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})^{-1}\\
&= i^{p+q}\det(\langle\frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})^{-1}\\
&= i^{\dim(\mathsf{U}/\mathsf{V})}\det(\langle\frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})^{-1}.
\end{aligned}$$

This, together with Lemma 48, shows that the right hand side of (146) is equal to

$$\gamma(\tilde{q}_{g_1,g_2})^{-2}\left(2^{-\dim\mathsf{V}}|\det(\tilde{L})|\right)^2,$$

which, by Proposition 46, coincides with $C(g_1,g_2)^{-2}$. □

Definition 51. *Let*

$$\widetilde{\mathrm{Sp}}(\mathrm{W})=\{(g,\xi);\ g\in\mathrm{Sp}(\mathrm{W}),\ \xi\in\mathbb{C}^\times,\ \xi^2=\Theta^2(g)\}.$$

where $\Theta^2(g)$ *is defined by Definition 49.*

Lemma 52. $\widetilde{\mathrm{Sp}}(\mathrm{W})$ *is a group with the multiplication defined by*

$$(g_1,\xi_1)(g_2,\xi_2)=(g_1g_2,\xi_1\xi_2C(g_1,g_2))\qquad(g_1,g_2\in\mathrm{Sp}(\mathrm{W}))\tag{147}$$

the identity equal to $(1,1)$ *and the inverse given by*

$$(g,\xi)^{-1}=(g^{-1},\overline{\xi})\qquad(g\in\mathrm{Sp}(\mathrm{W})).$$

Proof. Lemma 50 shows that the right hand side of (147) belongs to $\widetilde{\mathrm{Sp}}(\mathrm{W})$. A standard computation, as in [22, page 366] together with Proposition 46, shows that $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a group with the multiplication given by (147), the identity equal to $(1,C(1,1)^{-1})$ and

$$(g,\xi)^{-1}=(g^{-1},C(g^{-1},g)^{-1}\xi^{-1}).$$

Since, by Proposition 46, $C(1,1)=1$, it remains to check that

$$C(g^{-1},g)^{-1}\xi^{-1}=\overline{\xi}.$$

But, as in the proof of Lemma 48,

$$\begin{aligned}
C(g^{-1},g) &= 2^{\dim\mathsf{V}}|\det(\tilde{L})|^{-1}\\
&= |\det(g^-:\mathrm{W}/\mathrm{Ker}(g^-)\to g^-\mathrm{W})|=|\Theta^2(g)|^{-1}=|\xi|^{-2}.
\end{aligned}$$

This completes the proof. □

Notice that the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W})\ni(g,\xi)\to g\in\mathrm{Sp}(\mathrm{W})$$

is a group homomorphism with the kernel consisting of two elements. Thus $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a central extension of $\mathrm{Sp}(\mathrm{W})$ by the two element group $\mathbb{Z}/2\mathbb{Z}$:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Sp}(\mathrm{W}) \to 1. \tag{148}$$

Proposition 53. *The extension* (148) *does not split.*

Proof. Pick a two-dimensional symplectic subspace $\mathrm{W}_1 \subseteq \mathrm{W}$ and let $\mathrm{W}_2 = \mathrm{W}_1^{\perp}$, so that

$$\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2.$$

Define an element $g \in \mathrm{Sp}(\mathrm{W})$ by

$$g(w_1 + w_2) = -w_1 + w_2 \qquad (w_1 \in \mathrm{W}_1, w_2 \in \mathrm{W}_2).$$

Then

$$\Theta^2(g) = i^2 \det(-2 : \mathrm{W}_1 \to \mathrm{W}_1)^{-1} = (i/2)^2$$

and

$$C(g, g) = 2^2 \cdot 1 \cdot 1 = 2^2.$$

Let $\tilde{g} = (g, i/2)$. Then $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and

$$\tilde{g}^2 = (g^2, (i/2)^2 C(g, g)) = (1, -1) \text{ and } \tilde{g}^4 = (1, 1).$$

Thus the subgroup of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ generated by $\tilde{g}$ is cyclic of order 4. The subgroup of $\mathrm{Sp}(\mathrm{W})$ generated by g is cyclic of order 2. Hence the extension (148) does not split over that subgroup. □

Corollary 54. *Up to an equivalence of central group extensions, as in* [22, sec. 6.10], (148) *is the only non-trivial central extension of* $\mathrm{Sp}(\mathrm{W})$ *by* $\mathbb{Z}/2\mathbb{Z}$.

Proof. Since, as is well known (see [27, Theorems 5.10 and 11.1 (b)]),

$$H^2(\mathrm{Sp}(\mathrm{W}), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}),$$

the claim follows. □

Let

$$\phi^*(w) = \overline{\phi(-w)} \text{ and } u^*(\phi) = \overline{u(\phi^*)} \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ u \in \mathcal{S}^*(\mathrm{W}),\ w \in \mathrm{W}).$$

Lemma 55. *For any* $g \in \mathrm{Sp}(\mathrm{W})$, $t(g)^* = t(g^{-1})$.

Proof. By the definition (123),

$$t(g)^* = \left(\chi_{c(g)}\mu_{g^-\mathrm{W}}\right)^* = \overline{\chi_{c(g)}}\mu_{g^-\mathrm{W}} = \chi_{-c(g)}\mu_{g^-\mathrm{W}}.$$

Since $g^-\mathrm{W} = (g^{-1} - 1)\mathrm{W}$, it will suffice to check that for any $w \in \mathrm{W}$

$$-c(g)g^-w = c(g^{-1})g^-w.$$

The left hand side is equal to $-g^+w$. The right hand side is equal to

$$-c(g^{-1})(g^{-1} - 1)gw = -(g^{-1} - 1)gw = -g^+w.$$

□

Definition 56. *For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ define*

$$\Theta(\tilde{g}) = \xi \quad \textit{and} \quad T(\tilde{g}) = \Theta(\tilde{g}) t(g). \tag{149}$$

Lemma 57. *With the notation of (149), the following formulas hold*

$$T(\tilde{g}_1) \natural T(\tilde{g}_2) = T(\tilde{g}_1 \tilde{g}_2) \qquad (\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})), \tag{150}$$

$$T(\tilde{g})^* = T(\tilde{g}^{-1}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})). \tag{151}$$

Proof. By Proposition 46 the left hand side of (150) is equal to

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1 \tilde{g}_2)} C(g_1, g_2) T(\tilde{g}_1 \tilde{g}_2).$$

Lemma 52 shows that

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1 \tilde{g}_2)} C(g_1, g_2) = 1. \tag{152}$$

This verifies (150).

The equality (151) follows from Lemma 52 and Lemma 55:

$$T(\tilde{g})^* = \overline{\Theta(\tilde{g})} t(g)^* = \Theta(\tilde{g}^{-1}) t(g^{-1}) = T(\tilde{g}^{-1}).$$

□

Notice that Sp(W) is a connected Lie group. As such it has a unique connected double cover (up to an isomorphism of covers). See [3, sec. 16.30]. This way we may view $\widetilde{\mathrm{Sp}}(\mathrm{W})$, the metaplectic group, as a connected Lie group.

Lemma 58. *The map $T : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})$ is injective and continuous.*

Proof. The injectivity of T follows from the injectivity of $t : \mathrm{Sp}(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})$, which is obvious. Let

$$\mathrm{Sp}^c(\mathrm{W}) = \{g \in \mathrm{Sp}(\mathrm{W});\ \det g^- \neq 0\}.$$

Lemma 8 shows that

$$\mathrm{Sp}(\mathrm{W}) = \bigcup_{h \in \mathrm{Sp}(\mathrm{W})} \mathrm{Sp}^c(\mathrm{W}) h. \tag{153}$$

Let $\widetilde{\mathrm{Sp}}^c(\mathrm{W}) \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W})$ be the preimage of $\mathrm{Sp}^c(\mathrm{W})$. Then

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \bigcup_{\tilde{h} \in \widetilde{\mathrm{Sp}}(\mathrm{W})} \widetilde{\mathrm{Sp}}^c(\mathrm{W}) \tilde{h}.$$

By Lemma 57, we have

$$T(\tilde{g}) = T(\tilde{g}\tilde{h}^{-1}) \natural T(\tilde{h}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}^c(\mathrm{W}) \tilde{h})$$

Thus for $\phi \in \mathcal{S}(\mathrm{W})$,

$$T(\tilde{g}) \natural \phi = T(\tilde{g}\tilde{h}^{-1}) \natural (T(\tilde{h}) \natural \phi).$$

By Proposition 44, the map

$$\mathcal{S}(\mathrm{W}) \ni \phi \to T(\tilde{h}) \natural \phi \in \mathcal{S}(\mathrm{W})$$

is continuous. Hence it will suffice to check that the restriction of T to $\widetilde{\mathrm{Sp}}^c(\mathrm{W})$ is continuous. But this is obvious. □

4.7. **The conjugation property.** Let $\mathrm{L}^2(\mathrm{W})$ denote the Hilbert space of the Lebesgue measurable functions $\phi : \mathrm{W} \to \mathbb{C}$, with the norm given by

$$\parallel \phi \parallel_2^2 = \int_{\mathrm{W}} |\phi(w)|^2 \, d\mu_{\mathrm{W}}(w).$$

Lemma 57 shows that for any $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and any $\phi \in \mathcal{S}(\mathrm{W})$

$$\parallel T(\tilde{g})\natural\phi \parallel_2^2 = (T(\tilde{g})\natural\phi)^*\natural(T(\tilde{g})\natural\phi)(0) = \phi^*\natural T(\tilde{g})^*\natural T(\tilde{g})\natural\phi(0) = \phi^*\natural\phi(0) = \parallel \phi \parallel_2^2 .$$

Hence, the continuous linear map

$$\mathcal{S}(\mathrm{W}) \ni \phi \to T(\tilde{g})\natural\phi \in \mathcal{S}(\mathrm{W})$$

extends by continuity to an isometry

$$\mathrm{L}^2(\mathrm{W}) \ni \phi \to T(\tilde{g})\natural\phi \in \mathrm{L}^2(\mathrm{W}).$$

Furthermore, the formula

$$\omega_{1,1}(g)\phi(w) = \phi(g^{-1}w) \qquad (g \in \mathrm{Sp}(\mathrm{W}),\ \phi \in \mathrm{L}^2(\mathrm{W})).$$

defines a unitary representation $\omega_{1,1}$ of the symplectic group $\mathrm{Sp}(\mathrm{W})$ on $\mathrm{L}^2(\mathrm{W})$.

Proposition 59. *For any $\phi \in \mathrm{L}^2(\mathrm{W})$ and $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ in the preimage of $g \in \mathrm{Sp}(\mathrm{W})$, $T(\tilde{g})\natural\phi\natural T(\tilde{g}^{-1}) = \omega_{1,1}(g)\phi$.*

Proof. Since $T(\tilde{g})$ is a bounded operator, we may assume that $\phi \in \mathcal{S}(\mathrm{W})$. Lemma 42 says that

$$t(g)\natural\delta_w = \delta_{gw}\natural t(g) \qquad (w \in \mathrm{W}).$$

Therefore

$$T(\tilde{g})\natural\delta_w = \delta_{gw}\natural T(\tilde{g}) \qquad (w \in \mathrm{W}).$$

Since,

$$\phi = \int_{\mathrm{W}} \phi(w)\delta_w \, d\mu_{\mathrm{W}}(w) \text{ and } \int_{\mathrm{W}} \phi(w)\delta_{gw} \, d\mu_{\mathrm{W}}(w) = \omega_{1,1}(g)\phi,$$

we see that

$$T(\tilde{g})\natural\phi = (\omega_{1,1}(g)\phi)\natural T(\tilde{g}).$$

□

4.8. **The Weyl transform and the Weil representation.** Pick a complete polarization

$$\mathsf{W} = \mathsf{X} \oplus \mathsf{Y} \tag{154}$$

and recall that our normalization of measures is such that $d\mu_{\mathsf{W}}(x+y) = d\mu_{\mathsf{X}}(x)d\mu_{\mathsf{Y}}(y)$. Recall the the Weyl transform

$$\begin{aligned} &\mathcal{K}\colon \mathcal{S}^*(\mathsf{W}) \to \mathcal{S}^*(\mathsf{X}\times\mathsf{X}), \\ &\mathcal{K}(f)(x,x') = \int_{\mathsf{Y}} f(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y), \end{aligned} \tag{155}$$

This is an isomorphism of linear topological spaces, which restricts to an isometry

$$\mathcal{K}\colon \mathrm{L}^2(\mathsf{W}) \to \mathrm{L}^2(\mathsf{X}\times\mathsf{X}). \tag{156}$$

Each element $K \in \mathcal{S}^*(\mathsf{X}\times\mathsf{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))$ by

$$\mathrm{Op}(K)v(x) = \int_{\mathsf{X}} K(x,x')v(x')\, d\mu_{\mathsf{X}}(x'). \tag{157}$$

Schwartz Kernel Theorem, [46, statement (51.7), page 531], implies that

$$\mathrm{Op} : \mathcal{S}^*(\mathsf{X}\times\mathsf{X}) \to \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X})) \tag{158}$$

is an isomorphism of linear topological vector spaces. (One may prove it using [14, Theorem 5.2.1].) A straightforward computation shows that $\mathrm{Op}\circ\mathcal{K}$ transforms the twisted convolution of distributions (when it makes sense) into the composition of the corresponding operators. Also,

$$(\mathrm{Op}\circ\mathcal{K}(f))^* = \mathrm{Op}\circ\mathcal{K}(f^*) \qquad (f \in \mathcal{S}^*(\mathrm{W})) \tag{159}$$

and

$$\mathrm{tr}\ \mathrm{Op}\circ\mathcal{K}(f) = \int_{\mathsf{X}} \mathcal{K}(f)(x,x)\, d\mu_{\mathsf{X}}(x) = f(0) \tag{160}$$

if $\mathrm{Op}\circ\mathcal{K}(f)$ is of trace class, [18, Theorem 3.5.4]. Hence, the map

$$\mathrm{Op}\circ\mathcal{K} : \mathrm{L}^2(\mathsf{W}) \to \mathrm{H.S.}(\mathrm{L}^2(\mathsf{X})) \tag{161}$$

is an isometry, which is a well known fact [18, Theorem 1.4.1]. (Here $\mathrm{H.S.}(\mathrm{L}^2(\mathsf{X}))$ stands for the space of the Hilbert-Schmidt operators on $\mathrm{L}^2(\mathsf{X})$.)

Let $\mathrm{U}(\mathrm{L}^2(\mathsf{X}))$ denote the group of the unitary operators on the Hilbert space $\mathrm{L}^2(\mathsf{X})$.

Theorem 60. *Let $\omega = \mathrm{Op}\circ\mathcal{K}\circ T$. Then*

$$\omega\colon \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X}))$$

is an injective group homomorphism. For each $v \in \mathrm{L}^2(\mathsf{X})$, the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde{g} \to \omega(\tilde{g})v \in \mathrm{L}^2(\mathsf{X})$$

is continuous, so that $(\omega, \mathrm{L}^2(\mathsf{X}))$ is a unitary representation of the metaplectic group. The function Θ coincides with the character of this representation:

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} = \mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} \qquad (\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W})),$$

where the integral on the left is absolutely convergent. (Here $d\tilde{g}$ stands for any Haar measure on $\widetilde{\mathrm{Sp}}(\mathrm{W})$.) Moreover,

$$\omega(\tilde{g})\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(\tilde{g}^{-1})=\mathrm{Op}\circ\mathcal{K}(\omega_{1,1}(g)\phi)\qquad(\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W}),\ \phi\in\mathrm{L}^2(\mathrm{W})).$$

Proof. We see from the discussion in section 4.7 that the left multiplication by $\omega(\tilde{g})$ is an isometry on $\mathrm{H.S.}(\mathrm{L}^2(\mathsf{X}))$. This implies that $\omega(\tilde{g})$ is a unitary operator.

We see from (158) that for any two function $v_1, v_2\in\mathcal{S}(\mathsf{X})$ there is $\phi\in\mathcal{S}(\mathrm{W})$ such that

$$\int_{\mathsf{X}}\omega(\tilde{g})v_1(x)\overline{v_2(x)}\,d\mu_{\mathsf{X}}(x)=T(\tilde{g})(\phi)\qquad(\tilde{g}\in\mathrm{Sp}(\mathrm{W})).$$

Hence Lemma 58 shows that the left hand side is a continuous function of $\tilde{g}$. Since the operators $\omega(\tilde{g})$ are uniformly bounded (by 1), we see that the left hand side is a continuous function of $\tilde{g}$ for any $v_1, v_2\in\mathrm{L}^2(\mathsf{X})$. This implies the strong continuity of ω, see [49, Lemma 1.1.3] or [50, Proposition 4.2.2.1].

Lemmas 57 and 58 show that the $\omega:\widetilde{\mathrm{Sp}}(\mathrm{W})\to\mathrm{U}(\mathrm{L}^2(\mathsf{X}))$ is an injective group homomorphism.

It is not difficult to check that the function

$$\frac{\det(\mathrm{Ad}(g)-1)}{\det g^-}\qquad(g\in\mathrm{Sp}(\mathrm{W}))$$

is locally bounded. Furthermore, as shown by Harish-Chandra [11, Lemma 53, page 504], the function

$$|\det(\mathrm{Ad}(g)-1)|^{-1/2}\qquad(g\in\mathrm{Sp}(\mathrm{W}))$$

is locally integrable. Hence the function,

$$|\Theta(\tilde{g})|=|\det g^-|^{-1/2}\qquad(\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W}))\tag{162}$$

is locally integrable.

Notice that for any $\Psi\in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$,

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})}T(\tilde{g})\Psi(\tilde{g})\,d\tilde{g}\in\mathcal{S}(\mathrm{W}).\tag{163}$$

Indeed, since the Zariski topology on $\mathrm{Sp}(\mathrm{W})$ is noetherian the covering (153) contains a finite subcovering (see for example [13, Exercise 1.7(b)]). Hence, there are elements $\tilde{h}_1$, $\tilde{h}_2$, …, $\tilde{h}_m$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ such that

$$\widetilde{\mathrm{Sp}}(\mathrm{W})=\bigcup_{j=1}^m\widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h}_j.$$

Therefore Proposition 44, Lemma 57 and a standard partition of the unity argument reduces the proof of (163) to the case when $\Psi\in C_c^\infty(\widetilde{\mathrm{Sp}}^c(\mathrm{W}))$. In this case (163) is equal to

$$\int_{\mathfrak{sp}(\mathrm{W})}e^{\frac{\pi i}{2}\langle xw,w\rangle}\psi(x)\,dx\tag{164}$$

where $\psi \in C_c^\infty(\mathfrak{sp}(\mathrm{W}))$ and dx is a Lebesgue measure on $\mathfrak{sp}(\mathrm{W})$. The function (164) is equal to the pullback of a Fourier transform $\hat{\psi}$ of ψ from $\mathfrak{sp}^*(\mathrm{W})$ to W via the unnormalized moment map

$$\tau : \mathrm{W} \to \mathfrak{sp}^*(\mathrm{W}),\ \tau(w)(x) = \langle xw, w\rangle \qquad (x \in \mathfrak{sp}(\mathrm{W}),\ w \in \mathrm{W}). \tag{165}$$

Since $\hat{\psi} \in \mathcal{S}(\mathfrak{sp}(\mathrm{W}))$ and since τ is a polynomial map with uniformly bounded fibers,

$$\hat{\psi} \circ \tau \in \mathcal{S}(\mathrm{W}).$$

This verifies (163). Hence, we may compute the trace as follows:

$$\begin{aligned} \mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} &= \left(\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}\right)(0) = \left(\int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}\right)(0) \\ &= \int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})(0)\Psi(\tilde{g})\, d\tilde{g} = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}. \end{aligned}$$

The last formula is a direct consequence of Proposition 59. □

We end this section by recalling some well known formulas for the action of $\omega(\tilde{g})$ for some special elements $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$.

Proposition 61. *Let* $\mathrm{M} \subseteq \mathrm{Sp}(\mathrm{W})$ *be the subgroup of all the elements that preserve* X *and* Y. *Let* $\mathrm{M}^c = \{g \in \mathrm{M} : \det g^- \neq 0\}$. *Set*

$$\det_{\mathsf{X}}^{-1/2}(\tilde{g}) = \Theta(\tilde{g})|\det(\frac{1}{2}(c(g|_{\mathsf{X}}) + 1))|^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c).$$

Then

$$\left(\det_{\mathsf{X}}^{-1/2}(\tilde{g})\right)^2 = \det(g|_{\mathsf{X}})^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c), \tag{166}$$

the function $\det_{\mathsf{X}}^{-1/2} : \widetilde{\mathrm{M}}^c \to \mathbb{C}^\times$ *extends to a continuous group homomorphism*

$$\det_{\mathsf{X}}^{-1/2} : \widetilde{\mathrm{M}} \to \mathbb{C}^\times$$

and

$$\omega(\tilde{g})v(x) = \det_{\mathsf{X}}^{-1/2}(\tilde{g})v(g^{-1}x) \qquad (\tilde{g} \in \widetilde{\mathrm{M}},\ v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}). \tag{167}$$

Proof. Set $n = \dim \mathsf{X}$. Fix an element $g \in \mathrm{M}^c$. Then

$$\begin{aligned} \Theta^2(g) &= \det(ig^-)^{-1} = (-1)^n \det(g|_{\mathsf{X}} - 1)^{-1} \det(g|_{\mathsf{Y}} - 1)^{-1} \\ &= \det(g|_{\mathsf{X}} - 1)^{-1} \det(1 - g|_{\mathsf{Y}})^{-1} \\ &= \det(g|_{\mathsf{X}} - 1)^{-1} \det(1 - (g|_{\mathsf{X}})^{-1})^{-1} \\ &= \det(g|_{\mathsf{X}} - 1)^{-2} \det(g|_{\mathsf{X}}). \end{aligned}$$

Also,

$$\begin{aligned} &|\det(\frac{1}{2}(c(g|_{\mathsf{X}}) + 1))|^{-1} = |\det((g|_{\mathsf{X}})(g|_{\mathsf{X}} - 1)^{-1})|^{-1} \\ = \ &|\det(g|_{\mathsf{X}} - 1)||\det(g|_{\mathsf{X}})|^{-1}. \end{aligned}$$

This verifies (166).

Let $x, x' \in \mathsf{X}$ and let $y \in \mathsf{Y}$. Then

$$\begin{aligned}
\mathcal{K}(t(g))(x,x') &= \int_{\mathsf{Y}} t(g)(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\
&= \int_{\mathsf{Y}} \chi(\frac{1}{2}\langle c(g)(x-x'), y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\
&= \delta_0(\frac{1}{2}c(g)(x-x') - x - x') = \delta_0(\frac{1}{2}((c(g)-1)x - (c(g)+1)x'))\\
&= |\det(\frac{1}{2}(c(g|_{\mathsf{X}})+1))|^{-1}\delta_0(g^{-1}x - x').
\end{aligned}$$

Therefore

$$\mathcal{K}(T(\tilde{g}))(x,x') = \det{}_{\mathsf{X}}^{-1/2}(\tilde{g})\delta_0(g^{-1}x - x').$$

Thus we have (167) for $\tilde{g} \in \widetilde{\mathrm{M}}^c$. Since ω is a representation of $\widetilde{\mathrm{M}}$, the remaining claims follow. ☐

Proposition 62. *Suppose $g \in \mathrm{Sp(W)}$ acts trivially on Y and on W/Y. Then $\det((-g)-1) \neq 0$ and*

$$\omega(\tilde{g})v(x) = \pm\chi_{c(-g)}(2x)v(x) \qquad (v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}).$$

Proof. Since $-g$ acts as minus the identity on Y and on W/Y, $\det((-g)-1) \neq 0$ and $z := c(-g) \in \mathfrak{sp}(\mathrm{W})$ is well defined. We have

$$z(w) = (-g)^+((-g)^-)^{-1}(w) \quad (w \in \mathrm{W}).$$

Since g acts trivially on Y and on W/Y, we get, for every $x \in \mathsf{X}$ and every $y \in \mathsf{Y}$:

$$g(x+y) = x + y + y_x, \quad \text{where } y_x \in \mathsf{Y}.$$

It gives $(-g)^-(x+y) = -2x - 2y - y_x$. Also, $(-g)^-y = -2y$, so that $((-g)^-)^{-1}y = -\frac{1}{2}y$. Hence,

$$((-g)^-)^{-1}(x+y) = -\frac{1}{2}(x+y+((-g)^-)^{-1}y_x) = -\frac{1}{2}(x+y) + \frac{1}{4}y_x.$$

We obtain

$$z(x+y) = (-g)^+(-\frac{1}{2}(x+y) + \frac{1}{4}y_x),$$

that is,

$$z(x+y) = z(x) = \frac{1}{2}y_x. \tag{168}$$

In particular, we have

$$z\colon \mathsf{X} \to \mathsf{Y} \to 0.$$

Also, $\det(z-1) \neq 0$ and $c(z)$ is well defined. On the other hand, we have $(z-1)(x+y) = -(x+y) + \frac{1}{2}y_x$ and $(z-1)y = -y$. It follows that

$$(z-1)^{-1}(x+y) = -(x+y) - \frac{1}{2}y_x.$$

Hence,

$$c(z)(x+y) = (z+1)\left(-(x+y) - \frac{1}{2}y_x\right) = -\frac{1}{2}y_x - (x+y) - \frac{1}{2}y_x,$$

that is,

$$c(z)(w) = -w - 2z(w), \quad \text{for every } w \in \mathrm{W}. \tag{169}$$

We have $c(z) \in \mathrm{Sp(W)}$. Indeed, for any $w, w' \in \mathrm{W}$, writing $w = x + y$ and $w' = x' + y'$, with $x, x' \in \mathsf{X}$ and $y, y' \in \mathsf{Y}$, we have

$$\langle c(z)(w), c(z)(w')\rangle = \langle -w - y_x, -w' - y_{x'}\rangle = \langle w, w'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle.$$

However, since g is in $\mathrm{Sp(W)}$, we have

$$\langle x, x'\rangle = \langle gx, gx'\rangle = \langle x + y_x, x' + y_{x'}\rangle = \langle x, x'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle,$$

which gives

$$\langle x, y_{x'}\rangle + \langle y_x, x'\rangle = 0.$$

Hence,

$$\begin{aligned}
\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi_{-z}(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\
&= \chi_{-2z}(x-x')\delta_0(\frac{1}{2}(x+x')) = 2^n\, \chi_{-2z}(x-x')\,\delta_0(x+x') \\
\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi_{-z}(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\
&= \chi_{-z}(x-x')\delta_0(\frac{1}{2}(x+x')) = 2^n\chi_{-z}(x-x')\delta_0(x+x').
\end{aligned}$$

Moreover,

$$\Theta^2(c(z)) = \left(\frac{i}{2}\right)^{2n},$$

since $\dim\left((c(z)-1)(\mathrm{W})\right) = \dim \mathrm{W} = 2n$, and,

$$\det\left(c(z)-1\right) = (-2)^{2n}.$$

Thus

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') = \pm i^n \chi_{-z}(x-x')\delta_0(x+x').$$

Since Proposition 61 shows that

$$\omega(\widetilde{(-1)})v(x) = \zeta(\widetilde{-1})v(-x). \tag{170}$$

the proof is complete. □

Proposition 63. *Suppose $g \in \mathrm{Sp(W)}$ acts trivially on X and on W/X. Then* $\det((-g)-1) \neq 0$ *so that $z = c(-g) \in \mathfrak{sp}(\mathrm{W})$ is well defined and $z\colon \mathsf{Y} \to \mathsf{X} \to 0$. Assume $z(\mathsf{Y}) = \mathsf{X}$. Then*

$$\omega(\tilde{g})v(x) = \pm\frac{e^{\frac{\pi i}{4}\operatorname{sgn}\langle z,\ \rangle|_{\mathsf{Y}}}}{|\det(z\colon \mathsf{Y}\to\mathsf{X})|^{1/2}}\int_{\mathsf{X}} \chi_{z^{-1}}(x-x')v(x')\,d\mu_{\mathsf{X}}(x') \qquad (v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}),$$

where $z^{-1}\colon \mathsf{X} \to \mathsf{Y}$ *is the inverse of* $z\colon \mathsf{Y} \to \mathsf{X}$.

Proof. The existence of z and its properties are verified as in the proof of Proposition 62. In particular, for all $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, we have

$$g(x+y) = x + y + x_y, \quad \text{where } x_y \in \mathsf{X}.$$

Similarly to the proof of Proposition 62, we get

$$z(x+y) = z(y) = \frac{1}{2}x_y. \tag{171}$$

and

$$c(z)(x+y) = -(x+y) - x_y, \tag{172}$$

that is,

$$c(z)(w) = -w - 2z(w), \quad \text{for every } w \in \mathrm{W}. \tag{173}$$

From (171)and (172), we obtain

$$\langle c(z)(w), w\rangle = \langle -w - 2z(w), w\rangle = -2\langle z(w), w\rangle. \tag{174}$$

With notation (116), it gives

$$\chi_{c(z)}(w) = \chi\left(\frac{1}{4}\langle c(z)(w), w\rangle\right) = \chi\left(-\frac{1}{2}\langle z(w), w\rangle\right) = \chi_{-2z}(w). \tag{175}$$

Let

$$q(y, y') = \frac{1}{2}\langle zy, y'\rangle \qquad (y, y' \in \mathsf{Y}).$$

Then, in terms of Lemma 40 and the identification (114),

$$q^*(x, x') = -2\langle z^{-1}x, x'\rangle \qquad (x, x' \in \mathsf{X})$$

and

$$\gamma(q) = \frac{e^{\frac{\pi i}{4}\operatorname{sgn}\langle z, \rangle|_{\mathsf{Y}}}}{|\det(\frac{1}{2}z\colon \mathsf{Y} \to \mathsf{X})|^{1/2}}.$$

Indeed, using notation of Eqn.(11),

$$\langle y', \Phi(y)\rangle = \Phi(y)(y') = q(y, y') = \frac{1}{2}\langle zy, y'\rangle = \langle y', -\frac{1}{2}zy\rangle.$$

Hence, $\Phi(y) = x$ if and only if $-\frac{1}{2}zy = x$. Thus $\Phi^{-1}(x) = -2z^{-1}x$. Therefore

$$q^*(x, x') = x'(\Phi^{-1}(x)) = \langle \Phi^{-1}(x), x'\rangle = \langle -2z^{-1}x, x'\rangle.$$

Hence, by the definition of $\mathcal{K}$ (155), the assumption that z annihilates X and maps Y into X and Lemma 40, we obtain

$$\begin{aligned}\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -z(x-x'+y), x-x'+y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\ &= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -zy, y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\ &= \int_{\mathsf{Y}} \chi(\frac{1}{2}q(y,y))\chi(-\langle y, -\frac{1}{2}(x+x')\rangle)\, d\mu_{\mathsf{Y}}(y)\\ &= \gamma(q)\chi(-\frac{1}{2}q^*(-\frac{1}{2}(x+x'), -\frac{1}{2}(x+x')))\\ &= \gamma(q)\chi(\langle z^{-1}(-\frac{1}{2}(x+x')), -\frac{1}{2}(x+x')\rangle) = \gamma(q)\chi_{z^{-1}}(x+x').\end{aligned}$$

Therefore

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') \;=\; \Theta(\widetilde{c(z)})\gamma(q)\chi_{z^{-1}}(x+x').$$

But $\Theta(\widetilde{c(z)}) = \pm\left(\frac{i}{2}\right)^n$ (where $\dim \mathrm{W} = 2n$), so that

$$\Theta(\widetilde{c(z)})\gamma(q) = \pm\left(\frac{i}{2}\right)^n \frac{e^{\frac{\pi i}{4}\mathrm{sgn}\langle z\cdot,\cdot\rangle|_{\mathsf{Y}}}}{|\det(\frac{1}{2}z\colon \mathsf{Y}\to\mathsf{X})|^{1/2}} = \pm i^n \frac{e^{\frac{\pi i}{4}\mathrm{sgn}\langle z\cdot,\ \cdot\rangle|_{\mathsf{Y}}}}{|\det(z\colon \mathsf{Y}\to\mathsf{X})|^{1/2}}.$$

Furthermore, by Proposition 61,

$$\mathcal{K}(T(\widetilde{-1}))(x',x'') = \pm i^n\delta_0(x'-x'').$$

Hence, the formula for $\omega(\tilde{g}) = \omega(\widetilde{c(z)}(\widetilde{-1}))$ follows. □

4.9. **An extension of ω to $\widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H}(\mathrm{W})$.** By the Heisenberg group we understand the direct product $\mathrm{H}(\mathrm{W}) = \mathrm{W}\times\mathbb{R}$ with the multiplication given by

$$(w,r)(w',r') = (w+w', r+r'+\frac{1}{2}\langle w,w'\rangle) \qquad ((w,r),(w',r')\in \mathrm{H}(\mathrm{W})).$$

Set

$$T(w,r) = \chi(r)\delta_w \qquad ((w,r)\in\mathrm{H}(\mathrm{W})). \tag{176}$$

Then

$$T:\mathrm{H}(\mathrm{W})\to\mathcal{S}^*(\mathrm{W})$$

is a continuous embedding of the Heisenberg group into the space of the tempered distributions on W.

Theorem 64. *Let $\omega = \mathrm{Op}\circ\mathcal{K}\circ T$. Then*

$$\omega\colon \mathrm{H}(\mathrm{W})\to \mathrm{U}(\mathrm{L}^2(\mathsf{X}))$$

is an injective group homomorphism. For each $v\in\mathrm{L}^2(\mathsf{X})$, the map

$$\mathrm{H}(\mathrm{W})\ni\tilde{g}\to\omega(\tilde{g})v\in\mathrm{L}^2(\mathsf{X})$$

is continuous, so that $(\omega, \mathrm{L}^2(\mathsf{X}))$ *is a unitary representation of the group. Explicitly, for* $v \in \mathrm{L}^2(\mathsf{X})$ *and* $x \in \mathsf{X}$,

$$\omega(x_0, r)v(x) = \chi(r)v(x - x_0) \quad (x_0 \in \mathsf{X}, r \in \mathbb{R})\,, \tag{177}$$
$$\omega(y_0, r)v(x) = \chi(r)\chi(\langle y_0, x\rangle)v(x) \quad (y_0 \in \mathsf{Y}, r \in \mathbb{R})\,,$$

Hence, the representation $(\omega, \mathrm{L}^2(\mathsf{X}))$ *of* $\mathrm{H(W)}$ *is irreducible. Similarly, the representation* $(\omega, \mathcal{S}(\mathsf{X}))$ *of* $\mathrm{H(W)}$ *is irreducible, so that*

$$\mathrm{End}_{\mathrm{H(W)}}(\mathcal{S}(\mathsf{X})) = \mathbb{C} \cdot I. \tag{178}$$

Corollary 65. *For any continuous endomorphism* F *of* $\mathcal{S}(\mathsf{W})$ *which commutes all the twisted convolutions by* δ_w, $w \in \mathsf{W}$, *from the right, there is a unique tempered distribution* $f \in \mathcal{S}^*(\mathsf{W})$ *such that*

$$F : \mathcal{S}(\mathsf{W}) \ni \phi \to f \natural \phi \in \mathcal{S}(\mathsf{W})\,. \tag{179}$$

In other words,

$$\mathrm{End}_{\natural T(H(\mathsf{W}))}(\mathcal{S}(\mathsf{W})) = \mathcal{S}^*(\mathsf{W})\natural\,. \tag{180}$$

Proof. Recall the isomorphisms of linear topological spaces

$$\mathcal{S}(\mathsf{W}) \overset{\mathcal{K}}{\to} \mathcal{S}(\mathsf{X} \times \mathsf{X}) \overset{\mathrm{Op}}{\to} \mathrm{Hom}(\mathcal{S}^*(\mathsf{X}), \mathcal{S}(\mathsf{X})) = \mathcal{S}(\mathsf{X}) \otimes \mathcal{S}(\mathsf{X})\,,$$

where in the last equality we used the reflexivity of the Schwartz space $\mathcal{S}(\mathsf{X})$. Also,

$$\omega : H(\mathsf{W}) \overset{T}{\to} \mathcal{S}^*(\mathsf{W}) \overset{\mathcal{K}}{\to} \mathcal{S}^*(\mathsf{X} \times \mathsf{X}) \overset{\mathrm{Op}}{\to} \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))\,.$$

With the appropriate tensor product, we have

$$\mathrm{End}(\mathcal{S}(\mathsf{X}) \otimes \mathcal{S}(\mathsf{X})) = \mathrm{End}(\mathcal{S}(\mathsf{X})) \otimes \mathrm{End}(\mathcal{S}(\mathsf{X}))\,. \tag{181}$$

It is easy to check that, in these terms, for a fixed $h \in H(\mathsf{W})$, the post-multiplication by $\omega(h)$ on $\mathrm{Hom}(\mathcal{S}^*(\mathsf{X}), \mathcal{S}(\mathsf{X}))$ coincides with $I \otimes \omega(h)$. Hence the conjugation by that post-multiplication is given by $I \otimes Ad(\omega(h))$. Therefore (178) implies that the subspace of $\mathrm{End}(\mathcal{S}(\mathsf{X}) \otimes \mathcal{S}(\mathsf{X}))$ invariant under conjugations by all post-multiplication by $\omega(h)$, $h \in H(\mathsf{W})$, is equal to

$$\mathrm{End}(\mathcal{S}(\mathsf{X})) \otimes I\,. \tag{182}$$

These are the pre-multiplications by elements of $\mathrm{End}(\mathcal{S}(\mathsf{X}))$ on $\mathcal{S}(\mathsf{X}) \otimes \mathcal{S}(\mathsf{X})$.

Since $\mathrm{End}(\mathcal{S}(\mathsf{X})) \subseteq \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))$, and since the twisted convolution translates to the composition of operators, (182) implies the statement of the Corollary. □

Since the metaplectic group acts on the Heisenberg group via automorphisms

$$\tilde{g}(w, r) = (gw, r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}), (w, r) \in \mathrm{H(W)}),$$

we have the semidirect product $\widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H(W)}$, which we embed into the space of the tempered distributions by

$$T(\tilde{g}, (w, r)) = T(\tilde{g})\natural T(w, r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}), (w, r) \in \mathrm{H(W)}). \tag{183}$$

Lemma 42 shows that

$$T(\tilde{g})\natural T(w, r)\natural T(\tilde{g}^{-1}) = T(gw, r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}), (w, r) \in \mathrm{H(W)}). \tag{184}$$

Theorem 66. *Let* $\omega = \mathrm{Op} \circ \mathcal{K} \circ T$. *Then*

$$\omega\colon \widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X}))$$

is an injective group homomorphism. For each $v \in \mathrm{L}^2(\mathsf{X})$, *the map*

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H}(\mathrm{W}) \ni \tilde{g} \to \omega(\tilde{g})v \in \mathrm{L}^2(\mathsf{X})$$

is continuous, so that $(\omega, \mathrm{L}^2(\mathsf{X}))$ *is a unitary representation of the group. In particular,*

$$\omega(\tilde{g})\omega(w,r)\omega(\tilde{g}^{-1}) = \omega(gw,r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}),\ (w,r) \in \mathrm{H}(\mathrm{W})). \tag{185}$$

For a test function $\Phi \in C_c^\infty(\mathrm{H}(\mathrm{W}))$ *define a partial Fourier transform*

$$\Phi_\chi(w) = \int_{\mathbb{R}} \Phi(w,r)\chi(r)\,dr \qquad (w \in \mathrm{W}, r \in \mathbb{R}).$$

Then

$$\mathrm{tr}\,\omega(\Phi) = \Phi_\chi(0). \tag{186}$$

Thus the character of $\omega|_{\mathrm{H}(\mathrm{W})}$ *is equal to the the tensor product* $\delta_0 \otimes \chi$ *of the Dirac delta on* W *and the character* χ *multiplied by the Lebesgue measure on* $\mathbb{R}$.

For test functions $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$ *and* $\Phi \in C_c^\infty(\mathrm{H}(\mathrm{W}))$,

$$\mathrm{tr}\,(\omega(\Psi)\omega(\Phi)) = (T(\Psi)\natural\Phi_\chi)(0) = T(\Psi)(\Phi_\chi). \tag{187}$$

Proof. This is straightforward. For the irreduciblility it is convenient to check that the only bounded operator on $\mathrm{L}^2(\mathsf{X})$ that commutes with the action of the Heisenberg group is a constant multiple of the identity. ☐

4.10. **The oscillator semigroup.** Let $\mathsf{W}_{\mathbb{C}} \ni w \to \overline{w} \in \mathsf{W}_{\mathbb{C}}$ denote the complex conjugation with respect to $\mathsf{W} \subseteq \mathsf{W}_{\mathbb{C}}$. It is easy to check that the formula

$$i\langle w, \overline{w'}\rangle \qquad (w, w' \in \mathsf{W}_{\mathbb{C}}) \tag{188}$$

defines a hermitian form on $\mathsf{W}_{\mathbb{C}}$.

Lemma 67. *Let* $x, y \in \mathfrak{sp}(\mathsf{W})$ *and let* $z = x + iy$ *with* $\det(z-1) \neq 0$. *Then*

$$i\langle w, \overline{w}\rangle > i\langle c(z)w, \overline{c(z)w}\rangle \qquad (w \in \mathsf{W}_{\mathbb{C}} \setminus \{0\}) \tag{189}$$

if and only if

$$\langle yw, w\rangle > 0 \qquad (w \in \mathsf{W} \setminus \{0\})\,. \tag{190}$$

Later we shall abbreviate the condition (190) as $\langle y\ ,\ \rangle > 0$.

Proof. Notice that

$$\begin{aligned}
1 - \overline{c(z)}^{-1}c(z) &= 1 - ((\overline{z}+1)(\overline{z}-1)^{-1})^{-1}(z+1)(z-1)^{-1} \\
&= 1 - (\overline{z}+1)^{-1}(\overline{z}-1)(z+1)(z-1)^{-1} \\
&= (\overline{z}+1)^{-1}((\overline{z}+1)(z-1) - (\overline{z}-1)(z+1))(z-1)^{-1} \\
&= 4i(\overline{z}+1)^{-1}y(z-1)^{-1}\,.
\end{aligned}$$

Hence

$$\begin{aligned} i\langle w,\overline{w}\rangle - i\langle c(z)w,\overline{c(z)w}\rangle &= i\langle (1-\overline{c(z)}^{-1}c(z))w,\overline{w}\rangle \\ &= -4\langle y(z-1)^{-1}w,(-\overline{z}+1)^{-1}\overline{w}\rangle \\ &= 4\langle yw'',\overline{w''}\rangle\,, \end{aligned}$$

where $w''=(z-1)^{-1}w$. Clearly, $w''\neq 0$ if and only if $w\neq 0$. Also,

$$\langle yw'',\overline{w''}\rangle = \langle y\,Re(w''),Re(w'')\rangle + \langle y\,Im(w''),Im(w'')\rangle\,,$$

where $Re(w'')=\frac{1}{2}(w''+\overline{w''})$ and $Im(w'')=\frac{1}{2i}(w''-\overline{w''})$. This completes the proof. □

Corollary 68. *Define*

$$\begin{aligned} &\mathfrak{sp}(\mathsf{W}_{\mathbb{C}})^{++} \\ =\ &\{z=x+iy,\ x,y\in\mathfrak{sp}(\mathsf{W}),\ \det(z-1)\neq 0,\ \langle yw,w\rangle>0,\ w\in\mathsf{W}\setminus\{0\}\} \end{aligned} \tag{191}$$

and

$$\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++} = \{g\in\mathrm{Sp}(\mathsf{W}_{\mathbb{C}});\ \ i\langle w,\overline{w}\rangle > i\langle gw,\overline{gw}\rangle,\ w\in\mathsf{W}_{\mathbb{C}}\setminus\{0\}\}\,. \tag{192}$$

Then $c(\mathfrak{sp}(\mathsf{W}_{\mathbb{C}})^{++})=\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$. Moreover, if $g\in\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$ then $\overline{g}^{-1}\in\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$.

Proof. Notice that the condition

$$i\langle w,\overline{w}\rangle > i\langle gw,\overline{gw}\rangle,\ w\in\mathsf{W}_{\mathbb{C}}\setminus\{0\}$$

implies that 1 is not an eigenvalue of g. Thus

$$\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++} = \{g\in\mathrm{Sp}(\mathsf{W}_{\mathbb{C}});\ \det(g-1)\neq 0,\ i\langle w,\overline{w}\rangle > i\langle gw,\overline{gw}\rangle,\ w\in\mathsf{W}_{\mathbb{C}}\setminus\{0\}\}\,.$$

Since $\det(z-1)\neq 0$ implies $\det(c(z)-1)\neq 0$ and since $c\circ c$ is the identity, the equality $c(\mathfrak{sp}(\mathsf{W}_{\mathbb{C}})^{++})=\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$ follows from Lemma 67. Since, $\overline{c(x+iy)}^{-1}=c(-x+iy)$, the last claim follows. □

Lemma 69. *Consider an element $g\in\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$. If λ is an eigenvalue of g then $|\lambda|\neq 1$. In particular*

$$\det(\frac{1}{2}((\overline{g}^{-1}g)^2+1)) > 1\,. \tag{193}$$

Proof. If w is a g-eigenvector, i.e. $gw=\lambda w$, then $\overline{\lambda}^{-1}\overline{w}=\overline{g}^{-1}\overline{w}$. Also,

$$i\langle w,\overline{w}\rangle > i\langle gw,\overline{gw}\rangle = |\lambda|^2 i\langle w,\overline{w}\rangle\,.$$

Hence $i\langle w,\overline{w}\rangle\neq 0$ and $|\lambda|^2\neq 1$. This verifies the first statement. Since

$$i\langle gw,\overline{gw}\rangle = i\langle\overline{g}^{-1}gw,\overline{w}\rangle$$

we see that the eigenvalues of $\overline{g}^{-1}g$ are real. Therefore the eigenvalues of $(\overline{g}^{-1}g)^2$ are positive. But $(\overline{g}^{-1}g)^2\in\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$. Hence, by the first statement, the eigenvalues of $(\overline{g}^{-1}g)^2$ are positive and come in pairs $\nu\neq\nu^{-1}$. Therefore the determinant (193) is the product of numbers that look like

$$\frac{1}{4}(\nu+1)(\nu^{-1}+1)$$

and therefore are greater than 1. ☐

Lemma 70. *The set* $\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++} \subseteq \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})$ *is a subsemigroup without the identity, closed under the operation* $g \to \overline{g}^{-1}$. *Furthermore*

$$\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\mathrm{Sp}(\mathsf{W}) = \mathrm{Sp}(\mathsf{W})\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++} \subseteq \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++} \tag{194}$$

and

$$\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++} \cup \mathrm{Sp}(\mathsf{W}) \subseteq \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})\,. \tag{195}$$

is a sub-semigroup. Moreover, $\mathrm{Sp}(\mathsf{W})$ *is contained in the closure of* $\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$.

Proof. Everything except the last statement follows from the definition (192). Ler $g \in \mathrm{Sp}(\mathsf{W})$. Then

$$g = c(0)(-1)g = \lim_{y\to 0,\ \langle y\ ,\ \rangle > 0} c(iy)(-1)g,$$

where $c(iy)(-1)g \in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$, by (194). ☐

Notice that the set

$$\{\langle -iz\ ,\ \rangle;\ z = x+iy,\ \langle y\ ,\ \rangle > 0\}$$

coincides with the set of symmetric bilinear forms on $\mathsf{W}_{\mathbb{C}}$ with positive definite real part

$$Re\langle -iz\ ,\ \rangle|_{\mathsf{W}} = \langle y\ ,\ \rangle|_{\mathsf{W}} > 0.$$

Since the determinant of such a form may be identified with $\det(-iz)$, we see that there is a unique holomorphic function

$$\det{}^{1/2}(-iz) \qquad (z \in \mathfrak{sp}(\mathsf{W}_{\mathbb{C}})^{++}) \tag{196}$$

such that

$$\det{}^{1/2}(y) > 0 \qquad (y \in \mathfrak{sp}(\mathsf{W}),\ \langle y\ ,\ \rangle > 0). \tag{197}$$

Set

$$\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++} \;=\; \{(g,\xi);\ g \in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++},\ \xi^2 = \det(i(g-1))^{-1}\} \tag{198}$$

and

$$C(g_1,g_2) \;=\; \det{}^{-1/2}(\frac{1}{2i}(c(g_1)+c(g_2)) \qquad (g_1,g_2 \in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++})\,. \tag{199}$$

Lemma 71. *The set* $\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$ *with the multiplication*

$$(g_1,\xi_1)(g_2,\xi_2) = (g_1g_2, \xi_1\xi_2 C(g_1,g_2)) \tag{200}$$

is a semigroup.

This is the normalized oscillator semigroup of Howe, [20, (11.4)]

Proof. Notice that

$$\begin{aligned}&\frac{1}{2}(c(g_1)+c(g_2)) = \frac{1}{2}(g_1-1)^{-1}((g_1+1)(g_2-1)+(g_1-1)(g_2+1))(g_2-1)^{-1}\\ = \ &(g_1-1)^{-1}(g_1g_2-1)(g_2-1)^{-1}\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\det(i(g_1-1))^{-1}\det(i(g_2-1))^{-1}\det(-\frac{i}{2}(c(g_1)+c(g_2)))^{-1}\\ = \ &\det(i(g_1-1))^{-1}\det(i(g_2-1))^{-1}\det(-i(g_1-1)^{-1}(g_1g_2-1)(g_2-1)^{-1})^{-1}\\ = \ &\det(i(g_1g_2-1))^{-1}.\end{aligned}$$

Hence the product of two elements of $\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$ is contained in $\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$. One still have to check the associativity, i.e. the cocycle property of C:

$$C(g_1,g_2)C(g_1g_2,g_3) = C(g_1,g_2g_3)C(g_2,g_3) \qquad (g_1,g_2,g_3\in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++})\,. \tag{201}$$

Since both sides are holomorphic functions of the three variables we may assume that $g_j = c(iy_j)$, $y_j\in\mathfrak{sp}(\mathsf{W})$, $\langle y_j\ ,\ \rangle > 0$, $j=1,2,3$. Then each function in (201) is positive, so we'll be done as soon as we show that

$$(C(g_1,g_2)C(g_1g_2,g_3))^{-2} = (C(g_1,g_2g_3)C(g_2,g_3))^{-2} \qquad (g_1,g_2,g_3\in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++})\,. \tag{202}$$

In terms of the determinants (202) looks as follows

$$\det(y_1+y_2)\det(-ic(c(iy_1)c(iy_2))+y_3) = \det(y_1-ic(c(iy_2)c(iy_3))\det(y_2+y_3)\,. \tag{203}$$

Since

$$c(c(a)c(b)) = (b-1)(a+b)^{-1}(a-1)+1$$

(203) reduces to

$$\begin{aligned}&\det(y_1+y_2)\det((y_2+i)(y_1+y_2)^{-1}(y_1+i)+y_3-i)\\ = \ &\det(y_1-i+(y_3+i)(y_2+y_3)^{-1}(y_2+i))\det(y_2+y_3)\,.\end{aligned} \tag{204}$$

Further

$$\begin{aligned}&(y_2+i)(y_1+y_2)^{-1}(y_1+i)+y_3-i\\ = \ &(y_2+i)(y_1+y_2)^{-1}(y_1+i+(y_1+y_2)(y_2+i)^{-1}(y_3-i))\end{aligned}$$

and

$$\begin{aligned}&y_1-i+(y_3+i)(y_2+y_3)^{-1}(y_2+i)\\ = \ &((y_1-i)(y_2+i)^{-1}(y_2+y_3)+y_3+i)(y_2+y_3)^{-1}(y_2+i)\,.\end{aligned}$$

Therefore (204) is equivalent to

$$\begin{aligned}&\det(y_1+i+(y_1+y_2)(y_2+i)^{-1}(y_3-i))\\ = \ &\det((y_1-i)(y_2+i)^{-1}(y_2+y_3)+y_3+i)\,.\end{aligned} \tag{205}$$

But

$$(y_1+y_2)(y_2+i)^{-1} = (y_1-i+y_2+i)(y_2+i)^{-1} = (y_1-i)(y_2+i)^{-1}+1$$

and similarly

$$(y_2+i)^{-1}(y_2+y_3)=1+(y_2+i)^{-1}(y_3-i)\,.$$

Hence (205) is equivalent to

$$\begin{aligned}&\det(y_1+(y_1-i)(y_2+i)^{-1}(y_3-i)+y_3)\\ =\ &\det((y_1-i)(y_2+i)^{-1}(y_3-i)+y_1+y_3)\,,\end{aligned}\tag{206}$$

which is true. Thus (202) follows. □

Recall the following holomorphic function

$$chc(x+iy)=\int_{\mathsf{W}}\chi_{x+iy}(w)\,dw=\det{}^{-1/2}(\frac{1}{2i}(x+iy))\qquad (x,y\in\mathfrak{sp}(\mathsf{W}),\ \langle y\cdot,\cdot\rangle>0)\,.\tag{207}$$

This is the reciprocal of the unique holomorphic square root of the determinant of

$$\frac{1}{2i}(x+iy)=\frac{1}{2}(y-ix)\,,$$

which is positive for $x=0$, see (196).

Lemma 72. *The cocycles* (131) *and* (199) *match to form a continuous function*

$$C:\big(\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\cup\mathrm{Sp}(\mathsf{W})\big)\times\big(\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\cup\mathrm{Sp}(\mathsf{W})\big)\to\mathbb{C}^{\times}\,.$$

Proof. Let

$$C':\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\times\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\to\mathbb{C}^{\times}$$

denote the function (131) and let

$$C'':\mathrm{Sp}(\mathsf{W})\times\mathrm{Sp}(\mathsf{W})\to\mathbb{C}^{\times}$$

denote the function (199). We know that both are continuous. Hence it will suffice to check that they match on a dense subset consisting of the pairs

$$(g_1,g_2)\in\big(\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\cup\mathrm{Sp}(\mathsf{W})\big)\times\big(\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\cup\mathrm{Sp}(\mathsf{W})\big)$$

such that

$$\det(g_1-1)\det(g_2-1)\det(g_1g_2-1)\neq 0\,.\tag{208}$$

Since

$$\frac{1}{2}(c(g_1)+c(g_2))=(g_1-1)^{-1}(g_1g_2-1)(g_2-1)^{-1}\,,$$

it is clear from (199) that C' extends to a continuous function on the indicated subset. On the other hand, in terms of Proposition 46, $|\det(\tilde{L})|=1$ and by (207) and (135),

$$chc(c(g_1)+c(g_2))=2^{\dim V}\gamma(\tilde{q}_{g_1,g_2})$$

which shows that

$$C'(g_1,g_2)=C''(g_1,g_2)$$

for $g_1,g_2\in\mathrm{Sp}(\mathsf{W})$ satisfying (208). □

Define $\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{+}=\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}\cup\mathrm{Sp}(\mathsf{W})$ and

$$\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+}=\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}\cup\widetilde{\mathrm{Sp}(\mathsf{W})}\,.\tag{209}$$

Corollary 73. *The set* (209) *with the multiplication*

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2)) \tag{210}$$

is a semigroup. Also, the map

$$\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^+ \ni (g, \xi) \to (g, \xi)^* = (\overline{g}^{-1}, \overline{\xi}) \in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^+ \tag{211}$$

is a well defined involution.

Proof. Only the second claim needs checking, which is easy. □

This is the normalized oscillator semigroup extended by the metaplectic group as in [20, sec. 16].

Corollary 74. *The function*

$$\Theta : \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++} \ni (g, \xi) \to \xi \in \mathbb{C} \tag{212}$$

is holomorphic. It extends to a function

$$\Theta : \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+} \ni (g, \xi) \to \xi \in \mathbb{C}\,, \tag{213}$$

which satisfies the equality

$$\frac{\Theta(\tilde{g}_1 \tilde{g}_2)}{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)} = C(g_1, g_2) \qquad (g_1, g_2 \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+})\,. \tag{214}$$

Moreover, for $g_1, g_2 \in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^+$ *with* $g_1 - 1$ *and* $g_2 - 1$ *invertible*

$$C(g_1, g_2) \;=\; \det{}^{-1/2}(\frac{1}{2i}(c(g_1) + c(g_2)) = chc(c(g_1) + c(g_2))\,. \tag{215}$$

Furthermore, for any test function $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathsf{W}))$

$$\int_{\widetilde{\mathrm{Sp}}(\mathsf{W})} \Theta(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} = \lim_{\tilde{p}\to 1} \int_{\widetilde{\mathrm{Sp}}(\mathsf{W})} \Theta(\tilde{p}\tilde{g})\Psi(\tilde{g})\, d\tilde{g}\,, \tag{216}$$

where the Θ *on the left hand side is defined by* (149)*, the* Θ *on the right hand side by* (213) *and* $\tilde{p} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$.

Notice that the topology of the metaplectic group is not the one inherited from the embedding

$$\widetilde{\mathrm{Sp}}(\mathsf{W}) \subseteq \mathrm{Sp}(\mathsf{W}) \times \mathbb{C}^\times\,,$$

see Definition 51, because the function $\Theta : \widetilde{\mathrm{Sp}}(\mathsf{W}) \to \mathbb{C}^\times$ is not continuous.

Proof. The first claim is obvious. The equality (214) follows from Corollary 73. The first equality in (215) follows from from (199) and the second one from (207).

Fix an element $\widetilde{(-1)} \in \widetilde{\mathrm{Sp}}(\mathsf{W})$ in the preimage of -1 and let $\tilde{c}$ be a real analytic lift of the Cayley transform c such that $\widetilde{(-1)}\tilde{c}(0) = 1$. In order to prove (216) we may assume that Ψ is supported in a neighborhood of 1 contained in the range of $\widetilde{(-1)}\tilde{c}$. Then the integral on the right hand side of (216) is equal to

$$\begin{aligned}\int_{\widetilde{\mathrm{Sp}}(\mathsf{W})} \Theta(\tilde{p}\tilde{g})\Psi(\tilde{g})\,d\tilde{g} &= \int_{\widetilde{\mathrm{Sp}}(\mathsf{W})} \Theta(\tilde{p}\widetilde{(-1)}\tilde{g})\Psi(\widetilde{(-1)}\tilde{g})\,d\tilde{g}\\ &= \int_{\widetilde{\mathrm{Sp}}(\mathsf{W})} \Theta(\tilde{c}(iy)\tilde{g})\Psi(\widetilde{(-1)}\tilde{g})\,d\tilde{g}\\ &= \int_{\mathfrak{sp}(\mathsf{W})} \Theta(\tilde{c}(iy)\tilde{c}(x))\Psi(\widetilde{(-1)}\tilde{c}(x))Jac(x)\,dx\,,\end{aligned}$$

where $Jac(x)$ is the jacobian and, by Corollary 68, $\tilde{p}\widetilde{(-1)} = \tilde{c}(iy)$, with $\langle y\ ,\ \rangle > 0$. We see from (213) and (214) that

$$\Theta(\tilde{c}(iy)\tilde{c}(x)) = \Theta(\tilde{c}(iy))\Theta(\tilde{c}(x))chc(x+iy)\,.$$

Therefore the integral on the right hand side of (216) is equal to

$$\int_{\mathfrak{sp}(\mathsf{W})} \Theta(\tilde{c}(iy))chc(x+iy)\psi(x)\,dx\,, \tag{217}$$

where $\psi(x) = \Theta(\tilde{c}(x))\Psi(\tilde{c}(x))Jac(x)$ is a smooth compactly supported function. Here $\lim_{y\to 0}\Theta(\tilde{c}(iy)) = \Theta(\tilde{c}(0))$ exists. Also,

$$\begin{aligned}\int_{\mathfrak{sp}(\mathsf{W})} chc(x+iy)\psi(x)\,dx &= \int_{\mathfrak{sp}(\mathsf{W})}\int_{\mathsf{W}} \chi_{x+iy}(w)\psi(x)\,dw\,dx\\ &= \int_{\mathsf{W}}\int_{\mathfrak{sp}(\mathsf{W})} \chi_{x+iy}(w)\psi(x)\,dx\,dw\\ &= \int_{\mathsf{W}}\chi_{iy}(w)\int_{\mathfrak{sp}(\mathsf{W})} \chi_{x}(w)\psi(x)\,dx\,dw\,.\end{aligned}$$

Furthermore

$$\int_{\mathfrak{sp}(\mathsf{W})} \chi_x(w)\psi(x)\,dx = \hat{\psi}(\frac{1}{4}\tau(w))\,,$$

where τ is the unnormalized moment map defined in (165) and

$$\hat{\psi}(\xi) = \int_{\mathfrak{sp}(\mathsf{W})} \psi(x)e^{2\pi i\xi(x)}\,dx \qquad (\xi\in\mathfrak{sp}(\mathsf{W})^*)$$

is the Fourier transform of ψ. Also

$$\chi_{iy}(w) = e^{-\frac{\pi}{2}\langle yw,w\rangle} \le 1 \qquad (w\in\mathsf{W})\,.$$

Therefore

$$\lim_{y\to 0}\int_{\mathfrak{sp}(\mathsf{W})} chc(x+iy)\psi(x)\,dx = \int_{\mathsf{W}}\hat{\psi}(\frac{1}{4}\tau(w))\,dw\,, \tag{218}$$

exists.

The function $chc(x)$, $x \in \mathfrak{sp}(\mathsf{W})$, is locally integrable on $\mathfrak{sp}(\mathsf{W})$. This can be seen by checking, on Cartan subsalgebras, that the invariant function

$$\frac{\det(ad(x))}{\det(x)} \qquad (x \in \mathfrak{sp}(\mathsf{W}))$$

is locally bounded and using a theorem of Harish-Chadra saying that

$$|\det(ad(x)-1)|^{-\frac{1}{2}} \qquad (x \in \mathfrak{sp}(\mathsf{W}))$$

is locally integrable.

Notice that the matrix of the symmetric form $\langle J\ ,\ \rangle$ with respect to the basis $e_1, ..., e_n, f_1, ..., f_n$ (235) is the identity I. Also, let B denote the matrix of the form $\langle x\ ,\ \rangle$ with respect to the same basis. Then for $t > 0$, $|chc(x+itJ)|$ is a constant multiple of $|\det(-iB+tI)|^{1/2}$. By diagonalizing B we see that

$$|\det(-iB+tI)| \leq |\det(iB)|\,.$$

Hence

$$|chc(x+itJ)| \leq |chc(x)|\,.$$

Therefore

$$\underset{t\to 0+}{lim}\int_{\mathfrak{sp}(\mathsf{W})} chc(x+itJ)\psi(x)\,dx = \int_{\mathfrak{sp}(\mathsf{W})} chc(x)\psi(x)\,dx\,, \tag{219}$$

where the integral on the right converges absolutely. We see from (218) that the limit on the left hand side of (219) does exist, but since at this point we don't know the Fourier transform of chc, we need (219) to conclude that

$$\underset{y\to 0}{lim}\int_{\mathfrak{sp}(\mathsf{W})} chc(x+iy)\psi(x)\,dx = \int_{\mathfrak{sp}(\mathsf{W})} chc(x)\psi(x)\,dx\,, \tag{220}$$

and (216) follows. □

Theorem 75. *In addition to* (123), *define*

$$t(g) = \chi_{c(g)}\mu_{\mathsf{W}} \qquad (g \in \mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}) \tag{221}$$

and let

$$T(\tilde{g}) = \Theta(\tilde{g})t(g) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+})\,. \tag{222}$$

Then

$$T : \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+} \to \mathcal{S}^*(\mathsf{W}) \tag{223}$$

is a continuous injective map such that

$$T(\tilde{g}_1)\natural T(\tilde{g}_2) \;=\; T(\tilde{g}_1\tilde{g}_2) \qquad (\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+}) \tag{224}$$

and

$$T(\tilde{g}^*) \;=\; T(\tilde{g})^* \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+})\,. \tag{225}$$

Proof. We know from Lemma 57 that the equalities (224) and (225) hold for $\tilde{g}_1, \tilde{g}_2, \tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W})}$. Suppose $\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}(\mathsf{W})}^{++}$. Then the condition (208) holds. Hence a straightforward computation using (207) shows that

$$t(g_1)\natural t(g_2) = chc(c(g_1) + c(g_2))t(g_1 g_2)\,.$$

This together with (215) and (214) verifies (224) for $\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}(\mathsf{W})}^{++}$. Verifying (225) for $\tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W})}$ is straightforward.

The map (223) is clearly injective. Thus we'll be done as soon as we show that it is continuous. The restrictions to $\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$ is obviously continuous and so is the restriction to $\widetilde{\mathrm{Sp}(\mathsf{W})}$, by Lemma (58). Thus it will suffice that T restricted to the set of the $\tilde{g}$ with $\det(g-1) \neq 0$ is continuous, but this is straightforward. □

Lemma 76. *For any $\tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$ the following inequalities hold*

$$0 < \int_{\mathsf{W}} T(\tilde{g}\tilde{g}^*)(w)\,dw \tag{226}$$

and

$$0 < \int_{\mathsf{W}} T((\tilde{g}\tilde{g}^*)^2)(w)\,dw \leq 1\,. \tag{227}$$

Proof. Notice that, by (207),

$$\int_{\mathsf{W}} T(\tilde{g}\tilde{g}^*)(w)\,dw = \Theta(\tilde{g}\tilde{g}^*)chc(c(g\overline{g}^{-1}))\,. \tag{228}$$

We see from (214) that

$$\Theta(\tilde{g}\tilde{g}^*) = \Theta(\tilde{g})\Theta(\tilde{g}^*)C(g, \overline{g}^{-1})$$

and from and (215) that

$$C(g, \overline{g}^{-1}) = chc(c(g\overline{g}^{-1})) > 0\,,$$

because $c(g\overline{g}^{-1}) \in i\mathfrak{sp}(\mathsf{W})$. Since, by (211), $\Theta(\tilde{g}^*) = \overline{\Theta(\tilde{g})}$, the inequality (226) follows. The first inequality in (227) is a particular case of (226).

On the other hand, a straightforward computation shows that

$$|\Theta(\tilde{g})||chc(c(g))| = |\det(\frac{1}{2}(g+1))|^{-\frac{1}{2}} \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++})\,. \tag{229}$$

Hence, the (228) is equal to

$$|\det(\frac{1}{2}((g\overline{g}^{-1})^2 + 1))|^{-\frac{1}{2}} \tag{230}$$

which is greater than 1, by (193). □

Let $\mathrm{C}(\mathrm{L}^2(\mathsf{X}))$ denote the semigroup of contractions on the Hilbert space $\mathrm{L}^2(\mathsf{X})$.

Theorem 77. *Let* $\omega = \mathrm{Op} \circ \mathcal{K} \circ T$. *Then*

$$\omega \colon \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{+} \to \mathrm{C}(\mathrm{L}^2(\mathsf{X}))$$

is an injective semigroup homomorphism. Also, for any $\tilde{p} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$, *the operator* $\omega(\tilde{p})$ *is of trace class with*

$$\mathrm{tr}\, \omega(\tilde{p}) = \Theta(\tilde{p}) \,.$$

In particular

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{p}\tilde{g})\Psi(\tilde{g})\, d\tilde{g} = \mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{p}\tilde{g})\Psi(\tilde{g})\, d\tilde{g} \qquad (\tilde{p} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++},\ \Psi \in C_c^{\infty}(\widetilde{\mathrm{Sp}}(\mathrm{W}))\,.$$

Proof. Let $\tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$. Recall that for each $w \in \mathsf{W}$, the twisted convolution with the Dirac delta δ_w, $\phi \to \delta_w \natural \phi$, is a unitary operator on $\mathrm{L}^2(\mathsf{W})$. Since

$$T(\tilde{g})\natural\phi = \int_{\mathsf{W}} T(\tilde{g})(w)\delta_w\natural\phi\, d\mu_{\mathsf{W}}(w)\,,$$

the norm of the operator

$$\mathrm{L}^2(\mathsf{W}) \ni \phi \to T(\tilde{g})\natural\phi \in \mathrm{L}^2(\mathsf{W})$$

is bounded by the L^1 norm of $T(\tilde{g})$. Lemma 76 shows that L^1 norm of $T((\tilde{g}\tilde{g}^*)^2)$ is at most 1. But

$$T((\tilde{g}\tilde{g}^*)^2)\natural = T(\tilde{g})\natural T(\tilde{g})^*\natural T(\tilde{g})\natural T(\tilde{g})^*\natural$$

Since we work in the C^* algebra of the bounded operators on the Hilbert space $\mathrm{L}^2(\mathsf{W})$, we see that $\parallel T(\tilde{g})\natural \parallel \leq 1$. Hence, $\parallel \omega(\tilde{g}) \parallel \leq 1$. Also, $\omega(\tilde{g})$ is unitary for any $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W})$. Hence the first claim follows.

Since for $\tilde{p} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$, $\mathrm{tr}\, \omega(\tilde{p}) = T(\tilde{p})(0) = \Theta(\tilde{p})$, the second claim is clear. □

For completeness we mention the following theorem proven in [20, sec. 25].

Theorem 78. *Any element* $\tilde{g} \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$ *can be decomposed as*

$$\tilde{g} = \tilde{u}\tilde{p}\,,$$

where $\tilde{u} \in \widetilde{\mathrm{Sp}}(\mathsf{W})$ *and* $\tilde{p} = \tilde{p}^* \in \widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$ *has positive eigenvalues. Then the operator* $\omega(\tilde{u})$ *is unitary and* $\omega(\tilde{p}) = \omega(\tilde{p})^*$ *is positive, so that*

$$\omega(\tilde{g}) = \omega(\tilde{u})\omega(\tilde{p})$$

is the polar decomposition of $\omega(\tilde{g})$.

4.11. **Fock model: a formula of Robinson and Rawnsley.** Fix a positive complex structure J on W and let us view W as a complex vector space the action of J defines the multiplication by $i \in \mathbb{C}$. Then the formula

$$(w, w') = \langle Jw, w' \rangle + i\langle w, w' \rangle \qquad (w, w' \in \mathsf{W}) \tag{231}$$

defines a positive definite Hermitian form on W.

Suppose for a moment that $\dim \mathsf{W} = 2$. Then we may choose a basis e, f of W so that

$$\langle e, e \rangle = \langle f, f \rangle = 0, \quad \langle e, f \rangle = 1, \quad Je = -f, \quad Jf = e. \tag{232}$$

Then the map

$$\mathsf{W} \ni w \to z(w) = \langle e, w \rangle + i\langle w, f \rangle \in \mathbb{C} \tag{233}$$

is a $\mathbb{C}$-linear isomorphism and

$$(w, w') = z(w)\overline{z(w')}\,.$$

By computing in polar coordinates in $\mathbb{C} = \mathbb{R}^2$ we see that the functions

$$f_m(z) = \sqrt{\frac{\pi^m}{m!}}\,\overline{z}^m e^{-\frac{\pi}{2}|z|^2} \qquad (z \in \mathbb{C}, m = 0, 1, 2, ...) \tag{234}$$

form an orthonormal set in $\mathrm{L}^2(\mathbb{C})$. By studying the Taylor expansion of the entire functions centered at zero, we see that in fact these functions form an orthonormal basis of the Hilbert space of the square integrable functions of the form

$$f(z) = h(z)e^{-\frac{\pi}{2}|z|^2},$$

where h is antiholomorphic.

Now we consider the general case with $\dim \mathsf{W} = 2n$. We may choose a basis $e_1, e_2, ...,$ $f_1, f_2, ...$ of W so that for $1 \le j, k \le n$

$$\langle e_j, e_k \rangle = \langle f_j, f_k \rangle = 0, \quad \langle e_k, f_k \rangle = \delta_{j,k}, \quad Je_j = -f_j, \quad Jf_j = e_j. \tag{235}$$

Then the map

$$\begin{aligned} \mathsf{W} \ni w \to z(w) &= (z_1(w), z_2(w), ..., z_n(w)) \in \mathbb{C}^n\,, \\ z_j(w) &= \langle e_j, w \rangle + i\langle w, f_j \rangle \end{aligned} \tag{236}$$

is a $\mathbb{C}$-linear isomorphism and

$$(w, w') = \sum_{j=1}^{n} z_j(w)\overline{z_j(w')} \qquad (w, w' \in \mathsf{W})\,.$$

Let $\mathcal{H}$ denote the space of all the functions of the form $h(z(w))$, where h is antiholomorphic and

$$\int_{\mathsf{W}} |h(z(w))|^2 e^{-\pi(w,w)}\, dw < \infty.$$

This is a Hilbert space with the norm equal to the square root of the above integral. Notice that

$$\chi_{iJ}(w) = e^{-\frac{\pi}{2}(w,w)} \qquad (w, w' \in \mathsf{W})\,.$$

Hence,

$$\mathcal{H}\chi_{iJ} \subseteq \mathrm{L}^2(\mathsf{W})$$

is a closed subspace. Furthermore the map

$$\mathcal{H} \ni \phi \to f = \phi\chi_{iJ} \in \mathcal{H}\chi_{iJ} \tag{237}$$

is is a Hilbert space isomorphism. Also, using the case $\dim \mathsf{W} = 2$ we see that the functions

$$f_\alpha(w) = \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}}\overline{z(w)}^\alpha e^{-\frac{\pi}{2}(w,w)} \qquad (w \in \mathsf{W}, \alpha \in \mathbb{Z}^n_{\geq 0}) \tag{238}$$

form an orthonormal basis of $\mathcal{H}\chi_{iJ}$ and that the functions

$$\phi_\alpha(w) = \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}}\overline{z(w)}^\alpha \qquad (w \in \mathsf{W}, \alpha \in \mathbb{Z}^n_{\geq 0}) \tag{239}$$

form an orthonormal basis of $\mathcal{H}$.

Lemma 79. *Let $\phi \in \mathcal{H}$ and let $f = \phi\chi_{iJ}$. Then for $w, w_0 \in \mathsf{W}$,*

$$\begin{aligned} \delta_{w_0}\natural f(w) &= f(w-w_0)\chi(\frac{1}{2}\langle w_0, w\rangle) \\ &= \phi(w-w_0)e^{-\frac{\pi}{2}(w_0,w_0)}e^{\pi(w_0,w)}\chi_{iJ}(w)\,. \end{aligned} \tag{240}$$

In particular the formula

$$\sigma(w_0,t) = \phi(w-w_0)e^{-\frac{\pi}{2}(w_0,w_0)}e^{\pi(w_0,w)}\chi(t) \tag{241}$$

defines a unitary representation of the Heisenberg group $\mathrm{H}(\mathsf{W})$ *on* $\mathcal{H}$.

Proof. The first equality is obvious from the definition of the twisted convolution, (121). For the second equality we compute

$$\begin{aligned} & \chi(\frac{1}{4}\langle iJ(w-w_0), w-w_0\rangle + \frac{1}{2}\langle w_0, w\rangle) \\ = \; & \chi(\frac{1}{4}\langle iJw, w\rangle + \frac{1}{4}\langle iJw_0, w_0\rangle - \frac{1}{2}\langle iJw_0, w\rangle + \frac{1}{2}\langle w_0, w\rangle) \\ = \; & e^{\frac{\pi}{2}(-\langle Jw,w\rangle - \langle Jw_0,w_0\rangle + 2\langle Jw_0,w\rangle + 2i\langle w_0,w\rangle)} \\ = \; & \chi_{iJ}(w)e^{-\frac{\pi}{2}(w_0,w_0)}e^{\pi(w_0,w)}\,. \end{aligned}$$

Since

$$\| \delta_{w_0}\natural f \| = \| f \|\,,$$

we the lemma follows. ☐

Let X denote the span of the f_j and let Y denote the span of the e_j. The $\mathsf{W} = \mathsf{X} \oplus \mathsf{Y}$ is a complete polarization and we have the corresponding Weyl transform $\mathcal{K}$, as in (155).

Lemma 80. *For $f \in \mathcal{H}\chi_{iJ}$ the function*

$$\mathcal{K}(f)(x,x')e^{\pi x'^2} \qquad (x, x' \in \mathsf{X})\,, \tag{242}$$

where $x'^2 = (x', x')$, does not depend on x'.

Proof. We may assume that $\dim \mathsf{W} = 2$ and that $f = f_m$ is one of the basis elements, as in (234). Let us identify $\mathsf{W} = \mathbb{C}$ by the isomorphism (233). Notice that

$$e^{-\pi x^2}\partial_x e^{\pi x^2} = 2\pi x + \partial_x \tag{243}$$

and that, as a representation of the Lie algebra,

$$\omega(f - ie) = \omega(f) - i\omega(e) = -\partial_x + 2\pi x\,. \tag{244}$$

Furthermore, with the notation (125), for $v \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}&\int_{\mathbb{R}}((2\pi x' + \partial_{x'})\mathcal{K}(f_m)(x,x'))v(x')\,dx' \\ = &\int_{\mathbb{R}}\mathcal{K}(f_m)(x,x')((2\pi x' - \partial_{x'})v(x'))\,dx' \\ = &\int_{\mathbb{R}}\mathcal{K}(f_m\natural(\partial_f - i\partial_e))(x,x')v(x')\,dx'\,.\end{aligned} \tag{245}$$

But

$$\begin{aligned}(f_m\natural(\partial_f - i\partial_e))(w) &= -\pi i\langle f - ie, w\rangle f_m(w) + (\partial_f - i\partial_e) * f_m(w) \\ &= (-\pi(x - iy) - \partial_x + i\partial_y)f_m(x + iy) \\ &= -e^{-\frac{\pi}{2}(x^2+y^2)}(\partial_x - i\partial_y)e^{\frac{\pi}{2}(x^2+y^2)}f_m(x + iy)\,,\end{aligned}$$

where the last equality follows from (243). Since

$$(\partial_x - i\partial_y)(x - iy)^m = 0$$

by the proof is complete. □

Recall the Hermite polynomials $H_k(x)$,

$$H_k(x) = e^{x^2}(-\partial_x)e^{-x^2} = \sum_{j=0}^{k/2}\frac{k!}{j!(k-2j)!}(-1)^j(2x)^{k-2j} \qquad (x \in \mathbb{R},\ k = 0, 1, 2, ...)\,.$$

Lemma 81. *For any $m = 0, 1, 2, ...$,*

$$\mathcal{K}(f_m)(x,x') = \frac{2^{1/4}}{\sqrt{2^m m!}}H_m(\sqrt{2\pi}x)e^{-\pi x^2}\cdot 2^{1/4}e^{-\pi x'^2}\,.$$

Proof. By the definition of the Weyl transform, the left hand side is equal to

$$\begin{aligned}
& \sqrt{\frac{\pi^m}{m!}} \int_{\mathbb{R}} (x-x'-iy)^m e^{-\frac{\pi}{2}((x-x')^2+y^2)} e^{i\pi y(x+x')}\, dy \\
= \quad & \sqrt{\frac{\pi^m}{m!}} e^{-\frac{\pi}{2}(x-x')^2} \sum_{k=0}^{m} \binom{m}{k} (x-x')^{m-k} (-\frac{1}{\pi}\partial_x)^k \int_{\mathbb{R}} e^{-\frac{\pi}{2}y^2} e^{i\pi y(x+x')}\, dy \\
= \quad & \sqrt{\frac{\pi^m}{m!}} e^{-\frac{\pi}{2}(x-x')^2} \sum_{k=0}^{m} \binom{m}{k} (x-x')^{m-k} (-\frac{1}{\pi}\partial_x)^k 2^{1/2} e^{-\frac{\pi}{2}(x+x')^2} \\
= \quad & 2^{1/2}\sqrt{\frac{\pi^m}{m!}} e^{-\pi(x^2+x'^2)} \sum_{k=0}^{m} \binom{m}{k} (x-x')^{m-k} \pi^{-k} e^{\frac{\pi}{2}(x+x')^2} (-\partial_x)^k e^{-\frac{\pi}{2}(x+x')^2} \\
= \quad & 2^{1/2}\sqrt{\frac{\pi^m}{m!}} e^{-\pi(x^2+x'^2)} \sum_{k=0}^{m} \binom{m}{k} (x-x')^{m-k} (\sqrt{2\pi})^{-k} H_k(\sqrt{\frac{\pi}{2}}(x+x')) \\
= \quad & 2^{1/2}\sqrt{\frac{\pi^m}{m!}} e^{-\pi x^2} \sum_{k=0}^{m} \binom{m}{k} x^{m-k} (\sqrt{2\pi})^{-k} H_k(\sqrt{\frac{\pi}{2}}x)\,,
\end{aligned}$$

where the last equality follows from Lemma 80. Furthermore,

$$\begin{aligned}
& \sum_{k=0}^{m} \binom{m}{k} x^{m-k} (\sqrt{2\pi})^{-k} H_k(\sqrt{\frac{\pi}{2}}x) \\
= \quad & \sum_{k=0}^{m} \binom{m}{k} x^{m-k} \pi^{-k} \left(\sqrt{\frac{\pi}{2}}\right)^k H_k(\sqrt{\frac{\pi}{2}}x) \\
= \quad & \sum_{k=0}^{m} \binom{m}{k} x^{m-k} (\sqrt{2\pi})^{-k} \sum_{j=0}^{k/2} \frac{k!}{j!(k-2j)!} (-1)^j (\sqrt{2\pi}x)^{k-2j} \\
= \quad & \sum_{k=0}^{m} \sum_{j=0}^{k/2} \binom{m}{k} \frac{k!}{j!(k-2j)!} (-2\pi)^{-j} x^{m-2j} \\
= \quad & \sum_{j=0}^{m/2} \sum_{2j\le k\le m} \binom{m}{k} \frac{k!}{j!(k-2j)!} (-2\pi)^{-j} x^{m-2j} \\
= \quad & \sum_{j=0}^{m/2} \sum_{2j\le k\le m} \binom{m}{k} \frac{k!}{j!(k-2j)!} (-2\pi)^{-j} x^{m-2j} \\
= \quad & \sum_{j=0}^{m/2} \sum_{2j\le k\le m} \frac{(m-2j)!}{(m-k)!(k-2j)!} \frac{m!}{j!(m-2j)!} (-2\pi)^{-j} x^{m-2j}
\end{aligned}$$

$$= \sum_{j=0}^{m/2} 2^{m-2j} \frac{m!}{j!(m-2j)!}(-2\pi)^{-j} x^{m-2j}$$
$$= (\sqrt{2\pi})^{-m} H_m(\sqrt{2\pi}x)\,,$$

and the formula follows. □

Let

$$v_m(x) = \frac{2^{1/4}}{\sqrt{2^m m!}} H_m(\sqrt{2\pi}x) e^{-\pi x^2} \qquad (x \in \mathbb{R},\ m = 0, 1, 2, ...)\,.$$

Then Lemma 81 shows that

$$\mathcal{K}(f_m)(x, x') = v_m(x) v_0(x') \qquad (x, x' \in \mathsf{X})\,.$$

Hence

$$v_m = \mathrm{Op} \circ \mathcal{K}(f_m) v_0 \qquad (m = 0, 1, 2, ...)\,. \tag{246}$$

Let

$$v_\alpha(x) = v_{\alpha_1}(x_1) v_{\alpha_2}(x_2) ... v_{\alpha_n}(x_n) \qquad (\alpha \in \mathbb{Z}^n_{\geq 0}, x \in \mathbb{R}^n)\,. \tag{247}$$

Let us identify X with $\mathbb{R}^n$ by

$$\mathbb{R}^n \ni x \to x_1 f_1 + x_2 f_2 + ... + x_n f_n \in \mathsf{X}.$$

Then the v_α are functions on X. Also, we see from (238) and (246) that

$$v_\alpha = \mathrm{Op} \circ \mathcal{K}(f_\alpha) v_0 \qquad (\alpha \in \mathbb{Z}^n_{\geq 0})\,. \tag{248}$$

Theorem 82. *The functions* (247) *form an orthonormal basis of the space* $\mathrm{L}^2(\mathsf{X})$*, the functions* (238) *form an orthonormal basis of the space* $\mathcal{H}\chi_{iJ}$ *and the functions* (239) *form an orthonormal basis of the space* $\mathcal{H}$*. The maps*

$$\mathcal{H}\chi_{iJ} \ni f \to \mathrm{Op} \circ \mathcal{K}(f) v_0 \in \mathrm{L}^2(\mathsf{X}) \tag{249}$$

and

$$\mathcal{H} \ni \phi \to \mathrm{Op} \circ \mathcal{K}(\phi\chi_{iJ}) v_0 \in \mathrm{L}^2(\mathsf{X}) \tag{250}$$

are $\mathrm{H}(\mathsf{W})$*-intertwining isometries.*

Proof. As we noticed before, the map

$$\mathcal{H} \ni \phi \to \phi\chi_{iJ} \in \mathcal{H}\chi_{iJ}$$

is an $\mathrm{H}(\mathsf{W})$-intertwining isometry. The left twisted convolution results in in the left multiplication of the operators. Hence (250) is an $\mathrm{H}(\mathsf{W})$-intertwining isometry. Thus the range of the map (250) is an $\mathrm{H}(\mathsf{W})$-invariant closed subspace. Since, by Theorem 66, the group $\mathrm{H}(\mathsf{W})$ acts irreducibly on $\mathrm{L}^2(\mathsf{X})$, the range is equal to $\mathrm{L}^2(\mathsf{X})$. However this range is spanned by the functions v_α. Hence they form an orthonormal basis of $\mathrm{L}^2(\mathsf{X})$. □

Let $\mathcal{H}^{\mathrm{finite}}$ denote the space spanned by finite linear combinations the the basis elements (239) and let

$$\mathcal{H}^\infty = \{\phi \in \mathcal{H};\ \phi\chi_{iJ} \in \mathcal{S}(\mathsf{W})\}\,. \tag{251}$$

Then $\mathcal{H}^{\mathrm{finite}} \subseteq \mathcal{H}^\infty \subseteq \mathcal{H}$ are dense subspace.

Lemma 83. *For $f \in \mathcal{H}^{\infty}\chi_{iJ}$ and for $w, w' \in \mathsf{W}$*

$$\begin{aligned}\delta_{w'}\natural f(w) &= \int_{\mathsf{W}} f(w'')e^{-\frac{\pi}{2}(w'',w'')}e^{-\frac{\pi}{2}(w,w)}e^{\pi(w'',w)} \\ &\qquad e^{-\frac{\pi}{2}(w',w')}e^{\pi((w',w)-(w'',w'))}\,dw''\,,\end{aligned}$$

where the integral is absolutely convergent.

Proof. Write

$$f(w) = \sum_{\alpha}\int_{\mathsf{W}} f(w'')\overline{f_\alpha(w'')}\,dw'' f_\alpha(w) = \int_{\mathsf{W}} f(w'')\left(\sum_{\alpha}\overline{f_\alpha(w'')}f_\alpha(w)\right)\,dw''\,.$$

Then

$$\delta_{w'}\natural f(w) = \int_{\mathsf{W}} f(w'')\left(\sum_{\alpha}\overline{f_\alpha(w'')}f_\alpha(w-w')\chi(\frac{1}{2}\langle w'.w\rangle)\right)\,dw''\,.$$

Furthermore, the term in the parenthesis,

$$\begin{aligned}&\sum_{\alpha}\overline{f_\alpha(w'')}f_\alpha(w-w')\chi(\frac{1}{2}\langle w'.w\rangle)\\ =\;& \sum_{\alpha}\frac{\pi^{|\alpha|}}{\alpha!}e^{-\frac{\pi}{2}(w'',w'')}z(w'')^{\alpha}e^{-\frac{\pi}{2}(w-w',w-w')}z(w-w')^{\alpha}e^{-\frac{\pi}{2}((w,w')-(w',w))}\\ =\;& e^{-\frac{\pi}{2}(w'',w'')}e^{-\frac{\pi}{2}(w,w)}e^{\pi(w'',w-w')}e^{-\frac{\pi}{2}(w',w')}e^{-\frac{\pi}{2}((w,-w')+(-w',w)+(w,w')-(w',w))}\\ =\;& e^{-\frac{\pi}{2}(w'',w'')}e^{-\frac{\pi}{2}(w,w)}e^{\pi(w'',w)}e^{-\frac{\pi}{2}(w',w')}e^{\pi((w',w)-(w'',w'))}\,,\end{aligned}$$

and the formula follows. The integral is absolutely convergent because the function of w'' under the integral is dominated by $e^{-\frac{\pi}{2}(w'',w'')}$. □

Lemma 84. *For $x \in \mathfrak{sp}(\mathsf{W})$, $f \in \mathcal{H}^{\infty}\chi_{iJ}$ and $w \in \mathsf{W}$,*

$$\begin{aligned}\chi_x\natural f(w) &= \int_{\mathsf{W}} f(w'')e^{-\frac{\pi}{2}(w'',w'')}e^{-\frac{\pi}{2}(w,w)}e^{\pi(w'',w)}\\ &\qquad chc(x+iJ)\chi_{(x+iJ)^{-1}}((1+iJ)w+(1-iJ)w'')\,dw''\,,\end{aligned}$$

where the integral is absolutely convergent.

Proof. This follows from Lemma 83. Indeed,

$$\begin{aligned}&\chi_x\natural f(w) = \int_{\mathsf{W}}\chi_x(w')\delta_{w'}\natural f(w)\,dw'\\ =\;& \int_{\mathsf{W}}\chi_x(w')\int_{\mathsf{W}} f(w'')e^{-\frac{\pi}{2}(w'',w'')}e^{-\frac{\pi}{2}(w,w)}e^{\pi(w'',w)}e^{-\frac{\pi}{2}(w',w')}e^{\pi((w',w)-(w'',w'))}\,dw''\,dw'\,.\end{aligned}$$

The function under the double integral is dominated by

$$e^{-\frac{\pi}{2}(w'',w'')}e^{-\frac{\pi}{2}(w',w')}\,.$$

Hence we may change the order of integration and use the formula

$$\begin{aligned}
&\int_{\mathsf{W}} \chi_x(w')e^{-\frac{\pi}{2}(w',w')}e^{\pi((w',w)-(w'',w'))}\,dw' \\
= &\int_{\mathsf{W}} \chi_{x+iJ}(w')\chi(\frac{1}{2}\langle w',(1+iJ)w+(1-iJ)w''\rangle)\,dw' \\
= &\ chc(x+iJ)\chi_{(x+iJ)^{-1}}((1+iJ)w+(1-iJ)w'')\,.
\end{aligned}$$

which follows from (118). □

For $g\in \mathrm{End}(\mathsf{W})$, let $g^J = JgJ^{-1}$.

Lemma 85. *The following inequality holds*

$$\det(\frac{1}{2}(g+g^J))\geq 1 \qquad (g\in \mathrm{Sp}(\mathsf{W}))\,.$$

Proof. Let $g=kp$ be the polar decomposition of g with respect to the positive definite form $\langle J\cdot,\cdot\rangle$, with k orthogonal and p positive. (Since the conjugation by J is a Cartan involution on the group $\mathrm{Sp}(\mathsf{W})$, this is the Cartan decomposition of g.) Then the eigenvalues of p are positive and $p^J=p^{-1}$. Also $t+t^{-1}\geq 2$ for any positive t. Hence,

$$\det(\frac{1}{2}(g+g^J))=\det(\frac{1}{2}(p+p^J))=\det(\frac{1}{2}(p+p^{-1}))\geq 1\,.$$

□

Set

$$C(g)=\frac{1}{2}(g+g^J),\quad A(g)=\frac{1}{2}(g-g^J)\qquad (g\in \mathrm{Sp}(\mathsf{W}))\,. \tag{252}$$

Lemma 86. *Let $x\in\mathfrak{sp}(\mathsf{W})$ be in the domain of the Cayley transform and let $g=c(x)\in$ $\mathrm{Sp}(\mathsf{W})$. Then for $w,w''\in\mathsf{W}$,*

$$\begin{aligned}
&\chi_{(x+iJ)^{-1}}((1+iJ)w+(1-iJ)w'') \qquad (253)\\
= &\ e^{-\frac{\pi}{2}(C(g^{-1})^{-1}A(g^{-1})w,w)}e^{-\frac{\pi}{2}(w'',C(g)^{-1}A(g)w'')}e^{\pi(w'',C(g)^{-1}w)}e^{-\pi(w'',w)}\,.
\end{aligned}$$

Proof. The phase function of the left hand side is equal to

$$\begin{aligned}
&\langle (x+iJ)^{-1}((1+iJ)w+(1-iJ)w''),(1+iJ)w+(1-iJ)w''\rangle \qquad (254)\\
= &\ \langle (1-iJ)(x+iJ)^{-1}(1+iJ)w,w\rangle\\
+ &\ \langle (1+iJ)(x+iJ)^{-1}(1-iJ)w'',w''\rangle\\
+ &\ 2\langle (1+iJ)(x+iJ)^{-1}(1+iJ)w,w''\rangle\,.
\end{aligned}$$

Let $(x+iJ)^{-1}=A+iB$, with A and B real. Then

$$\begin{aligned}
i(1-iJ)(x+iJ)^{-1}(1+iJ) &= J(A+JAJ+JB-BJ)+i(A+JAJ+JB-BJ)\,,\\
i(1+iJ)(x+iJ)^{-1}(1-iJ) &= J(-A-JAJ+JB-BJ)-i(-A-JAJ-JB+BJ)\,,\\
i(1+iJ)(x+iJ)^{-1}(1+iJ) &= J(-A+JAJ+JB+BJ)+i(A-JAJ-JB-BJ)\,.
\end{aligned}$$

Hence the left hand side of (253) is equal to

$$e^{\frac{\pi}{2}((A+JAJ+JB-BJ)w,w)}e^{\frac{\pi}{2}(w'',(-A-JAJ+JB-BJ)w'')}e^{\pi(w'',(-A+JAJ+JB+BJ)w)}\,. \tag{255}$$

Thus in order to complete the proof we need to verify the following equalities

$$-A+JAJ+JB+BJ+1=C(g)^{-1}\,, \tag{256}$$

$$-(A+JAJ+JB-BJ)=C(g^{-1})^{-1}A(g^{-1})\,, \tag{257}$$

$$A+JAJ-JB+BJ=C(g)^{-1}A(g)\,. \tag{258}$$

A straightforward computation shows that

$$A=-x^J(1-xx^J)^{-1} \quad \text{and} \quad B=(1-x^Jx)^{-1}J\,.$$

Notice that

$$x^J(xx^J-1)^{-1}=(x^Jx-1)^{-1}x^J\,. \tag{259}$$

Indeed the difference between the left hand side and the right hand side is equal to

$$(x^Jx-1)^{-1}((x^Jx-1)x^J-x^J(xx^J-1))(xx^J-1)^{-1}=0\,.$$

The left hand side of (256) is equal to

$$\begin{aligned}
& (x^J-1)(1-xx^J)^{-1}+(x-1)(1-x^Jx)^{-1}+1\\
= & \left(1-(x^J-1)(xx^J-1)^{-1}\right)-(x-1)(x^Jx-1)^{-1}\\
= & (x-1)x^J(xx^J-1)^{-1}-(x-1)(x^Jx-1)^{-1}\\
= & (x-1)(x^Jx-1)^{-1}x^J-(x-1)(x^Jx-1)^{-1}\\
= & (x-1)(x^Jx-1)^{-1}(x^J-1)\,,
\end{aligned}$$

where the third equality follows from (259). The right hand side of (256) is equal to

$$\begin{aligned}
& \frac{1}{2}((x+1)(x-1)^{-1}+(x^J-1)^{-1}(x^J+1))^{-1}\\
= & \frac{1}{2}(x-1)((x^J-1)(x+1)+(x^J+1)(x-1))^{-1}(x^J-1)\\
= & (x-1)(x^Jx-1)^{-1}(x^J-1)\,,
\end{aligned}$$

and (256) follows.

The left hand side of (257) is equal to

$$(x+1)(x^Jx-1)^{-1}-(x^J+1)(xx^J-1)^{-1}$$

and the right hand side to

$$(x+1)(x^Jx-1)^{-1}((x+1)-(x^J+1))(x+1)^{-1}$$

Hence, the equality (257) is equivalent to

$$(x^J+1)(xx^J-1)^{-1}=(x+1)(x^Jx-1)^{-1}(x^J+1)(x+1)^{-1}\,,$$

or

$$(x^J+1)(xx^J-1)^{-1}x^J(x+1)=(x+1)(x^Jx-1)^{-1}(x^J+1)\,. \tag{260}$$

The formula (259) implies that (260) is equivalent to

$$x^J(xx^J-1)^{-1}x+(xx^J-1)^{-1}=x(x^Jx-1)^{-1}x^J+(x^Jx-1)^{-1}\,,$$

which is equivalent to

$$x^J(xx^J-1)^{-1}x-(x^Jx-1)^{-1}=x(x^Jx-1)^{-1}x^J-(xx^J-1)^{-1}\,. \tag{261}$$

Again, by (259), (261) is equivalent to

$$(x^Jx-1)^{-1}x^Jx-(x^Jx-1)^{-1}=(xx^J-1)^{-1}xx^J-(xx^J-1)^{-1}\,,$$

which is true. This verifies (257).

Replacing x by $-x$ results in replacing g by g^{-1}. Hence (258) follows from (257). □

Lemma 87. *For $\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathsf{W})$ over $g\in\mathrm{Sp}(\mathsf{W})$ with $g-1$ invertible*

$$(\Theta(\tilde{g})chc(c(g)+iJ))^2=\left(\det(C(g))_{\mathsf{W}_{\mathbb{C},J=-i}}\right)^{-1}\,.$$

Proof. The complexification of W splits into the direct sum of the eigenspaces for J:

$$\mathsf{W}_{\mathbb{C}}=\mathsf{W}_{\mathbb{C},J=-i}\oplus\mathsf{W}_{\mathbb{C},J=i}$$

and the maps

$$\begin{aligned}p^-:\mathsf{W}_{\mathbb{C}}\ni w\to\frac{1}{2}(1+iJ)w\in\mathsf{W}_{\mathbb{C},J=-i}\,,\\ p^+:\mathsf{W}_{\mathbb{C}}\ni w\to\frac{1}{2}(1-iJ)w\in\mathsf{W}_{\mathbb{C},J=i}\end{aligned}$$

are the corresponding projections. The map gp^-+p^+ preserves $\mathsf{W}_{\mathbb{C},J=i}$ and acts on it as the identity. Therefore

$$\det(gp^-+p^+)=\det(p^-gp^-)_{\mathsf{W}_{\mathbb{C},J=-i}}\,.$$

But

$$\begin{aligned}p^-gp^- &= \frac{1}{4}(g+iJg)(1+iJ)\\ &= \frac{1}{4}(g+iJg+giJ+g^J)\,,\end{aligned}$$

so

$$\begin{aligned}p^-gp^-|_{\mathsf{W}_{\mathbb{C},J=-i}} &= \frac{1}{4}(g+g^JiJ+giJ+g^J)\\ &= \frac{1}{4}(2g+2g^J)=C(g)\,.\end{aligned}$$

Therefore

$$\det(gp^-+p^+)=\det(C(g))_{\mathsf{W}_{\mathbb{C},J=-i}}\,. \tag{262}$$

On the other hand

$$\begin{aligned}(\Theta(\tilde{g})chc(c(g)+iJ))^2 &= \det{}^{-1}(i(g-1))\det{}^{-1}(\frac{1}{2i}(c(g)+iJ))\\ &= \det{}^{-1}(i(g-1)\frac{1}{2i}(c(g)+iJ))\\ &= \det{}^{-1}(\frac{1}{2}(g+1+(g-1)iJ))\\ &= \big(\det(gp^-+p^+)\big)^{-1}\,.\end{aligned}$$

This combined with (262) completes the proof. □

Set

$$\| w \|^2 = (w,w) \qquad (w \in \mathsf{W})\,.$$

Lemma 88. *For any* $g \in \mathrm{Sp}(\mathsf{W})$*,* $C(g)^{-1}A(g) \in \mathfrak{sp}(\mathsf{W})$*, and for any* $w \in \mathsf{W}$*,*

$$(w,w) \pm (C(g)^{-1}A(g)w,w) \geq 0\,. \tag{263}$$

Proof. We follow [40, sec.1]. A straightforward computation, using the definition (231), shows that

$$(C(g)w,w') = (w, C(g^{-1})w') \qquad (w,w' \in \mathsf{W}).$$

In other words, the hermitian conjugate $C(g)^*$ of $C(g)$ with respect to the form (231) is equal to $C(g^{-1})$.

We see from (258) that for g with $g-1$ invertible, $C(g)^{-1}A(g) \in \mathfrak{sp}(\mathsf{W})$. Hence by Lemma 84 and by continuity, the claim holds for all $g \in \mathrm{Sp}(\mathsf{W})$. Hence, using the definition (231) again, we check that

$$(C(g)^{-1}A(g)w,w') = \overline{(w,(C(g)^{-1}A(g)w')} \qquad (w,w' \in \mathsf{W})\,.$$

Hence $((C(g)^{-1}A(g))^2)^* = (C(g)^{-1}A(g))^2$.

Using cross multiplication we check that

$$C(g^{-1})^{-1}A(g^{-1}) = -A(g)C(g)^{-1}\,.$$

Hence

$$(1-C(g)^{-1}A(g))C(g^{-1}) = g^{-1}\,.$$

Therefore

$$\begin{aligned}C(g)(1-(C(g)^{-1}A(g))^2)C(g)^* &= C(g)(1+C(g)^{-1}A(g))(1-C(g)^{-1}A(g))C(g^{-1})\\ &= (C(g)+A(g))g^{-1} = 1\,.\end{aligned}$$

Hence

$$\begin{aligned}(w,w) &= (C(g)(1-(C(g)^{-1}A(g))^2)C(g)^*w,w)\\ &= ((1-(C(g)^{-1}A(g))^2)C(g)^*w, C(g)^*w)\,.\end{aligned}$$

Replacing w by $C(g)^{*-1}w = C(g^{-1})^{-1}w$ we see that

$$((1-(C(g)^{-1}A(g))^2)w,w) = (C(g^{-1})^{-1}w, C(g^{-1})^{-1}w)\,.$$

Hence

$$\| C(g)^{-1}A(g)w \|= \sqrt{\| w \|^2 - \| C(g^{-1})^{-1}w \|^2}\,.$$

Cauchy's inequality

$$|(C(g)^{-1}A(g)w, w)| \leq \| C(g)^{-1}A(g)w \| \| w \|$$

shows that

$$\begin{aligned}(w,w) - (C(g)^{-1}A(g)w, w) &\geq (\| w \| - \| C(g)^{-1}A(g)w \|) \| w \| \\ &= (\| w \| - \sqrt{\| w \|^2 - \| C(g^{-1})^{-1}w \|^2}) \| w \|\,,\end{aligned}$$

which verifies the inequality. □

Here is the main theorem of this section

Theorem 89. *The function*

$$\Theta(\tilde{g})chc(c(g)+iJ) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W}))\,, \tag{264}$$

defined by (149) and (207) for $\tilde{g}$ with $g-1$ invertible, extends to a continuous function on the whole group $\widetilde{\mathrm{Sp}}(\mathsf{W})$. Motivated by the formula of Lemma 87, we shall denote this extended function by

$$\det{}^{-1/2}(C(\tilde{g}))_{\mathsf{W}_{\mathbb{C},J=-i}} \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W}))\,. \tag{265}$$

This is the unique continues function whose square is equal to

$$\det{}^{-1}(C(g))_{\mathsf{W}_{\mathbb{C},J=-i}} \qquad (g \in \mathrm{Sp}(\mathsf{W}))\,.$$

Then, for $f \in \mathcal{H}^{\mathrm{finite}}\chi_{iJ}$ and $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathsf{W})$,

$$\begin{aligned}T(\tilde{g})\natural f(w) &= \det{}^{-1/2}(C(\tilde{g}))_{\mathsf{W}_{\mathbb{C},J=-i}} e^{-\frac{\pi}{2}(w,w)} \int_{\mathsf{W}} f(w'')e^{-\frac{\pi}{2}(w'',w'')} \\ &\quad e^{-\frac{\pi}{2}(C(g^{-1})^{-1}A(g^{-1})w,w)} e^{-\frac{\pi}{2}(w'',C(g)^{-1}A(g)w'')} e^{\pi(w'',C(g)^{-1}w)}\, dw''\,,\end{aligned} \tag{266}$$

where the integral is absolutely convergent.

Proof. Lemmas 84 and 86 and 87 imply that (266) holds for g with $g-1$ invertible. By Lemma 58 the left hand side extends to a continuous function. By Lemma 88, the integral on the right does too. Hence (265) follows from Lemma 87 and we get (266) in general. □

In the following corollary we recover a result of Robinson and Rawnsley, [40, (2.4)].

Corollary 90. *The unitary representation $(\omega, \mathrm{L}^2(X))$ is equivalent to $(\sigma, \mathcal{H})$, where for $\phi \in \mathcal{H}^{\mathrm{finite}}$,*

$$\begin{aligned}&\sigma(\tilde{g})\phi(w) = \det{}^{-1/2}(C(\tilde{g}))_{\mathsf{W}_{\mathbb{C},J=-i}} \\ &\int_{\mathsf{W}} \phi(w'')e^{-\frac{\pi}{2}(C(g^{-1})^{-1}A(g^{-1})w,w)} e^{-\frac{\pi}{2}(w'',C(g)^{-1}A(g)w'')} e^{\pi(w'',C(g)^{-1}w)} e^{-\pi(w'',w'')}\, dw''\,,\end{aligned} \tag{267}$$

where the integral is absolutely convergent. In particular, if $g^J = g$, then

$$\sigma(\tilde{g})\phi(w) = \det{}^{-1/2}(\tilde{g})_{\mathsf{W}_{\mathbb{C},J=-i}}\phi(g^{-1}w)\,. \tag{268}$$

Proof. The first part follows from Theorem 89. Since $\sigma(1)\phi = \phi$, i.e.

$$\phi(w) = \int_{\mathsf{W}} \phi(w'')e^{\pi(w'',w)}e^{-\pi(w'',w'')}\,dw''\,,$$

we see that (267) implies (268). □

4.12. **The action of the Lie algebra in the Fock model.** Suppose $x \in \mathfrak{sp}(\mathsf{W})$ is such that $x^J = -x$, in other words x anticommutes with J. Then the function

$$\mathsf{W} \times \mathsf{W} \ni w \to (w, xw) \in \mathbb{C}$$

is holomorphic, in fact a quadratic polynomial. Hence there are unique complex numbers x_{jk} such that

$$(w, xw) = \sum_{j,k=1}^{n} x_{jk} z_j(w) z_k(w) \qquad (w \in \mathsf{W}). \tag{269}$$

Corollary 91. *Let us identify* $\mathsf{W} = \mathbb{C}^n$ *as in* (236)*. Suppose* $x \in \mathfrak{sp}(\mathsf{W})$ *is such that* $x^J = -x$*. Then*

$$\sigma(x) = \frac{\pi}{2}\sum_{j,k=1}^{n} \overline{x}_{jk}\overline{z}_j\overline{z}_k - \frac{1}{2\pi}\sum_{j,k=1}^{n} x_{jk}\partial_{\overline{z}_j}\partial_{\overline{z}_k}\,.$$

Proof. Notice that for $t \in \mathbb{R}$

$$C(\exp(tx)) = \frac{1}{2}(\exp(tx) + \exp(-tx)) = \mathrm{ch}(tx)\,,$$

$$A(\exp(tx)) = \frac{1}{2}(\exp(tx) - \exp(-tx)) = \mathrm{sh}(tx)\,,$$

and

$$C(\exp(tx))^{-1}A(\exp(tx)) = \mathrm{th}(tx)\,.$$

Further,

$$\begin{aligned} &\frac{d}{dt}\left(e^{-\frac{\pi}{2}(\mathrm{th}(-tx)w,w)}e^{-\frac{\pi}{2}(w'',\mathrm{th}(tx)w'')}e^{\pi(w'',ch(tx)w)}\right)_{t=0} \\ = \ &\frac{\pi}{2}((xw,w) - (w'',xw''))e^{\pi(w'',w)} \end{aligned}$$

and, by the chain rule,

$$\frac{d}{dt}\left(\det{}^{1/2}(\mathrm{ch}(tx))_{\mathsf{W}_{\mathbb{C},J=i}}\right)_{t=0} = 0\,.$$

Let $\phi \in \mathcal{H}$ be a finite linear combination of the elements (239) of the basis of $\mathcal{H}$. Then the above calculations show that

$$\sigma(x)\phi(w) = \int_{\mathsf{W}} \phi(w'')\frac{\pi}{2}((xw,w) - (w'',xw''))e^{\pi(w'',w)}e^{-\pi(w'',w'')}\,dw''\,.$$

Notice that

$$\begin{aligned}(w'', xw'')e^{\pi(w'',w)} &= \sum_{j,k=1}^{n} x_{jk} z_j(w'') z_k(w'') e^{\pi \sum_{j=1}^{n} z_j(w'')\overline{z}_j(w'')} \\ &= \sum_{j,k=1}^{n} x_{jk} z_j z_k e^{\pi \sum_{j=1}^{n} z_j \overline{z}_j} \\ &= \frac{1}{\pi^2} \sum_{j,k=1}^{n} x_{jk} \partial_{\overline{z}_j} \partial_{\overline{z}_k} e^{\pi \sum_{j=1}^{n} z_j \overline{z}_j}\end{aligned}$$

and recall that

$$\phi(w) = \int_{\mathsf{W}} \phi(w'') e^{\pi(w'',w)} e^{-\pi(w'',w'')}\, dw''\,.$$

Thus the formula follows. □

Corollary 92. *Let us identify* $\mathsf{W} = \mathbb{C}^n$ *as in* (236)*. Then the image of the complexified Lie algebra* $\mathfrak{sp}(\mathsf{W}_{\mathbb{C}})$ *under the representation* σ *is the* $\mathbb{C}$*-linear span of the following operators*

$$\overline{z}_j \overline{z}_k\,, \quad \partial_{\overline{z}_j} \partial_{\overline{z}_k}\,, \quad \overline{z}_j \partial_{\overline{z}_k} + \partial_{\overline{z}_k} \overline{z}_j \qquad (1 \leq j, k \leq n)$$

If we rename $\overline{z}_j$ to z_j the we obtain the holomorphic functions rather than the anti-holomorphic functions and the usual description of the Fock model, as in [21, (2.2)].

4.13. **Restriction to the maximal compact subgroup $\widetilde{\mathrm{Sp}}(\mathsf{W})^J$ in the Fock model.** In this section we study the restriction of the character Θ to a maximal compact subgroup and to a compact Cartan subgroup as a distribution and as a function and we relate these restrictions to the traces of the restricted representation.

Let G be a real reductive group and let $\mathrm{K} \subseteq \mathrm{G}$ be a maximal compact subgroup. Consider an finite sum Π of irreducible unitary representations of G with the distribution character Θ_Π. Then the intersection of the wave front set of Θ_Π with the conormal bundle to the embedding $\mathrm{K} \to \mathrm{G}$ is empty. This is because there are no non-zero nilpotent elements in $\mathfrak{g}$ which belong to the -1 eigenspace of the Cartan involution, and the fibers of the wave front set consist of nilpotent elements. Hence the restriction

$$\Theta_\Pi|_{\mathrm{K}} \in D'(\mathrm{G}) \tag{270}$$

is well defined. In fact we have the following Lemma

Lemma 93. *For any* $f \in C^\infty(\mathrm{K})$ *the operator* $\Pi(f)$ *is of trace class and*

$$\Theta_\Pi|_{\mathrm{K}}(f) = \mathrm{tr}\ \Pi(f)\,.$$

Proof. The first claim is well known and the proof is similar to the argument we used in the proof of Lemma 107, see [49, 8.1.1]. The second one follows from [14, Theorem 8.2.3], as we shall explain below.

Let $\phi_k \in C_c^\infty(\mathrm{G})$ be approximative identity in the sense that $\phi_k \geq 0$, $\int_{\mathrm{G}} \phi_k(g)\, dg = 1$ and $\operatorname{supp}\phi_k$ tends to the identity if $k \to \infty$. Let $\psi_j \in C_c^\infty(\mathrm{G})$ be such that $\psi_j = 1$ on any compact set in G for large j. Theorem 8.2.3 in [14] implies that

$$\Theta_\Pi|_{\mathrm{K}}(f) = \lim_{j,k\to\infty} \Theta_\Pi(\psi_j(f * \phi_k))\,. \tag{271}$$

For j large ψ_j is equal to 1 on the support of $f * \phi_k$. Hence

$$\Theta_\Pi|_{\mathrm{K}}(f) = \lim_{k\to\infty} \Theta_\Pi(f * \phi_k)\,. \tag{272}$$

Fix $\epsilon > 0$. Then there is $f_\epsilon \in C^\infty(\mathrm{K})$ such that the trace norm $\| f_\epsilon \|_1 < \epsilon$ and $\Pi(f - f_\epsilon)$ is of finite rank. Then

$$\begin{aligned}\Theta_\Pi(f * \phi_k) - \Theta_\Pi(f) &= \operatorname{tr}(\Pi(f)(\Pi(\phi_k) - I)) \\ &= \operatorname{tr}(\Pi(f_\epsilon)(\Pi(\phi_k) - I)) + \operatorname{tr}(\Pi(f - f_\epsilon))(\Pi(\phi_k) - I))\end{aligned}$$

where

$$|\operatorname{tr}(\Pi(f_\epsilon)(\Pi(\phi_k) - I))| \leq \| \Pi(f_\epsilon) \|_1 \| \Pi(\phi_k) - I \| \leq \epsilon \cdot 2$$

and

$$\lim_{k\to\infty} \operatorname{tr}(\Pi(f - f_\epsilon))(\Pi(\phi_k) - I)) = 0\,.$$

□

Corollary 94. *In terms of distributions on* K*,*

$$\Theta_\Pi|_{\mathrm{K}} = \sum_{\sigma\in\hat{\mathrm{K}}} m_\sigma \Theta_\sigma\,,$$

where m_σ is the multiplicity of σ in Π. In particular

$$m_\sigma = \Theta_\Pi|_{\mathrm{K}}(\check{\Theta}_\sigma)\,,$$

where $\check{\Theta}_\sigma(k) = \Theta_\sigma(k^{-1})$.

Suppose G has a Cartan subgroup $\mathrm{T} \subseteq \mathrm{K}$. There are examples where the intersection of the conormal bundle of the embeding $\mathrm{T} \subseteq \mathrm{G}$ with the wave front set of Θ_Π is non-empty. Hence the restriction $\Theta_\Pi|_{\mathrm{T}}$ may not exist. Nevertheless, we'll see below that it does exist when G is the metaplectic group and Π is the Weil representation. However in general, Harish-Chandra's regularity theorem impies that $\Theta_\Pi|_{\mathrm{G}^{\mathrm{reg}}}$ is a function. Hence the restriction $\Theta_\Pi|_{\mathrm{T}^{\mathrm{reg}}}$ exists, as a function.

Corollary 95. *Suppose the rank of* K *is equal to the rank of* G*. Then the following series converges*

$$\Theta_\Pi(k) = \sum_{\sigma\in\hat{\mathrm{K}}} m_\sigma \Theta_\sigma(k) \qquad (k \in \mathrm{K} \cap \mathrm{G}^{\mathrm{reg}})\,. \tag{273}$$

Proof. The restriction of Θ_Π to $\mathrm{G}^{\mathrm{reg}}$ is a function, which may be further restricted to $\mathrm{K}\cap\mathrm{G}^{\mathrm{reg}}$. By the equality of rank assumption, this set is open in K and not empty. We know from Corollary 94 that for any test function $f\in C^\infty(\mathrm{K})$

$$\Theta_\Pi|_{\mathrm{K}}(f)=\sum_{\sigma\in\hat{\mathrm{K}}} m_\sigma \int_{\mathrm{K}} \Theta_\sigma(k) f(k)\,dk\,.$$

In particular, for $f\in C^\infty(\mathrm{K}\cap\mathrm{G}^{\mathrm{reg}})$,

$$\int_{\mathrm{K}} \Theta_\Pi(k) f(k)\,dk=\sum_{\sigma\in\hat{\mathrm{K}}} m_\sigma \int_{\mathrm{K}} \Theta_\sigma(k) f(k)\,dk\,.$$

Since Θ_Π and each Θ_σ are real analytic on $\mathrm{K}\cap\mathrm{G}^{\mathrm{reg}}$, taking f close to the Dirac delta at k implies (273). □

Notice that the following equation

$$\Theta_\Pi|_{\mathrm{K}}(f)=\int_{\mathrm{K}\cap\mathrm{G}^{\mathrm{reg}}} \Theta_\Pi(k) f(k)\,dk \qquad (f\in C^\infty(\mathrm{K}))\,. \tag{274}$$

is not true in general. In fact the integral on the right may not be convergent. The holomorphic discrete series of $\mathrm{G}=\mathrm{SL}_2(\mathbb{R})$ provides such an example.

From now on we study the case $\mathrm{G}=\widetilde{\mathrm{Sp}}(\mathsf{W})$, $\mathrm{K}=\widetilde{\mathrm{Sp}}(\mathsf{W})^J$ and $\Pi=\omega$. Let

$$\mathsf{W}=\bigoplus_{j=1}^n \mathsf{W}_j \tag{275}$$

be a direct sum orthogonal (with respect to the symplectic form) decomposition preserved by J, with $\dim\mathsf{W}_j=2$. Denote by J_j the restriction of J to W_j. Then

$$\mathfrak{t}=\bigoplus_{j=1}^n \mathbb{R}J_j \tag{276}$$

is an elliptic Cartan subalgebra of $\mathfrak{sp}(\mathsf{W})$. Denote by

$$\exp:\mathfrak{sp}(\mathsf{W})\to\mathrm{Sp}(\mathsf{W}) \ \text{ and } \ \widetilde{\exp}:\mathfrak{sp}(\mathsf{W})\to\widetilde{\mathrm{Sp}}(\mathsf{W}) \tag{277}$$

the exponential maps. Then $\mathrm{T}=\exp(\mathfrak{t})\subseteq\mathrm{Sp}(\mathsf{W})$ and $\widetilde{\mathrm{T}}=\widetilde{\exp}(\mathfrak{t})\subseteq\widetilde{\mathrm{Sp}}(\mathsf{W})$ are compact Cartan subgroups.

Lemma 96. *Suppose $w\in\mathsf{W}$ is such that*

$$\langle xw,w\rangle=0 \qquad (x\in\mathfrak{t})\,.$$

Then $w=0$.

Proof. Let $w=\sum_{j=1}^n w_j$ according to the decomposition (275) and let $x=\sum_{j=1}^n x_jJ_j$, as in (277). Then

$$0=\langle xw,w\rangle=\sum_j\langle xw_j,w_j\rangle=\sum_j x_j\langle J_jw_j,w_j\rangle$$

Hence

$$\langle J_j w_j, w_j\rangle = 0 \qquad j = 1, 2, ..., n)\,.$$

But $\langle J_j\ ,\ \rangle > 0$. Hence each $w_j = 0$ □

Corollary 97. *The intersection of the conormal bundle of the embedding* $\widetilde{\mathrm{T}} \subseteq \widetilde{\mathrm{Sp}}(\mathsf{W})$ *with the wave front set of* Θ *is empty. Hence the restriction* $\Theta|_{\widetilde{\mathrm{T}}}$ *is a well defined distribution on* $\widetilde{\mathrm{T}}$.

Proof. We identify the cotangent bundle to $\widetilde{\mathrm{Sp}}(\mathsf{W})$ with $\widetilde{\mathrm{Sp}}(\mathsf{W}) \times \mathfrak{sp}(\mathsf{W})^*$ as in [19, (1.7)] and use the fact that the wave front set of Θ is contained in $\widetilde{\mathrm{Sp}}(\mathsf{W}) \times \tau(\mathsf{W})$, proven in [38]. Then the intersection in question consists of points $(\tilde{t}, \tau(w)) \in \widetilde{\mathrm{T}} \times \tau(\mathsf{W})$ such that $\tau(w) \neq 0$ and $\tau(w)|_{\mathfrak{t}} = 0$. But then, according to Lemma 96, $\tau(w) = 0$. □

The following proposition describes the multiplication in the maximal compact subgroup $\widetilde{\mathrm{Sp}}(\mathsf{W})^J \subseteq \widetilde{\mathrm{Sp}}(\mathsf{W})$.

Proposition 98. *The map*

$$\widetilde{\mathrm{Sp}}(\mathsf{W})^J \ni \tilde{g} \to (g, \det{}^{-1/2}(\tilde{g})_{\mathsf{W}_{\mathbb{C}, J=-i}}) \in \mathrm{Sp}(\mathsf{W})^J \times \mathbb{C}^\times \tag{278}$$

is an injective group homomorphism which justifies the following identification

$$\widetilde{\mathrm{Sp}}(\mathsf{W})^J = \{(g, \zeta) \in \mathrm{Sp}(\mathsf{W})^J \times \mathbb{C}^\times;\ \zeta^2 = \det{}^{-1}(g)_{\mathsf{W}_{\mathbb{C}, J=-i}}\}\,. \tag{279}$$

In particular $\widetilde{\mathrm{Sp}}(\mathsf{W})^J$ *is an algebraic group.*

Proof. This is clear from Corollary 90. □

Notice that

$$\left(\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}\right)^J$$

is the complexification of $\widetilde{\mathrm{Sp}}(\mathsf{W})^J$ and

$$\left(\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}\right)^{\mathrm{T}}$$

is the complexification of $\widetilde{\mathrm{T}}$.

Corollary 99. *In terms of distributions equal to limits of holomorphic functions,*

$$\Theta|_{\widetilde{\mathrm{Sp}}(\mathsf{W})^J}(\tilde{k}) = \lim_{\tilde{p}\to 1} \Theta(\tilde{p}\tilde{k}) \qquad (\tilde{k} \in \widetilde{\mathrm{Sp}}(\mathsf{W})^J,\ \tilde{p} \in \left(\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}\right)^J) \tag{280}$$

and

$$\Theta|_{\widetilde{\mathrm{T}}}(\tilde{t}) = \lim_{\tilde{p}\to 1} \Theta(\tilde{p}\tilde{t}) \qquad (\tilde{t} \in \widetilde{\mathrm{T}},\ \tilde{p} \in \left(\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}\right)^{\mathrm{T}})\,. \tag{281}$$

Proof. Theorems 3.1.15 and 8.2.4 in [14] imply that if a distribution is the limit of a holomorphic function, then the restriction of it to a real analytic manifold is equal to the limit of the holomorphic function restricted to the local complexification of that manifold. Hence the formulas follow. □

Lemma 100. *Define* $\mathfrak{t}_{\mathbb{C}}^{++}=\mathfrak{t}_{\mathbb{C}}\cap\mathfrak{sp}(\mathsf{W}_{\mathbb{C}})^{++}$, $\mathrm{T}_{\mathbb{C}}^{++}=\mathrm{T}_{\mathbb{C}}\cap\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})^{++}$ *and* $\widetilde{\mathrm{T}}_{\mathbb{C}}^{++}=\widetilde{\mathrm{T}}_{\mathbb{C}}\cap\widetilde{\mathrm{Sp}(\mathsf{W}_{\mathbb{C}})}^{++}$. *Then*

$$\mathfrak{t}_{\mathbb{C}}^{++}=\{\sum_{j=1}^{n}(x_j+iy_j)J_j,\ y_j>0,\ 1\leq j\leq n\}\,, \tag{282}$$

$$\mathrm{T}_{\mathbb{C}}^{++}=\exp(\mathfrak{t}_{\mathbb{C}}^{++}) \tag{283}$$

and

$$\widetilde{\mathrm{T}}_{\mathbb{C}}^{++}=\widetilde{\exp}(\mathfrak{t}_{\mathbb{C}}^{++})\,. \tag{284}$$

Proof. The equality (282) is obvious from definitions. In order to verify the remaining equalities we may assume that $n=1$. Corollary 68 shows that $\mathrm{T}_{\mathbb{C}}^{++}=c(\mathfrak{t}_{\mathbb{C}}^{++})$. For a complex number $z\neq\pm i$ we have

$$\begin{aligned} c(zJ) &= (zJ+1)(zJ-1)^{-1}=(zJ+1)\frac{1}{z^2+1}(-zJ-1)\\ &= \frac{z^2-1}{z^2+1}I-\frac{2z}{z^2+1}J \end{aligned}$$

and for another complex number u,

$$\exp(uJ)\ =\ \cos(u)I+\sin(u)J\,.$$

Hence $\exp(uJ)=c(zJ)$ is equivalent to

$$\cos(u)=\frac{z^2-1}{z^2+1}\quad\text{and}\quad\sin(u)=-\frac{2z}{z^2+1}\,, \tag{285}$$

which implies

$$e^{iu}=\cos(u)+i\sin(u)=\frac{z^2-i2z-1}{z^2+1}=\frac{z-i}{z+i}\,. \tag{286}$$

Recall that given $z\neq i$ in the upper half plane, $\frac{z-i}{z+i}$ is in the unit disc and therefore there is u in the upper half plane such that (286) holds. Since

$$\cos(u)=\frac{1}{2}(e^{iu}+e^{-iu})=\frac{1}{2}\left(\frac{z-i}{z+i}+\frac{z+i}{z-i}\right)=\frac{z^2-1}{z^2+1}$$

and

$$\sin(u)=\frac{1}{2i}(e^{iu}-e^{-iu})=\frac{1}{2i}\left(\frac{z-i}{z+i}-\frac{z+i}{z-i}\right)=\frac{-2z}{z^2+1}\,,$$

the equality (285) holds for this u. Thus the left hand side of (283) is contained in the right hand side.

Conversely, given u in the upper half plane, we solve (286) for z in the upper half plane and get the equality (285). This verifies (283).

Since $\widetilde{\exp}:\mathfrak{t}_{\mathbb{C}}\to\widetilde{\mathrm{T}}_{\mathbb{C}}$ is surjective, (283) implies (284). □

Proposition 101. *For $z \in \mathfrak{t}_{\mathbb{C}}^{++}$, with $z = \sum_{j=1}^{n} z_j J_j$, $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $y_j > 0$, the following formula holds,*

$$\Theta(\widetilde{\exp}(z)) = \frac{\prod_{j=1}^{n} e^{i\frac{z_j}{2}}}{\prod_{j=1}^{n}(1 - e^{iz_j})}\,.$$

(Here iz_j are the eigenvalues of z on $\mathsf{W}_{\mathbb{C},J=i}$.)

Proof. We see from (284) that

$$\begin{aligned}
(\Theta(\widetilde{\exp}(z)))^2 &= \det(i(\exp(z) - 1))^{-1} \\
&= \left(\det(i(\exp(z) - 1))_{\mathsf{W}_{\mathbb{C},J=-i}} \det(i(\exp(z) - 1))_{\mathsf{W}_{\mathbb{C},J=i}}\right)^{-1} \\
&= \left(\prod_{j=1}^{n}(i(e^{-iz_j} - 1)(e^{iz_j} - 1))\right)^{-1} \\
&= \prod_{j=1}^{n} e^{iz_j} \left(\prod_{j=1}^{n}(1 - e^{iz_j}))\right)^{-2}\,.
\end{aligned}$$

Since the function $\Theta(\widetilde{\exp}(z))$ is holomorphic and since the set $\mathfrak{t}_{\mathbb{C}}^{++}$ is simply connected, we must have either

$$\Theta(\widetilde{\exp}(z)) = \prod_{j=1}^{n} e^{\frac{iz_j}{2}} \left(\prod_{j=1}^{n}(1 - e^{iz_j}))\right)^{-1}$$

or

$$\Theta(\widetilde{\exp}(z)) = -\prod_{j=1}^{n} e^{\frac{iz_j}{2}} \left(\prod_{j=1}^{n}(1 - e^{iz_j}))\right)^{-1}\,.$$

Let $y = \sum_{j=1}^{n} y_j J_j$, with all $y_j > 0$ and let $\tilde{p} = \widetilde{\exp}(iy)$. Then $\tilde{p}^2 \in \widetilde{\mathrm{T}}_{\mathbb{C}}^{++}$ and (214) shows that

$$\frac{\Theta(\tilde{p}^2)}{\Theta(\tilde{p})^2} = chc(2c(p)) > 0\,.$$

As we just computed, $\Theta(\tilde{p})^2 > 0$. Hence $\Theta(\tilde{p}^2) > 0$ and the formula follows. □

Corollary 102. *Let $t, t' \in \mathrm{T}_{\mathbb{C}}^{++}$ have eigenvalues t_j and t'_j on the space $\mathsf{W}_{\mathbb{C},J=i}$. Let $\tilde{t}, \tilde{t}' \in \widetilde{\mathrm{T}}_{\mathbb{C}}^{++}$ be some elements in the preimage under the covering map. We assume that they are determined by the ambiguity of the square roots*

$$\left(\prod_{j=1}^{n} t_j\right)^{\frac{1}{2}} \quad and \quad \left(\prod_{j=1}^{n} t'_j\right)^{\frac{1}{2}}$$

so that

$$\Theta(\tilde{t}) = \frac{\left(\prod_{j=1}^{n} t_j\right)^{\frac{1}{2}}}{\prod_{j=1}^{n}(1 - t_j)} \quad and \quad \Theta(\tilde{t}') = \frac{\left(\prod_{j=1}^{n} t'_j\right)^{\frac{1}{2}}}{\prod_{j=1}^{n}(1 - t'_j)}\,.$$

Then

$$\Theta(\tilde{t}\tilde{t}') = \frac{\left(\prod_{j=1}^n t_j\right)^{\frac{1}{2}} \left(\prod_{j=1}^n t_j'\right)^{\frac{1}{2}}}{\prod_{j=1}^n (1-t_jt_j')}\,.$$

Proof. We see from (214) that the last formula will follow if we check that

$$chc(c(t)+c(t')) = \frac{\prod_{j=1}^n (1-t_j)(1-t_j')}{\prod_{j=1}^n (1-t_jt_j')}. \tag{287}$$

In terms of Proposition 101 we have

$$\begin{aligned}
\Theta(\widetilde{\exp}(z)) &= \frac{\prod_{j=1}^n e^{i\frac{z_j}{2}}}{\prod_{j=1}^n (1-e^{iz_j})}\,,\\
\Theta(\widetilde{\exp}(z')) &= \frac{\prod_{j=1}^n e^{i\frac{z_j'}{2}}}{\prod_{j=1}^n (1-e^{iz_j'})}\,,\\
\Theta(\widetilde{\exp}(z+z')) &= \frac{\prod_{j=1}^n e^{i\frac{z_j+z_j'}{2}}}{\prod_{j=1}^n (1-e^{i(z_j+z_j')})}\,.
\end{aligned}$$

Hence, by (214),

$$chc(z+z') = \prod_{j=1}^n \frac{(1-e^{iz_j})(1-e^{iz_j'})}{1-e^{iz_j}e^{iz_j'}}\,,$$

which coincides with (287). □

We normalize the Haar measure on the group $\widetilde{\mathrm{T}}$ so that the total mass is 1. Then Proposition 101 and (281) imply the following corollary.

Corollary 103. *For any $\Psi \in \mathbb{C}^\infty(\widetilde{\mathrm{T}})$,*

$$\Theta|_{\widetilde{\mathrm{T}}}(\Psi) = \lim_{y\to 0} \frac{1}{(4\pi)^n} \int_0^{4\pi} \dots \int_0^{4\pi} \Psi(\widetilde{\exp}(\sum_{j=1}^n x_jJ_j)) \frac{\prod_{j=1}^n e^{\frac{ix_j-y_j}{2}}}{\prod_{j=1}^n (1-e^{ix_j-y_j})}\,,$$

where all $y_j > 0$.

The space $\mathcal{H}^{\mathrm{finite}}$, defined just before (251), is equal to the space of the antiholomorphic polynomials on W. Let $\mathcal{H}^{(m)} \subseteq \mathcal{H}^{\mathrm{finite}}$ denote the subspace of the polynomials homogeneous of degree $m = 0, 1, 2, ...$ Let ρ denote the permutation action of the group $\mathrm{Sp}(\mathsf{W})^J$ on these spaces:

$$\rho(g)\phi(w) = \phi(g^{-1}w)\,.$$

Define

$$\lambda_j : \mathfrak{t} \ni x = \sum_{j=1}^n x_jJ_j \to x_j \in \mathbb{R}\,.$$

We choose the positive root system of $\mathfrak{t}$ in $\mathfrak{sp}(\mathsf{W}_\mathbb{C})^J$ so that in the resulting order, $\lambda_1 > \lambda_2 > ... > \lambda_n$.

Lemma 104. *The representation $(\rho, \mathcal{H}^{(m)})$ is irreducible, with the highest weight $mi\lambda_1$. Moreover $(\rho|_{\mathrm{T}}, \mathcal{H}^{(m)})$ decomposes into the direct sum of one-dimensional representation each occurring with multiplicity one,*

$$\mathcal{H}^{(m)} = \sum_{|a|=m} \mathcal{H}_a^{(m)} ,$$

where $a = (a_1, ..., a_n)$, with all a_j non-negative integers, $|a| = a_1 + ... + a_n$, and

$$\rho(\exp(x))v = e^{ia_1x_1+...+ia_nx_n} v \qquad (v \in \mathcal{H}_\alpha^{(m)}) .$$

Proof. We may assume that the decomposition (275) is consistent with (236). Then we see from (236) that

$$\overline{z_j(\exp(-x_jJ_j)w_j)} = e^{ix_j}\overline{z_j(w_j)} .$$

Hence, in terms of (239),

$$\phi_a(\exp(-x)w) = e^{iax_1+...+ia_nx_n}\phi_a(w) .$$

Thus ϕ_a is a weight vector with weight $a_1i\lambda_1 + ... + a_ni\lambda_n$. The polynomials ϕ_a with $|a| = m$ form a basis of the space $\mathcal{H}^{(m)}$. Any non-zero $\mathrm{Sp}(\mathsf{W})^J$-invariant subspace is a sum of the weight spaces and they are permuted by the Weyl group $W(\mathrm{T})$, permuting the W_j. Hence the representation is irreducible. Clearly $mi\lambda_1$ is the highest weight. □

Recall the representation $(\sigma, \mathcal{H})$, defined and proven to be unitarily equivalent to $(\omega, \mathrm{L}^2(\mathsf{X}))$ in Corollary 90.

Corollary 105. *The representation $(\sigma|_{\widetilde{\mathrm{Sp}(\mathsf{W})^J}}, \mathcal{H}^{(m)})$ is irreducible, with the highest weight*

$$mi\lambda_1 + \frac{1}{2}\sum_{j=1}^{n} i\lambda_j .$$

Moreover $(\sigma|_{\widetilde{\mathrm{T}}}, \mathcal{H}^{(m)})$ decomposes into the direct sum of one-dimensional representation each occurring with multiplicity one,

$$\mathcal{H}^{(m)} = \sum_{|a|=m} \mathcal{H}_a^{(m)} ,$$

where $a = (a_1, ..., a_n)$, with all a_j non-negative integers, $|a| = a_1 + ... + a_n$, and

$$\sigma(\widetilde{\exp}(x))v = e^{i(a_1+\frac{1}{2})x_1+...+i(a_n+\frac{1}{2})x_n} v \qquad (v \in \mathcal{H}_a^{(m)}) .$$

Lemma 106. *For any $\Psi \in C^\infty(\widetilde{\mathrm{T}})$, $\sigma(\Psi) \in \mathrm{End}(\mathcal{H})$ is a trace class operator and*

$$\Theta_{\widetilde{\mathrm{T}}}(\Psi) = \mathrm{tr}\,\sigma(\Psi) = \sum_{|a|=m} \hat{\Psi}(a_1 + \frac{1}{2}, ..., a_n + \frac{1}{2}) , \tag{288}$$

where

$$\hat{\Psi}(b_1, ..., b_n) = \frac{1}{(4\pi)^n}\int_0^{4\pi} ... \int_0^{4\pi} \Psi(\widetilde{\exp}(\sum_{j=1}^{n} x_j J_j))e^{i(b_1x_1+...+i(b_nx_n)}\, dx_1...dx_n \qquad (289)$$

is the Forier coefficient of the periodic function at the indicated point.

Proof. Since the a occur with multiplicity one, Plancherel formula implies that $\sigma(\Psi)$ is a Hilbert-Schmidt operator, and hence of trace class. The second equality in (288) is obvious. The first one is verified as in the proof of Lemma 93. □

The following lemma is a particular case of the first part of Lemma 93. We include a proof for completeness of our study of the Weil representation.

Lemma 107. *For any smooth function Ψ on the group $\mathrm{Sp}(\mathsf{W})^J$, the operator $\rho(\Psi) \in \mathrm{End}(\mathcal{H})$ is of trace class.*

Proof. Recall the Harish-Chandra isomorphism

$$\gamma : \mathcal{U}(\mathfrak{sp}(\mathsf{W})^J_{\mathbb{C}})^{\mathrm{Sp}(\mathsf{W})^J} \to \mathcal{U}(\mathfrak{t}_{\mathbb{C}})^{W(\mathrm{T})} = \mathbb{C}[\mathfrak{t}^*_{\mathbb{C}}]^{W(\mathrm{T})}\,,$$

[10, Lemma 19]. Let $\mathcal{C} \in \mathcal{U}(\mathfrak{sp}(\mathsf{W})^J_{\mathbb{C}})^{\mathrm{Sp}(\mathsf{W})^J}$ be such that

$$\gamma(\mathcal{C})(z_1 i\lambda_1 + ... + z_n i\lambda_n) = (2z_1)^2 + ... + (2z_n)^2\,.$$

The sum of the positive roots multiplied by $\frac{1}{2}$ is equal to

$$i\delta = \sum_{j=1}^{n} \frac{n+1-2j}{2} i\lambda_j\,.$$

In particular

$$mi\lambda_1 + i\delta = \frac{n-1+2m}{2} i\lambda_1 + \sum_{j=2}^{n} \frac{n+1-2j}{2} i\lambda_j\,.$$

Hence $\mathcal{C}$ acts on $\mathcal{H}^{(m)}$ via multiplication by

$$(n-1+2m)^2 + \sum_{j=2}^{n} (n+1-2j)^2\,.$$

Also,

$$\dim \mathcal{H}^{(m)} = \frac{(m+n-1)!}{m!(n-1)!}$$

is a polynomial of degree $n-1$ in the variable m. Therefore

$$\sum_{m=0}^{\infty} \frac{\dim \mathcal{H}^{(m)}}{\rho(\mathcal{C}^n)|_{\mathcal{H}^{(m)}}} < \infty\,.$$

In other words, $\rho(\mathcal{C}^n)^{-1}$ is a trace class operator on $\mathcal{H}$. Since

$$\rho(\Psi) = \rho(\mathcal{C}^n)^{-1}\rho(C^n\Psi)$$

and since $\rho(C^n\Psi)$ is a bounded operator, we see that the operator $\rho(\Psi)$ is of trace class. □

5. Rossmann's formula for Θ.

The group of the sign changes $\{1,-1\}^n$ and the group of the permutations S_n act on the Lie algebra $\mathfrak{t}$ by

$$\begin{aligned}\epsilon x &= \epsilon\left(\sum_{j=1}^n x_j J_j\right) = \sum_{j=1}^n \epsilon_j x_j J_j, \qquad (\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_n) \in \{1,-1\}^n)\\ \sigma x &= \sigma\left(\sum_{j=1}^n x_j J_j\right) = \sum_{j=1}^n x_{\sigma^{-1}(j)} J_j, \qquad (\sigma \in S_n)\end{aligned}$$

and hence so does the semidirect product $S_n \ltimes \{1,-1\}^n$, wich coincides with the compact Weyl group $W(\mathrm{T})$,

$$(\epsilon\sigma)x = \sum_{j=1}^n \epsilon_j x_{\sigma^{-1}(j)} J_j\,.$$

The action on the dual $\mathfrak{t}^*$ is defined as usual

$$(\sigma\epsilon)\lambda(x) = \lambda((\epsilon^{-1}\sigma^{-1})(x)) = \sum_{j=1}^n \epsilon_j x_{\sigma(j)} J_j \qquad (x \in \mathfrak{t},\ \lambda \in \mathfrak{t}^*)\,.$$

For $x \in \mathfrak{sp}(\mathsf{W})$ with the eigenvalues ν of $ad(x)$ satisfying $|\nu| < \pi$, define

$$p(x) = \left(\frac{e^{ad(x/2)} - e^{-ad(x/2)}}{ad(x/2)}\right)^{\frac{1}{2}}.$$

Then,

$$p(x) = \prod_{\alpha>0} \frac{e^{\alpha(x/2)} - e^{-\alpha(x/2)}}{\alpha(x/2)} \qquad (x \in \mathfrak{t})\,, \tag{290}$$

where the product is over the positive roots, in some order of roots. We know from Lemma 104 and Proposition 101 that the weigts of $\mathfrak{t}$ in the Weil representation are

$$\left(\frac{1}{2}+a_1\right)i\lambda_1 + \left(\frac{1}{2}+a_2\right)i\lambda_2 + ... + \left(\frac{1}{2}+a_n\right)i\lambda_n \qquad (a_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq n)\,.$$

We choose the the following elements to be positive roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{sp}(\mathsf{W}_{\mathbb{C}})$

$$i\lambda_j - i\lambda_k, \quad i\lambda_j + i\lambda_k, \quad i2\lambda_l \qquad (1 \leq j < k \leq n,\ 1 \leq l \leq n)\,.$$

Then the Weil representation is a lowest weight representation with the lowest weight

$$\frac{1}{2}i\lambda_1 + \frac{1}{2}i\lambda_2 + ... + \frac{1}{2}i\lambda_n\,.$$

The sum of the positive roots multiplied by $\frac{1}{2}$ is equal to

$$\sum_{j=1}^n (n+1-j)i\lambda_j\,.$$

Hence the lowest weight minus the above sum is equal to

$$i\lambda = -\sum_{j=1}^{n}(n-j+\frac{1}{2})i\lambda_j\,.$$

This element represents the infinitesimal character of th Weil representation. Let Θ_{even} denote the charcter of the even part of the Weil representation and let Θ_{odd} denote the charcter of the odd part.

Proposition 108. *There is a constant C such that for any regular element $x \in \mathfrak{t}$*

$$\begin{aligned}
p(x)\Theta(\widetilde{\exp}(x)) &= C\sum_{\epsilon\in\{1,-1\}^n}\mathrm{sgn}(\epsilon)\frac{\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)e^{i(\epsilon\lambda)(\sigma^{-1}(x))}}{\prod_{\alpha>0}\alpha(x)}\,,\\
p(x)\Theta_{even}(\widetilde{\exp}(x)) &= C\sum_{\epsilon\in\{1,-1\}^n,\ \mathrm{sgn}(\epsilon)=1}\frac{\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)e^{i(\epsilon\lambda)(\sigma^{-1}(x))}}{\prod_{\alpha>0}\alpha(x)}\,,\\
p(x)\Theta_{odd}(\widetilde{\exp}(x)) &= -C\sum_{\epsilon\in\{1,-1\}^n,\ \mathrm{sgn}(\epsilon)=-1}\frac{\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)e^{i(\epsilon\lambda)(\sigma^{-1}(x))}}{\prod_{\alpha>0}\alpha(x)}\,.
\end{aligned}$$

Proof. Proposition 101 and (290) shows that

$$\Theta(\widetilde{\exp}(x)) = \prod_{\alpha>0,\ \alpha\ \mathrm{long}}\frac{e^{\alpha(x)/4}}{1-e^{\alpha(x)/2}} = \prod_{\alpha>0,\ \alpha\ \mathrm{long}}\frac{1}{e^{-\alpha(x)/4}-e^{\alpha(x)/4}}\,.$$

Hence,

$$\begin{aligned}
&\prod_{\alpha>0}\alpha(x/2)\cdot p(x)\cdot\Theta(\widetilde{\exp}(x)) \qquad (291)\\
=\ &\prod_{\alpha>0}(e^{\alpha(x)/2}-e^{-\alpha(x)/2})\cdot\prod_{\alpha>0,\ \alpha\ \mathrm{long}}\frac{1}{e^{-\alpha(x)/4}-e^{\alpha(x)/4}}\\
=\ &\prod_{j<k}(e^{i(x_j-x_k)/2}-e^{-i(x_j-x_k)/2})\cdot\prod_{j<k}(e^{i(x_j+x_k)/2}-e^{-i(x_j+x_k)/2})\\
\cdot\ &\prod_{l}(-e^{ix_l/2}-e^{-ix_l/2})\,.
\end{aligned}$$

Modulo the multiplication by $(-1)^n$ we recognize here the Weyl denominator for SO_{2n+1}, with the sum of the positive roots is equal to $-i\lambda$. Since $(-1)^n\mathrm{sgn}(\epsilon) = \mathrm{sgn}(-\epsilon)$, we see that (291) is equal to

$$(-1)^n\sum_{\epsilon,\sigma}\mathrm{sgn}(\sigma\epsilon)e^{-i(\sigma\epsilon)\lambda(x)} = \sum_{\epsilon,\sigma}\mathrm{sgn}(\sigma\epsilon)e^{i(\sigma\epsilon)\lambda(x)}$$

and the first formula follows.

Lemma 104 implies that

$$\begin{aligned}\Theta_{even}(\widetilde{\exp}(x)) &= \frac{1}{2}\left(\prod_{j-1}^{n}\frac{e^{ix_j/2}}{1-e^{ix_j}}+\prod_{j-1}^{n}\frac{e^{ix_j/2}}{1+e^{ix_j}}\right)\\ &= \frac{1}{2}\left(\Theta(\widetilde{\exp}(x))+e^{-i\frac{\pi}{2}n}\Theta(\widetilde{\exp}(x+\pi))\right)\,,\end{aligned}$$

where $x+\pi=\sum_{j=1}^{n}(x_j+\pi)J_j$. Let $q(x)=\prod_{\alpha>0}(e^{\alpha(x)/2}-e^{-\alpha(x)/2})$. Then a straightforward computation shows that

$$q(x+\pi)=(-1)^{\frac{n(n+1)}{2}}q(x)\,.$$

Hence,

$$q(x)\Theta_{even}(\widetilde{\exp}(x))=\frac{1}{2}\left(q(x)\Theta(\widetilde{\exp}(x))+(-1)^{\frac{n(n+1)}{2}}e^{-i\frac{\pi}{2}n}q(x+\pi)\Theta(\widetilde{\exp}(x+\pi)\right)\,.$$

On th other hand,

$$e^{i(\epsilon\lambda)((x+\pi))}=e^{-i\sum_{j=1}^{n}(n-j+\frac{1}{2})\epsilon_j(x_j+\pi)}=(-1)^{\frac{n(n-1)}{2}}\prod_{j=1}^{n}e^{-i\frac{\pi}{2}\epsilon_j}\cdot e^{i(\epsilon\lambda)(x)}$$

and

$$\begin{aligned}(-1)^{\frac{n(n+1)}{2}}e^{-i\frac{\pi}{2}n}\cdot(-1)^{\frac{n(n-1)}{2}}\prod_{j=1}^{n}e^{-i\frac{\pi}{2}\epsilon_j} &= (-1)^{n}e^{-i\frac{\pi}{2}n}\prod_{j=1}^{n}e^{-i\frac{\pi}{2}\epsilon_j}\\ = e^{i\frac{\pi}{2}n}\prod_{j=1}^{n}e^{-i\frac{\pi}{2}\epsilon_j} = \prod_{j=1}^{n}e^{i\frac{\pi}{2}(1-\epsilon_j)} &= \mathrm{sgn}(\epsilon)\,.\end{aligned}$$

Therefore the formulas for Θ_{even} and Θ_{odd} follow from the formula for Θ. □

Let $\mu_{\epsilon\lambda}$ be the tempered distribution on the Lie algebra $\mathfrak{sp}(\mathsf{W})$ equal to the appropriately normalized orbital integral on the $\mathrm{Sp}(\mathsf{W})$-orbit through $\epsilon\lambda$, as defined in [41, (5)]. Let $\hat{\mu}_{\epsilon\lambda}$ be the Fourier transform of $\mu_{\epsilon\lambda}$, denoted by $\theta_{\epsilon\lambda}$ in [41, (5)]. Thanks to Harish-Chandra's Regularity Theorem, $\hat{\mu}_{\epsilon\lambda}$ is a function. The following corollary follows from Proposition 108 and [41, Corollary, page 217].

Corollary 109. *There is a constant C such that for any regular element $x\in\mathfrak{t}$*

$$\begin{aligned}p(x)\Theta(\widetilde{\exp}(x)) &= C\sum_{\epsilon\in\{1,-1\}^n}\mathrm{sgn}(\epsilon)\hat{\mu}_{\epsilon\lambda}(x)\,,\\ p(x)\Theta_{even}(\widetilde{\exp}(x)) &= C\sum_{\epsilon\in\{1,-1\}^n,\ \mathrm{sgn}(\epsilon)=1}\hat{\mu}_{\epsilon\lambda}(x)\,,\\ p(x)\Theta_{odd}(\widetilde{\exp}(x)) &= -C\sum_{\epsilon\in\{1,-1\}^n,\ \mathrm{sgn}(\epsilon)=-1}\hat{\mu}_{\epsilon\lambda}(x)\,.\end{aligned}$$

Our direct proof may be replaced by a short argument using [4.5, Corollary 2.3].

6. The Weil representation over a p-adic field

Let $\mathbb{F}$ be a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$. (In fact our argument works for all non Archimedean fields of characteristic other than 2 till the statement (362) below. Hence our additional assumption.)

Let $\chi(r)$, $r \in \mathbb{F}$, be a character of the additive group $\mathbb{F}$ such that the kernel of χ is equal to $\mathfrak{o}_{\mathbb{F}}$. In this section we provide a construction of the corresponding the Weil representation, [51].

6.1. **The Fourier transform.** Let U be a finite dimensional vector space over $\mathbb{F}$ and let $\mathcal{L}$ be a lattice in U. We normalize the Haar measure μ_{U} on U so that the volume of the lattice $\mathcal{L}$ is 1. Let $\mathcal{L}* \subseteq \mathsf{U}^*$ be the dual lattice. Denote by μ_{U^*} the corresponding Haar measure.

Let $\mathcal{S}(\mathsf{U})$ be the Schwartz-Bruhat space on U, *i.e.,* the space of complex-valued locally constant functions with compact support on U. (Recall that a function ϕ on U is called locally constant if for each $u \in \mathsf{U}$ there is an open neighborhood $\mathcal{U}$ of u such that ϕ is constant on $\mathcal{U}$.) For $\phi \in \mathcal{S}(\mathsf{U})$ let

$$\mathcal{F}\phi(u^*) = \int_{\mathsf{U}} \phi(u)\chi(-u^*(u))\, d\mu_{\mathsf{U}}(u) \qquad (u^* \in \mathsf{U}^*) \tag{292}$$

be the Fourier transform of ϕ. Then, as is well known, $\mathcal{F}\phi \in \mathcal{S}(\mathsf{U}^*)$ and

$$\phi(u) = \int_{\mathsf{U}^*} \mathcal{F}\phi(u^*)\chi(u^*(u))\, d\mu_{\mathsf{U}^*}(u^*) \qquad (u \in \mathsf{U}), \tag{293}$$

see [52, Corollary 1, page 107].

As a linear topological space, $\mathcal{S}(\mathsf{U})$ is the inductive limit of the finite dimensional subspaces spanned by the characteristic functions of finite collections of open compact subsets. Let $\mathcal{S}^*(\mathsf{U})$ denote the linear topological dual of $\mathcal{S}(\mathsf{U})$. It corresponds to the space of the tempered distributions on U in the real case. When convenient we shall identify any bounded locally integrable function $f\colon \mathsf{U} \to \mathbb{C}$ with the tempered distribution $f\mu_{\mathsf{U}}$. In particular, $\mathcal{S}(\mathsf{U}) \subseteq \mathcal{S}^*(\mathsf{U})$. Then the Fourier transform

$$\mathcal{F}\colon \mathcal{S}(\mathsf{U}) \to \mathcal{S}(\mathsf{U}^*)$$

extends to

$$\mathcal{F}\colon \mathcal{S}^*(\mathsf{U}) \to \mathcal{S}^*(\mathsf{U}^*).$$

In fact, if we identify $\mathsf{U}^{**} = \mathsf{U}$ then the Fourier transform (292) is given by

$$\mathcal{F}\psi(u) = \int_{\mathsf{U}^*} \psi(u^*)\chi(-u^*(u))\, d\mu_{\mathsf{U}^*}(u^*) \qquad (\psi \in \mathcal{S}(\mathsf{U}^*),\ u \in \mathsf{U}) \tag{294}$$

and the inverse (293) by

$$\psi(u^*) = \int_{\mathsf{U}} \mathcal{F}\psi(u)\chi(u^*(u))\, d\mu_{\mathsf{U}}(u) \qquad (\psi \in \mathcal{S}(\mathsf{U}^*),\ u^* \in \mathsf{U}^*). \tag{295}$$

Therefore

$$\mathcal{F}(f)(\phi) = f(\mathcal{F}(\phi)) \qquad (f \in \mathcal{S}^*(\mathsf{U}),\ \phi \in \mathcal{S}(\mathsf{U}^*)).$$

Indeed, if $f \in \mathcal{S}(\mathsf{U})$, then

$$\begin{aligned}\mathcal{F}(f\mu_{\mathsf{U}})(\phi) &= ((\mathcal{F}f)\mu_{\mathsf{U}^*})(\phi)\\ &= \int_{\mathsf{U}^*}\int_{\mathsf{U}} f(u)\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u)\phi(u^*)\,d\mu_{\mathsf{U}^*}(u^*)\\ &= \int_{\mathsf{U}} f(u)\mathcal{F}\phi(u)\,d\mu_{\mathsf{U}}(u).\end{aligned}$$

Let $\mathsf{V} \subseteq \mathsf{U}$ be a non-zero subspace. Then $\mathcal{L} \cap \mathsf{V}$ is a lattice in V which determines the Haar measure μ_{V}. We may view μ_{V} as a tempered distribution on U by

$$\mu_{\mathsf{V}}(\phi) = \int_{\mathsf{V}} \phi(v)\,d\mu_{\mathsf{V}}(v) \qquad (\phi \in \mathcal{S}(\mathsf{U})).$$

In the case when V is zero, $\mu_{\mathsf{V}} = \mu_0$ is the unit measure at 0. In other words $\mu_0 = \delta_0$ is the Dirac delta at 0,

$$\mu_0(\phi) = \delta_0(\phi) = \phi(0) \qquad (\phi \in \mathcal{S}(\mathsf{U})).$$

Also, for future reference, let $\delta_u \in \mathcal{S}(\mathsf{U})$ be the Dirac delta at $u \in \mathsf{U}$,

$$\delta_u(\phi) = \phi(u) \qquad (\phi \in C(\mathsf{U})).$$

For an arbitrary subspace $\mathsf{V} \subseteq \mathsf{U}$, let $\mathsf{V}^\perp \subseteq \mathsf{U}^*$ be the annihilator of V. Then,

$$\mathcal{F}\mu_{\mathsf{V}} = \mu_{\mathsf{V}^\perp}. \tag{296}$$

Indeed, the formula (294) implies that (296) holds if $\mathsf{V} = \{0\}$.

The quotient space U/V contains the lattice $(\mathcal{L} + \mathsf{V})/\mathsf{V}$, which determines the normalization of the Haar measure $\mu_{\mathsf{U}/\mathsf{V}}$. Then for $\phi \in \mathcal{S}(\mathsf{U})$ we have $\tilde{\phi} \in \mathcal{S}(\mathsf{U}/\mathsf{V})$ defined by

$$\tilde{\phi}(u + \mathsf{V}) = \int_{\mathsf{V}} \phi(u + v)\,d\mu_{\mathsf{V}}(v).$$

Since (296) holds for the Fourier transform on U/V, with $(\mathsf{U}/\mathsf{V})^* = \mathsf{V}^\perp$ and the left hand side being the evaluation of the Fourier transform of a test function at zero, we have, with $\phi = \mathcal{F}\psi$,

$$\begin{aligned}\mu_{\mathsf{V}}(\mathcal{F}\psi) &= \mu_{\mathsf{V}}(\phi) = \int_{\mathsf{V}} \phi(v)\,d\mu_{\mathsf{V}}(v) = \tilde{\phi}(0) = \int_{\mathsf{V}^\perp} \mathcal{F}\tilde{\phi}(u^*)\,d\mu_{\mathsf{V}^\perp}(u^*)\\ &= \int_{\mathsf{V}^\perp}\int_{\mathsf{U}/\mathsf{V}} \tilde{\phi}(u+\mathsf{V})\chi(-u^*(u))\,d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\,d\mu_{\mathsf{V}^\perp}(u^*)\\ &= \int_{\mathsf{V}^\perp}\int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}} \phi(u+v)\,d\mu_{\mathsf{V}}(v)\chi(-u^*(u))\,d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})\,d\mu_{\mathsf{V}^\perp}(u^*)\\ &= \int_{\mathsf{V}^\perp}\int_{\mathsf{U}} \phi(u)\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u)\,d\mu_{\mathsf{V}^\perp}(u^*)\\ &= \int_{\mathsf{V}^\perp} \mathcal{F}\phi(u^*)\,d\mu_{\mathsf{V}^\perp}(u^*)\\ &= \mu_{\mathsf{V}^\perp}(\mathcal{F}(\phi)) = \mu_{\mathsf{V}^\perp}(\mathcal{F}^2(\psi)) = \mu_{\mathsf{V}^\perp}(\psi),\end{aligned}$$

where the last equality follows from the fact that $\mathcal{F}^2\psi(u) = \psi(-u)$, which is a simple consequence of (294) and (293). This completes the proof (296).

Consider two vector spaces U', U'' over $\mathbb{F}$ of the same dimension equipped with lattices $\mathcal{L}'$, $\mathcal{L}''$ respectively. Let $u_1', u_2', \dots, u_n'$ be a $\mathcal{L}'$-orthonormal basis of U' and let $u_1'', u_2'', \dots, u_n''$ be a $\mathcal{L}''$-orthonormal basis of U''. Suppose $L\colon \mathsf{U}' \to \mathsf{U}''$ is a linear bijection. Denote by M the matrix of L with respect to the two ordered basis:

$$Lu_j' = \sum_{i=1}^{n} M_{i,j} u_i'' \qquad (j = 1, 2, \dots, n).$$

Then $|\det(M)|_{\mathbb{F}}$ does not depend on the choice of the orthonormal basis. Thus we may define $|\det(L)|_{\mathbb{F}} = |\det(M)|_{\mathbb{F}}$ (see section 2.6).

Lemma 110. *With the above notation we have*

$$\int_{\mathsf{U}'} \phi(L(u'))\, d\mu_{\mathsf{U}'}(u')\, |\det(L)|_{\mathbb{F}} = \int_{\mathsf{U}''} \phi(u'')\, d\mu_{\mathsf{U}''}(u'') \qquad (\phi \in \mathcal{S}(\mathsf{U}'')). \tag{297}$$

Proof. This follows from Lemma 17. Indeed, let ϕ be the indicator function of $\mathcal{L}''$. Then the right hand side of the equation (108) is equal to 1. Hence we need to show that

$$\int_{\mathsf{U}'} \phi(L(u'))\, d\mu_{\mathsf{U}'}(u')\, |\det(L)|_{\mathbb{F}} = 1.$$

However, $\phi \circ L$ is the indicator function of $L^{-1}(\mathcal{L}'')$. Thus the problem is to check that

$$\mu_{\mathsf{U}'}(L^{-1}(\mathcal{L}''))\, |\det(L)|_{\mathbb{F}} = 1.$$

Fix an $\mathcal{L}'$-orthonormal basis $u_1',\ u_2', \dots$ of U' and an $\mathcal{L}''$-orthonormal basis $u_1'',\ u_2'', \dots$ of U''. Let T be the endomorphism of U' defined by

$$T(L^{-1}(u_j')) = u_j'' \qquad (j = 1, 2, \dots). \tag{298}$$

Then

$$T(L^{-1}(\mathcal{L}'')) = \mathcal{L}'.$$

Hence, by Lemma 17,

$$\mu_{\mathsf{U}'}(L^{-1}(\mathcal{L}''))\, |\det(T)|_{\mathbb{F}} = \mu_{\mathsf{U}'}(T(L^{-1}(\mathcal{L}''))) = \mu_{\mathsf{U}'}(\mathcal{L}')) = 1.$$

But (298) implies that $|\det(T)|_{\mathbb{F}} = |\det(L)|_{\mathbb{F}}$. Hence the claim follows. □

Let X and U be two finite dimensional vector spaces over $\mathbb{F}$ equipped with lattices and the corresponding normalized Haar measures μ_{X} and μ_{U}. Let $L\colon \mathsf{X} \to \mathsf{U}$ be a surjective linear map. Suppose f is a bounded function on U so that $f\mu_{\mathsf{U}} \in \mathcal{S}^*(\mathsf{U})$. Define $L^*(f\mu_{\mathsf{U}}) := (f \circ L)\mu_{\mathsf{X}}$. Thus for a test function $\phi \in \mathcal{S}(\mathsf{U})$,

$$L^*(f\mu_{\mathsf{U}})(\phi) = \int_{\mathsf{X}} f(L(x))\phi(x)\, d\mu_{\mathsf{X}}(x). \tag{299}$$

Choose a subspace $X' \subseteq X$ complementary to $\mathrm{Ker}(L)$ so that $X = \mathrm{Ker}(L) \oplus X'$. Let $\mu_{\mathrm{Ker}(L)}$ and $\mu_{X'}$ denote the corresponding normalized Haar measures on $\mathrm{Ker}(L)$ and X' respectively. Then (299) may be rewritten as

$$\int_{X'} \int_{\mathrm{Ker}(L)} f(L(x'+x''))\phi(x'+x'')\, d\mu_{\mathrm{Ker}(L)}(x'')\, d\mu_{X'}(x'). \tag{300}$$

Let L' denote the restriction of L to X'. Then $L' : X' \to \mathsf{U}$ is a bijection and Lemma 110 shows that (300) may be rewritten as

$$\int_{\mathsf{U}} f(u) L_*(\phi)(u)\, d\mu_{\mathsf{U}}(u). \tag{301}$$

where

$$L_*(\phi)(u) = \int_{\mathrm{Ker}(L)} \phi(L'^{-1}(u)+x'')\, d\mu_{\mathrm{Ker}(L)}(x'')\, |\det(L')|_{\mathbb{F}}^{-1}. \tag{302}$$

Notice that $L_* : \mathcal{S}(X) \to \mathcal{S}(\mathsf{U})$ is a continuous map. Hence we have the notion of a pullback of a distribution

$$L^*(f)(\phi) = f(L_*(\phi)) \qquad (\phi \in \mathcal{S}(X),\ f \in \mathcal{S}^*(\mathsf{U})) \tag{303}$$

which is consistent with [14, Theorem 6.1.2].

Lemma 111. *Let* X *and* U *are two finite dimensional vector space over* $\mathbb{F}$ *equipped with lattices and the corresponding normalized Haar measures* μ_{X} *and* μ_{U}. *Let* $L : \mathsf{X} \to \mathsf{U}$ *be a surjective linear map. Let*

$$\tilde{L} : \mathsf{X}/L^{-1}(\mathsf{V}) \to \mathsf{U}/\mathsf{V}$$

be the induced bijection. Then

$$L^*(\mu_{\mathsf{V}}) = |\det(\tilde{L})|_{\mathbb{F}}^{-1} \mu_{L^{-1}(\mathsf{V})}.$$

Proof. Let $\mathsf{X}' \subseteq \mathsf{X}$ be the orthogonal complement of $\mathrm{Ker}(L)$. Denote by L' the restriction of L to X' and by L'' the restriction of L to $\mathsf{X}' \cap L^{-1}(\mathsf{V})$. Then

$$L' \colon \mathsf{X}' \to \mathsf{U} \text{ and } L'' \colon \mathsf{X}' \cap L^{-1}(\mathsf{V}) \to \mathsf{V}$$

are bijections.

According to (303), for a test function $\phi \in \mathcal{S}(X)$ we have

$$L^*(\mu_{\mathsf{V}})(\phi) = \int_{\mathrm{Ker}(L)} \int_{\mathsf{V}} \phi(x + L'^{-1}(v))\, d\mu_{\mathsf{V}}(v)\, d\mu_{Ker(L)}(x)\, |\det(L')|_{\mathbb{F}}^{-1}. \tag{304}$$

Then the right hand side of (304) is equal to

$$\begin{aligned} &\int_{\mathrm{Ker}(L)} \int_{L''^{-1}(\mathsf{V})} \phi(x+y)\, d\mu_{L''^{-1}(\mathsf{V})}(y)\, d\mu_{\mathrm{Ker}(L)}(x)\, |\det(L'')|_{\mathbb{F}}\, |\det(L')|_{\mathbb{F}}^{-1} \\ = &\int_{L^{-1}(\mathsf{V})} \phi(z)\, d\mu_{L^{-1}(\mathsf{V})}(z)\, |\det(L'')|_{\mathbb{F}}\, |\det(L')|_{\mathbb{F}}^{-1}. \end{aligned}$$

Since $|\det(L'')|_{\mathbb{F}}^{-1}\, |\det(L')|_{\mathbb{F}} = |\det(\tilde{L})|_{\mathbb{F}}$, we are done. ☐

6.2. **Gaussians on $\mathbb{F}^n$.** Let B be the usual dot product on $\mathbb{F}^n$,

$$B(x,y) = x^t y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \qquad (x, y \in \mathbb{F}^n).$$

Then the Haar measure associated to the lattice $\mathfrak{o}_{\mathbb{F}}^n \subseteq \mathbb{F}^n$, $d\mu_{\mathbb{F}^n}(x) = dx_1dx_2\ldots dx_n$, is the n-fold direct product of Lebesgue measure dx_i on $\mathbb{F}$, such that $\int_{\mathfrak{o}_{\mathbb{F}}} dx_i = 1$.

For a symmetric matrix $A \in \mathrm{GL}(\mathbb{F}^n)$ define the corresponding Gaussian γ_A by

$$\gamma_A(x) := \chi(\frac{1}{2}x^tAx) \qquad (x \in \mathbb{F}^n).$$

Also, let

$$\gamma(A) = \mathcal{F}\gamma_A(0) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^tAx)\, dx.$$

In particular, taking $n = 1$, we have

$$\gamma(a) = \int_{\mathbb{F}} \chi(\frac{1}{2}ax^2)\, dx, \qquad (a \in \mathbb{F}^\times).$$

Let γ_W be the gamma factor defined by Weil in [51, n°14 cor. 2]. It is related to γ by the equality

$$\gamma(A) \;=\; |\det(A)|_{\mathbb{F}}^{-1/2}\, \gamma_W(A). \tag{305}$$

We set

$$\gamma_W(q) := \gamma_W(Q),$$

if q is a quadratic form with associated symmetric matrix Q as in Eq. (12). Then γ_W defines a unitary character of the Witt group of $\mathbb{F}$. The scalar $\gamma_W(a)$ is the gamma factor attached to the quadratic form $x \mapsto ax^2$ ($a \in \mathbb{F}^\times$). It depends only on the class of a modulo $(\mathbb{F}^\times)^2$. In particular, we have

$$\gamma_W(a^2) = \gamma_W(1) \quad \text{for all } a \in \mathbb{F}^\times. \tag{306}$$

Of course Eqn. (306) would not be true with γ instead of γ_W: we get $\gamma(a^2) = |a|_{\mathbb{F}}^{-1}\gamma_W(1)$. Note that $\gamma_W(1)$ and $\gamma(1)$ are equal.

Recall the *Hilbert symbol* (,): for any $a, b \in \mathbb{F}^\times$,

$$(a,b) := \begin{cases} 1 & \text{if } z^2 = ax^2 + by^2 \text{ has a non-zero solution } (x,y,z) \in \mathbb{F}^3, \\ -1 & \text{otherwise.} \end{cases}$$

Equivalently, if A is a central simple algebra over $\mathbb{F}$ with a basis i, j, and $k = ij = -ji$ such that $i^2 = a$ and $j^2 = b$, then $(a, b) = -1$ if A is a division algebra and $(a, b) = 1$ if it is isomorphic to the algebra of 2 by 2 matrices. It is related to the above γ factor as follows:

Proposition 112. *For any $a, b \in \mathbb{F}^\times$, we have*

$$(a,b) = \frac{\gamma(ab)\,\gamma(1)}{\gamma(a)\gamma(b)}. \tag{307}$$

Equivalently

$$(a,b) = \frac{\frac{\gamma(ab)}{\gamma(1)}}{\frac{\gamma(a)}{\gamma(1)}\frac{\gamma(b)}{\gamma(1)}}\,. \tag{308}$$

Also

$$\gamma_W(a)^8 = \left(|a|_{\mathbb{F}}^{1/2}\gamma(a)\right)^8 = 1 \qquad (a \in \mathbb{F}^\times)\,. \tag{309}$$

Proof. It follows from [51, n°25 prop. 3 and n°28 prop. 4] that

$$(a,b) = \frac{\gamma_W(ab)\,\gamma_W(1)}{\gamma_W(a)\gamma_W(b)}\,. \tag{310}$$

This is the formula at the bottom of page 176 in [51]. Also, a proof of (309) is on the pages 176 and 177 in [51].

Then the equality (307) is an immediate consequence of the equality $\gamma(a) = \gamma_W(a)\,|a|_{\mathbb{F}}^{-1/2}$. (See also [30, Corollary 2.16, page 440].) □

Corollary 113. *The function*

$$a \mapsto \mathfrak{s}(a) := |a|_{\mathbb{F}}\,\frac{\gamma(a)^2}{\gamma(1)^2} = \frac{\gamma_W(a)^2}{\gamma_W(1)^2}$$

is a character of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$.

Remark. The function $a \mapsto \frac{\gamma(a)^2}{\gamma(1)^2}$ is a character of $\mathbb{F}^\times$. However it does not have trivial restriction to $(\mathbb{F}^\times)^2$.

Remark. The character $\mathfrak{s}$ will play a similar role to that of the character s which was defined in Lemma 25 in the case of finite fields, and of $a \mapsto \dfrac{|a|}{a}$ in the case of $\mathbb{R}$.

In these terms we have the following theorem due to Weil.

Theorem 114. *For any symmetric matrix* $A \in \mathrm{GL}(\mathbb{F}^n)$*,*

$$\mathcal{F}\gamma_A = \gamma(A)\,\gamma_{-A^{-1}}, \tag{311}$$

and

$$\gamma(A) = \pm\gamma(1)^{n-1}\gamma(\det(A)). \tag{312}$$

Here the $\pm$ *function is invariant under the natural action of* $\mathrm{GL}(\mathbb{F}^n)$ *on the symmetric matrices. In particular*

$$\left(\gamma(1)^{-n}\gamma(A)\right)^2 = \left(\gamma(1)^{-1}\gamma(\det(A))\right)^2\,. \tag{313}$$

Proof. Suppose A is similar to

$$\begin{pmatrix} a_1 & 0 & 0 & ... & 0 \\ 0 & a_2 & 0 & ... & 0 \\ .. & .. & .. & ... & .. \\ 0 & 0 & 0 & ... & a_n \end{pmatrix}.$$

Since γ_W is a homomorphism from the Witt group of the symmetric forms to $\mathbb{C}$ we have

$$\gamma_W(A) = \gamma_W(a_1)\gamma_W(a_2)...\gamma_W(a_n)\,.$$

On the other hand (310) implies inductively that

$$\begin{aligned}\gamma_W(a_1a_2...a_n) &= (a_1a_2...a_{n-1}, a_n)\gamma_W(1)^{-1}\gamma_W(a_1a_2...a_{n-1})\gamma_W(a_n) \qquad (314)\\ &= (a_1, a_2)(a_1a_2, a_3)...(a_1a_2...a_{n-1}, a_n)\gamma_W(1)^{1-n}\gamma_W(a_1)\gamma_W(a_2)...\gamma_W(a_n)\,.\end{aligned}$$

Thus

$$\gamma(\det(A)) = (a_1, a_2)(a_1a_2, a_3)...(a_1a_2...a_{n-1}, a_n)\gamma(1)^{1-n}\gamma(A)\,,$$

or equivalently,

$$\gamma(A) = (a_1, a_2)(a_1a_2, a_3)...(a_1a_2...a_{n-1}, a_n)\gamma(1)^{n-1}\gamma(\det(A)) \qquad (315)$$

and (312) follows.

Equivalently, we have (see [51, Chap. I Théorème 2 and Chap. II § 26])

$$\gamma_W(A) = \pm\gamma(1)^{n-1}\,\gamma_W(\det A). \qquad (316)$$

Hence, from Eqn. (305) we obtain

$$\gamma(A) = \pm\gamma(1)^{n-1}\,\gamma(\det A). \qquad (317)$$

Then the first equation in the statement of the theorem follows from [51, Eqn. (17) and Théorème 2, I. § 14] applied to the character of second degree $x \mapsto \gamma_A(x)$. □

6.3. **Gaussians on a vector space.** Let U be a finite dimensional vector space over $\mathbb{F}$ with a lattice $\mathcal{L} \subseteq \mathsf{U}$. Suppose q is a non-degenerate symmetric bilinear form on U. Let $\gamma(q) = \gamma(Q)$, where Q is the matrix obtained from any $N_{\mathcal{L}}$-orthonormal basis $u_1, u_2, \ldots, u_n$ of U by

$$Q_{i,j} = q(u_i, u_j) \qquad (1 \le i, j \le n).$$

Also, we define $\gamma(0) = 1$.

Lemma 115. *If q is a non-degenerate symmetric bilinear form on U, then*

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)) \qquad (u^* \in \mathsf{U}^*).$$

Proof. Fix a $N_{\mathcal{L}}$-orthonormal basis $u_1, u_2, \ldots, u_n$ of U and let $u_1^*, u_2^*, \ldots, u_n^*$ be the dual basis of U^*. This is a $N_{\mathcal{L}_*}$-orthonormal basis. As we have seen in the proof of Lemma 27, if Q is the matrix corresponding to q, as above, then Q^{-1} corresponds to q^*.

Let $x_i = u_i^*(u)$ and let $y_j = u^*(u_j)$. Then

$$\begin{aligned}&\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^tQx)\chi(-x^ty)\,dx\\ &= \gamma(Q)\chi(-\frac{1}{2}y^tQ^{-1}y) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)),\end{aligned}$$

where the second equality follows from Theorem 114. □

6.4. **Gaussians on a symplectic space.** Let W be a finite dimensional vector space over $\mathbb{F}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$. We shall identify W with the dual W^* by

$$w^*(w) = \langle w, w^*\rangle \qquad (w, w^* \in \mathrm{W}). \tag{318}$$

The identification (318) provides to the following isomorphisms

$$\mathsf{U}^* = \mathrm{W}/\mathsf{U}^\perp \text{ and } (\mathsf{U}/\mathsf{V})^* = \mathsf{V}^\perp/\mathsf{U}^\perp, \tag{319}$$

where the orthogonal complements are taken in W, with respect to the symplectic form $\langle\ ,\ \rangle$. Let $\{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$ be a symplectic basis of W, that is:

$$\langle e_i, e_j\rangle = \langle e_{-i}, e_{-j}\rangle = 0 \quad \text{and} \quad \langle e_i, e_{-j}\rangle = \delta_{ij}, \quad \text{for all } 1 \le i, j \le n.$$

Let $\mathcal{L} := \sum_{j=-n}^{n} \mathfrak{o}_{\mathbb{F}} e_j$. Then $\mathcal{L}$ is a self-dual lattice in W, *i.e.,*

$$\{w \in \mathrm{W};\ \langle u, w\rangle \in \mathfrak{o}_{\mathbb{F}} \text{ for all } u \in \mathcal{L}\} = \mathcal{L}.$$

Moreover,

$$\{\langle w_1, w_2\rangle;\ w_1, w_2 \in \mathcal{L}\} = \mathfrak{o}_{\mathbb{F}}.$$

As explained in section 6.1, $\mathcal{L}$ leads to a normalization of the Haar measures on any subspace of $\mathsf{U} \subseteq \mathrm{W}$ and on any quotient U/V, where V is a subspace of U.

Lemma 116. *Suppose* $x \in \mathrm{Hom}(\mathsf{U}, \mathrm{W}/\mathsf{U}^\perp)$ *is such that*

$$\langle xu, v\rangle = \langle xv, u\rangle \qquad (u, v \in \mathsf{U}).$$

Set

$$q(u, v) = q_x(u, v) = \frac{1}{2}\langle xu, v\rangle \qquad (u, v \in \mathsf{U}).$$

Let V *be the radical of* q *and let* $\tilde{q}$ *be the induced non-degenerate form on* U/V. *Then*

(a) *If* $x \in \mathfrak{sp}(\mathsf{W})$ *is invertible, i.e.* $\mathsf{U} = \mathsf{W}$, *then*

$$chc(x) = \int_{\mathsf{W}} \chi(\frac{1}{4}\langle xw, w\rangle)\, d\mu_{\mathsf{W}}(w) = \pm\gamma(1)^{\dim_{\mathbb{F}} \mathsf{W} - 1}\gamma(\det(\frac{1}{2}x)),$$

where the $\pm$ *function is invariant under the adjoint action of the symplectic group.*

(b) $\mathsf{V} = \mathrm{Ker}(x)$*;*

(c) *The element* x *determines a bijection*

$$\underline{x} : \mathsf{U}/\mathsf{V} \to \mathsf{V}^\perp/\mathsf{U}^\perp,$$

with the inverse

$$\underline{x}^{-1} : \mathsf{V}^\perp/\mathsf{U}^\perp \to \mathsf{U}/\mathsf{V};$$

(d) *Let* $x^{-1} : \mathsf{V}^\perp \to \mathsf{U}/\mathsf{V}$ *be the composition of* $\underline{x}^{-1}$ *with the quotient map* $\mathsf{V}^\perp \to \mathsf{V}^\perp/\mathsf{U}^\perp$. *Define*

$$\chi_x(u) = \chi(\frac{1}{4}\langle xu, u\rangle) \qquad (u \in \mathsf{U}),$$

$$\chi_{x^{-1}}(w) = \chi(\frac{1}{4}\langle x^{-1}w, w\rangle) \qquad (w \in \mathsf{V}^\perp).$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned}
&\int_{\mathsf{U}}\int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathsf{U}}(u) \qquad (320)\\
=\;& \gamma(\tilde{q})\int_{\mathsf{V}^\perp} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathsf{V}^\perp}(w)\\
=\;& \gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \chi_{x^{-1}}(w+\mathsf{U}^\perp)\int_{\mathsf{U}^\perp}\phi(w+v)\, d\mu_{\mathsf{U}^\perp}(v)\, d\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp}(w+\mathsf{U}^\perp).
\end{aligned}$$

Also, for any $\phi \in \mathcal{S}(\mathrm{W}/\mathsf{U}^\perp)$,

$$\begin{aligned}
&\int_{\mathsf{U}}\int_{\mathrm{W}/\mathsf{U}^\perp} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{U}}(u) \qquad (321)\\
=\;& \gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \chi_{\underline{x}^{-1}}(w)\phi(w+\mathsf{U}^\perp)\, d\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp}(w+\mathsf{U}^\perp).
\end{aligned}$$

Proof. For part (a) set $\{u_1, \dots, u_{2n}\} = \{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$ and notice that

$$\begin{aligned}
chc(x) &= \int_{\mathsf{W}} \chi(\frac{1}{2}q_x(w))\, d\mu_{\mathsf{W}}(w) = \gamma(q_x) = \pm\gamma(1)^{\dim_{\mathbb{F}}\mathsf{W}-1}\gamma(\det(\langle \frac{1}{2}xu_j, u_k\rangle))\\
&= \pm\gamma(1)^{\dim_{\mathbb{F}}\mathsf{W}-1}\gamma(\det(\frac{1}{2}x))\,,
\end{aligned}$$

where the last equality follows from Lemma 20.

Part (b) is obvious. Part (c) means that $\mathrm{Ker}(x)^\perp = \mathrm{Im}(x)$, which is true. For $\phi \in \mathcal{S}(\mathrm{W})$ we have,

$$\begin{aligned}
&\int_{\mathsf{U}}\int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathsf{U}}(u)\\
=\;& \int_{\mathrm{W}} \mathcal{F}(\gamma_q\mu_{\mathsf{U}})(\frac{1}{2}w)\phi(w)\, d\mu_{\mathrm{W}}(w)\\
=\;& \int_{\mathrm{W}} \mathcal{F}(\gamma_q\mu_{\mathsf{U}})(w)\phi(2w)\, d\mu_{\mathrm{W}}(w)\, |2^{\dim \mathrm{W}}|_{\mathbb{F}}\\
=\;& \int_{\mathrm{W}} \mathcal{F}(\gamma_q\mu_{\mathsf{U}})(w)\phi(2w)\, d\mu_{\mathrm{W}}(w)\\
=\;& \gamma(\tilde{q})\int_{\mathsf{V}^\perp} \gamma_{-\tilde{q}^*}(w)\phi(2w)\, d\mu_{\mathsf{V}^\perp}(w)\\
=\;& \gamma(\tilde{q})\int_{\mathsf{V}^\perp} \gamma_{-\tilde{q}^*}(\frac{1}{2}w)\phi(w)\, d\mu_{\mathsf{V}^\perp}(w)\\
=\;& \gamma(\tilde{q})\int_{\mathsf{V}^\perp} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathsf{V}^\perp}(w).
\end{aligned}$$

This verifies (320). For $\phi \in \mathcal{S}(\mathrm{W}/\mathsf{U}^\perp)$ we have,

$$\int_{\mathsf{U}}\int_{\mathrm{W}/\mathsf{U}^\perp} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w+\mathsf{U}^\perp)\,d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\,d\mu_{\mathsf{U}}(u)$$

$$= \int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}}\int_{\mathrm{W}/\mathsf{U}^\perp} \chi_{\underline{x}}(u+\mathsf{V})\chi(\frac{1}{2}\langle u+v, w\rangle)\phi(w+\mathsf{U}^\perp)\,d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\,d\mu_{\mathsf{V}}(v)\,d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})$$

$$= \int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}}\int_{\mathrm{W}/\mathsf{U}^\perp} \gamma_{\tilde{q}}(u+\mathsf{V})\chi(\langle u+v, w\rangle)\phi(2w+\mathsf{U}^\perp)\,d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\,d\mu_{\mathsf{V}}(v)\,d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})$$

$$= \int_{\mathsf{U}/\mathsf{V}}\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \gamma_{\tilde{q}}(u+\mathsf{V})\chi(\langle u, w\rangle)\phi(2w+\mathsf{U}^\perp)\,d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)\,d\mu_{\mathsf{U}/\mathsf{V}}(u+\mathsf{V})$$

$$= \gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \gamma_{-\tilde{q}^*}(w+\mathsf{U}^\perp)\phi(2w+\mathsf{U}^\perp)\,d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)$$

$$= \gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \gamma_{-\tilde{q}^*}(\frac{1}{2}w+\mathsf{U}^\perp)\phi(w+\mathsf{U}^\perp)\,d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp)$$

$$= \gamma(\tilde{q})\int_{\mathsf{V}^\perp/\mathsf{U}^\perp} \chi_{\underline{x}^{-1}}(w+\mathsf{U}^\perp)\phi(w+\mathsf{U}^\perp)\,d\mu_{\mathrm{W}/\mathsf{U}^\perp}(w+\mathsf{U}^\perp).$$

This verifies (321). □

By a Gaussian on the symplectic space W we shall understand any non-zero constant multiple of the tempered distribution

$$\chi_x\mu_{\mathsf{U}} \in \mathcal{S}^*(\mathrm{W}) \qquad (322)$$

where the function χ_x is defined in Lemma 116. In these terms Lemma 116 says that the Fourier transform of a Gaussian is another Gaussian.

6.5. **Twisted convolution of Gaussians.** Recall the twisted convolution of two Schwartz functions $\psi, \phi \in \mathcal{S}(\mathrm{W})$:

$$\psi\natural\phi(w) = \int_{\mathrm{W}} \psi(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\,d\mu_{\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \qquad (323)$$

It is easy to see that the above integral converges and that $\psi\natural\phi \in \mathcal{S}(\mathrm{W})$. Also, the twisted convolutions

$$\delta_{w_0}\natural\phi(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w_0, w\rangle) \text{ and } \phi\natural\delta_{w_0}(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w, w_0\rangle) \qquad (324)$$

are well defined for any continuous function ϕ.

Let

$$t(g) = \chi_{c(g)}\mu_{g^-\mathrm{W}}. \qquad (325)$$

For any $\phi \in \mathcal{S}(\mathrm{W})$, the twisted convolution $t(g)\natural\phi$ is a continuous function given by the following absolutely convergent integral

$$t(g)\natural\phi(w) = \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\,d\mu_{g^-\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \qquad (326)$$

Lemma 117. *For any $g \in \mathrm{Sp}(\mathrm{W})$,*

$$t(g)\natural(\delta_{w_0}\natural\phi) = \delta_{gw_0}\natural(t(g)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ w_0 \in \mathrm{W}).$$

Proof. The left hand side evaluated at $w \in \mathrm{W}$ is equal to

$$\begin{aligned}
& \int_{g^-\mathrm{W}} \chi_{c(g)}(u)(\delta_{w_0}\natural\phi)(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
= & \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u-w_0)\chi(\frac{1}{2}\langle w_0, w-u\rangle)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
= & \int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)u, u\rangle + 2\langle w_0, w-u\rangle + 2\langle u, w\rangle))\, d\mu_{g^-\mathrm{W}}(u)
\end{aligned}$$

and the right hand side is equal to

$$\begin{aligned}
& (t(g)\natural\phi)(w-gw_0)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
= & \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-gw_0-u)\chi(\frac{1}{2}\langle u, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
= & \int_{g^-\mathrm{W}} \chi_{c(g)}(u-g^-w_0)\phi(w-gw_0-(u-g^-w_0))\\
& \chi(\frac{1}{2}\langle u-g^-w_0, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
= & \int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)(u-g^-w_0), u-g^-w_0\rangle\\
+ & \ 2\langle u-g^-w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle))\, d\mu_{g^-\mathrm{W}}(u).
\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}
& \langle c(g)(u-g^-w_0), u-g^-w_0\rangle + 2\langle u-g^-w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle\\
- & \ (\langle c(g)u, u\rangle + 2\langle w_0, w-u\rangle + 2\langle u, w\rangle) = 0.
\end{aligned}$$

Hence, the two sides are equal. ☐

Lemma 118. *Fix an element $g \in \mathrm{Sp}(\mathrm{W})$. Let $\mathsf{U} = g^-\mathrm{W}$. The map*

$$\mathsf{U} \ni u \to \langle\ , (1-c(g))u\rangle \in \mathsf{U}^* = \mathrm{W}/\mathsf{U}^\perp = \mathrm{W}/\mathrm{Ker}(g^-) \tag{327}$$

is bijective.

Fix a complement Z of U in W so that

$$\mathrm{W} = \mathsf{U} \oplus \mathsf{Z}.$$

We shall denote the elements of U by u and elements of Z by z. In particular every $w \in \mathrm{W}$ has a unique decomposition

$$w = u + z.$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$ *and any* $w' = u' + z' \in \mathrm{W}$,

$$\begin{aligned} & t(g)\natural\phi(w') & (328) \\ = \ & \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle) \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(u+z')\chi(-\frac{1}{2}\langle u, (1-c(g))u' + z'\rangle)\, d\mu_{\mathsf{U}}(u). \end{aligned}$$

In particular, (328) and (327) imply that $t(g)\natural\phi \in \mathcal{S}(\mathrm{W})$ *and that the map*

$$\mathcal{S}(\mathrm{W}) \ni \phi \to t(g)\natural\phi \in \mathcal{S}(\mathrm{W})$$

is continuous.

Proof. Suppose $\langle\ , (1-c(g))u\rangle = 0$. Then $(1-c(g))u \in \mathrm{Ker}\, g^-$. There is $u_0 \in \mathrm{W}$ such that $u = g^- u_0$. Therefore

$$\begin{aligned} 0 &= g^-(1-c(g))u = g^-(1-c(g))g^-u_0 \\ &= g^-(g^-)u_0 - g^-g^+u_0 = g^-(g^-)u_0 - g^+g^-u_0 \\ &= (g^- - g^-)g^-u_0 = -2g^-u_0 = -2u. \end{aligned}$$

This verifies (327).

The left hand side of (328) is equal to

$$\begin{aligned} & t(g)\natural\phi(w') = \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(w'-u)\chi(\frac{1}{2}\langle u, w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ = \ & \int_{\mathsf{U}} \chi_{c(g)}(u+u')\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ = \ & \int_{\mathsf{U}} \chi_{c(g)}(u')\chi_{c(g)}(u)\chi(\frac{1}{2}\langle c(g)u', u\rangle)\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ = \ & \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle) \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(z'-u)\chi(\frac{1}{2}\langle u, w' - c(g)u'\rangle)\, d\mu_{\mathsf{U}}(u), \end{aligned}$$

which coincides with the right hand side. □

In particular Lemma 118 shows that for any two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ there is a tempered distribution $t(g_1)\natural t(g_2) \in \mathcal{S}^*(\mathrm{W})$ such that

$$(t(g_1)\natural t(g_2))\natural\phi = t(g_1)\natural(t(g_2)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W})). \tag{329}$$

Proposition 119. *Fix two elements* $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$. *Let* $\mathsf{U}_1' \subseteq \mathsf{U}_1$ *be the* $N_{\mathcal{L}}$*-orthogonal complement of* U, *so that*

$$\mathsf{U}_1 = \mathsf{U}_1' \oplus \mathsf{U}.$$

Then the map

$$L : \mathsf{U}_1' + \mathsf{U}_2 \ni u_1' + u_2 \to c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 + \mathsf{U}^\perp \in \mathrm{W}/\mathsf{U}^\perp$$

is well defined, surjective and $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) = \mathsf{U}_{12}$. *Denote by*

$$\begin{aligned} & \tilde{L} : (\mathsf{U}_1 + \mathsf{U}_2)/\mathsf{U}_{12} \ni u_1 + u_2 + \mathsf{U}_{12} \to c(g_1)u_1 - c(g_2)u_2 - u_1 - u_2 + \mathsf{V}^\perp \in \mathrm{W}/\mathsf{V}^\perp \\ = \ & (\mathrm{W}/\mathsf{U}^\perp)/(\mathsf{V}^\perp/\mathsf{U}^\perp) \end{aligned}$$

the induced bijection and set

$$C(g_1, g_2) = \gamma(\tilde{q}_{g_1,g_2})|\det(\tilde{L})|_{\mathbb{F}}^{-1}. \tag{330}$$

Then C is a cocycle, with $C(1,1) = 1$, and

$$t(g_1)\natural t(g_2) = C(g_1, g_2)t(g_1 g_2). \tag{331}$$

Proof. Since $\mathsf{V}^\perp/\mathsf{U}^\perp = (c(g_1) + c(g_2))\mathsf{U}$, the map $\tilde{L}$ is well defined. Suppose $u_1' \in \mathsf{U}_1'$ and $u_2 \in \mathsf{U}_2$ are such that $L(u_1' + u_2) \in \mathsf{V}^\perp/\mathsf{U}^\perp$. Then there is $u \in \mathsf{U}$ such that

$$(c(g_1) + c(g_2))u + c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 \in \mathsf{U}^\perp.$$

Let

$$u = g_1^- v_1 = g_2^- v_2,\ v = u_1' = g_1^- w_1,\ w - v = u_2 = g_2^- w_2.$$

Then

$$(c(g_1) + c(g_2))u + c(g_1)v + c(g_2)(v - w) - w \in \mathsf{U}^\perp.$$

Hence, the computation (87) - (89) shows that $w = (g_1 g_2)^-(w_2 - v_2) \in \mathsf{U}_{12}$. Therefore $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) \subseteq \mathsf{U}_{12}$. But (327) implies that L is surjective and Lemma 7 (b) shows that $\dim((\mathsf{U}_1 + \mathsf{U}_2)/\mathsf{U}_{12}) = \dim((\mathrm{W}/\mathsf{U}^\perp)/(\mathsf{V}^\perp/\mathsf{U}^\perp))$. Thus $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) = \mathsf{U}_{12}$.

The computation (89) - (93) shows that, if $u_1' + u_2 \in \mathsf{U}_{12}$ then

$$\begin{aligned} & \langle c(g_1)u_1', u_1'\rangle + \langle c(g_2)u_2, u_2\rangle + 2\langle u_1', u_2\rangle + \langle (c(g_1) + c(g_2))^{-1}L(u_1' + u_2), L(u_1' + u_2)\rangle \\ = \ & \langle c(g_1 g_2)(u_1' + u_2), u_1 + u_2\rangle \end{aligned}$$

so that

$$\chi_{c(g_1)}(u_1')\chi_{c(g_2)}(u_2)\chi(\frac{1}{2}\langle u_1', u_2\rangle)\chi_{(c(g_1)+c(g_2))^{-1}}(L(u_1' + u_2)) = \chi_{c(g_1 g_2)}(u_1' + u_2). \tag{332}$$

Any $u_1 \in \mathsf{U}_1$ has a unique decomposition $u_1 = u_1' + u$, where $u_1' \in \mathsf{U}_1'$ and $u \in \mathsf{U}$. With this notation, Lemma 118 shows that for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned} & t(g_1)\natural(t(g_2)\natural\phi)(0) \qquad (333) \\ = \ & \int_{\mathsf{U}_1} \chi_{c(g_1)}(u_1)t(g_2)\natural\phi(u_1)\, d\mu_{\mathsf{U}_1}(u_1) \\ = \ & \int_{\mathsf{U}_1}\int_{\mathsf{U}_2} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2) - 1)u\rangle) \\ & \chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2, u_1'\rangle)\phi(u_2 + u_1')\, d\mu_{\mathsf{U}_2}(u_2)\, d\mu_{\mathsf{U}_1}(u_1) \\ = \ & \int_{\mathsf{U}}\int_{\mathsf{U}_1'}\int_{\mathsf{U}_2} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2) - 1)u\rangle) \\ & \chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2, u_1'\rangle)\phi(u_2 + u_1')\, d\mu_{\mathsf{U}_2}(u_2)\, d\mu_{\mathsf{U}_1'}(u_1')\, d\mu_{\mathsf{U}}(u) \end{aligned}$$

The formula (321) applied with $x = c(g_1) + c(g_2)$ shows that

$$\int_{\mathsf{U}} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2)-1)u\rangle)\, d\mu_{\mathsf{U}}(u) \tag{334}$$

$$= \chi_{c(g_1)}(u_1') \int_{\mathsf{U}} \chi_{c(g_1)+c(g_2)}(u)\chi(\frac{1}{2}\langle u, c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2\rangle)\, d\mu_{\mathsf{U}}(u)$$

$$= 2^{\dim \mathsf{V}}\gamma(\tilde{q}_{g_1,g_2})\chi_{c(g_1)}(u_1')(\chi_{\underline{(c(g_1)+c(g_2))}^{-1}}\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp})(c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2).$$

Furthermore, Lemma 111 shows that, for $u_1' + u_2 \in \mathsf{U}_{12}$,

$$\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp}(c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2) = L^*(\mu_{\mathsf{V}^\perp/\mathsf{U}^\perp})(u_1' + u_2) \tag{335}$$

$$= |\det(\tilde{L})|^{-1}\mu_{\mathsf{U}_{12}}(u_1' + u_2).$$

The formula (331) follows directly from (332) - (335).

We see from (324) that

$$t(g_1)\natural(t(g_2)\natural\phi)(w) = (t(g_1)\natural(t(g_2)\natural\phi))\natural\delta_{-w}(0) = (t(g_1)\natural(t(g_2)\natural(\phi\natural\delta_{-w})))(0)$$

$$= ((t(g_1)\natural t(g_2))\natural(\phi\natural\delta_{-w}))(0) = ((t(g_1)\natural t(g_2))\natural\phi)\natural\delta_{-w})(0) = (t(g_1)\natural t(g_2))\natural\phi(w).$$

Therefore

$$(t(g_1)\natural t(g_2))\natural\phi = t(g_1)\natural(t(g_2)\natural\phi).$$

Hence, $t(g_1)\natural t(g_2)$ coincides with the composition of $t(g_1)$ and $t(g_2)$ as elements of the associative algebra $\mathrm{End}(\mathcal{S}(\mathrm{W}))$. Therefore the function C is a cocycle. □

6.6. **Normalization of Gaussians and the metaplectic group.** For an element $h \in \mathrm{End}(\mathrm{W})$ define $h^\# \in \mathrm{End}(\mathrm{W})$ by

$$\langle hw, w'\rangle = \langle w, h^\# w'\rangle \qquad (w, w' \in \mathrm{W}). \tag{336}$$

Then $(\mathrm{Ker}\, h^\#)^\perp = h\mathrm{W}$.

Lemma 120. *Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and assume that $K_1 = \mathrm{Ker}\, g_1^- = 0$. Then*

$$|\det(\frac{1}{2}\tilde{L}))|_{\mathbb{F}} = |\det(g_2^- : K_{12} \to \mathsf{V}|_{\mathbb{F}})^{-1}.$$

Proof. Since, by Lemma 7 (c), $g_2^- K_{12} = \mathsf{V}$, the right hand side of the equation we need to prove makes sense. A straightforward computation shows that

$$\frac{1}{2}\tilde{L}\colon \mathrm{W}/\mathsf{U}_{12} \ni w + \mathsf{U}_{12} \to \frac{1}{2}(c(g_1)-1)w + \mathsf{V}^\perp = g_1^{-1}w + \mathsf{V}^\perp \in \mathrm{W}/\mathsf{V}^\perp.$$

Hence,

$$\det(\frac{1}{2}\tilde{L})^{-1} = \det(g_1^- : \mathrm{W}/\mathsf{V}^\perp \to \mathrm{W}/\mathsf{U}_{12}).$$

Notice that $g_1^{-1} - 1 = g_1^\#$. Since $\mathsf{V} = g_2^- K_{12}$ and $\mathsf{U}_{12} = K_{12}^\perp$, Lemma 22 shows that

$$\det(g_1^- : \mathrm{W}/\mathsf{V}^\perp \to \mathrm{W}/\mathsf{U}_{12}) = \det(g_1^{-1} - 1\colon K_{12} \to \mathsf{V}).$$

Since the restrictions of g_1^{-1} and g_2 to K_{12} are equal, we are done. □

Let B be a non-degenerate (not necessarily symmetric) bilinear form on a finite dimensional vector space over $\mathbb{F}$. Define the discriminant of B as

$$\mathrm{dis}(B) = \frac{\gamma_W(\det(A))}{\gamma(1)}, \tag{337}$$

where A is the matrix obtained from a basis u_1, u_2, …, u_n of the space by

$$A_{i,j} = B(u_i, u_j) \qquad (1 \le i, j \le n).$$

Clearly the discriminant does not depend on the choice of the basis.

We have

$$\mathrm{dis}(B)^2 = \mathfrak{s}(\det(A)). \tag{338}$$

For any $g \in \mathrm{Sp(W)}$ the formula

$$\langle g^- w, w' \rangle \qquad (w, w' \in \mathrm{W})$$

defines a bilinear form whose left and right radicals coincide with $\mathrm{Ker}(g^-)$. Hence we get a non-degenerate bilinear form B_g on the quotient $\mathrm{W}/\mathrm{Ker}(g^-)$. Then

$$\mathrm{dis}(B_g) = \frac{\gamma_W(\det(\langle g^- w_i, w_j\rangle_{1\le i,j\le r}))}{\gamma(1)},$$

where $w_1 + \mathrm{Ker}(g^-)$, $w_2 + \mathrm{Ker}(g^-)$, …, $w_r + \mathrm{Ker}(g^-)$ is a basis of $\mathrm{W}/\mathrm{Ker}(g^-)$.

For $g \in \mathrm{Sp(W)}$ define

$$\theta(g) := \gamma(1)^{\dim g^- \mathrm{W}}\ \mathrm{dis}(B_g). \tag{339}$$

Lemma 121. *Let $g_1, g_2 \in \mathrm{Sp(W)}$. Assume that $K_1 = \mathrm{Ker}\, g_1^- = \{0\}$. Then*

$$\gamma_W(\tilde{q}_{g_1,g_2})^2 \ = \ \frac{\theta(g_1 g_2)^2}{\theta(g_1)^2\, \theta(g_2)^2}, \tag{340}$$

where $\tilde{q}_{g_1,g_2}$ is the non-degenerate symmetric form defined in Notation 6.

Proof. Let h be the element in $\mathrm{GL(W)}$ defined in Eqn. (24). Then since $\mathfrak{s}$ is a character, it follows from Eqns. (338) and (26) that

$$\mathfrak{s}(\det(\langle (g_1g_2)^- w_i, hw_j\rangle_{a<i,j}) = \mathrm{dis}(\tilde{q}_{g_1,g_2})^2\ \mathfrak{s}(\det(\langle g_1^- w_i, hw_j\rangle_{b<i,j})). \tag{341}$$

But

$$\mathfrak{s}(\det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j})) = \mathrm{dis}(B_{g_1g_2})^2.$$

Therefore (341) may be rewritten as

$$\mathrm{dis}(B_{g_1g_2})^2\ \mathfrak{s}(\det(h)) = \mathrm{dis}(\tilde{q}_{g_1,g_2})^2. \tag{342}$$

Notice that

$$\begin{aligned} \mathrm{dis}(B_{g_1})^2 \ &= \ \mathfrak{s}(\det g_1^-) = \mathfrak{s}(\det(g_1(g_1^{-1} - 1))) = \mathfrak{s}(\det(g_1^{-1} - 1)) \\ &= \ \mathfrak{s}(\det(g_1^{-1} - 1))^{-1}. \end{aligned}$$

Then, from (27), we obtain

$$\mathrm{dis}(B_{g_1})^{-2}\ \mathfrak{s}(\det(h)) = \mathfrak{s}(-1)^{\dim \mathsf{U}}\ \ \mathrm{dis}(B_{g_2})^2.$$

Therefore

$$\mathfrak{s}(\det(h)) = \mathfrak{s}(-1)^{\dim \mathsf{U}}\ \operatorname{dis}(B_{g_1})^2\ \operatorname{dis}(B_{g_2})^2. \tag{343}$$

By combining (342) and (343) we see that

$$\begin{aligned} \operatorname{dis}(\tilde{q}_{g_1,g_2})^2 &= \operatorname{dis}(B_{g_1g_2})^2\, \mathfrak{s}(-1)^{\dim \mathsf{U}}\ \operatorname{dis}(B_{g_1})^2\ \operatorname{dis}(B_{g_2})^2 \\ &= \mathfrak{s}(-1)^{\dim \mathsf{U}}\, \frac{\operatorname{dis}(B_{g_1g_2})^2}{\operatorname{dis}(B_{g_1})^2 \operatorname{dis}(B_{g_2})^2}. \end{aligned} \tag{344}$$

We see from (316) that

$$\gamma_W(\tilde{q}_{g_1,g_2})^2 = \gamma(1)^{2\dim \mathsf{U} - 2\dim \mathsf{V}} \operatorname{dis}(\tilde{q}_{g_1,g_2})^2 = \mathfrak{s}(-1)^{\dim \mathsf{U}}\gamma(1)^{-2\dim \mathsf{U} - 2\dim \mathsf{V}} \operatorname{dis}(\tilde{q}_{g_1,g_2})^2,$$

because $\gamma(1)^4 = \mathfrak{s}(-1)$, which follows from the equality $\gamma(1)\gamma(-1) = 1$. Therefore (344) implies (340). □

Definition 122. *For $g \in \mathrm{Sp}(\mathrm{W})$ define*

$$\begin{aligned} \Theta^2(g) &:= \gamma(1)^{2\dim g^-\mathrm{W} - 2}\, (\gamma(\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W}))^2 \\ &= \theta^2(g) |\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})|_{\mathbb{F}}^{-1}, \end{aligned}$$

where

$$\theta^2(g) = \gamma(1)^{2\dim g^-\mathrm{W}}\, \mathfrak{s}(\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})).$$

(Here $\mathfrak{s}$ was defined in Corollary 113.)

Lemma 123. *We have*

$$\frac{\Theta^2(g_1g_2)}{\Theta^2(g_1)\Theta^2(g_2)} = C(g_1,g_2)^2 \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})). \tag{345}$$

Proof. Both sides of the equality (345) are cocycles. Hence, Lemma 8 shows that we may assume that $K_1 = \{0\}$. Therefore the equality (345) is equivalent to

$$\begin{aligned} &\frac{\det((g_1g_2)^- \colon \mathrm{W}/K_{12} \to \mathsf{U}_{12})}{\det(g_1^- \colon \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})} \\ &= (-1)^{\dim \mathsf{U}} \det(\langle \tfrac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})\, (\det(g_2^- \colon K_{12} \to \mathsf{V}))^{-2} \end{aligned} \tag{346}$$

In particular

$$\begin{aligned} &\frac{|\det((g_1g_2)^- \colon \mathrm{W}/K_{12} \to \mathsf{U}_{12})|_{\mathbb{F}}}{|\det(g_1^- \colon \mathrm{W} \to \mathrm{W})|_{\mathbb{F}} |\det(g_2^- : \mathrm{W}/K_2 \to \mathsf{U})|_{\mathbb{F}}} \\ &= |\det(\langle \tfrac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})|_{\mathbb{F}}\, |\det(g_2^- \colon K_{12} \to \mathsf{V})|_{\mathbb{F}}^{-2} \end{aligned} \tag{347}$$

This, together with Lemma 120, shows that the right hand side of (347) is equal to

$$|\det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathsf{U}/\mathsf{V}})|_{\mathbb{F}} \left(2^{-\dim \mathsf{V}} |\det(\tilde{L})|_{\mathbb{F}}\right)^2,$$

which, by Proposition 119, coincides with $|C(g_1,g_2)|^{-2}$. Hence, the absolute values of the two sides of (345) are equal. Hence, (345) (without the absolute values) follows from Lemma 121. □

Definition 124. *Let*

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) := \{(g,\xi);\ g \in \mathrm{Sp}(\mathrm{W}),\ \xi \in \mathbb{C}^\times,\ \xi^2 = \Theta^2(g)\},$$

where $\Theta^2(g)$ *is as in Definition 122.*

Lemma 125. $\widetilde{\mathrm{Sp}}(\mathrm{W})$ *is a group with the multiplication defined by*

$$(g_1,\xi_1)(g_2,\xi_2) = (g_1g_2, \xi_1\xi_2\, C(g_1,g_2)) \qquad (g_1,g_2 \in \mathrm{Sp}(\mathrm{W})) \tag{348}$$

the identity equal to $(1,1)$ *and the inverse given by*

$$(g,\xi)^{-1} = (g^{-1},\overline{\xi}) \qquad (g \in \mathrm{Sp}(\mathrm{W})).$$

Proof. Lemma 123 shows that the right hand side of (348) belongs to $\widetilde{\mathrm{Sp}}(\mathrm{W})$. A standard computation, as in [22, page 366], shows that $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a group with the multiplication given by (348), the identity equal to $(1, C(1,1)^{-1})$ and

$$(g,\xi)^{-1} = (g^{-1}, C(g^{-1},g)^{-1}\xi^{-1}).$$

Since, by Proposition 119, $C(1,1) = 1$, it remains to check that

$$C(g^{-1},g)^{-1}\xi^{-1} = \overline{\xi}.$$

But, as in the proof of Lemma 120,

$$\begin{aligned} C(g^{-1},g) &= 2^{\dim \mathsf{V}}|\det(\tilde{L})|_{\mathbb{F}}^{-1} \\ &= |\det(g^-: \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})|_{\mathbb{F}} = |\Theta^2(g)|_{\mathbb{F}}^{-1} = |\xi|_{\mathbb{F}}^{-2}. \end{aligned}$$

This completes the proof. □

Notice that the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni (g,\xi) \to g \in \mathrm{Sp}(\mathrm{W})$$

is a group homomorphism with the kernel consisting of two elements. Thus $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a central extension of $\mathrm{Sp}(\mathrm{W})$ by the two element group $\mathbb{Z}/2\mathbb{Z}$:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Sp}(\mathrm{W}) \to 1. \tag{349}$$

Proposition 126. *The extension* (349) *does not split.*

Proof. Pick a two-dimensional symplectic subspace $\mathrm{W}_1 \subseteq \mathrm{W}$ and let $\mathrm{W}_2 = \mathrm{W}_1^\perp$, so that

$$\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2.$$

Define an element $g \in \mathrm{Sp}(\mathrm{W})$ by

$$g(w_1 + w_2) = -w_1 + w_2 \qquad (w_1 \in \mathrm{W}_1, w_2 \in \mathrm{W}_2).$$

Then $g^-|_{\mathrm{W}_1} = (a-1)\,\mathrm{I}_2$ and $g^-|_{\mathrm{W}_2} = 0$. Hence $\mathrm{Ker}(g^-) = \mathrm{W}_2$ and $g^-(\mathrm{W}) = \mathrm{W}_1$. We get

$$\begin{aligned} \Theta^2(g) &= \gamma(1)^4\,\mathfrak{s}(\det(g^-: \mathrm{W}_1 \to \mathrm{W}_1))\,|\det g^-: \mathrm{W}_1 \to \mathrm{W}_1))|_{\mathbb{F}}^{-1} \\ &= \gamma(1)^4\,\mathfrak{s}(4)\,|-4|_{\mathbb{F}}^{-1} \\ &= \frac{\gamma(1)^4}{|-4|_{\mathbb{F}}^2}. \end{aligned}$$

We have $g^2 - 1 = 0$, and Eqn. (345) gives

$$C(g,g)^2 = \frac{1}{(\Theta^2(g))^2} = |-2|_{\mathbb{F}}^4.$$

Let $\tilde{g} = \left(g, \frac{\gamma(1)^2}{|-2|_{\mathbb{F}}}\right)$. Then $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, and

$$\tilde{g}^2 = (g^2, \Theta^2(g)\, C(g,g)) = (g^2, \gamma(1)^4) \quad \text{and} \quad \tilde{g}^4 = (g^4, \Theta^2(g^2)\, C(g^2, g^2)) = (g^4, 1).$$

Thus the subgroup of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ generated by $\tilde{g}$ is cyclic of order 4. The subgroup of $\mathrm{Sp}(\mathrm{W})$ generated by g is cyclic of order 2. Hence the extension (349) does not split over that subgroup. □

Corollary 127. *Up to an equivalence of central group extensions, as in* [22, sec. 6.10], (349) *is the only non-trivial central extension of* $\mathrm{Sp}(\mathrm{W})$ *by* $\mathbb{Z}/2\mathbb{Z}$.

Proof. Since, as is well known (see [27, Theorems 5.10 and 11.1 (b)]),

$$H^2(\mathrm{Sp}(\mathrm{W}), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}),$$

the claim follows. □

Let

$$\phi^*(w) = \overline{\phi(-w)} \text{ and } u^*(\phi) = u(\phi^*) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ u \in \mathcal{S}^*(\mathrm{W}),\ w \in \mathrm{W}).$$

Lemma 128. *For any* $g \in \mathrm{Sp}(\mathrm{W})$, $t(g)^* = t(g^{-1})$.

Proof. By the definition (325),

$$t(g)^* = \left(\chi_{c(g)}\mu_{g^-\mathrm{W}}\right)^* = \overline{\chi_{c(g)}}\mu_{g^-\mathrm{W}} = \chi_{-c(g)}\mu_{g^-\mathrm{W}}.$$

Since $g^-\mathrm{W} = (g^{-1}-1)\mathrm{W}$, it will suffice to check that for any $w \in \mathrm{W}$

$$-c(g)g^-w = c(g^{-1})g^-w.$$

The left hand side is equal to $-g^+w$. The right hand side is equal to

$$-c(g^{-1})(g^{-1}-1)gw = -(g^{-1}-1)gw = -g^+w.$$

□

Definition 129. *For* $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ *define*

$$\Theta(\tilde{g}) = \xi \text{ and } T(\tilde{g}) = \Theta(\tilde{g})t(g). \tag{350}$$

Lemma 130. *With the notation of (350), the following formulas hold*

$$T(\tilde{g}_1)\natural T(\tilde{g}_2) = T(\tilde{g}_1\tilde{g}_2) \qquad (\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})), \tag{351}$$

$$T(\tilde{g})^* = T(\tilde{g}^{-1}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})). \tag{352}$$

Proof. By Proposition 119 the left hand side of (351) is equal to

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1\tilde{g}_2)}C(g_1, g_2)T(\tilde{g}_1\tilde{g}_2).$$

Lemma 125 shows that

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1\tilde{g}_2)}C(g_1, g_2) = 1.$$

This verifies (351).

The equality (352) follows from Lemma 125 and Lemma 128:

$$T(\tilde{g})^* = \overline{\Theta(\tilde{g})}t(g)^* = \Theta(\tilde{g}^{-1})t(g^{-1}) = T(\tilde{g}^{-1}).$$

□

Lemma 131. *The map* $T : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})$ *is injective and continuous.*

Proof. The injectivity of T follows from the injectivity of $t : \mathrm{Sp}(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})$, which is obvious. Let

$$\mathrm{Sp}^c(\mathrm{W}) = \{g \in \mathrm{Sp}(\mathrm{W});\ \det g^- \neq 0\}.$$

Lemma 8 shows that

$$\mathrm{Sp}(\mathrm{W}) = \bigcup_{h\in \mathrm{Sp}(\mathrm{W})} \mathrm{Sp}^c(\mathrm{W})h. \tag{353}$$

Let $\widetilde{\mathrm{Sp}}^c(\mathrm{W}) \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W})$ be the preimage of $\mathrm{Sp}^c(\mathrm{W})$. Then

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \bigcup_{\tilde{h}\in \widetilde{\mathrm{Sp}}(\mathrm{W})} \widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h}.$$

By Lemma 130, we have

$$T(\tilde{g}) = T(\tilde{g}\tilde{h}^{-1})\natural T(\tilde{h}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h})$$

Thus for $\phi \in \mathcal{S}(\mathrm{W})$,

$$T(\tilde{g})\natural\phi = T(\tilde{g}\tilde{h}^{-1})\natural(T(\tilde{h})\natural\phi).$$

By Lemma 118, the map

$$\mathcal{S}(\mathrm{W}) \ni \phi \to T(\tilde{h})\natural\phi \in \mathcal{S}(\mathrm{W})$$

is continuous. Hence it will suffice to check that the restriction of T to $\widetilde{\mathrm{Sp}}^c(\mathrm{W})$ is continuous. But this is obvious. □

6.7. **The conjugation property.** Let $\mathrm{L}^2(\mathrm{W})$ denote the Hilbert space of the Lebesgue measurable functions $\phi\colon \mathrm{W}\to\mathbb{C}$, with the norm given by

$$\|\ \phi\ \|_2^2=\int_{\mathrm{W}}|\phi(w)|^2\,d\mu_{\mathrm{W}}(w).$$

Lemma 57 shows that for any $\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W})$ and any $\phi\in\mathcal{S}(\mathrm{W})$

$$\|\ T(\tilde{g})\natural\phi\ \|_2^2=(T(\tilde{g})\natural\phi)^*\natural(T(\tilde{g})\natural\phi)(0)=\phi^*\natural T(\tilde{g})^*\natural T(\tilde{g})\natural\phi(0)=\phi^*\natural\phi(0)=\|\ \phi\ \|_2^2\,.$$

Hence, the continuous linear map

$$\mathcal{S}(\mathrm{W})\ni\phi\to T(\tilde{g})\natural\phi\in\mathcal{S}(\mathrm{W})$$

extends by continuity to an isometry

$$\mathrm{L}^2(\mathrm{W})\ni\phi\to T(\tilde{g})\natural\phi\in\mathrm{L}^2(\mathrm{W}).$$

Furthermore, the formula

$$\omega_{1,1}(g)\phi(w)=\phi(g^{-1}w)\qquad(g\in\mathrm{Sp}(\mathrm{W}),\ \phi\in\mathrm{L}^2(\mathrm{W})).$$

defines a unitary representation $\omega_{1,1}$ of the symplectic group $\mathrm{Sp}(\mathrm{W})$ on $\mathrm{L}^2(\mathrm{W})$.

Proposition 132. *For any $\phi\in\mathrm{L}^2(\mathrm{W})$ and $\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W})$ in the preimage of $g\in\mathrm{Sp}(\mathrm{W})$, $T(\tilde{g})\natural\phi\natural T(\tilde{g}^{-1})=\omega_{1,1}(g)\phi$.*

Proof. Since $T(\tilde{g})\natural$ is a bounded operator, we may assume that $\phi\in\mathcal{S}(\mathrm{W})$. Lemma 42 says that

$$t(g)\natural\delta_w=\delta_{wg}\natural t(g)\qquad(w\in\mathrm{W}).$$

Therefore

$$T(\tilde{g})\natural\delta_w=\delta_{wg}\natural T(\tilde{g})\qquad(w\in\mathrm{W}).$$

Since,

$$\phi=\int_{\mathrm{W}}\phi(w)\delta_w\,d\mu_{\mathrm{W}}(w)\ \text{and}\ \int_{\mathrm{W}}\phi(w)\delta_{gw}\,d\mu_{\mathrm{W}}(w)=\omega_{1,1}(g)\phi,$$

we see that

$$T(\tilde{g})\natural\phi=(\omega_{1,1}(g)\phi)\natural T(\tilde{g}).$$

□

6.8. **The Weyl transform and the Weil representation.** Pick a complete polarization

$$\mathsf{W}=\mathsf{X}\oplus\mathsf{Y}\tag{354}$$

and recall that our normalization of measures is such that $d\mu_{\mathsf{W}}(x+y)=d\mu_{\mathsf{X}}(x)d\mu_{\mathsf{Y}}(y)$. Recall the Weyl transform

$$\mathcal{K}\colon\mathcal{S}^*(\mathsf{W})\to\mathcal{S}^*(\mathsf{X}\times\mathsf{X}),\tag{355}$$
$$\mathcal{K}(f)(x,x')=\int_{\mathsf{Y}}f(x-x'+y)\chi(\frac{1}{2}\langle y,x+x'\rangle)\,d\mu_{\mathsf{Y}}(y),$$

This is an isomorphism of linear topological spaces, which restricts to an isometry

$$\mathcal{K}\colon \mathrm{L}^2(\mathsf{W}) \to \mathrm{L}^2(\mathsf{X}\times\mathsf{X}). \tag{356}$$

Each element $K \in \mathcal{S}^*(\mathsf{X}\times\mathsf{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))$ by

$$(\mathrm{Op}(K)(v))(u) = K(u\otimes v) \qquad (u, v \in \mathcal{S}(\mathsf{X})). \tag{357}$$

Since the map

$$\mathcal{S}(\mathsf{X})\times\mathcal{S}(\mathsf{X}) \ni (u,v) \to u\otimes v \in \mathcal{S}(\mathsf{X}\times\mathsf{X})$$

is continuous, (357) defines a continuous injection

$$\mathrm{Op}: \mathcal{S}^*(\mathsf{X}\times\mathsf{X}) \to \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X})). \tag{358}$$

Conversely, if $S \in \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))$, then

$$S(v)(u) \qquad (u, v \in \mathcal{S}(\mathsf{X}))$$

defines a continuous linear map on $\mathcal{S}(\mathsf{X})\otimes\mathcal{S}(X) = \mathcal{S}(\mathsf{X}\times\mathsf{X})$. Hence the map (358) is bijective and thus a linear topological isomorphism.
A straightforward computation shows that $\mathrm{Op}\circ\mathcal{K}$ transforms the twisted convolution of distributions (when it makes sense) into the composition of the corresponding operators. Also,

$$(\mathrm{Op}\circ\mathcal{K}(f))^* = \mathrm{Op}\circ\mathcal{K}(f^*) \qquad (f\in\mathcal{S}^*(\mathrm{W})) \tag{359}$$

and

$$\mathrm{tr}\ \mathrm{Op}\circ\mathcal{K}(f) = \int_{\mathsf{X}} \mathcal{K}(f)(x,x)\, d\mu_{\mathsf{X}}(x) = f(0) \tag{360}$$

if $\mathrm{Op}\circ\mathcal{K}(f)$ is of trace class, [18, Theorem 3.5.4] (More precisely the same proof works). Hence, the map

$$\mathrm{Op}\circ\mathcal{K}\colon \mathrm{L}^2(\mathsf{W}) \to \mathrm{H.S.}(\mathrm{L}^2(\mathsf{X})) \tag{361}$$

is an isometry, which is a well known fact [18, Theorem 1.4.1]. (Here $\mathrm{H.S.}(\mathrm{L}^2(\mathsf{X}))$ stands for the space of the Hilbert-Schmidt operators on $\mathrm{L}^2(\mathsf{X})$.)

Let $\mathrm{U}(\mathrm{L}^2(\mathsf{X}))$ denote the group of the on the Hilbert space $\mathrm{L}^2(\mathsf{X})$.

Theorem 133. *Let $\omega = \mathrm{Op}\circ\mathcal{K}\circ T$. Then*

$$\omega\colon \mathrm{Sp}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X}))$$

is an injective group homomorphism. For each $v\in\mathrm{L}^2(\mathsf{X})$, the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde{g} \to \omega(\tilde{g})v \in \mathrm{L}^2(\mathsf{X})$$

is continuous, so that $(\omega, \mathrm{L}^2(\mathsf{X}))$ is a unitary representation of the metaplectic group. The function Θ coincides with the character of this representation:

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} = \mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} \qquad (\Psi\in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W})),$$

where the integral on the left is absolutely convergent. (Here $d\tilde{g}$ stands for any Haar measure on $\widetilde{\mathrm{Sp}}(\mathrm{W})$.) Moreover,

$$\omega(\tilde{g})\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(\tilde{g}^{-1}) = \mathrm{Op}\circ\mathcal{K}(\omega_{1,1}(g)\phi) \qquad (\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W}),\ \phi\in\mathrm{L}^2(\mathrm{W})).$$

Proof. We see from the discussion in section 6.7 that the left multiplication by $\omega(\tilde{g})$ is an isometry on H.S.$(\mathrm{L}^2(\mathsf{X}))$. This implies that $\omega(\tilde{g})$ is a unitary operator.

We see from (358) that for any two function $v_1, v_2 \in \mathcal{S}(\mathsf{X})$ there is $\phi \in \mathcal{S}(\mathrm{W})$ such that

$$\int_{\mathsf{X}} \omega(\tilde{g})v_1(x)\overline{v_2(x)}\, d\mu_{\mathsf{X}}(x) = T(\tilde{g})(\phi) \qquad (\tilde{g} \in \mathrm{Sp}(\mathrm{W})).$$

Hence Lemma 131 shows that the left hand side is a continuous function of $\tilde{g}$. Since the operators $\omega(\tilde{g})$ are uniformly bounded (by 1), we see that the left hand side is a continuous function of $\tilde{g}$ for any $v_1, v_2 \in \mathrm{L}^2(\mathsf{X})$. This implies the strong continuity of ω, see [49, Lemma 1.1.3] or [50, Proposition 4.2.2.1].

Lemmas 130 and 131 show that the $\omega : \mathrm{Sp}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X}))$ is an injective group homomorphism.

It is not difficult to check that the function

$$\frac{\det(\mathrm{Ad}(g) - 1)}{\det g^-} \qquad (g \in \mathrm{Sp}(\mathrm{W}))$$

is locally bounded. Furthermore, as shown by Harish-Chandra [12, Section 8], the function

$$|\det(\mathrm{Ad}(g) - 1)|_{\mathbb{F}}^{-1/2} \qquad (g \in \mathrm{Sp}(\mathrm{W})) \tag{362}$$

is locally integrable. Hence the function,

$$|\Theta(\tilde{g})| = |\det g^-|_{\mathbb{F}}^{-1/2} \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}))$$

is locally integrable. (We would like to thank Alan Roche for the reference, [12].)

Notice that for any $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$,

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} \in \mathcal{S}(\mathrm{W}). \tag{363}$$

Indeed, since the Zariski topology on Sp(W) is noetherian the covering (153) contains a finite subcovering (see for example [13, Exercise 1.7(b)]). Hence, there are elements $\tilde{h}_1$, $\tilde{h}_2$, ..., $\tilde{h}_m$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ such that

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \bigcup_{j=1}^{m} \widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h}_j.$$

Therefore Lemma 130 and a standard partition of the unity argument reduces the proof of (163) to the case when $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}^c(\mathrm{W}))$. In this case (163) is equal to

$$\int_{\mathfrak{sp}(\mathrm{W})} \chi_x(w)\psi(x)\, dx \tag{364}$$

where $\psi \in C_c^\infty(\mathfrak{sp}(\mathrm{W}))$ and dx is a Haar measure on $\mathfrak{sp}(\mathrm{W})$. The function (364) is equal to the pullback of a Fourier transform $\hat{\psi}$ of ψ from $\mathfrak{sp}^*(\mathrm{W})$ to W via the unnormalized moment map

$$\tau : \mathrm{W} \to \mathfrak{sp}^*(\mathrm{W}),\ \tau(w)(x) = \langle xw, w\rangle \qquad (x \in \mathfrak{sp}(\mathrm{W}),\ w \in \mathrm{W}). \tag{365}$$

Since $\hat{\psi} \in \mathcal{S}(\mathfrak{sp}(\mathrm{W}))$ and since τ is a polynomial map with uniformly bounded fibers,

$$\hat{\psi} \circ \tau \in \mathcal{S}(\mathrm{W}).$$

This verifies (363). Hence, we may compute the trace as follows:

$$\begin{aligned}
\mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} &= \left(\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}\right)(0) = \left(\int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}\right)(0) \\
&= \int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})(0)\Psi(\tilde{g})\, d\tilde{g} = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}.
\end{aligned}$$

The last formula is a direct consequence of Proposition 132. □

We end this section by recalling some well known formulas for the action of $\omega(\tilde{g})$ for some special elements $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$.

Proposition 134. *Let* $\mathrm{M} \subseteq \mathrm{Sp}(\mathrm{W})$ *be the subgroup of all the elements that preserve* X *and* Y. *Let* $\mathrm{M}^c := \{g \in \mathrm{M} \,:\, \det g^- \neq 0\}$. *Set*

$$\zeta(\tilde{g}) := \Theta(\tilde{g})\, |\det(\frac{1}{2}(c(g|_{\mathsf{X}}) + 1))|_{\mathbb{F}}^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c).$$

Then

$$(\zeta(\tilde{g}))^2 = (\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\, |\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c), \tag{366}$$

the function $\zeta\colon \widetilde{\mathrm{M}}^c \to \mathbb{C}^\times$ *extends to a continuous group homomorphism*

$$\zeta\colon \widetilde{\mathrm{M}} \to \mathbb{C}^\times$$

and

$$\omega(\tilde{g})v(x) = \zeta(\tilde{g})v(g^{-1}x) \qquad (\tilde{g} \in \widetilde{\mathrm{M}},\ v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}). \tag{367}$$

Proof. Set $n = \dim \mathsf{X}$. Fix an element $g \in \mathrm{M}^c$. Observe that

$$\det(g|_{\mathsf{Y}} - 1) = \det((g|_{\mathsf{X}})^{-1} - 1) = \det((g|_{\mathsf{X}})^{-1})\, \det(1 - g|_{\mathsf{X}}).$$

Then it follows from Definition 122 that

$$\begin{aligned}
\Theta^2(g) &= \gamma(1)^{4n}\, \mathfrak{s}(\det g^-)\, |\det g^-|_{\mathbb{F}}^{-1} \\
&= \gamma(1)^{4n}\, \mathfrak{s}(\det(g|_{\mathsf{X}} - 1)\det(g|_{\mathsf{Y}} - 1))\, |\det(g|_{\mathsf{X}} - 1)^{-1}\det(g|_{\mathsf{Y}} - 1)|_{\mathbb{F}}^{-1} \\
&= \gamma(1)^{4n}\, \mathfrak{s}(\det(g|_{\mathsf{X}} - 1))\, \mathfrak{s}(\det(g|_{\mathsf{Y}} - 1)\, |\det(g|_{\mathsf{X}} - 1)|_{\mathbb{F}}^{-1}\, |\det(g|_{\mathsf{Y}} - 1)|_{\mathbb{F}}^{-1} \\
&= \gamma(1)^{4n}\, \mathfrak{s}(\det(g|_{\mathsf{X}} - 1)^2)\, \mathfrak{s}(\det(-(g|_{\mathsf{X}})^{-1}))\, |\det(g|_{\mathsf{X}} - 1)|_{\mathbb{F}}^{-2}\, |\det(g|_{\mathsf{X}})|_{\mathbb{F}} \\
&= \gamma(1)^{4n}\, \mathfrak{s}((-1)^n)\, (\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\, |\det(g|_{\mathsf{X}} - 1)|_{\mathbb{F}}^{-2}\, |\det(g|_{\mathsf{X}})|_{\mathbb{F}} \\
&= \gamma(1)^{4n}\, (\mathfrak{s}(-1))^n\, (\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\, |\det(g|_{\mathsf{X}} - 1)|_{\mathbb{F}}^{-2}\, |\det(g|_{\mathsf{X}})|_{\mathbb{F}}.
\end{aligned}$$

Also,

$$\begin{aligned}
|\det(\frac{1}{2}(c(g|_{\mathsf{X}}) + 1))|_{\mathbb{F}}^{-1} &= |\det((g|_{\mathsf{X}})(g|_{\mathsf{X}} - 1)^{-1})|_{\mathbb{F}}^{-1} \\
&= |\det(g|_{\mathsf{X}} - 1))|_{\mathbb{F}}\, |\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}(\zeta(\tilde{g}))^2 &= \gamma(1)^{4n}\,(\mathfrak{s}(-1))^n(\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\,|\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1}\\ &= \gamma(1)^{2n}\,\gamma(-1)^{2n}\,(\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\,|\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1}\\ &= (\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\,|\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1}.\end{aligned}$$

This verifies (366).

Let $x, x' \in \mathsf{X}$ and let $y \in \mathsf{Y}$. Then

$$\begin{aligned}\mathcal{K}(t(g))(x,x') &= \int_{\mathsf{Y}} t(g)(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y)\\ &= \int_{\mathsf{Y}} \chi(\frac{1}{2}\langle c(g)(x-x'), y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y)\\ &= \delta_0(\frac{1}{2}c(g)(x-x') - x - x') = \delta_0(\frac{1}{2}((c(g)-1)x - (c(g)+1)x'))\\ &= |\det(\frac{1}{2}(c(g|_{\mathsf{X}})+1))|_{\mathbb{F}}^{-1}\delta_0(g^{-1}x - x').\end{aligned}$$

Therefore

$$\mathcal{K}(T(\tilde{g}))(x,x') = \zeta(\tilde{g})\delta_0(g^{-1}x - x').$$

Thus we have (367) for $\tilde{g} \in \widetilde{\mathrm{M}}^c$. Since ω is a representation of $\widetilde{\mathrm{M}}$, the remaining claims follow. □

Proposition 135. *Suppose $g \in \mathrm{Sp(W)}$ acts trivially on Y and on W/Y. Then $\det((-g)-1) \neq 0$ and*

$$\omega(\tilde{g})v(x) = \xi_0\,\chi_{c(-g)}(2x)\,v(x) \qquad (v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}), \quad \textit{where } \xi_0^2 = (\mathfrak{s}(2))^{2n}.$$

Proof. Since $-g$ acts as minus the identity on Y and on W/Y, $\det((-g)-1) \neq 0$ and $z = c(-g) \in \mathfrak{sp}(\mathrm{W})$ is well defined. We have

$$z(w) = (-g)^+((-g)^-)^{-1}(w) \quad (w \in \mathrm{W}).$$

Since g acts trivially on Y and on W/Y, we get, for every $x \in \mathsf{X}$ and every $y \in \mathsf{Y}$:

$$g(x+y) = x + y + y_x, \quad \text{where } y_x \in \mathsf{Y}.$$

It gives $(-g)^-(x+y) = -2x - 2y - y_x$. Hence

$$((-g)^-)^{-1}(x+y) = -\frac{1}{2}(x+y) + \frac{1}{4}y_x.$$

We obtain

$$z(x+y) = (-g)^+(-\frac{1}{2}(x+y) + \frac{1}{4}y_x) = \frac{1}{2}y_x.$$

In particular, we have

$$z\colon \mathsf{X} \to \mathsf{Y} \to 0.$$

Also, $\det(z-1)\neq 0$ and $c(z)$ is well defined. On the other hand, we have $(z-1)(x+y) = -(x+y)+\frac{1}{2}y_x$. It follows that

$$(z-1)^{-1}(x+y) = -(x+y) - \frac{1}{2}y_x.$$

Hence,

$$c(z)(x+y) = (z+1)\left(-(x+y)-\frac{1}{2}y_x\right) = -\frac{1}{2}y_x - (x+y) - \frac{1}{2}y_x,$$

that is,

$$c(z)(x+y) = -(x+y) - y_x. \tag{368}$$

We have $c(z)\in \mathrm{Sp(W)}$. Indeed, for any $w, w'\in \mathrm{W}$, writing $w = x+y$ and $w' = x'+y'$, with $x, x'\in \mathsf{X}$ and $y, y'\in \mathsf{Y}$, we have

$$\langle c(z)(w), c(z)(w')\rangle = \langle -w-y_x, -w'-y_{x'}\rangle = \langle w, w'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle.$$

However, since g is in $\mathrm{Sp(W)}$, we have

$$\langle x, x'\rangle = \langle gx, gx'\rangle = \langle x+y_x, x'+y_{x'}\rangle = \langle x, x'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle,$$

which gives

$$\langle x, y_{x'}\rangle + \langle y_x, x'\rangle = 0.$$

We obtain

$$\begin{aligned}\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi_z(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\ &= \chi_z(x-x')\delta_0(\frac{1}{2}(x+x')) = 2^n\,\chi_z(x-x')\,\delta_0(x+x').\end{aligned}$$

We have $\dim\left((c(z)-1)(\mathrm{W})\right) = \dim \mathrm{W} = 2n$, and,

$$\det\left(c(z)-1\right) = (-2)^{2n}.$$

We get

$$\begin{aligned}\Theta^2(c(z)) &= \gamma(1)^{4n}\ (\mathfrak{s}(-2))^{2n}\ 2^{-2n}\\ &= \gamma(1)^{4n}\ (\mathfrak{s}(-1)\mathfrak{s}(2))^{2n}\ 2^{-2n}\\ &= \gamma(1)^{4n}\ (\mathfrak{s}(-1))^{2n}\,\frac{\gamma(2)^{4n}}{\gamma(1)^{4n}},\end{aligned}$$

since $\mathfrak{s}(-1) = \gamma(1)^4$, and $\gamma(1)^8 = 1$. Hence,

$$\Theta^2(c(z)) = \gamma(2)^{4n}. \tag{369}$$

Thus

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') = 2^n\,\xi_0'\,\chi_z(x-x')\delta_0(x+x'),\quad \text{where } (\xi_0')^2 = \gamma(2)^{4n}.$$

Proposition 134 shows that

$$\omega((\widetilde{-1}))v(x) = \zeta(\widetilde{-1})v(-x).$$

We have

$$\left(\zeta(\widetilde{-1})\right)^2 = \mathfrak{s}((-1)^n)^{-1} = (\mathfrak{s}(-1))^{-n} = \gamma(1)^{-4n}.$$

Since

$$(\mathfrak{s}(2))^{2n} = 2^{2n}\left(\frac{\gamma(2)^2}{\gamma(1)^2}\right)^{2n},$$

the proof is complete. □

Proposition 136. *Suppose $g \in \mathrm{Sp}(\mathrm{W})$ acts trivially on X and on W/X. Then $\det((-g)-1) \neq 0$ so that $z = c(-g) \in \mathfrak{sp}(\mathrm{W})$ is well defined and $z : \mathsf{Y} \to \mathsf{X} \to 0$. Assume $z(\mathsf{Y}) = \mathsf{X}$. Then*

$$\omega(\tilde{g})v(x) = \pm\left(\frac{\mathfrak{s}(2)}{2}\right)^n \gamma(q)\int_{\mathsf{X}} \chi_{z^{-1}}(x-x')v(x')\,d\mu_{\mathsf{X}}(x') \qquad (v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}),$$

where $z^{-1} : \mathsf{X} \to \mathsf{Y}$ is the inverse of $z\colon \mathsf{Y} \to \mathsf{X}$. (The explicit computation of $\gamma(q)$ may be found in [36, Appendix]*.*

Proof. The existence of z and its properties are verified as in the proof of Proposition 135. In particular, for all $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, we have

$$g(x+y) = x + y + x_y, \quad \text{where } x_y \in \mathsf{X}.$$

Similarly to the proof of Proposition 135, we get

$$z(x+y) = z(y) = \frac{1}{2}x_y. \tag{370}$$

and

$$c(z)(x+y) = -(x+y) - x_y, \tag{371}$$

that is,

$$c(z)(w) = -w - 2z(w), \quad \text{for every } w \in \mathrm{W}. \tag{372}$$

Let

$$q(y,y') = \frac{1}{2}\langle zy, y'\rangle \qquad (y, y' \in \mathsf{Y}).$$

Then, in terms of Lemma 115 and the identification (318),

$$q^*(x,x') = -2\langle z^{-1}x, x'\rangle \qquad (x, x' \in \mathsf{X}).$$

Hence, by the definition of $\mathcal{K}$ (355), the assumption that z annihilates X and maps Y into X and Lemma 115, we obtain

$$\begin{aligned}
\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -z(x-x'+y), x-x'+y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\
&= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -zy, y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y) \\
&= \int_{\mathsf{Y}} \chi(\frac{1}{2}q(y,y))\chi(-\langle y, -\frac{1}{2}(x+x')\rangle)\, d\mu_{\mathsf{Y}}(y) \\
&= \gamma(q)\chi(-\frac{1}{2}q^*(-\frac{1}{2}(x+x'), -\frac{1}{2}(x+x'))) \\
&= \gamma(q)\chi(\langle z^{-1}(-\frac{1}{2}(x+x')), -\frac{1}{2}(x+x')\rangle) = \gamma(q)\chi_{z^{-1}}(x+x').
\end{aligned}$$

Therefore

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') = \Theta(\widetilde{c(z)})\gamma(q)\chi_{z^{-1}}(x+x').$$

But $\Theta(\widetilde{c(z)})^2 = \pm\gamma(2)^{4n}$ (see Eqn. (369)), where $\dim \mathrm{W} = 2n$. Furthermore, by Proposition 134,

$$\mathcal{K}(T(\widetilde{-1}))(x',x'') = \zeta(\widetilde{-1})\,\delta_0(x'-x''),$$

where $(\zeta(\widetilde{-1}))^2 = \gamma(1)^{-4n}$. Hence, the formula for $\omega(\tilde{g})$ follows. □

6.9. **An extension of ω to $\widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H}(\mathrm{W})$.** By the Heisenberg group we understand the direct product $\mathrm{H}(\mathrm{W}) = \mathrm{W} \times \mathbb{F}$ with the multiplication given by

$$(w,r)(w',r') = (w+w', r+r'+\frac{1}{2}\langle w,w'\rangle) \qquad ((w,r),(w',r') \in \mathrm{H}(\mathrm{W})).$$

Set

$$T(w,r) = \chi(r)\delta_w \qquad ((w,r) \in \mathrm{H}(\mathrm{W})). \tag{373}$$

Then

$$T : \mathrm{H}(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})$$

is a continuous embedding of the Heisenberg group into the space of the tempered distributions on W. Since the metaplectic group acts on the Heisenberg group via automorphisms

$$\tilde{g}(w,r) = (gw,r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}), (w,r) \in \mathrm{H}(\mathrm{W})),$$

we have the semidirect product $\widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H}(\mathrm{W})$, which we embed into the space of the tempered distributions by

$$T(\tilde{g},(w,r)) = T(\tilde{g})\natural T(w,r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}), (w,r) \in \mathrm{H}(\mathrm{W})). \tag{374}$$

Lemma 117 shows that

$$T(\tilde{g})\natural T(w,r)\natural T(\tilde{g}^{-1}) = T(gw,r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}), (w,r) \in \mathrm{H}(\mathrm{W})). \tag{375}$$

Theorem 137. *Let* $\omega = \mathrm{Op} \circ \mathcal{K} \circ T$. *Then*

$$\omega\colon \widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathsf{X}))$$

is an injective group homomorphism. For each $v \in \mathrm{L}^2(\mathsf{X})$, *the map*

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ltimes \mathrm{H}(\mathrm{W}) \ni \tilde{g} \to \omega(\tilde{g})v \in \mathrm{L}^2(\mathsf{X})$$

is continuous, so that $(\omega, \mathrm{L}^2(\mathsf{X}))$ *is a unitary representation of the group. In particular,*

$$\omega(\tilde{g})\omega(w,r)\omega(\tilde{g}^{-1}) = \omega(gw,r) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}),\ (w,r) \in \mathrm{H}(\mathrm{W})). \tag{376}$$

Explicitly, for $v \in \mathrm{L}^2(\mathsf{X})$ *and* $x \in \mathsf{X}$,

$$\begin{aligned} \omega(x_0,r)v(x) &= \chi(r)v(x-x_0) \quad (x_0 \in \mathsf{X}, r \in \mathbb{F})\,, \\ \omega(y_0,r)v(x) &= \chi(r)\chi(\langle y_0, x\rangle)v(x) \quad (y_0 \in \mathsf{Y}, r \in \mathbb{F})\,, \end{aligned} \tag{377}$$

Hence, the restriction of ω *to* $\mathrm{H}(\mathrm{W})$ *is irreducible.*

For a test function $\Phi \in C_c^\infty(\mathrm{H}(\mathrm{W}))$ *define a partial Fourier transform*

$$\Phi_\chi(w) = \int_{\mathbb{R}} \Phi(w,r)\chi(r)\,dr \qquad (w \in \mathrm{W}, r \in \mathbb{F}).$$

Then

$$\mathrm{tr}\,\omega(\Phi) = \Phi_\chi(0). \tag{378}$$

Thus the character of $\omega|_{\mathrm{H}(\mathrm{W})}$ *is equal to the the tensor product* $\delta_0 \otimes \chi$ *of the Dirac delta on* W *and the charcter* χ *multiplied by the Lebesgue measure on* $\mathbb{F}$.

For test functions $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$ *and* $\Phi \in C_c^\infty(\mathrm{H}(\mathrm{W}))$,

$$\mathrm{tr}\,(\omega(\Psi)\omega(\Phi)) = (T(\Psi)\natural\Phi_\chi)(0) = T(\Psi)(\Phi_\chi). \tag{379}$$

Proof. This is straightforward. For the irreduciblility it is convenient to check that the only bounded operator on $\mathrm{L}^2(\mathsf{X})$ that commutes with the action of the Heisenberg group is a constant multiple of the identity. ☐

6.10. **The lattice model.** Let $\mathbb{I}_{\mathcal{L}}$ denote the indicator function of the lattice $\mathcal{L} \subseteq \mathrm{W}$. Then the twisted convolution (323),

$$\delta_w \natural \mathbb{I}_{\mathcal{L}}(w') = \mathbb{I}_{w+\mathcal{L}}(w')\chi(\frac{1}{2}\langle w, w'\rangle) \qquad (w, w' \in \mathsf{W})\,. \tag{380}$$

Since we asume that the kernel of the character χ is equal to $\mathfrak{o}_{\mathbb{F}}$, we see from above that

$$\delta_{w+l}\natural\mathbb{I}_{\mathcal{L}} = \chi(\frac{1}{2}\langle l, w\rangle)\delta_w\natural\mathbb{I}_{\mathcal{L}} \qquad (w \in \mathsf{W},\ l \in \mathcal{L})\,. \tag{381}$$

Thus $\delta_{w+l}\natural\mathbb{I}_{\mathcal{L}}$ is a constant multiple of $\delta_w\natural\mathbb{I}_{\mathcal{L}}$. Select a set of representatives $\mathcal{A} \subseteq \mathsf{W}$ of the cosets $\mathsf{W}/\mathcal{L}$ so that we have the disjoint union decomposition of W,

$$\mathsf{W} = \bigcup_{\alpha \in \mathcal{A}} (\alpha + \mathcal{L})\,. \tag{382}$$

We shall assume tht $0 \in \mathcal{A}$. Let

$$f_\alpha = \delta_\alpha \natural \mathbb{I}_{\mathcal{L}} \qquad (\alpha \in \mathcal{A})\,. \tag{383}$$

The functions (383) form an orthonormal subset of $\mathrm{L}^2(\mathsf{W})$. Let $\mathcal{H} \subseteq \mathrm{L}^2(\mathsf{W})$ be the Hilbert subspace spanned by them,

$$\mathcal{H} = \left\{ \sum_{\alpha \in \mathcal{A}} c_\alpha f_\alpha;\ \sum_{\alpha \in \mathcal{A}} |c_\alpha|^2 < \infty \right\}. \tag{384}$$

This subspace is invariant under the action of the Heisenberg group H(W) by left twisted convolutions:

$$\mathcal{H} \ni f \to T(w,r) \natural f \in \mathcal{H}, \tag{385}$$

where $T(w,r)$ was defined in (373).

We may assume that the complete polarization (354) is $N_{\mathcal{L}}$-orthogonal, so that

$$\mathcal{L} = \mathsf{X} \cap \mathcal{L} + \mathsf{Y} \cap \mathcal{L} \tag{386}$$

so that

$$\mathbb{I}_{\mathcal{L}}(x+y) = \mathbb{I}_{\mathsf{X}\cap\mathcal{L}}(x)\mathbb{I}_{\mathsf{Y}\cap\mathcal{L}}(y) \qquad (x \in \mathsf{X}, y \in \mathsf{Y}). \tag{387}$$

Then a straightforward computation shows that

$$\mathcal{K}(\mathbb{I}_{\mathcal{L}})(x,x') = \mathbb{I}_{\mathsf{X}\cap\mathcal{L}}(x)\mathbb{I}_{\mathsf{X}\cap\mathcal{L}}(x') \qquad (x, x' \in \mathsf{X}). \tag{388}$$

With some more effort, using (380), we compute that

$$\mathcal{K}(\delta_w \natural \mathbb{I}_{\mathcal{L}})(x,x') = v_w(x)\mathbb{I}_{\mathsf{X}\cap\mathcal{L}}(x') \qquad (x, x' \in \mathsf{X}, w \in \mathsf{W}), \tag{389}$$

where

$$v_w(x) = \mathbb{I}_{\mathsf{X}\cap\mathcal{L}+x_w}(x)\chi(\langle w, x\rangle)\chi(\frac{1}{2}\langle x_w, y_w\rangle) \tag{390}$$

with $w = x_w + y_w$ according to the decomposition (354). In particular

$$v_w = \omega((w,0))\mathbb{I}_{\mathsf{X}\cap\mathcal{L}} \qquad (w \in \mathsf{W}). \tag{391}$$

Thus

$$\mathcal{K}(f_\alpha)(x,x') = v_\alpha(x)v_0(x') \qquad (x, x' \in \mathsf{X}, \alpha \in \mathcal{A}). \tag{392}$$

Hence the set of the v_α is orthonormal in $\mathrm{L}^2(\mathsf{X})$. Furthermore the map

$$\mathcal{H} \ni f \to \mathrm{Op} \circ \mathcal{K}(f) v_0 \in \mathrm{L}^2(\mathsf{X}) \tag{393}$$

intertwines the action of the Heisenberg group $H(\mathrm{W})$. We notice that, since $\mathrm{L}^2(\mathsf{X})$ is irreducible under this action, so is $\mathcal{H}$, and the v_α form an orthonormal basis of $\mathrm{L}^2(\mathsf{X})$.

Let $\mathrm{Sp}(\mathsf{W})_{\mathcal{L}} \subseteq \mathrm{Sp}(\mathsf{W})$ denote the stabilizer of the lattice. This is a maximal compact subgroup of the symplectic group.

Lemma 138. *The indicator function of the lattice $\mathcal{L}$ is an eigenvector for the action of $\mathrm{Sp}(\mathsf{W})_{\mathcal{L}}$ via left twisted convolution:*

$$t(g)\natural\mathbb{I}_{\mathcal{L}} = \int_{((g-1)\mathsf{W})\cap\mathcal{L}} \chi_{c(g)}(u)\, d\mu_{(g-1)\mathsf{W}}(u) \cdot \mathbb{I}_{\mathcal{L}} \qquad (g \in \mathrm{Sp}(\mathsf{W})_{\mathcal{L}}). \tag{394}$$

Proof. Let $\mathsf{U} = (g-1)\mathsf{W}$. In Lemma 118 we may chose the complementary subspace Z to be $N_{\mathcal{L}}$-orthogonal to U, so that

$$\mathcal{L} = \mathsf{U}\cap\mathcal{L} + \mathsf{Z}\cap\mathcal{L}\,.$$

Then the formula (328) applied to $\phi = \mathbb{I}_{\mathcal{L}}$ and $w' = u' + z'$ reads

$$\begin{aligned} & t(g)\natural\mathbb{I}_{\mathcal{L}}(w') \\ = \;& \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', z'\rangle)\int_{\mathsf{U}} \chi_{c(g)}(u)\mathbb{I}_{\mathsf{U}\cap\mathcal{L}}(u)\mathbb{I}_{\mathsf{Z}\cap\mathcal{L}}(z')\chi(-\frac{1}{2}\langle u, (1-c(g))u' + z'\rangle)\, d\mu_{\mathsf{U}}(u)\,. \end{aligned}$$

Here $u \in \mathcal{L}$ and $z' \in \mathcal{L}$. Hence $\chi(-\frac{1}{2}\langle u, z'\rangle) = 1$. Thus the above quantity is equal to

$$\chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', z'\rangle)\mathbb{I}_{\mathsf{Z}\cap\mathcal{L}}(z')\int_{\mathsf{U}\cap\mathcal{L}} \chi_{c(g)}(u)\chi(-\frac{1}{2}\langle u, (1-c(g))u'\rangle)\, d\mu_{\mathsf{U}}(u)\,.$$

Notice that $(g-1)\mathcal{L} \subseteq \mathsf{U}\cap\mathcal{L}$ is an open compact subgroup. Also, for $u \in (g-1)\mathcal{L}$ and for $r \in \mathsf{U}\cap\mathcal{L}$ we have $\langle c(g)u, r\rangle \in \mathfrak{o}_{\mathbb{F}}$ and $\langle c(g)u, u\rangle \in \mathfrak{o}_{\mathbb{F}}$. Hence

$$\begin{aligned} & \int_{\mathsf{U}\cap\mathcal{L}} \chi_{c(g)}(u)\chi(-\frac{1}{2}\langle u, (1-c(g))u'\rangle)\, d\mu_{\mathsf{U}}(u) \\ = \;& \sum_{r\in\mathsf{U}\cap\mathcal{L}/(g-1)\mathcal{L}} \int_{(g-1)\mathcal{L}} \chi_{c(g)}(r+u)\chi(-\frac{1}{2}\langle r+u, (1-c(g))u'\rangle)\, d\mu_{\mathsf{U}}(u) \\ = \;& \sum_{r\in\mathsf{U}\cap\mathcal{L}/(g-1)\mathcal{L}} \chi_{c(g)}(r)\chi(-\frac{1}{2}\langle r, (1-c(g))u'\rangle)\int_{(g-1)\mathcal{L}} \chi(-\frac{1}{2}\langle u, (1-c(g))u'\rangle)\, d\mu_{\mathsf{U}}(u)\,. \end{aligned}$$

But

$$\langle u, (1-c(g))u'\rangle = \langle (1+c(g))u, u'\rangle = 2\langle g(g-1)^{-1}u, u'\rangle = 2\langle (g-1)^{-1}u, g^{-1}u'\rangle\,.$$

Thus

$$\begin{aligned} & \int_{(g-1)\mathcal{L}} \chi(-\frac{1}{2}\langle u, (1-c(g))u'\rangle)\, d\mu_{\mathsf{U}}(u) = \int_{(g-1)\mathcal{L}} \chi(-\langle (g-1)^{-1}u, g^{-1}u'\rangle)\, d\mu_{\mathsf{U}}(u) \\ = \;& |\det(g-1 : \mathsf{W}/\mathrm{Ker}(g-1) \to (g-1)\mathsf{W})| \int_{\mathcal{L}} \chi(\langle (w, -g^{-1}u'\rangle)\, d\mu_{\mathsf{W}}(w) \\ = \;& |\det(g-1 : \mathsf{W}/\mathrm{Ker}(g-1) \to (g-1)\mathsf{W})| \cdot \mathbb{I}_{\mathcal{L}}(-g^{-1}u') \\ = \;& |\det(g-1 : \mathsf{W}/\mathrm{Ker}(g-1) \to (g-1)\mathsf{W})| \cdot \mathbb{I}_{\mathcal{L}}(u')\,. \end{aligned}$$

Therefore

$$\begin{aligned}
& t(g)\natural\mathbb{I}_{\mathcal{L}}(w') \\
= \ & |\det(g-1:\mathsf{W}/\mathrm{Ker}(g-1)\to(g-1)\mathsf{W})|\cdot\mathbb{I}_{\mathsf{U}\cap\mathcal{L}}(u')\mathbb{I}_{\mathsf{Z}\cap\mathcal{L}}(z')\cdot \\
& \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u',z'\rangle)\sum_{r\in\mathsf{U}\cap\mathcal{L}/(g-1)\mathcal{L}}\chi_{c(g)}(r)\chi(-\frac{1}{2}\langle r,(1-c(g))u'\rangle) \\
= \ & |\det(g-1:\mathsf{W}/\mathrm{Ker}(g-1)\to(g-1)\mathsf{W})|\cdot\mathbb{I}_{\mathcal{L}}(w') \\
& \chi_{c(g)}(u')\sum_{r\in\mathsf{U}\cap\mathcal{L}/(g-1)\mathcal{L}}\chi_{c(g)}(r)\chi(-\frac{1}{2}\langle c(g)u,r\rangle) \\
= \ & \sum_{r\in\mathsf{U}\cap\mathcal{L}/(g-1)\mathcal{L}}\chi_{c(g)}(u'-r)|\det(g-1:\mathsf{W}/\mathrm{Ker}(g-1)\to(g-1)\mathsf{W})|\cdot\mathbb{I}_{\mathcal{L}}(w') \\
= \ & \int_{\mathsf{U}\cap\mathcal{L}}\chi_{c(g)}(u'-u)\,d\mu_{\mathsf{U}}(u)\cdot\mathbb{I}_{\mathcal{L}}(w')
\end{aligned}$$

and (394) follows. □

Corollary 139. *Denote by* $\widetilde{\mathrm{Sp}}(\mathsf{W})_{\mathcal{L}}\subseteq\widetilde{\mathrm{Sp}}(\mathsf{W})$ *the preimage in the metaplectic group. Then*

$$T(\tilde{g})\natural\mathbb{I}_{\mathcal{L}}=\Theta_{\mathcal{L}}(\tilde{g})\mathbb{I}_{\mathcal{L}}\qquad(\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathsf{W})_{\mathcal{L}}),\tag{395}$$

where

$$\Theta_{\mathcal{L}}(\tilde{g})=\Theta(\tilde{g})\int_{((g-1)\mathsf{W})\cap\mathcal{L}}\chi_{c(g)}(u)\,d\mu_{(g-1)\mathsf{W}}(u)\qquad(\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathsf{W})_{\mathcal{L}}).$$

The map $\Theta_{\mathcal{L}}:\widetilde{\mathrm{Sp}}(\mathsf{W})_{\mathcal{L}}\to\mathbb{C}^\times$ *is a group homomorphism.*

Lemma 117 shows that,

$$T(\tilde{g})\natural\delta_\alpha\natural\mathbb{I}_{\mathcal{L}}=T(\tilde{g})\natural\delta_\alpha\natural T((\tilde{g})^{-1})\natural T(\tilde{g})\natural\mathbb{I}_{\mathcal{L}}=T(\tilde{g})\natural\delta_\alpha\natural T((\tilde{g})^{-1})\Theta_{\mathcal{L}}(\tilde{g})\mathbb{I}_{\mathcal{L}}=\delta_{g\alpha}\Theta_{\mathcal{L}}(\tilde{g})\mathbb{I}_{\mathcal{L}}$$

Therefore (see Proposition 132),

$$T(\tilde{g})\natural f=\Theta_{\mathcal{L}}(\tilde{g})\omega_{1,1}(g)f\qquad(\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathsf{W})_{\mathcal{L}},f\in\mathcal{H}).\tag{396}$$

Hence the map

$$\mathcal{H}\ni f\to\mathrm{Op}\circ\mathcal{K}(f)v_0\ni\mathrm{L}^2(\mathsf{X})\tag{397}$$

is a unitary equivalence of the representation $(\Theta_{\mathcal{L}}\otimes\omega_{1,1},\mathcal{H})$ and $(\omega,\mathrm{L}^2(\mathsf{X}))$ of the group $\widetilde{\mathrm{Sp}}(\mathsf{W})_{\mathcal{L}}$.

The entire group $\widetilde{\mathrm{Sp}}(\mathsf{W})$ acts on the right hand side of (393). Hence it does on the left hand side. The resulting representation of $\widetilde{\mathrm{Sp}}(\mathsf{W})$ is called the *lattice model* of the Weil representation $(\omega,\mathrm{L}^2(\mathsf{X}))$. As we just computed, its restriction to $\widetilde{\mathrm{Sp}}(\mathsf{W})_{\mathcal{L}}$ is equal to $(\Theta_{\mathcal{L}}\otimes\omega_{1,1},\mathcal{H})$.

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