Cent. Eur. J. Math. • 12(10) • 2014 • 1500-1585
DOI: 10.2478/s11533-014-0428-8

VERSITA

Central European Journal of **Mathematics**

A reverse engineering approach to the Weil representation

Research Article

Anne-Marie Aubert[1]*, Tomasz Przebinda[2]†

1 I.M.J.-P.R.G. (U.M.R. 7586 du C.N.R.S.), Université Pierre et Marie Curie, 4 place Jussieu, 75005 Paris, France

2 Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA

Received 15 January 2013; accepted 19 November 2013

Abstract: We describe a new approach to the Weil representation attached to a symplectic group over a finite or a local field. We dissect the representation into small pieces, study how they work, and put them back together. This way, we obtain a reversed construction of that of T. Thomas, skipping most of the literature on which the latter is based.

MSC: 22E45, 20C33, 22E46

Keywords: Weil representation • Oscillator representation • Metaplectic representation

1. Introduction

The Weil representation is a magnificent structure which keeps appearing in a variety of places throughout Mathematics and Physics. This is evident from a simple google or mathscinet search for "oscillator representation", "Weil representation", "Howe correspondence" or "local theta correspondence". The last two terms refer to a correspondence of irreducible representation for certain pairs of groups, conjectured to exist in [16], proven to exist over the reals in [19], over p-adic fields (p odd) in [39] and essentially proven not to exist over finite fields in [1]. A concise description of the Weil representation may be found in [37]. Anyone interested in a short and complete presentation should read that paper and stop right there. That work is really hard to improve upon. In this article we take the opposite approach. We dissect the Weil representation into small pieces, study how they work, and put them back together, in effect checking

* *E-mail: aubert@math.jussieu.fr*
† *E-mail: tprzebin@gmail.com*

that the formulas of [37, Theorem C] are correct, thus reversing Thomas' proofs and skipping most of the literature on which it is based. Hence the title of this article. The methods we use are elementary, i.e. contained in a graduate curriculum of an average university in the USA. In contrast, a reader well versed in Algebraic Geometry will certainly enjoy [6, 7] or [8]. In the real case one should also mentions some classics, such as [24] or [4].

The Weil representation concerns a symplectic group defined over a field or over the adeles (or, more recently, over a ring [2, 9, 21], or a finite abelian group [32]). The field could be finite or local. We always assume that the characteristic is not 2, skip the case of the complex numbers as not interesting, and the adeles, the rings and the finite abelian groups as very interesting but requiring more energy, which we have just exhausted. Here is a brief description of what we do.

Let $\mathbb{F}$ be a finite field of odd characteristic and let W be a finite dimensional vector space over $\mathbb{F}$ equipped with a non-degenerate symplectic form $\langle\ ,\ \rangle$. The symplectic form induces a *twisted convolution* $\natural$ on the space $\mathsf{L}^2(\mathsf{W})$, making it into an associative algebra with identity over $\mathbb{C}$. One may think of it as of "the essential part" of the group algebra of the Heisenberg group attached to $(\mathsf{W}, \langle\ ,\ \rangle)$. For any subspace $\mathsf{X} \subseteq \mathsf{W}$, define a measure μ_{X} on W by

$$\int_{\mathsf{X}} \psi(x)\, d\mu_{\mathsf{X}}(x) := |\mathsf{X}|^{-1/2} \sum_{x \in \mathsf{X}} \psi(x),$$

where $|\mathsf{X}|$ is the cardinality of X and $\psi\colon \mathsf{X} \to \mathbb{C}$ is a function. Fix a non-trivial character χ of the additive group $\mathbb{F}$. Then the twisted convolution (with respect to χ) of two functions $\phi, \psi\colon \mathsf{W} \to \mathbb{C}$ is defined as

$$\phi\natural\psi(w) := \int_{\mathsf{W}} \phi(u)\psi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{\mathsf{W}}(u) \qquad (w \in \mathsf{W}). \tag{1}$$

The algebra $\mathrm{H.S.}(\mathsf{L}^2(\mathsf{X}))$ of the Hilbert-Schmidt operators on $\mathsf{L}^2(\mathsf{X})$ may be identified with $\mathsf{L}^2(\mathsf{X} \times \mathsf{X})$ by assigning the integral kernel $K \in \mathsf{L}^2(\mathsf{X} \times \mathsf{X})$ to each operator $\mathrm{Op}(K) \in \mathrm{H.S.}(\mathsf{L}^2(\mathsf{X}))$ by setting

$$\mathrm{Op}(K)v(x) := \int_{\mathsf{X}} K(x, x')v(x')\, d\mu_{\mathsf{X}}(x').$$

Suppose that X is a part of a complete polarization $\mathsf{W} = \mathsf{X} \oplus \mathsf{Y}$. Let $\mathcal{K}\colon \mathsf{L}^2(\mathsf{W}) \to \mathsf{L}^2(\mathsf{X} \times \mathsf{X})$ be the corresponding the Weyl transform:

$$\mathcal{K}(\phi)(x, x') = \int_{\mathsf{Y}} \phi(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathsf{Y}}(y).$$

Then we have the following sequence of algebra isomorphisms:

$$\mathsf{L}^2(\mathsf{W}) \overset{\mathcal{K}}{\to} \mathsf{L}^2(\mathsf{X} \times \mathsf{X}) \overset{\mathrm{Op}}{\to} \mathrm{H.S.}(\mathsf{L}^2(\mathsf{X})). \tag{2}$$

Let $\mathrm{Sp}(\mathsf{W})$ denote the symplectic group, that is the isometry group of the form $\langle\ ,\ \rangle$. The main result of [37, Theorem C] gives an explicit formula for a map $T\colon \mathrm{Sp}(\mathsf{W}) \to \mathsf{L}^2(\mathsf{W})$ such that the resulting composition

$$\omega\colon \mathrm{Sp}(\mathsf{W}) \overset{T}{\to} \mathsf{L}^2(\mathsf{W}) \overset{\mathcal{K}}{\to} \mathsf{L}^2(\mathsf{X} \times \mathsf{X}) \overset{\mathrm{Op}}{\to} \mathrm{H.S.}(\mathsf{L}^2(\mathsf{X})), \tag{3}$$

is an injective group homomorphism of the symplectic group into the group $\mathrm{U}(\mathsf{L}^2(\mathsf{X}))$ of the unitary operators on $\mathsf{L}^2(\mathsf{X})$,

$$\omega : \mathrm{Sp}(\mathsf{W}) \to \mathrm{U}(\mathsf{L}^2(\mathsf{X})), \tag{4}$$

which has the following "conjugation property"

$$\left(\omega(g)\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(g^{-1})\right)(w) = \phi(g^{-1}w) \qquad (g \in \mathrm{Sp}(\mathsf{W}),\ \phi \in \mathsf{L}^2(\mathsf{W})). \tag{5}$$

A less explicit formula for $T(g)$ occurred already in [15, Theorem 2.9]. The missing ingredient was the description of the trace $\operatorname{tr}(\omega(g))$, which was done in [36] and led to [37, Theorem C]. A proof of the existence of ω satisfying (4) and (5) is also available in [5, Theorem 2.4]. In Section 3 we check, via a straightforward but non-trivial computation, that the ω given in [37, Theorem C] is indeed a group homomorphism.

Our approach is the following. For any $g \in \mathrm{Sp}(\mathrm{W})$, the left and right radicals of the bilinear form $(w, w') \mapsto \langle (g-1)w, w'\rangle$ coinciding with $\operatorname{Ker}(g-1)$, we get a non-degenerate bilinear form B_g on the quotient $\mathrm{W}/\operatorname{Ker}(g-1)$. Let $\operatorname{dis}(B_g)$ denote its discriminant. We set

$$\Theta(g) := |\operatorname{Ker}(g-1)|^{1/2}\, \gamma(1)^{\dim (g-1)\mathrm{W}}\, \operatorname{dis}(B_g), \tag{6}$$

where

$$\gamma(1) = \int_{\mathbb{F}} \chi(x^t x)\, d\mu_{\mathbb{F}}(x).$$

Then we define $T(g)$ by

$$T(g) := \Theta(g)\, \chi_{c(g)}\, \mathbb{I}_{(g-1)W},$$

where for $u \in (g-1)\mathrm{W}$

$$\chi_{c(g)}(u) = \chi(\tfrac{1}{4}\langle c(g)u, u\rangle), \tag{7}$$

$c(g)\colon (g-1)\mathrm{W} \to \mathrm{W}/\operatorname{Ker}(g-1)$ denoting the Cayley transform, and $\mathbb{I}_{(g-1)W}$ is the indicator function of $(g-1)W$.

Our first main result (Theorem 3.8) asserts that

$$T(g_1)\natural T(g_2) = T(g_1 g_2), \qquad \text{for any } g_1, g_2 \in \mathrm{Sp}(\mathrm{W}). \tag{8}$$

Let $\omega := \mathrm{Op} \circ \mathcal{K} \circ T$. Our second main result (Theorem 3.10) asserts that ω is an injective group homomorphism from $\mathrm{Sp}(\mathrm{W})$ to $\mathrm{U}(\mathrm{L}^2(\mathrm{X}))$, that the function Θ coincides with the character of the resulting representation, and that Eqn. (5) holds true.

In the case $\mathbb{F} = \mathbb{R}$, the reals, one has to deal with the "smog overspreading the infinite field" [15, page 2]. In particular the first two Hilbert spaces which occur in (2) have to be replaced by the spaces of tempered distributions. Hence, the algebra structure breaks down, but enough of it survives to make sense out of the formulas like

$$T(\tilde{g}_1)\natural T(\tilde{g}_2) = T(\tilde{g}_1 \tilde{g}_2), \tag{9}$$

where $\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, a double cover of $\mathrm{Sp}(\mathrm{W})$ (see below). The resulting representation ω of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ appeared first in [35], as a natural development in Quantum Mechanics, [38]. Explicit formulas for $\omega(\tilde{g})$, $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, may be found in [33, Theorem 5.3] and for $T(\tilde{g})$ in [26]. Furthermore, if one thinks of $\omega(\tilde{g})$ as of a pseudo-differential operator, then its Weyl symbol, see [14], is $T(\tilde{g})$.

Our approach consists of defining first, for $g \in \mathrm{Sp}(\mathrm{W})$,

$$\Theta^2(g) := \gamma(1)^{2\dim (g-1)\mathrm{W}} \left(\det(g-1\colon \mathrm{W}/\operatorname{Ker}(g-1) \to (g-1)\mathrm{W})\right)^{-1}, \tag{10}$$

setting next

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) := \{(g, \xi);\ g \in \mathrm{Sp}(\mathrm{W}),\ \xi \in \mathbb{C}^\times,\ \xi^2 = \Theta^2(g)\},$$

and finally

$$\Theta(\tilde{g}) := \xi, \qquad \text{for } \tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W}).$$

Let $\chi(r) = \exp(2\pi i r)$ for $r \in \mathbb{R}$. Define $\chi_{c(g)}$ as in (7). Then we set

$$T(\tilde{g}) := \Theta(\tilde{g})\, \chi_{c(g)}\, \mu_{(g-1)W},$$

where $\mu_{(g-1)W}$ is an appropriately normalized Haar measure on $(g-1)W$, and prove that the formula (9) is satisfied.

Similar difficulties as for the reals occur when $\mathbb{F}$ is a p-adic field, with some new ones, see Section 5 for details. The representation ω was constructed in [42] and the explicit formulas for $\omega(\tilde{g})$, $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, may be found in [31]. Our construction in the p-adic case occurs to be a mixed version of the finite and the reals cases, as shows the definition of $\Theta(\tilde{g})^2$ (see Definition 5.15).

Checking the equality (8) (or (9)) requires some effort. First we compute the twisted convolution of the unnormalized Gaussians $\chi_{c(g)}\,\mathbb{I}_{(g-1)W}$ (or $\chi_{c(g)}\,\mu_{(g-1)W}$) and obtain a cocycle $C(g_1, g_2)$. This is straightforward, but not easy in the sense that one has to keep track of various determinants, which are explained in Section 2. Then we "guess" the normalization factor $\Theta(g)$ (or $\Theta(\tilde{g})$) and verify (8) (or (9)). This second step is more difficult. "Guessing" the normalizing factor, which happens to be the distribution character of the Weil representation, was done for us by Teruji Thomas in the finite case and others in the remaining two cases. We show that the normalized Gaussians form a group by a direct computation involving the cocycle. The point is that this computation is the same in all three cases (finite, real and p-adic) and avoids the holomorphic continuation to the oscillator semigroup studied in [18, 27] or [29]. In a sense, we replace analytic difficulties by some convoluted linear algebra of Section 2. Our methods are equivalent, but not equal, to those used in [24, sec. 1.4-1.7] where the authors describe the cocycle $C(g_1, g_2)/|C(g_1, g_2)|$ and give a formula for the Weil representation acting in some Schrödinger model. Proving that $C(g_1, g_2)/|C(g_1, g_2)|$ is a cocycle relies on Kashiwara's description of Maslov index associated to three maximal isotropic subspaces of W. We deduce this fact from the associativity of the twisted convolution of the Gaussians. Thus our "convoluted linear algebra" replaces the beautiful theory of Maslov index. (Another justification for the title of our article.)

Weil's construction covers the cases of all locally compact non-discrete fields (including the reals) and adeles and gives applications to the theory of automorphic forms. Hence the name "Weil representation", taking away some of the credit from David Shale - a student of Erza Segal. Possibly in an attempt to find a middle ground Roger Howe proposed the name "the oscillator representation", [15, page 1]. The names "Segal-Shale-Weil representation", [22], "metaplectic representation", [30], and "spin representation of the symplectic group", [23] have also been used. Since, as the reader will see, understanding the Fourier transform of a Gaussian is the only prerequisite to follow our reverse engineering process, a name like "Gauss-Fourier-Segal-Shale-Weil representation" is another option. (In fact many researchers have been (and most likely will be) fascinated by the Gaussians and wrote volumes about them, see for example [28].) We chose to use the name "Weil representation", because it is the shortest one.

2. Linear algebra preliminaries

The first aim of this Section is to collect various results, valid for arbitrary commutative fields of characteristic not equal to 2, that we will use in each of the three next sections. It is the object of the subsections 2.1 to 2.4. The two other subsections are devoted to determinants over the reals, and over a p-adic field, respectively; the main result is Lemma 2.11 (resp. Lemma 2.23), which will be used in the proof of Lemma 4.17 (resp. Lemma 5.16).

2.1. General results on quadratic forms

Let $\mathbb{F}$ be a commutative field of characteristic not equal to 2. Let U be a finite dimensional vector space over $\mathbb{F}$. Suppose q is a non-degenerate symmetric bilinear form on U. Then the formula

$$\Phi(u)(v) = q(u, v) \qquad (u, v \in \mathrm{U}) \tag{11}$$

defines a linear isomorphism $\Phi\colon \mathrm{U} \to \mathrm{U}^*$, where U^* is the vector space dual to U. The form q^* dual to q is given by

$$q^*(u^*, v^*) = v^*(\Phi^{-1}(u^*)) \quad (u^*, v^* \in \mathrm{U}^*).$$

Let Q be the matrix obtained from any basis $u_1, u_2, \dots, u_n$ of U by

$$Q_{i,j} = q(u_i, u_j) \qquad (1 \le i, j \le n). \tag{12}$$

Lemma 2.1.
If Q is the matrix corresponding to q and a basis $u_1, u_2, \dots, u_n$ of U, as above, then Q^{-1} corresponds to q^* and the dual basis $u_1^*, u_2^*, \dots, u_n^*$ of U^*.

Proof. Suppose $\Phi(u) = u^*$. Then for any $v \in \mathrm{U}$,

$$u^*(v) = q(u, v) = \sum_{i,j=1}^{n} u_i^*(u) q(u_i, u_j) u_j^*(v).$$

Thus

$$u^* = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} u_i^*(u) q(u_i, u_j) \right) u_j^*.$$

Therefore

$$u^*(u_j) = \sum_{i=1}^{n} u_i^*(u) q(u_i, u_j) \qquad (1 \leq j \leq n).$$

In matrix form the above equations may be written as

$$(u^*(u_1), u^*(u_2), \dots, u^*(u_n)) = (u_1^*(u), u_2^*(u), \dots, u_n^*(u))Q.$$

Hence,

$$(u^*(u_1), u^*(u_2), \dots, u^*(u_n))Q^{-1} = (u_1^*(u), u_2^*(u), \dots, u_n^*(u)).$$

Thus

$$u = \sum_{j=1}^{n} u_j^*(u) u_j = \sum_{j=1}^{n} \sum_{i=1}^{n} u^*(u_i)(Q^{-1})_{i,j} u_j.$$

Therefore,

$$q^*(u^*, u^*) = \sum_{j=1}^{n} \sum_{i=1}^{n} u^*(u_i)(Q^{-1})_{i,j} u^*(u_i).$$

In other words,

$$q^*(u_i^*, u_j^*) = (Q^{-1})_{i,j}.$$ □

2.2. Symplectic spaces

Let W be a finite dimensional vector space over $\mathbb{F}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$ and let $\mathrm{U} \subseteq \mathrm{W}$ be a subspace. We shall identify W with the dual W^* by

$$w^*(w) = \langle w, w^* \rangle \qquad (w, w^* \in \mathrm{W}). \tag{13}$$

Then

$$\mathrm{U}^* = \mathrm{W}/\mathrm{U}^\perp \text{ and } (\mathrm{U}/\mathrm{V})^* = \mathrm{V}^\perp/\mathrm{U}^\perp, \tag{14}$$

where the orthogonal complements are taken in W, with respect to the symplectic form $\langle\ ,\ \rangle$.

Lemma 2.2.
Let $\mathrm{V}_1, \mathrm{V}_2 \subseteq \mathrm{W}$ be two subspaces and let $w \in \mathrm{W}$ be such that $\mathrm{V}_1 \cap (\mathrm{V}_2 + w) \neq \emptyset$. Then for any $v \in \mathrm{V}_1 \cap (\mathrm{V}_2 + w)$,

$$\mathrm{V}_1 \cap (\mathrm{V}_2 + w) = \mathrm{V}_1 \cap \mathrm{V}_2 + v.$$

Proof. There are vectors $v_1 \in \mathsf{V}_1$ and $v_2 \in \mathsf{V}_2$ such that

$$v = v_1 = v_2 + w.$$

Then

$$\mathsf{V}_1 \cap (\mathsf{V}_2 + w) - v = \mathsf{V}_1 \cap (\mathsf{V}_2 + w) - v_1 \subseteq \mathsf{V}_1 - v_1 = \mathsf{V}_1$$

and

$$\mathsf{V}_1 \cap (\mathsf{V}_2 + w) - v = \mathsf{V}_1 \cap (\mathsf{V}_2 + w) - (v_2 + w) \subseteq (\mathsf{V}_2 + w) - (v_2 + w) = \mathsf{V}_2.$$

Hence,

$$\mathsf{V}_1 \cap (\mathsf{V}_2 + w) - v \subseteq \mathsf{V}_1 \cap \mathsf{V}_2.$$

Conversely, let $\mathsf{V}_1 \ni v_1' = v_2' \in \mathsf{V}_2$. Then

$$v_1' + v = v_1' + v_1 \in \mathsf{V}_1 \text{ and } v_2' + v = v_2' + v_2 + w \in \mathsf{V}_2 + w.$$

Therefore

$$\mathsf{V}_1 \cap \mathsf{V}_2 + v \subseteq \mathsf{V}_1 \cap (\mathsf{V}_2 + w).$$ □

Let $\mathrm{Sp}(W)$ denote the isometry group of $\langle\ ,\ \rangle$:

$$\mathrm{Sp}(\mathsf{W}) = \{g \in \mathrm{GL}(\mathsf{W}) : \langle gw, gw' \rangle = \langle w, w' \rangle \quad \forall w, w' \in \mathsf{W}\}.$$

Let $\dim(\mathsf{W}) = 2n$. Then there is a group isomorphism

$$\mathrm{Sp}(\mathsf{W}) \simeq \mathrm{Sp}_{2n}(\mathbb{F}) := \{A \in \mathrm{GL}_{2n}(\mathbb{F}) : A^t J' A = J'\}, \tag{15}$$

where A^t means the transpose of A, and

$$J' = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

The Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{F})$ is equal to

$$\mathfrak{sp}_{2n}(\mathbb{F}) = \{X \in \mathfrak{gl}_{2n}(\mathbb{F}) : X^t J' + J' X = 0\}.$$

Matrices which belong to $\mathrm{Sp}_{2n}(\mathbb{F})$ are called *symplectic matrices*. It clearly follows from (15) that the square of the determinant of any symplectic matrix is 1. In fact, the determinant itself is always 1. Indeed, the determinant of any antisymmetric matrix can be expressed as the square of a polynomial in the entries of the matrix. This polynomial Pf is called the Pfaffian. The following identity holds true: $\mathrm{Pf}(A^t J' A) = \det(A)\,\mathrm{Pf}(J')$. Since $A^t J' A = J'$, we get $\det(A) = 1$.

2.3. The Cayley transform

For $g \in \mathrm{Sp}(\mathsf{W})$, we set

$$g^{\pm} := g \pm 1, \tag{16}$$

and define the Cayley transform by

$$c(g)\colon g^- \mathsf{W} \ni g^- w \to g^+ w + \mathrm{Ker}(g^-) \in \mathsf{W}/\mathrm{Ker}(g^-). \tag{17}$$

Then the bilinear form

$$\langle c(g)u', u''\rangle = \langle g^+w', g^-w''\rangle \qquad (u' = g^-w',\ u'' = g^-w'',\ w', w'' \in \mathrm{W}) \tag{18}$$

on the space $g^-\mathrm{W}$ is well defined and symmetric.

Lemma 2.3.
For any $g \in \mathrm{Sp}(\mathrm{W})$ the map

$$g^+ : \mathrm{Ker}(g^-) \to \mathrm{Ker}(g^-)$$

is bijective.

Also, for any $u = g^-w$, with $w \in \mathrm{W}$, the preimage of $c(g)u \in \mathrm{W}/\mathrm{Ker}(g^-)$ under the quotient map $\mathrm{W} \to \mathrm{W}/\mathrm{Ker}(g^-)$ is equal to $g^+w + \mathrm{Ker}(g^-)$.

Proof. Since g^+ commutes with g^-, g^+ preserves $\mathrm{Ker}(g^-)$. Suppose $w \in \mathrm{Ker}(g^-)$ and $g^+w = 0$. Then

$$g^-w = 0 \text{ and } g^+w = 0,$$

which implies $w = 0$. The second statement is obvious. □

Notation 2.4.
For $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$, let

$$\mathrm{U}_1 := g_1^-\mathrm{W}, \quad \mathrm{U}_2 := g_2^-\mathrm{W} \quad \text{and} \quad \mathrm{U}_{12} := (g_1g_2)^-\mathrm{W},$$
$$K_1 := \mathrm{Ker}\, g_1^-, \quad K_2 := \mathrm{Ker}\, g_2^- \quad \text{and} \quad K_{12} := \mathrm{Ker}(g_1g_2)^-.$$

Lemma 2.5.
Let $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and let $w, v \in \mathrm{W}$ be such that

$$v \in \mathrm{U}_1 \cap (\mathrm{U}_2 + w).$$

Then for any $u' \in \mathrm{U}_1 \cap \mathrm{U}_2$

$$\begin{aligned}&\langle c(g_1)(u'+v), u'+v\rangle + \langle c(g_2)(w-u'-v), w-u'-v\rangle + 2\langle u'+v, w\rangle\\ &\quad = \langle (c(g_1)+c(g_2))u', u'\rangle - 2\langle u', c(g_1)v - c(g_2)(w-v) - w\rangle + \langle c(g_1)v, v\rangle + \langle c(g_2)(w-v), w-v\rangle + 2\langle v, w\rangle.\end{aligned}$$

Proof. Notice that all the terms in the above expression make sense. Also,

$$\langle c(g_1)(u'+v), u'+v\rangle = \langle c(g_1)u', u'\rangle + 2\langle c(g_1)u', v\rangle + \langle c(g_1)v, v\rangle$$

and

$$\langle c(g_2)(w-u'-v), w-u'-v\rangle = \langle c(g_2)(w-v), w-v\rangle - 2\langle c(g_2)(w-v), u'\rangle + \langle c(g_2)u', u'\rangle.$$

Hence

$$\begin{aligned}&\langle c(g_1)(u'+v), u'+v\rangle + \langle c(g_2)(w-u'-v), w-u'-v\rangle\\ &\quad = \langle (c(g_1)+c(g_2))u', u'\rangle + \langle c(g_1)v, v\rangle + \langle c(g_2)(w-v), w-v\rangle + 2\langle c(g_1)u', v\rangle - 2\langle c(g_2)(w-v), u'\rangle.\end{aligned}$$

Furthermore

$$\langle c(g_1)u', v\rangle - \langle c(g_2)(w-v), u'\rangle = -\langle u', c(g_1)v\rangle + \langle u', c(g_2)(w-v)\rangle = -\langle u', c(g_1)v - c(g_2)(w-v)\rangle$$

and the desired equality follows. □

Notation 2.6.

For two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$, let $\mathrm{U} := \mathrm{U}_1 \cap \mathrm{U}_2$, and let q_{g_1,g_2} denote the following symmetric form on U:

$$q_{g_1,g_2}(u', u'') = \frac{1}{2}\left(\langle c(g_1)u', u''\rangle + \langle c(g_2)u', u''\rangle\right) \qquad (u', u'' \in \mathrm{U}). \tag{19}$$

Let $\mathrm{V} \subseteq \mathrm{U}$ be the radical of q_{g_1,g_2} and let $\tilde{q}_{g_1,g_2}$ be the corresponding non-degenerate form on the quotient U/V.

Lemma 2.7.

Let g_1, g_2, U and V be as in Notation 2.6. Then

(a) $\dim(K_1 \cap K_2) + \dim \mathrm{V} = \dim K_{12}$;

(b) $\dim \mathrm{W} - \dim \mathrm{U} - \dim \mathrm{V} = \dim K_1 + \dim K_2 - \dim K_{12}$;

(c) $\dim \mathrm{U}_1 + \dim \mathrm{U}_2 - \dim \mathrm{U}_{12} = \dim \mathrm{U} + \dim \mathrm{V}$;

(d) $\mathrm{V} = g_2^- K_{12} = (g_1^{-1} - 1)K_{12}$.

Proof. It is easy to check that the kernel of the following map

$$\mathrm{W} \oplus \mathrm{W} \ni (w_1, w_2) \to (a, b, c) \in \mathrm{W} \oplus \mathrm{W} \oplus \mathrm{W}$$

where

$$a = g_1^- w_1 - g_2^- w_2, \quad b = g_1^- w_1 + g_2^- w_2 \ \text{ and } c = g_1^+ w_1 + g_2^+ w_2,$$

is equal to

$$\{(w, -w);\ w \in K_1 \cap K_2\} \tag{20}$$

and that the set of the pairs (w_1, w_2) such that $a = 0$ and $c = 0$ is equal to

$$\{(-g_2 w_2, w_2);\ w_2 \in K_{12}\}. \tag{21}$$

Let $u \in \mathrm{U}$. Then there are $w_1, w_2 \in \mathrm{W}$ such that $u = g_1^- w_1 = g_2^- w_2$. In particular the element "a" is zero. The condition that $u \in \mathrm{V}$ means that

$$g_1^+ w_1 + g_2^+ w_2 \in \mathrm{U}^\perp. \tag{22}$$

Since $\mathrm{U}^\perp = K_1 + K_2$, there are elements $x_1 \in K_1$ and $x_2 \in K_2$ such that

$$g_1^+ w_1 + g_2^+ w_2 = x_1 + x_2.$$

Lemma 2.3 shows that there are unique elements $y_1 \in K_1$ and $y_2 \in K_2$ such that $g_1^+ y_1 = -x_1$ and $g_2^+ y_1 = -x_2$. Let $w_1' = w_1 + y_1$ and $w_2' = w_2 + y_2$. Then

$$g_1^+ w_1' + g_2^+ w_2' = 0 \ \text{ and } \ u = g_1^- w_1' = g_2^- w_2'.$$

Therefore V is equal to the projection on the "b component" of the set (21).

Hence, $\dim \mathrm{V}$ is equal to the dimension of the set (21) minus the dimension of the kernel (20):

$$\dim \mathrm{V} = \dim K_{12} - \dim(K_1 \cap K_2).$$

This verifies (a).

Since

$$\dim \mathsf{U}^{\perp} = \dim(K_1 + K_2) = \dim K_1 + \dim K_2 - \dim(K_1 \cap K_2)$$

and since $\dim \mathsf{U}^{\perp} = \dim \mathsf{W} - \dim \mathsf{U}$, (b) follows from (a).

We have

$$\begin{aligned}\dim \mathsf{U}_1 + \dim \mathsf{U}_2 - \dim \mathsf{U}_{12} &= (\dim \mathsf{W} - \dim K_1) + (\dim \mathsf{W} - \dim K_2) - (\dim \mathsf{W} - \dim K_{12}) \\ &= \dim \mathsf{W} + \dim K_{12} - \dim K_1 - \dim K_2 = \dim \mathsf{U} + \dim \mathsf{V},\end{aligned}$$

because of (b). It proves (c).

As we already noticed,

$$\begin{aligned}\mathsf{V} &= \{g_1^-(-g_2 w_2) + g_2^- w_2;\ w_2 \in K_{12}\} = \{g_1^-(-g_1^{-1} w_2) + g_2^- w_2;\ w_2 \in K_{12}\} \\ &= \{(g_1^{-1} - 1)w_2 + g_2^- w_2;\ w_2 \in K_{12}\} = \{2g_2^- w_2;\ w_2 \in K_{12}\} \\ &= \{g_2^- w_2;\ w_2 \in K_{12}\} = \{(g_1^{-1} - 1)w_2;\ w_2 \in K_{12}\}.\end{aligned}$$

This verifies (d). □

Lemma 2.8.
Let $g \in \mathsf{Sp}(\mathsf{W})$. Then there is a direct sum decomposition

$$\mathsf{W} = \mathsf{X} \oplus \mathsf{W}_0 \oplus \mathsf{Y} \oplus \mathsf{W}_1$$

such that the subspaces X and Y are isotropic,

$$\begin{aligned}&(\mathsf{X}+\mathsf{Y})^{\perp} = \mathsf{W}_0 + \mathsf{W}_1,\ \ \mathsf{X} \oplus \mathsf{W}_0 \oplus \mathsf{Y} = \mathsf{W}_1^{\perp}, \\ &\mathsf{X} \oplus \mathsf{W}_0 = \mathsf{Im}(g^-),\ \ \mathsf{X} \oplus \mathsf{W}_1 = \mathsf{Ker}(g^-),\ \ \textit{and}\ \mathsf{X} = \mathsf{Ker}(g^-) \cap \mathsf{Ker}(g^-)^{\perp},\end{aligned}$$

where $\mathsf{Im}(g^-) = g^-\mathsf{W}$. Furthermore, there are unique elements

$$g_0 \in \mathsf{Sp}(\mathsf{W}_0),\ \ T \in \mathsf{Hom}(\mathsf{W}_0, \mathsf{X}),\ \ S \in \mathsf{Hom}(\mathsf{Y}, \mathsf{X})$$

such that for $x \in \mathsf{X}$, $w_0 \in \mathsf{W}_0$, $y \in \mathsf{Y}$ and $w_1 \in \mathsf{W}_1$

$$g(x + w_0 + y + w_1) = (x + Tw_0 + Sy) + (g_0 w_0 - g_0 T^* y) + y + w_1,$$

where $T^* \in \mathsf{Hom}(\mathsf{Y}, \mathsf{W}_0)$ is the conjugate of T with respect to the pairing $\langle\ ,\ \rangle$, and the map

$$\mathsf{W}_0 \oplus \mathsf{Y} \ni w_0 + y \to (Tw_0 + Sy) + ((g_0 - 1)w_0 - T^* y) \in \mathsf{X} \oplus \mathsf{W}_0$$

is invertible.

In particular if $g_1 \in \mathsf{End}(\mathsf{W})$ is defined by

$$g_1(x + w_0 + y + w_1) = -x - g_0^{-1} w_0 - y - w_1,$$

then $g_1 \in \mathsf{Sp}(\mathsf{W})$ and $\mathsf{Ker}(g_1 g^-) = \mathsf{Ker}(g g_1^-) = 0$.

Proof. Clearly $\mathsf{X} = \operatorname{Ker}(g^-) \cap \operatorname{Ker}(g^-)^\perp$ is an isotropic subspace. Let $\mathsf{Y} \subseteq \mathsf{W}$ be another isotropic subspace such that the restriction of the symplectic form to the sum $\mathsf{X} + \mathsf{Y}$ is non-degenerate. Define $\mathsf{W}' = (\mathsf{X} + \mathsf{Y})^\perp$. Then we have $\mathsf{W} = \mathsf{X} \oplus \mathsf{W}' \oplus \mathsf{Y}$.

Also,

$$\mathsf{X} \oplus \mathsf{W}' = \mathsf{X}^\perp = \operatorname{Ker}(g^-) + \operatorname{Ker}(g^-)^\perp \supseteq \operatorname{Ker}(g^-).$$

Set $\mathsf{W}_1 = \operatorname{Ker}(g^-) \cap \mathsf{W}'$. Then the above inclusion implies that $\operatorname{Ker}(g^-) = \mathsf{X} \oplus \mathsf{W}_1$. Let $\mathsf{W}_0 = \mathsf{W}_1^\perp \cap \mathsf{W}'$. Then

$$\mathsf{W}' = \mathsf{W}_0 \oplus \mathsf{W}_1 \quad \text{and} \quad \operatorname{Im}(g^-) = \operatorname{Ker}(g^-)^\perp = \mathsf{X} \oplus \mathsf{W}_0.$$

Since g acts as the identity on W_1, g preserves $\mathsf{W}_1^\perp$. Then $g|_{\mathsf{W}_1^\perp}$ acts as the identity on X. Also, the stabilizer of X in $\operatorname{Sp}(\mathsf{W})$ is a parabolic subgroup. Hence the formula for g follows from the well known structure of these subgroups.

Clearly the element g_1 belongs to $\operatorname{Sp}(\mathsf{W})$. Let $w = x + w_0 + y + w_1$ as in the lemma.

Suppose $g_1 g w = w$. Then

$$x = -x - Tw_0 - Sy,\ w_0 = -w_0 + g_0^{-1}T^*y,\ y = -y \text{ and } w_1 = -w_1.$$

Since the characteristic of the field $\mathbb{F}$ is not 2, we see that $w = 0$.

Suppose $gg_1 w = w$. Then

$$x = -x - Tg_0^{-1}w_0 - Sy,\ w_0 = -w_0 + T^*y,\ y = -y \text{ and } w_1 = -w_1.$$

Again, since the characteristic of the field $\mathbb{F}$ is not 2, we see that $w = 0$. □

2.4. More lemmas

Assume from now on till the end of this subsection that $K_1 = \operatorname{Ker} g_1^- = \{0\}$.

In this case $\mathsf{U} = g_2^- \mathsf{W}$. Then

$$K_2 \cap K_{12} = K_1 \cap K_2 = \{0\}.$$

Hence there is a subspace $\mathsf{W}_2 \subseteq \mathsf{W}$ such that

$$\mathsf{W} = K_{12} \oplus \mathsf{W}_2 \oplus K_2. \tag{23}$$

Pick a subspace $\mathsf{U}' \subseteq \mathsf{W}$ such that

$$\mathsf{W} = \mathsf{U} \oplus \mathsf{U}'.$$

Then $\mathsf{U} = K_2^\perp$ and $\dim \mathsf{U}' = \dim K_2$. Fix a basis $w_{b+1}, w_{b+2}, \dots$ of K_2 and let $w'_{b+1}, w'_{b+2}, \dots$ be the dual basis of U' in the sense that

$$\langle w_i, w'_j \rangle = \delta_{i,j} \qquad (b < i, j).$$

Define an element $h \in \operatorname{GL}(\mathsf{W})$ by

$$h|_{K_{12} \oplus \mathsf{W}_2} = (g_1^{-1} - 1)^{-1} g_2^-,\ hw_i = (g_1^{-1} - 1)^{-1} w'_i,\ b < i. \tag{24}$$

Let us extend the basis w_i of K_2 to a basis of W so that $w_i \in K_{12}$ if $i \le a$ and $w_i \in \mathsf{W}_2$ if $a < i \le b$. Then

$$hw_i = w_i \qquad (i \le a). \tag{25}$$

Lemma 2.9.
The following equalities hold:

$$\begin{aligned}\det(\langle (g_1g_2)^- w_i, hw_j\rangle_{a<i,j}) &= \det(\langle \tfrac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\le b})\\ &= \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j})\det(h).\end{aligned}$$

Moreover, we have

$$\det(\langle w_i, (g_1^{-1}-1)hw_j\rangle_{i,j}) = (-1)^{\dim \mathrm{U}}\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}). \tag{26}$$

Proof. Notice that both $c(g_1)$ and $c(g_2)$ are well defined on the space U and

$$g_1^- \frac{1}{2}(c(g_1)+c(g_2))g_2^- = \frac{1}{2}(g_1^+g_2^- + g_1^-g_2^+) + g_1^- K_2 = (g_1g_2)^- + g_1^- K_2. \tag{27}$$

Suppose $a < i, j \le b$. Then (27) shows that

$$\begin{aligned}\langle (g_1g_2)^- w_i, hw_j\rangle &= \langle (g_1g_2)^- w_i, (g_1^{-1}-1)^{-1}g_2^- w_j\rangle\\ &= \langle g_1^{-1}(g_1g_2)^- w_i, g_2^- w_j\rangle\\ &= \langle g_1^{-1}g_1^- \frac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle\\ &= \langle \tfrac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle.\end{aligned} \tag{28}$$

Suppose $j \le b < i$. Then $(g_1g_2)^- w_i = g_1^- w_i$. Hence,

$$\begin{aligned}\langle (g_1g_2)^- w_i, hw_j\rangle &= \langle g_1^- w_i, (g_1^{-1}-1)^{-1}g_2^- w_j\rangle\\ &= \langle w_i, g_2^- w_j\rangle\\ &= \langle (g_2^{-1}-1)w_i, w_j\rangle\\ &= \langle -g_2^{-1}g_2^- w_i, w_j\rangle\\ &= \langle 0, w_j\rangle\\ &= 0.\end{aligned} \tag{29}$$

If $b < i, j$, then

$$\langle (g_1g_2)^- w_i, hw_j\rangle = \langle g_1^- w_i, hw_j\rangle. \tag{30}$$

Notice that

$$\det(\langle g_1^- w_i, hw_j\rangle_{b<i,j}) = \det(\langle w_i, (g_1^{-1}-1)hw_j\rangle_{b<i,j}) = \det(\langle w_i, w_j'\rangle_{b<i,j}) = 1. \tag{31}$$

The first equality in (26) follows from relations (28), (29), (30) and (31).

Since h preserves the subspace K_{12}, it makes sense to define $\tilde{h} \in \mathrm{GL}(\mathrm{W}/K_{12})$ by

$$\tilde{h}(w+K_{12}) = hw \qquad (w \in \mathrm{W}).$$

Then

$$\det(\langle (g_1g_2)^- w_i, hw_j\rangle_{a<i,j}) = \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j})\det(\tilde{h}).$$

But (25) implies $\det(\tilde{h}) = \det(h)$. Hence the second equality in (26) follows.

Also, if $j \leq b < i$, then

$$\langle w_i, (g_1^{-1} - 1)hw_j\rangle = \langle w_i, g_2^- w_j\rangle = 0$$

because $K_2 \perp \mathsf{U}$. Hence,

$$\begin{aligned}\det(\langle w_i, (g_1^{-1} - 1)hw_j\rangle_{i,j}) &= \det(\langle w_i, (g_1^{-1} - 1)hw_j\rangle_{i,j\leq b}) \det(\langle w_i, (g_1^{-1} - 1)hw_j\rangle_{b<i,j})\\ &= \det(\langle w_i, (g_1^{-1} - 1)hw_j\rangle_{i,j\leq b})\\ &= \det(\langle w_i, g_2^- w_j\rangle_{i,j\leq b})\\ &= (-1)^{\dim K_{12}+\dim \mathsf{W}_2} \det(\langle g_2^- w_i, w_j\rangle_{i,j\leq b})\\ &= (-1)^{\dim \mathsf{U}} \det(\langle g_2^- w_i, w_j\rangle_{i,j\leq b}),\end{aligned}$$

This verifies (26). □

Corollary 2.10.
With the above notation we have

$$\det(\langle \frac{1}{2}(c(g_1) + c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\leq b}) = (-1)^{\dim \mathsf{U}} \frac{\det(\langle (g_1 g_2)^- w_i, w_j\rangle_{a<i,j})}{\det(\langle g_1^- w_i, w_j\rangle_{i,j}) \det(\langle g_2^- w_i, w_j\rangle_{i,j\leq b})^{-1}}.$$

2.5. Determinants over the reals

Consider two vector spaces U', U'' over $\mathbb{R}$ of the same dimension equipped with positive definite bilinear symmetric forms B', B'' respectively. Let $u_1', u_2', \ldots, u_n'$ be a B'-orthonormal basis of U' and let $u_1'', u_2'', \ldots, u_n''$ be a B''-orthonormal basis of U''. Suppose $L\colon \mathsf{U}' \to \mathsf{U}''$ is a linear bijection. Denote by M the matrix of L with respect to the two ordered basis:

$$Lu_j' = \sum_{i=1}^{n} M_{i,j} u_i'' \qquad (j = 1, 2, \ldots, n).$$

Then $(\det(M))^2$ does not depend on the choice of the orthonormal basis. (Indeed, if we change the orthonormal bases in the two spaces, we get two matrices $P = (P^t)^{-1}$ and $Q = (Q^t)^{-1}$, so that the new matrix is $M' = PMQ$. Thus $\det(M') = \det(P)\det(M)\det(Q)$. Since $(\det(P))^2 = (\det(Q))^2 = 1$, we see that $(\det(M'))^2 = (\det(M))^2$.) Thus we may define $(\det(L))^2 := (\det(M))^2$.

We shall also need a notion of a determinant for a linear map between two vector spaces (under some additional assumptions of course). For that reason we fix an element $J \in \mathrm{Sp}(\mathsf{W})$ and the corresponding positive definite symmetric bilinear form B, that is,

$$B(w, w') = \langle J(w), w'\rangle \qquad (w, w' \in \mathsf{W}). \tag{32}$$

Then every subspace of W has a B-orthonormal basis.

For a subset $S \subseteq \mathsf{W}$ let $S^{\perp_B} \subseteq \mathsf{W}$ be the B-orthogonal complement of S. It is easy to see that

$$S^{\perp_B} = J^{-1}S^{\perp} = JS^{\perp}. \tag{33}$$

For an element $h \in \mathrm{End}(\mathsf{W})$ define $h^\# \in \mathrm{End}(\mathsf{W})$ by

$$\langle hw, w'\rangle = \langle w, h^\# w'\rangle \qquad (w, w' \in \mathsf{W}). \tag{34}$$

Then $(\mathrm{Ker}\, h^\#)^\perp = h\mathsf{W}$.

Consider an element $h \in \mathrm{End}(\mathrm{W})$ such that $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$. (In our applications h will be equal to g^-, where $g \in \mathrm{Sp}(\mathrm{W})$. Then $g^{\#} = g^{-1} - 1 = -g^{-1}g^-$ has the same kernel as g^-.) Let $L = J^{-1}h$. Denote by L^* the adjoint to L with respect to B, $(B(Lw, w') = B(w, L^*w'))$. Then $L^* = Jh^{\#}$. Hence $Ker\, L = Ker\, L^*$. Since B is anisotropic, L maps $(Ker\, L)^{\perp_B} = L\mathrm{W}$ bijectively onto itself. Thus it makes sense to talk about $\det(L|_{L\mathrm{W}})$, the determinant of the restriction of L to $L\mathrm{W}$. If $w_1, w_2, \dots, w_m$ is a B-orthonormal basis of $(Ker\, L)^{\perp_B}$, then

$$\det(L|_{L\mathrm{W}}) = \frac{\det(B(Lw_i, w_j)_{1\le i,j\le m})}{\det(B(w_i, w_j)_{1\le i,j\le m})} = \det(B(Lw_i, w_j)_{1\le i,j\le m}) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le m}). \tag{35}$$

Under the condition $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$, we define $\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})$ to be the quantity (35).

Suppose $\mathrm{U} \subseteq \mathrm{W}$ is a subspace and $x \in \mathrm{Hom}(\mathrm{U}, \mathrm{W})$ is a linear map such that the formula

$$\langle xu, u'\rangle \qquad (u, u' \in \mathrm{U})$$

defines a symmetric bilinear form on U with the radical $\mathrm{V} \subseteq \mathrm{U}$. The form B induces a positive definite form on the quotient U/V. Pick a B-orthonormal basis $u_1 + \mathrm{V}, \dots, u_k + \mathrm{V} \in \mathrm{U}/\mathrm{V}$ and set

$$\det(\langle x\ ,\ \rangle_{\mathrm{U}/\mathrm{V}}) = \det(\langle xu_i, u_j\rangle_{1\le i,j\le k}). \tag{36}$$

It is easy to see that the quantity (36) does not depend on the choice of the B-orthonormal basis.

Lemma 2.11.

Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and assume that $K_1 = \{0\}$. Then

$$\frac{\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W})\det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} = (-1)^{\dim \mathrm{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathrm{U}/\mathrm{V}})\,(\det(g_2^- : K_{12} \to \mathrm{V}))^{-2}.$$

Proof. Let $\mathrm{W}_2 \subseteq \mathrm{W}$ be the B-orthogonal complement of $K_{12} + K_2$. Then (23) holds, because B is anisotropic. Let $w_1, w_2, \dots$ be a basis of W such that $w_1, w_2, \dots, w_a$ is a B-orthonormal basis of K_{12}, $w_{a+1}, w_{a+2}, \dots, w_b$ is a B-orthonormal basis of W_2 and $w_{b+1}, w_{b+2}, \dots$ is a B-orthonormal basis of K_2. Let $Q \in \mathrm{GL}(\mathrm{W})$ be such that

$Qw_1, Qw_2, \dots$ is a B-orthonormal basis of W,
$Qw_i = w_i$ if $i \le b$,
$Qw_i \perp_B K_{12} + \mathrm{W}_2$ if $b < i$.

Define the matrix elements $Q_{j,i}$ by

$$Qw_i = \sum_j Q_{j,i}w_j.$$

Then

$$Q_{j,i} = \delta_{j,i} \text{ if } i \le b.$$

Hence,

$$\det(Q) = \det((Q_{j,i})_{1\le j,i}) = \det((Q_{j,i})_{b<j,i}) = \det((Q_{j,i})_{a<j,i})$$

and

$$1 = \det(J^{-1}) = \det(B(J^{-1}Qw_i, Qw_j)_{1\le i,j}) = \det(\langle Qw_i, Qw_j\rangle_{1\le i,j}) = (\det(Q))^2\ \det(\langle w_i, w_j\rangle_{1\le i,j}).$$

Therefore

$$\det((Q_{j,i})_{a<j,i})^2\ \det(\langle w_i, w_j\rangle_{1\le i,j}) = 1. \tag{37}$$

Let $u_1, u_2, \dots, u_b$ be B-orthogonal basis of U such that $u_1, u_2, \dots, u_a$ span V. Define the matrix elements $(g_2^-)_{k,i}$ by

$$g_2^- w_i = \sum_{k=1}^{b} (g_2^-)_{k,i} u_k \qquad (1 \le i \le b).$$

Since $g_2^- K_{12} = \mathsf{V}$, we see that

$$(g_2^-)_{k,i} = 0 \text{ if } i \le a < k.$$

Hence

$$\det(((g_2^-)_{k,i})_{1\le k,i\le b}) = \det(((g_2^-)_{k,i})_{1\le k,i\le a}) \det(((g_2^-)_{k,i})_{a<k,i\le b}). \tag{38}$$

Also,

$$\begin{aligned} &(\det(g_2^- : K_{12} \to \mathsf{V}))^2 = (\det(((g_2^-)_{k,i})_{1\le k,i\le a}))^2 \text{ and} \\ &(\det(g_2^- : \mathsf{W}_2 \to \mathsf{U}/\mathsf{V}))^2 = (\det(((g_2^-)_{k,i})_{a<k,i\le b}))^2. \end{aligned} \tag{39}$$

Define $h \in \mathrm{GL}(\mathsf{W})$ as in (24). Then (26) shows that

$$\det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}) \det(h) = \det(\langle \tfrac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\le b}). \tag{40}$$

Furthermore, by (26),

$$\begin{aligned} \det(h) = \det((g_1^{-1}-1)^{-1}(g_1^{-1}-1)h) &= \det(g_1^{-1}-1)^{-1} \det((g_1^{-1}-1)h) \\ &= \det(g_1^{-1}-1)^{-1} \det(\langle w_i, (g_1^{-1}-1)hw_j\rangle_{1\le i,j}) \det(\langle w_i, w_j\rangle_{1\le i,j})^{-1} \\ &= \det(g_1^{-1}-1)^{-1}(-1)^{\dim \mathsf{U}} \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(\langle w_i, w_j\rangle_{1\le i,j})^{-1} \end{aligned} \tag{41}$$

Also,

$$\det(\langle \tfrac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\le b}) = \det(\langle \tfrac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2.$$

By (35),

$$\det((g_1g_2)^- : \mathsf{W}/K_{12} \to \mathsf{U}_{12}) = \det(\langle (g_1g_2)^- Qw_i, Qw_j\rangle_{a<i,j}) = \det((Q_{i,j})_{a<i,j})^2 \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}).$$

Define an element $q \in \mathrm{GL}(\mathsf{W})$ by

$$\begin{aligned} qw_i &= J^{-1}u_i \text{ if } i \le b, \\ qw_i &= w_i \text{ if } b < i. \end{aligned}$$

Then $qw_1, qw_2, \dots, qw_b$ is a B-orthonormal basis of $J^{-1}\mathsf{U} = K_2^{\perp_B}$ so that

$$\det(g_2^- : \mathsf{W}/K_2 \to \mathsf{U}) = \det(\langle g_2^- qw_i, qw_j\rangle_{i,j\le b}).$$

Define the coefficients $q_{i,j}$ by

$$qw_i = \sum_j q_{j,i} w_j.$$

Then

$$q_{j,i} = \delta_{j,i} \text{ if } b < i$$

so that

$$\det(q) = \det((q_{j,i})_{1\le i,j}) = \det((q_{j,i})_{1\le i,j\le b}).$$

Also,

$$g_2^- q w_i = \sum_j q_{j,i} g_2^- w_j = \sum_{j\le b} q_{j,i} g_2^- w_j \qquad (i \le b).$$

Therefore,

$$\det(\langle g_2^- q w_i, q w_j\rangle_{i,j\le b}) = \det(q)^2 \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}).$$

Define the coefficients $q_{i,j}^{-1}$ of the inverse map q^{-1} by

$$w_i = q^{-1}(q w_i) = \sum_j q_{i,j}^{-1} q w_j.$$

Since, the $q w_i$ form an orthonormal basis of W,

$$q_{i,j}^{-1} = B(q^{-1} q w_i, q w_j) = B(w_i, q w_j) = B(q w_j, w_i),$$

so that

$$q_{i,j}^{-1} = \begin{cases} \langle u_j, w_i\rangle \text{ if } j \le b, \\ B(w_j, w_i) \text{ if } j > b, \\ B(w_j, w_i) = \delta_{i,j} \text{ if } i, j > b. \end{cases}$$

In particular, $q_{i,j}^{-1} = 0$ if $j \le b < i$ so that

$$\det(q)^{-1} = \det(q^{-1}) = \det((q_{i,j}^{-1})_{i,j\le b}) = \det(\langle u_j, w_i\rangle_{i,j\le b}).$$

Thus

$$\begin{aligned} &\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U}) = (\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}))^2 \det(q)^2 \qquad (42)\\ &= (\det(\langle \sum_{k=1}^{b} (g_2^-)_{k,i} u_k, w_j\rangle_{i,j\le b}))^2 \det(q)^2 = (\det((g_2^-)_{k,i})_{k,i\le b}) \det(\langle u_k, w_j\rangle_{k,j\le b}))^2 \det(q)^2 \\ &= (\det((g_2^-)_{k,i})_{k,i\le b}))^2 = (\det(g_2^- : K_{12} \to \mathrm{V}))^2 (\det(g_2^- : \mathrm{W}_2 \to \mathrm{U/V}))^2, \end{aligned}$$

where the last equality follows from (38) and (39). The formula (37) follows from (37) – (42) via a straightforward computation:

$$\begin{aligned} &\frac{\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= \frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= \frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det((g_(2^-)_{k,i})_{a<k,i\le b})^2}{\det(h) \det g_1^- \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= \frac{(-1)^{\dim \mathrm{U}} \det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2}{\det(g_1^{-1}-1)^{-1} \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(\langle w_i, w_j\rangle_{1\le i,j})^{-1} \det g_1^- \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= \frac{(-1)^{\dim \mathrm{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2}{\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= \frac{(-1)^{\dim \mathrm{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathrm{U/V}}) (\det(g_2^- : \mathrm{W}_2 \to \mathrm{U/V}))^2}{\det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= \frac{(-1)^{\dim \mathrm{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathrm{U/V}}) (\det(g_2^- : \mathrm{W}_2 \to \mathrm{U/V}))^2}{(\det(g_2^- : K_{12} \to \mathrm{V}))^2 (\det(g_2^- : \mathrm{W}_2 \to \mathrm{U/V}))^2} \\ &= \frac{(-1)^{\dim \mathrm{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathrm{U/V}})}{(\det(g_2^- : K_{12} \to \mathrm{V}))^2}. \end{aligned}$$

(Here the second equality follows from (40) and (42), and the third one from (41).) □

2.6. Determinants over p-fields

Let $\mathbb{F}$ be a commutative p-field in the terminology of [43, Def 2, page 12], that is, $\mathbb{F}$ is a local non Archimedean field with finite residue field. Hence $\mathbb{F}$ is a finite extension of either the p-adic field $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$ (the fraction field of the ring $\mathbb{F}_p[[t]]$ of formal power series in one indeterminate t with coefficient in $\mathbb{F}_p$).

Denote by $|\ |_{\mathbb{F}}$ the module on $\mathbb{F}$, as in [43, page 4]. Then $\mathfrak{o}_{\mathbb{F}} = \{a \in \mathbb{F} \ : \ |a|_{\mathbb{F}} \leq 1\}$ is the ring of integers of $\mathbb{F}$, and we have $\mathfrak{o}_{\mathbb{F}}^{\times} = \{a \in \mathbb{F} \ : \ |a|_{\mathbb{F}} = 1\}$ as in [43, page 12].

Being locally compact, $\mathbb{F}$ has a real-valued Haar measure: the unique translation invariant measure $\mu_{\mathbb{F}}$ with the properties

$$d\mu(ax) = |a|_{\mathbb{F}}\, d\mu(x) \quad (x \in \mathbb{F},\ a \in \mathbb{F}^{\times}),$$

$$\mu_{\mathbb{F}}(\mathfrak{o}_{\mathbb{F}}) = \int_{|x|_{\mathbb{F}} \leq 1} d\mu(x) = 1.$$

Let $r \in \mathbb{Z}$. One has

$$\mu_{\mathbb{F}}(\varpi_{\mathbb{F}}^{r}\, \mathfrak{o}_{\mathbb{F}}) = \int_{|x|_{\mathbb{F}} \leq q^r} d\mu_{\mathbb{F}}(x) = q^r. \tag{43}$$

Then Eqn. (43) gives

$$\int_{|x|_{\mathbb{F}} = q^r} d\mu_{\mathbb{F}}(x) = \int_{|x|_{\mathbb{F}} \leq q^r} d\mu_{\mathbb{F}}(x) - \int_{|x|_{\mathbb{F}} \leq q^{r-1}} d\mu_{\mathbb{F}}(x) = q^r(1 - q^{-1}). \tag{44}$$

More generally, let $r, R \in \mathbb{Z}$ with $r \leq R$. One gets

$$\int_{q^r \leq |x|_{\mathbb{F}} \leq q^R} d\mu_{\mathbb{F}}(x) = \int_{|x|_{\mathbb{F}} \leq q^R} d\mu_{\mathbb{F}}(x) - \int_{|x|_{\mathbb{F}} \leq q^r} d\mu_{\mathbb{F}}(x) = q^R - q^r. \tag{45}$$

Let $\mathbf{r} = (r_1, r_2, \dots, r_n) \in \mathbb{Z}^n$ and $\mathbf{R} = (R_1, R_2, \dots, R_n) \in \mathbb{Z}^n$ where $r_i \leq R_i$ for every $i \in \{1, \dots, n\}$. We set

$$B(\mathbf{r}, \mathbf{R}) := \left\{ \mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{F}^n \ : \ q^{r_i} \leq |x_i|_{\mathbb{F}} \leq q^{R_i} \text{ for } i = 1, \dots, n \right\}.$$

It follows from (45) that

$$\mu_{\mathbb{F}^n}(B(\mathbf{r}, \mathbf{R})) = \prod_{i=1}^{n} (q^{R_i} - q^{r_i}). \tag{46}$$

The following Lemma relates the volume of the linear image of the set in $\mathbb{F}^n$ to the volume of the set itself.

Lemma 2.12.
Let $L\colon \mathbb{F}^n \to \mathbb{F}^n$ be an invertible linear transformation then

$$\mu_{\mathbb{F}^n}(L(B)) = |\det(L)|_{\mathbb{F}}\ \mu_{\mathbb{F}^n}(B), \qquad \textit{for all } B \in \mathfrak{B}(\mathbb{F}^n). \tag{47}$$

Proof. Call $B(\mathbf{r}, \mathbf{R})^t := \{\mathbf{x}^t \ : \ x \in B(\mathbf{r}, \mathbf{R})\}$ a cell in $\mathbb{F}^n$. (Here $\mathbf{x}^t$ means the transpose of $\mathbf{x}$.) We will first check that the relation (47) for every cell $B(\mathbf{r}, \mathbf{R})^t$. The matrix representing L can be written as a product of elementary matrices, and since determinant preserves products, it is sufficient to show that the relation (47) holds for elementary matrices.

Let $i \in \{1, \dots, n\}$, let $y \in \mathbb{F}^{\times}$ and let $E_i(y)$ be the elementary matrix obtained by multiplying by y the i-th row of the identity $n \times n$ matrix. We have $\det(E_i(y)) = y$ and

$$\begin{aligned} E_i(y) \cdot B(\mathbf{r}, \mathbf{R})^t &= \left\{ (x_1, \dots, x_{i-1}, y x_i, x_{i+1}, \dots, x_n)^t \ : \ q^{r_k} \leq |x_k|_{\mathbb{F}} \leq q^{R_k} \text{ for } k = 1, \dots, n \right\} \\ &= \left\{ (x_1, \dots, x_n) \ : \ q^{r_k} \leq |x_k|_{\mathbb{F}} \leq q^{R_k} \text{ for } k \neq i,\ \ |y|_{\mathbb{F}_q}^{r_i} \leq |x_i|_{\mathbb{F}} \leq |y|_{\mathbb{F}_q}^{R_i} \right\}, \end{aligned}$$

since $|yx_i|_{\mathbb{F}} = |y|_{\mathbb{F}} \cdot |x_i|_{\mathbb{F}}$. Hence

$$\mu_{\mathbb{F}^n}(E_i(y) \cdot B(\mathbf{r},\mathbf{R})^t) = |y|_{\mathbb{F}} \cdot \prod_{k=1}^{n}(q^{R_k} - q^{r_k}) = |\det(E_i(y))|_{\mathbb{F}} \cdot \mu_{\mathbb{F}^n}(B(\mathbf{r},\mathbf{R})^t).$$

Let $i, j \in \{1, \dots, n\}$. Let $E_{i,j}$ be the elementary matrix corresponding to the interchange of row i with row j. We have $\det(E_{i,j}) = -1$ and

$$E_{i,j} \cdot B(\mathbf{r},\mathbf{R})^t = \left\{ (x_1, \dots, x_n) \ : \ \begin{cases} q^{r_k} \leq |x_k|_{\mathbb{F}} \leq q^{R_k} & \text{for } k \neq i, j \\ q^{r_i} \leq |x_j|_{\mathbb{F}} \leq q^{R_i} \\ q^{r_j} \leq |x_i|_{\mathbb{F}} \leq q^{R_j} \end{cases} \right\}.$$

Hence $|\det(E_{i,j})|_{\mathbb{F}} = 1$ and $\mu_{\mathbb{F}^n}(E_{i,j} \cdot B(\mathbf{r},\mathbf{R})^t) = \mu_{\mathbb{F}^n}(B(\mathbf{r},\mathbf{R})^t)$.

Let $E_{i\cup j}$ be the elementary matrix obtained by replacing row i by the sum of row i and row j. By multiplying by the matrix $E_{i,1}$ if necessary, we may assume that $i = 1$. We have $E_{1\cup j}(x_1, \dots, x_n)^t = (x_1 + x_j, x_2 \dots, x_n)^t$. Hence $\det(E_{1\cup j}) = 1$. We can view $\mathbb{F}^n$ as the Cartesian product $\mathbb{F} \times \mathbb{F}^{n-1}$. For every $\mathbf{x}' = (x_2, \dots, x_n)^t \in \mathbb{F}^{n-1}$, let

$$B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'} := \left\{ z \in \mathbb{F} \ : \ (z, x_2, \dots, x_n)^t \in B(\mathbf{r},\mathbf{R})^t \right\}$$

and similarly

$$(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'} := \left\{ z + x_j \in \mathbb{F} \ : \ (z + x_j, x_2, \dots, x_n)^t \in E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t \right\}.$$

We have

$$(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'} = \{ z \in \mathbb{F} \ : \ (z, x_2, \dots, x_n)^t \in B(\mathbf{r},\mathbf{R})^t \} \ + \ x_j$$

that is,

$$(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'} = B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'} + x_j.$$

Thus, for all $\mathbf{x}' \in \mathbb{F}^{n-1}$, $(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'}$ is a translation of $B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'}$ and since, the measure $\mu_{\mathbb{F}}$ is translation-invariant, we have $\mu_{\mathbb{F}}((E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'}) = \mu_{\mathbb{F}}(B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'})$. On the other hand, by Fubini's Theorem, we get

$$\mu_{\mathbb{F}^n}(E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t) = \int_{\mathbb{F}^{n-1}} \mu_{\mathbb{F}}(((E_{1\cup j} \cdot B(\mathbf{r},\mathbf{R})^t)_{\mathbf{x}'}) d\mu_{\mathbb{F}^{n-1}}(\mathbf{x}') = \int_{\mathbb{F}^{n-1}} \mu_{\mathbb{F}} B(\mathbf{r},\mathbf{R})^t_{\mathbf{x}'}) d\mu_{\mathbb{F}^{n-1}}(\mathbf{x}') = \mu_{\mathbb{F}^n}(B(\mathbf{r},\mathbf{R})^t).$$

Every open set in $\mathbb{F}^n$ can be written as a countable union of cells in $\mathbb{F}^n$ and therefore, by the countable additivity of the Haar measure on $\mathbb{F}$, the measure $\mu_{\mathbb{F}^n}$ satisfies the relation (47) is for any open set. Then the regularity of $\mu_{\mathbb{F}^n}$ implies that (47) holds for any Borel set. □

Lemma 2.12 shows that Lemma 4.2 is still valid on the local nonarchimedean field $\mathbb{F}$ with the pullback $L^*(\mu_V)$ defined as in Eqn. (174) up to replacing the absolute value $|\ \ |$ by $|\ \ |_{\mathbb{F}}$, that is, we obtain here:

$$L^*(\mu_V) = |\det(\tilde{L})|_{\mathbb{F}}^{-1} \, \mu_{L^{-1}(V)}.$$

Let W be a finite dimensional vector space over $\mathbb{F}$ and let $\mathcal{L} \subseteq \mathrm{W}$ be a lattice, [43, page 28]. Let $\mathrm{W}^* = \mathrm{Hom}(\mathrm{W}, \mathbb{F})$ be the dual vector space and let

$$\mathcal{L}_* = \{ w^* \in \mathrm{W}^* : \ w^*(w) \in \mathfrak{o}_{\mathbb{F}} \text{ for all } w \in \mathcal{L} \}.$$

This is the lattice dual to $\mathcal{L}$.

Lemma 2.13.

For any subspace $\mathrm{U} \subseteq \mathrm{W}$, *the restriction map*

$$\mathrm{W}^* \ni w^* \to w^*|_{\mathrm{U}} \in \mathrm{U}^*$$

induces the following short exact sequence

$$0 \to \mathcal{L}_* \cap \mathrm{U}^\perp \to \mathcal{L}_* \to (\mathcal{L} \cap \mathrm{U})_* \to 0,$$

where $\mathrm{U}^\perp \subseteq \mathrm{W}^*$ *is the annihilator of* U. *In particular we have the isomorphisms of lattices*

$$(\mathcal{L}_* + \mathrm{U}^\perp)/\mathrm{U}^\perp = \mathcal{L}_*/\mathcal{L}_* \cap \mathrm{U}^\perp = (\mathcal{L} \cap \mathrm{U})_*.$$

Proof. By [43, Theorem 1, page 29], there is a basis $w_1, \dots, w_m, \dots$ of W such that $w_1, \dots, w_m$ is a basis of U and $\mathcal{L} = \mathfrak{o}_{\mathbb{F}} w_1 + \mathfrak{o}_{\mathbb{F}} w_2 + \dots$. Hence, $\mathcal{L} \cap \mathrm{U} = \mathfrak{o}_{\mathbb{F}} w_1 + \dots + \mathfrak{o}_{\mathbb{F}} w_m$. Let $w_1^*, \dots, w_m^*, \dots$ be the dual basis of W^* ($w_i^*(w_j) = \delta_{i,j}$). Then $\mathcal{L}_* = \mathfrak{o}_{\mathbb{F}} w_1^* + \dots + \mathfrak{o}_{\mathbb{F}} w_m^* + \dots$ and $(\mathcal{L} \cap \mathrm{U})_* = \mathfrak{o}_{\mathbb{F}} w_1^* + \dots + \mathfrak{o}_{\mathbb{F}} w_m^*$. Hence the restriction map is surjective. The rest is obvious. $\square$

Recall the notion of a norm, [43, page 24], and the norm associated to a lattice

$$N_{\mathcal{L}}(w) = \inf\{|x|_{\mathbb{F}}^{-1} :\ x \in \mathbb{F}^\times,\ xw \in \mathcal{L}\} \qquad (w \in \mathrm{W}),$$

[43, page 28]. Then $\mathcal{L} = \{w \in \mathrm{W} :\ N_{\mathcal{L}}(w) \leq 1\}$. The following fact is stated in [43, page 29]

Lemma 2.14.

Let N *be a norm on* W. *The* $N = N_{\mathcal{L}}$ *if and only if*

$$\mathcal{L} = \{w \in \mathrm{W} :\ N(w) \leq 1\} \tag{48}$$

and

$$\{N(w) :\ w \in \mathrm{W}\} = \{|x|_{\mathbb{F}} :\ x \in \mathbb{F}\}. \tag{49}$$

Let N be a norm on W. As in [43, p. 26], we shall say that two subspaces $\mathrm{W}', \mathrm{W}''$ of W are *N-orthogonal* to each other whenever $\mathrm{W} = \mathrm{W}' \oplus \mathrm{W}''$, and $N(w' + w'') = \sup(N(w'), N(w''))$ for all $w' \in \mathrm{W}'$ and all $w'' \in \mathrm{W}''$.

Lemma 2.15.

Let $\mathrm{V} \subseteq \mathrm{W}$ *be a subspace. Then*

$$N_{(\mathcal{L}+\mathrm{V})/\mathrm{V}}(w + \mathrm{V}) = \inf\{N_{\mathcal{L}}(w + v) :\ v \in \mathrm{V}\} \qquad (w \in \mathrm{W}). \tag{50}$$

Proof. [43, Theorem 1, page 29] implies that there is a subspace $\mathrm{V}' \subseteq \mathrm{W}$ which is $N_{\mathcal{L}}$-orthogonal to V and such that

$$\mathrm{W} = \mathrm{V}' \oplus \mathrm{V} \tag{51}$$

and

$$\mathcal{L} = \mathcal{L} \cap \mathrm{V}' \oplus \mathcal{L} \cap \mathrm{V}. \tag{52}$$

Let $N(w + \mathrm{V})$ denote the right hand side of (50). For $w \in \mathrm{W}$ let $w' \in \mathrm{V}'$ denote the V'-component of w, according to the decomposition (51). Then clearly

$$N(w + \mathrm{V}) = N_{\mathcal{L}}(w') \qquad (w \in \mathrm{W}).$$

In particular N is a norm on W/V. Also, the range of N coincides with the range of $N_{\mathcal{L}}$. Hence Lemma 2.14 implies that $N = N_{\mathcal{L}'}$, where $\mathcal{L}' = \{w + \mathrm{V} \in \mathrm{W}/\mathrm{V} :\ N(w+\mathrm{V}) \le 1\}$. The condition $N(w+\mathrm{V}) \le 1$ means that $N_{\mathcal{L}}(w') \le 1$, which is equivalent to $w' \in \mathcal{L}'$. Thus

$$\mathcal{L}' = \{w + \mathrm{V} \in \mathrm{W}/\mathrm{V} :\ w' \in \mathcal{L}\}.$$

But (52) shows that the condition $w' \in \mathcal{L}$ is equivalent to $w \in \mathcal{L} + \mathrm{V}$. (Indeed, if $w' \in \mathcal{L}$ then $w \in \mathcal{L} + \mathrm{V}$. Conversely, suppose $w \in \mathcal{L} + \mathrm{V}$. Then there is $w_0 \in \mathcal{L}$ and $v \in \mathrm{V}$ such that $w = w_0 + v$. Hence, $w' = w_0'$. But $w_0' \in \mathcal{L} \cap \mathrm{V}'$ by (52). Thus $w' \in \mathcal{L}$.) Therefore

$$\mathcal{L}' = (\mathcal{L} + \mathrm{V})/\mathrm{V}. \qquad \square$$

Corollary 2.16.

Under the identifications of Lemma 2.13, the following equalities hold for any $w^* \in \mathrm{W}^*$:

$$N_{(\mathcal{L}\cap\mathrm{U})_*}(w^*|_{\mathrm{U}}) = N_{(\mathcal{L}_*+\mathrm{U}^\perp)/\mathrm{U}^\perp}(w^* + \mathrm{U}^\perp) = \inf\{N_{\mathcal{L}_*}(w^* + w_0^*) :\ w_0^* \in \mathrm{U}^\perp\} = \max\{|w^*(u)|_{\mathbb{F}} :\ u \in \mathcal{L} \cap \mathrm{U}\}.$$

(The second equality means that the norm on the quotient is the usual quotient norm.)

Proof. The first equality amounts to the last identification of Lemma 2.13. The second equality follows from Lemma 2.15 with W, $\mathcal{L}$ and V replaced by W^*, $\mathcal{L}_*$ and $\mathrm{U}^\perp$ respectively. The third equality follows from the fact that

$$N_{\mathcal{L}_*}(w^*) = \max\{|w^*(w)|_{\mathbb{F}} :\ w \in \mathcal{L}\}. \tag{53}$$

One may verify the equality (53) as follows. The right hand side of (53) defines a norm on W^* whose range coincides with the range of $|\ |_{\mathbb{F}}$. The set of the w^* such that the right hand side is less or equal than 1 coincides with the set of the w^* such that $w^*(w) \in \mathfrak{o}_{\mathbb{F}}$ for all $w \in \mathcal{L}$. But this is $\mathcal{L}_*$. Hence Lemma 2.14 implies (53). $\square$

Let $\mathcal{L} \subseteq \mathrm{W}$ be a lattice. We know from [43, Theorem1, page 29], that there is a basis $w_1, w_2, \dots$ of W such that

$$\mathcal{L} = \mathfrak{o}_{\mathbb{F}} w_1 + \mathfrak{o}_{\mathbb{F}} w_2 + \dots. \tag{54}$$

In particular the spaces $\mathbb{F}w_1, \mathbb{F}w_2, \dots$ are $N_{\mathcal{L}}$-orthogonal and $1 = N_{\mathcal{L}}(w_1) = N_{\mathcal{L}}(w_2) = \dots$. Thus we may define a basis of W to be $N_{\mathcal{L}}$-orthonormal if the condition (54) holds.

Let $w_1^*, w_2^*, \dots$ be the dual basis of W^*. Then

$$\mathcal{L}_* = \mathfrak{o}_{\mathbb{F}} w_1^* + \mathfrak{o}_{\mathbb{F}} w_2^* + \dots.$$

Hence the basis $w_1^*, w_2^*, \dots$ is $N_{\mathcal{L}_*}$-orthonormal.

Suppose W′ is another finite dimensional vector space over $\mathbb{F}$ with a lattice $\mathcal{L}'$ and an $\mathrm{N}_{\mathcal{L}'}$-orthonormal basis $w_1', w_2', \dots$. Given $h \in \mathrm{Hom}(\mathrm{W}, \mathrm{W}')$, there is the corresponding matrix

$$M(h) = [h_{ji}],\ \ \mathfrak{h}(w_i) = \sum_j h_{ji} w_j'.$$

The determinant $\det(M(h))$ does depend on the choice of the bases, but the quantity $\det(M(h))(\mathfrak{o}_{\mathbb{F}}^\times)^2$ does not. Hence we may define

$$\det(h : \mathrm{W} \to \mathrm{W}') = \det(M(h))(\mathfrak{o}_{\mathbb{F}}^\times)^2 \in \mathbb{F}^\times/(\mathfrak{o}_{\mathbb{F}}^\times)^2 \tag{55}$$

and

$$|\det(h : \mathrm{W} \to \mathrm{W}')|_{\mathbb{F}} = |\det(M(h))|_{\mathbb{F}} \in \mathbb{R}. \tag{56}$$

Lemma 2.17.

Let $h \in \mathrm{Hom}(\mathrm{W}, \mathrm{W}')$ *and let* $h^* \in \mathrm{Hom}(\mathrm{W}'^*, \mathrm{W}^*)$ *be the adjoint map. Then*

$$\det(h : \mathrm{W} \to \mathrm{W}') = \det(h^* : \mathrm{W}'^* \to \mathrm{W}^*).$$

Proof. Let $w_1^*, w_2^*, \dots \in \mathrm{W}^*$ be the basis dual to $w_1, w_2, \dots$ and let $w_1'^*, w_2'^*, \dots \in \mathrm{W}'^*$ be the basis dual to w_1', $w_2', \dots$. Then,

$$h(w_i) = \sum_j h_{ji} w_j' \text{ if and only if } h^*(w_j'^*) = \sum_i h_{ji} w_i^*,$$

because

$$h_{ji} = w_j'^*(\sum_j h_{ji} w_j') = w_j'^*(h(w_i)) = h^*(w_j'^*)(w_i).$$

Hence, the matrix $M(h^*)$ is the transpose of the matrix $M(h)$ and the claim follows. □

Lemma 2.18.

For any $T \in \mathrm{End}(\mathrm{W})$ *any Haar measure* μ *on the additive group* W *and any measurable set* $B \subseteq \mathrm{W}$

$$\mu(T(B)) = |\det(T)|_{\mathbb{F}}\, \mu(B).$$

Proof. This is a direct consequence of Lemma 2.12. □

Lemma 2.19.

Suppose $w_1, w_2, \dots$ *is an* $N_{\mathcal{L}}$*-orthonormal basis of* W *and* $T \in \mathrm{End}(\mathrm{W})$ *is such that* $Tw_1, Tw_2, \dots$ *is also an* $N_{\mathcal{L}}$*-orthonormal basis of* W. *Then* $|\det(T)|_{\mathbb{F}} = 1$.

Proof. Since, by the assumption, $T(\mathcal{L}) = \mathcal{L}$, the map T preserves the Haar measure on W. Hence, Lemma 2.18 shows that $|\det(T)|_{\mathbb{F}} = 1$. □

From now on we assume that the space W is equipped with a non-degenerate symplectic form $\langle\ ,\ \rangle$. We shall identify W with the dual W^* by

$$w(u) = \langle u, w\rangle \qquad (u, w \in \mathrm{W}). \tag{57}$$

Then, for a subspace $\mathrm{U} \subseteq \mathrm{W}$ the annihilator $\mathrm{U}^\perp$ coincides with the $\langle\ ,\ \rangle$-orthogonal complement. We shall say that the lattice $\mathcal{L}$ is self-dual in the sense that $\mathcal{L} = \mathcal{L}_*$. Let us fix a self-dual lattice $\mathcal{L} \subseteq \mathrm{W}$.

For any two subspaces $\mathrm{V} \subseteq \mathrm{U} \subseteq \mathrm{W}$, N shall denote the quotient norm of $N_{\mathcal{L}}$:

$$N(u + \mathrm{V}) = \inf\{N_{\mathcal{L}}(u + v) :\ v \in \mathrm{V}\} \qquad (u \in \mathrm{U}). \tag{58}$$

For an element $h \in \mathrm{End}(\mathrm{W})$ define $h^\# \in \mathrm{End}(\mathrm{W})$ by

$$\langle hw, w'\rangle = \langle w, h^\# w'\rangle \qquad (w, w' \in \mathrm{W}).$$

Then $(\mathrm{Im}\, h)^\perp = \mathrm{Ker} h^\#$. Hence, if $\mathrm{Ker} h = \mathrm{Ker} h^\#$ then we have the following short exact sequence

$$0 \to (\mathrm{Im}\, h)^\perp \to \mathrm{W} \to \mathrm{Im}\, h \to 0. \tag{59}$$

In the next Lemma, we shall consider $\mathrm{Im}\, h$ as the quotient $\mathrm{W}/(\mathrm{Im}\, h)^\perp$, and N will be the corresponding quotient norm as defined in (58).

Lemma 2.20.

Suppose $h \in \mathrm{End}(\mathrm{W})$ is such that $\mathrm{Ker}\, h = \mathrm{Ker}\, h^{\#}$. Let $u_1, \dots, u_k$ be an N-orthonormal basis of $\mathrm{Im}\, h$ and let $w_1 + \mathrm{Ker} h$, $\dots$, $w_k + \mathrm{Ker} h$ be the dual basis of $\mathrm{W}/\mathrm{Ker} h$. Let $M = M(h)$ be the matrix of the induced bijection $h\colon \mathrm{W}/\mathrm{Ker} h \to \mathrm{Im}\, h$ with respect to these two ordered basis,

$$hw_i = \sum_j M_{j,i}\, u_j.$$

Then

$$\det(M(h)) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le k}).$$

Also, we may choose the elements $w_1, \dots, w_k$ so that the spaces $\mathbb{F}w_1, \dots, \mathbb{F}w_k$, $\mathrm{Ker}\, h$ are N-orthogonal.

Proof. Since

$$\langle hw_i, w_j\rangle = \langle \sum_l M_{l,i} u_l, w_j\rangle = M_{j,i},$$

the formula for the determinant follows. The last statement follows from Lemma 2.15 and Corollary 2.16. □

Notice that if $u_1', \dots, u_k'$ is another N-orthonormal basis of $\mathrm{Im}\, h$, with dual basis $w_1' + \mathrm{Ker} h, \dots, w_k' + \mathrm{Ker} h$, then

$$\det(\langle hw_i', w_j'\rangle_{1\le i,j\le k}) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le k})\, a^2,$$

where $a \in \mathbb{F}^\times$ is the determinant of the transition matrix from $u_1, \dots, u_k$ to $u_1', \dots, u_k'$ (which is also the determinant of the transition matrix from the corresponding dual basis). We know from Lemma 2.19 that $|a|_{\mathbb{F}} = 1$. Hence without any ambiguity we may define

$$\det(h\colon \mathrm{W}/\mathrm{Ker} h \to \mathrm{Im}\, h) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le k})\,(\mathfrak{o}_{\mathbb{F}}^\times)^2 \tag{60}$$

as an element of $\mathbb{F}^\times/(\mathfrak{o}_{\mathbb{F}}^\times)^2$. Also, without any ambiguity we may define

$$|\det(h\colon \mathrm{W}/\mathrm{Ker} h \to \mathrm{Im}\, h)|_{\mathbb{F}} = |\det(\langle hw_i, w_j\rangle_{1\le i,j\le k})|_{\mathbb{F}} \tag{61}$$

as a positive real number.

Similarly, if $\mathrm{U} \subset \mathrm{W}$ is a subspace and $x \in \mathrm{Hom}(\mathrm{U}, \mathrm{W})$ is such that the bilinear form

$$\langle xu, u'\rangle \qquad (u, u' \in \mathrm{U})$$

is symmetric, with the radical $\mathrm{V} \subseteq \mathrm{U}$, we define

$$\det(\langle x\ ,\ \rangle_{\mathrm{U/V}}) = \det(\langle xu_i, u_j\rangle_{1\le i,j\le k})\,(\mathfrak{o}_{\mathbb{F}}^\times)^2 \tag{62}$$

and

$$|\det(\langle x\ ,\ \rangle_{\mathrm{U/V}})|_{\mathbb{F}} = |\det(\langle xu_i, u_j\rangle_{1\le i,j\le k})|_{\mathbb{F}}, \tag{63}$$

where $u_1 + \mathrm{V}, u_2 + \mathrm{V}, \dots,$ is an N-orthonormal basis of $\mathrm{U/V}$.

Lemma 2.21.

If $w_1, w_2, \dots$ is a $N_{\mathcal{L}}$-orthonormal basis of W, then

$$|\det(\langle w_i, w_j\rangle_{1\le i,j})|_{\mathbb{F}} = 1.$$

Proof. Since $\langle w_i, w_j\rangle \in \mathfrak{o}_{\mathbb{F}}$,

$$|\det((\langle w_i, w_j\rangle_{1\le i,j})|_{\mathbb{F}} \le 1.$$

Since the lattice $\mathcal{L}$ is self-dual the same inequality holds for the dual basis. The product of the two matrices is 1. Hence the equality follows. □

With the notation of (60), suppose $\mathrm{Ker} h = 0$. Let $h_{k,i}$ be the matrix coefficients of h with respect to the basis w_1, w_2, . . . :

$$hw_i = \sum_k h_{k,i} w_k.$$

Then

$$\langle hw_i, w_j\rangle = \sum_k h_{k,i}\langle w_k, w_j\rangle$$

and

$$\det((h_{k,i})_{1\le k,i}) = \det(h)$$

is the usual determinant of h. Hence, by Lemma 2.21, the determinant defined in (60) satisfies the following equation

$$\det(h : \mathrm{W} \to \mathrm{W}) = \det(h)\,(\mathfrak{o}_{\mathbb{F}}^{\times})^2. \tag{64}$$

More generally, suppose $K, \mathrm{V} \subseteq \mathrm{W}$ are two subspaces of the same dimension and $h \in \mathrm{Hom}(K, \mathrm{V})$ is a bijection. Choose an N-orthonormal basis of $\mathbb{F}$, an N-orthonormal basis of V and let $h_{k,i}$ denote the corresponding matrix coefficients of h. Then, by Lemma 2.21,

$$\det((h_{k,i})_{1\le k,i})\,(\mathfrak{o}_{\mathbb{F}}^{\times})^2$$

does not depend on the choice of the bases. Therefore we may define

$$\det(h : K \to \mathrm{V}) = \det((h_{k,i})_{1\le k,i})\,(\mathfrak{o}_{\mathbb{F}}^{\times})^2 \tag{65}$$

as an element of $\mathbb{F}^{\times}/(\mathfrak{o}_{\mathbb{F}}^{\times})^2$ and

$$|\det(h : K \to \mathrm{V})|_{\mathbb{F}} = |\det((h_{k,i})_{1\le k,i})|_{\mathbb{F}} \tag{66}$$

as an element of $\mathbb{R}^+$. Notice that, via the identification (57), the definitions (65) and (66) are consistent with (55) and (56). Also, Lemma 2.17 may be rephrased as

Lemma 2.22.
Let $h \in \mathrm{End}(\mathrm{W})$ and let $K \subseteq \mathrm{W}$ be a subspace. Then

$$h^{\#}((hK)^{\perp}) \subseteq K^{\perp}, \tag{67}$$

$$\det(h : K \to hK) = \det(h^{\#} : \mathrm{W}/(hK)^{\perp} \to \mathrm{W}/K^{\perp}) \tag{68}$$

and

$$|\det(h : K \to hK)|_{\mathbb{F}} = |\det(h^{\#} : \mathrm{W}/(hK)^{\perp} \to \mathrm{W}/K^{\perp})|_{\mathbb{F}}. \tag{69}$$

Proof. The point is that $\mathrm{W}/K^{\perp} = K^*$, $\mathrm{W}/(hK)^{\perp} = (hK)^*$ and $h^{\#} = h^*$. □

In the next Lemma, we keep the notation defined in Notation 2.4 and Notation 2.6, that is, for $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$,

$$\mathrm{U} = \mathrm{U}_1 \cap \mathrm{U}_2 = g_1^{-}\mathrm{W} \cap g_2^{-}\mathrm{W} \quad \text{and} \quad \mathrm{U}_{12} = (g_1g_2)^{-}\mathrm{W},$$

$$K_1 = \mathrm{Ker}\, g_1^{-}, \quad K_2 = \mathrm{Ker}\, g_2^{-} \quad \text{and} \quad K_{12} = \mathrm{Ker}(g_1g_2)^{-}.$$

Lemma 2.23.
Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ *and assume that* $K_1 = \{0\}$. *Then*

$$\frac{\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W})\det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} = (-1)^{\dim \mathrm{U}} \det(\langle \frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathrm{U/V}})\,(\det(g_2^- : K_{12} \to \mathrm{V}))^{-2}$$

and

$$\frac{|\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12})|_{\mathbb{F}}}{|\det(g_1^- : \mathrm{W} \to \mathrm{W})|_{\mathbb{F}}|\det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})|_{\mathbb{F}}} = |\det(\langle \frac{1}{2}(c(g_1)+c(g_2))\ ,\ \rangle_{\mathrm{U/V}})|_{\mathbb{F}}\,|\det(g_2^- : K_{12} \to \mathrm{V})|_{\mathbb{F}}^{-2}$$

Proof. Clearly (70) follows from (70). We shall verify (70). Let $\mathrm{W}_2 \subseteq \mathrm{W}$ be the $N_{\mathcal{L}}$-orthogonal complement of $K_{12}+K_2$. Then (23) holds. Let $w_1, w_2, \dots$ be a basis of W such that $w_1, w_2, \dots, w_a$ is a $N_{\mathcal{L}}$-orthonormal basis of K_{12}, $w_{a+1}, w_{a+2}, \dots, w_b$ is a $N_{\mathcal{L}}$-orthonormal basis of W_2 and $w_{b+1}, w_{b+2}, \dots$ is a $N_{\mathcal{L}}$-orthonormal basis of K_2. Then $w_1, w_2, \dots, w_b$ is $N_{\mathcal{L}}$-orthonormal basis of $K_{12}+\mathrm{W}_2$. Theorem 1 on page 29 in [43] implies that we may extend it to an $N_{\mathcal{L}}$-orthonormal basis of W:

$$w_1, \dots, w_b, w'_{b+1}, w'_{b+2}, \dots .$$

Define an element $Q \in \mathrm{GL}(\mathrm{W})$ by

$$Q(w_i) = \begin{cases} w_i & \text{if } i \le b, \\ w'_i & \text{if } i > b. \end{cases}$$

Then

$$\begin{aligned} &Qw_1, Qw_2, \dots \ \text{ is a } N_{\mathcal{L}}\text{-orthonormal basis of W},\\ &Qw_i = w_i \text{ if } i \le b,\\ &\mathbb{F}Qw_{b+1} + \mathbb{F}Qw_{b+2} + \dots \ \text{ is } N_{\mathcal{L}}\text{-orthogonal to } K_{12}+\mathrm{W}_2. \end{aligned}$$

We see from Lemma 2.21 that

$$|\det(\langle Qw_i, Qw_j\rangle_{1\le i,j})|_{\mathbb{F}} = 1.$$

Hence, we may replace one of the w_i by a suitable $(\mathfrak{o}_{\mathbb{F}})^\times$-multiple of it so that

$$\det(\langle Qw_i, Qw_j\rangle_{1\le i,j}) = 1. \tag{70}$$

Define the matrix elements $Q_{j,i}$ by

$$Qw_i = \sum_j Q_{j,i} w_j.$$

Then

$$Q_{j,i} = \delta_{j,i} \text{ if } i \le b.$$

In particular the matrix $((Q_{j,i})_{1\le j,i})$ looks as follows

$$((Q_{j,i})_{1\le j,i}) = \begin{pmatrix} \mathrm{I} & * \\ 0 & ((Q_{j,i})_{b<j,i}) \end{pmatrix},$$

where I is the identity matrix of size b. Hence,

$$\det(Q) = \det((Q_{j,i})_{1\le j,i}) = \det((Q_{j,i})_{b<j,i}) = \det((Q_{j,i})_{a<j,i}).$$

Therefore (70) implies

$$\det((Q_{j,i})_{a<j,i})^2 \ \det(\langle w_i, w_j\rangle_{1\le i,j}) = 1. \tag{71}$$

Let $u_1, u_2, \dots, u_b$ be a $N_{\mathcal{L}}$-orthogonal basis of U such that $u_1, u_2, \dots, u_a$ span V. (The existence of such a basis follows from [43, Theorem 1, page 29].) Define the matrix elements $(g_2^-)_{k,i}$ by

$$g_2^- w_i = \sum_{k=1}^{b} (g_2^-)_{k,i} u_k \qquad (1 \le i \le b).$$

Since $g_2^- K_{12} = \mathrm{V}$, we see that

$$(g_2^-)_{k,i} = 0 \text{ if } i \le a < k.$$

Hence

$$\det(((g_2^-)_{k,i})_{1\le k,i\le b}) = \det(((g_2^-)_{k,i})_{1\le k,i\le a}) \det(((g_2^-)_{k,i})_{a<k,i\le b}).$$

Define $h \in \mathrm{GL(W)}$ as in (24). Then (26) shows that

$$\det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}) \det(h) = \det(\langle \frac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\le b}).$$

Furthermore, by (26),

$$\begin{aligned}\det(h) = \det((g_1^{-1}-1)^{-1}(g_1^{-1}-1)h) &= \det(g_1^{-1}-1)^{-1} \det((g_1^{-1}-1)h)\\ &= \det(g_1^{-1}-1)^{-1} \det(\langle w_i, (g_1^{-1}-1)^{-1} h w_j\rangle_{1\le i,j}) \det(\langle w_i, w_j\rangle_{1\le i,j})^{-1}\\ &= \det(g_1^{-1}-1)^{-1}(-1)^{\dim \mathrm{U}} \det(\langle g_2^- w_i, w_j\rangle_{i,j\le b}) \det(\langle w_i, w_j\rangle_{1\le i,j})^{-1}\end{aligned}$$

Also,

$$\det(\langle \frac{1}{2}(c(g_1)+c(g_2))g_2^- w_i, g_2^- w_j\rangle_{a<i,j\le b}) = \det(\langle \frac{1}{2}(c(g_1)+c(g_2))u_k, u_l\rangle_{a<k,l\le b}) \det(((g_2^-)_{k,i})_{a<k,i\le b})^2.$$

By (60),

$$\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12}) = \det(\langle (g_1g_2)^- Qw_i, Qw_j\rangle_{a<i,j}) (\mathfrak{o}_{\mathbb{F}}^{\times})^2 = \det((Q_{i,j})_{a<i,j})^2 \det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j}) (\mathfrak{o}_{\mathbb{F}}^{\times})^2.$$

We see from Lemma 2.20 that there are elements $qw_i \in \mathrm{W}$, $i \le b$, such that

$$\langle u_j, qw_i\rangle = \delta_{j,i} \qquad (j, i \le b) \tag{72}$$

and the spaces $\mathbb{F}qW_1, \dots, \mathbb{F}qw_b, K_2$ are N-orthogonal. Define an element $q \in \mathrm{GL(W)}$ by

$$\begin{aligned} q(w_i) &= qw_i \text{ if } i \le b,\\ q(w_i) &= w_i \text{ if } b < i. \end{aligned}$$

Then

$$\det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U}) = \det(\langle g_2^- qw_i, qw_j\rangle_{i,j\le b}) (\mathfrak{o}_{\mathbb{F}}^{\times})^2. \tag{73}$$

Define the coefficients $q_{i,j}$ by

$$qw_i = \sum_j q_{j,i} w_j.$$

Then

$$q_{j,i} = \delta_{j,i} \text{ if } b < i$$

so that

$$\det(q) = \det((q_{j,i})_{1\le i,j}) = \det((q_{j,i})_{1\le i,j\le b}).$$

Also,

$$g_2^- q w_i = \sum_j q_{j,i} g_2^- w_j = \sum_{j \le b} q_{j,i} g_2^- w_j \qquad (i \le b).$$

Therefore,

$$\det(\langle g_2^- q w_i, q w_j \rangle_{i,j \le b}) = \det(q)^2 \det(\langle g_2^- w_i, w_j \rangle_{i,j \le b}). \tag{74}$$

Define the coefficients $q_{i,j}^{-1}$ of the inverse map q^{-1} by

$$w_i = q^{-1}(q w_i) = \sum_j q_{i,j}^{-1} q w_j.$$

Then, by (72),

$$q_{i,j}^{-1} = \begin{cases} \langle u_j, w_i \rangle \text{ if } j \le b, \\ \delta_{i,j} \text{ if } i > b. \end{cases}$$

Hence,

$$\det(q)^{-1} = \det(q^{-1}) = \det((q_{i,j}^{-1})_{i,j \le b}) = \det(\langle u_j, w_i \rangle_{i,j \le b}).$$

Thus

$$\begin{aligned} \det(\langle g_2^- w_i, w_j \rangle_{i,j \le b}) \det(g_2^- : \mathsf{W}/K_2 \to \mathsf{U}) &= (\det(\langle g_2^- w_i, w_j \rangle_{i,j \le b}))^2 \det(q)^2 (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ &= (\det(\langle \sum_{k=1}^{b} (g_2^-)_{k,i} u_k, w_j \rangle_{i,j \le b}))^2 \det(q)^2 (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ &= (\det((g_2^-)_{k,i})_{k,i \le b})^2 \det(\langle u_k, w_j \rangle_{k,j \le b}))^2 \det(q)^2 (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ &= (\det((g_2^-)_{k,i})_{k,i \le b}))^2 (\mathfrak{o}_{\mathbb{F}}^\times)^2 \\ &= (\det(g_2^- : K_{12} \to \mathsf{V}))^2 (\det(g_2^- : \mathsf{W}_2 \to \mathsf{U}/\mathsf{V}))^2 (\mathfrak{o}_{\mathbb{F}}^\times)^2, \end{aligned}$$

where the first equality follows from (73) combined with (74), and the last equality follows from (72). Now the formula (70) may be verified via a straightforward computation, where we ignore the factor $(\mathfrak{o}_{\mathbb{F}}^\times)^2$ for convenience:

$$\begin{aligned} &\frac{\det((g_1 g_2)^- : \mathsf{W}/K_{12} \to \mathsf{U}_{12})}{\det(g_1^- : \mathsf{W} \to \mathsf{W}) \det(g_2^- : \mathsf{W}/K_2 \to \mathsf{U})} \\ =\ &\frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle (g_1 g_2)^- w_i, w_j \rangle_{a<i,j})}{\det(g_1^- : \mathsf{W} \to \mathsf{W}) \det(g_2^- : \mathsf{W}/K_2 \to \mathsf{U})} \\ =\ &\frac{\det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) u_k, u_l \rangle_{a<k,l \le b}) \det(((g_2^-)_{k,i})_{a<k,i \le b})^2}{\det(h) \det g_1^- \det(g_2^- : \mathsf{W}/K_2 \to \mathsf{U})} \\ =\ &\frac{(-1)^{\dim \mathsf{U}} \det((Q_{i,j})_{a<i,j})^2 \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) u_k, u_l \rangle_{a<k,l \le b}) \det(((g_2^-)_{k,i})_{a<k,i \le b})^2}{\det(g_1^{-1} - 1)^{-1} \det(\langle g_2^- w_i, w_j \rangle_{i,j \le b}) \det(\langle w_i, w_j \rangle_{1 \le i,j})^{-1} \det g_1^- \det(g_2^- : \mathsf{W}/K_2 \to \mathsf{U})} \\ =\ &\frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) u_k, u_l \rangle_{a<k,l \le b}) (\det((g_2^-)_{k,i})_{a<k,i \le b}))^2}{\det(\langle g_2^- w_i, w_j \rangle_{i,j \le b}) \det(g_2^- : \mathsf{W}/K_2 \to \mathsf{U})} \\ =\ &\frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \ ,\ \rangle_{\mathsf{U}/\mathsf{V}}) (\det((g_2^-)_{k,i})_{a<k,i \le b})^2}{(\det(g_2^- : K_{12} \to \mathsf{V}))^2 (\det(g_2^- : \mathsf{W}_2 \to \mathsf{U}/\mathsf{V}))^2} \\ =\ &\frac{(-1)^{\dim \mathsf{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \ ,\ \rangle_{\mathsf{U}/\mathsf{V}})}{(\det(g_2^- : K_{12} \to \mathsf{V}))^2}. \end{aligned}$$

(Here the first equality follows from (72), the second equality from (72) and (72), the third from (72), the forth from (71) and the fifth from (75).) □

3. The Weil representation over a finite field of odd characteristic

Let $\mathbb{F}$ be a finite field of odd characteristic and let $\chi : \mathbb{F} \to \mathbb{C}^\times$ be a non-trivial character of the additive group $\mathbb{F}$. In this Section we provide an elementary construction of the corresponding the Weil representation, [5].

3.1. The Fourier transform

Let U be a finite dimensional vector space over $\mathbb{F}$. Define a measure μ_{U} on U by

$$\int_{\mathrm{U}} \phi(u)\, d\mu_{\mathrm{U}}(u) = |\mathrm{U}|^{-1/2} \sum_{u\in\mathrm{U}} \phi(u),$$

where $|\mathrm{U}|$ is the cardinality of U and $\phi : \mathrm{U} \to \mathbb{C}$ is a function. For E a subset of U let denote by $\mathbb{I}_E$ the indicator function of E, that is, the normalized characteristic function of E:

$$\mathbb{I}_E(u) := \begin{cases} |E|^{-1} & \text{if } u \in E; \\ 0 & \text{otherwise.} \end{cases}$$

Define the Fourier transform $\mathcal{F}$ by

$$\mathcal{F}\phi(u^*) = \int_{\mathrm{U}} \phi(u)\chi(-u^*(u))\, d\mu_{\mathrm{U}}(u) \qquad (u^* \in \mathrm{U}^*).$$

Then μ_{U^*} is the measure dual to μ_{U} in the sense that

$$\phi(u) = \int_{\mathrm{U}^*} \mathcal{F}\phi(u^*)\chi(u^*(u))\, d\mu_{\mathrm{U}^*}(u^*) \qquad (u \in \mathrm{U}).$$

We record by the way the following, easy to verify, formula

$$\mathcal{F}\mathbb{I}_{\mathrm{V}} = |\mathrm{V}||\mathrm{U}|^{-1/2}\mathbb{I}_{\mathrm{V}^\perp}, \tag{75}$$

where $V \subseteq \mathrm{U}$ is a vector subspace with the orthogonal complement $\mathrm{V}^\perp \subseteq \mathrm{U}^*$.

3.2. Gaussians on $\mathbb{F}^n$

For a symmetric matrix $A \in GL(\mathbb{F}^n)$ define the corresponding Gaussian γ_A by

$$\gamma_A(x) = \chi(\frac{1}{2}x^tAx) \qquad (x \in \mathbb{F}^n),$$

where we view the x as a column vector. Also, let

$$\gamma(A) = \mathcal{F}\gamma_A(0) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^tAx)\, d\mu_{\mathbb{F}^n}(x).$$

Lemma 3.1.
If we identify $\mathbb{F}^n$ with the dual $(\mathbb{F}^n)^*$ by

$$y(x) = x^ty \qquad (x, y \in \mathbb{F}^n),$$

then

$$\mathcal{F}\gamma_A = \gamma(A)\gamma_{-A^{-1}}.$$

Proof. Notice that
$$\frac{1}{2}x^tAx = \frac{1}{2}(x - A^{-1}y)^tA(x - A^{-1}y) - \frac{1}{2}y^tA^{-1}y + x^ty.$$
Hence,
$$\mathcal{F}\gamma_A(y) = \int_{\mathbb{F}^n} \gamma_A(x)\chi(-x^ty)\, d\mu_{\mathbb{F}^n}(x) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}(x - A^{-1}y)^tA(x - A^{-1}y))\, d\mu_{\mathbb{F}^n}(x)\chi(-\frac{1}{2}y^tA^{-1}y)$$
$$= \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^tAx)\, d\mu_{\mathbb{F}^n}(x)\chi(-\frac{1}{2}y^tA^{-1}y). \qquad \square$$

Lemma 3.2.

Suppose $n = 1$. Then

(a) $\gamma(a) = \gamma(ab^2) \qquad (a, b \in \mathbb{F}^\times)$,

(b) $\gamma(-a) = \overline{\gamma(a)} = \gamma(a)^{-1} \qquad (a \in \mathbb{F}^\times)$,

(c) *the function*
$$a \mapsto s(a) = \gamma(a)\gamma(-1) \qquad (a \in \mathbb{F}^\times)$$
coincides with the unique non-trivial character of the group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$.

Proof. Part (a) and the first equation in (b) are obvious. Let us extend the character s to $\mathbb{F}$ by letting $s(0) = 0$. Then, since $\frac{1}{2}a \neq 0$, we see from (75) that
$$\begin{aligned}
\gamma(a) &= \int_{\mathbb{F}} (1 + s)(y)\chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y)\\
&= \int_{\mathbb{F}} \chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y) + \int_{\mathbb{F}} s(y)\chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y)\\
&= \int_{\mathbb{F}^\times} s(y)\chi(\frac{1}{2}ay)\, d\mu_{\mathbb{F}}(y) = \int_{\mathbb{F}^\times} s(a^{-1}y)\chi(\frac{1}{2}y)\, d\mu_{\mathbb{F}}(y)\\
&= s(a^{-1}) \int_{\mathbb{F}^\times} s(y)\chi(\frac{1}{2}y)\, d\mu_{\mathbb{F}}(y) = s(a)\gamma(1).
\end{aligned}$$
Also,
$$\begin{aligned}
\gamma(1)\overline{\gamma(1)} &= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(y)s(z)\chi(\frac{1}{2}(y - z))\, d\mu_{\mathbb{F}}(y)\, d\mu_{\mathbb{F}}(z)\\
&= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(yz)\chi(\frac{1}{2}(y - z))\, d\mu_{\mathbb{F}}(y)\, d\mu_{\mathbb{F}}(z)\\
&= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(y)\chi(\frac{1}{2}(y - 1)z)\, d\mu_{\mathbb{F}}(y)\, d\mu_{\mathbb{F}}(z)\\
&= \int_{\mathbb{F}^\times}\int_{\mathbb{F}^\times} s(y)\chi((y - 1)z)\, d\mu_{\mathbb{F}}(z)\, d\mu_{\mathbb{F}}(y)\\
&= \int_{\mathbb{F}^\times} s(y) \left(\int_{\mathbb{F}} \chi((y - 1)z)\, d\mu_{\mathbb{F}}(z) - |\mathbb{F}|^{-1/2} \right) d\mu_{\mathbb{F}}(y)\\
&= \int_{\mathbb{F}^\times} s(y)|\mathbb{F}|^{1/2}\mathbb{I}_0(y - 1)\, d\mu_{\mathbb{F}}(y) - |\mathbb{F}|^{-1/2} \int_{\mathbb{F}^\times} s(y)\, d\mu_{\mathbb{F}}(y) = s(1),
\end{aligned}$$
because the restriction of $\mu_{\mathbb{F}}$ to $\mathbb{F}^\times$ is a Haar measure on $\mathbb{F}^\times$ and s is a non-trivial character of the abelian group $\mathbb{F}^\times$. Since $s(1) = 1$, we see that
$$\gamma(1)\overline{\gamma(1)} = 1.$$
In particular $|\gamma(1)| = 1$. Therefore the first computation in this proof shows that $|\gamma(a)| = 1$ for all $a \in \mathbb{F}^\times$. This implies the second equality in (b). Finally
$$s(a) = \gamma(a)\gamma(1)^{-1} = \gamma(a)\gamma(-1),$$
as claimed in (c). $\square$

Corollary 3.3.

For arbitrary $n \geq 1$ and a symmetric matrix $A \in \mathrm{GL}(\mathbb{F}^n)$,

$$\gamma(A) = \gamma(1)^n\, s(\det(A)).$$

Proof. There is $g \in \mathrm{GL}(\mathbb{F}^n)$ and a diagonal matrix $D = \mathrm{diag}(a_1, a_2, \dots, a_n) \in \mathrm{GL}(\mathbb{F}^n)$ such that $A = g^t D g$. Hence,

$$\begin{aligned}\gamma(A) &= \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^t A x)\, d\mu_{\mathbb{F}^n}(x) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^t D x)\, d\mu_{\mathbb{F}^n}(x)\\ &= \int_{\mathbb{F}^n} \prod_{j=1}^{n} \chi(\frac{1}{2}a_j x_j^2)\, d\mu_{\mathbb{F}^n}(x) = \prod_{j=1}^{n} \gamma(a_j) = \prod_{j=1}^{n} (\gamma(1) s(a_j))\\ &= \gamma(1)^n s(\prod_{j=1}^{n} a_j) = \gamma(1)^n s(\det(D)) = \gamma(1)^n s(\det(A)).\end{aligned}$$

□

3.3. Gaussians on a vector space

Let $\gamma(q) = \gamma(Q)$, where Q is defined as in Eq. (12).

Lemma 3.4.

If q is a non-degenerate symmetric bilinear form on U, then

$$\int_{\mathrm{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\, d\mu_{\mathrm{U}}(u) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)) \qquad (u^* \in \mathrm{U}^*).$$

Proof. Let $x_i = u_i^*(u)$ and let $y_j = u^*(u_j)$. Then

$$\int_{\mathrm{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\, d\mu_{\mathrm{U}}(u) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^t Q x)\chi(-x^t y)\, d\mu_{\mathbb{F}^n}(x) = \gamma(Q)\chi(-\frac{1}{2}y^t Q^{-1} y) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)),$$

where the second equality follows from Lemma 3.1 and the last one follows from Lemma 2.1. □

Corollary 3.5.

Let q be a symmetric form on U with the radical V. Denote by $\tilde{q}$ the induced non-degenerate form on U/V. Then, for any $u^* \in \mathrm{U}^*$,

$$\int_{\mathrm{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\, d\mu_{\mathrm{U}}(u) = |\mathrm{V}|^{1/2}\gamma(\tilde{q})\mathbb{I}_{\mathrm{V}^\perp}(u^*)\chi(-\frac{1}{2}\tilde{q}^*(u^*,u^*)),$$

where we identify $\mathrm{V}^\perp = (\mathrm{U}/\mathrm{V})^*$.

Proof. The left hand side is equal to

$$\begin{aligned}&\int_{\mathrm{U}/\mathrm{V}} \int_{\mathrm{V}} \chi(\frac{1}{2}q(u+v,u+v))\chi(-u^*(u+v))\, d\mu_{\mathrm{V}}(v)\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\ &= \int_{\mathrm{U}/\mathrm{V}} \chi(\frac{1}{2}\tilde{q}(u+\mathrm{V},u+\mathrm{V})) \left(\int_{\mathrm{V}} \chi(-u^*(u+v))\, d\mu_{\mathrm{V}}(v)\right) d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\ &= \int_{\mathrm{U}/\mathrm{V}} \chi(\frac{1}{2}\tilde{q}(u+\mathrm{V},u+\mathrm{V})) \left(\chi(-u^*(u))|\mathrm{V}|^{1/2}\mathbb{I}_{\mathrm{V}^\perp}(u^*)\right) d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\ &= |\mathrm{V}|^{1/2}\mathbb{I}_{\mathrm{V}^\perp}(u^*) \int_{\mathrm{U}/\mathrm{V}} \chi(\frac{1}{2}\tilde{q}(u+\mathrm{V},u+\mathrm{V}))\chi(-u^*(u))\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\ &= |\mathrm{V}|^{1/2}\mathbb{I}_{\mathrm{V}^\perp}(u^*)\gamma(\tilde{q})\chi(-\frac{1}{2}\tilde{q}^*(u^*,u^*)).\end{aligned}$$

□

3.4. Gaussians on a symplectic space

Lemma 3.6.
Suppose $x \in \mathrm{Hom}(\mathrm{U}, \mathrm{W}/\mathrm{U}^{\perp})$ *is such that*

$$\langle xu, v\rangle = \langle xv, u\rangle \qquad (u, v \in \mathrm{U}).$$

Set

$$q(u, v) = \frac{1}{2}\langle xu, v\rangle \qquad (u, v \in \mathrm{U}).$$

Let V *be the radical of* q *and let* $\tilde{q}$ *be the induced non-degenerate form on* U/V*. Then*

(a) $\mathrm{V} = \mathrm{Ker}(x)$*;*

(b) *for any* $w \in \mathrm{V}^{\perp}$ *there is* $u \in \mathrm{U}$ *such that* $xu + (w + \mathrm{U}^{\perp}) = 0$*;*

(c) *for any* $w \in \mathrm{W}$

$$\int_{\mathrm{U}} \chi(\frac{1}{4}\langle xu', u'\rangle)\chi(-\frac{1}{2}\langle u', w\rangle)\, d\mu_{\mathrm{U}}(u') = |\mathrm{V}|^{1/2}\gamma(\tilde{q})\mathbb{I}_{\mathrm{V}^{\perp}}(w)\chi(-\frac{1}{4}\langle u, w\rangle)$$

where $u \in \mathrm{U}$ *is such that* $xu + (w + \mathrm{U}^{\perp}) = 0$.

Proof. Part (a) is obvious. Part (b) means that $\mathrm{Ker}(x)^{\perp} = \mathrm{Im}(x)$, which is true. We know from Corollary 3.5 that the left hand side of (c) is equal to

$$|\mathrm{V}|^{1/2}\gamma(\tilde{q})\mathbb{I}_{\mathrm{V}^{\perp}}(w)\chi(-\frac{1}{2}\tilde{q}^*(\frac{1}{2}w, \frac{1}{2}w)).$$

Hence we may assume that $w \in \mathrm{V}^{\perp}$. Recall the map $\Phi\colon \mathrm{U}/\mathrm{V} \to (\mathrm{U}/\mathrm{V})^* = \mathrm{V}^{\perp}/\mathrm{U}^{\perp}$:

$$\Phi(u + \mathrm{V})(u' + \mathrm{V}) = \tilde{q}(u' + \mathrm{V}, u + \mathrm{V}) = \frac{1}{2}\langle xu', u\rangle.$$

Suppose $u \in \mathrm{U}$ is such that $\Phi(u + \mathrm{V}) = \frac{1}{2}w + \mathrm{U}^{\perp}$. Then, by the above,

$$\langle u', \frac{1}{2}w\rangle = \frac{1}{2}\langle xu', u\rangle = \langle u', -\frac{1}{2}xu\rangle \qquad (u' \in \mathrm{U}).$$

Therefore, $xu + \frac{1}{2}w \in \mathrm{U}^{\perp}$. In other words, $xu + (w + \mathrm{U}^{\perp}) = 0$ and we see that

$$\tilde{q}^*(\frac{1}{2}w + \mathrm{U}^{\perp}, \frac{1}{2}w + \mathrm{U}^{\perp}) = \langle u, \frac{1}{2}w\rangle,$$

so that

$$-\frac{1}{2}\tilde{q}^*(\frac{1}{2}w + \mathrm{U}^{\perp}, \frac{1}{2}w + \mathrm{U}^{\perp}) = -\frac{1}{4}\langle u, w\rangle.$$

The formula (c) follows. □

3.5. Twisted convolution of Gaussians

Recall the twisted convolution of two functions $\phi, \psi : \mathrm{W} \to \mathbb{C}$:

$$\phi \natural \psi(w) = \int_{\mathrm{W}} \phi(u)\psi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{76}$$

Let

$$\chi_{c(g)}(u) = \chi(\frac{1}{4}\langle c(g)u, u\rangle) \qquad (u \in g^{-}\mathrm{W}).$$

More generally, for x and U as in Lemma 3.6, let

$$\chi_x(u) = \chi(\frac{1}{4}\langle xu, u\rangle) \qquad (u \in \mathrm{U}).$$

By a Gaussian we understand the following function,

$$\mathbb{I}_{g^{-}\mathrm{W}}(w)\chi_{c(g)}(w) \qquad (w \in \mathrm{W}). \tag{77}$$

The goal of this subsection is to verify the following proposition.

Proposition 3.7.
For any $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$,

$$\left(\mathbb{I}_{\mathrm{U}_1}\chi_{c(g_1)}\right) \natural \left(\mathbb{I}_{\mathrm{U}_2}\chi_{c(g_2)}\right) = C(g_1, g_2)\, \mathbb{I}_{\mathrm{U}_{12}}\, \chi_{c(g_1 g_2)},$$

where

$$C(g_1, g_2) = \frac{|K_{12}|^{1/2}}{|K_1|^{1/2}|K_2)|^{1/2}}\, \gamma(\tilde{q}_{g_1, g_2}).$$

Proof. Notice first that, by the definition of the twisted convolution (76),

$$\left(\mathbb{I}_{\mathrm{U}_1}\chi_{c(g_1)}\right) \natural \left(\mathbb{I}_{\mathrm{U}_2}\chi_{c(g_2)}\right)(w) = 0$$

if $(\mathrm{U}_1 \cap (\mathrm{U}_2 + w) = \emptyset$. Therefore we may assume that there is $v \in \mathrm{U}_1$ such that $w - v \in \mathrm{U}_2$. Lemmas 2.2 and 2.5 plus a straightforward computation show that

$$\begin{aligned}\left(\mathbb{I}_{\mathrm{U}_1}\chi_{c(g_1)}\right) \natural \left(\mathbb{I}_{\mathrm{U}_2}\chi_{c(g_2)}\right)(w) = \frac{|\mathrm{U}|^{1/2}}{|\mathrm{W}|^{1/2}} \int_{\mathrm{U}} \chi_{c(g_1)+c(g_2)}(u')\chi(-\frac{1}{2}\langle u', c(g_1)v + c(g_2)(v-w) - w\rangle)\, d\mu_{\mathrm{U}}(u') \\ \cdot \chi_{c(g_1)}(v)\chi_{c(g_2)}(v-w)\chi(\frac{1}{2}\langle v, w\rangle).\end{aligned}$$

Since $\mathrm{V}^{\perp} = \mathrm{Ker}(c(g_1) + c(g_2))^{\perp}$ is the image of $c(g_1) + c(g_2)$, we see from Lemma 3.6 that the expression (78) is not zero if and only if there is $u \in \mathrm{U}$ such that

$$(c(g_1) + c(g_2))u + (c(g_1)v + c(g_2)(v-w) - w) \in \mathrm{U}^{\perp}. \tag{78}$$

Let

$$u = g_1^{-}v_1 = g_2^{-}v_2, \;\; v = g_1^{-}w_1 \text{ and } w - v = g_2^{-}w_2. \tag{79}$$

Then,

$$g_1^{+}v_1 + g_2^{+}v_2 + g_1^{+}w_1 - g_2^{+}w_2 - w \in \mathrm{U}^{\perp} = K_1 + K_2.$$

Hence, Lemma 2.3 shows that, without changing v or $w-v$, we may choose w_1 and w_2 in (79) so that

$$g_1^+v_1 + g_2^+v_2 + (g_1^+)w_1 - g_2^+w_2 - w = 0. \tag{80}$$

Multiplying (80) by g_1^- we get

$$g_1^-g_1^+v_1 + g_1^-g_2^+v_2 + g_1^-g_1^+w_1 - g_1^-g_2^+w_2 - g_1^-w = 0.$$

Since, $g_1^-(g_1^+)v_1 = (g_1^+)g_1^-v_1 = (g_1^+)g_2^-v_2$, we see that

$$g_1^+g_2^-v_2 + g_1^-g_2^+v_2 + g_1^+g_1^-w_1 - g_1^-g_2^+w_2 - g_1^-w = 0.$$

But, by (79), $g_1^-w_1 = w - g_2^-w_2$. Hence,

$$g_1^+g_2^-v_2 + g_1^-g_2^+v_2 + g_1^+w - g_1^+g_2^-w_2 - g_1^-g_2^+w_2 - g_1^-w = 0.$$

Thus

$$(g_1^+g_2^- + g_1^-g_2^+)(v_2 - w_2) + 2w = 0.$$

Therefore

$$w = (g_1g_2)^-(w_2 - v_2). \tag{81}$$

Hence, $w \in (g_1g_2)^-\mathrm{W}$.

Conversely, suppose $w = (g_1g_2)^-w_0$ for some $w_0 \in \mathrm{W}$. Then

$$w = g_1^-g_2w_0 + g_2^-w_0.$$

Let $w_1 = g_2w_0$ and let $w_2 = w_0$, so that

$$v = g_1^-w_1 \text{ and } w - v = g_2^-w_2,$$

as in (79). Then,

$$c(g_1)v + c(g_2)(v-w) - w = g_1^+w_1 - g_2^+w_2 = (g_1^+g_2 - g_2^+ - (g_1g_2)^-)w_0 = 0 \in \mathrm{U}^\perp.$$

Therefore (78) holds with $u = 0$. Thus we have the indicator function $\mathbb{I}_{(g_1g_2)^-\mathrm{W}}$ in the formula of Proposition 3.7.

Furthermore, with u as in Lemma 3.6 (b),

$$\begin{aligned}
&-\langle u, c(g_1)v + c(g_2)(v-w) - w\rangle + \langle c(g_1)v, v\rangle + \langle c(g_2)(v-w), v-w\rangle + 2\langle v, w\rangle \\
&= \langle g_2^-v_2, -g_1^+w_1 + g_2^+w_2 + w\rangle + \langle g_1^+w_1, g_1^-w_1\rangle + \langle g_2^+w_2, g_2^-w_2\rangle + 2\langle g_1^-w_1, w\rangle \\
&= \langle g_2^-v_2, -g_1^+w_1 + g_2^+w_2 + w\rangle + \langle g_1^+w_1, w - g_2^-w_2\rangle + \langle g_2^+w_2, g_2^-w_2\rangle + 2\langle w - g_2^-w_2, w\rangle \\
&= \langle g_2^-v_2, -g_1^+w_1 + g_2^+w_2 + w\rangle + \langle g_1^+w_1, w - g_2^-w_2\rangle + \langle g_2^+w_2, g_2^-w_2\rangle + 2\langle w, g_2^-w_2\rangle \\
&= \langle g_2^-v_2, g_2^+w_2 + w\rangle + \langle g_1^+w_1, g_2^-v_2\rangle + \langle g_1^+w_1, w - g_2^-w_2\rangle + \langle g_2^+w_2, g_2^-w_2\rangle + 2\langle w, g_2^-w_2\rangle.
\end{aligned} \tag{82}$$

Notice that

$$\begin{aligned}
&\langle g_1^+w_1, g_2^-v_2\rangle = \langle g_1^+w_1, g_1^-v_1\rangle = \langle (g_1^{-1} - 1)g_1^+w_1, v_1\rangle \\
&= -\langle g_1^{-1}g_1^-(g_1^+)w_1, v_1\rangle = -\langle g_1^{-1}g_1^+g_1^-w_1, v_1\rangle = -\langle g_1^{-1}g_1^+(w - g_2^-w_2), v_1\rangle \\
&= -\langle (1 + g_1^{-1})(w - g_2^-w_2), v_1\rangle = -\langle w - g_2^-w_2, g_1^+v_1\rangle = \langle g_1^+v_1, w - g_2^-w_2\rangle.
\end{aligned}$$

Hence, (82) is equal to

$$
\begin{aligned}
&\langle g_2^- v_2, g_2^+ w_2 + w\rangle + \langle g_1^+ v_1, w - g_2^- w_2\rangle + \langle g_1^+ w_1, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle \\
&\quad = \langle g_2^- v_2, g_2^+ w_2 + w\rangle + \langle g_1^+ w_1 + g_1^+ v_1, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle.
\end{aligned}
\tag{83}
$$

Now we compute $g_1^+ w_1 + g_1^+ v_1$ from (80) and substitute in (83) to see that (83) is equal to

$$
\begin{aligned}
&\langle g_2^- v_2, g_2^+ w_2 + w\rangle + \langle w + g_2^+ w_2 - g_2^+ v_2, w - g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle \\
&= \langle g_2^- v_2, g_2^+ w_2\rangle + \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle - \langle w, g_2^- w_2\rangle \\
&\quad -\langle g_2^+ w_2, g_2^- w_2\rangle + \langle g_2^+ v_2, g_2^- w_2\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle w, g_2^- w_2\rangle \\
&= \langle g_2^- v_2, g_2^+ w_2\rangle + \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle + \langle g_2^+ v_2, g_2^- w_2\rangle + \langle w, g_2^- w_2\rangle \\
&= \langle (g_2 - g_2^{-1}) v_2, w_2\rangle + \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle + \langle (g_2^{-1} - g_2) v_2, w_2\rangle + \langle w, g_2^- w_2\rangle \\
&= \langle g_2^- v_2, w\rangle + \langle g_2^+ w_2, w\rangle - \langle g_2^+ v_2, w\rangle - \langle g_2^- w_2, w\rangle = \langle 2(w_2 - v_2), w\rangle.
\end{aligned}
\tag{84}
$$

But we know from (81) that $w = (g_1 g_2)^-(w_2 - v_2)$. Hence, (84) is equal to

$$
\begin{aligned}
&\langle 2(w_2 - v_2), (g_1 g_2)^-(w_2 - v_2)\rangle = \langle (g_1 g_2)^-(w_2 - v_2) + 2(w_2 - v_2), (g_1 g_2)^-(w_2 - v_2)\rangle \\
&= \langle (g_1 g_2^+)(w_2 - v_2), (g_1 g_2)^-(w_2 - v_2)\rangle = \langle c(g_1 g_2) w, w\rangle.
\end{aligned}
$$

(Notice that the computation (82) - (85) may be simplified as follows. We already know from (81) that $w = (g_1 g_2)^- w_0$ for some $w_0 \in \mathsf{W}$. Hence, we may choose $w_1 = g_2 w_0$, $v = g_1^- g_2 w_0$ and $w_2 = w_0$ in (79). Then

$$
c(g_1)v + c(g_2)(v - w) - w = 0
$$

and therefore it will suffice to show that

$$
\langle c(g_1)v, v\rangle + \langle c(g_2)(w - v), w - v\rangle + 2\langle v, w - v\rangle = \langle c(g_1 g_2) w, w\rangle.
\tag{85}
$$

The left hand side of (85) is equal to

$$
\begin{aligned}
&\langle g_1^+ w_1, g_1^- w_1\rangle + \langle g_2^+ w_2, g_2^- w_2\rangle + 2\langle g_1^- w_1, g_2^- w_2\rangle \\
&= \langle (g_1 g_2 + g_2) w_0, (g_1 g_2 - g_2) w_0\rangle + \langle g_2^+ w_0, g_2^- w_0\rangle + 2\langle (g_1 g_2 - g_2) w_0, g_2^- w_0\rangle \\
&= 2\langle w_0, g_1 g_2 w_0\rangle = \langle (g_1 g_2^+) w_0, (g_1 g_2)^- w_0\rangle,
\end{aligned}
$$

which coincides with the right hand side.) Therefore Lemma 3.6 shows that for $w \in \mathsf{V}^\perp$,

$$
\begin{aligned}
\int_{\mathsf{U}} \chi_{c(g_1)+c(g_2)}(u')\chi(-\frac{1}{2}\langle u', c(g_1)v + c(g_2)(v - w) - w\rangle)\, d\mu_{\mathsf{U}}(u') \cdot \chi_{c(g_1)}(v)\chi_{c(g_2)}(v - w)\chi(\frac{1}{2}\langle v, w\rangle) \\
= |\mathsf{V}|^{1/2}\gamma(\tilde{q}_{g_1,g_2})\chi_{c(g_1 g_2)}(w).
\end{aligned}
$$

By combining this with (78) we see that

$$
\left(\mathbb{I}_{g_1^-\mathsf{W}}\chi_{c(g)}\right) \natural \left(\mathbb{I}_{g_2^-\mathsf{W}}\chi_{c(g)}\right) = \frac{|\mathsf{U}|^{1/2}|\mathsf{V}|^{1/2}}{|\mathsf{W}|^{1/2}}\gamma(\tilde{q}_{g_1,g_2})\mathbb{I}_{(g_1 g_2)^-\mathsf{W}}\chi_{c(g_1 g_2)}
$$

But Lemma 2.7 implies

$$
\frac{|\mathsf{U}|^{1/2}|\mathsf{V}|^{1/2}}{|\mathsf{W}|^{1/2}} = \frac{|K_{12}|^{1/2}}{|K_1|^{1/2}|K_2|^{1/2}}.
$$

□

3.6. Normalization of Gaussians

Let B be a non-degenerate bilinear form on a finite dimensional vector space over $\mathbb{F}$. Define the discriminant of B as

$$\mathrm{dis}(B) = s(\det(A)), \tag{86}$$

where A is the matrix obtained from a basis $u_1, u_2, \dots, u_n$ of the space by

$$A_{i,j} = B(u_i, u_j) \qquad (1 \le i, j \le n).$$

Clearly the discriminant does not depend on the choice of the basis.

For any $g \in \mathrm{Sp}(\mathrm{W})$ the formula

$$\langle g^- w, w' \rangle \qquad (w, w' \in \mathrm{W})$$

defines a bilinear form whose left and right radicals coincide with $\mathrm{Ker}(g^-)$. Hence we get a non-degenerate bilinear form B_g on the quotient $\mathrm{W}/\mathrm{Ker}(g^-)$. Then, for $g \neq 1$,

$$\mathrm{dis}(B_g) = s(\det(\langle g^- w_i, w_j \rangle_{1 \le i,j \le r})),$$

where $w_1 + \mathrm{Ker}(g^-)$, $w_2 + \mathrm{Ker}(g^-)$, $\dots$, $w_r + \mathrm{Ker}(g^-)$ is a basis of $\mathrm{W}/\mathrm{Ker}(g^-)$. For completeness set $\mathrm{dis}(B_1) = 1$.

For $g \in \mathrm{Sp}(\mathrm{W})$ define

$$\begin{aligned} \Theta(g) &= |\mathrm{Ker}(g^-)|^{1/2}\, \gamma(1)^{\dim g^-\mathrm{W}}\, \mathrm{dis}(B_g), \\ T(g) &= \Theta(g)\, \mathbb{I}_{g^-\mathrm{W}}\, \chi_{c(g)}. \end{aligned} \tag{87}$$

Theorem 3.8.
For any $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$,

$$T(g_1) \natural T(g_2) = T(g_1 g_2).$$

Proof. Proposition 3.7 implies that we'll be done as soon as we show that

$$C(g_1, g_2) = \frac{\Theta(g_1 g_2)}{\Theta(g_1)\Theta(g_2)} \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})). \tag{88}$$

Also, we see from Proposition 3.7 that the absolute values of both sides of (88) are equal. Hence, (88) is equivalent to

$$\gamma(\tilde{q}_{g_1,g_2}) = \frac{\theta(g_1 g_2)}{\theta(g_1)\theta(g_2)} \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})), \tag{89}$$

where

$$\theta(g) = \gamma(1)^{\dim g^-\mathrm{W}}\, \mathrm{dis}(B_g) \qquad (g \in \mathrm{Sp}(\mathrm{W})).$$

Since the twisted convolution is associative, the function $C(g_1, g_2)$ is a cocycle:

$$C(g_1, g_2) C(g_1 g_2, g_3) = C(g_1, g_2 g_3) C(g_2, g_3) \qquad (g_1, g_2, g_3 \in \mathrm{Sp}(\mathrm{W})).$$

Recall the non-degenerate symmetric form $\tilde{q}_{g_1,g_2}$ defined in Notation 2.6. Hence, by the formula for $C(g_1, g_2)$ in Proposition 3.7, the function $\gamma(\tilde{q}_{g_1,g_2})$ is also a cocycle:

$$\gamma(\tilde{q}_{g_1,g_2}) \gamma(\tilde{q}_{g_1 g_2, g_3}) = \gamma(\tilde{q}_{g_1, g_2 g_3}) \gamma(\tilde{q}_{g_2,g_3}) \qquad (g_1, g_2, g_3 \in \mathrm{Sp}(\mathrm{W})).$$

Let

$$C'(g_1, g_2) = \frac{\theta(g_1 g_2)}{\theta(g_1)\theta(g_2)} \qquad (g_1, g_2 \in \mathrm{Sp(W)}).$$

This is also a cocycle. Fix two elements $g_2, g_3 \in \mathrm{Sp(W)}$. We have seen in Lemma 2.8 that there is $g_1 \in \mathrm{Sp(W)}$ such that $K_1 = \mathrm{Ker}\, g_1^- = \{0\}$ and $K_{12} = \mathrm{Ker}(g_1 g_2)^- = \{0\}$. Assume that (89) holds when $K_1 = \{0\}$. Then

$$\gamma(\tilde{q}_{g_2,g_3}) = \frac{\gamma(\tilde{q}_{g_1,g_2})\gamma(\tilde{q}_{g_1g_2,g_3})}{\gamma(\tilde{q}_{g_1,g_2g_3})} = \frac{C'(g_1, g_2)C'(g_1g_2, g_3)}{C'(g_1, g_2g_3)} = C'(g_2, g_3).$$

Hence, in order to verify (89) we may assume that $K_1 = \{0\}$. Then Corollary 2.10 implies

$$\mathrm{dis}(\tilde{q}_{g_1,g_2}) = \mathrm{dis}(B_{g_1g_2}) s(-1)^{\dim \mathrm{U}} \,\mathrm{dis}(B_{g_1})\,\mathrm{dis}(B_{g_2}) = s(-1)^{\dim \mathrm{U}} \frac{\mathrm{dis}(B_{g_1g_2})}{\mathrm{dis}(B_{g_1})\,\mathrm{dis}(B_{g_2})}.$$

But, it follows from Lemma 2.7 that

$$\frac{\gamma(1)^{\dim \mathrm{U}_{12}}}{\gamma(1)^{\dim \mathrm{U}_1}\,\gamma(1)^{\dim \mathrm{U}_2}} = \gamma(1)^{(-\dim \mathrm{U} - \dim \mathrm{V})}. \tag{90}$$

On the other hand, we see from Corollary 3.3 that

$$\gamma(\tilde{q}_{g_1,g_2}) = \gamma(1)^{\dim \mathrm{U} - \dim \mathrm{V}}\,\mathrm{dis}(\tilde{q}_{g_1,g_2}) = s(-1)^{\dim \mathrm{U}}\gamma(1)^{-\dim \mathrm{U} - \dim \mathrm{V}}\,\mathrm{dis}(\tilde{q}_{g_1,g_2}),$$

because $\gamma(1)^2 = s(-1)$. Therefore (90) implies (89). □

3.7. The conjugation property

Let $\omega_{1,1}$ denote the permutation representation of $\mathrm{Sp(W)}$ on $\mathrm{L}^2(\mathrm{W})$:

$$\omega_{1,1}(g)\phi(w) = \phi(g^{-1}w) \qquad (g \in \mathrm{Sp(W)},\ \phi \in \mathrm{L}^2(\mathrm{W})).$$

Also, let

$$\phi^*(w) = \overline{\phi(-w)} \qquad (w \in \mathrm{W},\ \phi \in \mathrm{L}^2(\mathrm{W})).$$

Proposition 3.9.
For any $\phi \in \mathrm{L}^2(\mathrm{W})$ and $g \in \mathrm{Sp(W)}$ we have

(a) $T(1)\natural\phi = \phi\natural T(1) = \phi$,

(b) $T(g)\natural\phi\natural T(g^{-1}) = \omega_{1,1}(g)\phi$,

(c) $T(g)^* = T(g^{-1})$.

Proof. Since,

$$T(1) = |\mathrm{W}|^{1/2}\mathbb{I}_{\{0\}}$$

part (a) is easy to check. We see from (87) that the equality (c) is equivalent to

$$\gamma(1)^{-\dim g^-\mathrm{W}}\,\mathrm{dis}(B_g) = \gamma(1)^{\dim g^-\mathrm{W}}\,\mathrm{dis}(B_{g^{-1}}),$$

which is the same as

$$s(-1)^{\dim g^-\mathrm{W}} = \mathrm{dis}(B_g)\,\mathrm{dis}(B_{g^{-1}}).$$

But the last equality holds because

$$\mathrm{dis}(B_{g^{-1}}) = \mathrm{dis}(-B_g) = s(-1)^{\dim g^- \mathrm{W}}\, \mathrm{dis}(B_g).$$

Thus it remains to prove the equality (b), which is equivalent to

$$T(g)\natural \mathbb{I}_{w_0} = \mathbb{I}_{gw_0}\natural T(g). \tag{91}$$

The left hand side of (91) evaluated at w' is equal to

$$|\mathrm{W}|^{-1/2}\Theta(g)\mathbb{I}_{g^-\mathrm{W}}(w'-w_0)\chi(\frac{1}{4}(\langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle w'-w_0, w'\rangle))$$

and the right hand side is equal to

$$|\mathrm{W}|^{-1/2}\Theta(g)\mathbb{I}_{g^-\mathrm{W}}(w'-gw_0)\chi(\frac{1}{4}(\langle c(g)(w'-gw_0), w'-gw_0\rangle + 2\langle gw_0, w'\rangle)).$$

Since,

$$w'-gw_0 = (w'-w_0) - g^- w_0$$

both sides have the same support. Also,

$$\begin{aligned}
&\langle c(g)(w'-gw_0), w'-gw_0\rangle + 2\langle gw_0, w'\rangle - (\langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle w'-w_0, w'\rangle)\\
&= \langle c(g)(w'-gw_0), w'-gw_0\rangle - \langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle g^+ w_0, w'\rangle\\
&= \langle c(g)((w'-w_0) - g^- w_0, (w'-w_0) - g^- w_0\rangle - \langle c(g)(w'-w_0), w'-w_0\rangle + 2\langle g^+ w_0, w'\rangle\\
&= \langle c(g) g^- w_0, g^- w_0\rangle - 2\langle c(g) g^- w_0, w'-w_0\rangle + 2\langle g^+ w_0, w'\rangle\\
&= \langle g^+ w_0, g^- w_0\rangle - 2\langle g^+ w_0, w'-w_0\rangle + 2\langle g^+ w_0, w'\rangle\\
&= \langle g^+ w_0, g^- w_0\rangle + 2\langle g^+ w_0, w_0\rangle = \langle (g^{-1}-1)g^+ w_0, w_0\rangle + 2\langle g^+ w_0, w_0\rangle\\
&= \langle (g^{-1}-g)w_0, w_0\rangle + 2\langle gw_0, w_0\rangle = 0.
\end{aligned}$$

Therefore the two sides of (91) are equal. □

3.8. The Weyl transform and the Weil representation

Pick a complete polarization

$$\mathrm{W} = \mathrm{X} \oplus \mathrm{Y} \tag{92}$$

and recall that our normalization of measures is such that $d\mu_{\mathrm{W}}(x+y) = d\mu_{\mathrm{X}}(x)d\mu_{\mathrm{Y}}(y)$. Recall the Weyl transform

$$\begin{aligned}
&\mathcal{K}\colon \mathrm{L}^2(\mathrm{W}) \to \mathrm{L}^2(\mathrm{X}\times\mathrm{X}),\\
&\mathcal{K}(\phi)(x,x') = \int_{\mathrm{Y}} \phi(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathrm{Y}}(y).
\end{aligned} \tag{93}$$

Each element $K \in \mathrm{L}^2(\mathrm{X}\times\mathrm{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathrm{L}^2(\mathrm{X}), \mathrm{L}^2(\mathrm{X}))$ by

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x,x')v(x')\, d\mu_{\mathrm{X}}(x'). \tag{94}$$

A straightforward computation shows that $\mathrm{Op}\circ\mathcal{K}$ transforms the twisted convolution of functions into the composition of the corresponding operators. Also,

$$\mathrm{tr}\ \mathrm{Op}\circ\mathcal{K}(\phi)=\int_{\mathsf{X}}\mathcal{K}(\phi)(x,x)\,d\mu_{\mathsf{X}}(x)=\phi(0)\ \text{and}\ (\mathrm{Op}\circ\mathcal{K}(\phi))^*=\mathrm{Op}\circ\mathcal{K}(\phi^*). \tag{95}$$

Hence, the map

$$\mathrm{Op}\circ\mathcal{K}\colon \mathsf{L}^2(\mathsf{W})\to \mathsf{H.S.}(\mathsf{L}^2(\mathsf{X})) \tag{96}$$

is an isometry. (Here $\mathsf{H.S.}(\mathsf{L}^2(\mathsf{X}))$ stands for the space of the Hilbert-Schmidt operators on $\mathsf{L}^2(\mathsf{X})$.) Let $\mathsf{U}(\mathsf{L}^2(\mathsf{X}))$ denote the group of the unitary operators on the Hilbert space $\mathsf{L}^2(\mathsf{X})$.

By combining (92) - (96) with Theorem 3.8 and Proposition 3.9 we deduce the following theorem.

Theorem 3.10.

Let $\omega=\mathrm{Op}\circ\mathcal{K}\circ T$. Then

$$\omega\colon \mathrm{Sp}(\mathsf{W})\to\mathsf{U}(\mathsf{L}^2(\mathsf{X}))$$

is an injective group homomorphism. The function Θ coincides with the character of the resulting representation:

$$\Theta(g)=\mathrm{tr}\ \omega(g)\qquad (g\in\mathrm{Sp}(\mathsf{W})).$$

Moreover,

$$\omega(g)\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(g^{-1})=\mathrm{Op}\circ\mathcal{K}(\omega_{1,1}(g)\phi)\qquad (g\in\mathrm{Sp}(\mathsf{W}),\ \ \phi\in\mathsf{L}^2(\mathsf{W})).$$

We end this Section by recalling some well known formulas for the action of $\omega(g)$ for some special elements $g\in\mathrm{Sp}(\mathsf{W})$.

Proposition 3.11.

Let $\mathsf{M}\subseteq\mathrm{Sp}(\mathsf{W})$ be the subgroup of all the elements that preserve X and Y. Then the restriction to X defines a group isomorphism $\mathsf{M}\ni g\to g|_{\mathsf{X}}\in\mathrm{GL}(\mathsf{X})$ and

$$\omega(g)v(x)=s(\det(g|_{\mathsf{X}}))v(g^{-1}x)\qquad (g\in\mathsf{M},\ \ v\in\mathsf{L}^2(\mathsf{X}),\ \ x\in\mathsf{X}). \tag{97}$$

Proof. Fix an element $g\in\mathsf{M}$. Let $x_1, x_2, \dots, x_k$ be elements of X such that the vectors $x_1+\mathrm{Ker}(g^-)|_{\mathsf{X}}$, $x_2+\mathrm{Ker}(g^-)|_{\mathsf{X}}$, $\dots$, $x_k+\mathrm{Ker}(g^-)|_{\mathsf{X}}$ form a basis of the vector space $\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}$. Pick $y_1, y_2, \dots, y_k$ in Y so that $\langle x_i, y_j\rangle=1$. Then the vectors $y_1+\mathrm{Ker}(g^-)|_{\mathsf{Y}}$, $y_2+\mathrm{Ker}(g^-)|_{\mathsf{Y}}$, $\dots$, $y_k+\mathrm{Ker}(g^-)|_{\mathsf{Y}}$ form a basis of the vector space $\mathsf{Y}/\mathrm{Ker}(g^-)|_{\mathsf{Y}}$. Let $w_1:=x_1$, $\dots$, $w_{2k}:=y_k$. Then $w_1+\mathrm{Ker}(g^-)$, $\dots$, $w_{2k}+\mathrm{Ker}(g^-)$ for a basis of $\mathsf{W}/\mathrm{Ker}(g^-)$. Furthermore g defines an endomorphism $g^{-1}|_{\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}}$ of the space $\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}$ and

$$\begin{aligned}
\det(\langle g^-w_i, w_j\rangle_{1\le i,j\le 2k}) &= (-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(\langle g^-x_i, y_j\rangle_{1\le i,j\le k})\det(\langle g^-y_i, x_j\rangle_{1\le i,j\le k})\\
&= (-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(\langle g^-x_i, y_j\rangle_{1\le i,j\le k})\det(\langle y_i, (g^{-1}-1)x_j\rangle_{1\le i,j\le k})\\
&= (-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(\langle g^-x_i, y_j\rangle_{1\le i,j\le k})\det(\langle g^{-1}g^-x_j, y_i\rangle_{1\le i,j\le k})\\
&= (-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\left(\det(\langle g^-x_i, y_j\rangle_{1\le i,j\le k})\right)^2\det(g^{-1}|_{\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}}).
\end{aligned}$$

But $\det(g^{-1}|_{\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}}})=\det(g|_{\mathsf{X}}^{-1})$. Therefore

$$\begin{aligned}
\Theta(g) &= |\mathrm{Ker}(g^-)|^{\frac12}\cdot\gamma(1)^{\dim g^-\mathsf{W}}\cdot s\left((-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(g|_{\mathsf{X}}^{-1})\right)\\
&= \left(\frac{|\mathsf{W}|}{|g^-\mathsf{W}|}\right)^{\frac12}\cdot\gamma(1)^{2\dim g^-\mathsf{X}}\cdot s\left((-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(g|_{\mathsf{X}}^{-1})\right)\\
&= \frac{|\mathsf{Y}|}{|g^-\mathsf{Y}|}\cdot(s(-1))^{\dim g^-\mathsf{X}}\cdot s\left((-1)^{\dim(\mathsf{X}/\mathrm{Ker}(g^-)|_{\mathsf{X}})}\det(g|_{\mathsf{X}}^{-1})\right)\\
&= \frac{|\mathsf{Y}|}{|g^-\mathsf{Y}|}\cdot s\left(\det(g|_{\mathsf{X}}^{-1})\right).
\end{aligned}$$

Let $x, x' \in \mathsf{X}$ and let $y \in \mathsf{Y}$ be such that $x - x' + y \in g^-\mathsf{W}$. Then $x - x' \in g^-\mathsf{X}$ and $y \in g^-\mathsf{Y}$. Moreover,

$$\frac{1}{4}\langle c(g)(x-x'+y), x-x'+y\rangle = \frac{1}{2}\langle c(g)(x-x'), y\rangle.$$

Hence, (75) shows that

$$\begin{aligned}\int_{g^-\mathsf{Y}} \chi_{c(g)}(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y) &= \left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} \int_{g^-\mathsf{Y}} \chi(\frac{1}{2}\langle y, x+x'-c(g)(x-x')\rangle)\,d\mu_{g^-\mathsf{Y}}(y)\\ &= \left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} (|g^-\mathsf{Y}|)^{\frac{1}{2}}\, \mathbb{I}_{\mathrm{Ker}(g^-)|_{\mathsf{X}}}\left(\frac{1}{2}(x+x'-c(g)(x-x'))\right),\end{aligned}$$

because the annihilator of $g^-\mathsf{Y}$ in X coincides with $\mathrm{Ker}(g^-)|_{\mathsf{X}}$. But the condition $x + x' - c(g)(x-x') \in \mathrm{Ker}(g^-)|_{\mathsf{X}}$ means that $x' = g^{-1}x$. Indeed, if $x - x' = g^-\tilde{x}$, then

$$0 = g^-(x+x'-c(g)(x-x')) = g^-(x+x'-g^+\tilde{x}) = g^-(x+x') - g^+(x-x') = 2(gx'-x).$$

Therefore,

$$\begin{aligned}\mathcal{K}(T(g))(x,x') &= \Theta(g)\left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} (|g^-\mathsf{Y}|)^{\frac{1}{2}}\,\delta_0(g^{-1}x-x')\\ &= \frac{|\mathsf{Y}|}{|g^-\mathsf{Y}|}\, s\left(\det(g|_{\mathsf{X}}^{-1})\right)\left(\frac{|g^-\mathsf{Y}|}{|\mathsf{Y}|}\right)^{\frac{1}{2}} (|g^-\mathsf{Y}|)^{\frac{1}{2}}\,\delta_0(g^{-1}x-x')\\ &= |\mathsf{Y}|^{\frac{1}{2}} s\left(\det(g|_{\mathsf{X}}^{-1})\right)\delta_0(g^{-1}x-x')\end{aligned}$$

and the formula for $\omega(g)$ follows. □

Proposition 3.12.

Suppose $g \in \mathrm{Sp}(\mathsf{W})$ acts trivially on Y and on W/Y. Then $\det((-g)-1) \neq 0$ and

$$\omega(g)v(x) = \chi_{c(-g)}(2x)v(x) \qquad (v \in \mathrm{L}^2(\mathsf{X}),\ x \in \mathsf{X}).$$

Proof. Since $-g$ acts as minus the identity on Y and on W/Y, $\det((-g)-1) \neq 0$ and $z := c(-g) \in \mathfrak{sp}(\mathsf{W})$ is well defined. Furthermore

$$z\colon \mathsf{X} \to \mathsf{Y} \to 0.$$

Hence,

$$\begin{aligned}&\int_{\mathsf{Y}} \chi_z(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y) = \int_{\mathsf{Y}} \chi_z(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y)\\ &= \chi_z(x-x')|Y|^{\frac{1}{2}}\delta_0(\frac{1}{2}(x+x')) = \chi_z(2x)|Y|^{\frac{1}{2}}\delta_0(x+x')\end{aligned}$$

Moreover,

$$\Theta(-g) = \gamma(1)^{\dim(\mathsf{W})} s(\det(-2)) = s(-1)^{\dim(\mathsf{X})} s((-2)^{\dim(\mathsf{W})}) = s(-1)^{\dim(\mathsf{X})}.$$

Thus

$$\mathcal{K}(T(-g))(x,x') = s(-1)^{\dim(\mathsf{X})}\chi_z(2x)|Y|^{\frac{1}{2}}\delta_0(x+x').$$

Therefore,

$$\omega(-g)v(x) = s(-1)^{\dim(\mathsf{X})}\chi_z(2x)v(-x).$$

Since, by Proposition 3.11,

$$\omega(-1)v(x) = s(-1)^{\dim(\mathsf{X})}v(-x),$$

the formula for $\omega(g)$ follows. □

Proposition 3.13.

Suppose $g \in \mathrm{Sp}(\mathrm{W})$ maps X bijectively onto Y and Y onto X and $g^2 = -1$. Then

$$\omega(g)v(x) = \gamma(1)^{\dim(\mathrm{X})} \int_{\mathrm{X}} \chi(\langle gx, x'\rangle)v(x')\, d\mu_{\mathrm{X}}(x') \qquad (v \in \mathrm{L}^2(\mathrm{X}),\ x \in \mathrm{X}).$$

(Thus $\omega(g)$ is a Fourier transform on $\mathrm{L}^2(\mathrm{X})$.)

Proof. The formula

$$(gx, x') \qquad (x, x' \in \mathrm{X})$$

defines a non-degenerate symmetric bilinear form on X. Hence, there is a basis $x_1, x_2, \dots, x_n$ of X and scalars $a_j \in \mathbb{F}^\times$ such that

$$(gx_i, x_j) = a_j \delta_{i,j} \qquad (1 \le i, j \le n).$$

Set $y_j := -a_j^{-1} x_j$, $1 \le j \le n$. Then $y_1, y_2, \dots, y_n$ is a basis of Y and $\langle x_i, y_j \rangle = \delta_{i,j}$ for all $1 \le i, j \le n$. We have

$$g^- x_i = -a_j y_j - x_j \quad \text{and} \quad g^- y_i = a_j^{-1} x_j - x_j - y_j.$$

Set $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$. Then, with $\mathrm{I} = \mathrm{I}_n$,

$$\det\left(g^-\right) = \det\begin{pmatrix} -\mathrm{I} & A^{-1} \\ -A & -\mathrm{I} \end{pmatrix} = \det\begin{pmatrix} \mathrm{I} & 0 \\ -A & \mathrm{I} \end{pmatrix}\begin{pmatrix} -\mathrm{I} & A^{-1} \\ -A & -\mathrm{I} \end{pmatrix} = \det\begin{pmatrix} -\mathrm{I} & A^{-1} \\ 0 & -2\mathrm{I} \end{pmatrix} = 2^n \neq 0.$$

Thus $\mathrm{Ker}(g^-) \neq 0$ so that $g^-\mathrm{W} = \mathrm{W}$. Moreover, with $w_i = x_i$ and $w_{n+i} = y_i$ for $i = 1, 2, \dots, n$, we have

$$\det\left(\langle g^- w_i, w_j\rangle_{1 \le i,j \le 2n}\right) = \det\begin{pmatrix} -\mathrm{I} & A^{-1} \\ -A & -2\mathrm{I} \end{pmatrix}^t \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} = 2^n \det\begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} = 2^n.$$

Thus

$$\mathrm{dis}(B_g) = s(2^n).$$

Hence,

$$\Theta(g) = \gamma(1)^{2n}\, s(2^n) = s(-1)^n\, s(2^n) = s(-2)^n.$$

Since $g^+ = g^-(-g)$, we see that $c(g) = -g$. Further,

$$\langle (c(g)(x - x' + y), x - x' + y\rangle = \langle -g(x - x' + y), x - x' + y\rangle = \langle -g(x - x'), x - x'\rangle + \langle -gy, y\rangle.$$

Therefore,

$$\begin{aligned}\int_{\mathrm{Y}} \chi_{c(g)}(x - x' + y)\chi(\tfrac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathrm{Y}}(y) &= \chi_{-g}(x - x') \int_{\mathrm{Y}} \chi_{-g}(y)\chi(\tfrac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathrm{Y}}(y) \\ &= \chi_{-g}(x - x')\, \gamma(\tilde{q})\, \chi_g(x + x') = \gamma(\tilde{q})\, \chi(\langle gx, x'\rangle),\end{aligned}$$

where $\tilde{q}$ is the following symmetric bilinear form on Y

$$\tilde{q}(y, y') = \frac{1}{2}\langle -gy, y'\rangle \qquad (y, y' \in \mathrm{Y}).$$

Since,

$$\det(\tilde{q}(y_i, y_j)_{1 \le i,j \le n}) = \left(-\frac{1}{2}\right)^n,$$

we see that

$$\gamma(\tilde{q}) = \gamma(1)^n s\left(-\frac{1}{2}\right)^n.$$

Therefore,

$$\mathcal{K}(T(g))(x, x') = s(-2)^n\, \gamma(1)^n\, s\left(-\frac{1}{2}\right)^n \chi(\langle gx, x'\rangle) = \gamma(1)^n\, \chi(\langle gx, x'\rangle). \qquad \square$$

4. The Weil representation over $\mathbb{R}$

Let $\chi(r) = \exp(2\pi i r)$, $r \in \mathbb{R}$. This is a non-trivial character of the additive group $\mathbb{R}$. In this Section we provide a construction of the corresponding Weil representation, [35], [42].

4.1. The Fourier transform

Let U be a finite dimensional vector space over $\mathbb{R}$ and let B be a positive definite scalar product on U. We normalize the Lebesgue measure μ_{U} on U so that the volume of the unit cube (with respect to B) is 1. The formula

$$\Phi(u)(v) = B(u, v) \qquad (u, v \in \mathsf{U})$$

defines a linear isomorphism $\Phi : \mathsf{U} \to \mathsf{U}^*$. The form B^* dual to B is given by

$$B^*(u^*, v^*) = v^*(\Phi^{-1}(u^*)) \quad (u^*, v^* \in \mathsf{U}^*).$$

This is a symmetric positive definite bilinear form on U^*. Denote by μ_{U^*} the corresponding Lebesgue measure.

Let $\mathcal{S}(\mathsf{U})$ be the Schwartz space on U, [13, Definition 7.1.2]. For $\phi \in \mathcal{S}(\mathsf{U})$ let

$$\mathcal{F}\phi(u^*) = \int_{\mathsf{U}} \phi(u)\chi(-u^*(u))\, d\mu_{\mathsf{U}}(u) \qquad (u^* \in \mathsf{U}^*)$$

be the Fourier transform of ϕ. Then, as is well known, $\mathcal{F}\phi \in \mathcal{S}(\mathsf{U}^*)$ and

$$\phi(u) = \int_{\mathsf{U}^*} \mathcal{F}\phi(u^*)\chi(u^*(u))\, d\mu_{\mathsf{U}^*}(u^*) \qquad (u^* \in \mathsf{U}^*),$$

see [13, Theorem 7.1.5].

Let $\mathcal{S}^*(\mathsf{U})$ denote the space of the tempered distributions on U, [13, Definition 7.1.7]. When convenient we shall identify any bounded locally integrable function $f : \mathsf{U} \to \mathbb{C}$ with the tempered distribution $f\mu_{\mathsf{U}}$. In particular, $\mathcal{S}(\mathsf{U}) \subseteq \mathcal{S}^*(\mathsf{U})$. Then the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathsf{U}) \to \mathcal{S}(\mathsf{U}^*)$$

extends to

$$\mathcal{F} : \mathcal{S}^*(\mathsf{U}) \to \mathcal{S}^*(\mathsf{U}^*),$$

[13, Definition 7.1.9].

Let $\mathsf{V} \subseteq \mathsf{U}$ be a non-zero subspace. The form B restricts to V and determines the Lebesgue measure μ_{V}. We may view μ_{V} as a tempered distribution on U by

$$\mu_{\mathsf{V}}(\phi) = \int_{\mathsf{V}} \phi(v)\, d\mu_{\mathsf{V}}(v) \qquad (\phi \in \mathcal{S}(\mathsf{U})).$$

In the case when V is zero we define $\mu_{\mathsf{V}} = \mu_0$ to be the unit measure at 0. In other words $\mu_0 = \delta_0$ is the Dirac delta at 0,

$$\mu_0(\phi) = \delta_0(\phi) = \phi(0) \qquad (\phi \in C(\mathsf{U})).$$

Also, for future reference, let $\delta_u \in \mathcal{S}(\mathsf{U})$ be the Dirac delta at $u \in \mathsf{U}$,

$$\delta_u(\phi) = \phi(u) \qquad (\phi \in C(\mathsf{U})).$$

For an arbitrary subspace $\mathsf{V} \subseteq \mathsf{U}$, let $\mathsf{V}^{\perp} \subseteq \mathsf{U}^*$ be the annihilator of V. Then,

$$\mathcal{F}\mu_{\mathsf{V}} = \mu_{\mathsf{V}^{\perp}}, \tag{98}$$

see [13, Theorem 7.1.25].

The quotient space U/V may be identified with the B-orthogonal complement of V in U. Hence it inherits the natural scalar product.

Consider two real vector spaces U', U'' of the same dimension equipped with scalar products B', B'' respectively. Let $u_1', u_2', \dots, u_n'$ be a B'-orthonormal basis of U' and let $u_1'', u_2'', \dots, u_n''$ be a B''-orthonormal basis of U''. Suppose $L\colon \mathsf{U}' \to \mathsf{U}''$ is a linear bijection. Denote by M the matrix of L with respect to the two ordered basis:

$$Lu_j' = \sum_{i=1}^{n} M_{i,j} u_i'' \qquad (j = 1, 2, \dots, n).$$

Then $|\det(M)|$ does not depend on the choice of the orthonormal basis. Thus we may define $|\det(L)| = |\det(M)|$ (see Section 2.5).

Lemma 4.1.

With the above notation we have

$$\int_{\mathsf{U}'} \phi(L(u'))\, d\mu_{\mathsf{U}'}(u')\, |\det(L)| = \int_{\mathsf{U}''} \phi(u'')\, d\mu_{\mathsf{U}''}(u'') \qquad (\phi \in \mathcal{S}(\mathsf{U}'')). \tag{99}$$

Proof. Since $\int_0^1 \int_0^1 \cdots \int_0^1 dx_n \cdots dx_2\, dx_1 = 1$ and by definition of $\mu_{\mathsf{U}'}$, $\mu_{\mathsf{U}'}([0,1]u_1' + [0,1]u_2' + \cdots + [0,1]u_n') = 1$,

$$\int_{\mathsf{U}'} \phi(u')\, d\mu_{\mathsf{U}'}(u') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(x_1'u_1' + x_2'u_2' + \cdots + x_n'u_n')\, dx_n' \cdots dx_2'\, dx_1',$$

and similarly for U''. Therefore the right hand side of (99) equals

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\sum_{i=1}^{n} x_i'' u_i'')\, dx_n'' \cdots dx_2''\, dx_1'' &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\sum_{i=1}^{n} \sum_{j=1}^{n} M_{i,j} x_j' u_i'')\, dx_n' \cdots dx_2'\, dx_1' |\det(M)| \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\sum_{j=1}^{n} x_j' L(u_j'))\, dx_n' \cdots dx_2'\, dx_1' |\det(M)| \end{aligned}$$

which coincides with the left hand side. □

Lemma 4.2.

Suppose X is a finite dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form and $L\colon \mathsf{X} \to \mathsf{U}$ is a surjective linear map. Let

$$\tilde{L} : \mathsf{X}/L^{-1}(\mathsf{V}) \to \mathsf{U}/\mathsf{V}$$

be the induced bijection. Then

$$L^*(\mu_{\mathsf{V}}) = |\det(\tilde{L})|^{-1} \mu_{L^{-1}(\mathsf{V})},$$

where the pullback $L^*(\mu_{\mathsf{V}})$ is defined as in [13, Theorem 6.1.2].

Proof. Let $\mathsf{X}' \subseteq \mathsf{X}$ be the orthogonal complement of $\mathrm{Ker}(L)$. Denote by L' the restriction of L to X' and by L'' the restriction of L to $\mathsf{X}' \cap L^{-1}(\mathsf{V})$. Then

$$L' \colon \mathsf{X}' \to \mathsf{U} \text{ and } L'' \colon \mathsf{X}' \cap L^{-1}(\mathsf{V}) \to \mathsf{V}$$

are bijections.

According to [13, Theorem 6.1.2], for a test function ϕ we have

$$L^*(\mu_{\mathsf{V}})(\phi) = \int_{\mathrm{Ker}(L)} \int_{\mathsf{V}} \phi(x + L'^{-1}(v))\, d\mu_{\mathsf{V}}(v)\, d\mu_{\mathrm{Ker}(L)}(x)\, |\det(L')|^{-1}. \tag{100}$$

Lemma 4.1 shows that the right hand side of (100) is equal to

$$\int_{\mathrm{Ker}(L)} \int_{L''^{-1}(\mathsf{V})} \phi(x + y)\, d\mu_{L''^{-1}(\mathsf{V})}(y)\, d\mu_{\mathrm{Ker}(L)}(x)\, |\det(L'')|\, |\det(L')|^{-1} = \int_{L^{-1}(\mathsf{V})} \phi(z)\, d\mu_{L^{-1}(\mathsf{V})}(z)\, |\det(L'')|\, |\det(L')|^{-1}.$$

Since $|\det(L'')|^{-1}\, |\det(L')| = |\det(\tilde{L})|$, we are done. □

4.2. Gaussians on $\mathbb{R}^n$

Let B be the usual dot product on $\mathbb{R}^n$,

$$B(x, y) = x^t y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \qquad (x, y \in \mathbb{R}^n).$$

Then $d\mu_{\mathbb{R}^n}(x) = dx$ is the usual Lebesgue measure on $\mathbb{R}^n$, see [34, Theorem 10.33].

For a symmetric matrix $A \in \mathrm{GL}(\mathbb{R}^n)$ define the corresponding Gaussian γ_A by

$$\gamma_A(x) = \chi(\frac{1}{2} x^t A x) \qquad (x \in \mathbb{R}^n).$$

Also, let

$$\gamma(A) = \mathcal{F}\gamma_A(0) = \int_{\mathbb{R}^n} \chi(\frac{1}{2} x^t A x)\, dx.$$

As customary, we shall identify $\mathbb{R}^n$ with the dual $(\mathbb{R}^n)^*$ via the dot product. In these terms we have the following theorem, [13, Theorem 7.6.1].

Theorem 4.3.
For any symmetric matrix $A \in \mathrm{GL}(\mathbb{R}^n)$,

$$\mathcal{F}\gamma_A = \frac{e^{\frac{\pi i}{4}\mathrm{sgn}(A)}}{\sqrt{|\det A|}} \gamma_{-A^{-1}},$$

where $\mathrm{sgn}(A)$ is the number of the positive eigenvalues of A (counted with the multiplicities) minus the number of the negative eigenvalues of A (counted with the multiplicities). In particular,

$$\gamma(A) = \frac{e^{\frac{\pi i}{4}\mathrm{sgn}(A)}}{\sqrt{|\det A|}}. \tag{101}$$

Remark 4.4.
Eqn.(101) follows also from [42, Chap. I Théorème 2 and Chap. II § 26].

Remark 4.5.

Eqn.(101) implies that

$$\gamma(A) = \pm\gamma(1)^{n-1}\,\gamma(\det A), \tag{102}$$

which can be viewed as the analog on $\mathbb{R}$ of Corollary 3.3.

Indeed, by applying Eqn.(101) to both 1 and $\det A$, we get

$$\gamma(1) = e^{\frac{\pi i}{4}} \quad \text{and} \quad \gamma(\det A) = \frac{e^{\frac{\pi i}{4}\operatorname{sign}(\det A)}}{\sqrt{|\det A|}},$$

where $\operatorname{sign}(\det A)$ is the sign of the determinant of A. Hence we are reduced to compare the congruence modulo 4 of $\operatorname{sgn}(A)$ with those of $n-1+\operatorname{sign}(\det A)$. Let p (resp. q) denote the number of the positive (resp. negative) eigenvalues of A. We have $\operatorname{sgn}(A) = p - q$ and $n = p + q$. It follows that

$$n - 1 + \operatorname{sign}(\det A) - \operatorname{sgn}(A) = 2q - 1 + \operatorname{sign}(\det A) \equiv 0 \pmod 4,$$

since $\operatorname{sign}(\det A) = (-1)^q$.

Remark 4.6.

It is easy to see from (102) that

$$\left(\frac{\gamma(a)}{\gamma(1)}\right)^2 = \frac{1}{a} \qquad (a \in \mathbb{R}^\times). \tag{103}$$

4.3. Gaussians on a vector space

Let U be a finite dimensional vector space over $\mathbb{R}$ with a symmetric positive definite bilinear form B. Suppose q is a non-degenerate symmetric bilinear form on U. Let $\gamma(q) = \gamma(Q)$, where Q is the matrix obtained from any B-orthonormal basis u_1, u_2, u_n of U by

$$Q_{i,j} = q(u_i, u_j) \qquad (1 \le i, j \le n).$$

Also, we define $\gamma(0) = 1$.

Lemma 4.7.

If q is a non-degenerate symmetric bilinear form on U, then

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)) \qquad (u^* \in \mathsf{U}^*).$$

Proof. Fix a B-orthonormal basis u_1, u_2, $\ldots$, u_n of U and let u_1^*, u_2^*, $\ldots$, u_n^* be the dual basis of U^*. This is a B^*-orthonormal basis. As we have seen in the proof of Lemma 3.4, if Q is the matrix corresponding to q, as above, then Q^{-1} corresponds to q^*.

Let $x_i = u_i^*(u)$ and let $y_j = u^*(u_j)$. Then

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u) = \int_{\mathbb{R}^n} \chi(\frac{1}{2}x^tQx)\chi(-x^ty)\,dx = \gamma(Q)\chi(-\frac{1}{2}y^tQ^{-1}y) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)),$$

where the second equality follows from Theorem 4.3. $\square$

4.4. Gaussians on a symplectic space

Let W be a finite dimensional vector space over $\mathbb{R}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Fix a positive definite compatible complex structure J on W. In other words, $J \in \mathfrak{sp}(\mathrm{W})$, $J^2 = -\mathrm{I}$ and the form

$$B(w, w') = \langle J(w), w'\rangle \qquad (w, w' \in \mathrm{W}) \tag{104}$$

is positive definite. As explained in Section 4.1, this leads to a normalization of the Lebesgue measures on any subspace of $\mathrm{U} \subseteq \mathrm{W}$ and on any quotient U/V, where V is a subspace of U.

We shall identify W with the dual W^* by

$$w^*(w) = \langle w, w^*\rangle \qquad (w, w^* \in \mathrm{W}). \tag{105}$$

Then

$$\mathrm{U}^* = \mathrm{W}/\mathrm{U}^\perp \text{ and } (\mathrm{U}/\mathrm{V})^* = \mathrm{V}^\perp/\mathrm{U}^\perp, \tag{106}$$

where the orthogonal complements are taken in W, with respect to the symplectic form $\langle\ ,\ \rangle$.

Lemma 4.8.
Suppose $x \in \mathrm{Hom}(\mathrm{U}, \mathrm{W}/\mathrm{U}^\perp)$ is such that

$$\langle xu, v\rangle = \langle xv, u\rangle \qquad (u, v \in \mathrm{U}).$$

Set

$$q(u, v) = \frac{1}{2}\langle xu, v\rangle \qquad (u, v \in \mathrm{U}).$$

Let V be the radical of q and let $\tilde{q}$ be the induced non-degenerate form on U/V. Then

(a) $\mathrm{V} = \mathrm{Ker}(x)$;

(b) *The element x determines a bijection*

$$\underline{x} : \mathrm{U}/\mathrm{V} \to \mathrm{V}^\perp/\mathrm{U}^\perp,$$

with the inverse

$$\underline{x}^{-1} : \mathrm{V}^\perp/\mathrm{U}^\perp \to \mathrm{U}/\mathrm{V};$$

(c) *Let $x^{-1} : \mathrm{V}^\perp \to \mathrm{U}/\mathrm{V}$ be the composition of $\underline{x}^{-1}$ with the quotient map $\mathrm{V}^\perp \to \mathrm{V}^\perp/\mathrm{U}^\perp$. Define*

$$\chi_x(u) = \chi(\frac{1}{4}\langle xu, u\rangle) \qquad (u \in \mathrm{U}), \tag{107}$$

$$\chi_{x^{-1}}(w) = \chi(\frac{1}{4}\langle x^{-1}w, w\rangle) \qquad (w \in \mathrm{V}^\perp). \tag{108}$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\int_{\mathrm{U}}\int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathrm{U}}(u) = 2^{\dim(\mathrm{V})}\gamma(\tilde{q})\int_{\mathrm{V}^\perp} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathrm{V}^\perp}(w)$$
$$= 2^{\dim(\mathrm{V})}\gamma(\tilde{q})\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \chi_{x^{-1}}(w + \mathrm{U}^\perp)\int_{\mathrm{U}^\perp} \phi(w + v)\, d\mu_{\mathrm{U}^\perp}(v)\, d\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp}(w + \mathrm{U}^\perp). \tag{109}$$

Also, for any $\phi \in \mathcal{S}(\mathrm{W}/\mathrm{U}^\perp)$,

$$\int_{\mathrm{U}}\int_{\mathrm{W}/\mathrm{U}^\perp} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w + \mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w + \mathrm{U}^\perp)\, d\mu_{\mathrm{U}}(u) \tag{110}$$
$$= 2^{\dim(\mathrm{V})}\gamma(\tilde{q})\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \chi_{\underline{x}^{-1}}(w)\phi(w + \mathrm{U}^\perp)\, d\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp}(w + \mathrm{U}^\perp).$$

Proof. Part (a) is obvious. Part (b) means that $\mathrm{Ker}(x)^{\perp} = \mathrm{Im}(x)$, which is true. For $\phi \in \mathcal{S}(\mathrm{W})$ we have,

$$
\begin{aligned}
\int_{\mathrm{U}} \int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathrm{U}}(u) &= \int_{\mathrm{W}} \mathcal{F}(\gamma_q \mu_{\mathrm{U}})(\frac{1}{2}w)\phi(w)\, d\mu_{\mathrm{W}}(w)\\
&= \int_{\mathrm{W}} \mathcal{F}(\gamma_q \mu_{\mathrm{U}})(w)\phi(2w)\, d\mu_{\mathrm{W}}(w)\, 2^{\dim \mathrm{W}}\\
&= \gamma(\tilde{q}) \int_{\mathrm{V}^{\perp}} \gamma_{-\tilde{q}^*}(w)\phi(2w)\, d\mu_{\mathrm{V}^{\perp}}(w)\, 2^{\dim \mathrm{W}}\\
&= \gamma(\tilde{q}) \int_{\mathrm{V}^{\perp}} \gamma_{-\tilde{q}^*}(\frac{1}{2}w)\phi(w)\, d\mu_{\mathrm{V}^{\perp}}(w)\, 2^{\dim \mathrm{W} - \dim \mathrm{V}^{\perp}}\\
&= \gamma(\tilde{q}) \int_{\mathrm{V}^{\perp}} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathrm{V}^{\perp}}(w)\, 2^{\dim \mathrm{V}}.
\end{aligned}
$$

This verifies (109). For $\phi \in \mathcal{S}(\mathrm{W}/\mathrm{U}^{\perp})$ we have,

$$
\begin{aligned}
&\int_{\mathrm{U}} \int_{\mathrm{W}/\mathrm{U}^{\perp}} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{W}/\mathrm{U}^{\perp}}(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{U}}(u)\\
&= \int_{\mathrm{U}/\mathrm{V}} \int_{\mathrm{V}} \int_{\mathrm{W}/\mathrm{U}^{\perp}} \chi_{\underline{x}}(u+\mathrm{V})\chi(\frac{1}{2}\langle u+v, w\rangle)\phi(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{W}/\mathrm{U}^{\perp}}(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{V}}(v)\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\
&= \int_{\mathrm{U}/\mathrm{V}} \int_{\mathrm{V}} \int_{\mathrm{W}/\mathrm{U}^{\perp}} \gamma_{\tilde{q}}(u+\mathrm{V})\chi(\langle u+v, w\rangle)\phi(2w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{W}/\mathrm{U}^{\perp}}(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{V}}(v)\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\
&\quad 2^{\dim \mathrm{W}/\mathrm{U}^{\perp}}\\
&= \int_{\mathrm{U}/\mathrm{V}} \int_{\mathrm{V}^{\perp}/\mathrm{U}^{\perp}} \gamma_{\tilde{q}}(u+\mathrm{V})\chi(\langle u, w\rangle)\phi(2w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{W}/\mathrm{U}^{\perp}}(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\, 2^{\dim \mathrm{W}/\mathrm{U}^{\perp}}\\
&= \gamma(\tilde{q}) \int_{\mathrm{V}^{\perp}/\mathrm{U}^{\perp}} \gamma_{-\tilde{q}^*}(w+\mathrm{U}^{\perp})\phi(2w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{W}/\mathrm{U}^{\perp}}(w+\mathrm{U}^{\perp})\, 2^{\dim \mathrm{W}/\mathrm{U}^{\perp}}\\
&= \gamma(\tilde{q}) \int_{\mathrm{V}^{\perp}/\mathrm{U}^{\perp}} \gamma_{-\tilde{q}^*}(\frac{1}{2}w+\mathrm{U}^{\perp})\phi(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{W}/\mathrm{U}^{\perp}}(w+\mathrm{U}^{\perp})\, 2^{\dim \mathrm{W}/\mathrm{U}^{\perp} - \dim \mathrm{V}^{\perp}/\mathrm{U}^{\perp}}\\
&= \gamma(\tilde{q}) \int_{\mathrm{V}^{\perp}/\mathrm{U}^{\perp}} \chi_{\underline{x}^{-1}}(w+\mathrm{U}^{\perp})\phi(w+\mathrm{U}^{\perp})\, d\mu_{\mathrm{W}/\mathrm{U}^{\perp}}(w+\mathrm{U}^{\perp})\, 2^{\dim \mathrm{V}}.
\end{aligned}
$$

This verifies (110). □

By a Gaussian on the symplectic space W we shall understand any non-zero constant multiple of the tempered distribution

$$\chi_x \mu_{\mathrm{U}} \in \mathcal{S}^*(\mathrm{W}) \tag{111}$$

where the function χ_x is defined in Lemma 4.8. In these terms Lemma 4.8 says that the Fourier transform of a Gaussian is another Gaussian.

4.5. Twisted convolution of Gaussians

Recall the twisted convolution of two Schwartz functions $\psi, \phi \in \mathcal{S}(\mathrm{W})$:

$$\psi \natural \phi(w) = \int_{\mathrm{W}} \psi(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{112}$$

It is easy to see that the above integral converges and that $\psi \natural \phi \in \mathcal{S}(\mathrm{W})$. Also, the twisted convolutions

$$\delta_{w_0} \natural \phi(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w_0, w\rangle) \text{ and } \phi \natural \delta_{w_0}(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w, w_0\rangle) \tag{113}$$

are well defined for any continuous function ϕ.

Let

$$t(g) = \chi_{c(g)}\mu_{g^-\mathrm{W}}. \tag{114}$$

For any $\phi \in \mathcal{S}(\mathrm{W})$, the twisted convolution $t(g)\natural\phi$ is a continuous function given by the following absolutely convergent integral

$$t(g)\natural\phi(w) = \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{115}$$

Lemma 4.9.
For any $g \in \mathrm{Sp}(\mathrm{W})$,

$$t(g)\natural(\delta_{w_0}\natural\phi) = \delta_{gw_0}\natural(t(g)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ w_0 \in \mathrm{W}).$$

Proof. The left hand side evaluated at $w \in \mathrm{W}$ is equal to

$$\begin{aligned}
&\int_{g^-\mathrm{W}} \chi_{c(g)}(u)(\delta_{w_0}\natural\phi)(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
&= \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u-w_0)\chi(\frac{1}{2}\langle w_0, w-u\rangle)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
&= \int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)u, u\rangle + 2\langle w_0, w-u\rangle + 2\langle u, w\rangle))\, d\mu_{g^-\mathrm{W}}(u)
\end{aligned}$$

and the right hand side is equal to

$$\begin{aligned}
(t(g)\natural\phi)(w-gw_0)\chi(\frac{1}{2}\langle gw_0, w\rangle) &= \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-gw_0-u)\chi(\frac{1}{2}\langle u, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
&= \int_{g^-\mathrm{W}} \chi_{c(g)}(u-g^-w_0)\phi(w-gw_0-(u-g^-w_0))\\
&\quad \chi(\frac{1}{2}\langle u-g^-w_0, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
&= \int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)(u-g^-w_0), u-g^-w_0\rangle\\
&\quad +2\langle u-g^-w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle))\, d\mu_{g^-\mathrm{W}}(u).
\end{aligned}$$

A straightforward computation shows that

$$\langle c(g)(u-g^-w_0), u-g^-w_0\rangle + 2\langle u-g^-w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle - (\langle c(g)u, u\rangle + 2\langle w_0, w-u\rangle + 2\langle u, w\rangle) = 0.$$

Hence, the two sides are equal. □

Let

$$\partial_{w_0} = \lim_{t\to 0}\frac{\delta_{tw_0}-\delta_0}{t}.$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$ and $w_0 \in \mathrm{W}$,

$$\begin{aligned}
\partial_{w_0}\natural\phi(w) &= \pi i\langle w_0, w\rangle\phi(w) + \partial_{w_0} * \phi(w)\\
\phi\natural\partial_{w_0}(w) &= -\pi i\langle w_0, w\rangle\phi(w) + \partial_{w_0} * \phi(w)
\end{aligned} \tag{116}$$

where $\partial_{w_0} * \phi(w) = \frac{d}{dt}\phi(w-tw_0)|_{t=0}$ is the directional derivative in the direction of $-w_0$.

Corollary 4.10.

For any $g \in \mathrm{Sp}(\mathrm{W})$,

$$t(g)\natural(\partial_{w_0}\natural\phi) = \partial_{gw_0}\natural(t(g)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ w_0 \in \mathrm{W}).$$

Proposition 4.11.

For any $g \in \mathrm{Sp}(\mathrm{W})$ and $\phi \in \mathcal{S}(\mathrm{W})$, $t(g)\natural\phi \in \mathcal{S}(\mathrm{W})$. Moreover the map

$$\mathcal{S}(\mathrm{W}) \ni \phi \to t(g)\natural\phi \in \mathcal{S}(\mathrm{W})$$

is continuous.

Proof. We see from Corollary 4.10 with the formulas (116) that for any $w_0, w \in \mathrm{W}$,

$$2\pi i\langle w_0, w\rangle(t(g)\natural\phi)(w) = \partial_{w_0}\natural(t(g)\natural\phi)(w) - (t(g)\natural\phi)\natural\partial_{w_0}(w) = t(g)\natural(\partial_{g^{-1}w_0}\natural\phi - \phi\natural\partial_{w_0})(w)$$

and similarly

$$2\partial_{w_0} * (t(g)\natural\phi)(w) = t(g)\natural(\partial_{g^{-1}w_0}\natural\phi + \phi\natural\partial_{w_0})(w).$$

Hence, for any polynomial coefficient differential operator P on W there is a polynomial coefficient differential operator Q on W such that

$$P(t(g)\natural\phi) = t(g)\natural Q(\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W})). \tag{117}$$

Notice also that by the definition (115)

$$\| t(g)\natural\phi \|_\infty \leq \sup_{w\in\mathrm{W}} \int_{g^-\mathrm{W}} |\phi(w-u)|\, d\mu_{g^-\mathrm{W}}(u) < \infty \tag{118}$$

and that the right hand side is a continuous seminorm on $\mathcal{S}(\mathrm{W})$. The proposition clearly follows from these two facts. $\square$

In particular Proposition 4.11 shows that for any two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ there is a tempered distribution $t(g_1)\natural t(g_2) \in \mathcal{S}^*(\mathrm{W})$ such that

$$(t(g_1)\natural t(g_2))\natural\phi = t(g_1)\natural(t(g_2)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W})). \tag{119}$$

In order to verify Proposition 4.13 below, we shall need an explicit formula for the function $t(g)\natural\phi$ of Proposition 4.11. This is provided by the following Lemma.

Lemma 4.12.

Fix an element $g \in \mathrm{Sp}(\mathrm{W})$. Let $\mathrm{U} = g^-\mathrm{W}$. The map

$$\mathrm{U} \ni u \to \langle\ , (1 - c(g))u\rangle \in \mathrm{U}^* = \mathrm{W}/\mathrm{U}^\perp = \mathrm{W}/\mathrm{Ker}(g^-) \tag{120}$$

is bijective.

Fix a complement Z of U in W so that

$$\mathrm{W} = \mathrm{U} \oplus \mathrm{Z}.$$

We shall denote the elements of U by u and elements of Z by z. In particular every $w \in \mathrm{W}$ has a unique decomposition

$$w = u + z.$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$ and any $w' = u' + z' \in \mathrm{W}$,

$$t(g)\natural\phi(w') = \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle) \int_{\mathrm{U}} \chi_{c(g)}(u)\phi(u + z')\chi(-\frac{1}{2}\langle u, (1 - c(g))u' + z'\rangle)\, d\mu_{\mathrm{U}}(u). \tag{121}$$

In particular, (120) implies that $t(g)\natural\phi \in \mathcal{S}(\mathrm{W})$.

Proof. Suppose $\langle\ ,(1-c(g))u\rangle = 0$. Then $(1-c(g))u \in \operatorname{Ker} g^-$. There is $u_0 \in \mathsf{W}$ such that $u = g^- u_0$. Therefore

$$\begin{aligned} 0 &= g^-(1-c(g))u = g^-(1-c(g))g^-u_0 = g^-(g^-)u_0 - g^-g^+u_0 \\ &= g^-(g^-)u_0 - g^+g^-u_0 = (g^- - g^-)g^-u_0 = -2g^-u_0 = -2u. \end{aligned}$$

This verifies (120).

The left hand side of (121) is equal to

$$\begin{aligned} t(g)\natural\phi(w') &= \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(w'-u)\chi(\frac{1}{2}\langle u, w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ &= \int_{\mathsf{U}} \chi_{c(g)}(u+u')\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ &= \int_{\mathsf{U}} \chi_{c(g)}(u')\chi_{c(g)}(u)\chi(\frac{1}{2}\langle c(g)u', u\rangle)\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ &= \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle)\int_{\mathsf{U}} \chi_{c(g)}(u)\phi(z'-u)\chi(\frac{1}{2}\langle u, w'-c(g)u'\rangle)\, d\mu_{\mathsf{U}}(u), \end{aligned}$$

which coincides with the right hand side. $\square$

In the following proposition we use Notation 2.4 and Notation 2.6.

Proposition 4.13.
Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathsf{W})$. Let $\mathsf{U}_1' \subseteq \mathsf{U}_1$ be the orthogonal complement of U with respect to the positive definite form B, so that

$$\mathsf{U}_1 = \mathsf{U}_1' \oplus \mathsf{U}.$$

Then the map

$$L\colon \mathsf{U}_1' + \mathsf{U}_2 \ni u_1' + u_2 \to c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 + \mathsf{U}^\perp \in \mathsf{W}/\mathsf{U}^\perp$$

is well defined, surjective and $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) = \mathsf{U}_{12}$. Denote by

$$\tilde{L}\colon (\mathsf{U}_1+\mathsf{U}_2)/\mathsf{U}_{12} \ni u_1+u_2+\mathsf{U}_{12} \to c(g_1)u_1 - c(g_2)u_2 - u_1 - u_2 + \mathsf{V}^\perp \in \mathsf{W}/\mathsf{V}^\perp = (\mathsf{W}/\mathsf{U}^\perp)/(\mathsf{V}^\perp/\mathsf{U}^\perp)$$

the induced bijection and set

$$C(g_1, g_2) = \gamma(\tilde{q}_{g_1,g_2})2^{\dim \mathsf{V}}|\det(\tilde{L})|^{-1}.$$

Then C is a cocycle, with $C(g_1, 1) = C(1, g_2) = 1$, and

$$t(g_1)\natural t(g_2) = C(g_1, g_2)t(g_1g_2). \tag{122}$$

Here, and elsewhere in this paper, the determinant of the zero map on a zero vector space is by definition equal 1.

Proof. Since $\mathsf{V}^\perp/\mathsf{U}^\perp = (c(g_1)+c(g_2))\mathsf{U}$, the map $\tilde{L}$ is well defined. Suppose $u_1' \in \mathsf{U}_1'$ and $u_2 \in \mathsf{U}_2$ are such that $L(u_1'+u_2) \in \mathsf{V}^\perp/\mathsf{U}^\perp$. Then there is $u \in \mathsf{U}$ such that

$$(c(g_1)+c(g_2))u + c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 \in \mathsf{U}^\perp.$$

Let $u = g_1^- v_1 = g_2^- v_2$, $v = u_1' = g_1^- w_1$, and $w - v = u_2 = g_2^- w_2$. Then

$$(c(g_1)+c(g_2))u + c(g_1)v + c(g_2)(v-w) - w \in \mathsf{U}^\perp.$$

Hence, the computation (79) - (81) shows that $w = (g_1g_2)^-(w_2 - v_2) \in \mathrm{U}_{12}$. Therefore $L^{-1}(\mathrm{V}^\perp/\mathrm{U}^\perp) \subseteq \mathrm{U}_{12}$.

The map L is surjective. Indeed, for every $w \in W$, set $u_2 = g_2^- w_2$ with $w_2 = -\frac{1}{2}g_2^{-1}w$. Then

$$L(u_2) = -c(g_2)u_2 - u_1 + U^\perp = -g_2^+ w_2 - g_2^- w_2 + U^\perp = -2g_2 w_2 U^\perp = w + U^\perp .$$

Lemma 2.7 (b) shows that $\dim((\mathrm{U}_1 + \mathrm{U}_2)/\mathrm{U}_{12}) = \dim((\mathrm{W}/\mathrm{U}^\perp)/(\mathrm{V}^\perp/\mathrm{U}^\perp))$. Thus $L^{-1}(\mathrm{V}^\perp/\mathrm{U}^\perp) = \mathrm{U}_{12}$.

Here is a direct proof of this last equality. We already know that $L^{-1}(\mathrm{V}^\perp/\mathrm{U}^\perp) \subseteq \mathrm{U}_{12}$. Therefore it will suffice to show that $L(U_{12}) \subset V^\perp/U^\perp$. This is true, because one can show (as was done in the first part of the proof), that for $u = (g_1g_2)^- w = u_1 + u_2 = (u_1' + u') + u_2$, with $u_1 = g_1^- g_2 w$ and $u_2 = g_2^- w$, one has:

$$\begin{aligned} L(u) &= c(g_1)g_1^- g_2 w - c(g_2)g_2^- w - (c(g_1) + c(g_2))u' - (u_1 + u_2) + U^\perp \\ &= (g_1^+)g_2 w - g_2^+ w - u - (c(g_1) + c(g_2))u' + U^\perp \\ &= (g_1g_2)^- w - u - (c(g_1) + c(g_2))u' + U^\perp \\ &= u - u - (c(g_1) + c(g_2))u' + U^\perp \in V^\perp/U^\perp. \end{aligned}$$

The computation (81) - (85) shows that, if $u_1' + u_2 \in \mathrm{U}_{12}$ then

$$\langle c(g_1)u_1', u_1'\rangle + \langle c(g_2)u_2, u_2\rangle + 2\langle u_1', u_2\rangle + \langle (c(g_1) + c(g_2))^{-1}L(u_1' + u_2), L(u_1' + u_2)\rangle = \langle c(g_1g_2)(u_1' + u_2), u_1 + u_2\rangle$$

so that

$$\chi_{c(g_1)}(u_1')\chi_{c(g_2)}(u_2)\chi(\frac{1}{2}\langle u_1', u_2\rangle)\chi_{(c(g_1)+c(g_2))^{-1}}(L(u_1' + u_2)) = \chi_{c(g_1g_2)}(u_1' + u_2). \tag{123}$$

Any $u_1 \in \mathrm{U}_1$ has a unique decomposition $u_1 = u_1' + u$, where $u_1' \in \mathrm{U}_1'$ and $u \in \mathrm{U}$. With this notation, Lemma 4.12 shows that for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned} t(g_1)\natural(t(g_2)\natural\phi)(0) &= \int_{\mathrm{U}_1} \chi_{c(g_1)}(u_1)t(g_2)\natural\phi(u_1)\, d\mu_{\mathrm{U}_1}(u_1) \qquad (124)\\ &= \int_{\mathrm{U}_1}\int_{\mathrm{U}_2} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2) - 1)u\rangle) \\ &\quad \chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2, u_1'\rangle)\phi(u_2 + u_1')\, d\mu_{\mathrm{U}_2}(u_2)\, d\mu_{\mathrm{U}_1}(u_1) \\ &= \int_{\mathrm{U}}\int_{\mathrm{U}_1'}\int_{\mathrm{U}_2} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2) - 1)u\rangle) \\ &\quad \chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2, u_1'\rangle)\phi(u_2 + u_1')\, d\mu_{\mathrm{U}_2}(u_2)\, d\mu_{\mathrm{U}_1'}(u_1')\, d\mu_{\mathrm{U}}(u) \end{aligned}$$

The formula (110) applied with $x = c(g_1) + c(g_2)$ shows that

$$\begin{aligned} &\int_{\mathrm{U}} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2) - 1)u\rangle)\, d\mu_{\mathrm{U}}(u) \qquad (125)\\ &= \chi_{c(g_1)}(u_1')\int_{\mathrm{U}} \chi_{c(g_1)+c(g_2)}(u)\chi(\frac{1}{2}\langle u, c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2\rangle)\, d\mu_{\mathrm{U}}(u) \\ &= 2^{\dim \mathrm{V}}\gamma(\tilde{q}_{g_1,g_2})\chi_{c(g_1)}(u_1')(\chi_{\underline{(c(g_1)+c(g_2))}^{-1}}\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp})(c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2). \end{aligned}$$

Furthermore, Lemma 4.2 shows that, for $u_1' + u_2 \in \mathrm{U}_{12}$,

$$\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp}(c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2) = L^*(\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp})(u_1' + u_2) = |\det(\tilde{L})|^{-1}\mu_{\mathrm{U}_{12}}(u_1' + u_2). \tag{126}$$

The formula (122) follows directly from (123) - (126).

We see from (113) that

$$t(g_1)\natural(t(g_2)\natural\phi)(w) = (t(g_1)\natural(t(g_2)\natural\phi))\natural\delta_{-w}(0) = (t(g_1)\natural(t(g_2)\natural(\phi\natural\delta_{-w})))(0)$$
$$= ((t(g_1)\natural t(g_2))\natural(\phi\natural\delta_{-w}))(0) = ((t(g_1)\natural t(g_2))\natural\phi)\natural\delta_{-w})(0) = (t(g_1)\natural t(g_2))\natural\phi(w).$$

Therefore

$$(t(g_1)\natural t(g_2))\natural\phi = t(g_1)\natural(t(g_2)\natural\phi).$$

Hence, $t(g_1)\natural t(g_2)$ coincides with the composition of $t(g_1)$ and $t(g_2)$ as elements of the associative algebra $\mathrm{End}(\mathcal{S}(\mathsf{W}))$. Therefore the function C is a cocycle. □

4.6. Normalization of Gaussians and the metaplectic group

For a subset $S \subseteq \mathsf{W}$ let $S^{\perp_B} \subseteq \mathsf{W}$ be the B-orthogonal complement of S and for an element $h \in \mathrm{End}(\mathsf{W})$ let $h^{\#} \in \mathrm{End}(\mathsf{W})$ be as in (34). In particular, $(\mathrm{Ker}\, h^{\#})^{\perp} = h\mathsf{W}$.

Lemma 4.14.
Let $h \in \mathrm{End}(\mathsf{W})$ *and let* $K \subseteq \mathsf{W}$ *be a subspace. Then*

$$h^{\#}((hK)^{\perp}) \subseteq K^{\perp} \tag{127}$$

and

$$|\det(h : K \to hK)| = |\det(h^{\#} : \mathsf{W}/(hK)^{\perp} \to \mathsf{W}/K^{\perp})|. \tag{128}$$

Proof. The inclusion (127) follows directly from (34).

Let $w_1, \ldots, w_a$ be a B-orthonormal basis of K and let $u_1, \ldots, u_a$ be a B-orthonormal basis of hK. Since J is a B-isometry, $Jw_1, \ldots, Jw_a \in JK$ and $Ju_1, \ldots, Ju_a \in JhK$ are B-orthonormal basis. Define a matrix $(h_{k,i})_{1\le k,i\le a}$ by

$$hw_i = \sum_{k=1}^{a} h_{k,i}u_k \qquad (1 \le i \le a).$$

Then

$$|\det(h : K \to hK)| = |\det((h_{k,i})_{1\le k,i\le a})|. \tag{129}$$

We see from (33) that

$$JhK = (hK)^{\perp\perp_B} \text{ and } JK = K^{\perp\perp_B}.$$

Therefore

$$|\det(h^{\#} : \mathsf{W}/(hK)^{\perp} \to \mathsf{W}/K^{\perp})| = |\det((h^{\#}_{k,i})_{1\le k,i\le a})|, \tag{130}$$

where

$$h^{\#}Ju_i \in \sum_{k=1}^{a} h^{\#}_{k,i}Jw_k + K^{\perp} \qquad (1 \le i \le a).$$

But,

$$h_{j,i} = \sum_{k=1}^{a} h_{k,i}B(u_j, u_k) = -\sum_{k=1}^{a} h_{k,i}\langle u_k, Ju_j\rangle = -\langle hw_i, Ju_j\rangle$$
$$= -\langle w_i, h^{\#}Ju_j\rangle = -\langle w_i, \sum_{k=1}^{a} h^{\#}_{k,j}Jw_k\rangle = -\sum_{k=1}^{a} h^{\#}_{k,j}\langle w_i, Jw_k\rangle = \sum_{k=1}^{a} h^{\#}_{k,j}B(w_k, w_i) = h^{\#}_{i,j}.$$

Hence, (128) follows from (129) and (130). □

Lemma 4.15.
Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ *and assume that* $K_1 = \mathrm{Ker}\, g_1^- = 0$. *Then*

$$2^{-\dim \mathrm{V}}|\det(\tilde{L})| = |\det(g_2^- : K_{12} \to \mathrm{V})|^{-1}.$$

Proof. Since, by Lemma 2.7 (c), $\mathrm{V} = g_2^- K_{12} = (g_1^{-1} - 1)K_{12}$, the right hand side of the equation we need to prove makes sense. Also,

$$2^{-\dim \mathrm{V}}|\det(\tilde{L})| = |\det(\frac{1}{2}\tilde{L})|$$

and a straightforward computation shows that

$$\frac{1}{2}\tilde{L}\colon \mathrm{W}/\mathrm{U}_{12} \ni w + \mathrm{U}_{12} \to \frac{1}{2}(c(g_1) - 1)w + \mathrm{V}^\perp = (g_1^-)^{-1}w + \mathrm{V}^\perp \in \mathrm{W}/\mathrm{V}^\perp.$$

Hence,

$$|\det(\frac{1}{2}\tilde{L})|^{-1} = |\det(g_1^- : \mathrm{W}/\mathrm{V}^\perp \to \mathrm{W}/\mathrm{U}_{12})|.$$

Notice that $g_1^{-1} - 1 = g_1^\#$. Since $\mathrm{V} = g_2^- K_{12}$ and $\mathrm{U}_{12} = K_{12}^\perp$, Lemma 4.14 shows that

$$|\det(g_1^- : \mathrm{W}/\mathrm{V}^\perp \to \mathrm{W}/\mathrm{U}_{12})| = |\det(g_1^{-1} - 1\colon K_{12} \to \mathrm{V})|.$$

Since the restrictions of g_1^{-1} and g_2 to K_{12} are equal, we are done. □

Consider an element $h \in \mathrm{End}(\mathrm{W})$ such that $\mathrm{Ker}\, h = \mathrm{Ker}\, h^\#$. (In our applications h will be equal to g^-, where $g \in \mathrm{Sp}(\mathrm{W})$. Then $g^\# = g^{-1} - 1 = -g^{-1}g^-$ has the same kernel as g^-.) Let $L = J^{-1}h$. Denote by L^* the adjoint to L with respect to B (i.e. $B(Lw, w') = B(w, L^*w')$). Then $L^* = Jh^\#$. Hence $\mathrm{Ker}\, L = \mathrm{Ker}\, L^*$. Therefore L maps $(\mathrm{Ker}\, L)^{\perp_B} = L\mathrm{W}$ bijectively onto itself. Thus it makes sense to talk about $\det(L|_{L\mathrm{W}})$, the determinant of the restriction of L to $L\mathrm{W}$. If $w_1, w_2, \dots, w_m$ is a B-orthonormal bais of $(\mathrm{Ker}\, L)^{\perp_B}$, then

$$\det(L|_{L\mathrm{W}}) = \det(B(Lw_i, w_j)_{1\le i,j\le m}) = \det(\langle hw_i, w_j\rangle_{1\le i,j\le m}). \tag{131}$$

Under the condition $\mathrm{Ker}\, h = \mathrm{Ker}\, h^\#$, we define $\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})$ to be the quantity (131).

Since

$$B(Jw_i, Jw_j) = \langle JJw_i, Jw_j\rangle = \langle Jw_i, w_j\rangle = B(w_i, w_j),$$

$Jw_1, Jw_2, \dots, Jw_m$ is a B-orthonormal basis of $h\mathrm{W}$ ($=JL\mathrm{W}$). Further, if the coefficients $h_{j,i}$ are defined by

$$hw_i = \sum_j h_{j,i}Jw_j,$$

then

$$\det(\langle hw_i, w_j\rangle_{1\le i,j\le m}) = \det(\langle\sum_k h_{k,i}Jw_k, w_j\rangle_{1\le i,j\le m})$$
$$= \det((h_{k,i})_{1\le k,i\le m})\det(\langle Jw_k, w_j\rangle_{1\le k,j\le m}) = \det((h_{k,i})_{1\le k,i\le m})\det(B(w_k, w_j)_{1\le k,j\le m}) = \det((h_{k,i})_{1\le k,i\le m}).$$

Thus $|\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})| = |\det((h_{k,i})_{1\le k,i\le m})|$ coincides with the absolute value of the determinant defined previously in Section 4.1. In particular,

$$\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W}) = \mathrm{sgn}(\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W}))|\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})|. \tag{132}$$

Hence, if we identify $\mathbb{R}^\times/(\mathbb{R}^\times)^2$ with $\{\pm 1\}$ via the sgn, then $\det(h : \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})$ is equal to the discriminant of the bilinear form induced by $\langle h\ ,\ \rangle$ on the quotient $\mathrm{W}/\mathrm{Ker} h$ times $|\det(h\colon \mathrm{W}/\mathrm{Ker}\, h \to h\mathrm{W})|$.

Definition 4.16.
For $g \in \mathrm{Sp}(\mathrm{W})$ define

$$\begin{aligned}\Theta^2(g) &:= \gamma(1)^{2\dim g^- \mathrm{W}} \det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})^{-1} \\ &= \gamma(1)^{2\dim g^- \mathrm{W}-2} (\gamma(\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W}))^2.\end{aligned}$$

(Here the second equality follows from (103).)

Lemma 4.17.
We have

$$\frac{\Theta^2(g_1 g_2)}{\Theta^2(g_1)\Theta^2(g_2)} = C(g_1, g_2)^2 \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})). \tag{133}$$

Proof. Both sides of the equality (133) are cocycles. Hence, Lemma 2.8 shows that we may assume that $K_1 = \{0\}$. In terms of the notation of Lemma 2.11 we have

$$-\dim \mathrm{U}_{12} + \dim \mathrm{W} + \dim \mathrm{U} = \dim K_{12} + \dim \mathrm{U} = \dim \mathrm{V} + \dim \mathrm{U} = -\dim(\mathrm{U}/\mathrm{V}) + 2\dim \mathrm{U}.$$

Hence,

$$\gamma(1)^{2(-\dim \mathrm{U}_{12}+\dim \mathrm{W}+\dim \mathrm{U})} = \gamma(1)^{4\dim U} \gamma(1)^{-2\dim(U/V)} = (-1)^{\dim \mathrm{U}} \gamma(1)^{-2\dim(\mathrm{U}/\mathrm{V})}. \tag{134}$$

Therefore the equality (37) is equivalent to

$$\begin{aligned}&\frac{\gamma(1)^{-2\dim \mathrm{U}_{12}} \det((g_1 g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12})}{\gamma(1)^{-2\dim \mathrm{W}} \det(g_1^- : \mathrm{W} \to \mathrm{W})\gamma(1)^{-2\dim \mathrm{U}} \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= \gamma(1)^{-2\dim(\mathrm{U}/\mathrm{V})} \det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathrm{U}/\mathrm{V}}) \det(g_2^- : K_{12} \to \mathrm{V})^{-2}.\end{aligned} \tag{135}$$

By Remark 4.5, we get

$$\gamma(\tilde{q}_{g_1,g_2}) = e^{\frac{\pi i}{4}\mathrm{sgn}(\tilde{q}_{g_1,g_2})} |\det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathrm{U}/\mathrm{V}})|^{-1/2}$$

and

$$\mathrm{sgn}(\tilde{q}_{g_1,g_2}) = p - q,$$

where p is the dimension of the maximal subspace of U/V on which the form $\langle c(g_1) + c(g_2)\ ,\ \rangle$ is positive definite and q is the dimension of the maximal subspace of U/V on which the form $\langle c(g_1) + c(g_2)\ ,\ \rangle$ is negative definite. Hence,

$$\begin{aligned}\gamma(\tilde{q}_{g_1,g_2})^2 &= i^{p-q} |\det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathrm{U}/\mathrm{V}})|^{-1} \\ &= i^{p-q}(-1)^q \det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathrm{U}/\mathrm{V}})^{-1} \\ &= i^{p+q} \det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathrm{U}/\mathrm{V}})^{-1} \\ &= i^{\dim(\mathrm{U}/\mathrm{V})} \det(\langle \frac{1}{2}(c(g_1) + c(g_2))\ ,\ \rangle_{\mathrm{U}/\mathrm{V}})^{-1}.\end{aligned}$$

This, together with Lemma 4.15, shows that the right hand side of (135) is equal to

$$\gamma(\tilde{q}_{g_1,g_2})^{-2} \left(2^{-\dim \mathrm{V}} |\det(\tilde{L})|\right)^2,$$

which, by Proposition 4.13, coincides with $C(g_1, g_2)^{-2}$. □

Definition 4.18.

Let

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \{(g, \xi);\ g \in \mathrm{Sp}(\mathrm{W}),\ \xi \in \mathbb{C}^\times,\ \xi^2 = \Theta^2(g)\}.$$

where $\Theta^2(g)$ is defined by Definition 4.16.

Lemma 4.19.

$\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a group with the multiplication defined by

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2)) \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})) \tag{136}$$

the identity equal to $(1, 1)$ and the inverse given by

$$(g, \xi)^{-1} = (g^{-1}, \overline{\xi}) \qquad (g \in \mathrm{Sp}(\mathrm{W})).$$

Proof. Lemma 4.17 shows that the right hand side of (136) belongs to $\widetilde{\mathrm{Sp}}(\mathrm{W})$. A standard computation, as in [20, page 366], shows that $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a group with the multiplication given by (136), the identity equal to $(1, C(1, 1)^{-1})$ and

$$(g, \xi)^{-1} = (g^{-1}, C(g^{-1}, g)^{-1}\xi^{-1}).$$

Since, by Proposition 4.13, $C(1, 1) = 1$, it remains to check that

$$C(g^{-1}, g)^{-1}\xi^{-1} = \overline{\xi}.$$

But, as in the proof of Lemma 4.15,

$$C(g^{-1}, g) = 2^{\dim \mathrm{V}} |\det(\tilde{L})|^{-1} = |\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})| = |\Theta^2(g)|^{-1} = |\xi|^{-2}.$$

This completes the proof. □

Notice that the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni (g, \xi) \to g \in \mathrm{Sp}(\mathrm{W})$$

is a group homomorphism with the kernel consisting of two elements. Thus $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a central extension of $\mathrm{Sp}(\mathrm{W})$ by the two element group $\mathbb{Z}/2\mathbb{Z}$:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Sp}(\mathrm{W}) \to 1. \tag{137}$$

Proposition 4.20.

The extension (137) does not split.

Proof. Pick a two-dimensional symplectic subspace $\mathrm{W}_1 \subseteq \mathrm{W}$ and let $\mathrm{W}_2 = \mathrm{W}_1^\perp$, so that

$$\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2.$$

Define an element $g \in \mathrm{Sp}(\mathrm{W})$ by

$$g(w_1 + w_2) = -w_1 + w_2 \qquad (w_1 \in \mathrm{W}_1, w_2 \in \mathrm{W}_2).$$

Then

$$\Theta^2(g) = i^2 \det(-2 : \mathrm{W}_1 \to \mathrm{W}_1)^{-1} = (i/2)^2$$

and

$$C(g,g) = 2^2 \cdot 1 \cdot 1 = 2^2.$$

Let $\tilde{g} = (g, i/2)$. Then $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and

$$\tilde{g}^2 = (g^2, (i/2)^2 C(g,g)) = (1,-1) \text{ and } \tilde{g}^4 = (1,1).$$

Thus the subgroup of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ generated by $\tilde{g}$ is cyclic of order 4. The subgroup of $\mathrm{Sp}(\mathrm{W})$ generated by g is cyclic of order 2. Hence the extension (137) does not split over that subgroup. □

Corollary 4.21.

Up to an equivalence of central group extensions, as in [20, sec. 6.10], (137) is the only non-trivial central extension of $\mathrm{Sp}(\mathrm{W})$ *by* $\mathbb{Z}/2\mathbb{Z}$.

Proof. Since, as is well known (see [25, Theorems 5.10 and 11.1 (b)]),

$$H^2(\mathrm{Sp}(\mathrm{W}), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}),$$

the claim follows. □

Let

$$\phi^*(w) = \overline{\phi(-w)} \text{ and } u^*(\phi) = \overline{u(\phi^*)} \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ u \in \mathcal{S}^*(\mathrm{W}),\ w \in \mathrm{W}).$$

Lemma 4.22.

For any $g \in \mathrm{Sp}(\mathrm{W})$, $t(g)^* = t(g^{-1})$.

Proof. By the definition (114),

$$t(g)^* = \left(\chi_{c(g)} \mu_{g^-\mathrm{W}}\right)^* = \overline{\chi_{c(g)}} \mu_{g^-\mathrm{W}} = \chi_{-c(g)} \mu_{g^-\mathrm{W}}.$$

Since $g^-\mathrm{W} = (g^{-1} - 1)\mathrm{W}$, it will suffice to check that for any $w \in \mathrm{W}$

$$-c(g) g^- w = c(g^{-1}) g^- w.$$

The left hand side is equal to $-g^+ w$. The right hand side is equal to

$$-c(g^{-1})(g^{-1} - 1) g w = -(g^{-1} - 1) g w = -g^+ w. \qquad \square$$

Definition 4.23.

For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ *define*

$$\Theta(\tilde{g}) = \xi \quad \textit{and} \quad T(\tilde{g}) = \Theta(\tilde{g}) t(g). \tag{138}$$

Lemma 4.24.

With the notation of (138), the following formulas hold

$$T(\tilde{g}_1) \natural T(\tilde{g}_2) = T(\tilde{g}_1 \tilde{g}_2) \qquad (\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})), \tag{139}$$

$$T(\tilde{g})^* = T(\tilde{g}^{-1}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})). \tag{140}$$

Proof. By Proposition 4.13 the left hand side of (139) is equal to

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1\tilde{g}_2)}C(g_1,g_2)T(\tilde{g}_1\tilde{g}_2).$$

Lemma 4.19 shows that

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1\tilde{g}_2)}C(g_1,g_2)=1.$$

This verifies (139).

The equality (140) follows from Lemma 4.19 and Lemma 4.22:

$$T(\tilde{g})^*=\overline{\Theta(\tilde{g})}t(g)^*=\Theta(\tilde{g}^{-1})t(g^{-1})=T(\tilde{g}^{-1}).$$ □

Notice that Sp(W) is a connected Lie group. As such it has a unique connected double cover (up to an isomorphism of covers). See [3, sec. 16.30]. This way we may view $\widetilde{\mathrm{Sp}}(\mathrm{W})$, the metaplectic group, as a connected Lie group.

Lemma 4.25.

The map $T:\widetilde{\mathrm{Sp}}(\mathrm{W})\to\mathcal{S}^*(\mathrm{W})$ is injective and continuous.

Proof. The injectivity of T follows from the injectivity of $t:\mathrm{Sp}(\mathrm{W})\to\mathcal{S}^*(\mathrm{W})$, which is obvious. Let

$$\mathrm{Sp}^c(\mathrm{W})=\{g\in\mathrm{Sp}(\mathrm{W});\ \det g^-\neq 0\}.$$

Lemma 2.8 shows that

$$\mathrm{Sp}(\mathrm{W})=\bigcup_{h\in\mathrm{Sp}(\mathrm{W})}\mathrm{Sp}^c(\mathrm{W})h. \tag{141}$$

Let $\widetilde{\mathrm{Sp}}^c(\mathrm{W})\subseteq\widetilde{\mathrm{Sp}}(\mathrm{W})$ be the preimage of $\mathrm{Sp}^c(\mathrm{W})$. Then

$$\widetilde{\mathrm{Sp}}(\mathrm{W})=\bigcup_{\tilde{h}\in\widetilde{\mathrm{Sp}}(\mathrm{W})}\widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h}.$$

By Lemma 4.24, we have

$$T(\tilde{g})=T(\tilde{g}\tilde{h}^{-1})\natural T(\tilde{h})\qquad(\tilde{g}\in\widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h})$$

Thus for $\phi\in\mathcal{S}(\mathrm{W})$,

$$T(\tilde{g})\natural\phi=T(\tilde{g}\tilde{h}^{-1})\natural(T(\tilde{h})\natural\phi).$$

By Proposition 4.11, the map

$$\mathcal{S}(\mathrm{W})\ni\phi\to T(\tilde{h})\natural\phi\in\mathcal{S}(\mathrm{W})$$

is continuous. Hence it will suffice to check that the restriction of T to $\widetilde{\mathrm{Sp}}^c(\mathrm{W})$ is continuous. But this is obvious. □

4.7. The conjugation property

Let $\mathrm{L}^2(\mathrm{W})$ denote the Hilbert space of the Lebesgue measurable functions $\phi : \mathrm{W} \to \mathbb{C}$, with the norm given by

$$\| \phi \|_2^2 = \int_{\mathrm{W}} |\phi(w)|^2 \, d\mu_{\mathrm{W}}(w).$$

Lemma 4.24 shows that for any $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and any $\phi \in \mathcal{S}(\mathrm{W})$

$$\| T(\tilde{g})\natural\phi \|_2^2 = (T(\tilde{g})\natural\phi)^*\natural(T(\tilde{g})\natural\phi)(0) = \phi^*\natural T(\tilde{g})^*\natural T(\tilde{g})\natural\phi(0) = \phi^*\natural\phi(0) = \| \phi \|_2^2 .$$

Hence, the continuous linear map

$$\mathcal{S}(\mathrm{W}) \ni \phi \to T(\tilde{g})\natural\phi \in \mathcal{S}(\mathrm{W})$$

extends by continuity to an isometry

$$\mathrm{L}^2(\mathrm{W}) \ni \phi \to T(\tilde{g})\natural\phi \in \mathrm{L}^2(\mathrm{W}).$$

Furthermore, the formula

$$\omega_{1,1}(g)\phi(w) = \phi(g^{-1}w) \qquad (g \in \mathrm{Sp}(\mathrm{W}),\ \phi \in \mathrm{L}^2(\mathrm{W})).$$

defines a unitary representation $\omega_{1,1}$ of the symplectic group $\mathrm{Sp}(\mathrm{W})$ on $\mathrm{L}^2(\mathrm{W})$.

Proposition 4.26.
For any $\phi \in \mathrm{L}^2(\mathrm{W})$ and $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ in the preimage of $g \in \mathrm{Sp}(\mathrm{W})$, $T(\tilde{g})\natural\phi\natural T(\tilde{g}^{-1}) = \omega_{1,1}(g)\phi$.

Proof. Since $T(\tilde{g})$ is a bounded operator, we may assume that $\phi \in \mathcal{S}(\mathrm{W})$. Lemma 4.9 says that

$$t(g)\natural\delta_w = \delta_{wg}\natural t(g) \qquad (w \in \mathrm{W}).$$

Therefore

$$T(\tilde{g})\natural\delta_w = \delta_{wg}\natural T(\tilde{g}) \qquad (w \in \mathrm{W}).$$

Since,

$$\phi = \int_{\mathrm{W}} \phi(w)\delta_w \, d\mu_{\mathrm{W}}(w) \text{ and } \int_{\mathrm{W}} \phi(w)\delta_{gw} \, d\mu_{\mathrm{W}}(w) = \omega_{1,1}(g)\phi,$$

we see that

$$T(\tilde{g})\natural\phi = (\omega_{1,1}(g)\phi)\natural T(\tilde{g}).$$

□

4.8. The Weyl transform and the Weil representation

Pick a complete polarization

$$\mathrm{W} = \mathrm{X} \oplus \mathrm{Y} \tag{142}$$

and recall that our normalization of measures is such that $d\mu_{\mathrm{W}}(x+y) = d\mu_{\mathrm{X}}(x)d\mu_{\mathrm{Y}}(y)$. Recall the the Weyl transform

$$\mathcal{K}\colon \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\mathrm{X} \times \mathrm{X}), \tag{143}$$
$$\mathcal{K}(f)(x, x') = \int_{\mathrm{Y}} f(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle) \, d\mu_{\mathrm{Y}}(y),$$

This is an isomorphism of linear topological spaces, which restricts to an isometry

$$\mathcal{K}\colon \mathrm{L}^2(\mathrm{W}) \to \mathrm{L}^2(\mathrm{X} \times \mathrm{X}). \tag{144}$$

Each element $K \in \mathcal{S}^*(\mathrm{X}\times\mathrm{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$ by

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x,x')v(x')\, d\mu_{\mathrm{X}}(x'). \tag{145}$$

Schwartz Kernel Theorem, [13, Theorem 5.2.1], implies that

$$\mathrm{Op} : \mathcal{S}^*(\mathrm{X}\times\mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X})) \tag{146}$$

is an isomorphism of linear topological vector spaces. A straightforward computation shows that $\mathrm{Op}\circ\mathcal{K}$ transforms the twisted convolution of distributions (when it makes sense) into the composition of the corresponding operators. Also,

$$(\mathrm{Op}\circ\mathcal{K}(f))^* = \mathrm{Op}\circ\mathcal{K}(f^*) \qquad (f\in\mathcal{S}^*(\mathrm{W})) \tag{147}$$

and

$$\mathrm{tr}\ \mathrm{Op}\circ\mathcal{K}(f) = \int_{\mathrm{X}} \mathcal{K}(f)(x,x)\, d\mu_{\mathrm{X}}(x) = f(0) \tag{148}$$

if $\mathrm{Op}\circ\mathcal{K}(f)$ is of trace class, [17, Theorem 3.5.4]. Hence, the map

$$\mathrm{Op}\circ\mathcal{K} : \mathrm{L}^2(\mathrm{W}) \to \mathrm{H.S.}(\mathrm{L}^2(\mathrm{X})) \tag{149}$$

is an isometry, which is a well known fact [17, Theorem 1.4.1]. (Here $\mathrm{H.S.}(\mathrm{L}^2(\mathrm{X}))$ stands for the space of the Hilbert-Schmidt operators on $\mathrm{L}^2(\mathrm{X})$.)

Let $\mathrm{U}(\mathrm{L}^2(\mathrm{X}))$ denote the group of the unitary operators on the Hilbert space $\mathrm{L}^2(\mathrm{X})$.

Theorem 4.27.

Let $\omega = \mathrm{Op}\circ\mathcal{K}\circ T$. Then

$$\omega\colon \mathrm{Sp}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathrm{X}))$$

is an injective group homomorphism. For each $v\in\mathrm{L}^2(\mathrm{X})$, the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde g \to \omega(\tilde g)v \in \mathrm{L}^2(\mathrm{X})$$

is continuous, so that $(\omega, \mathrm{L}^2(\mathrm{X}))$ is a unitary representation of the metaplectic group. The function Θ coincides with the character of this representation:

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde g)\Psi(\tilde g)\, d\tilde g = \mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde g)\Psi(\tilde g)\, d\tilde g \qquad (\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W})),$$

where the integral on the left is absolutely convergent. (Here $d\tilde g$ stands for any Haar measure on $\widetilde{\mathrm{Sp}}(\mathrm{W})$.) Moreover,

$$\omega(\tilde g)\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(\tilde g^{-1}) = \mathrm{Op}\circ\mathcal{K}(\omega_{1,1}(g)\phi) \qquad (\tilde g\in\widetilde{\mathrm{Sp}}(\mathrm{W}),\ \phi\in\mathrm{L}^2(\mathrm{W})).$$

Proof. We see from the discussion in Section 4.7 that the left multiplication by $\omega(\tilde g)$ is an isometry on $\mathrm{H.S.}(\mathrm{L}^2(\mathrm{X}))$. This implies that $\omega(\tilde g)$ is a unitary operator.

We see from (146) that for any two function $v_1, v_2\in\mathcal{S}(\mathrm{X})$ there is $\phi\in\mathcal{S}(\mathrm{W})$ such that

$$\int_{\mathrm{X}} \omega(\tilde g)v_1(x)\overline{v_2(x)}\, d\mu_{\mathrm{X}}(x) = T(\tilde g)(\phi) \qquad (\tilde g\in\mathrm{Sp}(\mathrm{W})).$$

Hence Lemma 4.25 shows that the left hand side is a continuous function of $\tilde{g}$. Since the operators $\omega(\tilde{g})$ are uniformly bounded (by 1), we see that the left hand side is a continuous function of $\tilde{g}$ for any $v_1, v_2 \in \mathrm{L}^2(\mathrm{X})$. This implies the strong continuity of ω, see [40, Lemma 1.1.3] or [41, Proposition 4.2.2.1].

Lemmas 4.24 and 4.25 show that the $\omega : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathrm{X}))$ is an injective group homomorphism.

It is not difficult to check that the function

$$\frac{\det(\mathrm{Ad}(g)-1)}{\det g^-} \qquad (g \in \mathrm{Sp}(\mathrm{W}))$$

is locally bounded. Furthermore, as shown by Harish-Chandra [10, Lemma 53, page 504], the function

$$|\det(\mathrm{Ad}(g)-1)|^{-1/2} \qquad (g \in \mathrm{Sp}(\mathrm{W}))$$

is locally integrable. Hence the function,

$$|\Theta(\tilde{g})| = |\det g^-|^{-1/2} \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W}))$$

is locally integrable.

Notice that for any $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$,

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} \in \mathcal{S}(\mathrm{W}). \tag{150}$$

Indeed, since the Zariski topology on $\mathrm{Sp}(\mathrm{W})$ is noetherian the covering (141) contains a finite subcovering (see for example [12, Exercise 1.7(b)]). Hence, there are elements $\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_m$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ such that

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \bigcup_{j=1}^{m} \widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h}_j.$$

Therefore Proposition 4.11, Lemma 4.24 and a standard partition of the unity argument reduces the proof of (150) to the case when $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}^c(\mathrm{W}))$. In this case (150) is equal to

$$\int_{\mathfrak{sp}(\mathrm{W})} e^{\frac{\pi i}{2}\langle xw, w\rangle}\psi(x)\, dx \tag{151}$$

where $\psi \in C_c^\infty(\mathfrak{sp}(\mathrm{W}))$ and dx is a Lebesgue measure on $\mathfrak{sp}(\mathrm{W})$. The function (151) is equal to the pullback of a Fourier transform $\hat{\psi}$ of ψ from $\mathfrak{sp}^*(\mathrm{W})$ to W via the unnormalized moment map

$$\tau : \mathrm{W} \to \mathfrak{sp}^*(\mathrm{W}),\ \ \tau(w)(x) = \langle xw, w\rangle \qquad (x \in \mathfrak{sp}(\mathrm{W}),\ \ w \in \mathrm{W}). \tag{152}$$

Since $\hat{\psi} \in \mathcal{S}(\mathfrak{sp}(\mathrm{W}))$ and since τ is a polynomial map with uniformly bounded fibers,

$$\hat{\psi} \circ \tau \in \mathcal{S}(\mathrm{W}).$$

This verifies (150). Hence, we may compute the trace as follows:

$$\begin{aligned}\operatorname{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{g})\Psi(\tilde{g})\, d\tilde{g} &= \left(\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}\right)(0) = \left(\int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}\right)(0)\\ &= \int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})(0)\Psi(\tilde{g})\, d\tilde{g} = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\Psi(\tilde{g})\, d\tilde{g}.\end{aligned}$$

The last formula is a direct consequence of Proposition 4.26. $\square$

We end this Section by recalling some well known formulas for the action of $\omega(\tilde{g})$ for some special elements $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$.

Proposition 4.28.

Let $\mathrm{M} \subseteq \mathrm{Sp}(\mathrm{W})$ be the subgroup of all the elements that preserve X and Y. Let $\mathrm{M}^c = \{g \in \mathrm{M} : \det g^- \neq 0\}$. Set

$$\det_{\mathrm{X}}^{-1/2}(\tilde{g}) = \Theta(\tilde{g}) |\det(\frac{1}{2}(c(g|_{\mathrm{X}}) + 1))|^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c).$$

Then

$$\left(\det_{\mathrm{X}}^{-1/2}(\tilde{g})\right)^2 = \det(g|_{\mathrm{X}})^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c), \tag{153}$$

the function $\det_{\mathrm{X}}^{-1/2} \colon \widetilde{\mathrm{M}}^c \to \mathbb{C}^\times$ extends to a continuous group homomorphism

$$\det_{\mathrm{X}}^{-1/2} \colon \widetilde{\mathrm{M}} \to \mathbb{C}^\times$$

and

$$\omega(\tilde{g})v(x) = \det_{\mathrm{X}}^{-1/2}(\tilde{g}) v(g^{-1}x) \qquad (\tilde{g} \in \widetilde{\mathrm{M}},\ v \in \mathcal{S}(\mathrm{X}),\ x \in \mathrm{X}). \tag{154}$$

Proof. Set $n = \dim \mathrm{X}$. Fix an element $g \in \mathrm{M}^c$. Then

$$\begin{aligned}\Theta^2(g) &= \det(ig^-)^{-1} = (-1)^n \det(g|_{\mathrm{X}} - 1)^{-1} \det(g|_{\mathrm{Y}} - 1)^{-1} = \det(g|_{\mathrm{X}} - 1)^{-1} \det(1 - g|_{\mathrm{Y}})^{-1} \\ &= \det(g|_{\mathrm{X}} - 1)^{-1} \det(1 - (g|_{\mathrm{X}})^{-1})^{-1} = \det(g|_{\mathrm{X}} - 1)^{-2} \det(g|_{\mathrm{X}}).\end{aligned}$$

Also,

$$|\det(\frac{1}{2}(c(g|_{\mathrm{X}}) + 1))|^{-1} = |\det((g|_{\mathrm{X}})(g|_{\mathrm{X}} - 1)^{-1})|^{-1} = |\det(g|_{\mathrm{X}} - 1))| |\det(g|_{\mathrm{X}})|^{-1}.$$

This verifies (153).

Let $x, x' \in \mathrm{X}$ and let $y \in \mathrm{Y}$. Then

$$\begin{aligned}\mathcal{K}(t(g))(x, x') &= \int_{\mathrm{Y}} t(g)(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathrm{Y}}(y) \\ &= \int_{\mathrm{Y}} \chi(\frac{1}{2}\langle c(g)(x - x'), y\rangle)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, d\mu_{\mathrm{Y}}(y) \\ &= \delta_0(\frac{1}{2}c(g)(x - x') - x - x') = \delta_0(\frac{1}{2}((c(g) - 1)x - (c(g) + 1)x')) \\ &= |\det(\frac{1}{2}(c(g|_{\mathrm{X}}) + 1))|^{-1} \delta_0(g^{-1}x - x').\end{aligned}$$

Therefore

$$\mathcal{K}(T(\tilde{g}))(x, x') = \det_{\mathrm{X}}^{-1/2}(\tilde{g}) \delta_0(g^{-1}x - x').$$

Thus we have (154) for $\tilde{g} \in \widetilde{\mathrm{M}}^c$. Since ω is a representation of $\widetilde{\mathrm{M}}$, the remaining claims follow. □

Proposition 4.29.

Suppose $g \in \mathrm{Sp}(\mathrm{W})$ acts trivially on Y and on W/Y. Then $\det((-g) - 1) \neq 0$ and

$$\omega(\tilde{g})v(x) = \pm\chi_{c(-g)}(2x)v(x) \qquad (v \in \mathcal{S}(\mathrm{X}),\ x \in \mathrm{X}).$$

Proof. Since $-g$ acts as minus the identity on Y and on W/Y, $\det((-g)-1)\neq 0$ and $z := c(-g) \in \mathfrak{sp}(\mathsf{W})$ is well defined. We have

$$z(w) = (-g)^{+}((-g)^{-})^{-1}(w) \quad (w \in \mathsf{W}).$$

Since g acts trivially on Y and on W/Y, we get, for every $x \in \mathsf{X}$ and every $y \in \mathsf{Y}$:

$$g(x+y) = x + y + y_x, \quad \text{where } y_x \in \mathsf{Y}.$$

It gives $(-g)^{-}(x+y) = -2x - 2y - y_x$. Also, $(-g)^{-}y = -2y$, so that $((-g)^{-})^{-1}y = -\frac{1}{2}y$. Hence,

$$((-g)^{-})^{-1}(x+y) = -\frac{1}{2}(x+y+((-g)^{-})^{-1}y_x) = -\frac{1}{2}(x+y) + \frac{1}{4}y_x.$$

We obtain

$$z(x+y) = (-g)^{+}(-\frac{1}{2}(x+y) + \frac{1}{4}y_x),$$

that is,

$$z(x+y) = z(x) = \frac{1}{2}y_x. \tag{155}$$

In particular, we have

$$z\colon \mathsf{X} \to \mathsf{Y} \to 0.$$

Also, $\det(z-1)\neq 0$ and $c(z)$ is well defined. On the other hand, we have $(z-1)(x+y) = -(x+y) + \frac{1}{2}y_x$ and $(z-1)y = -y$. It follows that

$$(z-1)^{-1}(x+y) = -(x+y) - \frac{1}{2}y_x.$$

Hence,

$$c(z)(x+y) = (z+1)\left(-(x+y) - \frac{1}{2}y_x\right) = -\frac{1}{2}y_x - (x+y) - \frac{1}{2}y_x,$$

that is,

$$c(z)(w) = -w - 2z(w), \quad \text{for every } w \in \mathsf{W}. \tag{156}$$

We have $c(z) \in \mathrm{Sp}(\mathsf{W})$. Indeed, for any $w, w' \in \mathsf{W}$, writing $w = x+y$ and $w' = x'+y'$, with $x, x' \in \mathsf{X}$ and $y, y' \in \mathsf{Y}$, we have

$$\langle c(z)(w), c(z)(w')\rangle = \langle -w - y_x, -w' - y_{x'}\rangle = \langle w, w'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle.$$

However, since g is in $\mathrm{Sp}(\mathsf{W})$, we have

$$\langle x, x'\rangle = \langle gx, gx'\rangle = \langle x + y_x, x' + y_{x'}\rangle = \langle x, x'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle,$$

which gives

$$\langle x, y_{x'}\rangle + \langle y_x, x'\rangle = 0.$$

Hence,

$$\begin{aligned}
\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi_{-z}(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\
&= \chi_{-2z}(x-x')\delta_0(\frac{1}{2}(x+x')) = 2^n\, \chi_{-2z}(x-x')\, \delta_0(x+x')\\
\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi_{-z}(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\
&= \chi_{-z}(x-x')\delta_0(\frac{1}{2}(x+x')) = 2^n\chi_{-z}(x-x')\delta_0(x+x').
\end{aligned}$$

Moreover,

$$\Theta^2(c(z)) = \left(\frac{i}{2}\right)^{2n},$$

since $\dim((c(z)-1)(\mathsf{W})) = \dim \mathsf{W} = 2n$, and,

$$\det(c(z)-1) = (-2)^{2n}.$$

Thus

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') = \pm i^n \chi_{-z}(x-x')\delta_0(x+x').$$

Since Proposition 4.28 shows that

$$\omega(\widetilde{(-1)})v(x) = \pm i^n v(-x),$$

the proof is complete. □

Proposition 4.30.
Suppose $g \in \mathrm{Sp}(\mathsf{W})$ acts trivially on X and on W/X. Then $\det((-g)-1) \neq 0$ so that $z = c(-g) \in \mathfrak{sp}(\mathsf{W})$ is well defined and $z\colon \mathsf{Y} \to \mathsf{X} \to 0$. Assume $z(\mathsf{Y}) = \mathsf{X}$. Then

$$\omega(\tilde{g})v(x) = \pm \frac{e^{\frac{\pi i}{4}\,\mathrm{sgn}\langle z\,,\ \rangle|_{\mathsf{Y}}}}{|\det(z\colon \mathsf{Y}\to\mathsf{X})|^{1/2}} \int_{\mathsf{X}} \chi_{z^{-1}}(x-x')v(x')\,d\mu_{\mathsf{X}}(x') \qquad (v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}),$$

where $z^{-1}\colon \mathsf{X} \to \mathsf{Y}$ is the inverse of $z\colon \mathsf{Y}\to\mathsf{X}$.

Proof. The existence of z and its properties are verified as in the proof of Proposition 4.29. In particular, for all $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, we have

$$g(x+y) = x + y + x_y, \quad \text{where } x_y \in \mathsf{X}.$$

Similarly to the proof of Proposition 4.29, we get

$$z(x+y) = z(y) = \frac{1}{2}x_y. \tag{157}$$

and

$$c(z)(x+y) = -(x+y) - x_y, \tag{158}$$

that is,

$$c(z)(w) = -w - 2z(w), \quad \text{for every } w \in \mathsf{W}. \tag{159}$$

From (157)and (158), we obtain

$$\langle c(z)(w), w\rangle = \langle -w - 2z(w), w\rangle = -2\langle z(w), w\rangle. \tag{160}$$

With notation (107), it gives

$$\chi_{c(z)}(w) = \chi\left(\frac{1}{4}\langle c(z)(w), w\rangle\right) = \chi\left(-\frac{1}{2}\langle z(w), w\rangle\right) = \chi_{-2z}(w). \tag{161}$$

Let

$$q(y,y') = \frac{1}{2}\langle zy, y'\rangle \qquad (y, y' \in \mathsf{Y}).$$

Then, in terms of Lemma 4.7 and the identification (105),

$$q^*(x,x') = -2\langle z^{-1}x, x'\rangle \qquad (x,x' \in \mathsf{X})$$

and

$$\gamma(q) = \frac{e^{\frac{\pi i}{4}\,\mathrm{sgn}\langle z,\,\rangle|_{\mathsf{Y}}}}{|\det(\frac{1}{2}z\colon \mathsf{Y}\to\mathsf{X})|^{1/2}}.$$

Indeed, using notation of Eqn.(11),

$$\langle y', \Phi(y)\rangle = \Phi(y)(y') = q(y,y') = \frac{1}{2}\langle zy, y'\rangle = \langle y', -\frac{1}{2}zy\rangle.$$

Hence, $\Phi(y) = x$ if and only if $-\frac{1}{2}zy = x$. Thus $\Phi^{-1}(x) = -2z^{-1}x$. Therefore

$$q^*(x,x') = x'(\Phi^{-1}(x)) = \langle \Phi^{-1}(x), x'\rangle = \langle -2z^{-1}x, x'\rangle.$$

Hence, by the definition of $\mathcal{K}$ (143), the assumption that z annihilates X and maps Y into X and Lemma 4.7, we obtain

$$\begin{aligned}
\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -z(x-x'+y), x-x'+y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\
&= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -zy, y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathsf{Y}}(y)\\
&= \int_{\mathsf{Y}} \chi(\frac{1}{2}q(y,y))\chi(-\langle y, -\frac{1}{2}(x+x')\rangle)\, d\mu_{\mathsf{Y}}(y)\\
&= \gamma(q)\chi(-\frac{1}{2}q^*(-\frac{1}{2}(x+x'), -\frac{1}{2}(x+x')))\\
&= \gamma(q)\chi(\langle z^{-1}(-\frac{1}{2}(x+x')), -\frac{1}{2}(x+x')\rangle) = \gamma(q)\chi_{z^{-1}}(x+x').
\end{aligned}$$

Therefore

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') = \Theta(\widetilde{c(z)})\gamma(q)\chi_{z^{-1}}(x+x').$$

But $\Theta(\widetilde{c(z)}) = \pm\left(\frac{i}{2}\right)^n$ (where $\dim \mathsf{W} = 2n$), so that

$$\Theta(\widetilde{c(z)})\gamma(q) = \pm\left(\frac{i}{2}\right)^n \frac{e^{\frac{\pi i}{4}\,\mathrm{sgn}\langle z,\,\rangle|_{\mathsf{Y}}}}{|\det(\frac{1}{2}z\colon \mathsf{Y}\to\mathsf{X})|^{1/2}} = \pm i^n \frac{e^{\frac{\pi i}{4}\,\mathrm{sgn}\langle z,\,\rangle|_{\mathsf{Y}}}}{|\det(z\colon \mathsf{Y}\to\mathsf{X})|^{1/2}}.$$

Furthermore, by Proposition 4.28,

$$\mathcal{K}(T(\widetilde{-1}))(x',x'') = \pm i^n \delta_0(x'-x'').$$

Hence, the formula for $\omega(\tilde{g}) = \omega(\widetilde{c(z)}(\widetilde{-1}))$ follows. □

5. The Weil representation over a p-adic field

Let $\mathbb{F}$ be a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$. (In fact our argument works for all non Archimedean fields of characteristic other than 2 till the statement (221) below. Hence our additional assumption.)

Let $\chi(r)$, $r \in \mathbb{F}$, be a character of the additive group $\mathbb{F}$ such that the kernel of χ is equal to $\mathfrak{o}_F$. In this Section we provide a construction of the corresponding the Weil representation, [42].

5.1. The Fourier transform

Let U be a finite dimensional vector space over $\mathbb{F}$ and let $\mathcal{L}$ be a lattice in U. We normalize the Haar measure μ_{U} on U so that the volume of the lattice $\mathcal{L}$ is 1. Let $\mathcal{L}* \subseteq \mathrm{U}^*$ be the dual lattice. Denote by μ_{U^*} the corresponding Haar measure.

Let $\mathcal{S}(\mathrm{U})$ be the Schwartz-Bruhat space on U, *i.e.,* the space of complex-valued locally constant functions with compact support on U. (Recall that a function ϕ on U is called locally constant if for each $u \in \mathrm{U}$ there is an open neighborhood $\mathcal{U}$ of u such that ϕ is constant on $\mathcal{U}$.) For $\phi \in \mathcal{S}(\mathrm{U})$ let

$$\mathcal{F}\phi(u^*) = \int_{\mathrm{U}} \phi(u)\chi(-u^*(u))\, d\mu_{\mathrm{U}}(u) \qquad (u^* \in \mathrm{U}^*) \tag{162}$$

be the Fourier transform of ϕ. Then, as is well known, $\mathcal{F}\phi \in \mathcal{S}(\mathrm{U}^*)$ and

$$\phi(u) = \int_{\mathrm{U}^*} \mathcal{F}\phi(u^*)\chi(u^*(u))\, d\mu_{\mathrm{U}^*}(u^*) \qquad (u^* \in \mathrm{U}^*), \tag{163}$$

see [43, Corollary 1, page 107].

As a linear topological space, $\mathcal{S}(\mathrm{U})$ is the inductive limit of the finite dimensional subspaces spanned by the characteristic functions of finite collections of open compact subsets.

Let $\mathcal{S}^*(\mathrm{U})$ denote the linear topological dual of $\mathcal{S}(\mathrm{U})$. It corresponds to the space of the tempered distributions on U in the real case. When convenient we shall identify any bounded locally integrable function $f\colon \mathrm{U} \to \mathbb{C}$ with the tempered distribution $f\mu_{\mathrm{U}}$. In particular, $\mathcal{S}(\mathrm{U}) \subseteq \mathcal{S}^*(\mathrm{U})$. Then the Fourier transform

$$\mathcal{F}\colon \mathcal{S}(\mathrm{U}) \to \mathcal{S}(\mathrm{U}^*)$$

extends to

$$\mathcal{F}\colon \mathcal{S}^*(\mathrm{U}) \to \mathcal{S}^*(\mathrm{U}^*).$$

In fact, if we identify $\mathrm{U}^{**} = \mathrm{U}$ then the Fourier transform (162) is given by

$$\mathcal{F}\psi(u) = \int_{\mathrm{U}^*} \psi(u^*)\chi(-u^*(u))\, d\mu_{\mathrm{U}^*}(u^*) \qquad (\psi \in \mathcal{S}(\mathrm{U}^*),\ u \in \mathrm{U}) \tag{164}$$

and the inverse (163) by

$$\psi(u^*) = \int_{\mathrm{U}} \mathcal{F}\psi(u)\chi(u^*(u))\, d\mu_{\mathrm{U}}(u) \qquad (\psi \in \mathcal{S}(\mathrm{U}^*),\ u^* \in \mathrm{U}^*). \tag{165}$$

Therefore

$$\mathcal{F}(f)(\phi) = f(\mathcal{F}(\phi)) \qquad (f \in \mathcal{S}^*(\mathrm{U}),\ \phi \in \mathcal{S}(\mathrm{U}^*)).$$

Indeed, if $f \in \mathcal{S}(\mathrm{U})$, then

$$\begin{aligned}
\mathcal{F}(f\mu_{\mathrm{U}})(\phi) &= ((\mathcal{F}f)\mu_{\mathrm{U}^*})(\phi)\\
&= \int_{\mathrm{U}^*}\int_{\mathrm{U}} f(u)\chi(-u^*(u))\, d\mu_{\mathrm{U}}(u)\phi(u^*)\, d\mu_{\mathrm{U}^*}(u^*)\\
&= \int_{\mathrm{U}} f(u)\mathcal{F}\phi(u)\, d\mu_{\mathrm{U}}(u).
\end{aligned}$$

Let $\mathrm{V} \subseteq \mathrm{U}$ be a non-zero subspace. Then $\mathcal{L} \cap \mathrm{V}$ is a lattice in V which determines the Haar measure μ_{V}. We may view μ_{V} as a tempered distribution on U by

$$\mu_{\mathrm{V}}(\phi) = \int_{\mathrm{V}} \phi(v)\, d\mu_{\mathrm{V}}(v) \qquad (\phi \in \mathcal{S}(\mathrm{U})).$$

In the case when V is zero, $\mu_{\mathrm{V}} = \mu_0$ is the unit measure at 0. In other words $\mu_0 = \delta_0$ is the Dirac delta at 0,

$$\mu_0(\phi) = \delta_0(\phi) = \phi(0) \qquad (\phi \in \mathcal{S}(\mathrm{U})).$$

Also, for future reference, let $\delta_u \in \mathcal{S}(\mathrm{U})$ be the Dirac delta at $u \in \mathrm{U}$,

$$\delta_u(\phi) = \phi(u) \qquad (\phi \in C(\mathrm{U})).$$

For an arbitrary subspace $\mathrm{V} \subseteq \mathrm{U}$, let $\mathrm{V}^\perp \subseteq \mathrm{U}^*$ be the annihilator of V. Then,

$$\mathcal{F}\mu_{\mathrm{V}} = \mu_{\mathrm{V}^\perp}. \tag{166}$$

Indeed, the formula (164) implies that (166) holds if $\mathrm{V} = \{0\}$.

The quotient space U/V contains the lattice $(\mathcal{L} + \mathrm{V})/\mathrm{V}$, which determines the normalization of the Haar measure $\mu_{\mathrm{U/V}}$. Then for $\phi \in \mathcal{S}(\mathrm{U})$ we have $\tilde{\phi} \in \mathcal{S}(\mathrm{U/V})$ defined by

$$\tilde{\phi}(u + \mathrm{V}) = \int_{\mathrm{V}} \phi(u + v)\, d\mu_{\mathrm{V}}(v).$$

Since (166) holds for the Fourier transform on U/V, with $(\mathrm{U/V})^* = \mathrm{V}^\perp$ and the left hand side being the evaluation of the Fourier transform of a test function at zero, we have, with $\phi = \mathcal{F}\psi$,

$$\begin{aligned}
\mu_{\mathrm{V}}(\mathcal{F}\psi) = \mu_{\mathrm{V}}(\phi) &= \int_{\mathrm{V}} \phi(v)\, d\mu_{\mathrm{V}}(v) = \tilde{\phi}(0) = \int_{\mathrm{V}^\perp} \mathcal{F}\tilde{\phi}(u^*)\, d\mu_{\mathrm{V}^\perp}(u^*) \\
&= \int_{\mathrm{V}^\perp} \int_{\mathrm{U/V}} \tilde{\phi}(u + \mathrm{V})\chi(-u^*(u))\, d\mu_{\mathrm{U/V}}(u + \mathrm{V})\, d\mu_{\mathrm{V}^\perp}(u^*) \\
&= \int_{\mathrm{V}^\perp} \int_{\mathrm{U/V}} \int_{\mathrm{V}} \phi(u + v)\, d\mu_{\mathrm{V}}(v)\chi(-u^*(u))\, d\mu_{\mathrm{U/V}}(u + \mathrm{V})\, d\mu_{\mathrm{V}^\perp}(u^*) \\
&= \int_{\mathrm{V}^\perp} \int_{\mathrm{U}} \phi(u)\chi(-u^*(u))\, d\mu_{\mathrm{U}}(u)\, d\mu_{\mathrm{V}^\perp}(u^*) \\
&= \int_{\mathrm{V}^\perp} \mathcal{F}\phi(u^*)\, d\mu_{\mathrm{V}^\perp}(u^*) \\
&= \mu_{\mathrm{V}^\perp}(\mathcal{F}(\phi)) = \mu_{\mathrm{V}^\perp}(\mathcal{F}^2(\psi)) = \mu_{\mathrm{V}^\perp}(\psi),
\end{aligned}$$

where the last equality follows from the fact that $\mathcal{F}^2\psi(u) = \psi(-u)$, which is a simple consequence of (164) and (163). This completes the proof (166).

Consider two vector spaces U', U'' over $\mathbb{F}$ of the same dimension equipped with lattices $\mathcal{L}'$, $\mathcal{L}''$ respectively. Let $u_1', u_2', \dots, u_n'$ be a $\mathcal{L}'$-orthonormal basis of U' and let $u_1'', u_2'', \dots, u_n''$ be a $\mathcal{L}''$-orthonormal basis of U''. Suppose $L\colon \mathrm{U}' \to \mathrm{U}''$ is a linear bijection. Denote by M the matrix of L with respect to the two ordered basis:

$$Lu_j' = \sum_{i=1}^{n} M_{i,j} u_i'' \qquad (j = 1, 2, \dots, n).$$

Then $|\det(M)|_{\mathbb{F}}$ does not depend on the choice of the orthonormal basis. Thus we may define $|\det(L)|_{\mathbb{F}} = |\det(M)|_{\mathbb{F}}$ (see Section 2.6).

Lemma 5.1.
With the above notation we have

$$\int_{\mathrm{U}'} \phi(L(u'))\, d\mu_{\mathrm{U}'}(u')\, |\det(L)|_{\mathbb{F}} = \int_{\mathrm{U}''} \phi(u'')\, d\mu_{\mathrm{U}''}(u'') \qquad (\phi \in \mathcal{S}(\mathrm{U}'')). \tag{167}$$

Proof. This follows from Lemma 2.18. Indeed, let ϕ be the indicator function of $\mathcal{L}''$. Then the right hand side of the equation (99) is equal to 1. Hence we need to show that

$$\int_{\mathrm{U}'} \phi(L(u'))\, d\mu_{\mathrm{U}'}(u')\, |\det(L)|_{\mathbb{F}} = 1.$$

However, $\phi \circ L$ is the indicator function of $L^{-1}(\mathcal{L}'')$. Thus the problem is to check that

$$\mu_{\mathrm{U}'}(L^{-1}(\mathcal{L}''))\, |\det(L)|_{\mathbb{F}} = 1.$$

Fix an $\mathcal{L}'$-orthonormal basis $u_1',\ u_2', \dots$ of U' and an $\mathcal{L}''$-orthonormal basis $u_1'',\ u_2'', \dots$ of U''. Let T be the endomorphism of U' defined by

$$T(L^{-1}(u_j')) = u_j'' \qquad (j = 1, 2, \dots). \tag{168}$$

Then

$$T(L^{-1}(\mathcal{L}'')) = \mathcal{L}'.$$

Hence, by Lemma 2.18,

$$\mu_{\mathrm{U}'}(L^{-1}(\mathcal{L}''))\, |\det(T)|_{\mathbb{F}} = \mu_{\mathrm{U}'}(T(L^{-1}(\mathcal{L}''))) = \mu_{\mathrm{U}'}(\mathcal{L}')) = 1.$$

But (168) implies that $|\det(T)|_{\mathbb{F}} = |\det(L)|_{\mathbb{F}}$. Hence the claim follows. □

Let X and U be two finite dimensional vector spaces over $\mathbb{F}$ equipped with lattices and the corresponding normalized Haar measures μ_X and μ_{U}. Let $L\colon X \to \mathrm{U}$ be a surjective linear map. Suppose f is a bounded function on U so that $f\mu_{\mathrm{U}} \in \mathcal{S}^*(\mathrm{U})$. Define $L^*(f\mu_{\mathrm{U}}) := (f \circ L)\mu_{\mathrm{X}}$. Thus for a test function $\phi \in \mathcal{S}(\mathrm{U})$,

$$L^*(f\mu_{\mathrm{U}})(\phi) = \int_X f(L(x))\phi(x)\, d\mu(x). \tag{169}$$

Choose a subspace $X' \subseteq X$ complementary to $\mathrm{Ker}(L)$ so that $X = \mathrm{Ker}(L) \oplus X'$. Let $\mu_{\mathrm{Ker}(L)}$ and $\mu_{X'}$ denote the corresponding normalized Haar measures on $\mathrm{Ker}(L)$ and X' respectively. Then (169) may be rewritten as

$$\int_{X'} \int_{\mathrm{Ker}(L)} f(L(x' + x''))\phi(x' + x'')\, d\mu_{\mathrm{Ker}(L)}(x'')\, d\mu_{X'}(x'). \tag{170}$$

Let L' denote the restriction of L to X'. Then $L' : X' \to \mathrm{U}$ is a bijection and Lemma 5.1 shows that (170) may be rewritten as

$$\int_{\mathrm{U}} \int_{\mathrm{Ker}(L)} f(u) L_*(\phi)(u)\, d\mu_{\mathrm{U}}(u). \tag{171}$$

where

$$L_*(\phi)(u) = \int_{\mathrm{Ker}(L)} \phi(L'^{-1}(u) + x'')\, d\mu_{\mathrm{Ker}(L)}(x'')\, |\det(L')|_{\mathbb{F}}^{-1}. \tag{172}$$

Notice that $L_* : \mathcal{S}(X) \to \mathcal{S}(\mathrm{U})$ is a continuous map. Hence we have the notion of a pullback of a distribution

$$L^*(f)(\phi) = f(L_*(\phi)) \qquad (\phi \in \mathcal{S}(X),\ f \in \mathcal{S}^*(\mathrm{U})) \tag{173}$$

which is consistent with [13, Theorem 6.1.2].

Lemma 5.2.
Let X *and* U *are two finite dimensional vector space over* $\mathbb{F}$ *equipped with lattices and the corresponding normalized Haar measures* μ_X *and* μ_{U}*. Let* $L : X \to \mathsf{U}$ *be a surjective linear map. Let*

$$\tilde{L} : \mathsf{X}/L^{-1}(\mathsf{V}) \to \mathsf{U}/\mathsf{V}$$

be the induced bijection. Then

$$L^*(\mu_{\mathsf{V}}) = |\det(\tilde{L})|_{\mathbb{F}}^{-1} \mu_{L^{-1}(\mathsf{V})}.$$

Proof. Let $\mathsf{X}' \subseteq \mathsf{X}$ be the orthogonal complement of $\mathrm{Ker}(L)$. Denote by L' the restriction of L to X' and by L'' the restriction of L to $\mathsf{X}' \cap L^{-1}(\mathsf{V})$. Then

$$L' : \mathsf{X}' \to \mathsf{U} \text{ and } L'' : \mathsf{X}' \cap L^{-1}(\mathsf{V}) \to \mathsf{V}$$

are bijections.

According to (173), for a test function $\phi \in \mathcal{S}(X)$ we have

$$L^*(\mu_{\mathsf{V}})(\phi) = \int_{\mathrm{Ker}(L)} \int_{\mathsf{V}} \phi(x + L'^{-1}(v))\, d\mu_{\mathsf{V}}(v)\, d\mu_{Ker(L)}(x)\, |\det(L')|_{\mathbb{F}}^{-1}. \tag{174}$$

Then the right hand side of (174) is equal to

$$\int_{\mathrm{Ker}(L)} \int_{L''^{-1}(\mathsf{V})} \phi(x+y)\, d\mu_{L''^{-1}(\mathsf{V})}(y)\, d\mu_{\mathrm{Ker}(L)}(x)\, |\det(L'')|_{\mathbb{F}}\, |\det(L')|_{\mathbb{F}}^{-1} = \int_{L^{-1}(\mathsf{V})} \phi(z)\, d\mu_{L^{-1}(\mathsf{V})}(z)\, |\det(L'')|_{\mathbb{F}}\, |\det(L')|_{\mathbb{F}}^{-1}.$$

Since $|\det(L'')|_{\mathbb{F}}^{-1}\, |\det(L')|_{\mathbb{F}} = |\det(\tilde{L})|_{\mathbb{F}}$, we are done. $\square$

5.2. Gaussians on $\mathbb{F}^n$

Let B be the usual dot product on $\mathbb{F}^n$,

$$B(x,y) = x^t y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \qquad (x, y \in \mathbb{F}^n).$$

Then the Haar measure associated to the lattice $\mathfrak{o}_{\mathbb{F}}^n \subseteq \mathbb{F}^n$, $d\mu_{\mathbb{F}^n}(x) = dx_1 dx_2 \dots dx_n$, is the n-fold direct product of Lebesgue measure dx_i on $\mathbb{F}$, such that $\int_{\mathfrak{o}_{\mathbb{F}}} dx_i = 1$.

For a symmetric matrix $A \in \mathrm{GL}(\mathbb{F}^n)$ define the corresponding Gaussian γ_A by

$$\gamma_A(x) := \chi(\frac{1}{2} x^t A x) \qquad (x \in \mathbb{F}^n).$$

Also, let

$$\gamma(A) = \mathcal{F}\gamma_A(0) = \int_{F^n} \chi(\frac{1}{2} x^t A x)\, dx.$$

In particular, taking $n = 1$, we have

$$\gamma(a) = \int_F \chi(\frac{1}{2} x^t A x)\, dx, \qquad \text{for } a \in \mathbb{F}^\times.$$

Let γ_W be the gamma factor defined by Weil in [42, n°14 cor. 2]. It is related to γ by the equality

$$\gamma(A) \;=\; |\det(A)|_{\mathbb{F}}^{-1/2}\, \gamma_W(A). \tag{175}$$

We set

$$\gamma_W(q) := \gamma_W(Q),$$

if q is a quadratic form with associated symmetric matrix Q as in Eq. (12). Then γ_W defines a unitary character of the Witt group of $\mathbb{F}$. The scalar $\gamma_W(a)$ is the gamma factor attached to the quadratic form $x \mapsto ax^2$ ($a \in \mathbb{F}^\times$). It depends only on the class of a modulo $(\mathbb{F}^\times)^2$. In particular, we have

$$\gamma_W(a^2) = \gamma_W(1) \quad \text{for all } a \in \mathbb{F}^\times. \tag{176}$$

Of course Eqn. (176) would not be true with γ instead of γ_W: we get $\gamma(a^2) = |a|_{\mathbb{F}}^{-1}\gamma_W(1)$. Note that $\gamma_W(1)$ and $\gamma(1)$ are equal.

Recall the *Hilbert symbol* (,): for any $a, b \in \mathbb{F}^\times$,

$$(a,b) := \begin{cases} 1 & \text{if } z^2 = ax^2 + bx^2 \text{ has a non-zero solution } (x,y,z) \in \mathbb{F}^3, \\ -1 & \text{otherwise.} \end{cases}$$

It is related to the above γ factor as follows:

Proposition 5.3.

For any $a, b \in \mathbb{F}^\times$, we have

$$(a,b) = \frac{\gamma(ab)\,\gamma(1)}{\gamma(a)\gamma(b)}. \tag{177}$$

Proof. It follows from [42, n°25 prop. 3 and n°28 prop. 4] that

$$(a,b) = \frac{\gamma_W(ab)\,\gamma_W(1)}{\gamma_W(a)\gamma_W(b)}.$$

Then the equality (177) is an immediate consequence of the equality $\gamma(a) = \gamma_W(a)\,|a|^{-1/2}$. □

Corollary 5.4.

The function

$$a \mapsto \mathfrak{s}(a) := |a|_{\mathbb{F}}\,\frac{\gamma(a)^2}{\gamma(1)^2}$$

is a character of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$.

Remark 5.5.

The function $a \mapsto \frac{\gamma(a)^2}{\gamma(1)^2}$ is a character of $\mathbb{F}^\times$. However it does not have trivial restriction to $(K^\times)^2$.

Remark 5.6.

The character $\mathfrak{s}$ will play a similar role to that of the character s which was defined in Lemma 3.2 in the case of finite fields, and of $a \mapsto \dfrac{|a|}{a}$ in the case of $\mathbb{R}$.

In these terms we have the following theorem due to Weil.

Theorem 5.7.

For any symmetric matrix $A \in \mathrm{GL}(\mathbb{F}^n)$,

$$\mathcal{F}\gamma_A = \gamma(A)\,\gamma_{-A^{-1}}, \tag{178}$$

and

$$\gamma(A) = \pm\gamma(1)^{n-1}\gamma(\det(A)). \tag{179}$$

Proof. We have (see [42, Chap. I Théorème 2 and Chap. II § 26])

$$\gamma_W(A) = \pm\gamma(1)^{n-1}\,\gamma_W(\det A). \tag{180}$$

Hence, from Eqn. (175) we obtain

$$\gamma(A) = \pm\gamma(1)^{n-1}\,\gamma(\det A). \tag{181}$$

Then the first equation in the statement of the theorem follows from [42, Eqn. (17) and Théorème 2, I. § 14] applied to the character of second degree $x \mapsto \gamma_A(x)$. □

5.3. Gaussians on a vector space

Let U be a finite dimensional vector space over $\mathbb{F}$ with a lattice $\mathcal{L} \subseteq \mathsf{U}$. Suppose q is a non-degenerate symmetric bilinear form on U. Let $\gamma(q) = \gamma(Q)$, where Q is the matrix obtained from any $N_{\mathcal{L}}$-orthonormal basis $u_1, u_2, \ldots, u_n$ of U by

$$Q_{i,j} = q(u_i, u_j) \qquad (1 \le i, j \le n).$$

Also, we define $\gamma(0) = 1$.

Lemma 5.8.
If q is a non-degenerate symmetric bilinear form on U, then

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)) \qquad (u^* \in \mathsf{U}^*).$$

Proof. Fix a $N_{\mathcal{L}}$-orthonormal basis $u_1, u_2, \ldots, u_n$ of U and let $u_1^*, u_2^*, \ldots, u_n^*$ be the dual basis of U^*. This is a $N_{\mathcal{L}_*}$-orthonormal basis. As we have seen in the proof of Lemma 3.4, if Q is the matrix corresponding to q, as above, then Q^{-1} corresponds to q^*.

Let $x_i = u_i^*(u)$ and let $y_j = u^*(u_j)$. Then

$$\int_{\mathsf{U}} \chi(\frac{1}{2}q(u,u))\chi(-u^*(u))\,d\mu_{\mathsf{U}}(u) = \int_{\mathbb{F}^n} \chi(\frac{1}{2}x^tQx)\chi(-x^ty)\,dx = \gamma(Q)\chi(-\frac{1}{2}y^tQ^{-1}y) = \gamma(q)\chi(-\frac{1}{2}q^*(u^*,u^*)),$$

where the second equality follows from Theorem 5.7. □

5.4. Gaussians on a symplectic space

Let W be a finite dimensional vector space over $\mathbb{F}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$. We shall identify W with the dual W^* by

$$w^*(w) = \langle w, w^*\rangle \qquad (w, w^* \in \mathsf{W}). \tag{182}$$

The identification (182) provides to the following isomorphisms

$$\mathsf{U}^* = \mathsf{W}/\mathsf{U}^\perp \text{ and } (\mathsf{U}/\mathsf{V})^* = \mathsf{V}^\perp/\mathsf{U}^\perp, \tag{183}$$

where the orthogonal complements are taken in W, with respect to the symplectic form $\langle\ ,\ \rangle$. Let $\{e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}\}$ be a symplectic basis of W, that is:

$$\langle e_i, e_j\rangle = \langle e_{-i}, e_{-j}\rangle = 0 \quad \text{and} \quad \langle e_i, e_{-j}\rangle = \delta_{ij}, \quad \text{for all } 1 \le i, j \le n.$$

Let $\mathcal{L} := \sum_{j=-n}^{n} \mathfrak{o}_{\mathbb{F}} e_j$. Then $\mathcal{L}$ is a self-dual lattice in W, *i.e.,*

$$\{w \in \mathrm{W};\ \langle u, w\rangle \in \mathfrak{o}_{\mathbb{F}} \text{ for all } u \in \mathcal{L}\} = \mathcal{L}.$$

Moreover,

$$\{\langle w_1, w_2\rangle;\ w_1, w_2 \in \mathcal{L}\} = \mathfrak{o}_{\mathbb{F}}.$$

As explained in Section 5.1, $\mathcal{L}$ leads to a normalization of the Haar measures on any subspace of $\mathrm{U} \subseteq \mathrm{W}$ and on any quotient U/V, where V is a subspace of U.

Lemma 5.9.

Suppose $x \in \mathrm{Hom}(\mathrm{U}, \mathrm{W}/\mathrm{U}^\perp)$ *is such that*

$$\langle xu, v\rangle = \langle xv, u\rangle \qquad (u, v \in \mathrm{U}).$$

Set

$$q(u, v) = \frac{1}{2}\langle xu, v\rangle \qquad (u, v \in \mathrm{U}).$$

Let V *be the radical of* q *and let* $\tilde{q}$ *be the induced non-degenerate form on* U/V. *Then*

(a) $\mathrm{V} = \mathrm{Ker}(x)$;

(b) *The element* x *determines a bijection*

$$\underline{x} : \mathrm{U}/\mathrm{V} \to \mathrm{V}^\perp/\mathrm{U}^\perp,$$

with the inverse

$$\underline{x}^{-1} : \mathrm{V}^\perp/\mathrm{U}^\perp \to \mathrm{U}/\mathrm{V};$$

(c) *Let* $x^{-1} : \mathrm{V}^\perp \to \mathrm{U}/\mathrm{V}$ *be the composition of* $\underline{x}^{-1}$ *with the quotient map* $\mathrm{V}^\perp \to \mathrm{V}^\perp/\mathrm{U}^\perp$. *Define*

$$\chi_x(u) = \chi(\frac{1}{4}\langle xu, u\rangle) \qquad (u \in \mathrm{U}),$$
$$\chi_{x^{-1}}(w) = \chi(\frac{1}{4}\langle x^{-1}w, w\rangle) \qquad (w \in \mathrm{V}^\perp).$$

Then, for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned}
&\int_{\mathrm{U}}\int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathrm{U}}(u) \qquad (184)\\
&= 2^{\dim(\mathrm{V})}\gamma(\tilde{q})\int_{\mathrm{V}^\perp} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathrm{V}^\perp}(w)\\
&= 2^{\dim(\mathrm{V})}\gamma(\tilde{q})\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \chi_{x^{-1}}(w+\mathrm{U}^\perp)\int_{\mathrm{U}^\perp} \phi(w+v)\, d\mu_{\mathrm{U}^\perp}(v)\, d\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp}(w+\mathrm{U}^\perp).
\end{aligned}$$

Also, for any $\phi \in \mathcal{S}(\mathrm{W}/\mathrm{U}^\perp)$,

$$\begin{aligned}
&\int_{\mathrm{U}}\int_{\mathrm{W}/\mathrm{U}^\perp} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{U}}(u) \qquad (185)\\
&= 2^{\dim(\mathrm{V})}\gamma(\tilde{q})\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \chi_{\underline{x}^{-1}}(w)\phi(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp}(w+\mathrm{U}^\perp).
\end{aligned}$$

Proof. Part (a) is obvious. Part (b) means that $\mathrm{Ker}(x)^\perp = \mathrm{Im}(x)$, which is true. For $\phi \in \mathcal{S}(\mathrm{W})$ we have,

$$\begin{aligned}\int_{\mathrm{U}}\int_{\mathrm{W}} \chi_x(u)\chi(-\frac{1}{2}\langle u, w\rangle)\phi(w)\, d\mu_{\mathrm{W}}(w)\, d\mu_{\mathrm{U}}(u) &= \int_{\mathrm{W}} \mathcal{F}(\gamma_q\mu_{\mathrm{U}})(\frac{1}{2}w)\phi(w)\, d\mu_{\mathrm{W}}(w)\\ &= \int_{\mathrm{W}} \mathcal{F}(\gamma_q\mu_{\mathrm{U}})(w)\phi(2w)\, d\mu_{\mathrm{W}}(w)\, 2^{\dim \mathrm{W}}\\ &= \gamma(\tilde{q})\int_{\mathrm{V}^\perp} \gamma_{-\tilde{q}^*}(w)\phi(2w)\, d\mu_{\mathrm{V}^\perp}(w)\, 2^{\dim \mathrm{W}}\\ &= \gamma(\tilde{q})\int_{\mathrm{V}^\perp} \gamma_{-\tilde{q}^*}(\frac{1}{2}w)\phi(w)\, d\mu_{\mathrm{V}^\perp}(w)\, 2^{\dim \mathrm{W}-\dim \mathrm{V}^\perp}\\ &= \gamma(\tilde{q})\int_{\mathrm{V}^\perp} \chi_{x^{-1}}(w)\phi(w)\, d\mu_{\mathrm{V}^\perp}(w)\, 2^{\dim \mathrm{V}}.\end{aligned}$$

This verifies (184). For $\phi \in \mathcal{S}(\mathrm{W}/\mathrm{U}^\perp)$ we have,

$$\begin{aligned}&\int_{\mathrm{U}}\int_{\mathrm{W}/\mathrm{U}^\perp} \chi_x(u)\chi(\frac{1}{2}\langle u, w\rangle)\phi(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{U}}(u)\\ &= \int_{\mathrm{U}/\mathrm{V}}\int_{\mathrm{V}}\int_{\mathrm{W}/\mathrm{U}^\perp} \chi_{\underline{x}}(u+\mathrm{V})\chi(\frac{1}{2}\langle u+v, w\rangle)\phi(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{V}}(v)\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\ &= \int_{\mathrm{U}/\mathrm{V}}\int_{\mathrm{V}}\int_{\mathrm{W}/\mathrm{U}^\perp} \gamma_{\tilde{q}}(u+\mathrm{V})\chi(\langle u+v, w\rangle)\phi(2w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{V}}(v)\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\\ &\quad 2^{\dim \mathrm{W}/\mathrm{U}^\perp}\\ &= \int_{\mathrm{U}/\mathrm{V}}\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \gamma_{\tilde{q}}(u+\mathrm{V})\chi(\langle u, w\rangle)\phi(2w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{U}/\mathrm{V}}(u+\mathrm{V})\, 2^{\dim \mathrm{W}/\mathrm{U}^\perp}\\ &= \gamma(\tilde{q})\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \gamma_{-\tilde{q}^*}(w+\mathrm{U}^\perp)\phi(2w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, 2^{\dim \mathrm{W}/\mathrm{U}^\perp}\\ &= \gamma(\tilde{q})\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \gamma_{-\tilde{q}^*}(\frac{1}{2}w+\mathrm{U}^\perp)\phi(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, 2^{\dim \mathrm{W}/\mathrm{U}^\perp-\dim \mathrm{V}^\perp/\mathrm{U}^\perp}\\ &= \gamma(\tilde{q})\int_{\mathrm{V}^\perp/\mathrm{U}^\perp} \chi_{\underline{x}^{-1}}(w+\mathrm{U}^\perp)\phi(w+\mathrm{U}^\perp)\, d\mu_{\mathrm{W}/\mathrm{U}^\perp}(w+\mathrm{U}^\perp)\, 2^{\dim \mathrm{V}}.\end{aligned}$$

This verifies (185). □

By a Gaussian on the symplectic space W we shall understand any non-zero constant multiple of the tempered distribution

$$\chi_x\mu_{\mathrm{U}} \in \mathcal{S}^*(\mathrm{W}) \tag{186}$$

where the function χ_x is defined in Lemma 5.9. In these terms Lemma 5.9 says that the Fourier transform of a Gaussian is another Gaussian.

5.5. Twisted convolution of Gaussians

Recall the twisted convolution of two Schwartz functions $\psi, \phi \in \mathcal{S}(\mathrm{W})$:

$$\psi\natural\phi(w) = \int_{\mathrm{W}} \psi(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{187}$$

It is easy to see that the above integral converges and that $\psi\natural\phi \in \mathcal{S}(\mathrm{W})$. Also, the twisted convolutions

$$\delta_{w_0}\natural\phi(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w_0, w\rangle) \text{ and } \phi\natural\delta_{w_0}(w) = \phi(w-w_0)\chi(\frac{1}{2}\langle w, w_0\rangle) \tag{188}$$

are well defined for any continuous function ϕ.

Let

$$t(g) = \chi_{c(g)}\mu_{g^-\mathrm{W}}. \tag{189}$$

For any $\phi \in \mathcal{S}(\mathrm{W})$, the twisted convolution $t(g)\natural\phi$ is a continuous function given by the following absolutely convergent integral

$$t(g)\natural\phi(w) = \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u) \qquad (w \in \mathrm{W}). \tag{190}$$

Lemma 5.10.
For any $g \in \mathrm{Sp}(\mathrm{W})$,

$$t(g)\natural(\delta_{w_0}\natural\phi) = \delta_{gw_0}\natural(t(g)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ \ w_0 \in \mathrm{W}).$$

Proof. The left hand side evaluated at $w \in \mathrm{W}$ is equal to

$$\begin{aligned}
&\int_{g^-\mathrm{W}} \chi_{c(g)}(u)(\delta_{w_0}\natural\phi)(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
&= \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-u-w_0)\chi(\frac{1}{2}\langle w_0, w-u\rangle)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_{g^-\mathrm{W}}(u)\\
&= \int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)u, u\rangle + 2\langle w_0, w-u\rangle + 2\langle u, w\rangle))\, d\mu_{g^-\mathrm{W}}(u)
\end{aligned}$$

and the right hand side is equal to

$$\begin{aligned}
(t(g)\natural\phi)(w-gw_0)\chi(\frac{1}{2}\langle gw_0, w\rangle) &= \int_{g^-\mathrm{W}} \chi_{c(g)}(u)\phi(w-gw_0-u)\chi(\frac{1}{2}\langle u, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
&= \int_{g^-\mathrm{W}} \chi_{c(g)}(u-g^-w_0)\phi(w-gw_0-(u-g^-w_0))\\
&\quad \chi(\frac{1}{2}\langle u-g^-w_0, w-gw_0\rangle)\, d\mu_{g^-\mathrm{W}}(u)\chi(\frac{1}{2}\langle gw_0, w\rangle)\\
&= \int_{g^-\mathrm{W}} \phi(w-u-w_0)\chi(\frac{1}{4}(\langle c(g)(u-g^-w_0), u-g^-w_0\rangle\\
&\quad +2\langle u-g^-w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle))\, d\mu_{g^-\mathrm{W}}(u).
\end{aligned}$$

A straightforward computation shows that

$$\langle c(g)(u-g^-w_0), u-g^-w_0\rangle + 2\langle u-g^-w_0, w-gw_0\rangle + 2\langle gw_0, w\rangle - (\langle c(g)u, u\rangle + 2\langle w_0, w-u\rangle + 2\langle u, w\rangle) = 0.$$

Hence, the two sides are equal. □

Lemma 5.11.
Fix an element $g \in \mathrm{Sp}(\mathrm{W})$. Let $\mathrm{U} = g^-\mathrm{W}$. The map

$$\mathrm{U} \ni u \to \langle\ , (1-c(g))u\rangle \in \mathrm{U}^* = \mathrm{W}/\mathrm{U}^\perp = \mathrm{W}/\mathrm{Ker}(g^-) \tag{191}$$

is bijective.

Fix a complement Z of U in W so that

$$\mathrm{W} = \mathrm{U} \oplus \mathrm{Z}.$$

We shall denote the elements of U *by* u *and elements of* Z *by* z*. In particular every* $w \in \mathsf{W}$ *has a unique decomposition*

$$w = u + z.$$

Then, for any $\phi \in \mathcal{S}(\mathsf{W})$ *and any* $w' = u' + z' \in \mathsf{W}$,

$$t(g)\natural\phi(w') = \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle) \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(u+z')\chi(-\frac{1}{2}\langle u, (1-c(g))u' + z'\rangle)\, d\mu_{\mathsf{U}}(u).$$

In particular, (192) and (191) imply that $t(g)\natural\phi \in \mathcal{S}(\mathsf{W})$ *and that the map*

$$\mathcal{S}(\mathsf{W}) \ni \phi \to t(g)\natural\phi \in \mathcal{S}(\mathsf{W})$$

is continuous.

Proof. Suppose $\langle\ , (1-c(g))u\rangle = 0$. Then $(1-c(g))u \in \operatorname{Ker} g^-$. There is $u_0 \in \mathsf{W}$ such that $u = g^- u_0$. Therefore

$$\begin{aligned} 0 = g^-(1-c(g))u = g^-(1-c(g))g^-u_0 &= g^-(g^-)u_0 - g^-g^+u_0 = g^-(g^-)u_0 - g^+g^-u_0 \\ &= (g^- - g^-)g^-u_0 = -2g^-u_0 = -2u. \end{aligned}$$

This verifies (191).

The left hand side of (192) is equal to

$$\begin{aligned} t(g)\natural\phi(w') &= \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(w'-u)\chi(\frac{1}{2}\langle u, w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ &= \int_{\mathsf{U}} \chi_{c(g)}(u+u')\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ &= \int_{\mathsf{U}} \chi_{c(g)}(u')\chi_{c(g)}(u)\chi(\frac{1}{2}\langle c(g)u', u\rangle)\phi(z'-u)\chi(\frac{1}{2}\langle u+u', w'\rangle)\, d\mu_{\mathsf{U}}(u) \\ &= \chi_{c(g)}(u')\chi(\frac{1}{2}\langle u', w'\rangle) \int_{\mathsf{U}} \chi_{c(g)}(u)\phi(z'-u)\chi(\frac{1}{2}\langle u, w' - c(g)u'\rangle)\, d\mu_{\mathsf{U}}(u), \end{aligned}$$

which coincides with the right hand side. □

In particular Lemma 5.11 shows that for any two elements $g_1, g_2 \in \mathrm{Sp}(\mathsf{W})$ there is a tempered distribution $t(g_1)\natural t(g_2) \in \mathcal{S}^*(\mathsf{W})$ such that

$$(t(g_1)\natural t(g_2))\natural\phi = t(g_1)\natural(t(g_2)\natural\phi) \qquad (\phi \in \mathcal{S}(\mathsf{W})). \tag{192}$$

Proposition 5.12.

Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathsf{W})$*. Let* $\mathsf{U}_1' \subseteq \mathsf{U}_1$ *be the orthogonal complement of* U *with respect to the positive definite form* B*, so that*

$$\mathsf{U}_1 = \mathsf{U}_1' \oplus \mathsf{U}.$$

Then the map

$$L : \mathsf{U}_1' + \mathsf{U}_2 \ni u_1' + u_2 \to c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 + \mathsf{U}^\perp \in \mathsf{W}/\mathsf{U}^\perp$$

is well defined, surjective and $L^{-1}(\mathsf{V}^\perp/\mathsf{U}^\perp) = \mathsf{U}_{12}$*. Denote by*

$$\tilde{L} : (\mathsf{U}_1+\mathsf{U}_2)/\mathsf{U}_{12} \ni u_1 + u_2 + \mathsf{U}_{12} \to c(g_1)u_1 - c(g_2)u_2 - u_1 - u_2 + \mathsf{V}^\perp \in \mathsf{W}/\mathsf{V}^\perp = (\mathsf{W}/\mathsf{U}^\perp)/(\mathsf{V}^\perp/\mathsf{U}^\perp)$$

the induced bijection and set

$$C(g_1, g_2) = \gamma(\tilde{q}_{g_1,g_2})2^{\dim \mathsf{V}}|\det(\tilde{L})|_{\mathbb{F}}^{-1}.$$

Then C *is a cocycle, with* $C(1,1) = 1$*, and*

$$t(g_1)\natural t(g_2) = C(g_1, g_2)t(g_1g_2). \tag{193}$$

Proof. Since $\mathrm{V}^\perp/\mathrm{U}^\perp = (c(g_1)+c(g_2))\mathrm{U}$, the map $\tilde{L}$ is well defined. Suppose $u_1' \in \mathrm{U}_1'$ and $u_2 \in \mathrm{U}_2$ are such that $L(u_1'+u_2) \in \mathrm{V}^\perp/\mathrm{U}^\perp$. Then there is $u \in \mathrm{U}$ such that

$$(c(g_1)+c(g_2))u + c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2 \in \mathrm{U}^\perp.$$

Let

$$u = g_1^- v_1 = g_2^- v_2,\ \ v = u_1' = g_1^- w_1,\ \ w - v = u_2 = g_2^- w_2.$$

Then

$$(c(g_1)+c(g_2))u + c(g_1)v + c(g_2)(v-w) - w \in \mathrm{U}^\perp.$$

Hence, the computation (79) - (81) shows that $w = (g_1g_2)^-(w_2 - v_2) \in \mathrm{U}_{12}$. Therefore $L^{-1}(\mathrm{V}^\perp/\mathrm{U}^\perp) \subseteq \mathrm{U}_{12}$. But (191) implies that L is surjective and Lemma 2.7 (b) shows that $\dim((\mathrm{U}_1+\mathrm{U}_2)/\mathrm{U}_{12}) = \dim((\mathrm{W}/\mathrm{U}^\perp)/(\mathrm{V}^\perp/\mathrm{U}^\perp))$. Thus $L^{-1}(\mathrm{V}^\perp/\mathrm{U}^\perp) = \mathrm{U}_{12}$. The computation (81) - (85) shows that, if $u_1'+u_2 \in \mathrm{U}_{12}$ then

$$\langle c(g_1)u_1', u_1'\rangle + \langle c(g_2)u_2, u_2\rangle + 2\langle u_1', u_2\rangle + \langle (c(g_1)+c(g_2))^{-1}L(u_1'+u_2), L(u_1'+u_2)\rangle = \langle c(g_1g_2)(u_1'+u_2), u_1+u_2\rangle$$

so that

$$\chi_{c(g_1)}(u_1')\chi_{c(g_2)}(u_2)\chi(\frac{1}{2}\langle u_1', u_2\rangle)\chi_{(c(g_1)+c(g_2))^{-1}}(L(u_1'+u_2)) = \chi_{c(g_1g_2)}(u_1'+u_2). \tag{194}$$

Any $u_1 \in \mathrm{U}_1$ has a unique decomposition $u_1 = u_1' + u$, where $u_1' \in \mathrm{U}_1'$ and $u \in \mathrm{U}$. With this notation, Lemma 5.11 shows that for any $\phi \in \mathcal{S}(\mathrm{W})$,

$$\begin{aligned} t(g_1)\natural(t(g_2)\natural\phi)(0) &= \int_{\mathrm{U}_1} \chi_{c(g_1)}(u_1)t(g_2)\natural\phi(u_1)\,d\mu_{\mathrm{U}_1}(u_1) \\ &= \int_{\mathrm{U}_1}\int_{\mathrm{U}_2} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2)-1)u\rangle) \\ &\quad \chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2, u_1'\rangle)\phi(u_2+u_1')\,d\mu_{\mathrm{U}_2}(u_2)\,d\mu_{\mathrm{U}_1}(u_1) \\ &= \int_{\mathrm{U}}\int_{\mathrm{U}_1'}\int_{\mathrm{U}_2} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2)-1)u\rangle) \\ &\quad \chi_{c(g_2)}(u_2)\chi(-\frac{1}{2}\langle u_2, u_1'\rangle)\phi(u_2+u_1')\,d\mu_{\mathrm{U}_2}(u_2)\,d\mu_{\mathrm{U}_1'}(u_1')\,d\mu_{\mathrm{U}}(u) \end{aligned} \tag{195}$$

The formula (185) applied with $x = c(g_1)+c(g_2)$ shows that

$$\begin{aligned} &\int_{\mathrm{U}} \chi_{c(g_1)}(u_1)\chi_{c(g_2)}(u)\chi(\frac{1}{2}\langle u, u_1'\rangle)\chi(\frac{1}{2}\langle u_2, (c(g_2)-1)u\rangle)\,d\mu_{\mathrm{U}}(u) \\ &= \chi_{c(g_1)}(u_1')\int_{\mathrm{U}} \chi_{c(g_1)+c(g_2)}(u)\chi(\frac{1}{2}\langle u, c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2\rangle)\,d\mu_{\mathrm{U}}(u) \\ &= 2^{\dim \mathrm{V}}\gamma(\tilde{q}_{g_1,g_2})\chi_{c(g_1)}(u_1')(\chi_{\underline{(c(g_1)+c(g_2))}^{-1}}\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp})(c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2). \end{aligned} \tag{196}$$

Furthermore, Lemma 5.2 shows that, for $u_1'+u_2 \in \mathrm{U}_{12}$,

$$\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp}(c(g_1)u_1' - c(g_2)u_2 - u_1' - u_2) = L^*(\mu_{\mathrm{V}^\perp/\mathrm{U}^\perp})(u_1'+u_2) = |\det(\tilde{L})|^{-1}\mu_{\mathrm{U}_{12}}(u_1'+u_2).$$

The formula (193) follows directly from (194) - (197).

We see from (188) that

$$\begin{aligned} &t(g_1)\natural(t(g_2)\natural\phi)(w) = (t(g_1)\natural(t(g_2)\natural\phi))\natural\delta_{-w}(0) = (t(g_1)\natural(t(g_2)\natural(\phi\natural\delta_{-w})))(0) \\ &= ((t(g_1)\natural t(g_2))\natural(\phi\natural\delta_{-w}))(0) = ((t(g_1)\natural t(g_2))\natural\phi)\natural\delta_{-w})(0) = (t(g_1)\natural t(g_2))\natural\phi(w). \end{aligned}$$

Therefore

$$(t(g_1)\natural t(g_2))\natural\phi = t(g_1)\natural(t(g_2)\natural\phi).$$

Hence, $t(g_1)\natural t(g_2)$ coincides with the composition of $t(g_1)$ and $t(g_2)$ as elements of the associative algebra $\mathrm{End}(\mathcal{S}(\mathrm{W}))$. Therefore the function C is a cocycle. □

5.6. Normalization of Gaussians and the metaplectic group

For an element $h \in \mathrm{End}(\mathrm{W})$ define $h^{\#} \in \mathrm{End}(\mathrm{W})$ by

$$\langle hw, w' \rangle = \langle w, h^{\#} w' \rangle \qquad (w, w' \in \mathrm{W}). \tag{197}$$

Then $(\mathrm{Ker}\, h^{\#})^{\perp} = h\mathrm{W}$.

Lemma 5.13.
Fix two elements $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$ and assume that $K_1 = \mathrm{Ker}\, g_1^- = 0$. Then

$$2^{-\dim \mathrm{V}} \det(\tilde{L}) = \det(g_2^- : K_{12} \to \mathrm{V})^{-1}.$$

Proof. Since, by Lemma 2.7 (c), $g_2^- K_{12} = \mathrm{V}$, the right hand side of the equation we need to prove makes sense. Also,

$$2^{-\dim \mathrm{V}} \det(\tilde{L}) = \det(\frac{1}{2}\tilde{L})$$

and a straightforward computation shows that

$$\frac{1}{2}\tilde{L} \colon \mathrm{W}/\mathrm{U}_{12} \ni w + \mathrm{U}_{12} \to \frac{1}{2}(c(g_1) - 1)w + \mathrm{V}^{\perp} = g_1^{-1} w + \mathrm{V}^{\perp} \in \mathrm{W}/\mathrm{V}^{\perp}.$$

Hence,

$$\det(\frac{1}{2}\tilde{L})^{-1} = \det(g_1^- : \mathrm{W}/\mathrm{V}^{\perp} \to \mathrm{W}/\mathrm{U}_{12}).$$

Notice that $g_1^{-1} - 1 = g_1^{\#}$. Since $\mathrm{V} = g_2^- K_{12}$ and $\mathrm{U}_{12} = K_{12}^{\perp}$, Lemma 2.22 shows that

$$\det(g_1^- : \mathrm{W}/\mathrm{V}^{\perp} \to \mathrm{W}/\mathrm{U}_{12}) = \det(g_1^{-1} - 1 \colon K_{12} \to \mathrm{V}).$$

Since the restrictions of g_1^{-1} and g_2 to K_{12} are equal, we are done. □

Let B be a non-degenerate (not necessarily symmetric) bilinear form on a finite dimensional vector space over $\mathbb{F}$. Define the discriminant of B as

$$\mathrm{dis}(B) = \frac{\gamma_W(\det(A))}{\gamma(1)}, \tag{198}$$

where A is the matrix obtained from a basis $u_1, u_2, \dots, u_n$ of the space by

$$A_{i,j} = B(u_i, u_j) \qquad (1 \le i, j \le n).$$

Clearly the discriminant does not depend on the choice of the basis.

We have

$$\mathrm{dis}(B)^2 = \mathfrak{s}(\det(A)). \tag{199}$$

For any $g \in \mathrm{Sp}(\mathrm{W})$ the formula

$$\langle g^- w, w' \rangle \qquad (w, w' \in \mathrm{W})$$

defines a bilinear form whose left and right radicals coincide with $\mathrm{Ker}(g^-)$. Hence we get a non-degenerate bilinear form B_g on the quotient $\mathrm{W}/\mathrm{Ker}(g^-)$. Then

$$\mathrm{dis}(B_g) = \frac{\gamma_W(\det(\langle g^- w_i, w_j \rangle_{1 \le i,j \le r}))}{\gamma(1)},$$

where $w_1 + \mathrm{Ker}(g^-)$, $w_2 + \mathrm{Ker}(g^-)$, …, $w_r + \mathrm{Ker}(g^-)$ is a basis of $\mathrm{W}/\mathrm{Ker}(g^-)$.

For $g \in \mathrm{Sp}(\mathrm{W})$ define

$$\theta(g) := \gamma(1)^{\dim g^- \mathrm{W}}\, \mathrm{dis}(B_g). \tag{200}$$

Lemma 5.14.

Let $g_1, g_2 \in \mathrm{Sp}(\mathrm{W})$. Assume that $K_1 = \mathrm{Ker}\, g_1^- = \{0\}$. Then

$$\gamma_W(\tilde{q}_{g_1,g_2})^2 \;=\; \frac{\theta(g_1g_2)^2}{\theta(g_1)^2\,\theta(g_2)^2}, \tag{201}$$

where $\tilde{q}_{g_1,g_2}$ is the non-degenerate symmetric form defined in Notation 2.6.

Proof. Let h be the element in $\mathrm{GL}(\mathrm{W})$ defined in Eqn. (24). Then since $\mathfrak{s}$ is a character, it follows from Eqns. (199) and (26) that

$$\mathfrak{s}(\det(\langle (g_1g_2)^- w_i, hw_j\rangle_{a<i,j}) = \mathrm{dis}(\tilde{q}_{g_1,g_2})^2\ \mathfrak{s}(\det(\langle g_1^- w_i, hw_j\rangle_{b<i,j})). \tag{202}$$

But

$$\mathfrak{s}(\det(\langle (g_1g_2)^- w_i, w_j\rangle_{a<i,j})) = \mathrm{dis}(B_{g_1g_2})^2.$$

Therefore (202) may be rewritten as

$$\mathrm{dis}(B_{g_1g_2})^2\ \mathfrak{s}(\det(h)) = \mathrm{dis}(\tilde{q}_{g_1,g_2})^2. \tag{203}$$

Notice that

$$\mathrm{dis}(B_{g_1})^2 = \mathfrak{s}(\det g_1^-) = \mathfrak{s}(\det(g_1(g_1^{-1}-1))) = \mathfrak{s}(\det(g_1^{-1}-1)) = \mathfrak{s}(\det(g_1^{-1}-1))^{-1}.$$

Then, from (26), we obtain

$$\mathrm{dis}(B_{g_1})^{-2}\ \mathfrak{s}(\det(h)) = \mathfrak{s}(-1)^{\dim \mathrm{U}}\ \ \mathrm{dis}(B_{g_2})^2.$$

Therefore

$$\mathfrak{s}(\det(h)) = \mathfrak{s}(-1)^{\dim \mathrm{U}}\ \ \mathrm{dis}(B_{g_1})^2\ \mathrm{dis}(B_{g_2})^2. \tag{204}$$

By combining (203) and (204) we see that

$$\mathrm{dis}(\tilde{q}_{g_1,g_2})^2 = \mathrm{dis}(B_{g_1g_2})^2\,\mathfrak{s}(-1)^{\dim \mathrm{U}}\ \ \mathrm{dis}(B_{g_1})^2\ \ \mathrm{dis}(B_{g_2})^2 = \mathfrak{s}(-1)^{\dim \mathrm{U}}\ \frac{\mathrm{dis}(B_{g_1g_2})^2}{\mathrm{dis}(B_{g_1})^2\,\mathrm{dis}(B_{g_2})^2}.$$

We see from (180) that

$$\gamma_W(\tilde{q}_{g_1,g_2})^2 = \gamma(1)^{2\dim \mathrm{U}-2\dim \mathrm{V}}\,\mathrm{dis}(\tilde{q}_{g_1,g_2})^2 = \mathfrak{s}(-1)^{\dim \mathrm{U}}\gamma(1)^{-2\dim \mathrm{U}-2\dim \mathrm{V}}\,\mathrm{dis}(\tilde{q}_{g_1,g_2})^2,$$

because $\gamma(1)^4 = \mathfrak{s}(-1)$, which follows from the equality $\gamma(1)\gamma(-1) = 1$. Therefore (205) implies (201). □

Definition 5.15.

For $g \in \mathrm{Sp}(\mathrm{W})$ define

$$\Theta^2(g) := \gamma(1)^{2\dim g^-\mathrm{W}-2}\,(\gamma(\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W}))^2 = \theta^2(g)|\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})|_{\mathbb{F}}^{-1},$$

where

$$\theta^2(g) = \gamma(1)^{2\dim g^-\mathrm{W}}\,\mathfrak{s}(\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})).$$

(Here $\mathfrak{s}$ was defined in Lemma 5.4.)

Lemma 5.16.
We have

$$\frac{\Theta^2(g_1g_2)}{\Theta^2(g_1)\Theta^2(g_2)} = C(g_1, g_2)^2 \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})). \tag{205}$$

Proof. Both sides of the equality (205) are cocycles. Hence, Lemma 2.8 shows that we may assume that $K_1 = \{0\}$. Therefore the equality (70) is equivalent to

$$\begin{aligned} &\frac{\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12})}{\det(g_1^- : \mathrm{W} \to \mathrm{W}) \det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})} \\ &= (-1)^{\dim \mathrm{U}} \det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \ , \ \rangle_{\mathrm{U}/\mathrm{V}}) \left(\det(g_2^- : K_{12} \to \mathrm{V})\right)^{-2} \end{aligned} \tag{206}$$

In particular

$$\frac{|\det((g_1g_2)^- : \mathrm{W}/K_{12} \to \mathrm{U}_{12})|_{\mathbb{F}}}{|\det(g_1^- : \mathrm{W} \to \mathrm{W})|_{\mathbb{F}} |\det(g_2^- : \mathrm{W}/K_2 \to \mathrm{U})|_{\mathbb{F}}} = |\det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \ , \ \rangle_{\mathrm{U}/\mathrm{V}})|_{\mathbb{F}} |\det(g_2^- : K_{12} \to \mathrm{V})|_{\mathbb{F}}^{-2}$$

This, together with Lemma 5.13, shows that the right hand side of (207) is equal to

$$|\det(\langle \frac{1}{2}(c(g_1) + c(g_2)) \ , \ \rangle_{\mathrm{U}/\mathrm{V}})|_{\mathbb{F}} \left(2^{-\dim \mathrm{V}} |\det(\tilde{L})|_{\mathbb{F}}\right)^2 ,$$

which, by Proposition 5.12, coincides with $|C(g_1, g_2)|^{-2}$. Hence, the absolute values of the two sides of (205) are equal. Hence, (205) (without the absolute values) follows from Lemma 5.14. □

Definition 5.17.
Let

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) := \{(g, \xi);\ g \in \mathrm{Sp}(\mathrm{W}),\ \xi \in \mathbb{C}^\times,\ \xi^2 = \Theta^2(g)\},$$

where $\Theta^2(g)$ is as in Definition 5.15.

Lemma 5.18.
$\widetilde{\mathrm{Sp}}(\mathrm{W})$ *is a group with the multiplication defined by*

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1g_2, \xi_1\xi_2\, C(g_1, g_2)) \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W})) \tag{207}$$

the identity equal to $(1, 1)$ *and the inverse given by*

$$(g, \xi)^{-1} = (g^{-1}, \overline{\xi}) \qquad (g \in \mathrm{Sp}(\mathrm{W})).$$

Proof. Lemma 5.16 shows that the right hand side of (207) belongs to $\widetilde{\mathrm{Sp}}(\mathrm{W})$. A standard computation, as in [20, page 366], shows that $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a group with the multiplication given by (207), the identity equal to $(1, C(1,1)^{-1})$ and

$$(g, \xi)^{-1} = (g^{-1}, C(g^{-1}, g)^{-1}\xi^{-1}).$$

Since, by Proposition 5.12, $C(1, 1) = 1$, it remains to check that

$$C(g^{-1}, g)^{-1}\xi^{-1} = \overline{\xi}.$$

But, as in the proof of Lemma 5.13,

$$C(g^{-1}, g) = 2^{\dim \mathrm{V}} |\det(\tilde{L})|_{\mathbb{F}}^{-1} = |\det(g^- : \mathrm{W}/\mathrm{Ker}(g^-) \to g^-\mathrm{W})|_{\mathbb{F}} = |\Theta^2(g)|_{\mathbb{F}}^{-1} = |\xi|_{\mathbb{F}}^{-2}.$$

This completes the proof. □

Notice that the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni (g, \xi) \to g \in \mathrm{Sp}(\mathrm{W})$$

is a group homomorphism with the kernel consisting of two elements. Thus $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a central extension of $\mathrm{Sp}(\mathrm{W})$ by the two element group $\mathbb{Z}/2\mathbb{Z}$:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Sp}(\mathrm{W}) \to 1. \tag{208}$$

Proposition 5.19.

The extension (208) does not split.

Proof. Pick a two-dimensional symplectic subspace $\mathrm{W}_1 \subseteq \mathrm{W}$ and let $\mathrm{W}_2 = \mathrm{W}_1^{\perp}$, so that

$$\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2.$$

Define an element $g \in \mathrm{Sp}(\mathrm{W})$ by

$$g(w_1 + w_2) = -w_1 + w_2 \qquad (w_1 \in \mathrm{W}_1, w_2 \in \mathrm{W}_2).$$

Then $g^-|_{\mathrm{W}_1} = (a-1)\,\mathrm{I}_2$ and $g^-|_{\mathrm{W}_2} = 0$. Hence $\mathrm{Ker}(g^-) = \mathrm{W}_2$ and $g^-(\mathrm{W}) = \mathrm{W}_1$. We get

$$\Theta^2(g) = \gamma(1)^4\, \mathfrak{s}(\det(g^- : \mathrm{W}_1 \to \mathrm{W}_1))\,|\det g^- : \mathrm{W}_1 \to \mathrm{W}_1)|_{\mathbb{F}}^{-1} = \gamma(1)^4\, \mathfrak{s}(4)\,|-4|_{\mathbb{F}}^{-1} = \frac{\gamma(1)^4}{|-4|_{\mathbb{F}}^{\frac{1}{2}}}.$$

We have $g^2 - 1 = 0$, and Eqn. (205) gives

$$C(g,g)^2 = \frac{1}{(\Theta^2(g))^2} = |-2|_{\mathbb{F}}^{4}.$$

Let $\tilde{g} = \left(g, \dfrac{\gamma(1)^2}{|-2|_{\mathbb{F}}}\right)$. Then $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, and

$$\tilde{g}^2 = (g^2, \Theta^2(g)\, C(g,g)) = (g^2, \gamma(1)^4) \quad \text{and} \quad \tilde{g}^4 = (g^4, \Theta^2(g^2)\, C(g^2, g^2)) = (g^4, 1).$$

Thus the subgroup of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ generated by $\tilde{g}$ is cyclic of order 4. The subgroup of $\mathrm{Sp}(\mathrm{W})$ generated by g is cyclic of order 2. Hence the extension (208) does not split over that subgroup. □

Corollary 5.20.

Up to an equivalence of central group extensions, as in [20, sec. 6.10], (208) is the only non-trivial central extension of $\mathrm{Sp}(\mathrm{W})$ *by* $\mathbb{Z}/2\mathbb{Z}$.

Proof. Since, as is well known (see [25, Theorems 5.10 and 11.1 (b)]),

$$H^2(\mathrm{Sp}(\mathrm{W}), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}),$$

the claim follows. □

Let

$$\phi^*(w) = \overline{\phi(-w)} \text{ and } u^*(\phi) = u(\phi^*) \qquad (\phi \in \mathcal{S}(\mathrm{W}),\ u \in \mathcal{S}^*(\mathrm{W}),\ w \in \mathrm{W}).$$

Lemma 5.21.

For any $g \in \mathrm{Sp}(\mathrm{W})$, $t(g)^* = t(g^{-1})$.

Proof. By the definition (189),

$$t(g)^* = \left(\chi_{c(g)}\mu_{g^-\mathrm{W}}\right)^* = \overline{\chi_{c(g)}}\mu_{g^-\mathrm{W}} = \chi_{-c(g)}\mu_{g^-\mathrm{W}}.$$

Since $g^-\mathrm{W} = (g^{-1}-1)\mathrm{W}$, it will suffice to check that for any $w \in \mathrm{W}$

$$-c(g)g^-w = c(g^{-1})g^-w.$$

The left hand side is equal to $-g^+w$. The right hand side is equal to

$$-c(g^{-1})(g^{-1}-1)gw = -(g^{-1}-1)gw = -g^+w. \qquad \square$$

Definition 5.22.

For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ define

$$\Theta(\tilde{g}) = \xi \text{ and } T(\tilde{g}) = \Theta(\tilde{g})t(g). \tag{209}$$

Lemma 5.23.

With the notation of (138), the following formulas hold

$$T(\tilde{g}_1)\natural T(\tilde{g}_2) = T(\tilde{g}_1\tilde{g}_2) \qquad (\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})), \tag{210}$$

$$T(\tilde{g})^* = T(\tilde{g}^{-1}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})). \tag{211}$$

Proof. By Proposition 5.12 the left hand side of (210) is equal to

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1\tilde{g}_2)}C(g_1, g_2)T(\tilde{g}_1\tilde{g}_2).$$

Lemma 5.18 shows that

$$\frac{\Theta(\tilde{g}_1)\Theta(\tilde{g}_2)}{\Theta(\tilde{g}_1\tilde{g}_2)}C(g_1, g_2) = 1.$$

This verifies (210).

The equality (211) follows from Lemma 5.18 and Lemma 5.21:

$$T(\tilde{g})^* = \overline{\Theta(\tilde{g})}t(g)^* = \Theta(\tilde{g}^{-1})t(g^{-1}) = T(\tilde{g}^{-1}). \qquad \square$$

Lemma 5.24.

The map $T : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})$ is injective and continuous.

Proof. The injectivity of T follows from the injectivity of $t : \mathrm{Sp}(\mathrm{W}) \to \mathcal{S}^*(\mathrm{W})$, which is obvious. Let

$$\mathrm{Sp}^c(\mathrm{W}) = \{g \in \mathrm{Sp}(\mathrm{W});\ \det g^- \neq 0\}.$$

Lemma 2.8 shows that

$$\mathrm{Sp}(\mathrm{W}) = \bigcup_{h \in \mathrm{Sp}(\mathrm{W})} \mathrm{Sp}^c(\mathrm{W})h. \tag{212}$$

Let $\widetilde{\mathrm{Sp}}^c(\mathrm{W}) \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W})$ be the preimage of $\mathrm{Sp}^c(\mathrm{W})$. Then

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \bigcup_{\tilde{h}\in\widetilde{\mathrm{Sp}}(\mathrm{W})} \widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h}.$$

By Lemma 5.23, we have

$$T(\tilde{g}) = T(\tilde{g}\tilde{h}^{-1})\natural T(\tilde{h}) \qquad (\tilde{g} \in \widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h})$$

Thus for $\phi \in \mathcal{S}(\mathrm{W})$,

$$T(\tilde{g})\natural\phi = T(\tilde{g}\tilde{h}^{-1})\natural(T(\tilde{h})\natural\phi).$$

By Lemma 5.11, the map

$$\mathcal{S}(\mathrm{W}) \ni \phi \to T(\tilde{h})\natural\phi \in \mathcal{S}(\mathrm{W})$$

is continuous. Hence it will suffice to check that the restriction of T to $\widetilde{\mathrm{Sp}}^c(\mathrm{W})$ is continuous. But this is obvious. $\square$

5.7. The conjugation property

Let $\mathrm{L}^2(\mathrm{W})$ denote the Hilbert space of the Lebesgue measurable functions $\phi\colon \mathrm{W} \to \mathbb{C}$, with the norm given by

$$\| \phi \|_2^2 = \int_{\mathrm{W}} |\phi(w)|^2 \, d\mu_{\mathrm{W}}(w).$$

Lemma 4.24 shows that for any $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and any $\phi \in \mathcal{S}(\mathrm{W})$

$$\| T(\tilde{g})\natural\phi \|_2^2 = (T(\tilde{g})\natural\phi)^*\natural(T(\tilde{g})\natural\phi)(0) = \phi^*\natural T(\tilde{g})^*\natural T(\tilde{g})\natural\phi(0) = \phi^*\natural\phi(0) = \| \phi \|_2^2 .$$

Hence, the continuous linear map

$$\mathcal{S}(\mathrm{W}) \ni \phi \to T(\tilde{g})\natural\phi \in \mathcal{S}(\mathrm{W})$$

extends by continuity to an isometry

$$\mathrm{L}^2(\mathrm{W}) \ni \phi \to T(\tilde{g})\natural\phi \in \mathrm{L}^2(\mathrm{W}).$$

Furthermore, the formula

$$\omega_{1,1}(g)\phi(w) = \phi(g^{-1}w) \qquad (g \in \mathrm{Sp}(\mathrm{W}),\ \phi \in \mathrm{L}^2(\mathrm{W})).$$

defines a unitary representation $\omega_{1,1}$ of the symplectic group $\mathrm{Sp}(\mathrm{W})$ on $\mathrm{L}^2(\mathrm{W})$.

Proposition 5.25.
For any $\phi \in \mathrm{L}^2(\mathrm{W})$ and $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ in the preimage of $g \in \mathrm{Sp}(\mathrm{W})$, $T(\tilde{g})\natural\phi\natural T(\tilde{g}^{-1}) = \omega_{1,1}(g)\phi$.

Proof. Since $T(\tilde{g})\natural$ is a bounded operator, we may assume that $\phi \in \mathcal{S}(\mathrm{W})$. Lemma 4.9 says that

$$t(g)\natural\delta_w = \delta_{wg}\natural t(g) \qquad (w \in \mathrm{W}).$$

Therefore

$$T(\tilde{g})\natural\delta_w = \delta_{wg}\natural T(\tilde{g}) \qquad (w \in \mathrm{W}).$$

Since,

$$\phi = \int_{\mathrm{W}} \phi(w)\delta_w \, d\mu_{\mathrm{W}}(w) \text{ and } \int_{\mathrm{W}} \phi(w)\delta_{gw} \, d\mu_{\mathrm{W}}(w) = \omega_{1,1}(g)\phi,$$

we see that

$$T(\tilde{g})\natural\phi = (\omega_{1,1}(g)\phi)\natural T(\tilde{g}).$$

$\square$

5.8. The Weyl transform and the Weil representation

Pick a complete polarization

$$\mathrm{W} = \mathrm{X} \oplus \mathrm{Y} \tag{213}$$

and recall that our normalization of measures is such that $d\mu_{\mathrm{W}}(x+y) = d\mu_{\mathrm{X}}(x)d\mu_{\mathrm{Y}}(y)$. Recall the Weyl transform

$$\begin{aligned} &\mathcal{K}\colon \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\mathrm{X}\times\mathrm{X}),\\ &\mathcal{K}(f)(x,x') = \int_{\mathrm{Y}} f(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathrm{Y}}(y), \end{aligned} \tag{214}$$

This is an isomorphism of linear topological spaces, which restricts to an isometry

$$\mathcal{K}\colon \mathrm{L}^2(\mathrm{W}) \to \mathrm{L}^2(\mathrm{X}\times\mathrm{X}). \tag{215}$$

Each element $K \in \mathcal{S}^*(\mathrm{X}\times\mathrm{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$ by

$$(\mathrm{Op}(K)(v))(u) = K(u\otimes v) \qquad (u, v \in \mathcal{S}(\mathrm{X})). \tag{216}$$

Since the map

$$\mathcal{S}(\mathrm{X})\times\mathcal{S}(\mathrm{X}) \ni (u,v) \to u\otimes v \in \mathcal{S}(\mathrm{X}\times\mathrm{X})$$

is continuous, (216) defines a continuous injection

$$\mathrm{Op} : \mathcal{S}^*(\mathrm{X}\times\mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X})). \tag{217}$$

Conversely, if $S \in \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$, then

$$S(v)(u) \qquad (u, v \in \mathcal{S}(\mathrm{X}))$$

defines a continuous linear map on $\mathcal{S}(\mathrm{X})\otimes\mathcal{S}(X) = \mathcal{S}(\mathrm{X}\times\mathrm{X})$. Hence the map (217) is bijective and thus a linear topological isomorphism.

A straightforward computation shows that $\mathrm{Op}\circ\mathcal{K}$ transforms the twisted convolution of distributions (when it makes sense) into the composition of the corresponding operators. Also,

$$(\mathrm{Op}\circ\mathcal{K}(f))^* = \mathrm{Op}\circ\mathcal{K}(f^*) \qquad (f \in \mathcal{S}^*(\mathrm{W})) \tag{218}$$

and

$$\mathrm{tr}\ \mathrm{Op}\circ\mathcal{K}(f) = \int_{\mathrm{X}} \mathcal{K}(f)(x,x)\ d\mu_{\mathrm{X}}(x) = f(0) \tag{219}$$

if $\mathrm{Op}\circ\mathcal{K}(f)$ is of trace class, [17, Theorem 3.5.4] (More precisely the same proof works). Hence, the map

$$\mathrm{Op}\circ\mathcal{K}\colon \mathrm{L}^2(\mathrm{W}) \to \mathrm{H.S.}(\mathrm{L}^2(\mathrm{X})) \tag{220}$$

is an isometry, which is a well known fact [17, Theorem 1.4.1]. (Here $\mathrm{H.S.}(\mathrm{L}^2(\mathrm{X}))$ stands for the space of the Hilbert-Schmidt operators on $\mathrm{L}^2(\mathrm{X})$.)

Let $\mathrm{U}(\mathrm{L}^2(\mathrm{X}))$ denote the group of the unitary operators on the Hilbert space $\mathrm{L}^2(\mathrm{X})$.

Theorem 5.26.
Let $\omega = \mathrm{Op}\circ\mathcal{K}\circ T$*. Then*

$$\omega\colon \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathrm{X}))$$

is an injective group homomorphism. For each $v \in \mathrm{L}^2(\mathrm{X})$*, the map*

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde{g} \to \omega(\tilde{g})v \in \mathrm{L}^2(\mathrm{X})$$

is continuous, so that $(\omega, \mathrm{L}^2(\mathrm{X}))$ *is a unitary representation of the metaplectic group. The function* Θ *coincides with the character of this representation:*

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\Psi(\tilde{g})\,d\tilde{g} = \mathrm{tr}\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{g})\Psi(\tilde{g})\,d\tilde{g} \qquad (\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W})),$$

where the integral on the left is absolutely convergent. (Here $d\tilde{g}$ *stands for any Haar measure on* $\widetilde{\mathrm{Sp}}(\mathrm{W})$*.) Moreover,*

$$\omega(\tilde{g})\,\mathrm{Op}\circ\mathcal{K}(\phi)\,\omega(\tilde{g}^{-1}) = \mathrm{Op}\circ\mathcal{K}(\omega_{1,1}(g)\phi) \qquad (\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W}),\ \ \phi\in\mathrm{L}^2(\mathrm{W})).$$

Proof. We see from the discussion in Section 5.7 that the left multiplication by $\omega(\tilde{g})$ is an isometry on $\mathrm{H.S.}(\mathrm{L}^2(\mathrm{X}))$. This implies that $\omega(\tilde{g})$ is a unitary operator.

We see from (217) that for any two function $v_1, v_2 \in \mathcal{S}(\mathrm{X})$ there is $\phi \in \mathcal{S}(\mathrm{W})$ such that

$$\int_{\mathrm{X}} \omega(\tilde{g})v_1(x)\overline{v_2(x)}\,d\mu_{\mathrm{X}}(x) = T(\tilde{g})(\phi) \qquad (\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W})).$$

Hence Lemma 5.24 shows that the left hand side is a continuous function of $\tilde{g}$. Since the operators $\omega(\tilde{g})$ are uniformly bounded (by 1), we see that the left hand side is a continuous function of $\tilde{g}$ for any $v_1, v_2 \in \mathrm{L}^2(\mathrm{X})$. This implies the strong continuity of ω, see [40, Lemma 1.1.3] or [41, Proposition 4.2.2.1].

Lemmas 5.23 and 5.24 show that the $\omega : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{U}(\mathrm{L}^2(\mathrm{X}))$ is an injective group homomorphism.

It is not difficult to check that the function

$$\frac{\det(\mathrm{Ad}(g)-1)}{\det g^-} \qquad (g\in\mathrm{Sp}(\mathrm{W}))$$

is locally bounded. Furthermore, as shown by Harish-Chandra [11, Section 8], the function

$$|\det(\mathrm{Ad}(g)-1)|_{\mathbb{F}}^{-1/2} \qquad (g\in\mathrm{Sp}(\mathrm{W})) \tag{221}$$

is locally integrable. Hence the function,

$$|\Theta(\tilde{g})| = |\det g^-|_{\mathbb{F}}^{-1/2} \qquad (\tilde{g}\in\widetilde{\mathrm{Sp}}(\mathrm{W}))$$

is locally integrable. (We would like to thank Alan Roche for the reference, [11].)

Notice that for any $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$,

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\,d\tilde{g} \in \mathcal{S}(\mathrm{W}). \tag{222}$$

Indeed, since the Zariski topology on $\mathrm{Sp}(\mathrm{W})$ is noetherian the covering (141) contains a finite subcovering (see for example [12, Exercise 1.7(b)]). Hence, there are elements $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_m$ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ such that

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \bigcup_{j=1}^m \widetilde{\mathrm{Sp}}^c(\mathrm{W})\tilde{h}_j.$$

Therefore Lemma 5.23 and a standard partition of the unity argument reduces the proof of (150) to the case when $\Psi \in C_c^\infty(\widetilde{\mathrm{Sp}}^c(\mathrm{W}))$. In this case (150) is equal to

$$\int_{\mathfrak{sp}(\mathrm{W})} \chi_x(w)\psi(x)\,dx \tag{223}$$

where $\psi \in C_c^\infty(\mathfrak{sp}(\mathrm{W}))$ and dx is a Haar measure on $\mathfrak{sp}(\mathrm{W})$. The function (223) is equal to the pullback of a Fourier transform $\hat{\psi}$ of ψ from $\mathfrak{sp}^*(\mathrm{W})$ to W via the unnormalized moment map

$$\tau : \mathrm{W} \to \mathfrak{sp}^*(\mathrm{W}),\ \ \tau(w)(x) = \langle xw, w\rangle \qquad (x \in \mathfrak{sp}(\mathrm{W}),\ \ w \in \mathrm{W}). \tag{224}$$

Since $\hat{\psi} \in \mathcal{S}(\mathfrak{sp}(\mathrm{W}))$ and since τ is a polynomial map with uniformly bounded fibers,

$$\hat{\psi} \circ \tau \in \mathcal{S}(\mathrm{W}).$$

This verifies (222). Hence, we may compute the trace as follows:

$$\begin{aligned}
\mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \omega(\tilde{g})\Psi(\tilde{g})\,d\tilde{g} &= \left(\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\,d\tilde{g}\right)(0) = \left(\int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})\Psi(\tilde{g})\,d\tilde{g}\right)(0)\\
&= \int_{\widetilde{\mathrm{Sp}}^c(\mathrm{W})} T(\tilde{g})(0)\Psi(\tilde{g})\,d\tilde{g} = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\tilde{g})\Psi(\tilde{g})\,d\tilde{g}.
\end{aligned}$$

The last formula is a direct consequence of Proposition 5.25. □

We end this Section by recalling some well known formulas for the action of $\omega(\tilde{g})$ for some special elements $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$.

Proposition 5.27.

Let $\mathrm{M} \subseteq \mathrm{Sp}(\mathrm{W})$ *be the subgroup of all the elements that preserve* X *and* Y. *Let* $\mathrm{M}^c := \{g \in \mathrm{M} : \det g^- \neq 0\}$. *Set*

$$\zeta(\tilde{g}) := \Theta(\tilde{g})\,|\det(\frac{1}{2}(c(g|_\mathrm{X}) + 1))|_\mathbb{F}^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c).$$

Then

$$(\zeta(\tilde{g}))^2 = (\mathfrak{s}(\det(g|_\mathrm{X})))^{-1}\,|\det(g|_\mathrm{X})|_\mathbb{F}^{-1} \qquad (\tilde{g} \in \widetilde{\mathrm{M}}^c), \tag{225}$$

the function $\zeta\colon \widetilde{\mathrm{M}}^c \to \mathbb{C}^\times$ *extends to a continuous group homomorphism*

$$\zeta\colon \widetilde{\mathrm{M}} \to \mathbb{C}^\times$$

and

$$\omega(\tilde{g})v(x) = \zeta(\tilde{g})v(g^{-1}x) \qquad (\tilde{g} \in \widetilde{\mathrm{M}},\ \ v \in \mathcal{S}(\mathrm{X}),\ \ x \in \mathrm{X}). \tag{226}$$

Proof. Set $n = \dim \mathrm{X}$. Fix an element $g \in \mathrm{M}^c$. Observe that

$$\det(g|_\mathrm{Y} - 1) = \det((g|_\mathrm{X})^{-1} - 1) = \det((g|_\mathrm{X})^{-1})\,\det(1 - g|_\mathrm{X}).$$

Then it follows from Definition 5.15 that

$$\begin{aligned}
\Theta^2(g) &= \gamma(1)^{4n}\,\mathfrak{s}(\det g^-)\,|\det g^-|_\mathbb{F}^{-1}\\
&= \gamma(1)^{4n}\,\mathfrak{s}(\det(g|_\mathrm{X} - 1)\det(g|_\mathrm{Y} - 1))\,|\det(g|_\mathrm{X} - 1)^{-1}\det(g|_\mathrm{Y} - 1)|_\mathbb{F}^{-1}\\
&= \gamma(1)^{4n}\,\mathfrak{s}(\det(g|_\mathrm{X} - 1))\,\mathfrak{s}(\det(g|_\mathrm{Y} - 1)\,|\det(g|_\mathrm{X} - 1)|_\mathbb{F}^{-1}\,|\det(g|_\mathrm{Y} - 1)|_\mathbb{F}^{-1}\\
&= \gamma(1)^{4n}\,\mathfrak{s}(\det(g|_\mathrm{X} - 1)^2)\,\mathfrak{s}(\det(-(g|_\mathrm{X})^{-1}))\,|\det(g|_\mathrm{X} - 1)|_\mathbb{F}^{-2}\,|\det(g|_\mathrm{X})|_\mathbb{F}\\
&= \gamma(1)^{4n}\,\mathfrak{s}((-1)^n)\,(\mathfrak{s}(\det(g|_\mathrm{X})))^{-1}\,|\det(g|_\mathrm{X} - 1)|_\mathbb{F}^{-2}\,|\det(g|_\mathrm{X})|_\mathbb{F}\\
&= \gamma(1)^{4n}\,(\mathfrak{s}(-1))^n\,(\mathfrak{s}(\det(g|_\mathrm{X})))^{-1}\,|\det(g|_\mathrm{X} - 1)|_\mathbb{F}^{-2}\,|\det(g|_\mathrm{X})|_\mathbb{F}.
\end{aligned}$$

Also,

$$|\det(\frac{1}{2}(c(g|_{\mathsf{X}})+1))|_{\mathbb{F}}^{-1} = |\det((g|_{\mathsf{X}})(g|_{\mathsf{X}}-1)^{-1})|_{\mathbb{F}}^{-1} = |\det(g|_{\mathsf{X}}-1))|_{\mathbb{F}}\,|\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1}.$$

Hence

$$\begin{aligned}(\zeta(\tilde{g}))^2 &= \gamma(1)^{4n}\,(\mathfrak{s}(-1))^n(\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\,|\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1} = \gamma(1)^{2n}\,\gamma(-1)^{2n}\,(\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\,|\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1}\\ &= (\mathfrak{s}(\det(g|_{\mathsf{X}})))^{-1}\,|\det(g|_{\mathsf{X}})|_{\mathbb{F}}^{-1}.\end{aligned}$$

This verifies (225).

Let $x, x' \in \mathsf{X}$ and let $y \in \mathsf{Y}$. Then

$$\begin{aligned}\mathcal{K}(t(g))(x,x') &= \int_{\mathsf{Y}} t(g)(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y)\\ &= \int_{\mathsf{Y}} \chi(\frac{1}{2}\langle c(g)(x-x'), y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y)\\ &= \delta_0(\frac{1}{2}c(g)(x-x')-x-x') = \delta_0(\frac{1}{2}((c(g)-1)x-(c(g)+1)x'))\\ &= |\det(\frac{1}{2}(c(g|_{\mathsf{X}})+1))|_{\mathbb{F}}^{-1}\delta_0(g^{-1}x-x').\end{aligned}$$

Therefore

$$\mathcal{K}(T(\tilde{g}))(x,x') = \zeta(\tilde{g})\delta_0(g^{-1}x-x').$$

Thus we have (226) for $\tilde{g} \in \widetilde{\mathsf{M}}^c$. Since ω is a representation of $\widetilde{\mathsf{M}}$, the remaining claims follow. □

Proposition 5.28.

Suppose $g \in \mathrm{Sp}(\mathsf{W})$ acts trivially on Y and on W/Y. Then $\det((-g)-1) \neq 0$ and

$$\omega(\tilde{g})v(x) = \xi_0\,\chi_{c(-g)}(2x)\,v(x) \qquad (v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}), \quad \textit{where } \xi_0^2 = (\mathfrak{s}(2))^{2n}.$$

Proof. Since $-g$ acts as minus the identity on Y and on W/Y, $\det((-g)-1) \neq 0$ and $z = c(-g) \in \mathfrak{sp}(\mathsf{W})$ is well defined. We have

$$z(w) = (-g)^+((-g)^-)^{-1}(w) \quad (w \in \mathsf{W}).$$

Since g acts trivially on Y and on W/Y, we get, for every $x \in \mathsf{X}$ and every $y \in \mathsf{Y}$:

$$g(x+y) = x+y+y_x, \quad \text{where } y_x \in \mathsf{Y}.$$

It gives $(-g)^-(x+y) = -2x-2y-y_x$. Hence

$$((-g)^-)^{-1}(x+y) = -\frac{1}{2}(x+y)+\frac{1}{4}y_x.$$

We obtain

$$z(x+y) = (-g)^+(-\frac{1}{2}(x+y)+\frac{1}{4}y_x) = \frac{1}{2}y_x.$$

In particular, we have

$$z\colon \mathsf{X} \to \mathsf{Y} \to 0.$$

Also, $\det(z-1) \neq 0$ and $c(z)$ is well defined. On the other hand, we have $(z-1)(x+y) = -(x+y) + \frac{1}{2}y_x$. It follows that

$$(z-1)^{-1}(x+y) = -(x+y) - \frac{1}{2}y_x.$$

Hence,

$$c(z)(x+y) = (z+1)\left(-(x+y) - \frac{1}{2}y_x\right) = -\frac{1}{2}y_x - (x+y) - \frac{1}{2}y_x,$$

that is,

$$c(z)(x+y) = -(x+y) - y_x. \tag{227}$$

We have $c(z) \in \mathrm{Sp}(\mathrm{W})$. Indeed, for any $w, w' \in \mathrm{W}$, writing $w = x+y$ and $w' = x'+y'$, with $x, x' \in \mathrm{X}$ and $y, y' \in \mathrm{Y}$, we have

$$\langle c(z)(w), c(z)(w')\rangle = \langle -w - y_x, -w' - y_{x'}\rangle = \langle w, w'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle.$$

However, since g is in $\mathrm{Sp}(\mathrm{W})$, we have

$$\langle x, x'\rangle = \langle gx, gx'\rangle = \langle x + y_x, x' + y_{x'}\rangle = \langle x, x'\rangle + \langle x, y_{x'}\rangle + \langle y_x, x'\rangle,$$

which gives

$$\langle x, y_{x'}\rangle + \langle y_x, x'\rangle = 0.$$

We obtain

$$\mathcal{K}(t(c(z)))(x,x') = \int_{\mathrm{Y}} \chi_z(x-x')\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathrm{Y}}(y) = \chi_z(x-x')\delta_0(\frac{1}{2}(x+x')) = 2^n\, \chi_z(x-x')\,\delta_0(x+x').$$

We have $\dim\left((c(z)-1)(\mathrm{W})\right) = \dim \mathrm{W} = 2n$, and,

$$\det\left(c(z)-1\right) = (-2)^{2n}.$$

We get

$$\Theta^2(c(z)) = \gamma(1)^{4n}\ (\mathfrak{s}(-2))^{2n}\ 2^{-2n} = \gamma(1)^{4n}\ (\mathfrak{s}(-1)\mathfrak{s}(2))^{2n}\ 2^{-2n} = \gamma(1)^{4n}\ (\mathfrak{s}(-1))^{2n}\ \frac{\gamma(2)^{4n}}{\gamma(1)^{4n}},$$

since $\mathfrak{s}(-1) = \gamma(1)^4$, and $\gamma(1)^8 = 1$. Hence,

$$\Theta^2(c(z)) = \gamma(2)^{4n}. \tag{228}$$

Thus

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') = 2^n\ \xi_0'\,\chi_z(x-x')\delta_0(x+x'), \quad \text{where } (\xi_0')^2 = \gamma(2)^{4n}.$$

Proposition 5.27 shows that

$$\omega((\widetilde{-1}))v(x) = \zeta(\widetilde{-1})v(-x).$$

We have

$$\left(\zeta(\widetilde{-1})\right)^2 = \mathfrak{s}((-1)^n)^{-1} = (\mathfrak{s}(-1))^{-n} = \gamma(1)^{-4n}.$$

Since

$$(\mathfrak{s}(2))^{2n} = 2^{2n}\left(\frac{\gamma(2)^2}{\gamma(1)^2}\right)^{2n},$$

the proof is complete. □

Proposition 5.29.
Suppose $g \in \mathrm{Sp}(\mathsf{W})$ acts trivially on X and on W/X. Then $\det((-g)-1) \neq 0$ so that $z = c(-g) \in \mathfrak{sp}(\mathsf{W})$ is well defined and $z : \mathsf{Y} \to \mathsf{X} \to 0$. Assume $z(\mathsf{Y}) = \mathsf{X}$. Then

$$\omega(\tilde{g})v(x) = \pm\left(\frac{\mathfrak{s}(2)}{2}\right)^n \gamma(q) \int_{\mathsf{X}} \chi_{z^{-1}}(x-x')v(x')\,d\mu_{\mathsf{X}}(x') \qquad (v \in \mathcal{S}(\mathsf{X}),\ x \in \mathsf{X}),$$

where $z^{-1} : \mathsf{X} \to \mathsf{Y}$ is the inverse of $z : \mathsf{Y} \to \mathsf{X}$. (The explicit computation of $\gamma(q)$ may be found in [31, Appendix].

Proof. The existence of z and its properties are verified as in the proof of Proposition 5.28. In particular, for all $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, we have

$$g(x+y) = x+y+x_y, \quad \text{where } x_y \in \mathsf{X}.$$

Similarly to the proof of Proposition 5.28, we get

$$z(x+y) = z(y) = \frac{1}{2}x_y. \tag{229}$$

and

$$c(z)(x+y) = -(x+y) - x_y, \tag{230}$$

that is,

$$c(z)(w) = -w - 2z(w), \quad \text{for every } w \in \mathsf{W}. \tag{231}$$

Let

$$q(y,y') = \frac{1}{2}\langle zy, y'\rangle \qquad (y, y' \in \mathsf{Y}).$$

Then, in terms of Lemma 5.8 and the identification (182),

$$q^*(x,x') = -2\langle z^{-1}x, x'\rangle \qquad (x,x' \in \mathsf{X}).$$

Hence, by the definition of $\mathcal{K}$ (214), the assumption that z annihilates X and maps Y into X and Lemma 5.8, we obtain

$$\begin{aligned}
\mathcal{K}(t(c(z)))(x,x') &= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -z(x-x'+y), x-x'+y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y)\\
&= \int_{\mathsf{Y}} \chi(\frac{1}{4}\langle -zy, y\rangle)\chi(\frac{1}{2}\langle y, x+x'\rangle)\,d\mu_{\mathsf{Y}}(y)\\
&= \int_{\mathsf{Y}} \chi(\frac{1}{2}q(y,y))\chi(-\langle y, -\frac{1}{2}(x+x')\rangle)\,d\mu_{\mathsf{Y}}(y)\\
&= \gamma(q)\chi(-\frac{1}{2}q^*(-\frac{1}{2}(x+x'), -\frac{1}{2}(x+x')))\\
&= \gamma(q)\chi(\langle z^{-1}(-\frac{1}{2}(x+x')), -\frac{1}{2}(x+x')\rangle) = \gamma(q)\chi_{z^{-1}}(x+x').
\end{aligned}$$

Therefore

$$\mathcal{K}(T(\widetilde{c(z)}))(x,x') = \Theta(\widetilde{c(z)})\gamma(q)\chi_{z^{-1}}(x+x').$$

But $\Theta(\widetilde{c(z)})^2 = \pm\gamma(2)^{4n}$ (see Eqn. (228)), where $\dim \mathsf{W} = 2n$. Furthermore, by Proposition 5.27,

$$\mathcal{K}(T(\widetilde{-1}))(x',x'') = \zeta(\widetilde{-1})\,\delta_0(x'-x''),$$

where $(\zeta(\widetilde{-1}))^2 = \gamma(1)^{-4n}$. Hence, the formula for $\omega(\tilde{g})$ follows. □

Acknowledgements

We would like to thank Angela Pasquale and Allan Merino for their careful reading of this article, some corrections and suggestions. Also, we express our gratitude to the two referees for their time and guidance through some related literature.

The second author was partially supported by the NSA grant H98230-13-1-0205.

References

[1] Aubert A.-M., Michel J., Rouqier R., Correspondance de Howe pour les groupes réductifs sur les corps finis, Duke Math. J., 1996, 83, 353–397.

[2] Cliff G., McNeilly D., Szechtman F., Weil representations of symplectic groups over rings, J. London Math. Soc. (2), 2000, 62(2), 423–436.

[3] Dieudonné J., Éléments d'Analyse, Gauthier-Villars Éditeur, 1971.

[4] Folland G. B., Real analysis: Modern techniques and their applications, 2nd Edition. Books. Published: 26 October 2012.

[5] Gérardin P., The Weil representations associated to finite fields, J. of Algebra, 1977, 46, 54–101.

[6] Gurevich S., Hadani R., The geometric Weil representation. Selecta Math. N.S., 2007, 13, 465–481.

[7] Gurevich S., Hadani R., Quantization of symplectic vector spaces over finite fields, J. Symplectic Geom., 2009, 7, 475–502.

[8] Gurevich S., Hadani R., Howe R., Quadratic reciprocity and the sign of the Gauss sum via the finite the Weil representation. Int. Math. Res. Not. 2010, 3729–3745.

[9] Gutiérrez L., Pantoja J., Soto-Andrade J., On generalized Weil representations over involutive rings, Contemporary Mathematics 544, 2011, 109–123.

[10] Harish-Chandra. Invariant Eigendistributions on a Semisimple Lie groups. Trans. Amer. Math. Soc, 1965, 119, 457–508.

[11] Harish-Chandra, Harmonic analysis on reductive p-adic groups, 1970, Bull. Amer. Math. Soc., 76, 529–551.

[12] Hartshorne R., Algebraic Geometry. Springer-Verlag, 1977. Graduate Text in Mathematics: 52.

[13] Hörmander L., The analysis of linear partial differential operators I, Springer Verlag, 1983.

[14] Hörmander L., The analysis of linear partial differential operators III, Springer Verlag, 1985.

[15] Howe R., Invariant theory and duality for classical groups over finite fields, with applications to their singular representation theory, preprint, 1973.

[16] Howe R., θ-series and invariant theory, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, 275–285, Amer. Math. Soc., Providence, R.I., 1979.

[17] Howe R., Quantum mechanics and partial differential equations. J. Funct. Anal., 1980, 38, 188–254.

[18] Howe R., The oscillator semigroup, Proc. Symp. Pure Math., Amer. Math. Soc., 48, 61–132, 1988.

[19] Howe R., Transcending classical invariant theory, J. Amer. Math. Soc. 2, 1989, 2, 535-552.

[20] Jacobson N., Basic Algebra II, W. H. Freeman and Company, San Francisco, 1980.

[21] Johnson R., Pantoja J., Weil representation of $\mathrm{SL}_*(2, A)$ for a locally profinite ring A with involution, J. Lie Theory 14, 2004, 1–9.

[22] Kashiwara M., Vergne M., On the Segal-Shale-Weil representation and harmonic polynomials, Invent. Math., 1978, 44, 1–47.

[23] Kirillov A. A., Elements of the theory of representations, Nauka, Moscow, 1978.

[24] Lion G., Vergne M., The Weil representation, Maslov index and theta series. BirkhÃ©user, Boston, 1980.

[25] Matsumoto H., Sur les sous-groupes arithmétiques des groupes semi-simples déployés. Ann. Sci. École Norm. Sup. (4), 1969, 2(1), 1–62.

[26] Müller D., Ricci F., Analysis of second order differential operators on Heisenberg group I. Invent. Math., 1990, 101, 545–582.

[27] Nazarov M., Neretin Y., Olshanskii G., Semi-groupes engendrés par la représentation de Weil du groupe symplectique de dimension infinie. C. R. Acad. Sci. Paris Sr. I Math. 1989, 309, 443–446 (in French).
[28] Neretin Y., Lectures on Gaussian integral operators and classical groups. EMS Series of Lectures in Mathematics European Mathematical Society (EMS), Zurich, 2011.
[29] Neretin Y., On a semigroup of operators in the boson Fock space. (Russian) Funktsional. Anal. i Prilozhen. 24 (1990), 63-73, 96; translation in Funct. Anal. Appl. 24 (1990), 135-144.
[30] Neuhauser M., An explicit construction of the metaplectic representation over a finite field, Journal of Lie Theory 12, 2002, 15–30.
[31] Perrin P., Représentations de Schrödinger, Indice de Maslov et groupe métaplectique, Non Commutative Harmonic Analysis and Lie Groups 880 (1981), 370–407.
[32] Prasad A., On character values and decomposition of the Weil representation associated to a finite abelian group, J. Analysis, 17, 2009, 73–86.
[33] Ranga Rao R., On some explicit formulas in the theory of Weil representations. Pacific Journal of Mathematics, 1993, 157, 335â€Ş-371.
[34] Rudin W., Principles of mathematical analysis, McGraw-Hill, Inc, 1964.
[35] Shale D., Linear symmetries of free boson fields, Trans. Amer. Math. Soc., 1962, 340, 309–321.
[36] Thomas T., Character of the Weil representation, 2008, 77(2), 221 –239.
[37] Thomas T., The Weil representation, the Weyl transform, and transfer factor, 2009.
[38] Von Neumann J., Mathematical fundations of quantum mechanics, translated by Robert T. Beyer, Princeton University Press, 1955.
[39] Waldspurger J.-L., Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I, Israel Math. Conf. Proc., 2, 1989, 267–324.
[40] Wallach N., Real Reductive Groups I, Academic Press, 1988.
[41] Warner, G., Harmonic analysis on semisimple Lie groups. I, Springer-Verlag, 1972, Die Grundlehren der mathematischen Wissenschaften, Band 188.
[42] Weil A., Sur certains groupes d'opérateurs unitaires, 1964, Acta Math., 111, 143–211.
[43] Weil A., Basic Number Theory, Springer-Verlag, 1973. Classics in Mathematics.