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NOTATION (unexplained in the text):

$\mathbf{R}$ = the field of real numbers

$\mathbf{Z}$ = the ring of integers in $\mathbf{R}$

$\mathbf{Z}_+$ = $\{0,1,2,3...\}$ = the set of non-negative integers

$\mathbf{C}$ = the field of complex numbers

$\binom{p}{q} = \frac{p!}{q!(p-q)!}$ for $p,q \in \mathbf{Z}$, $p \geqslant q \geqslant 0$

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

$|\alpha| = \sum_{j=1}^{n} \alpha_j$ for $\alpha = (\alpha_1, \alpha_2, .., \alpha_n) \in Z_+^n$

$M_{p,q}(\mathbf{F})$ = the space of matrixes with p rows and q columns with entries in $\mathbf{F}$, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.

$\mathrm{col}(w_1, w_2, .., w_p)$ = a typical element of $M_{p,1}(\mathbf{F})$, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.

$\partial_{z_j} f = \frac{\partial f}{\partial z_j}$ for $f \in \mathbf{C}[z_1, z_2, .., z_n]$.

$[X,Y] = XY - YX$ for X,Y in any ring

$U(V)$ = the group of unitary operators on a Hilbert space V.

$\mathrm{Hom}_G(V,V^1)$ = the space of continuous operators from the linear topological vector space V to the linear topological vector space V^1 intertwining the (given) representations of the topological group G on V and on V^1 respectively

V^* = the space of continuous linear functionals on the linear topological vector space V.

G^0 = the connected component of identity in the topological group G.

$\hat{G}$ = the set unitary equivalence classes of irreducible unitary representations of the topological group G.

$\mathrm{Int}\, g(x) = g\, x\, g^{-1}$ for g,x in a group G

$\underline{g}$ = the Lie algebra of a Lie group G

$\underline{\mathbf{g}}$ = the complexification of $\underline{g}$

$\Delta(\underline{\mathbf{g}},\underline{\mathbf{h}})$ = the set of roots of the Cartan subalgebra $\underline{\mathbf{h}}$, in the complex, reductive Lie algebra $\underline{\mathbf{g}}$ [Wa, 111]

$\Sigma\,\Delta^+(\underline{\mathbf{g}},\underline{\mathbf{h}}) = \sum_{\alpha\in\Delta^+(\underline{\mathbf{g}},\underline{\mathbf{h}})} \alpha$, for $\Delta^+(\underline{\mathbf{g}},\underline{\mathbf{h}}) \subseteq \Delta(\underline{\mathbf{g}},\underline{\mathbf{h}})$

$\rho(\underline{n}) = \rho(\underline{n},\underline{a}) = \frac{1}{2}$ sum of the roots of $\underline{a}$ in $\underline{n}$,

where P = MAN is a Langlands decomposition of a parabolic subgroup of a reductive Lie group G. [Wa 1.2, Vl 0.2].

$GL(n,\mathbf{C})$, $GL(n,\mathbf{R})$, $O(p,q)$, $O(n)$, $Sp(n,\mathbf{C})$, $Sp(n,\mathbf{R})$, $U(n,n)$, $U(n)$ - are concrete matrix groups as defined in [He Ch IX §4]

I_n = the matrix in $M_{n,n}(\mathbf{R})$ with 1 on the diagonal and 0 elsewhere.

CONFLICTING NOTATION:

π is either a representation of a compact group or is the real number equal to the area of the disc with radius one.

is either a quasicharacter of a Cartan subgroup (2.3.11) or

$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,dx$ for $z \in \mathbf{C}$, Re $z > 0$

ABSTRACT

We calculate this correspondence and show that the unitary representations of $O(2, 2,)$ are mapped to unitary representations of $Sp(2, \mathbb{R})$.

Key words and phrases: Howe's correspondence, unitarity, example

INTRODUCTION

The theory of representations of real reductive groups enjoys extremely rapid growth these years. In the early seventies Langlands [La] has reduced the problem of classification of the (infinitesimal equivalence classes) of the irreducible admissible representations of such groups to the analogous problem for tempered representations. Later Knapp and Zuckerman [K-Z2] have described the tempered representations and provided a unitarity criterion [K-Z1] for admissible representations in Langlands picture.

In the early eighties Vogan [V4] and later Wallach [W1] have proven a theorem about unitarizibility of the Derived Functor Modules invented by Zuckerman. This led to the classification of the unitary dual of GL(n,D) for $D = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ [V3].

In the meantime Howe [H2] has shown that for a real reductive dual pair G,G' (1.1.2) there is a bijection between some irreducible admissible representations of $\tilde{G}$ (a double cover of G) and some such representations of $\tilde{G}'$ (a double cover of G') - see (1.2.15). We call this bijection the OSCILLATOR DUALITY CORRESPONDENCE.

The role of unitarity in this correspondence is still obscure. Adams [A] had identified the (unitary) representations of O(p,q) corresponding to the disprete series representations of $Sp(n,\mathbf{R})$ for $2n < \min\{p,q\}$. His methods rely (among others) on Repka's theorems about decomposition of the tensor product of holomorphic and antiholomorphic discrete series representations of $Sp(n,\mathbf{R})$[R].

The object of this paper is to provide the full description of the Oscillator Duality Correspondence for the pair O(2,2), $Sp(2,\mathbf{R})$. We do it in Chapter 3. Since Repka's results do not apply to our example, the computations are sometimes technically unpleasant. The result however is simple. Roughly speaking (for the precise statement see (3.6.5)).

(I.1) The Oscillator Duality Correspondence induces the identity map on the (appropriately identified) sets of character data for these groups - except the case when a discrete series representation of O(2,2) corresponds to a non-discrete series representation of $Sp(2,\mathbf{R})$.

Some other properties of this correspondence are summarized in (3.6.6) and (3.6.7). Moreover we would like to draw the reader's attention to the Theorem (C.7), in the Appendix C, where (again roughly speaking) we show that

(I.3) the determinant representation of the group $O(p,q)$ occurs in the pairing with $Sp(m,\mathbb{R})$ if and only if $p+q \leq m$.

In Chapter 2, §4, §5 we list the unitary duals of $Sp(2,\mathbb{R})$ and $O(2,2)$ respectively. The reader will find the necessary calculations based on [K-Z1] and [K-Sp] for $Sp(2,\mathbb{R})$ in the Appendix A. A surprising conclusion from a comparison of these lists with the description of the Oscillator Duality Correspondence is the fact (3.6.11) that

(I.2) the unitary representations of $O(2,2)$ correspond to unitary representations of $Sp(2,\mathbb{R})$.

We have also checked (not included in this paper) that (I.2) remains true if one replaces $O(2,2)$ by the Lorentz group $O(1,3)$. It is also known [P1] that for any real reductive dual pair the Oscillator Duality Correspondence maps hermitian representation of one group to hermitian representations of the other group (and vice versa).

Recently Jian Shu Li [Li] has shown, using Mackey's theory, that the Oscillator Duality Correspondence maps the whole unitary dual of $\tilde{O}(p,q)$ into the unitary dual of $\widetilde{Sp}(n,\mathbb{R})$ for $p+q \leq n$. This result holds for groups over any local fields!

All the above should convince the reader that the Oscillator Duality Correspondence may become an important tool for the, still distant, classification of the unitary dual of a real classical group (since all of them fit into the framework of Howe's theory of reductive dual pairs).

Finally I can't resist the temptation to mention that Stanislaw Ulam [The Scottish Book] has noticed the following phrase of Shakespeare's in Henry VIII: "Things done without example, in their issue are to be feared." Ulam's interpretation seems to be that this is an "anti - 'new math'" statement. Most probably our paper will add a drop to this controversy.

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CHAPTER 1

THE OSCILLATOR DUALITY CORRESPONDENCE

In sections 1 and 2 we recall the notion of a reductive dual pair (1.1.2), list the basic properties of the oscillator representation (1.2.6) and quote Howe's Duality Theorem (1.2.15). In paragraphs 3 and 4 we review various models of the oscillator representation, illustrate them on examples and relate them to each other.

Section 5 is devoted to a clarification of the notion of a lowest degree K-type (1.5.16) introduced by Howe in [H2].

§1. Reductive Dual Pairs

Let $(W, \langle\ ,\ \rangle)$ be a finite dimensional symplectic vector space over a commutative field. Denote by

(1.1.1) $Sp(W, \langle\ ,\ \rangle) = Sp(W) = Sp$ the isometry group of the form $\langle\ ,\ \rangle$.

(1.1.2) Def. [H3, H5]. A pair of subgroups G, G' of the symplectic group $Sp(W)$ is called a reductive dual pair if

(1.1.3) G' is the centralizer of G in $Sp(W)$ and vice versa, and

(1.1.4) both G and G' act reductively on W.

These pairs have been classified [H4, H5] over fields of characteristic different than two.

(1.1.5) Example. Let us inject any element g of the group $O(2, 2)$ into the group $Sp(8, \mathbf{R})$ (see "Notation" for the definition of these groups) by the map

(1.1.6) $g \to \mathrm{diag}\left(g, g, (g^t)^{-1}, (g^t)^{-1}\right)$.

Define an imbedding of $Sp(2, \mathbf{R})$ into $Sp(8, \mathbf{R})$ by lifting of the following homomorphism of Lie algebras:

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$$(1.1.7)\quad \begin{bmatrix} x_1 & x_2 & y_1 & y_2 \\ x_3 & x_4 & y_2 & y_4 \\ v_1 & v_2 & -x_1 & -x_3 \\ v_2 & v_4 & -x_2 & -x_4 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1I & 0 & x_2I & 0 & y_1I & 0 & y_2I & 0 \\ 0 & x_1I & 0 & x_2I & 0 & -y_1I & 0 & -y_2I \\ x_3I & 0 & x_4I & 0 & y_2I & 0 & y_4I & 0 \\ 0 & x_3I & 0 & x_4I & 0 & -y_2I & 0 & -y_4I \\ v_1I & 0 & v_2I & 0 & -x_1I & 0 & -x_3I & 0 \\ 0 & -v_1I & 0 & -v_2I & 0 & -x_1I & 0 & -x_3I \\ v_2I & 0 & v_4I & 0 & -x_2I & 0 & -x_4I & 0 \\ 0 & -v_2I & 0 & -v_4I & 0 & -x_2I & 0 & -x_4I \end{bmatrix}$$

Here $I = I_2$.

One checks easily that the images of $O(2, 2)$ and $Sp(2, \mathbf{R})$ form a reductive dual pair in $Sp(8, \mathbf{R})$. It will be convenient in some of our future calculations to use another orthogonal matrix group isomorphic to $O(2, 2)$. Namely let $O_{2,2}$ denote the group of real matrixes of size 4 preserving the symmetric bilinear form on $\mathbf{R}^4$ defined by the matrix

$$(1.1.8)\qquad F = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} .$$

This group is isomorphic to the group $O(2, 2)$ via

$$(1.1.9)\qquad \text{the conjugation by } c = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix} \text{ inside } GL(4, \mathbf{R}) .$$

The matrix

$$(1.1.10)\qquad C = \mathrm{diag}\,(c, c, c, c)$$

belongs to the maximal compact subgroup

$$(1.1.11)\qquad O(16) \cap Sp(8, \mathbf{R})$$

of $Sp(8, \mathbf{R})$.

We realize $O_{2,2}$, $Sp(2, \mathbf{R})$ as a reductive dual pair in $Sp(8, \mathbf{R})$ by the following commuting diagram of group homomorphisms:

(1.1.12)
$$\begin{tikzcd} O_{2,2} \arrow[r, "(1.1.6)"] \arrow[d, "(1.1.9)"] & Sp(8, \mathbf{R}) \arrow[d, "\mathrm{Int}\ C"] & Sp(2, \mathbf{R}) \arrow[l] \arrow[d, dashed, "\mathrm{id}"] \\ O(2,2) \arrow[r, "(1.1.6)"] & Sp(8, \mathbf{R}) & Sp(2, \mathbf{R}) \arrow[l, "(1.1.7)"'] \end{tikzcd}$$

where id stands for the identity map and the unmarked arrow for $(\mathrm{Int}\ C)^{-1} \circ (1.1.7) \circ \mathrm{id}$.

§2. The Oscillator Representation.

(1.2.1) <u>Def</u> [H4 ChI §3] . The Heisenberg group attached to the real symplectic vector space W is the smooth manifold

$$H(W) = W \oplus R$$

with multiplication

$$(w, r)(w', r') = (w + w', r + r' + \tfrac{1}{2} \langle w , w' \rangle)$$

where $w, w' \in W$ and $r, r' \in \mathbf{R}$.

The symplectic group $Sp(W)$ acts on $H(W)$ by automorphisms leaving the center $Z(H(W))$ of $H(W)$ pointwise fixed:

(1.2.2) $g(w, r) = (gw, r) \qquad (g \in Sp(W), (w, r) \in H(W))$.

Therefore

(1.2.3) the unique connected two fold covering group $\widetilde{Sp}$ of Sp, usually called the metaplectic group acts on $H(W)$ and we may form a semidirect product of Lie groups.

(1.2.4) $\widetilde{Sp} \times_s H(W)$.

Let

(1.2.5) χ be a non-trivial unitary character of the abelian group $Z(H(W))$.

(1.2.6) Theorem [H5]. Up to a unitary equivalence there is only one unitary representation ω of the group (1.2.4) satisfying the following conditions:

(1.2.7) the restriction of ω to $H(W)$ is irreducible,

(1.2.8) ω restricted to $Z(H(W))$ is a multiple of χ ,

(1.2.9) $\omega(\tilde{g})\omega(h)\omega(\tilde{g})^{-1} = \omega(g(h))$

for $h \in H(W)$ and $\tilde{g} \in \widetilde{Sp}$ covering $g \in Sp(W)$.

One of the most striking properties of ω is expressed in Howe's Duality Theorem (1.2.15). Before stating it we fix some additional notation.

For any group $G \subseteq Sp$ let

(1.2.10) $\tilde{G}$ be the preimage of G in $\widetilde{Sp}$.

If G is a reductive Lie group let

(1.2.11) $R(\tilde{G})$ be the set of infinitesimal equivalence classes of continuous irreducible admissible representations of $\tilde{G}$ on locally convex topological vector spaces.

Denote by ω^{∞} the smooth representation of $\widetilde{Sp}$ associated to ω and let

(1.2.12) $R(\tilde{G}, \omega)$ be the set of elements of $R(\tilde{G})$ which can be realized as quotients of the smooth representation ω^{∞} by $\omega^{\infty}(\tilde{G})$-invariant closed subspaces.

Consider a reductive dual pair G, G' in Sp . It is not hard to show that $\tilde{G}$ and $\tilde{G}'$ commute with one another. The identity maps from $\tilde{G}$ to itself and from $\tilde{G}'$ to itself induce a homomorphism

(1.2.13) $\tilde{G} \times \tilde{G}' \to \tilde{G} \cdot \tilde{G}' \subseteq \widetilde{Sp}$.

Hence we may regard $\omega^{\infty}|\tilde{G} \cdot \tilde{G}'$ as a representation of $\tilde{G} \times \tilde{G}$. It is well known [F] that $R(\tilde{G} \times \tilde{G}') \cong R(\tilde{G}) \times R(\tilde{G}')$. The identification associates to $\Pi \in R(\tilde{G})$ and $\Pi' \in R(\tilde{G}')$ the outer tensor product $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \times \tilde{G}')$. The topology of $\Pi \check{\otimes} \Pi'$ is not uniquely determined but the infinitesimal equivalence class is. In this sense

(1.2.14) $R(\tilde{G}\cdot \tilde{G}', \omega)$ is a subset of the cartesian product
$R(\tilde{G}, \omega) \times R(\tilde{G}', \omega)$.

It can be verified that the projections of $R(\tilde{G} \cdot \tilde{G}', \omega)$ onto $R(\tilde{G}, \omega)$ and $R(\tilde{G}', \omega)$ are surjective.

(1.2.15) Theorem [H2]. The set $R(\tilde{G} \cdot \tilde{G}', \omega)$ is the graph of bijection between $R(\tilde{G}, \omega)$ and $R(\tilde{G}', \omega)$. In other words, for each $\Pi \in R(\tilde{G}, \omega)$ there is a unique $\Pi' \in R(\tilde{G}', \omega)$ such that

(1.2.16) $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega)$

and vice versa. Moreover for Π and Π' as above

(1.2.17) $\dim \mathrm{Hom}_{\tilde{G} \cdot \tilde{G}}(\omega^\infty, \Pi \check{\otimes} \Pi') = 1$.

(1.2.18) Def. The function $\Pi \to \Pi'$ defined by (1.2.16) is called the Oscillator Duality Correspondence.

It is known that the representation ω^∞ can be realized on a Schwartz space S ([H4, H5],. (1.3.16)) and that the action of the group $\tilde{S}p$ extends from S to S*-the space of tempered distributions.

(1.2.19) Proposition. For each element $\Pi \check{\otimes} \Pi'$ of $R(\tilde{G} \cdot \tilde{G}', \omega)$ there is a unique irreducible $\tilde{G} \cdot \tilde{G}'$ submodule V of S* realizing $\Pi \check{\otimes} \Pi'$, and conversely each irreducible $\tilde{G} \cdot \tilde{G}'$ submodule V of S* corresponds to an element $\Pi \check{\otimes} \Pi'$ of $R(\tilde{G} \cdot \tilde{G}', \omega)$.

Proof: Let ω^c denote the contragradient oscillator representation of $\tilde{S}p$ realized on S . Assume that $V \subseteq S^*$ is a closed $\omega(\tilde{G} \cdot \tilde{G}')$-irreducible subspace. Then the topological dual V* is a quotient of S by a closed $\omega^c(\tilde{G} \cdot \tilde{G}')$-invariant subspace N . Since S* is a reflexive topological vector space, it follows from [Ke - Na, Ch. 5, 20.2 (ii)]that V is reflexive. Therefore V* does not contain any closed non-trivial subspaces invariant under the contragradient action of $\tilde{G} \cdot \tilde{G}'$. Thus

(1.2.20) V* represents an element of $R(\tilde{G} \cdot \tilde{G}', \omega^c)$.

Conversely, let $N \subseteq S$ be a $\omega^c(\tilde{G} \cdot \tilde{G}')$-invariant closed subspace such that S/N is irreducible. Denote by V the annihilator of N in S^* with the contragradient action of $\tilde{G} \cdot \tilde{G}'$. Then again by reflexivity of N

(1.2.21) V is a closed $\omega(\tilde{G} \cdot \tilde{G})$-irreducible subspace of S^*.

Combining (1.2.20) and (1.2.21) we see that

(1.2.22) the set of V^*'s, for V's described in the proposition, is a set of realizations of all elements of $R(\tilde{G} \cdot \tilde{G}', \omega^c)$.

Since by [P1 Theorem 5.5], $R(G \cdot G', \omega^c) = R(G \cdot G', \omega)^c$ = the set of infinitesimal equivalence classes of contragradient representations of $\tilde{G} \cdot \tilde{G}$, the statement (1.2.20) completes the proof.

Q.E.D.

§3. The Mixed and the Schrödinger Models

Let

(1.3.1) $W = W_1 \oplus W_2$ be a decomposition of W into an orthogonal sum of two symplectic spaces, and

(1.3.2) $W_1 = X \oplus Y$ be a complete polarization of W_1.

The statement (1.3.2) means that X, Y are maximal isotropic subspaces of W_1 and W_1 is a direct sum of them. Denote by

(1.3.3) P_Y the parabolic subgroup of $Sp(W)$ whose elements preserve Y, and by

(1.3.4) N_Y the subgroup of P_Y which acts trivially on Y and on $(Y \oplus W_2)/Y$.

As explained in [H4 Ch I §9], the center $Z(N_Y)$ of the group N_Y is isomorphic to the abelian additive group $S^{2*}(X)$ of the symmetric bilinear forms on X $\left(g(x, x') = \langle x, gx' \rangle,\ g \in Z(N_Y)\ x, x' \in X\right)$. Moreover if we define the map

$$* : \mathrm{Hom}\left(W_2\ Y\right) \to \mathrm{Hom}\left(X, W_2\right) \quad \text{by}$$

(1.3.5) $\langle Tw, x\rangle = \langle w, T^*x\rangle \qquad \left(x \in X\ ,\ w \in W_2\right)$

then a straightforward calculation shows that for $T, T' \in \mathrm{Hom}(W_2, Y)$ the map

(1.3.6) $$X \times X \ni (x, x') \longrightarrow \langle x, (TT'^* - T'T^*)x'\rangle \in R$$

is a symmetric bilinear form on X.
We can thus define a group structure on

(1.3.7) $$\mathrm{Hom}(W_2, Y) \times S^{2*}(X)$$

by introducing the following multiplication:

(1.3.8) $$(T, b)(T', b') = \left(T + T', b + b + \tfrac{1}{2}(TT'^* - T'T^*)\right)$$

with $T, T' \in \mathrm{Hom}(W_2, Y)$ and $b, b' \in S^{2*}(X)$. Having developed the above notation we can quote a result from [H4 Ch I §9] which says that

(1.3.9) $$N_Y \cong \mathrm{Hom}\ (W_2, Y) \times S^{2*}(X) .$$

Put

(1.3.10) $$M_{X,Y} = P_X \cap P_Y .$$

Then $M_{X,Y}$ preserves the decompositions (1.3.1), (1.3.2) and by Witt's theorem

(1.3.11) $$M_{X,Y} \cong GL(X) \times Sp(W_2)$$

by restriction. It is well known that

(1.3.12) $$P_Y = M_{X,Y} N_Y \quad \text{and} \quad M_{X,Y} \cap N_Y = \{1\} .$$

Combining (1.3.12), (1.3.11), and (1.3.9) we obtain an isomorphism

(1.3.13) $$P_Y \cong (GL(X) \times Sp(W_2)) \times_s (\mathrm{Hom}\ (W_2, Y) \times S^{2*}(X)) .$$

In the sense of (1.3.13) the preimage $\tilde{P}_Y$ of P_Y in the metaplectic group $\tilde{S}p(W)$ is the product in $\tilde{S}p$ of a group (isomorphic to) the metaplectic group $\tilde{S}p(W_2)$, the group

(1.3.14) $$\tilde{G}L(X) = \{(g, \xi) \mid g \in GL(X), \xi^2 = \det g\} ,$$

and the unipotent radical (1.3.9).

Let

(1.3.15) ω_2 be a realization of the oscillator representation of the group $\tilde{S}p(W_2) \times_s H(W_2)$ (1.2.6) on a Hilbert space V_2.

(1.3.16) Theorem [H4 Ch II §3]. The oscillator representation ω may be realized on the Hilbert space $V = L^2(X, V_2)$ and the restriction of ω to $\tilde{P}_Y$ is given by the following formulas:

(1.3.17) $\omega(g)\, f(x) = \omega_2(g)(f(x)) \qquad (g \in \tilde{S}p(W_2))$,

(1.3.18) $\omega(g, \xi)f(x) = \xi^{-1}f(g^{-1}x) \qquad ((g,\xi) \in \tilde{G}L(X))$,

(1.3.19) $\omega(b)f(x) = \chi(-\tfrac{1}{2}\, b(x,x))f(x) \qquad (b \in S^{2*}(X))$,

(1.3.20) $\omega(T)f(x) = \omega_2(T^*(x))(f(x)) \qquad (T \in \mathrm{Hom}\,(W_2, Y))$.

Moreover the space of smooth vectors

(1.3.21) $V^\infty = S(X, V_2^\infty)$ is the Schwartz space of X with values in V_2^∞ .

(1.3.22) Def [H4] The realization of ω described in the above theorem is called a mixed model of the oscillator representation adopted to the decomposition

(1.3.23) $W = (X \oplus Y) \oplus W_2$.

In particular, when $W_2 = 0$ we obtain a Schrödinger model.

(1.3.24) Example. Let $W = M_{1,2n}(\mathbb{R})$ be the space of row vectors $w = (w_1, w_2, \ldots, w_{2n})$, with the symplectic form

(1.3.25) $\langle w, w' \rangle = w_1 w'_{n+1} + \ldots + w_n w'_{n+n} - w_n w'_1 - \ldots - w_{n+n} w'_n$.

Then the group $Sp(n, \mathbb{R})$ acts on W by

(1.3.26) $g(w) = wg^t$.

Let X be the subspace of W consisting of vectors with zeros in the last n-places, and Y be the subspace whose vectors have zeros in the first n-places. Then the subgroup P_Y of $Sp(n, \mathbf{R})$ consists of matrices with zero in the upper right corner and the isomorphism (1.3.13) is given by the following maps:

(1.3.27) $$M_{X,Y} \ni \begin{bmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{bmatrix} \longleftrightarrow g \in GL(n, \mathbf{R})$$

(1.3.28) $$N_Y \ni \begin{bmatrix} I_n & 0 \\ b & I_n \end{bmatrix} \longleftrightarrow b \in S(n, \mathbf{R})$$

where $S(n,\mathbf{R})$ denotes the space of all symmetric real matrices of size n, which are identified with the bilinear symmetric forms on $X \cong M_{1,n}(R)$ as usual:

(1.3.29) $$b(x, x) = xbx^t \ .$$

For notational convenience let us identify X with $M_{n,1}(\mathbf{R})$ by

(1.3.30) $$X \ni (x,0) \to x^t \in M_{n,1}(\mathbf{R}) \ .$$

Then by (1.3.27), (1.3.28), the formulas (1.3.18), (1.3.19) may be rewritten as

(1.3.31)
$$\omega(g, \xi)f(x) = \xi^{-1}f(g^{-1}x) \qquad \left(g \in GL(n,\mathbf{R}),\ \xi^2 = \det g\right)$$
$$\omega(b)f(x) = \chi\left(-\tfrac{1}{2}\, x^t\, bx\right)f(x) \qquad \left(b \in S(m, R)\right) \ .$$

where

(1.3.32) $$\chi(r) = e^{4\pi i r} \qquad (r \in R) \ .$$

This character (1.3.32) of R is going to be fixed throughout this paper.

§4. The Fock Models

Let J be a compatible, positive, complex structure on W. This means that J belongs to $Sp(W)$, $J^2 = -I$ and (I = the identy map) $\langle Jw, w \rangle > 0$ for all $w \in W$, $w \neq 0$ [H4]. Since J generates a subfield of $\mathrm{End}_{\mathbf{R}}(W)$ isomorphic to $\mathbf{C}$ we can view W as a complex vector space where

$iw = J(w)$ for $w \in W$. The centralizer of J in $Sp(W)$ is a maximal compact subgroup U_J, which is isomorphic to $U(n)$ if $2n = \dim_{\mathbb{R}}(W)$. Define a hermitian form $(\ ,\ )_J$ or W by

(1.4.1) $$(w_1, w_2)_J = \langle Jw_1, w_2\rangle + i\langle w_1, w_2\rangle \qquad (w_1, w_2 \in W),$$

and the corresponding norm

(1.4.2) $$|w|_J^2 = (w, w)_J \qquad (w \in W).$$

Consider

(1.4.3) the Hilbert space V_J of holomorphic functions on W which are square integrable with respect to the measure

$$\exp(-|w|_J^2)dm(w)$$

where m is a Haar measure on the additive group W.

(1.4.4) According to [H4 Ch II §3] there is a realization of ω on the Hilbert space V_J. We shall refer to this realization as to

(1.4.5) the Fock model of ω adapted to the compatible positive structure J on W and describe it in a special case below.

(1.4.6) Example Let W be the $2n$ dimensional real vector space underlying $\mathbb{C}^n$. On W consider the symplectic form

(1.4.7) $$\langle w,w'\rangle = 2\mathrm{Im}\, w^t\, \overline{w}' \qquad (w, w' \in W)$$

Define $J(w) = iw$, where $i = \sqrt{-1}$ as usual. Then J is a compatible, positive complex structure on W. Since

$$|w|_J^2 = |w_1|^2 + |w_2|^2 + \dots + |w_n|^2 \qquad (w = \mathrm{col}(w_1, w_2, \dots, w_n)),$$

we see that $U_J = U(n)$.

We can realize W as a real symplectic subspace of the standard complex sympletic space C^{2n} [He Ch IX §4] by the map

$$(1.4.8) \quad W \equiv w \longrightarrow \begin{bmatrix} w \\ \bar{w} \end{bmatrix} \in \mathbf{C}^{2n} .$$

Then the group $Sp(W, \langle\ ,\ \rangle)$ is identified with the group $Sp(n, C) \cap U(n, n)$.

Recall that the Lie algebra $\underline{sp}(n,\mathbf{C})$ of $Sp(n,C)$ consists of matrices of the form

$$(1.4.9) \quad \begin{bmatrix} U & Z_1 \\ Z_2 & -U^t \end{bmatrix} \quad (Z_1 = Z_1^t\ ,\ Z_2 = Z_2^t\ ,\ U \in M_{n,n}(C))$$

Let $e_{j,k}$ denote a matrix with 1 in the j-th row and k-th column. Then, by (1.4.9), a basis of $\underline{sp}(n,\mathbf{C})$ consists of the linear combinations of

$$e_{j,n+k} + e_{k,n+j},\ e_{n+j,k} + e_{n+k,j},\ e_{j,k} - e_{n+k,n+j} \quad (1 \leqslant j,k \leqslant n).$$

We define a representation of $\underline{sp}(n,\mathbf{C})$ on $C[z_1, z_2,\dots,z_n]$ by

$$\begin{aligned} &\omega(e_{j,n+k} + e_{k,n+j}) = -z_j z_k \\ (1.4.10) \qquad &\omega(e_{n+j,k} + e_{n+k,j}) = \partial_{z_j} \partial_{z_k} \\ &\omega(e_{j,k} - e_{n+k,n+j}) = \tfrac{1}{2}(z_j \partial_{z_k} + \partial_{z_k} z_j) \end{aligned}$$

where $1 \leqslant j\ ,\ k \leqslant n$.

We introduce a scalar product in $C[z_1, z_2,\dots,z_n]$ by declaring the set of elements

$$(1.4.11) \quad \frac{z^\alpha}{\sqrt{\alpha!}} \quad (\alpha = (\alpha_1, \alpha_2,\dots,\alpha_n)\ ;\ \alpha_j = 0, 1, 2,\dots;\ j = 1, 2,\dots,n) .$$

to be the orthonormal basis. One can check [H7] that the Hilbert space norm on $\mathbf{C}[z_1, z_2,\dots z_n]$ obtained this way is

$$(1.4.12) \qquad \|f\|^2 = \int_{\mathbf{C}^n} |f(z)|^2 e^{-|z|^2} dm(z)$$

where m is a suitably normalized Haar measure on the additive group $\mathbf{C}^n$. By completion one obtains the Hilbert space of holomorphic functions $f : \mathbf{C}^n \to \mathbf{C}$ such that $\|f\| < \infty$ (1.4.12).

A straightforward calculation shows that

(1.4.13) $\underline{sp}(n, C) \cap \underline{u}(n, n)$ is mapped via ω into the space of skew-Hermitian operators.

The real Lie algebra (1.4.13) consists of matrices

$$(1.4.14) \qquad \begin{bmatrix} U & Z \\ \overline{Z} & \overline{U} \end{bmatrix} \qquad \left(U \in \underline{u}(n)\ ,\ Z = Z^t \in M_{n,n}(C)\right) .$$

Fix an injection of $U \in \underline{u}(n)$ into (1.4.14) by

$$(1.4.15) \qquad U \to \mathrm{diag}(U, \overline{U}) .$$

Define a double covering $\tilde{U}(n)$ of $U(n)$ by

$$(1.4.16) \qquad \tilde{U}(n) = \left\{(g,\xi) \mid g \in U(n)\ ,\ \xi^2 = \det g\right\}$$

with the covering map $(g, \xi) \to g$.

Let $\tilde{U}(n)$ act on $\mathbf{C}[z_1, z_2, \ldots, z_n]$ via the formula:

$$(1.4.17) \qquad \omega(g,\xi)f(z) = \xi f(zg) .$$

Here z denotes the row vector $\left(z_1, z_2, \ldots, z_n\right)$.
The formulas (1.4.10), (1.4.17), (1.4.15) make $\mathbf{C}[z_1, z_2 \ldots, z_n]$ into an $\left(\underline{sp}(n, C) \cap \underline{u}(n, n), \tilde{U}(n)\right)$-module . It is known [H2] that this is the Harish-Chandra module of $\left(\omega, V_J\right)$.

The lie algebra $\underline{sp}(n, \mathbf{R})$ is isomorphic to $\underline{sp}(n, \mathbf{C}) \cap \underline{u}(n, n)$ via the following map:

$$(1.4.18) \qquad \underline{sp}(n, \mathbf{R}) \ni X \to C_n X C_n^{-1} \in \underline{sp}(n, C) \cap \underline{u}(n, n) ,$$

where

$$(1.4.19) \qquad C_n = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_n & iI_n \\ I_n & iI_n \end{bmatrix} .$$

Let for $x \in R$ and $\pi = 3.14\ldots$,

$$(1.4.20) \qquad \Psi_m(x) = \left(\pi^m\, m!\right)^{-1/2} \left(-1/2\, \partial_x + \pi x\right)^m e^{-\pi x^2} , \ (m = 0, 1, 2, \ldots) .$$

Define an isometry from $L^2(\mathbb{R}^n)$ to the Hilbert space V_J by the linear extension of the following map of the orthonormal basis:

$$(1.4.21)\qquad \Psi_\alpha \to \frac{z^\alpha}{\sqrt{\alpha!}} \qquad\qquad (\alpha \in Z_+^n) ,$$

where $\Psi_\alpha(x_1, x_2, \ldots, x_n) = \Psi_{\alpha_1}(x_1)\, \Psi_{\alpha_2}(x_2) \cdots \Psi_{\alpha_n}(x_n)$.

Combining the lift of (1.4.18) with the pullback of (1.4.21) we get the following commutative diagram

$$(1.4.22)\qquad \begin{array}{ccc} \widetilde{Sp}(n, \mathbb{R}) & \longrightarrow & (Sp(n, \mathbb{C}) \cap U(n, n))^\sim \\ \big\downarrow & & \big\downarrow \omega \\ U(L^2(\mathbb{R}^n)) & \longleftarrow & U(V_J) \end{array}$$

where the left vertical arrow is defined by the three others.
This way we obtain a realization of the oscillator representation ω of $\underline{sp}(n, R)$ on $L^2(R^n)$. A straightforward calculation involving the properties [H7 (1.7.14)] of the Hermite functions Ψ_m (1.4.20) shows that this realization coincides with the one described in (1.3.24).

§5. The degree of a representation in the Oscillator Duality Correspondence.

Let J be a compatible, positive, complex structure on W as in §4. Let

(1.5.1) $\quad$ $\lg J$ be the element of the Lie algebra of the center of U_J such that

(1.5.2) $\quad$ $J = \exp(\lg J)$ and

(1.5.3) $\quad$ the norm of $\lg J$ is minimal with respect to this property.

In the example (1.4.6)

$$(1.5.4)\qquad \lg J = i\,\frac{\pi}{2}\begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} .$$

Since all the compatible, positive complex structures on W are conjugate via the adjoint action of Sp(W) [H4 Ch I 12.4] all the Fock models are equivalent as representations of $\widetilde{Sp}$. In particular the spectrum of the operator $\omega(\lg J)$ does not depend on the particular Fock model V_J. Therefore (1.5.4) and the last formula in (1.4.10) imply that for any Fock model V_J of ω

(1.5.5) V_J decomposes into a direct sum of finite dimensional eigenspaces for $\omega(\lg J)$ with eigenvalues

$$i \frac{\pi}{2} (d + \frac{n}{2}) \qquad (d = 0,1,2,..; \ 2n = \dim_{\mathbf{R}}(W)).$$

Here $\pi = 3.14...$.

In the example (1.4.6) the eigenspace with eigenvalue $i \frac{\pi}{2} (d + \frac{n}{2})$ (1.5.5) is simply the space of homogenous polynomials in $\mathbf{C}[z_1, z_2, .., z_n]$ of degree d. Let L be a closed subgroup of U_J. Following [H2] we introduce the

(1.5.6) Def. For $\sigma \in R(\tilde{L},\omega)$ let $\deg_J(\sigma)$ be the minimal eigenvalue of the operator

(1.5.7) $$(i \frac{\pi}{2})^{-1}(\omega(\lg J) - \frac{n}{2} I)$$

on the σ-isotypic component of V_J.

Assume that L,L' is a reductive dual pair in $Sp(W)$ and that J' is another compatible positive complex structure on W such that $L \subseteq U_{J'}$. Then L centralizes both J and J'. Therefore J and J' belong to L'. Moreover it follows from the known structure of the pairs L,L' [H4 Ch I §6] that

(1.5.8) $\lg J$ and $\lg J'$ are conjugate inside L'.

Indeed, a sketch of an argument for (1.5.8) looks as follows. One can, clearly, assume that L,L' is a, so called, irreducible pair [H5]. If L' is a symplectic group--the case of our interest in this paper--then (1.5.8) follows directly from [H4 Ch I §6]. In the remaining (two) cases an analysis similar to [H4 Ch I §6], the substance of which is that all maximal compact subgroups of L' are conjugate, yields (1.5.8).

This statement implies that for any $\sigma \in R(\tilde{L},\omega)$ the spectra of $\omega(\lg J)$ and $\omega(\lg J')$ on the σ-isotypic components of V_J and $V_{J'}$, respectively, coincide. In particular

(1.5.9) $\deg_J(\sigma) = \deg_{J'}(\sigma)$.

In the example (1.4.6)

(1.5.10) $\deg_J(\sigma)$ is the minimal degree of a non-zero polynomial in the σ-isotypic component of $\mathbf{C}[z_1, z_2, \ldots, z_n]$.

This explains the notation: $\deg_J(\sigma)$. The statement (1.5.9) implies that for a given reductive dual pair L, L' with L compact and for $\sigma \in R(\tilde{L}, \omega)$ one can define

(1.5.11) $\deg(\sigma)$ to be $\deg_J(\sigma)$ for any Fock model V_J of ω such that $L \subseteq U_J$.

Though our main concern in this paper is to compute the Oscillator Duality Correspondence for a small dual pair, it is easier to state and prove the following lemma in general.

Assume that $W = W_1 \oplus W_2$ as in (1.3.1). Let L_j, L_j' be a reductive dual pair in $Sp(W_j)$ with L_j compact ($j=1,2$). Then $L_1 \times L_2$ and $L_1' \times L_2'$ act on W coordinatewise, and form a reductive dual pair in $Sp(W)$. Let ω_j denote the oscillator representation of $\tilde{S}p(W_j)$, $j = 1,2$.

(1.5.12) Lemma. If $\sigma_j \in R(\tilde{L}_j, \omega_j)$ for $j=1,2$, then $\sigma_1 \check{\otimes} \sigma_2 \in R((L_1 \times L_2)^{\sim}, \omega)$ and

$$\deg(\sigma_1 \check{\otimes} \sigma_2) = \deg_1(\sigma_1) + \deg_2(\sigma_2).$$

Here $\deg_j$ refers to the pair L_j, L_j', and we identify $\sigma_1 \check{\otimes} \sigma_2$ with its pull to $\tilde{L}_1 \times \tilde{L}_2$.

Proof: It is known [H6] that the pull back of ω to $\tilde{S}p(W_1) \times \tilde{S}p(W_2)$ is isomorphic to $\omega_1 \check{\otimes} \omega_2$. We may choose the Fock models of ω_1 and ω_2 satisfying (1.5.11) for L_1 and L_2 in such a way that their Harish-Chandra modules are spaces P^1 and P^2 of polynomials as in (1.4.6), respectively. Then we have the following $\tilde{L}_1 \times \tilde{L}_2$-isotypic direct sum decomposition

$$(1.5.13) \qquad P^1 \otimes P^2 = \bigoplus_{\sigma_1,\sigma_2} P^1_{\sigma_1} \otimes P^2_{\sigma_2}$$

where σ_j varies over $R(\tilde{L}_j, \omega_j)$, $i = 1,2$. Since the degree of a product of two homogenous polynomials is equal to the sum of their degrees the Lemma follows from (1.5.13).

Q.E.D.

Let G,G' be any reductive dual pair in Sp(W). Modulo eventual conjugation inside Sp(W) we may assume that

(1.5.14) $\quad K = G \cap U_J$ and $K' = G' \cap U_J$ are maximal compact subgroups of G and G' respectively.

Let

$$(1.5.15) \qquad \Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}', \omega).$$

Since K (K') and its centralizer in Sp(W) form a reductive dual pair [H2 §5] the statement (1.5.11) applied to L = K (L = K') implies that independently of J satisfying (1.5.14) we may define

$$\deg \Pi = \min \{\deg \sigma \mid \sigma \text{ is a } \tilde{K}\text{-type of } \Pi\},$$

(1.5.16) $\quad D(\Pi) = \{\sigma \in (\tilde{K})^\wedge \mid \deg \sigma = \deg \Pi\}$, and analogously $\deg \Pi'$, $D(\Pi')$.

The number <u>$\deg \Pi$</u> will be called <u>the degree of the representation Π</u>, and the set <u>$D(\Pi)$ the set of lowest degree $\tilde{K}$-types of Π</u>. It follows from the Lemma 4.1 in [H2] that

(1.5.17) $\quad$ under the above assumptions $\deg \Pi = \deg \Pi'$.

Fix a Fock model V_J (1.4.3) of ω, and under the assumption (1.5.14) define [H2 (3.8)]

(1.5.18) $H(K)_\sigma$ to be the $\deg_\sigma$-eigenspace of the operator (1.5.7) on $(V_J)_\sigma$, and similarly $H(K')_{\sigma'}$.

In the example (1.4.6),

(1.5.19) $H(K)_\sigma$ is the subspace of $\mathbf{C}[z_1,z_2,..,z_n])_\sigma$ of homogenous polynomials of degree $\deg \sigma$.

By combining the Lemmas 3.3 and 4.1 in [H2] we deduce the following

(1.5.20) Theorem. If $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}',\omega)$ and $\sigma \in D(\Pi)$ then there is a unique $\sigma' \in D(\Pi')$ and a unique subspace $H_{\sigma,\sigma'}$ of V_J such that

(1.5.21) $H_{\sigma,\sigma'}$ is not annihilated by a quotient map from V_J^∞ to Π,

(1.5.22) $H_{\sigma,\sigma'}$ is contained in the $\deg \sigma$-eigenspace of (1.5.7),

(1.5.23) $H_{\sigma,\sigma'}$ is $\tilde{K} \cdot \tilde{K}$-invariant and is of type σ as a $\tilde{K}$-module.

Moreover

(1.5.24) $H_{\sigma,\sigma'} = H(K)_\sigma \cap H(K')_\sigma \cong \sigma \check{\otimes} \sigma'$ as a $\tilde{K} \times \tilde{K}'$-module.

(1.5.25) Corollary. Let $V \neq 0$ be a closed $\tilde{G} \cdot \tilde{G}'$-invariant subspace of the Hilbert space V_J. Assume that V is isotypic for the action of $\tilde{G}$. Then as a $\tilde{G} \cdot \tilde{G}'$-module

(1.5.26) $V \cong \Pi \check{\otimes} \Pi'$ for a $\Pi \in (\tilde{G})^\wedge$ and a $\Pi' \in (\tilde{G}')^\wedge$, and

(1.5.27) $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}',\omega)$.

Assume that the Fock model V_J and $Sp(W)$ are chosen as in the example (1.4.6). Suppose that V has a cyclic vector

(1.5.28) $$f(z) = \sum_\alpha a_\alpha \frac{z^\alpha}{\sqrt{\alpha!}} \qquad (\alpha \in \mathbf{Z}_+{}^n).$$

Let d be the minimal integer for which the d-homogenous component $f_d(z)$ of $f(z)$ is non-zero. Assume that $d < \deg \Pi$ and that the action of $\tilde{K}$ on $f_d(z)$ generates an irreducible subspace of type $\sigma \in (\tilde{K})^\wedge$. Then

(1.5.29) $\sigma \in D(\Pi)$, $d = \deg \Pi$, and the subspace generated by the $\tilde{K} \cdot \tilde{K}'$-action on $f_d(z)$ is equal to $H_{\sigma,\sigma'}$ (1.5.24).

Proof: Since under the oscillator representation ω of $\tilde{Sp}$ the von Neumann algebras generated by $\omega(\tilde{G})$ and $\omega(\tilde{G}')$ are mutual commutants [H2 Theorem 6.1] the space V is irreducible as a $\tilde{G} \cdot \tilde{G}'$-module and (1.5.26) holds. Let

(1.5.30) Pr be the projection operator from V_J onto V.

Then the restriction of Pr to V_J^∞ is continuous and we see that (1.5.27) is valid.

Since the scalar product

(1.5.31) $$\big(f(z),\ f_d(z)\big) = \sum_{|\alpha|=d} |a_\alpha|^2 > 0$$

the $\tilde{G} \cdot \tilde{G}'$-intertwining operator Pr does not annihilate the subspace H_d generated by the $\tilde{K} \cdot \tilde{K}'$-action on $f_d(z)$. In particular σ is a $\tilde{K}$-type of Π and since $d \leqslant \deg \Pi$, σ is a lowest degree $\tilde{K}$-type of Π. By the Theorem (1.5.20) there is a unique $\sigma' \in D(\Pi')$ such that $H_d = H_{\sigma,\sigma'}$.

Q.E.D.

(1.5.32) <u>Lemma</u> Let L' be the centralizer of K in Sp. Assume that $\Pi \check{\otimes} \Pi' \in R(\tilde{G} \cdot \tilde{G}',\omega)$, $\sigma \in D(\Pi)$ and that $\sigma \check{\otimes} \Pi'_{L'} \in R(\tilde{K} \cdot \tilde{L}',\omega)$. Suppose that $\tau' \in D(\Pi'_{L'})$ and $\sigma' \in D(\Pi')$ correspond to σ via (1.5.26). Then σ' is the unique $\tilde{K}'$-type of τ' which occurs in $R(\tilde{K}',\omega)$ and has degree equal to $\deg \sigma$.

Proof: This is just the statement (3.15) in [H2].

Q.E.D.

CHAPTER 2
A CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS

Since the goal of this paper is to describe the Oscillator Duality Correspondence (1.2.18) for the pair $O_{2,2}$, $Sp(2,\mathbb{R})$ we have to choose a parametrization of the set of infinitesimal equivalence classes of irreducible admissible representations of these groups. Vogan's classification by sets of character data (2.3.14) and lowest K-types (2.2.18) seems to be the best choice. The reader will find complete lists of irreducible admissible and unitary representations of these groups in §4 and §5.

In §1 we review two different norms (2.1.13), (2.1.21) on K-types introduced by Vogan and Carmona and compare them with the notion of degree (1.5.16) on examples.

The lemma (2.2.23) explains how, using the notions of lowest and of lowest degree K-types, one can attempt to identify representations being in the Oscillator Duality Correspondence.

§1. The maximal compact subgroups

Fix the following maximal compact subgroups

(2.1.1) $\qquad O_{2,2} \cap O(4)$, and

(2.1.2) $\qquad Sp(2,\mathbb{R}) \cap O(4)$

of $O_{2,2}$ (1.1.5) and $Sp(2,\mathbb{R})$ respectively. The isomorphism (1.1.9) maps (2.1.1) onto the group

(2.1.3) $\qquad O(2,2) \cap O(4)$

which is isomorphic to $O(2) \times O(2)$ via the map

(2.1.4) $\qquad O(2) \times O(2) \ni (g_1,g_2) \to \mathrm{diag}(g_1,g_2) \in O(2,2) \cap O(4)$.

The unitary dual of $O(2) \times O(2)$ consists of outer tensor products

$$(2.1.5) \qquad \pi_1 \check{\otimes} \pi_2 \qquad (\pi_1, \pi_2 \in \hat{O}(2)).$$

It is well known [V1 1.4.2] that

$$(2.1.6) \qquad \hat{O}(2) = \{1,\ \det,\ \mathrm{Ind}\ \chi_n \mid n = 1,2,3,\ldots\}$$

where 1 stands for the trivial representation

$$\mathrm{Ind} \text{ for } \mathrm{Ind}_{SO(2)}^{O(2)} \quad \text{and} \quad \chi_n\begin{pmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{pmatrix} = e^{inx}.$$

To shorten the notation let us write (for m, n = ±1,±2,±3,.., and p,q = 0,1)

$$(2.1.7) \qquad \begin{aligned} \pi_{m,n} &= \mathrm{Ind}\ \chi_m \check{\otimes} \mathrm{Ind}\ \chi_n, \\ \pi^p_{m,0} &= \mathrm{Ind}\ \chi_m \check{\otimes} (\det)^p, \\ \pi^p_{0,n} &= (\det)^p \check{\otimes} \mathrm{Ind}\ \chi_n, \\ \pi^{p,q}_{0,0} &= (\det)^p \check{\otimes} (\det)^q \end{aligned}$$

Since $\mathrm{Ind}\ \chi_n = \mathrm{Ind}\ \chi_{-n}$ we may replace m by -m or n by -n in (2.1.7) without changing the representation.

(2.1.8) Via a composition of the isomorphisms (2.1.4), (1.1.9) we shall use (2.1.7) to parametrize the unitary dual of (2.1.1).

The irreducible unitary representations of the group U(2) are well known. They can be described in terms of highest weights as follows. Each $\pi \in \hat{U}(2)$ defines an irreducible representation of the Lie algebra $\underline{u}(2)$. This representation can be extended to the complexification $\underline{gl}(2,\mathbf{C})$ of $\underline{u}(2)$.

For $m,n \in \mathbf{Z}$, $m \geq n$, let

(2.1.9) $\pi'_{m,n}$ be an irreducible unitary representation of U(2) which contains a vector v stabilized by the upper triangular Borel subalgebra of $\underline{gl}(2,\mathbf{C})$ and such that $\mathrm{diag}(u_1,u_2) \in \underline{gl}(2,\mathbf{C})$ acts on v via multiplication by $mu_1 + nu_2$.

Then

(2.1.10) $$\hat{U}(2) = \{\pi'_{m,n} \mid m,n \in \mathbf{Z},\ m \geq n\}.$$

Combining the isomorphisms (1.4.18), (1.4.15) we shall use (2.1.10) to parametrize the unitary dual of (2.1.2).

Let in the rest of this paragraph G be any real reductive Lie group in the sense of Vogan [V1 0.1.2] with a maximal compact subgroup K. Choose a maximal torus T of K and a positive root system $\Delta^+(\underline{\mathbf{k}},\underline{\mathbf{t}})$. Put

(2.1.11) $$2\rho_c = \Sigma\ \Delta^+(\underline{\mathbf{k}},\underline{\mathbf{t}}) \text{ (see "Notation").}$$

For a $\mu \in i\underline{t}^*$ put

(2.1.12) $$\|\mu\|_v^2 = (\mu+2\rho_c,\ \mu+2\rho_c)$$

where (,) denotes the inner product on $\underline{t}^*$ obtained from the Killing form on $\underline{g}$ by restriction to $\underline{t}$ and dualization.

(2.1.13) Def [V1 5.4.18]. If $\pi \in \hat{K}$ has a highest weight $\mu \in i\underline{t}^*$ define the norm of π

$$\|\pi\| = \|\mu\|_v.$$

(2.1.14) Example. For the group $G = Sp(2,\mathbf{R})$

$$\|\pi'_{m,n}\|^2 = (m+1)^2 + (n-1)^2 = \frac{1}{2}\left((m+n)^2 + (m-n+2)^2\right)$$

The group $0_{2,2}$ does not satisfy [V1 0.1.2b]. Notice however that the definition (2.1.12) does not depend on G containing K. We may regard $O(2) \times O(2)$ as a maximal compact subgroup of $GL(2) \times GL(2)$ which does satisfy [V1 0.1.2]. Doing so we have

$$\|\pi_{m,n}\|^2 = m^2+n^2$$

(2.1.15) $$\|\pi^p_{m,o}\| = \|\pi^p_{o,m}\| = |m|$$

$$\|\pi^{p,q}_{o,o}\| = 0 \qquad (2.1.9).$$

Let H be the centralizer of T in G. For any $\mu \in i\underline{t}^*$, which is dominant with respect to $\Delta^+(\underline{\mathbf{k}},\underline{\mathbf{t}})$ (i.e. $(\mu,\alpha) \geq 0$ for all $\alpha \in \Delta^+(\underline{\mathbf{k}},\underline{\mathbf{t}})$) there is a θ-invariant positive root system $\Delta^+(\underline{\mathbf{g}},\underline{\mathbf{h}})$ making $\mu + 2\rho_c$ dominant [V1 pp 239]. Let

(2.1.16) $$2\rho = \sum_{\alpha\in\Delta^+(\underline{\mathbf{g}},\underline{\mathbf{h}})} \alpha .$$

Then

(2.1.17) $$\mu+2\rho_c - \rho$$

is not necessarily dominant with respect to $\Delta^+(\underline{\mathbf{g}},\underline{\mathbf{h}})$. Define a closed Weyl chamber in $\underline{\mathbf{h}}^*$:

(2.1.18) $$\underline{\underline{C}} = \{\gamma \in \underline{\mathbf{h}}^* \mid \mathrm{Re}(\gamma,\alpha) \geq 0,\ \alpha \in \Delta^+(\underline{\mathbf{g}},\underline{\mathbf{h}})\}$$

Let

(2.1.19) $\lambda = \text{lambda}\ (\mu)$ be the point in $\underline{\underline{C}}$ closest to $\mu+2\rho_c - \rho$.

Here we measure distance using the usual scalar product on $\mathbf{h}^*$. (See the sentence after (2.3.10).)

Since $\underline{\underline{C}}$ is closed and convex, lambda (μ) is well defined. Put

(2.1.20) $$\|\mu\|_{\text{lambda}} = (\lambda,\lambda) \qquad (\mu \in it^*,\ \lambda = \text{lambda}\ (\mu)).$$

(2.1.21) Def. If $\pi \in \hat{K}$ has a highest weight $\mu \in \underline{\mathbf{t}}^*$, set

$$\|\pi\|_{\text{lambda}} = \|\mu\|_{\text{lambda}}.$$

(2.1.22) Example. A straightforward calculation based on the above definitions shows that for $G = Sp(2,\mathbf{R})$, in the notation (2.1.9),

$$\|\pi'_{m,n}\|_{\text{lambda}} = \begin{cases} ((m-1)^2+(n-2)^2)^{1/2} & m \geq n \geq 2 \\ m-1 & m \geq 1\ \&\ 0 \leq n \leq 2 \\ ((m-1)^2+n^2)^{1/2} & m \geq 1\ \&\ 1-m \leq n \leq 0 \\ \frac{\sqrt{2}}{2}(m-n-1) & m+n = 0,\ m-n \geq 1 \\ 0 & m = n = 0 \end{cases}$$

and $\|\pi'_{m,n}\|_{lambda} = \|\pi'_{-n,-m}\|_{lambda}$ $(m+n \geq 0)$.

Carmona [Ca §2] has shown that the construction (2.1.21) is the same as Vogan's [V1 5.3.3, 5.4.1].

Finally we notice that det $g = 1$ for g in the image of the group (2.1.1) or (2.1.2) in U(8). Therefore we may choose the following splitting

$$g \to (g,1)$$

of the covering (1.4.16) over this image. In this sense we shall talk about the degree (1.5.11) of the representations (2.1.7) and (2.1.9). It is known [H2], and easy to check, that the centralizer of the image of (2.1.1) in $Sp(8,\mathbb{R})$ (1.1.12) is isomorphic to $Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R})$. Similarly the centralizer of (2.1.2) in $Sp(8,\mathbb{R})$ is isomorphic to U(2,2). Since the Oscillator Duality Correspondence is well understood for the pairs U(2), U(2,2) and O(2), $Sp(2,\mathbb{R})$ (see [K-Ve III §6 and II §6]) we find that

$$\text{(2.1.23)} \qquad \deg(\pi'_{m,n}) = |m| + |n|$$

and (using (1.5.12))

$$\deg(\pi_{m,n}) = |m| + |n|$$

$$\text{(2.1.24)} \qquad \deg(\pi^{p}_{m,o}) = \deg(\pi^{p}_{o,m}) = |m| + 2p$$

$$\deg(\pi^{p,q}_{0,0}) = 2p + 2q$$

§2 The Cuspidal Data

As we have already noticed the group $O_{2,2}$ is not a real reductive Lie group in the sense of Vogan [V1, 0.1.2]. This is only a minor obstacle because the subgroup, of index two, $SO_{2,2} = O_{2,2} \cap SL(2,\mathbb{R})$ satisfies [V1 0.1.2]. Thus in the rest of this chapter we denote by G any group satisfying [V1 0.1.2] or $G \cong O_{2,2}$ and leave to the reader the verification of the theorems in this case (compare [P1]).

Let P be a parabolic subgroup of G with Langlands decomposition

(2.2.1) $P = MAN$ [Wa pp. 81].

(2.2.2) Example. For $G = O_{2,2}$ let

$$P_\alpha = M_\alpha A_\alpha N_\alpha \qquad (\alpha = 0,1)$$

where

(2.2.3) $$M_0 = \{\operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2) \mid \varepsilon_1, \varepsilon_2 = \pm 1\},$$

(2.2.4) $$A_0 = \{\operatorname{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1, a_2 > 0\},$$

(2.2.5) $$\underline{n}_0 = \left\{ \begin{bmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{bmatrix} \mid x, y \in \mathbf{R} \right\}, \text{ and}$$

(2.2.6) $$M_1 = \{\operatorname{diag}(g, (g^t)^{-1}) \mid g \in SL^{\pm}(2, \mathbf{R})\} \quad [\text{VI, ChI § 4}]$$

(2.2.7) $$A_1 = \{\operatorname{diag}(a, a, a^{-1}, a^{-1}) \mid a > 0\},$$

(2.2.8) $$\underline{n}_1 = \left\{ \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \mid y \in \underline{o}\,(2) \right\}$$

The groups P_0, P_1 and $O_{2,2}$ exhaust (up to conjugacy) all parabolic subgroups of $O_{2,2}$.

(2.2.9) Example. Let $G = Sp(2, \mathbf{R})$. Define

(2.2.10) $$P'_\alpha = M_\alpha A_\alpha N'_\alpha \qquad (\alpha = 0,1)$$

with M_α, A_α as in (2.2.2) and

(2.2.11) $$\underline{n}'_0 = \left\{ \begin{bmatrix} 0 & x & u & v \\ 0 & 0 & v & w \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{bmatrix} \mid x, u, v, w \in \mathbf{R} \right\},$$

(2.2.12) $$\underline{n}'_1 = \left\{ \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \mid y \in S(2, \mathbf{R}) \right\} \qquad (1.3.28).$$

There is exactly one more proper parabolic subgroup $P_2' = M_2 A_2 N_2$ in $Sp(2,\mathbb{R})$ not conjugate to (2.2.10) with $N_2 \subseteq N_0'$ and

(2.2.13) $$M_2 = \left\{ \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \varepsilon & 0 \\ 0 & c & 0 & d \end{bmatrix} \;\middle|\; \varepsilon = \pm 1;\ ad - bc = 1;\ a,b,c,d \in \mathbb{R} \right\},$$

(2.2.14) $$A_2 = \{\mathrm{diag}(a,1,a^{-1},1) \mid a > 0\}.$$

All P_α as P'_α listed above are cuspidal in the sense that M_α has a compact Cartan subgroup [B-W III §4.1].

Let K be a maximal compact subgroup of G, compatible with the Cartan decomposition of $\underline{g}$ as in [V1, 0.1.2].

(2.2.15) Def [K, VII §11]. An irreducible admissible representation δ of the group M (2.2.1) is called tempered iff all its K-finite matrix coefficients are in $L^p(M)$ for any $p > 2$. The representation δ is said to belong to the discrete series of M iff all these matrix coefficients are in $L^2(M)$.

Remark: Tempered representations are unitary.
For a fixed parabolic subgroup P (2.2.1), a linear functional $\nu \in \underline{a}^*$ and a discrete series representation δ of M let

(2.2.16) $\mathrm{Ind}_P^G(\delta \otimes \nu)$ denote the parabolically induced representation of G [V1 Ch4 §1] (This is normalized induction.)

(2.2.17) Def [V1 6.6.11] A set of cuspidal data for G is a triple (MA,δ,ν), where P = MAN is a cuspidal parabolic subgroup of G, δ is a discrete series representation of M and $\nu \in \underline{a}^*$.

(2.2.18) Def [V1 5.4.18]. Let Π be a representation of G.
A representation $\pi \in \hat{K}$ occurring in $\Pi|K$ is called a lowest K-type of Π iff $\|\pi\| \leq \|\sigma\|$ (2.1.13) for any K-type σ of Π.

Finally we are ready to state the classification.

(2.2.19) Theorem. Let Π be an irreducible admissible representation of G and let $\pi \in \hat{K}$ be a lowest K-type of Π. Then there is a set of cuspidal data (MA,δ,ν) such that

(2.2.20) π is a lowest K-type in $\mathrm{Ind}_{M\cap K}^{K}(\delta|_{M\cap K})$;

(2.2.21) for any parabolic subgroup P with Langlands decomposition $P = MAN$, Π is infinitesimally equivalent to the unique irreducible subquotient of (2.2.16) containing the K-type π;

(2.2.22) if $(M'A',\delta',\nu')$ is another set of cuspidal data satisfying (2.2.21), then $(M'A',\delta',\nu')$ is conjugate by K to (MA,δ,ν).

This is a simple consequence of the theorems [V1, 6.6.14, 6.6.15], [V2, 5.2] and the fact [V1, page 297] that for an irreducible representation Π, as in (2.2.18), the set of lowest K-types of Π is equal to the set of K-types π of Π with $\|\pi\|_{\text{lambda}}$ minimal.

Consider the pull back of the oscillator representation ω (1.3.24) to $G \times G'$, with $G = O_{2,2}$, $G' = Sp(2,\mathbb{R})$, obtained via (1.1.6), (1.1.12). Denote by K and K' the maximal compact subgroups (2.1.1) and (2.1.2) respectively. Then of course $K \times K'$ is a maximal compact subgroup of $G \times G'$. We identify G and G' with their images in $Sp(8,\mathbb{R})$.

(2.2.23) Lemma. Let (MA,δ,ν) and $(M'A',\delta',\nu')$ be sets of cuspidal data for G and G' respectively. Extend MA and $M'A'$ to two parabolic subgroups $P = MAN \subseteq G$ and $P' = M'A'N' \subseteq G'$. Assume that

(2.2.24) there is a continuous, linear, $G \times G'$ intertwining map

$$\omega^\infty \to \mathrm{Ind}_P^G(\delta \otimes \nu) \check{\otimes} \mathrm{Ind}_{P'}^{G'}(\delta' \otimes \nu'),$$

(2.2.25) $\sigma \check{\otimes} \sigma'$ is a lowest $K \times K'$-type (2.2.18) of the induced

representation (2.2.24), and the space $H_{\sigma,\sigma'}$ (1.5.24) is non-zero

(2.2.26) the operator (2.2.24) does not annihilate the subspace $H_{\sigma,\sigma'}$ (1.5.30) of ω^∞, (2.2.27) $\mathrm{Ind}_P^G(\delta \otimes \nu)$ has an irreducible quotient Π containing the K-type σ, and $\sigma \in D(\Pi)$ (1.5.16).

Then the representation Π' of G' corresponding to Π via the Oscillator Duality Correspondence (1.2.18)

(2.2.28) is the unique irreducible subquotient of

$\mathrm{Ind}_{P'}^{G'}(\delta' \otimes \nu')$ containing the K'-type σ', and $\sigma' \in D(\Pi')$.

Proof: By assumption we have the following sequence of $G \times G'$-intertwining maps

$$\begin{array}{ccl} \omega^\infty \xrightarrow{(2.2.24)} & \mathrm{Ind}_P^G(\delta \otimes \nu) \check{\otimes} \mathrm{Ind}_{P'}^{G'}(\delta' \otimes \nu') \\ & \big\downarrow Q \otimes \mathrm{id} \\ & \Pi \check{\otimes} \mathrm{Ind}_{P'}^{G'}(\delta' \otimes \nu') \end{array} \tag{2.2.29}$$

where Q denotes the quotient map (2.2.27). Since Q is faithful on the σ-isotypic component of the induced representation (2.2.24), the assumption (2.2.25) implies that $Q \check{\otimes} \mathrm{id} \circ (2.2.24) \neq 0$. The parabolically induced epresentation (2.2.28) has a finite composition series [V1 4.1.12, 0.3.21]. Therefore there is an irreducible quotient

$$\Pi \check{\otimes} \Pi' \in R(G \cdot G', \omega)$$

of the image of ω^∞ in $\Pi \check{\otimes} \mathrm{Ind}_{P'}^{G'}(\delta' \otimes \nu')$ under (2.2.24). By (2.2.27), Π' contains the K'-type σ'.

Q.E.D.

<u>Remark.</u> This lemma with the above proof holds for any reductive dual pair $\tilde{G}, \tilde{G}'$. To see this one has to verify (2.2.19) in this general situation (see [P1]).

§3. THE CHARACTER DATA

The Theorem (2.2.19) assumes understanding of the discrete series representations $\delta \in \hat{M}$. For the group $O_{2,2}$ this is easy because its identity component is covered by $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and the theory of highest weight representations is sufficient (see [V1 Ch I] or [K Ch II § 5]). The discrete series of $Sp(2,\mathbb{R})$ requires more attention. It can be obtained from some unitary characters of a compact Cartan subgroup by applying the Zuckerman functors [V1 6.6.9]. Using them one can state the classification theorem (2.2.19) in terms of lowest K-types and irreducible representations of Cartan subgroups of G.

To be more precise we need some definitions. For a cuspidal parabolic subgroup P = MAN (2.2.1) let

(2.3.1) T be a compact Cartan subgroup of M, and

(2.3.2) H = TA.

Then H is a θ-stable Cartan subgroup of G [V1 6.6.12].

(2.3.3) Example. The following two groups (2.3.4), (2.3.5), may be regarded as θ-stable Cartan subgroups of either $O_{2,2}$ or $Sp(2,\mathbb{R})$ (as defined explicitly in (1.1.5) and in "Notation"). We will use the fact that these θ-stable Cartans in the two groups coincide to identify the character data associated to them (see (2.3.14) for the definition of character data).

(2.3.4) $H_0 = M_0 A_0$ (2.2.3);

(2.3.5) $H_1 = T_1 A_1$, with

$$T_1 = \left\{\mathrm{diag}(g,g),\ g = \begin{pmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{pmatrix},\ x \in \mathbb{R}.\right\}$$

and A_1 as in (2.2.7).

Moreover there is a compact Cartan subgroup of $O_{2,2}$

(2.3.6) $$H_3 = \left\{\frac{1}{2}\begin{bmatrix} g_1+g_2 & g_1-g_2\\ g_1-g_2 & g_1+g_2\end{bmatrix} \mid g_1,\ g_2 \in SO(2)\right\}.$$

The groups (2.3.4), (2.3.5), (2.3.6) exhaust all conjugacy classes of the θ-stable Cartan subgroups of $O_{2,2}$.

In $Sp(2,\mathbf{R})$ there are two other conjugacy classes represented by

(2.3.7) $$H_3' = \left\{ \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & c_2 & 0 & s_2 \\ -s_1 & 0 & c_1 & 0 \\ 0 & -s_1 & 0 & c_2 \end{bmatrix} \middle| c_j = \cos(x_j), s_j = \sin(x_j),\ x_j \in R,\ j = 1,2 \right\},$$

and

(2.3.8) $H_2 = T_2 A_2$ with T_2 consisting of elements

(2.3.9) $$\begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & c & 0 & s \\ 0 & 0 & \varepsilon & 0 \\ 0 & -s & 0 & c \end{bmatrix} \quad (\varepsilon = \pm 1,\ c = \cos x,\ s = \sin x,\ x \in \mathbf{R}),$$

and A_2 as in (2.2.14).

Let

(2.3.10) $\underline{h} = \underline{t} \oplus \underline{a}$ be the Lie algebra of H (2.3.2).

By restriction and dualization the Killing form on $\underline{g}$ provides an inner product (,) on $\underline{\mathbf{h}}^*$, (real on $it^* + a^*$)

(2.3.11) Def. [V1 6.6.1] A regular character of H (2.3.2) is a pair (Γ,γ), where $\Gamma : H \to \mathbf{C}^\times$ is a group homomorphism and $\gamma \in \underline{\mathbf{h}}^*$, such that

(2.3.12) if $\alpha \in \Delta(\underline{\mathbf{m}},\underline{\mathbf{t}})$, then (α,γ) is real and not zero, and

(2.3.13) if $\Delta^+(\underline{\mathbf{m}},\underline{\mathbf{t}}) = \{\alpha \in \Delta(\underline{\mathbf{m}},\underline{\mathbf{t}}) \mid (\alpha,\gamma) > 0\}$
then the differential

$$d\Gamma = \gamma + \frac{1}{2} \Sigma\ \Delta^+(\underline{\mathbf{m}},\underline{\mathbf{t}}) - \Sigma\ \Delta^+(\mathbf{m} \cap \mathbf{k},\mathbf{t}).$$

(2.3.14) Def [V1 pp. 410] A set of character data for G is a triple (H,Γ,γ) where H is a θ-stable Cartan subgroup of G and (Γ,γ) is a regular character of H.

To each set of cuspidal data (MA,δ,ν) (2.2.17) we can associate a set of character data (H,Γ,γ) by choosing H as in (2.3.2) and defining

(2.3.15) $\Gamma|_T =$ the highest weight [V1 5.1.1] of the unique lowest $M \cap K$-type of δ,

(2.3.16) $d\Gamma|_{\underline{a}} = \gamma|_{\underline{a}} = \nu.$

This map is bijective on the level of K-conjugacy classes [V1 6.6.12].

Let also

(2.3.17) $A\,[H,\Gamma,\gamma] = A\,[H,\Gamma]$

be the set of lowest K-types in the induced representation (2.2.20)

This set depends only on the conjugacy class $[H,\Gamma,\gamma]$ of (H,Γ,γ)

(2.3.18) <u>Theorem</u> [V1 6.6] Each (infinitesimal equivalence class of) irreducible admissible representation Π of G corresponds to a unique K-conjugacy class $[H,\Gamma,\gamma]$ of character data for G and a subset $\underline{\underline{A}}$ of $\underline{\underline{A}}[H,\Gamma,\gamma]$ in such a way that $\underline{\underline{A}} = \underline{\underline{A}}(\Pi)$ is the set of lowest K-types of Π. For such a Π we shall write

(2.3.19) $\Pi = \Pi_G[H,\Gamma,\gamma](\underline{\underline{A}}) = \Pi[H,\Gamma,\gamma](\underline{\underline{A}})$

Remark: The equivalence of (2.2.19) and (2.3.18) is explained in [V1 6.6]

The element $\gamma \in \underline{\mathbf{h}}^*$ in the Theorem (2.3.18) represents the infinitesimal character [V1 0.3.18] of the representation Π (2.3.19) in terms of the Harish-Chandra isomorphism [V1 0.2.8]. The Theorem (2.3.18) asserts that to identify Π one has to know its infinitesimal character γ, a lowest K-type of Π and a character Γ of the associated Cartan subgroup H. This subgroup is determined (up to K-conjugacy) by any lowest K-type of Π.

If Π is tempered (2.2.15), the situation is simpler.

(2.3.20) <u>Theorem</u> [V1 6.1.5] Let (H,Γ,γ) be a set of character data for G with $\gamma = \lambda \oplus \nu$, $\lambda \in \underline{\mathbf{t}}^*$, $\nu \in \underline{a}^*$. Assume that $\pi \in \underline{\underline{A}} \subseteq A[H,\Gamma,\gamma]$ and that $\Pi = \Pi_G[H,\Gamma,\gamma](\underline{\underline{A}})$ as in (2.3.19). Then

(2.3.21) $\|\lambda\| = \|\pi\|_{\text{lambda}} = \min\{\|\sigma\|_{\text{lambda}} \mid \sigma \text{ is a K-type of } \Pi\}$.

•(2.3.22) Corollary. Let Π be a tempered (irreducible) representation of G with real infinitesimal character [V1 5.4.11]. Assume that $\underline{h}$ is a fundamental Cartan subalgebra of $\underline{g}$ and that $\gamma \in \underline{\mathbf{h}}^*$ represents the infinitesimal character of Π. Let $\pi \in \hat{K}$ be a K-type of Π. Then

(2.3.23) $\|\pi\|_{\text{lambda}} = \|\gamma\|$

implies that π is a lowest K-type of Π.

Proof: Since Π is tempered $\nu \in i\underline{a}^*$ (2.3.16), [Tr]. The assumption that Π has a real infinitesimal character implies that $\nu \in \underline{a}^*$. Therefore $\nu = 0$ and $\|\gamma\| = \|\lambda\|$ (2.3.21).

Q.E.D.

(2.3.24) Remark: For a fixed θ-stable Cartan subgroup H one defines the corresponding Weyl group

(2.3.25) $W(H) = W(G,H) =$ (the normalizer of H in G)/H

This group acts on H by conjugation and also on regular characters of H by

$$w\Gamma(g) = \Gamma(w^{-1}gw).$$

Using the theorem (2.3.18) one can thus parametrize the admissible dual of G by choosing a maximal family $H_0, H_1, H_2, \ldots$ of non-conjugate θ-stable Cartan subgroups and for each H_j specifying a fundamental domain for the action of $W(G,H_j)$ on the set of regular characters of H_j. By a representation attached to H_j we shall mean any representation (2.3.19) of the form

$$\Pi_G[H_j, \Gamma, \gamma](\underline{\underline{A}}).$$

For our groups $O_{2,2}$ and $Sp(2,\mathbf{R})$ we will use the Cartans defined in (2.3.4),..., (2.3.8).

§4. THE UNITARY DUAL OF $Sp(2,\mathbb{R})$.

In this paragraph we describe the admissible dual and the unitary dual of $Sp(2,\mathbb{R})$ in terms of character data (2.3.14). For the proofs we refer the reader to Appendix A. For a complex number z we will write

(2.4.1) $z \succ 0$ if and only if either $\mathrm{Re}\, z > 0$ or $\mathrm{Re}\, z = 0$ and $\mathrm{Im}\, z \geq 0$.

The representations attached to the split Cartan subgroup H_0. (2.3.4)

Let

(2.4.2) $$\sigma_1,\sigma_2 = 0,1;\ \nu_1,\nu_2 \in \mathbf{C};\ \nu_1,\nu_2,\ \nu_1-\nu_2 \succ 0;\ \sigma_2 \geq \sigma_1 \text{ if } \nu_1 = \nu_2;$$

$$\Gamma(\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_1,\varepsilon_2)) = \varepsilon_1^{\sigma_1}\varepsilon_2^{\sigma_2};$$

$$\Gamma(\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})) = a_1^{\nu_1}a_2^{\nu_2};$$

$$\gamma = d\Gamma$$

(2.4.3) Proposition. For the parameters (2.4.2)

$$\underline{\underline{A}}[H_0,\Gamma,\gamma] = \begin{cases} \{1\} & \text{if } \sigma_1 = \sigma_2 = 0; \\ \{\pi'_{1,0},\pi'_{0,-1}\} & \text{if } \sigma_1 = 1 \text{ and } \sigma_2 = 0, \text{ or } \sigma_1 = 0 \text{ and } \sigma_2 = 1; \\ \{\pi'_{1,1},\pi'_{-1,-1}\} & \text{if } \sigma_1 = \sigma_2 = 1. \end{cases}$$

(2.4.4) Theorem. The irreducible admissible representations of $Sp(2,\mathbb{R})$ attached to the Cartan subgroup H_0,

(2.4.5) $$\Pi = \Pi[H_0,\Gamma,\gamma](\underline{\underline{A}}),$$

fall into eight disjoint families described by the parameters Γ,γ (2.4.2) and $\underline{\underline{A}}$ (2.3.19) as follows:

(2.4.6) $\sigma_1 = \sigma_2 = 0$, $\underline{\underline{A}} = \{1\}$ ($\Pi = 1$ for $\nu_1 = 2$, $\nu_2 = 1$);

(2.4.7) $\sigma_1 = 1,\ \sigma_2 = 0,\ \nu_1 \neq 0,\ \underline{\underline{A}} = \{\pi'_{1,0}, \pi'_{0,-1}\}$;

(2.4.8) $\sigma_1 = 0,\ \sigma_2 = 1,\ \nu_2 \neq 0,\ \underline{\underline{A}} = \{\pi'_{1,0}, \pi'_{0,-1}\}$;

(2.4.9) $\sigma_1 = 0,\ \sigma_2 = 1,\ \nu_2 = 0,\ \underline{\underline{A}} = \{\pi'_{1,0}\}$;

(2.4.10) $\sigma_1 = 0,\ \sigma_2 = 1,\ \nu_2 = 0,\ \underline{\underline{A}} = \{\pi'_{0,-1}\}$;

(2.4.11) $\sigma_1 = \sigma_2 = 1,\ \nu_1 \neq 0,\ \nu_2 \neq 0,\ \underline{\underline{A}} = \{\pi'_{1,1}, \pi'_{-1,-1}\}$;

(2.4.12) $\sigma_1 = \sigma_2 = 1,\ \nu_2 = 0,\ \underline{\underline{A}} = \{\pi'_{1,1}\}$;

(2.4.13) $\sigma_1 = \sigma_2 = 1,\ \nu_2 = 0,\ \underline{\underline{A}} = \{\pi'_{-1,-1}\}$.

(2.4.14) Theorem. The representations listed in (2.4.4) are unitary if and only if the following additional assumptions are satisfied.

(2.4.15) $\nu_1, \nu_2 \in i\,\mathbf{R}$, or $\nu_1, \nu_2 \in \mathbf{R}$ and $\nu_1 + \nu_2 \leq 1$, or $\nu_1 - \nu_2 \in i\,\mathbf{R}$ and $\nu_1 + \nu_2 \in \mathbf{R}$ and $\nu_1 + \nu_2 \leq 1$ in (2.4.6) and in (2.4.11); $\nu_2 \in i\,\mathbf{R}$ and $\nu_1 \in \mathbf{R}$ and $\nu_1 \leq 1$, or $\nu_1 = 2$ and $\nu_2 = 1$ in (2.4.6);

(2.4.16) $\nu_1, \nu_2 \in i\,\mathbf{R}$ in (2.4.7) and (2.4.8); $\nu_2 \in i\,\mathbf{R}$ and $\nu_1 \in \mathbf{R}$ and $\nu_1 \leq 1$ in (2.4.8);

(2.4.17) $\nu_1 \in i\,\mathbf{R}$ or $\nu_1 \in \mathbf{R}$ and $\nu_1 \leq 1$ in (2.4.9), (2.4.10), (2.4.12), (2.4.13)

The representations attached to H_1 (2.3.5).

Let

(2.4.18) $n \in \mathbf{Z},\ n \geq 1,\ \nu \in C,\ \nu > 0$;

$$\Gamma(\mathrm{diag}(g,g)) = e^{i(n+1)x};$$

$$\Gamma(\mathrm{diag}(a,a,a^{-1},a^{-1})) = a^{\nu};$$

$$\gamma|_{\underline{a}} = d\Gamma|_{\underline{a}};$$

$$\gamma(\mathrm{diag}(X,X)) = inx, \quad X = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}, \quad x \in \mathbf{R}.$$

(2.4.19) Proposition. For Γ,γ as in (2.4.18) we have

$$\underset{=}{A}[H_1,\Gamma,\gamma] = \begin{cases} \{\pi'_{m,-m}\} & \text{if } n = 2m-1 \\ \{\pi'_{m+1,-m},\pi'_{m,-m-1}\} & \text{if } n = 2m \end{cases}$$

(2.4.20) Theorem. The irreducible admissible representations of $Sp(2,\mathbf{R})$ attached to the Cartan subgroup H_1 fall into four disjoint families described by the parameters Γ,γ (2.4.18) and $\underset{=}{A}$ (2.3.20) as follows:

(2.4.21) $n = 2m-1, \ \underset{=}{A} = \{\pi'_{m,-m}\}$;

(2.4.22) $n = 2m, \ \nu \neq 0, \ \underset{=}{A} = \{\pi'_{m+1,-m},\pi'_{m,-m-1}\}$;

(2.4.23) $n = 2m, \ \nu = 0, \ \underset{=}{A} = \{\pi'_{m+1,-m}\}$;

(2.4.24) $n = 2m, \ \nu = 0, \ \underset{=}{A} = \{\pi'_{m,-m-1}\}$.

(2.4.25) Theorem. The representations listed in (2.4.20) are unitary if and only if the following additional assumptions are satisfied:

(2.4.26) $\nu \in i\,\mathbf{R}$ or $\nu \in \mathbf{R}$ and $\nu < 1$ in (2.4.21).

(2.4.27) $\nu \in i\,\mathbf{R}$ in (2.4.22).

The representations (2.4.23), (2.4.24) are tempered.

The representations attached to H_2 (2.3.8).

Let

(2.4.28) $\sigma = 0,1; \ n \in \mathbf{Z}, \ n \neq 0, \ \nu \in \mathbf{C}, \ \nu > 0.$

$$\Gamma'\left(\begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & c & 0 & s \\ 0 & 0 & \varepsilon & 0 \\ 0 & -s & 0 & c \end{bmatrix}\right) = \varepsilon^{\sigma} e^{inx} e^{i\,\mathrm{sgn}(n)x}$$

$(c = \cos x,\ s = \sin x,\ x \in \mathbf{R})$

$$\Gamma'(\mathrm{diag}(a,1,a^{-1},1)) = a^{\nu}$$

$$\gamma_n'|_{\underline{a}_1} = d\Gamma|_{\underline{a}_1}$$

$$\gamma_n'\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix}\right) = inx,\ x \in \mathbf{R}.$$

(2.4.29) Proposition. Under the assumptions (2.4.28) the set of lowest K-types

$$\underline{\underline{A}}[H_2,\Gamma',\gamma_n'] = \begin{cases} \{\pi'_{-1,n-1}\} & \text{if } \sigma = 1,\ n < 0; \\ \{\pi'_{n+1,1}\} & \text{if } \sigma = 1,\ n > 0; \\ \{\pi'_{-2,n-1},\pi'_{0,n-1}\} & \text{if } \sigma = 0,\ n < 0; \\ \{\pi'_{n+1,2},\pi'_{n+1,0}\} & \text{if } \sigma = 0,\ n > 0. \end{cases}$$

(2.4.30) Theorem. The irreducible admissible representations of $Sp(2,\mathbf{R})$ attached to H_2 fall into eight families described by the parameters Γ',γ_n' (2.4.28) and $\underline{\underline{A}}$ as follows:

(2.4.31) $\sigma = 1,\ n < 0,\ \underline{\underline{A}} = [\pi'_{-1,n-1}\}$;

(2.4.32) $\sigma = 1,\ n > 0,\ \underline{\underline{A}} = \{\pi'_{n+1,1}\}$;

(2.4.33) $\sigma = 0,\ n < 0,\ \nu \neq 0,\ \underline{\underline{A}} = \{\pi'_{-2,n-1}\ \pi'_{0,n-1}\}$;

(2.4.34) $\sigma = 0,\ n > 0,\ \nu \neq 0,\ \underline{\underline{A}} = \{\pi'_{n+1,2},\pi'_{n+1,0}\}$;

(2.4.35) $\sigma = 0,\ n < 0,\ \nu = 0,\ \underline{\underline{A}} = \{\pi'_{-2},_{n-1}\}$;

(2.4.36) $\sigma = 0,\ n < 0,\ \nu = 0,\ \underline{\underline{A}} = \{\pi'_{0\ ,n-1}\}$;

(2.4.37) $\sigma = 0,\ n > 0,\ \nu = 0,\ \underline{\underline{A}} = \{\pi'_{n+1,2}\}$;

(2.4.38) $\sigma = 0,\ n > 0,\ \nu = 0,\ \underline{\underline{A}} = \{\pi'_{n+1,0}\}$.

(2.4.39) Theorem. The representations listed in (2.4.30) are unitary under the following additional assumptions:

(2.4.40) $\nu \in i\,\mathbb{R}$ or $\nu \in \mathbb{R}$ and $\nu \leq 1$ in (2.4.31), (2.4.32);

(2.4.41) $\nu \in i\,\mathbb{R}$ in (2.4.33) and (2.4.34).

The representations attached to H'_3 (2.3.7).

These are the discrete series representations (2.2.15). Let

(2.4.42) $m,n \in \mathbb{Z};\ m-n > 0;\ m,n,m+n \neq 0;$

$$\gamma'_{m,n}\left(\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \\ -x & 0 & 0 & 0 \\ 0 & -y & 0 & 0 \end{bmatrix}\right) = i(mx + ny) \qquad x,y \in \mathbb{R};$$

Γ' be associated to γ via (2.3.13).

(2.4.43) Theorem. The irreducible admissible representations of $Sp(2,\mathbb{R})$ attached to H'_3 are all unitary, belong the discrete series and have the form

$$\Pi[H'_3, \Gamma', \gamma'_{m,n})(\underline{\underline{A}})$$

with $\Gamma', \gamma'_{n,n}$ as in (2.4.42) and $\underline{\underline{A}}$ containing the single element

(2.4.44) $\pi'_{m+1,n+2}$ if $n > 0$;

(2.4.45) $\pi'_{m+1,n}$ if $m > -n,\ n < 0$;

(2.4.46) $\pi'_{m,n-1}$ if $m < -n,\ n < 0$;

(2.4.47) $\pi'_{m-2,n-1}$ if $m < 0$.

§5. THE UNITARY DUAL OF $O_{2,2}$.

Consider the following two embeddings of the group $SL(2,\mathbb{R})$ into $O_{2,2}$:

(2.5.1) $$SL(2,\mathbf{R}) \ni g \to \operatorname{diag}(g,(g^t)^{-1}) \in O_{2,2},$$

(2.5.2) $$SL(2,\mathbf{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & -d & 0 \\ c & 0 & 0 & d \end{bmatrix} \in O_{2,2}.$$

Then the image of $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ in $O_{2,2}$ under (2.5.1) × (2.5.2) is the connected component of the identity of $O_{2,2}$. Since $O(2) \times O(2)$ has four connected components, so does $O_{2,2}$ (1 1.9), (2.1.4). Using this and the known structure of the admissible dual of $SL(2,\mathbf{R})$ [V1 ChI], [L], one can, with some effort, figure out the admissible and the unitary duals of $O_{2,2}$. We present the results in this paragraph leaving the proofs to the reader.

<u>The representations attached to H_0 (2.3.4).</u>

(2.5.3) <u>Proposition.</u> Using the notation of (2.1.7) and (2.4.2)

$$\underline{\underline{A}}[H_0,\Gamma,\gamma] = \begin{cases} \{\pi^{1,1}_{0,0},1\} & \text{if } \sigma_1 = \sigma_2 = 0; \\ \{\pi^p_{1,0},\pi^p_{0,1};\ p = 0,1\} & \text{if } \sigma_1 \neq \sigma_2; \\ \{\pi^{1,0}_{0,0},\pi^{0,1}_{0,0}\} & \text{if } \sigma_1 = \sigma_2 = 1. \end{cases}$$

(2.5.4) <u>Theorem.</u> The irreducible admissible representations of $O_{2,2}$ attached to the Cartan subgroup H_0 fall into twelve disjoint families described by the parameters Γ,γ (2.4.1) and $\underline{\underline{A}}$ (2.3.19) as follows:

(2.5.5) $$\sigma_1 = \sigma_2 = 0,\ \nu_2 \neq 0,\ \underline{\underline{A}} = \{\pi^{1,1}_{0,0},1\};$$

(2.5.6) $$\sigma_1 = \sigma_2 = 0,\ \nu_2 = 0,\ \underline{\underline{A}} = \{1\}\ (\Pi = 1 \text{ for } \nu_1 = 1);$$

(2.5.7) $$\sigma_1 = \sigma_2 = 0,\ \nu_2 = 0,\ \underline{\underline{A}} = \{\pi^{1,1}_{0,0}\}\ (\Pi = \det \text{ for } \nu_1 = 1);$$

(2.5.8) $$\sigma_1 \neq \sigma_2,\ \nu_2 \neq 0,\ \underline{\underline{A}} = \underline{\underline{A}}[H_0,\Gamma,\gamma];$$

(2.5.9) $$\sigma_1 = 1,\ \sigma_2 = 0,\ \nu_1 \neq 0,\ \nu_2 = 0,\ \underline{\underline{A}} = \{\pi^0_{1,0},\pi^0_{0,1}\},$$

(2.5.21) $\nu_1 \in i\,\mathbf{R}$ in (2.5.9), (2.5.10), (2.5.12), (2.5.13).

<u>The representations attached to H_1 (2.3.5)</u>.

(2.5.22) <u>Proposition</u>. Under the assumptions (2.4.17)

$$\underline{\underline{A}}[H_1,\Gamma,\gamma] = \begin{cases} \{\pi_{m,m}\} & n = 2m-1, \\ \{\pi_{m+1,m},\pi_{m,m+1}\} & n = 2m \end{cases}$$

(2.5.23) <u>Theorem</u>. The irreducible admissible representations of $O_{2,2}$ attached to H_1 fall into four disjoint families described by the parameters Γ,γ (2.4.18) and $\underline{\underline{A}}$ (2.3.19) as follows:

(2.5.24) $n = 2m-1$, $\underline{\underline{A}} = \{\pi_{m,m}\}$, $\nu \in \mathbf{C}$, $\nu > 0$;

(2.5.25) $n = 2m$, $\nu \neq 0$, $\underline{\underline{A}} = \underline{\underline{A}}[H_1,\Gamma,\gamma]$;

(2.5.26) $n = 2m$, $\nu = 0$, $\underline{\underline{A}} = \{\pi_{m+1,m}\}$;

(2.5.27) $n = 2m$, $\nu = 0$, $\underline{\underline{A}} = \{\pi_{m,m+1}\}$.

(2.5.28) <u>Theorem</u>. The representations listed in (2.5.23) are unitary if and only if the following additional assumptions are satisfied:

(2.5.29) $\nu \in i\,\mathbf{R}$ or $\nu \in R$ and $\nu \leq 1$ in (2.5.24);

(2.5.30) $\nu \in i\,\mathbf{R}$ in (2.5.25).

<u>The representations attached to H_3 (2.3.6)</u>.

Let

(2.5.31) $m,n \in \mathbf{Z}$; $m-n \neq 0$, $m+n \neq 0$, $m > 0$; $n \leq 0$;

$$\gamma_{m,n}(\mathrm{diag}(X_1,X_2)) = imx_1 + inx_2, \text{ where}$$

$$X_j = \begin{bmatrix} 0 & x_j \\ -x_j & 0 \end{bmatrix}, \; x_j \in \mathbf{R},\; j = 1,2;$$

Then $\gamma_{m,n} \in \underline{t}_3^*$ via the isomorphism (1.1.9). Let Γ be associated to $\gamma_{m,n}$ via (2.3.13).

(2.5.32) <u>Theorem</u>. The irreducible admissible representations of $O_{2,2}$ attached to H_3 belong to the discrete series and have the form

$$\Pi[H_3, \Gamma, \gamma_{m,n}](\underline{\underline{A}})$$

with $\Gamma, \gamma_{m,n}$ as in (2.5.31) and $\underline{\underline{A}}$ containing the single element (see (2.1.7)).

(2.5.33) $\pi_{m+1,n}$ if $m > -n$, $m \neq 0$, $n \neq 0$;

(2.5.34) $\pi_{m,n-1}$ if $-n > m$, $m \neq 0$, $n \neq 0$;

(2.5.35) $\pi^0_{m+1,0}$ if $n = 0$;

(2.5.36) $\pi^1_{m+1,0}$ if $n = 0$;

(2.5.37) $\pi^0_{0,n-1}$ if $m = 0$;

(2.5.38) $\pi^1_{0,n-1}$ if $m = 0$.

Here one obtains (2.5.36) and (2.5.38) from (2.5.35) and (2.5.37), respectively, by tensoring with det.

CHAPTER 3. $R(O_{2,2} \cdot Sp(2,\mathbb{R}),\omega)$.

In this chapter we compute the above set. We begin with a smaller pair $O(2,2)$, $Sp(1,\mathbb{R})$ in §1 to establish some notation necessary in §2 and §3. Section 3, where we find all pairs $\Pi \check{\otimes} \Pi'$ being in the Oscillator Duality Correspondence such that Π' is a discrete series representation of $Sp(2,\mathbb{R})$, is the most technically involved part. We are forced to combine here Howe's L^P-estimates for the matrix coefficients of the oscillator representation (3.3.10), Vogan's classification of tempered representations (2.3.22) and Zuckerman's translation functors (3.3.54), (3.3.55). Sections 4 and 5 are technically easier. In §6 we show that our list of corresponding representations from §2, §3, §4, §5 is complete.

The conclusion of this chapter is theorem (3.6.1) which asserts that the Oscillator Duality Correspondence maps unitary representations of $O_{2,2}$ to unitary representations of $Sp(2,\mathbb{R})$.

§1. $O(2,2) \times Sp(1,\mathbb{R})$.

Put

(3.1.1) $$s = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad \text{and} \quad s_2 = \operatorname{diag}(s,s).$$

Then the conjugation Int s_2 inside $GL(4,\mathbf{C})$ transforms the Lie algebra $\underline{o}(2,2)$ into

(3.1.2) $$\underline{g} = \left\{ \begin{bmatrix} it_1 & 0 & \bar{z}_1 & \bar{z}_2 \\ 0 & -it_1 & z_2 & z_1 \\ \bar{z}_1 & z_2 & it_2 & 0 \\ \bar{z}_2 & z_1 & 0 & -it_2 \end{bmatrix} t_1, t_2 \in \mathbf{R};\ z_1, z_2 \in \mathbf{C} \right\}.$$

and the maximal compact subgroup (2.1.3) into $K = K^{\circ} \cdot K^{\#}$, where

$$K^{\circ} = \{\operatorname{diag}(u_1, u_1^{-1}, u_2, u_2^{-1}) \mid u_j \in \mathbf{C},\ |u_j| \in \mathbf{C},\ |u_j| = 1,\ j = 1,2\}.$$

(3.1.3)

$$K^{\#} = \{\mathrm{diag}(w^{\varepsilon_1}, w^{\varepsilon_2},) \mid \varepsilon_1, \varepsilon_2 = 0,1\}, \ w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Define the following root vectors in the complexification $\underline{\mathbf{g}}$ of $\underline{g}$ (3.1.2):

(3.1.4)
$$x_- = e_{1,3} + e_{4,2}, \qquad y_- = e_{2,4} + e_{3,1}$$
$$x_+ = e_{1,4} + e_{3,2}, \qquad y_+ = e_{2,3} + e_{4,1}$$

Put

(3.1.5)
$$h_- = e_{1,1} - e_{2,2} - e_{3,3} + e_{4,4},$$
$$h_+ = e_{1,1} - e_{2,2} + e_{3,3} - e_{4,4}.$$

Then clearly

(3.1.6) $$[x_\alpha, y_\alpha] = h_\alpha, \ [h_\alpha, x_\alpha] = 2x_\alpha, \ [h_\alpha, y_\alpha] = -2y_\alpha \ \text{ for } \alpha = +,-.$$

so that the decomposition

(3.1.7) $$\underline{\mathbf{g}} = (\mathbf{C}x_- + \mathbf{C}h_- + \mathbf{C}y_-) + (\mathbf{C}x_+ + \mathbf{C}h_+ + \mathbf{C}y_+)$$

provides an isomorphism of $\underline{\mathbf{g}}$ with the direct sum of two copies of $\underline{sl}(2,\mathbf{C})(= \underline{sp}(1,\mathbf{C}))$.

The Lie algebra (of 2 by 2 matrices)

(3.1.8) $$\underline{g}_2' = \underline{sp}(1,\mathbf{C})$$

has the following two root vectors:

(3.1.9) $$x' = i\, e_{1,2} \quad \text{and} \quad y' = -\, i\, e_{2,1}.$$

Let

(3.1.10) $$h' = e_{1,1} - e_{2,2}$$

Then x', h', y' satisfy the standard commutation relations (3.3.7). Put

(3.1.11) $\quad \underline{g}'_2 = \underline{sp}(1,\mathbf{C}) \cap u(1,1)$ and $K'_2 = U(1)$.

We embed $\underline{g}$, and g'_2, into $\underline{sp}(4,\mathbf{C}) \cap \underline{u}(4,4)$ by mapping the typical element of $\underline{g}$ (3.1.2) to

(3.1.12) $\quad \begin{bmatrix} U & Z \\ \overline{Z} & \overline{U} \end{bmatrix}$ with

(3.1.13) $\quad U = \mathrm{diag}(it_1,-it_1,-it_2,it_2)$, and

(3.1.14) $$Z = \begin{bmatrix} 0 & 0 & z_1 & z_2 \\ 0 & 0 & \overline{z}_2 & \overline{z}_1 \\ z_1 & \overline{z}_2 & 0 & 0 \\ z_2 & \overline{z}_1 & 0 & 0 \end{bmatrix};$$

and

(3.1.15) $\quad \underline{g}'_2 \ni \begin{bmatrix} it & z \\ \overline{z} & -it \end{bmatrix} \to$ (3.1.12), with

$$U = \mathrm{diag}(it,it,-it,-it),$$

$$Z = \begin{bmatrix} zs^2 & 0 \\ 0 & \overline{zs}^2 \end{bmatrix} \quad (s \text{ as in } (3.1.1)).$$

Moreover we

(3.1.16) $\quad$ inject K into $U(4)$ by the identity map, and

(3.1.17) $\quad K'_2$ into $U(4)$ by $K'_2 \ni u \to \mathrm{diag}(u,u,u^{-1},u^{-1})$.

This way we have embedded $(\underline{g},K) \times (g'_2, K'_2)$ into $(\underline{sp}(4,\mathbf{C}) \cap \underline{U}(4,4), u(4))$ or equivalently $G \times G'_2$ into $Sp(4,\mathbf{C}) \cap U(4,4)$, where

(3.1.18) $\quad G = O(2,2)$, $G'_2 = Sp(1,\mathbf{C}) \cap U(1,1)$.

The determinant (1.4.16) of an element of the image of K or K' in $U(4)$ under (3.1.16) or (3.1.17) is clearly equal to 1. Putting $\xi = 1$ in (1.4.17) we shall pull back the oscillator representation ω to the group $K \times K'$. The action of $\underline{g}$ in the Fock model (c.f. §1.4) $\mathbf{C}[z_1,z_2,z_3,z_4]$ is given by:

$$\begin{aligned}
\omega(x_-) &= -z_1 z_3 + \partial_{z_2} \partial_{z_4}\\
\omega(h_-) &= z_1\partial_{z_1} - z_2 \partial_{z_2} + z_3 \partial_{z_3} - z_4 \partial_{z_4}\\
\omega(y_-) &= -z_2 z_4 + \partial_{z_1} \partial_{z_3}\\
\omega(x_+) &= -z_1 z_4 + \partial_{z_2} \partial_{z_3}\\
\omega(h_+) &= z_1 \partial_{z_1} - z_2 \partial_{z_2} - z_3 \partial_{z_3} + z_4 \partial_{z_4}\\
\omega(y_+) &= -z_2 z_3 + \partial_{z_1} \partial_{z_4}
\end{aligned} \tag{3.1.19}$$

Let for $z = (z_1, z_2, z_3, z_4)$ and $n = 0,1,2,3\ldots,$

$$\begin{aligned}
f_n(z) &= \sum_{a=0}^{\infty} \frac{(z_1 z_2 z_3)^a}{a!} \frac{z_4^{a+n}}{(n+a)!}\\
f_{-n}(z) &= \sum_{a=0}^{\infty} \frac{z_1^{a+n}}{(a+n)!} \frac{(z_2 z_3 z_4)^a}{a!} .
\end{aligned} \tag{3.1.20}$$

Then the norm (1.4.12)

$$\|f_{\pm n}\|^2 = \sum_{a=0}^{\infty} \frac{a!}{(a+n)!} \text{ is finite for } n \geq 2. \tag{3.1.21}$$

A straight forward calculation shows that

$$\begin{aligned}
&\omega(x_-)f_n = \omega(y_+)f_n = 0,\\
&\omega(h_-)f_n = -nf_n, \ \omega(h_+)f_n = nf_n
\end{aligned} \tag{3.1.22}$$

and that

$$\begin{aligned}
&\omega(y_-)f_{-n} = \omega(y_+)f_{-n} = 0,\\
&\omega(h_-)f_{-n} = nf_{-n}, \ \omega(h_+)f_{-n} = nf_{-n}.
\end{aligned} \tag{3.1.23}$$

Similarly

$$\omega(x') = -z_1 z_2 + \partial_{z_3} \partial_{z_4},$$

(3.1.24) $\omega(h') = z_1 \partial_{z_1} + z_2 \partial_{z_2} - z_3 \partial_{z_3} - z_4 \partial_{z_4}$,

$\omega(y') = -z_3 z_4 + \partial_{z_1} \partial_{z_2}$,

and

$\omega(x')f_n = \omega(y')f_{-n} = 0$,

(3.1.25) $\omega(h')f_n = -nf_n$, $\omega(h')f_{-n} = nf_{-n}$.

(3.1.26) Theorem. The closed subspace of the Hilbert space of the oscillator representation ω (1.4.12) generated by the action of $(\underline{g},K) \times (\underline{g}_2', K_2')$ on f_{n+1} (f_{-n-1}), for $n \geqslant 1$, is irreducible as a $G \times G_2'$-module and isomorphic to

(3.1.27) $\Pi \check{\otimes} \Pi' \in R(G \cdot G_2', \omega)$

where

(3.1 28) Π is the discrete series representation of G with the lowest K-type $\pi^0_{0,-n-1}(\pi^0_{n+1,0})$;

(3.1.29) Π' is a discrete series representation of G_2' with the lowest K'-type π'_{-n} (π'_n).

$(\pi_b'(u) = u^b \quad u \in U(1), b \in \mathbf{Z})$

Moreover

(3.1.30) $\deg \Pi = n$, $D(\Pi) = \underline{\underline{A}}(\Pi)$, $D(\Pi') = \underline{\underline{A}}(\Pi')$.

Proof: The situation for f_{-n-1} is entirely analogous, to that of f_{n+1}, therefore we leave it to the reader. Let V be the Hilbert space generated by the action of $(\underline{g},K) \times (\underline{g}_2',K_2')$ on f_{n+1}. The formulas (3.1.25) imply that (g',K') acts on V via a unitary representation Π' of highest weight $-n-1$. There is only one such representation of G', namely the one described in (3.1.29), see [L Ch VI §6, Theorem 8].

Similarly G acts on V via a unitary representation Π, which according to the formulas (3.1.22) and the decomposition (3.1.7) must be the one described in (3.1.28). By the corollary (1.5.31), $V \cong \Pi \check{\otimes} \Pi'$.

It is apparent from (3.1.17) and (1.4.17) that for any K_2'-type π'_{-b} (3.1.29), $\deg \pi'_{-b} = |b|$. Therefore $\deg \Pi' = n$ and (3.1.30) holds (compare (1.5.16) and (2.1.13)).

Q.E.D.

Finally, for future use in §6, we record the following

(3.1.31) <u>Proposition</u>. Let $\Pi = \Pi_{0_{2,2}}[H_0,\Gamma,\gamma](A)$ as in (2.5.6) or (2.5.9) or, (2.5.11) with A containing $\pi^0_{1,0}$ or $\pi^0_{0,1}$. Then $\Pi \in R(G,\omega)$.

The proof of (3.1.31) is entirely analogous to the proof of the Theorem (3.4.31). Therefore we omit it.

§2. THE DISCRETE SERIES OF $0_{2,2}$, I.

Let us embed the groups $0(4) \cap 0(2,2)$ and $U(2)$ in $U(8)$ by the following maps

(3 2.1) $$\mathrm{diag}(g_1,g_2) \to \mathrm{diag}(g_1,g_2,g_1,g_2),\ (g_j \in 0(2),\ j = 1,2), \quad \text{and}$$

(3.2.2) $$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \to \begin{bmatrix} u_{11}I_2 & 0 & u_{12}I_2 & 0 \\ 0 & -u_{11}I_2 & 0 & -u_{21}I_2 \\ u_{21}I_2 & 0 & u_{22}I_2 & 0 \\ 0 & -u_{12}I_2 & 0 & -u_{22}I_2 \end{bmatrix}$$

respectively. Clearly the determinants of the images (3.2.1), (3.2.2) are one. Putting $\xi = 1$ in (1.4.17) we obtain, via (3.2.1) and (3.2.2) a pull back of the oscillator representation ω (1.4.6) to the group $0(4) \cap 0(2,2) \times U(2)$. This is consistant with Chapter 2, §1.

The differentials of the maps (3.2.1) and (3.2.2), when composed with (1.4.15) and then with the inverse of (1.4.18), provide the same injection of the Lie algebras $\underline{o}(4) \cap \underline{o}(2,2)$ and $\underline{o}(4) \cap \underline{sp}(2,\mathbb{R})$ in $\underline{sp}(8,\mathbb{R})$, respectively, as the diagram (1.1.12) does.

Thus via the identification (1.1.9) we have a well defined pull back of the Harish-Chandra module of ω to

$$(\underline{o}_{2,2},\ 0(4) \cap 0_{2,2}) \times (\underline{sp}(2,\mathbb{R}),\ 0(4) \cap Sp(2,\mathbb{R})).$$

Since the splitting of the covering $\widetilde{Sp}(8,\mathbf{R}) \to Sp(8,\mathbf{R})$ over the images of the maximal compact subgroups $O(4) \cap O_{2,2}$ and $O(4) \cap Sp(2,\mathbf{R})$ determines the splitting over the images of the groups $O_{2,2}$ and $Sp(2,\mathbf{R})$, the set

(3.2.3) $$R(O_{2,2} \cdot Sp(2,\mathbf{R}),\omega)$$

is well defined (see (1.2.14)).

(3.2.4) Theorem. Put $\sigma = 0$ in (2.4.28). Let in the notation (2.5.32)

(3.2.5) $$\begin{aligned} \Pi &= \Pi_{O_{2,2}}[H_3,\Gamma,\gamma_{0,n}](\pi^0_{0,n-1}), \\ \Pi' &= \Pi_{Sp(2,\mathbf{R})}[H_2,\Gamma',\gamma'_n](\pi'_{0,n-1}), \end{aligned}$$

for $n = -1,-2,-3,\ldots$, and

(3.2.6) $$\begin{aligned} \Pi &= \Pi_{O_{2,2}}[H_3,\Gamma,\gamma_{n,0}](\pi^0_{n+1,0}), \\ \Pi' &= \Pi_{Sp(2,\mathbf{R})}[H_2,\Gamma',\gamma'_n](\pi'_{n+1,0}), \end{aligned}$$

for $n = 1,2,3,\ldots$. Then

(3.2.7) $$\Pi \check{\otimes} \Pi' \in R(O_{2,2} \cdot Sp(2,\mathbf{R}),\omega)$$

and

(3.2.8) $$\deg \Pi = |n| + 1, \quad D(\Pi) = \underline{\underline{A}}(\Pi), \quad D(\Pi') = \underline{\underline{A}}(\Pi').$$

Proof: We identify $O_{2,2}$ with $O(2,2)$ via (1.1.9). The group $Sp(8,\mathbf{R})$ is the isometry group of the symplectic space $(W, \langle,\rangle)$ (1.3.24). Define the direct sum decomposition

(3.2.9) $$W = X \oplus Y \oplus W_2, \text{ where}$$

(3.2.10) $$X = \{(a,b,c,d,\underbrace{0,\ \ldots,0}_{12}) \mid a,b,c,d, \in \mathbf{R}\}$$

(3.2.11) $$Y = \{(\underbrace{0,\ldots,0}_{8},a,b,c,d,\underbrace{0,\ldots,0}_{4}) \mid a,b,c,d \in \mathbf{R}\},$$

(3.2.12) $$W_2 = \{(\underbrace{0,\ldots,0}_{4},a,b,c,d,\underbrace{0,\ldots,0}_{4},e,f,g,h) \mid a,b,\ldots,h \in \mathbf{R}\}.$$

Then the preimage of the parabolic subgroup $P_Y \subseteq Sp(8,\mathbf{R})$ (1.3.3) in $Sp(2,\mathbf{R})$ via (1.1.7) is equal to $\theta P_2'$ (2.2.13). Moreover the image of $O(2,2)$ under (1.1.6) is contained in $M_{X,Y}$ (1.3.10). Let us fix the isomorphism

(3.2.13) $$M_2^0 \ni \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(1,\mathbf{R})$$

Clearly $M_2 \cong (\mathbf{Z}/2\mathbf{Z}) \times Sp(1,\mathbf{R})$, (2.2.13).

Let ω_2 be the Fock model of the oscillator representation of $(Sp(4,\mathbf{C}) \cap U(4,4))^{\sim}$, (1.4.6). We pull back ω_2 to $O(2,2) \times M_2^0$ by the group isomorphism

(3.2.14) $$((3.1.12)\times(3.1.15))\circ(\mathrm{Int}\ s_2 \times (1.4.18)\circ(\mathrm{id} \times (3.2.13))$$

where s_2 is defined in (3.1.1).

Consider the mixed model, as in (1.3.22), of the oscillator representation ω of $\tilde{S}p(8,\mathbf{R})$ adapted to the decomposition (3.2.9) with ω_2 as a representation of $\tilde{S}p(W_2) = \tilde{S}p(4,\mathbf{R}) \cong (Sp(4,\mathbf{C}) \cap U(4,4))^{\sim}$ by (1.4.18). Define

(3.2.15) $$T_0\colon S(X,\omega_2^\infty) \to \omega_2^\infty \quad \text{by} \quad T(f) = f(0).$$

Since $\dim x = 4$, it follows immediately from the theorem (1.3.16) that

(3.2.16) $$T_0 \in \mathrm{Hom}_{\theta P_2}(\omega^\infty,(1 \otimes \omega_2^\infty) \otimes \rho(\theta\underline{n}_2) \otimes 1), \text{ and}$$

(3.2.17) $$T_0 \in \mathrm{Hom}_{O(2,2)}(\omega^\infty,\omega_2^\infty).$$

For each discrete series representation Π_2' (3.1.29) of M_2^0 (3.2.12), let

(3.2.18) T_2 denote the $O(2,2) \times M_2^0$ - intertwining operator from ω_2^∞ onto $\Pi \check{\otimes} \Pi_2'$ (3.1.27).

define $T = T_2 \circ T_0$. Then by (3.2.16), (3.2.17) we get

(3.2.19) $$T \in \mathrm{Hom}_{O(2,2)\times\theta P_2}(\omega^\infty, \check{\Pi} \otimes ((1 \otimes \Pi_2') \otimes \rho(\theta\underline{n}_2) \otimes 1)).$$

The smooth Frobenius reciprocity theorem (Appendix B) implies the existence of a non-zero, $O(2,2) \times Sp(2,\mathbf{R})$-intertwining operator

(3.2.20) $$\mathrm{Ind}\ T : \omega^\infty \to \check{\Pi} \otimes \mathrm{Ind}_{\theta P_2}^{Sp(2,\mathbf{R})}(\delta \otimes \nu) \quad \text{where}$$

(3.2.21) $$\delta = 1 \check{\otimes} \Pi_2' \quad \text{and} \quad \nu = 0.$$

Therefore there is an irreducible subquotient $\Pi' \in Sp(2,\mathbf{R})^\wedge$ of the induced representation (3.2.20) such that

(3.2.22) $$\Pi \check{\otimes} \Pi' \in R(O_{2,2} \cdot Sp(2,\mathbf{R}),\omega)$$

Assume that Π_2' has a lowest K_2'-type π_{n+1}' (3.1.29) with $n > 0$. Then Π contains the unique lowest degree K-type $\pi_{n+1,0}^0$ and has degree n+1 in the sense of (1.5.16) with respect to the pair $O(2,2)$, $Sp(1,\mathbf{R})$. The K-types of Π are $\pi_{n+1+r+s,\ r-s}$ $(r,s = 0,1,2,3,..,\ r\neq s)$ and $\pi_{n+1+2r,0}^0$ $(r=0,\ 1,2,...)$. Therefore (2.1.24) implies that $\pi_{n+1,0}^0$ is the unique lowest K-type of Π and that $\deg \Pi = n+1$ (with respect to the pair $O(2,2)$, $Sp(2,\mathbf{R})$). This and (1.5.20) shows (3.2.8).

We know already from the above discussion and from the theorem (1.5.20) that the map (3.2.20) does not annihilate the space $H_{\pi,\pi'}$, with $\pi = \pi_{n+1,0}^0$. On the other hand (3.2.20) implies that π' occurs in

(3.2.23) $$\mathrm{Ind}_{K'\cap M_2}^{K'}(1\otimes\pi_r') \text{ for an } r \geqslant n+1,\ r-n-1 \in 2\mathbf{Z}.$$

Since the degree of π' is n+1, (2.1.23) and (A.1.8) imply that $\pi' = \pi_{n+1,0}'$. The case $n < 0$ is completely analogous.

Q.E.D.

§3. THE DISCRETE SERIES OF $O_{2,2}$, II.

(3.3.1) Theorem. In the notation (2.5.32), (2.4.43) let

(3.3.2)
$$\Pi = \Pi_{O_{2,2}}[H_3,\Gamma,\gamma_{m,n}](\pi_{m+1,n}),$$
$$\Pi' = \Pi_{Sp(2,\mathbf{R})}[H_3',\Gamma',\gamma_{m,n}'](\pi_{m+1,n}')$$

for $m+n > 0$, $m > 0 > n$ and

(3.3.3)
$$\Pi = \Pi_{O_{2,2}}[H_3,\Gamma,\gamma_{m,n}](\pi_{m,n-1}],$$
$$\Pi' = \Pi_{Sp(2,\mathbf{R})}[H_3',\Gamma',\gamma_{m,n}'](\pi_{m}',_{n-1})$$

for $m+n < 0$, $m > 0 > n$. Then

(3.3.4) $\quad \Pi \check{\otimes} \Pi' \in R(O_{2,2} \cdot Sp(2,\mathbf{R}),\omega)$ and

(3.3.5) $\quad \deg \Pi = |m| + |n| + 1.$

In the proof of this theorem we shall use the following lemmas.

(3.3.6) Lemma: Let x,h,y be the standard basis of the Lie algebra $\underline{sl}(2,\mathbf{C})$ satisfying the commutation relations

(3.3.7) $\quad [x,y] = h,\ [h,x] = 2x,\ [h,y] = -2y.$

Assume that $\underline{sl}(2,\mathbf{C})$ acts on a vector space containing a vector v such that

(3.3.8) $\quad x\,v = 0,\ h\,v = -\,b\,v$

where $b = 1,2,3\ldots$. Then the span of the vectors y^jv $(j = 0,1,2\ldots)$ is invariant under $\underline{sl}\,(2,\mathbf{C})$ and

(3.3.9) $\quad (j+1)^{-1}(b+j)^{-1}\, xy^{j+1}\, v = -y^jv \qquad (j = 0,1,2\ldots).$

Proof: If $j = 0$ then (3.3.9) holds because

$$b^{-1}xyv = b^{-1}(yx + h)v = -v.$$

Assume that $j > 0$. Then, by induction on j,

$$xy^{1+j}v = (yxy^j + hy^j)v =$$

$$= -j(b + j - 1)y^j v + (-b - 2j)y^j v$$

$$= -(j+1)(b+j)y^j v.$$

Q.E.D.

(3.3.10) Lemma. If $\Pi \otimes \check{\Pi}' \in R(O_{2,2} \cdot Sp(2,\mathbb{R}), \omega)$ occurs as a closed subspace of the Hilbert space V of ω, then Π' is tempered (2.2.15).

Proof: It is known [H6 Prop. 8.1] that the matrix coefficients

$$\widetilde{S}p(n,\mathbb{R}) \ni g \to (\omega(g)v,v') \in \mathbf{C} \quad (v,v' \in V)$$

of the oscillator representation of the metaplectic group belong to $L^p(\widetilde{S}p(n,\mathbb{R}))$ for any $p > 4n$. In particular, when $n = 2$, they are in L^p with $p > 8$. Since the pull back (3.2.3), (1.1.12) of the oscillator representation ω of $\widetilde{S}p(8,\mathbb{R})$ to $Sp(2,\mathbf{R})$ is essentially a tensor product of two copies of the oscillator representation of $\widetilde{S}p(2,\mathbf{R})$ and two copies of the contragradient one, the Schwartz inequality implies that the matrix coefficients of this pull back are in $L^p(Sp(2,\mathbf{R}))$ for any $p > 2$. Therefore the matrix coefficients of Π' are in L^p for any $p > 2$.

Q.E.D.

(3.3.11) Lemma. Assume that $m,n \in \mathbf{Z}$, $n \geq m+2$, $m \geq 2$. Then

$$\sum_{j=0}^{\infty} \sum_{a,b=j}^{\infty} \left(\frac{(m-1+j)!}{j!(m-1)!}\right)^2 \frac{a!}{(a+m)!} \frac{b!}{(b+n)!} \tag{3.3.12}$$

is finite.

Proof: Since

$$\sum_{b=j}^{\infty} \frac{b!}{(b+n)!} \leq n^n \sum_{b=j}^{\infty} (b+n)^{-n} \leq \frac{n^n}{n-1} (n-1+j)^{1-n},$$

(3.3.12) can be estimated by a positive multiple of

$$\sum_{j=0}^{\infty} \left(\frac{(m-1+j)!}{j!}\right)^2 (m-1+j)^{1-m} (n-1+j)^{1-n}$$

$$\leqslant \sum_{j=0}^{\infty} \frac{(m-1+j)^{m-1}}{(n-1+j)^{n-1}} \leqslant \sum_{j=0}^{\infty} (n-1+j)^{m-n}$$

which is finite because $m-n \leqslant -2$.

Q.E.D.

Proof of the theorem (3.3.1): Let $\underline{g}$ and K be defined as in (3.1.2), (3.1.3). Put

(3.3.13) $G' = Sp(2,\mathbf{C}) \cap U(2,2)$, $\underline{g}' = \underline{sp}(2,\mathbf{C}) \cap \underline{u}(2,2)$ and $K' = U(2)$.

Fix the diagonal Cartan subgroup $\underline{t}'$ in $\underline{g}'$ and chose the following root vectors in the complexification $\mathbf{g}'$ of $\underline{g}'$:

(3.3.14)
$$x'_- = e_{1,2};\ x'_+ = -i(e_{1,4} + e_{2,3});\ x'_1 = -ie_{1,3};\ x'_2 = -ie_{2,4}$$
$$y'_- = e_{2,1};\ y'_+ = i(e_{4,1} + e_{3,2});\ y'_1 = ie_{3,1};\ y'_2 = ie_{4,2}.$$

Put

(3.3.15) $h'_\alpha = [x'_\alpha, y'_\alpha]$ for $\alpha = -,+,1,2.$

Then $x'_\alpha, h'_\alpha, y'_\alpha$ satisfy (3.1.7) for α as in (3.3.15). Define

(3.3.16) $s_4 = \mathrm{diag}(s,\bar{s},s,\bar{s})$ with s as in (3.1.1).

Then $s_4 \in U(8)$ acts on $sp(8,\mathbf{C})$ by conjugation (1.4.15). We pull back the oscillator representation ω of $\underline{sp}(8,\mathbf{C})$ (1.4.10) to $\mathbf{g} \oplus \mathbf{g}'$ via the injection

(3.3.17) $$\mathrm{Int}\ s_4 \circ (1.4.18) \circ \big((1.1.6)\oplus(1.1.7)\big) \circ (\mathrm{Int}\ s_2^{-1} \oplus (1.4.18)^{-1})$$

of $\mathbf{g} \oplus \mathbf{g}'$ into $\underline{sp}(8,\mathbf{C})$. This makes our notation compatible with (3.2.3), and yields the following formulas

(3.3.18)
$$\omega(x'_-) = z_1\partial_{z_5} + z_2\partial_{z_6} - z_7\partial_{z_3} - z_8\partial_{z_4}$$
$$\omega(h'_-) = z_1\partial_{z_1} + z_2\partial_{z_2} - z_3\partial_{z_3} - z_4\partial_{z_4} - z_5\partial_{z_5} - z_6\partial_{z_6} + z_7\partial_{z_7} + z_8\partial_{z_8}$$

$$\omega(y'_-) = z_5\partial_{z_1} + z_6\partial_{z_2} - z_3\partial_{z_7} - z_4\partial_{z_7}$$

(3.3.19) $$\omega(x'_+) = -z_1z_6 - z_2z_5 + \partial_{z_3}\partial_{z_8} + \partial_{z_4}\partial_{z_7}$$

$$\omega(h'_+) = z_1\partial_{z_1} + z_2\partial_{z_2} - z_3\partial_{z_3} - z_4\partial_{z_4} + z_5\partial_{z_5} + z_6\partial_{z_6} - z_7\partial_{z_7} - z_8\partial_{z_8}$$

$$\omega(y'_+) = -z_3z_8 - z_4z_7 + \partial_{z_1}\partial_{z_6} + \partial_{z_2}\partial_{z_5}$$

(3.3.20) $$\omega(x'_1) = -z_1z_2 + \partial_{z_3}\partial_{z_4}$$

$$\omega(h'_1) = z_1\partial_{z_1} + z_2\partial_{z_2} - z_3\partial_{z_3} - z_4\partial_{z_4}$$

$$\omega(y'_1) = -z_3z_4 + \partial_{z_1}\partial_{z_2}$$

(3.3.21) $$\omega(x'_2) = -z_5z_6 + \partial_{z_7}\partial_{z_8}$$

$$\omega(h'_2) = z_5\partial_{z_5} + z_6\partial_{z_6} - z_7\partial_{z_7} - z_8\partial_{z_8}$$

$$\omega(y'_2) = -z_7z_8 + \partial_{z_5}\partial_{z_6}$$

Similarly we get

(3.3.22) $$\omega(x_-) = -z_1z_3 + \partial_{z_2}\partial_{z_4} - z_5z_7 + \partial_{z_6}\partial_{z_8}$$

$$\omega(h_-) = z_1\partial_{z_1} - z_2\partial_{z_2} + z_3\partial_{z_3} - z_4\partial_{z_4} + z_5\partial_{z_5} - z_6\partial_{z_6} + z_7\partial_{z_7} - z_8\partial_{z_8}$$

$$\omega(y_-) = -z_2z_4 + \partial_{z_1}\partial_{z_3} - z_6z_8 + \partial_{z_5}\partial_{z_8}$$

(3.3.23) $$\omega(x_+) = -z_1z_4 + \partial_{z_2}\partial_{z_3} - z_5z_8 + \partial_{z_6}\partial_{z_7}$$

$$\omega(h_+) = z_1\partial_{z_1} - z_2\partial_{z_2} - z_3\partial_{z_3} + z_4\partial_{z_4} + z_5\partial_{z_5} - z_6\partial_{z_6} - z_7\partial_{z_7} + z_8\partial_{z_8}$$

$$\omega(y_+) = -z_2z_3 + \partial_{z_1}\partial_{z_4} - z_6z_7 + \partial_{z_5}\partial_{z_8}.$$

For two functions ϕ_1 and ϕ_2 of four complex variables let

(3.3.24) $$\phi_1 \otimes \phi_2(z_1, z_2, \ldots, z_8) = \phi_1(z_1, z_2, z_3, z_4)\phi_2(z_5, z_6, z_7, z_8).$$

Denote by ω_2 the oscillator representation (1.4.10) of $\underline{sp}(4,\mathbf{C}) \cap \underline{u}(4,4)$.

Then clearly

(3.3.25) $$\omega|_{\underline{g}} = \omega_2|_{\underline{g}} \otimes \omega_2|_{\underline{g}}$$

We want to show that (3.3.3) implies (3.3.4). Define

(3.3.26) $$f_{m,n} = \sum_{j=0}^{\infty} \frac{(n-1)!}{j!(n-1+j)!}\, \omega_2(x_-)^j f_{-m} \otimes \omega_2(y_-)^j f_n$$

for integers m,n with $n > m \geq 1$.

The number m will happen to be the one from (3.3.3), but the n in (3.3.26) will correspond to $-n-1$ in (3.3.3).

Here f_{-m}, f_n are as in (3.1.20). A straight forward calculation using (3.1.22) and the lemma (3.3.6) verifies that

(3.3.27) $$\omega(x_-)\, f_{m,n} = 0.$$

Since y_+ commutes with x_- and with y_-, (3.1.22) and (3.1.23) imply that

(3.3.28) $$\omega(y_+) f_{m,n} = 0.$$

Similarly

(3.3.29) $$\omega(h_-) f_{m,n} = (m-n) f_{m,n} \quad \text{and}$$

$$\omega(h_+) f_{m,n} = (m+n)\, f_{m,n}.$$

Straight from the definitions (3.1.19), (3.1.20), we get

(3.3.30) $$\omega_2(x_-) f_{-m} = -m\, z_3 f_{-m-1},$$

$$\omega_2(y_-) f_n = -n\, z_2\, f_{n+1}.$$

Iterating (3.3.30) we obtain the explicit formula

(3.3.31) $$f_{m,n} = \sum_{j,a,b=0}^{\infty} \binom{m-1+j}{j} \frac{z_1^{m+a+j} z_2^{a+j} (z_3 z_4)^a (z_5 z_6)^b z_7^{b+j} z_8^{n+b+j}}{(m+a+j)!\ a!\ b!\ (n+b+j)!}.$$

In particular, by the definition (1.4.11) and the lemma (3.3.11)

(3.3.32) $\|f_{m,n}\|^2 = \sum_{j=0}^{\infty} \sum_{a,b=j}^{\infty} \binom{m-1+j}{j}^2 \frac{a!}{(m+a)!} \frac{b!}{(n+b)!}$

is finite for $n \geq m+2$, $m \geq 2$.

Let

(3.3.33) $V_{m,n}$ be the closed subspace of the Hilbert space of ω generated by the action of $(\underline{g},K) \times (\underline{g}', K')$ on $f_{m,n}$ with $n \geq m+2$, $m \geq 2$; $m,n \in Z$.

The formulas (3.3.27), (3.3.28), (3.3.29) and the known structure of the highest weight $\underline{sl}(2,\mathbf{C})$-modules imply that the $\underline{\mathbf{g}}$-module $\omega(\underline{\mathbf{g}})\, f_{m,n}$ is irreducible. Therefore $V_{m,n}$ is isotypic as a $O_{2,2}$-module and by the Corollary (1.5.25)

(3.3.34) $V_{m,n} \simeq \Pi \check{\otimes} \Pi' \in (O_{2,2} \times Sp(2,\mathbf{R}))^{\wedge}$

where, according to our parametrization of $\hat{K}$ (2.1.8),

(3.3.35) Π belongs to the discrete series of $O_{2,2}$ and contains the unique lowest K-type $\pi_{m,n}$ (2.1.7).

From the known weight decomposition of the discrete series representations of SL(2,**R**) [L] and from the formulas (3.3.27), (3.3.28), (3.3.29) we deduce that (for $p,q > 0$)

(3.3.36) $\pi_{p,q}$ is a K-type of Π iff
$p+q \in m+n + 2\mathbf{Z}_+$ and $p-q \in m-n-2\mathbf{Z}_+$.

Therefore, by (2.1.24),

(3.3.37) $\pi_{m,n}$ is a lowest degree K-type of Π and $\deg \Pi = m+n$.

The lowest degree term of $f_{m,n}$ (3.3.31) is a constant multiple of $z_1^m z_8^n$. Since

(3.3.38) $\omega|_K = \omega_2|_K \otimes \omega_2|_K$ (see also (3.2.1) and (1.4.17))

(3.1.16) implies that the space generated by the action of K on $z_1^m z_8^n$ is

(3.3.39) $$Cz_1^m z_8^n + Cz_2^m z_8^n + Cz_1^m z_7^n + Cz_2^m z_7^n \cong \pi_{m,n} .$$

By (3.3.18) and (3.3.19),

(3.3.40) $$\omega(x'_-)\, z_1^m z_8^n = 0,\ \omega(h'_-) z_1^m z_8^n = (m+n) z_1^m z_8^n$$

$$\omega(h'_+) z_1^m z_8^n = (m-n)\ z_1^m z_8^n .$$

Therefore the Corollary (1.5.25) implies that

(3.3.41) Π' (3.3.34) contains the (lowest degree) K'-type $\pi'_{m,-n}$.

The formulas (3.3.27), (3.3.28), (3.3.29) imply that

(3 3.42) Π has infinitesimal character $\gamma_{m,-n+1}$ (2.5.32).

It follows from [P2] that

(3.3.43) Π' has infinitesimal character $\gamma'_{m,-n+1}$ (2.4.42).

In particular (2.1.22) implies that

(3.3.44) $$\|\pi'_{m,-n}\|_{\text{lambda}} = \|\gamma'_{m,-n+1}\|$$

Since $\|f_{m,n}\| < \infty$ (3.3.32), the lemma (3.3.10) ensures that Π' is tempered. Combining this with (3.3.44) and the corollary (2.3.22) we conclude that

(3.3.45) $\pi'_{m,-n}$ is a lowest K'-type (2.2.18) of Π'.

The representations (3.3.45) appear in the list of possible lowest K'-types ((2.4.3), (2.4.19), (2.4.29), (2.4.43)) only in the representations attached to H'_3. Therefore Π' is a discrete series representation. Combining the above with (3.3.35) we see that (3.3.3) implies (3.3.4) except the case of lowest $K \times K'$-types $\pi_{1,-n} \otimes \pi'_{1,-n}$ $(n = 3,4,5\ldots)$.

To complete the argument we employ the theory of translation functors [Z]. Let S* be the dual topological vector space to the space of smooth vectors S on which ω is realized (ω extends to S*). Denote by $V^0_{2,n}$ the

Harish-Chandra module of $V_{2,n}(n \geq 4)$.

Let

(3.3.46) Diff denote the Weyl algebra of polynomial coefficient differential operators acting on $\mathbf{C}[z_1, z_2, \ldots, z_8]$.

Then $(\underline{g},K)\times(\underline{g}',K')$ acts on Diff by conjugation and, in the spirit of the formula (1.2.9), we have a $(\underline{g},K)\times(\underline{g}',K')$-intertwining map

(3.3.47) $$Q : \text{Diff} \otimes V^0_{2,n} \to S^*, \; Q(u\otimes v) = u(v).$$

Let

(3.3.48) $$u = z_3 \, \partial_{z_5} - z_7 \, \partial_{z_1}.$$

We check easily that

(3.3.49) $$[\omega(x_-),u] = [\omega(y_-),u] = [\omega(h_-),u] = 0,$$
$$[\omega(y_+),u] = 0, \; [\omega(h_+),u] = -2u$$

(3.3.50) $$[\omega(y'_-),u] = [\omega(y'_+),u] = [\omega(y'_1),u] = [\omega(y'_2),u] = 0,$$
$$[\omega(x'_-),u] = [\omega(h'_-),u] = 0, \; [\omega(h'_+),u] = -2u.$$

Fix the Borel subalgebra of $\underline{\mathbf{g}} \oplus \underline{\mathbf{g}}'$ containing the vectors x_-, x_+ (3.1.4), x'_-, x'_+, x'_1, x'_2 (3.3.14). Denote by

(3.3.51) V the subspace of Diff generated by the action of $(\underline{g},K)\times(\underline{g}',K')$ on u.

It follows from (3.3.49) and (3.3.50) that

(3.3.52) u is a lowest weight vector in V with weight $\gamma_{-1,-1} \oplus \gamma'_{-1,-1}$ ((2.5.32), (2.4.42)) and V is irreducible.

According to our choice of Borel subalgebra in $\underline{\mathbf{g}} \oplus \underline{\mathbf{g}}'$ we consider

(3.3.53) $\gamma_{n-1,2} \oplus \gamma'_{n-1,2}$ as a representative for the infinitesimal character of the $(\underline{g},K)\times(\underline{g}',K')$-module $V^0_{2,n}$.

It follows from [Z1, §1] that

(3.3.54) $V \otimes V^0_{2,n}$ contains exactly one irreducible submodule W^0 with infinitesimal character

$$(\gamma_{n-1,2} \oplus \gamma'_{n-1,2}) + (\gamma_{-1,-1} \oplus \gamma'_{-1,-1}) = \gamma_{n-2,1} \oplus \gamma'_{n-2,1}.$$

Moreover [V1, 8.2.1] implies that

(3.3.55) W^0 is isomorphic to the Harish-Chandra module of the discrete series representation (of our pair of groups) with lowest $K\times K'$-type $\pi_{1,-n+1} \check{\otimes} \pi'_{1,-n+1}$.

Let

(3.3.56) W^0_1 be the subspace of $V \otimes V^0_{2,n}$ generated by the action of $(\underline{g},K)\times(\underline{g}',K')$ on $u \otimes f_{2,n}$.

The formulas (3.3.27), (3.3.28), (3.3.29) and (3.3.49), (3.3.50) imply that

(3.3.57 $(\underline{g},K)$ acts on W^0_1 with infinitesimal character $\gamma_{n-2,1}$.

Therefore $W^0_1 = W^0 \oplus W^0_2$ (3.3.54), where $(\underline{g}',K')$ acts on W^0_2 with infinitesimal character different than $\gamma'_{n-2,1}$. Since by [P2] the infinitesimal characters of representations which occur in the Oscillator Duality Correspondence, for our pair, must coincide we see that

(3.3.58) $Q(W^0_2) = 0$, and therefore $Q(W^0) = Q(W^0_1)$.

By a straightforward calculation we check that $u\, f_{2,n} \neq 0$. Thus $Q(W^0_1) \neq 0$ and by (3.3.58)

(3.3.59) $Q(W^0) \neq 0$.

Combining (3.3.62) and (3.3.55) we conclude the proof of "(3.3.4) if (3.3.3)." The case (3.3.2) can be verified in a similar way with $f_{m,n}$ (3.3.26) replaced by

(3.3.53) $$\phi_{m,n} = \sum_{j=0}^{\infty} \frac{(m-1)!}{j!(m-1+j)!}\, \omega_2(x_-)^j f_{-m} \otimes \omega_2(y_-)^j f_n \quad \text{for} \quad m > n \geq 1.$$

Q.E.D.

§4. THE PRINCIPAL SERIES

Let us identify $M_{8,1}(\mathbf{R})$ with $M_{4,2}(\mathbf{R})$ by

$$(3.4.1) \qquad \mathrm{col}(x_1,x_2,\ldots,x_8) \to \begin{bmatrix} x_1 & x_5 \\ x_2 & x_6 \\ x_3 & x_7 \\ x_4 & x_8 \end{bmatrix}.$$

Then the pull back of the oscillator representation to $O_{2,2} \times Sp(2,\mathbf{R})$ by (1.1.12) can be realized on the space of Schwartz functions $S(M_{4,2}(\mathbf{R})) = S$ and the formulas (1.3.31) imply that for $f \in S$

$$(3.4.2) \qquad \omega(g)\, f(x) = f(g^{-1}x) \qquad (g \in O_{2,2},\ x \in M_{4,2}(\mathbf{R}));$$

$$(3.4.3) \qquad \omega(h')\, f(x) = (\det h)^{-2}\ f(x(h^t)^{-1}),$$

where $h' = \mathrm{diag}(h,(h^t)^{-1}) \in Sp(2,\mathbf{R})$, $h \in GL(2,\mathbf{R})$;

$$(3.4.4) \qquad \omega(b')\, f(x) = \chi(-\tfrac{1}{2}\beta(x,x))f(x),$$

$$b' = \begin{bmatrix} I_2 & 0 \\ b & I_2 \end{bmatrix},\ \beta(x,x) = \mathrm{Tr}(x\ b\ x^t F) \ \text{(for } F \text{ see (1.1.8))}$$

For $\sigma = 0,1$ and for $\mu \in \mathbf{C}$ put

$$(3.4.5) \qquad \chi_{\sigma,\mu}(r) = (\mathrm{sgn}\ r)^{\sigma}|r|^{\mu} \qquad (r \in \mathbf{R}).$$

Define the following tempered distributions on $M_{4,2}(\mathbf{R})$:

$$(3.4.6) \qquad u(f) = \int f(\begin{bmatrix} x_1 & x_3 \\ 0 & x_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix})\ \chi_{\sigma_1,\mu_1}(x_1)\chi_{\sigma_2,\mu_2}(x_2)\ dx_1dx_2dx_3$$

where the integration is over $\mathbf{R}^3$ and
$\sigma_1,\sigma_2 = 0,1$; $\mu_1,\mu_2 \in \mathbf{C}$; $\mathrm{Re}\ \mu_1 > -1$; $\mathrm{Re}\ \mu_2 > -1$;
if $\mu_j = -1$ then $\sigma_j - \mu_j \in 2\mathbf{Z}$, $j = 1,2$.

It is classical that u is well defined [S-W, Ch VI Thm 3.1]. A simple consequence of the definition (3.4.6) and the formulas (3.4.2), (3.4.3) (3.4.4) is the fact that

$$(3.4.7) \qquad u \in \mathrm{Hom}_{P_0 \times \theta P'_0}(S, (\delta \otimes (\nu + \rho(\underline{n}_0))) \check{\otimes} (\delta' \otimes (\nu' + \rho(\theta \underline{n}'_0)))),$$

where P_0, P'_0 are defined in (2.2.2), (2.2.10), and

$$\delta(\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2)) = \varepsilon_1^{\sigma_1} \varepsilon_2^{\sigma_2}, \ \delta = \delta',$$

$$\nu(\mathrm{diag}(a_1, a_2, -a_1, -a_2)) = (\mu_1 + 1)a_1 + (\mu_2 + 1)a_2, \ \nu = \nu'.$$

By composing u with the inverse of the map (1.4.21) we obtain u as a continuous linear functional on the space V of Schwartz functions in the Fock model (1.4.6) [H7, Cor. 1.7.2]. We may expand it in a series

$$(3.4.8) \qquad u = \sum_\alpha u_\alpha z_\alpha / \sqrt{\alpha!}$$

where the summation is over all α in $\mathbf{Z}_+^8$ and

$$(3.4.9) \qquad u(f) = \sum_\alpha u_\alpha f_\alpha \quad \text{for any } f \text{ in } V \text{ with } f(z) = \sum_\alpha f_\alpha z^\alpha / \sqrt{\alpha!}.$$

(3.4.10) Lemma. The lowest degree term of u (3.4.8) is a non-zero multiple of

$$(3.4.11) \qquad z_1^{\sigma_1} z_6^{\sigma_2}.$$

Proof: By applying u (3.4.8) to $z^\alpha/\sqrt{a!}$ we find that (see (1.4.20) for ψ_m)

$$(3.4.12) \qquad u_\alpha = C(\alpha) \cdot \int_{\mathbf{R}} \psi_{\alpha_1}(x) \chi_{\sigma_1, \mu_1}(x) dx \int_{\mathbf{R}} \psi_{\alpha_5}(x) dx \int_{\mathbf{R}} \psi_{\alpha_6}(x) \chi_{\sigma_2, \mu_2}(x) dx,$$

where

$$C(\alpha) = \psi_{\alpha_2}(0)\, \psi_{\alpha_3}(0)\, \psi_{\alpha_4}(0)\, \psi_{\alpha_7}(0)\, \psi_{\alpha_8}(0).$$

We are looking for $u_\alpha \neq 0$ with $|\alpha|$ minimal. Clearly we can assume that $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_7 = \alpha_8 = 0$ (1.4.20). By a change of variables we check that

(3.4.13)
$$\int_{\mathbf{R}} \psi_0(x)\chi_{\sigma,\mu}(x)dx = \begin{cases} 0 & \text{for } \mu = -1,\ \sigma = 1 \\ \mathrm{const}_1\ (1+(-1)^{\sigma})\Gamma\left(\frac{\mu+1}{2}\right), & \text{otherwise} \end{cases}$$
$$\int_{\mathbf{R}} \psi_1(x)\chi_{\sigma,\mu}(x)dx = \mathrm{const}_2\ (1-(-1)^{\sigma})\Gamma\left(\frac{\mu+2}{2}\right)$$

where $(\sigma,\mu) = (\sigma_j,\mu_j)$ (3.4.6), const_1 and const_2 are positive constants independent of (σ,μ) and $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ The lemma follows immediately from (3.4.12), (3.4.13) and the fact that $\Gamma(z) \neq 0$ for $\mathrm{Re}\ z \geqslant 0$, $z \neq 0$ [Vi, Ch V §1.5].

Q.E.D.

According to the formulas (3.3.18), (3.3.19), (3.3.22), (3.1.16), (3.3.38), (1.4.17) the space V contains the following $K \times K'$ ((3.1.3), (3.3.13))-invariant and irreducible subspaces:

(3.4.14) $\mathbf{C}\cdot 1 \cong 1 \check{\otimes} 1$

(3.4.15) $\mathbf{C}z_1 + \mathbf{C}z_2 + \mathbf{C}z_5 + \mathbf{C}z_6 \cong \pi^0_{1,0} \check{\otimes} \pi'_{1,0}$,

(3.4.16) $\mathbf{C}z_3 + \mathbf{C}z_4 + \mathbf{C}z_7 + \mathbf{C}z_8 \cong \pi^0_{0,-1} \otimes \pi'_{0,-1}$,

(3.4.17) $\mathbf{C}(z_1z_6 - z_2z_5) \cong \pi^{1,0}_{0,0} \check{\otimes} \pi'_{1,1}$

(3.4.18) $\mathbf{C}(z_3z_8 - z_4z_7) \cong \pi^{0,1}_{0,0} \check{\otimes} \pi'_{-1,-1}$.

Here the parametrization of representations is as in (2.1.7), (2.1.9). Put

(3.4.19) Let c be as in (1.1.9), and s_4 as in (3.3.16). Set $c_0 = \mathrm{diag}\ (c,c)$. Let $\tilde{c}_0$, $\tilde{s}_4 \in \tilde{U}(8)$ map to c_0, s_4 respectively, via the covering map (1.4.16).

(3.4.20) <u>Lemma.</u> Let for u (3.4.6) and δ,ν (3.4.7)

$$(3.4.21) \qquad \mathrm{Ind}\, u : S \to \mathrm{Ind}_{P_0 \times \theta P_0'}^{O_{2,2} \times Sp(2,\mathbf{R})} \left((\delta \otimes \nu) \check{\otimes} (\delta' \otimes \nu') \right)$$

be the non-zero, continuous, $O_{2,2} \times Sp(2,\mathbf{R})$-intertwining operator provided by the smooth Frobenius reciprocity theorem (Appendix B). Then the operator

(3.4.22) $\quad$ Ind $u \circ (1.4.21)^{-1}$ from V to the induced representation (3.4.21)

does not annihilate the image under $\omega(g_4)$ (3.4.19) of the space

(3.4.14) if $\sigma_1 = \sigma_2 = 0$;

(3 4.15), (3.4.16) if $\sigma_1 \neq \sigma_2$,

(3.4.17), (3.4.18) if $\sigma_1 = \sigma_2 = 1$.

Moreover this image transforms under the maximal compact subgroup (2.1.1) × (2.1.2) of $O_{2,2} \times Sp(2,\mathbf{R})$ as indicated in (3.4.14),.., (3.4.18),

and

(3.4.23) $\quad$ the operator (3.4.21) annihilates every polynomial $f \in V$ of degree lower than $\sigma_1 + \sigma_2$.

Proof: We have the following commuting diagram of group isomorphisms:

$$\begin{tikzcd}
O(4)\cap O(2,2)\times O(4)\cap Sp(2,R) \arrow[r, "(1.1.6)\times(1.1.7)"] \arrow[d, "(1.1.9)\times \mathrm{id}"'] & O(16)\cap Sp(8,R) \arrow[r, "(1.4.15)^{-1}\circ(1.4.18)"] \arrow[d, phantom, "\mathrm{Int}\ C"] & U(8) \arrow[d, "\mathrm{Int}\ c_0"] \\
O(4)\cap O(2,2)\times O(4)\cap Sp(2,\mathbf{R}) \arrow[r, "(1.1.6)\times(1.1.7)"] \arrow[d, "\mathrm{Int}\ s_2\times(1.4.15)^{-1}\circ(1.4.18)"'] & O(16)\cap Sp(8,\mathbf{R}) \arrow[r, "(1.4.15)^{-1}\circ(1.4.18)"] & U(8) \arrow[d, "\mathrm{Int}\ s_4"] \\
K\times K' \arrow[rr] & & U(8)
\end{tikzcd}$$

where the lowest horizontal arrow makes $K\times K'$ act on the Fock model as in §2, and C is defined in (1.1.10), c_0 in (3.4.19), s_2 in (3.1.1) and s_4 in

(3.3.16). Since g_4 (3.4.19) covers $(s_4c_0)^{-1}$, the above diagram and our parametrization (2.1.8), (2.1.10) of the unitary dual of the group $O(4) \cap O_{2,2} \times O(4) \cap Sp(2,\mathbb{R})$ implies that the spaces (3.4.14),..., (3.4.18) transform under this group as indicated there. It follows easily from the formula (1.4.17) that $\omega(\tilde{s}_4^{-1})$ preserves the spaces (3.4.14),...., (3.4.18). Since the lowest degree term of u (3.4.8) is $z_1^{\sigma_1} a_6^{\sigma_2}$ (3.4.11), the lowest degree term of $u \circ \omega(\tilde{c}_0)$ is (a non-zero multiple of)

$$(3.4.24) \qquad (z_1 + z_3)^{\sigma_1}(z_6 + z_8)^{\sigma_2}$$

Therefore $u \circ \omega(\tilde{c}_0)$ does not vanish on 1 if $\sigma_1 = \sigma_2 = 0$; z_1 and z_3 if $\sigma_1 = 1$ and $\sigma_2 = 0$; z_6 and z_8 if $\sigma_1 = 0$ and $\sigma_2 = 1$; z_1z_6 and z_3z_8 if $\sigma_1 = \sigma_2 = 1$. From the definition of the induced map (B.5) we see that the above implies the lemma.

Q.E.D.

Define another tempered distribution on S

$$(3.4.25) \quad v(f) = \int f\left(\begin{bmatrix} 0 & x \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \chi_{\sigma,\mu}(x)\, dx \qquad \sigma = 0,1;\ \mu \in \mathbf{C},\ \mathrm{Re}\,\mu > -1).$$

A straight forward calculation shows that

$$(3.4.26) \qquad v \in \mathrm{Hom}_{P_0 \times \theta P_0'}(S, (\delta \otimes (\nu + \rho(\underline{n}_0)) \check{\otimes} (\delta' \otimes (\nu' + \rho(\underline{n}_0')))),$$

where $\delta(\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_1,\varepsilon_2)) = \varepsilon_1^{\sigma}$, $\delta'(\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_1,\varepsilon_2)) = \varepsilon_2^{\sigma}$,

$$\nu(\mathrm{diag}(a_1,a_2,-a_1,-a_2)) = \mu a_1, \quad \nu'(\mathrm{diag}(a_1,a_2 -a_1,-a_2)) = \mu a_2.$$

Notice that under the assumptions (3.4.6) we could not reach the parameters δ, ν (3.4.26) in (3.4.7). This is why we consider the distribution v (3.4.25).

As in the lemma (3.4.10) we check that in the Fock model the lowest degree term of v is z_5^{σ} (up to a non-zero multiple depending on μ). Therefore copying the proof of the lemma (3.4.20) we obtain

(3.4.27) Lemma. Let

$$\text{(3.4.28)} \qquad \mathrm{Ind}\, \nu : S \to \mathrm{Ind}_{P_0 \times \theta P_0'}^{0_{2,2} \times Sp(2,\mathbb{R})} ((\delta \otimes \nu) \otimes (\delta' \otimes \nu')^{\vee}) \qquad (3.4.26)$$

be the non-zero, continuous, $0_{2,2} \times Sp(2,\mathbb{R})$-intertwining operator provided by the smooth Frobenius reciprocity theorem (Appendix B). Then the operator

(3.4.29) $\mathrm{Ind}\, \nu \circ (1.4.21)^{-1}$ from V to the induced representation (3.4.28) does not annihilate the image under $\omega(g_4)$ (3.4.19) of the space

(3.4.14) if $\sigma = 0$

(3.4.15), (3.4.16) if $\sigma = 1$. Moreover

(3.4.30) the operator (3.4.29) annihilates every polynomial $f \in V$ of degree lower than σ.

(3.4.31) Theorem. Let

$$\text{(3.4.32)} \qquad \begin{aligned} \Pi &= \Pi_{0_{2,2}}[H_0, \Gamma, \gamma](\underline{\underline{A}}), \text{ and} \\ \Pi' &= \Pi_{Sp(2,\mathbb{R})}[H_0, \Gamma, \gamma](\underline{\underline{A}}') \end{aligned}$$

as in (2.5.4) and (2.4.5) respectively. Define the sets

$$\text{(3.4.33)} \qquad D = \{\pi^{0,0}_{0,0}, \pi^{0}_{1,0}, \pi^{0}_{0,-1}, \pi^{1,0}_{0,0}, \pi^{0,1}_{0,0}\} \subseteq \hat{K},$$

$$\text{(3.4.34)} \qquad D' = \{\pi'_{0,0}, \pi'_{1,0}, \pi'_{0,-1}, \pi'_{1,1}, \pi'_{-1,-1}\} \subseteq \hat{K}'$$

and the function $\partial : D \to D'$ by

$$\text{(3.4.35)} \qquad \partial(\pi^{0,0}_{0,0}) = \pi'_{0,0};\ \partial(\pi^{0}_{m,n}) = \pi'_{m,n};\ \partial(\pi^{1,0}_{0,0}) = \pi'_{1,1};\ \partial(\pi^{0,1}_{0,0}) = \pi'_{-1,-1}.$$

Then $\underline{\underline{A}} \cap D \neq \emptyset$ and $\underline{\underline{A}}' \cap \partial(\underline{\underline{A}} \cap D) \neq 0$ imply that

(3.4.36) $\quad \check{\Pi \otimes \Pi'} \in R(O_{2,2} \cdot Sp(2,\mathbf{R}),\omega)$ (3.2.3), and

(3.4.37) $\quad \deg \Pi = \sigma_1 + \sigma_2$

Proof. The images of the spaces (3.4.14),.., (3.4.18) under $\omega(g_4)$ (3.4.19) are clearly of the form $H_{\pi,\pi'}$ (1.5.24) with $\pi' = \partial(\pi)$ (3.4.35). The operators Ind u and Ind v constructed in (3.4.21) and (3.4.28) satisfy the assumptions (2.2.24), (2.2.26) of the lemma (2.2.23) and the corresponding $K \times K'$-types $\pi \check{\otimes} \partial(\pi)$ are lowest in the induced representations (3.4.21), (3.4.28)-see (2.4.3), (2.5.3). By the choice of parameters $\sigma_1, \sigma_2, \mu_1, \mu_2$ (3.4.6) and σ, μ (3.4.25) the representation

(3.4.38) $\quad \mathrm{Ind}_P^{O_{2,2}}(\delta \otimes \nu)$ ((3.4.7), (3.4.26))

has an irreducible quotient Π containing a K-type $\pi \in D$ (3.4.33). Clearly $\pi \in D(\Pi)$. Therefore the lemma (2.2.23) implies that

$$\check{\Pi \otimes \Pi'} \in R(O_{2,2} \cdot Sp(2,\mathbf{R}),\omega)$$

where $\partial(\pi) \in D(\Pi')$ and Π' is a subquotient of

$$\mathrm{Ind}_{\theta P_0}^{Sp(2,\mathbf{R})}(\delta' \otimes \nu')$$

Notice that the regular characters (Γ,γ) obtained from δ,ν and from δ',ν' coincide (modulo a conjugation by K'), and that by (3.4.6), (3.4.25) they exhaust all possible regular characters (2.4.2) (modulo the Weyl group). This shows (3.4.36). The last statement follows from (3.4.23) and (3.4.30).

Q.E.D.

§5. THE MAXIMAL PARABILICS.

In the notation (3.4.1).,,. (3.4.4) define the following linear map from S to the space of functions on $SL^{\pm}(2,\mathbf{R})$ [V1, I §4]:

(3.5.1) $\quad u(f)(g) = \int_0^\infty f\,\binom{rg}{0} r^\mu dr \qquad (f \in S,\ g \in SL^{\pm}(2,\mathbf{R}),\ \mu \in \mathbf{C},\ \mathrm{Re}\,\mu > 0).$

Let

(3.5.2) $$\|y\| = \max\{|\lambda|;\ \lambda \in \mathbf{C},\ \lambda \text{ is an eigenvalue of } y\} \quad (y \in M_{2,2}(\mathbf{R})).$$

Since f (3.5.1) is a Schwartz function, there are constants $N > \mathrm{Re}\,\mu+1$ and $C_N < \infty$ such that

(3.5.3) $$\left|f\begin{pmatrix} y \\ 0 \end{pmatrix}\right| \leq C_N(1+\|y\|)^{-N} \qquad \text{for all } y \in M_{4,4}(\mathbf{R}).$$

Therefore

(3.5.4) $$|u(f)(g)| \leq \int_0^\infty c_N(1+x\|g\|)^{-N}\, x^{\mathrm{Re}\,\mu+1}\, \frac{dx}{x}$$

$$= \left(C_N \int_0^\infty (1+x)^{-N}\, x^{\mathrm{Re}\,\mu} dx\right) \cdot \|g\|^{-\mathrm{Re}\,\mu-1}$$

Apply the Cartan decomposition [L, Ch VIII §2 Int 2] to the element g:

(3.5.5) $$g = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\varepsilon_1\sin\theta_1 & \varepsilon_1\cos\theta_1 \end{bmatrix} h_t \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\varepsilon_2\sin\theta_2 & \varepsilon_2\cos\theta_2 \end{bmatrix},$$

$$0 \leq \theta_1, \theta_2 < 2\pi;\ \varepsilon_1, \varepsilon_2 = \pm 1,\ h_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix};\ t \in R.$$

Since the norm (3.5.2) is an operator norm we see that

(3.5.6) $$\|g\| = \|h_t\| = e^{|t|}.$$

Therefore, for $r,\ p \in \mathbf{R}$

(3.5.7) $$\int_{SL^{\pm}(2,\mathbf{R})} \|g\|^{-pr}\, dg \leq \int_{SL^{\pm}(2,R)} e^{-pr|t|} dg = \int_0^\infty e^{-prt}\sinh(2t)dt.$$

This number is finite for $pr > 2$.

Combining (3.5.4), (3.5.7), we see that

(3.5.8) the map u (3.5.1) is a continuous linear operator from S to the space of smooth vectors in $L^2(SL^{\pm}(2,\mathbf{R}))$ if $\mathrm{Re}\,\mu > 0$ and in any $L^p(SL^{\pm}(2,\mathbf{R}))$, $p > 2$, if $\mathrm{Re}\,\mu \geq 0$.

It follows from [L, Ch IX §1] that

(3.5.9) the Harish-Chandra modules of the discrete series representations of $SL^{\pm}(2,\mathbf{R})$ are all contained in the intersection of all the spaces $L^q(SL^{\pm}(2,\mathbf{R}))$ for $q > 1$.

Let V be the Harish-Chandra module of the isotypic component in $L^2(SL^{\pm}(2,\mathbf{R}))$ of one discrete series representation of this group.

Denote by V^q the completion of V in $L^q = L^q(SL^{\pm}(2,\mathbf{R}))$, $q > 1$. By a theorem of Harish-Chandra [HC] V^q is irreducible with respect to the left and right action of the group $SL^{\pm}(2,\mathbf{R}) \times SL^{\pm}(2,\mathbf{R})$. Taking the adjoint of the injection $V^q \to L^q$ we obtain a surjection

(3.5.10) $$(L^q)^* \to (V^q)^*$$

which intertwines the contragradient actions of the group $SL^{\pm}(2,\mathbf{R}) \times SL^{\pm}(2,\mathbf{R})$ on both spaces. Since L^q is a reflexive linear topological vector space it follows from [Ke-Na Ch 5 §20.2 (ii)] that V^q is reflexive so that in particular $(V^q)^*$ is irreducible.

Let $\overline{V}$ be the space of complex conjugates of the functions from V. Then the image of $\overline{V}$ under the identification $(L^q)^* \cong L^p$, $\frac{1}{p} + \frac{1}{q} = 1$, and the restriction map (3.5.10) is not zero (by a straight forward integration). Therefore $(V^q)^*$ may be identified with $\overline{V}^p$, and we have a continuous intertwining map

$$L^p \to \overline{V}^p.$$

We summarize the above discussion in the following statement:

(3.5.11) for each discrete series representation δ of $SL^{\pm}(2,\mathbf{R})$ and for each $p \geq 2$ there exist an irreducible representation of $SL^{\pm}(2,\mathbf{R}) \times SL^{\pm}(2,\mathbf{R})$ on a Banach space $\overline{V}^p$, which is infinitesimally equivalent to $\delta \otimes \delta^c$, and a continuous non-zero, linear, intertwining map

$$T : L^p(SL^{\pm}(2,\mathbf{R})) \to \overline{V}^p.$$

Here δ^c denotes the contragradient of δ, and the group acts on L^p by the left and right translations.

(3.5.12) Lemma. the image of S under u (3.5.1) is dense in $L^p(SL^{\pm}(2,\mathbb{R}))$ for any $p > 2$.

Proof: Fix $\phi \in L^q(SL^{\pm}(2,\mathbf{R}))$, $\frac{1}{q} + \frac{1}{p} = 1$, and define a linear map

(3.5.13) $$S \ni f \to \int_{SL^{\pm}(2,\mathbf{R})} u(f)(s)\phi(s)ds \in \mathbf{C}$$

By (3.5.8), this map is a tempered distribution on S. Assume that the integral (3.5.13) is zero for all $f \in S$. Then

$$0 = \int_0^\infty \int_{SL^{\pm}(2,\mathbb{R})} g(xs)x^\mu \phi(s)dsdx \qquad \text{for all } g \in S(M_{2,2}(\mathbf{R})).$$

and therefore the function

$$R^+ \times SL^{\pm}(2,\mathbb{R}) \ni (x,s) \to x^\mu \phi(s) \in \mathbf{C}$$

is zero. Since $x^\mu \neq 0$ we conclude that $\phi = 0$.

Q.E.D.

(3.5.14) Corollary. For every discrete series representation δ of $SL^{\pm}(2,\mathbb{R})$, the operator

(3.5.15) $$v = T \circ u : S \to \overline{V}^p \qquad ((3.5.11), (3.5.1))$$

is non-zero.

Let, for $\mu \in \mathbf{C}$, $\nu \in \underline{a}^*_1$ (2.2.7) be defined by

(3.5.16) $$\nu(\mathrm{diag}(a,a,-a,-a)) = \mu a$$

We identify M_1 (2.2.6) with $SL^{\pm}(2,\mathbf{R})$ via

(3.5.17) $$SL^{\pm}(2,\mathbb{R}) \ni g \to \mathrm{diag}(g,(g^t)^{-1}) \in M_1.$$

(3.5.18) Lemma. Fix $\mu \in \mathbf{C}$ with $\mathrm{Re}\,\mu \geq 0$, define ν as in (3.5.16) and let δ be a discrete series representation of M_1 and v be as in (3.5.15). Then

$$v \in \mathrm{Hom}_{P_1 \times \theta P_1'}(S, (\delta \otimes (\nu + \rho(\underline{n}_1))) \check{\otimes} (\delta^c \otimes (\nu + \rho(\theta \underline{n}_1'))))$$

is non-zero.

Proof. The group $P_1 \times \theta P_1'$ acts on S by the formulas (3.4.2), (3.4.3), (3.4.4). Therefore one obtains the transformation properties of v with respect to $A_1N_1 \times A_1'\theta N_1'$ by a straight forward calculation. The intertwining properties of v with respect to $M_1 \times M_1$ and the fact that $v \neq 0$ are immediate from (3.5.11) and the Corollary (3.5.14).

Q.E.D.

(3.5.19) Theorem. Let

$$\Pi = \Pi_{O_{2,2}}[H_1, \Gamma, \gamma](\underline{\underline{A}}), \text{ and}$$

(3.5.20) $$\Pi' = \Pi_{Sp(2,\mathbf{R})}[H_1, \Gamma, \gamma](\underline{\underline{A}}')$$

as in (2.5.23) and (2.4.20) respectively. Define the following bijection

(3.5.21) $$\partial : \underline{\underline{A}}_{O_{2,2}}[H_1, \Gamma, \gamma] \to \underline{\underline{A}}_{Sp(2,\mathbf{R})}[H_1, \Gamma, \gamma]$$

by

(3.5.22) $$\partial(\pi_{m,-m}) = \pi'_{m,-m} \qquad \text{if } n = 2m-1,$$

(3.5.23) $$\partial(\pi_{m+1,-m}) = \pi'_{m+1,-m}, \ \partial(\pi_{m,-m-1}) = \pi'_{m,-m-1}$$

$$\text{if } n = 2m.$$

Then $\underline{\underline{A}}' = \partial(\underline{\underline{A}})$ implies that

(3.5.24) $$\Pi \check{\otimes} \Pi' \in R(O_{2,2} \cdot Sp(2,\mathbf{R}), \omega), \text{ and } \deg \Pi = n+1.$$

Proof: We notice that $GL(2,\mathbf{R}), GL(2,\mathbf{R})$ is a reductive dual pair in $Sp(4,\mathbf{R})$ and that via the obvious extension of the identification (3.5.17) to an isomorphism $GL(2,\mathbf{R}) \cong M_1A_1$,

(3.5.25) $$(\delta \otimes \nu) \check{\otimes} (\delta^c \otimes \nu) \in R(M_1A_1 \cdot M_1A_1, \omega_1), \qquad (3.5.18),$$

where ω_1 is the oscillator representation of $\widetilde{Sp}(4,\mathbb{R})$ (corresponding to the same character (1.3.32) as ω).

Let

$$\deg(\delta\otimes\nu) = n+1, \qquad (1.5.16) \tag{3.5.26}$$

Then n = 1,2,3,... depends on δ (is equal to the lambda norm of δ). As far as the compact groups are concerned, the Harish-Chandra module of ω_1 may be identified ((3.4.1), (1.4.21)) with the subspace

$$\mathbb{C}[z_1,z_2,z_5,z_6] \subseteq \mathbb{C}[z_1,\dots,z_8] \tag{3.5.27}$$

of the Harish-Chandra module of ω. Let σ be a lowest degree O(2)-type of $\delta\otimes\nu$ (3.5.25) and let

$$H'_{\sigma,\sigma'} \subseteq \mathbb{C}[z_1,z_2,z_5,z_6] \tag{3.5.28}$$

be the corresponding subspace (1.5.24) in the Fock model of ω_1. It follows immediately from the definition (B.5) that

(3.5.29) the induced map Ind v when restricted to the subspace $K \times K' \cdot H'_{\sigma,\sigma'}$, generated by the action of $K \times K'$ ((3.1.3), (3.3.13)) on $H'_{\sigma,\sigma'}$, is injective.

A straightforward calculation (using (1.5.32)) shows that

$$\begin{aligned} H_{\sigma,\sigma'} = {} & \mathbb{C}(z_1+iz_2+iz_5-z_6)^{n+1} + \mathbb{C}(iz_1+z_2-z_5+iz_6)^{n+1} \\ & + \mathbb{C}(iz_1-z_2+z_5+iz_6)^{n+1} + \mathbb{C}(-z_1+iz_2+iz_5+z_6)^{n+1} \end{aligned} \tag{3.5.30}$$

and that for g_4 defined in (3.4.19)

$$\begin{aligned} \omega(g_4^{-1})H_{\sigma,\sigma'} = {} & \mathbb{C}(iz_2+z_3-z_6+iz_7)^{n+1} + \mathbb{C}(iz_1+z_4-z_5+iz_8)^{n+1} \\ & + \mathbb{C}(-z_2+iz_3+iz_6+z_7)^{n+1} + \mathbb{C}(-z_1+iz_4+iz_5+z_8)^{n+1}. \end{aligned} \tag{3.5.31}$$

Applying $\frac{1}{\sqrt{2}}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$ as an element of K' (3.3.13) to the space (3.5.31) we obtain the space

$$(3.5.32)\qquad \mathbf{C}(z_1+z_8)^{n+1} + \mathbf{C}(z_2+z_7)^{n+1} + \mathbf{C}(z_3+z_6)^{n+1} + \mathbf{C}(z_4+z_5)^{n+1}$$

It follows easily from the formulas (3.3.18), (3.3.23), that for $f_0(z) = (z_1+z_8)^{n+1}$

$$(3.5.33)\qquad \omega(x'_-)f_0 = 0 \text{ and } \omega(h)f_0 = (n+1)f_0 \quad \text{for } h = h_+ \text{ or } h'_-.$$

The formulas (3.3.19), (3.3.15) imply that the space generated by the action of the center of K' on f_0 is

$$(3.5.33)\qquad \sum_{k=0}^{n+1} \mathbf{C}\, f_k,\quad f_k(z) = (z_1-z_8)^k(z_1+z_8)^{n+1-k}, \quad \text{and that}$$

$$(3.5.34)\qquad \omega(h'_+)f_k = \omega(h_-)\, f_k = kf_{k-1} + (n+1-k)f_{k+1}\ (k=0,1,..,n+1).$$

Assume that $n+1 = 2m$ is an even integer. Then (3.5.34) implies that

$$(3.5.35)\qquad \omega(h'_+)\Phi = \omega(h_-)\Phi = 0, \text{ where}$$

$$\Phi = \sum_{k=0}^{m} (-1)^k\binom{m}{k} f_{2k}$$

Since $\omega(h'_+)$ commutes with $\omega(h'_-)$, $\omega(x'_-)$, $\omega(h_+)$ the statements (3.5.33), (3.5.35) shows that

(3.5.36) the subspace generated by the action of $K \times K'$ on Φ is isomorphic to $\pi_{m,-m} \check{\otimes}\, \pi'_{m,-m}$.

Assume now that $n = 2m$, $m \in \mathbf{Z}$. For $\varepsilon = \pm 1$ define

$$(3.5.37)\qquad \Phi_\varepsilon = \sum_{k=0}^{n+1} a_k f_k, \text{ where } a_1 = a_n = \varepsilon \quad \text{and}$$

$$a_{k+1} = (k+\varepsilon-2-n)(k+1)^{-1}a_{k-1} \qquad (k = 1,2,3..,n).$$

Then by (3.5.34),

$$(3.5.38)\qquad \omega(h'_+)\Phi_\varepsilon = \omega(h_-)\Phi_\varepsilon = \varepsilon\, \Phi_\varepsilon \qquad\qquad \varepsilon = \pm 1.$$

Thus

(3.5.39) the subspace generated by the action of $K \times K'$ on Φ_ε is isomorphic to $\pi_{m+1,-m} \otimes \check{\pi}'_{m+1,-m}$ for $\varepsilon = 1$ and to $\pi_{m,-m-1} \otimes \check{\pi}'_{m,-m-1}$ for $\varepsilon = -1$.

By (2.1.23) and (2.1.24) the subspaces (3.5.36) and (3.5.39) occur in lowest degree i.e. are of the form $H_{\pi,\pi'}$, (1.5.24). Combining (3.5.29), (3.5.36), (3.5.39), (2.4.19), (2.5.22) we conclude that the operator

$$\mathrm{Ind}\ v : S \to \mathrm{Ind}_{P_1 \times \theta P_1}^{O_{2,2} \times Sp(2,\mathbf{R})} ((\delta\otimes\nu)\check{\otimes}(\delta^c\otimes\nu)),\ (3.5.29),$$

satisfies the assumptions of the lemma (2.2.23). By the choice of μ (3.5.1), (2.2.27) is valid, therefore the lemma (2.2.23) implies the theorem.

Q.E.D.

§6. COMPLETENESS OF THE LIST AND UNITARITY.

(3.6.1) Theorem. The representations listed in (3.2.4), (3.3.1), (3.4.31), (3.5.19) exhaust the set $R(O_{2,2} \cdot Sp(2,\mathbf{R}),\omega)$, (3.2.3).

Proof: Let ω_1 denote the representation of $O_{2,2}$ denoted by ω in (3.1.31). By inspection of the list of the admissible irreducible representations of $O_{2,2}$ (2.5.4), (2.5.23), (2.5.32) and the Theorems (3.1.26), (3.1.31) we check that the only representations of $O_{2,2}$ which do not occur in (3.2.4), (3.3.1), (3.4.31), (3.5.19) are of the form

(3.6.2) $\det \otimes \Pi$, where $\Pi \in R(O_{2,2},\omega)$ and Π is not equivalent to $\det \otimes \Pi$, and $\Pi \in R(O_{2,2}, \omega_1)$

Assume that $\det \otimes \Pi \in R(O_{2,2},\omega)$ - as above. Since every irreducible admissible representation of $O_{2,2}$ is self-contragradient we see that

(3.6.3) the trivial representation of $O_{2,2}$ is a quotient of $\Pi \otimes \Pi$ ($= \Pi \otimes \Pi^c$).

Therefore our assumption implies that

(3.6.4) $\det \in R(O_{2,2},\omega \otimes \omega_1)$.

This contradicts the theorem (C.7).

Q.E.D.

In order to clarify our description of the set $R(O_{2,2} \cdot Sp(2,\mathbb{R}), \omega)$ (3.6.1) we shall emphasize some of its properties.

(3.6.5) <u>Theorem</u>. Under the identification (2.3.3) of the sets of character data for $O_{2,2}$ and $Sp(2,\mathbb{R})$ the Oscillator Duality Correspondence induces the identity map on these sets - except the case when a discrete series representation of $O_{2,2}$ corresponds to a tempered, but not discrete series, representation of $Sp(2,\mathbb{R})$ (3.2.4).

(3.6.6) <u>Theorem</u>. There are some discrete series representations $\check{\Pi} \otimes \Pi' \in R(O_{2,2} \cdot Sp(2,\mathbb{R}),\omega)$ which occur in the Hilbert space of ω and some which don't.

This follows from the proof of (3.3.1). The point is that $f_{1,n}$ (3.3.26) does not belong to the Hilbert space of ω.

(3.6.7) <u>Theorem</u>. The representations $\Pi \in R(O_{2,2})$ which do not occur in $R(O_{2,2},\omega)$ have the property that

(3.6.8) Π is not equivalent to $\det \otimes \Pi$, and

(3.6.9) Π occurs in the Oscillator Duality Correspondence for the pair $O_{2,2}$, $Sp(1,\mathbb{R})$.

Moreover for any $\Pi \in R(O_{2,2})$

(3.6.10) either Π or $\det \otimes \Pi$ occurs in $R(O_{2,2},\omega)$.

Combining our description of $R(O_{2,2} \cdot Sp(2,\mathbb{R})\omega)$ (3.2.4), (3.3.1), (3.4.31), (3.5.19) with the classification theorems (2.4.14), (2.4.25), (2.5.17), (2.5.28) for the unitary duals of $O_{2,2}$ and $Sp(2,\mathbb{R})$ we obtain the following

(3.6.11) Theorem. The Oscillator Duality Correspondence maps $R(O_{2,2},\omega) \cap \hat{O}_{2,2}$ into $R(Sp(2,\mathbf{R}),\omega) \cap \hat{Sp}(2,\mathbf{R})$.

The converse is not true since the trivial representation of $Sp(2,\mathbf{R})$ corresponds to a non-unitary representation of $O_{2,2}$ - see (2.4.6), (3.4.31), (2.5.18).

Imitating the calculations of §4 one can check that the theorem (3.6.11) holds with $O_{2,2}$ replaced by $O(1,3)$.

APPENDIX A. THE UNITARY DEAL OF Sp(2,$\mathbf{R}$).

The results we prove here are known to experts for years. Since the facts we need about representations induced from maximal parabolics are available in the literature [K-B and B-K] we treat them marginally in §4. The only relevant computations are contained in §3 where we classify the unitarizible Langlands quotients (A.3.1) corresponding to the minimal parabolic subgroup of Sp(2,$\mathbf{R}$).

§1. THE LOWEST K'-types.

In this paragraph we prove the propositions (2.4.3), (2.4.19), (2.4.29).

Proof of (2.4.3). By the Frobenius reciprocity theorem for compact groups the condition $\pi'_{p,q} \in \underline{\underline{A}}[H_0,\Gamma,\gamma]$ is equivalent to the following requirement

(A.1.1) $\|\pi'_{p,q}\|$ (2.1.13) is minimal with respect to the property:

(A.1.2) $\pi'_{p,q}$ when restricted to the subgroup

$$\{\mathrm{diag}(\varepsilon_1,\varepsilon_2) \mid \varepsilon_1,\varepsilon_2 = \pm 1\} \subseteq U(2)$$

contains the representation

$$\mathrm{diag}(\varepsilon_1,\varepsilon_2) \to \varepsilon_1^{\sigma_1}\varepsilon_2^{\sigma_2}.$$

The weights of $\pi'_{p,q}$, on the diagonal Cartan subgroup of U(2), are

(A.1.3) $$\mathrm{diag}(u_1,u_2) \to u_1^{p-k}u_2^{q+k} \qquad (k=0,1,2,.., p-q).$$

Thus (A.1.2) means that

(A.1.4) $$p-k \in \sigma_1+2\mathbf{Z} \text{ and } q+k \in \sigma_2+2\mathbf{Z} \text{ for a } k=0,1,2,..,p-q.$$

Combining this with the formula (2.1.14) for the norm of $\pi'_{p,q}$ we get this result.

Q.E.D.

Proof of (2.4.19). Again by the Frobenius reciprocity theorem it follows from the known structure of the discrete series representations of $SL^{\pm}(2,\mathbf{R})$

[V1, Ch I §4] that $\pi'_{p,q} \in \underline{\underline{A}}[H_1,\Gamma,\gamma]$ iff $\|\pi'_{p,q}\|$ is minimal with respect to the property

(A.1.5) $\pi'_{p,q}$ when restricted to $SO(2) \subseteq U(2)$ contains a character χ_r (2.1.6) with $r \geq n+1$ and $r-n-1 \in 2\mathbf{Z}$.

The condition that $\pi'_{p,q}|_{SO(2)}$ contains χ_r implies $p-q \geq r$. Therefore (A.1.5) and (2.1.14) imply

(A.1.6) $$2\|\pi'_{p,q}\|^2 = (p+q)^2 + (p-q+2)^2 \geq (n+3)^2.$$

If $n+1 = 2m$ is even, then $\pi'_{m,-m}$ satisfies (A.1.5) and gives equality in (A.1.6).

If $n = 2m$, then $\pi'_{m+1,-m}$ and $\pi'_{m,-m-1}$ satisfy (A.1.5) and have minimal norm (A.1.6).

Q.E.D.

Proof of (2.4.29). Consider the case $n > 0$. Then $\pi'_{p,q} \in \underline{\underline{A}}[H_2,\Gamma,\gamma]$ if and only if $\|\pi'_{p,q}\|$ is minimal with respect to the property

(A.1.7) $\pi'_{p,q}$ when restricted to the subgroup

$$T_2 = \{\mathrm{diag}(\varepsilon,u) \mid \varepsilon = \pm 1,\ u \in \mathbf{C},\ |u| = 1\} \subseteq U(2)$$

contains a representation

$$\mathrm{diag}(\varepsilon,u) \to \varepsilon^{\sigma} u^{r} \qquad r \geq n+1,\ r-n-1 \in 2\mathbf{Z}.$$

Since we know the weights of $\pi'_{p,q}$ (A.1.3), (A.1.7) translates to

(A.1.8) there is $k=0,1,2,..,p-q$ such that $q+k \in \sigma+2\mathbf{Z}$ and $p-k \geq n+1$, $p-k-n-1 \in 2\mathbf{Z}$.

This implies that

(A.1.9) $$(p+1)^2 + (q-1)^2 \geq (n+2)^2 + (\sigma-1)^2.$$

Thus the formula (2.1.14) completes the proof for $n > 0$. The case $n < 0$ is entirely analogous.

Q.E.D.

§2. DECOMPOSITION OF THE PRINCIPAL SERIES.

The goal of this paragraph is to prove the theorem (2.4.4). We begin with the computation of the reducibility groups [V1, 4.3.13, 4.4.9].

Let us choose an orthonormal basis of $\underline{a}_0^*$ - the dual of the complexification of the Lie algebra of A_0 (2.2.4):

(A.2.1) $$e_j(\mathrm{diag}(a_1, a_2 - a_1, -a_2)) = a_j \qquad (j=1,2).$$

Then the set of positive roots with respect to the minimal parabolic subgroup P_0' (2.2.10) is

(A.2.2) $$\Delta^+ = \{e_1 - e_2, \; 2e_1, e_1 + e_2, \; 2e_2\}.$$

Denote by

(A.2.3) $w_\alpha \in W(H_0)$ (2.3.4) the reflection with respect to the root $\alpha \in \Delta^+$.

We shall parametrize the dual $\hat{M}_0$ of M_0 (2.2.3) by the pairs $\delta = (\delta_1, \delta_2)$ of numbers $\delta_1, \delta_2 = 0,1$:

(A.2.4) $$\delta(\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2)) = \varepsilon_1^{\delta_1} \varepsilon_2^{\delta_2}$$

Then for the simple reflections

(A.2.5)
$$w_{e_1-e_2}(\delta_1, \delta_2) = (\delta_2, \delta_1), \; w_{2e_2}\delta = \delta,$$
$$w_{e_1-e_2}(\nu_1 e_1 + \nu_2 e_2) = \nu_2 e_1 + \nu_1 e_2,$$
$$w_{2e_2}(\nu_1 e_1 + \nu_2 e_2) = \nu_1 e_1 - \nu_2 e_2.$$

Since Γ (2.4.2) is of the form $\delta \otimes e^{\check{\nu}}$, the statement (A.2.5) implies that (2.4.2) describes a fundamental domain for the action of $W(H_0)$ on the set of regular characters (2.3.11) on H_0. Following [V1, 4.3.6] for each $\alpha \in \Delta^+$ we fix an injection ϕ_α of the Lie algebra $\underline{sl}(2,\mathbb{R})$ into $\underline{sp}(2,\mathbb{R})$:

$$\phi_{e_1-e_2}(x) = \begin{bmatrix} x & 0 \\ 0 & -x^t \end{bmatrix} \tag{A.2.6}$$

$$\phi_{2e_1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.2.7}$$

$$\phi_{e_1+e_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & c & -d & 0 \\ c & 0 & 0 & -d \end{bmatrix} \tag{A.2.8}$$

$$\phi_{2e_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & d \end{bmatrix} \tag{A.2.9}$$

Let $m_\alpha = \exp(\phi_\alpha(\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}))$ for $\alpha \in \Delta^+$ and $\pi = 3.14\ldots$. Then

$$\begin{aligned} m_{e_1-e_2} &= \mathrm{diag}(-1,-1,-1,-1), \quad m_{2e_1} = \mathrm{diag}(-1,1,-1,1) \\ m_{e_1+e_2} &= m_{e_1-e_2}, \qquad m_{2e_2} = \mathrm{diag}(1,-1,1,-1). \end{aligned} \tag{A.2.10}$$

Having the m_α's we can identify the set of good roots [V1, 4.3.11]

$$\bar{\Delta}_\delta = \{\alpha \in \Delta^+ \mid \delta(m_\alpha) = 1\} \tag{A.2.11}$$

and the subgroup $W_\delta^0 \subseteq W(H_0)$ generated by w_α, $\alpha \in \bar{\Delta}_\delta$. Explicitly

$$W_\delta^0 = \begin{cases} W(H_0) & \text{for } \delta = (0,0) \\ \{1, w_{2e_2}\} & \text{for } \delta = (1,0) \\ \{1, w_{2e_1}\} & \text{for } \delta = (0,1) \\ \{1, w_{e_1-e_2}, w_{e_1+e_2}, w_{e_1-e_1}\, w_{e_1+e_2}\} & \text{for } \delta = (1,1) \end{cases} \tag{A.2.12}$$

The stabilizer of δ in $W(H_0)$

(A.2.13) $$W_\delta = \begin{cases} \{W(H_0)\} & \text{for } \delta_1 = \delta_2 \\ \{1, w_{2e_1}, w_{2e_2}, w_{2e_1} w_{2e_2}\} & \text{for } \delta_1 \neq \delta_2. \end{cases}$$

Let $\nu = \nu_1 e_1 + \nu_2 e_2$ belong to the closed positive Weyl chamber:

(A.2.14) $\mathrm{Re}(\nu_1 - \nu_2) \geq 0$, $\mathrm{Re}\, \nu_2 \geq 0$.

The stabilizer of ν in $W(H_0)$

(A.2.15) $W(\nu)$ is generated by w_α, $\alpha \in \Delta^+$, α orthogonal to ν.

Put

(A.2.16) $W^0_\delta(\nu) = W^0_\delta \cap W(\nu)$, $W_\delta(\nu) = W_\delta \cap W(\nu)$

then [V1, 4.4.9] the reducibility group

(A.2.17) $R_\delta(\nu) = W_\delta(\nu)/W^0_\delta(\nu)$.

(A.2.18) <u>Lemma</u>. Under the assumption (A.2.14) the group $R_\delta(\nu)$ has exactly two elements if either of the following conditions is satisfied

(A.2.19) $\nu = 0$ and $\delta \neq (0,0)$

(A.2.20) $\nu_1 \neq 0$, $\nu_2 = 0$ and $\delta_2 = 1$

(A.2.21) $\nu_1 = 0$, $\nu_2 \neq 0$ and $\delta_1 = 1$

Otherwise $R_\delta(\nu)$ is trivial.

(A.2.22) <u>Remark</u>. Since clearly $R_{w\delta}(w\nu) = R_\delta(\nu)$ for $w = w_{e_1 - e_2}$ the conditions (A.2.20) and (A.2.21) are equivalent.

Proof of the lemma: Clearly

(A.2.23) $R_\delta(\nu) \cong W(\nu) \cong \{1\}$ if $\nu_1 \neq \nu_2$, $\nu_1 \neq 0$, $\nu_2 \neq 0$,

and by (A.2.12), (A.2.13), (A.2.16)

(A.2.24) $R_\delta(\nu) \cong \{1\}$ for all ν (A.2.14) if $\delta = (0,0)$.

Moreover (A.2.12) and (A.2.13) imply that

(A.2.25) $\qquad R_\delta(0) \simeq \mathbf{Z}/2\mathbf{Z} \quad \text{if} \quad \delta \neq (0,0).$

In the cases complementary to (A.2.23) and to (A.2.25) we have (modulo the Remark (A.2.22))

(A.2.26)
$$W(\nu) = \begin{cases} \{1, w_{e_1 - e_2}\} & \text{if } \nu_1 = \nu_2 \neq 0 \\ \\ \{1, w_{2e_2}\} & \text{if } \nu_1 \neq 0,\ \nu_2 = 0 \end{cases}$$

Combining (A.2.12), (A.2.13), and (A.2.26) we see that

(A.2.27)
$$R_\delta(\nu) \simeq \begin{cases} \{1\} & \text{for all } \delta & \text{if } \nu_1 = \nu_2 \neq 0 \\ \mathbf{Z}/2\mathbf{Z} & \text{for } \delta_2 = 1 & \text{if } \nu_1 \neq 0,\ \nu_2 = 0 \\ \{1\} & \text{for } \delta = (1,0) & \text{if } \nu_1 \neq 0,\ \nu_2 = 0 \end{cases}$$

Chasing through (A.2.23), (A.2.24), (A.2.25) and (A.2.27) we obtain the lemma.

Q.E.D.

<u>Proof of the Theorem (2.4.4):</u> Consider the induced representation

(A.2.28) $\qquad \mathrm{Ind}_{P_0'}^{Sp(2,\mathbf{R})} (\delta \otimes \nu)$

where $\delta \in \hat{M}_0$ and ν satisfies (A.2.14). (2.4.4) is a theorem about the irreducible subquotients of (A.2.28) containing the lowest K'-types. It follows from [V1 4.3.15 c) and 4.4.10] that

(A.2.29) the number of irreducible subquotients of (A.2.28) containing a lowest K'-type from $\underline{\underline{A}}[H_0, \Gamma, \gamma]$ is equal to the cardinality of $R_\delta(\nu)$.

This and the lemma (A.2.18) imply (2.4.4) via a case by case verification.

Q.E.D.

§3. THE UNITARY REPRESENTATIONS ATTACHED TO H_0.

Here we prove the theorem (2.4.14), which describes the unitarizable quotients of the induced representation (A.2.28) containing K'-types from $\underline{\underline{A}}[H_0, \Gamma, \gamma]$ (2.4.3).

We begin by quoting from [K-Sp, §1] Langlands classification of the irreducible admissible representations of a connected semisimple Lie group G having a faithful matrix representation. Fix a minimal parabolic subgroup P_0 in G.

(A.3.1) Theorem [K-Sp, §1]. The (equivalence classes of) irreducible admissible representations of G stand in one to one correspondence with all triples (P,δ,ν) where

(A.3.2) $P = MAN$ is a parabolic subgroup of G containing the fixed minimal one P_0.

(A.3.3) δ is an irreducible tempered unitary representation (2.2.15) of M

(A.3.4) ν is a complex valued linear functional on the Lie algebra $\underline{a}$ of A with $\mathrm{Re}\,\nu$ in the open positive Weyl chamber.

The Langlands representation $J(P,\delta,\nu)$ is the unique irreducible quotient of the induced representation

(A.3.5) $$\mathrm{Ind}_P^G(\delta\otimes\nu)$$

and is given as the image of an explicit intertwining operator

(A.3.6) $$A(\theta P,P,\delta,\nu) : \mathrm{Ind}_P^G(\delta\otimes\nu) \to \mathrm{Ind}_{\theta P}^G(\delta\otimes\nu)$$

Here θ denotes a fixed Cartan involution on G. Let K be the corresponding maximal compact subgroup of G.

Right after this comes the

(A.3.7) Unitarizability Criterion [K-Z1, K-Sp §1]. $J(P,\delta,\nu)$ is infinitesimally unitary if and only if

(A.3.8) there exists a w in K normalizing A with
$wPw^{-1} = \theta P$, $w\delta \cong \delta$, $w\nu = -\bar{\nu}$; and

(A.3.9) the hermitian intertwining operator

$$B(P,w,\delta,\nu) = \delta(w)R(w)A(\theta P,P,\delta,\nu)$$

is positive or negative semidefinite.

Here $R(w)$ denotes the right translation of functions by w, and $\delta(w)$ is an extension of δ to the smallest subgroup of G containing M and w. Such an extension exists - see [K-S, Lemma 7.9]. Moreover the operator $B(P,w,\delta,\nu)$ is independent of a choice of a representative w in the normalizer of A in K [K, pp. 546].

(A.3.10) Example. Let $G = Sp(2,\mathbf{R})$. The representations listed in (2.4.4) are either tempered - if $\mathrm{Re}\,\nu_1 = \mathrm{Re}\,\nu_2 = 0$ or of the form $\Pi = J(P,\delta,\nu)$ (A.3.1) with $P \neq G$. Using the lemma (A.2.18) we verify the following interpretation of (2.4.4) in terms of (A.3.1):

(A.3.11) $\mathrm{Re}\,\nu_2 > 0$, $\mathrm{Re}(\nu_1-\nu_2) > 0$. Then $P = P_0'$ is minimal and Π, with $\nu = \nu_1 e_1+\nu_2 e_2$, is one of the representations (2.4.6), (2.4.7), (2.4.8), (2.4.11) depending on $\delta = (\sigma_1,\sigma_2)$, (A.2.4).

(A.3.12) $\mathrm{Re}\,\nu_1 = \mathrm{Re}\,\nu_2 > 0$, $P = P_1'$ (2.2.10) and Π is one of the representations (2.4.6), (2.4.7), (2.4.8), (2.4.11) depending on

$$\delta = \mathrm{Ind}_{M_1\cap P_0}^{M_1}(\sigma\otimes 0), \qquad \sigma = (\sigma_1,\sigma_2) \in \hat{M}_0.$$

Here e_1+e_2 is viewed as an element of $\underline{a}_1^*$ (2.2.7) in the obvious way and δ is irreducible [V1 Ch I §4].

(A.3.13) $\mathrm{Re}\,\nu_1 > 0$, $\mathrm{Re}\,\nu_2 = 0$, $P = P_2'$ (2.2.13). Here the situation is more complex. Let

$$\tau = \mathrm{Ind}_{M_2\cap P_0}^{M_2}(\sigma\otimes\nu_2 e_2)$$

where $\sigma = (\sigma_1,\sigma_2) \in \hat{M}_0$ and e_2 belongs to $\underline{a}_2^*$ in the obvious way. It follows from the well known properties of the principal series of $SL(2,\mathbf{R})$ [V1 Ch I §3] that

τ is irreducible if $\sigma_2 = 0$ or $\nu_2 \neq 0$, and

$\tau = \delta_+ \otimes \delta_-$ is a direct sum of two irreducible tempered representations if $\sigma_2 = 1$ and $\nu_2 = 0$.

Therefore Π with $\sigma_2 = 0$ and $\delta = \tau$ is one of the representations (2.4.6), (2.4.7) depending on σ_1. If $\nu_2 \neq 0$ and $\delta = \tau$ then Π is one of (2.4.6), (2.4.7) (2.4.8), (2.4.11) depending on (σ_1, σ_2). For $\sigma_2 = 1$ and $\nu_2 = 0$. $J(P_2, \delta_+, \nu_1 e_1)$ is one of (2.4.9), (2.4.12), and $J(P_2, \delta_-, \nu_1 e_1)$ one of (2.4.10), (2.4.13) depending on σ_1.

Here the last sentence actually determines δ_+ and δ_-. Also $e_1 \in \underline{\alpha}_1^*$ in the obvious way.

We shall investigate the unitarizability of the Langlands quotients (A.3.10) via a careful analysis of the intertwining operators (A.3.6). Let us begin by recalling some known facts about them from [K-S, K, W2] for a general connected, semisimple Lie group G.

Let P be a minimal parabolic subgroup of G with a Langlands decomposition $P = MAN$. For any element w in the normalizer of A in K and for $\nu \in \underline{a}^*$ with the real part in the open positive (with respect to N) Weyl chamber, and any $\delta \in \hat{M}$ one defines the G-intertwining operator

(A.3.14) $$A(w,\delta,\nu): \ \mathrm{Ind}_P^G(\delta \otimes \nu) \to \mathrm{Ind}_P^G(w\delta \otimes w\nu)$$

by the integral

(A.3.15) $$A(w,\delta,\nu)f(x) = \int_{w^{-1}Nw \cap \theta N} f(xwy)dy.$$

Here dy stands for an appropriately normalized measure on the indicated nilpotent group [K-S Ch I §2] and the above integral is convergent [K-S Prop. 4.1]. Recall that the Weyl group $W(A) = N_K(A)M/M$ is generated by simple reflections w_α ($\alpha \in \Delta(\underline{g},\underline{a})$ simple positive root) and that by the length of an element $w \in W(A)$ one understands the smallest possible number of w_α's such that w is a product of them. An important property the intertwining operators (A.3.14) have is the so-called cocycle relation: for $w_1, w_2 \in W(A)$

(A.3.16) if length $(w_1 w_2)$ = length (w_1) + length (w_2) then

$$A(w_1 w_2, \delta, \nu) = A(w_1, w_2\delta, w_2\nu)\, A(w_2, \delta, \nu),$$

[K-S, Prop 7. 8(iv)].

It is well known that the space of K-finite vectors in the induced representation $\mathrm{Ind}_P^G(\delta\otimes\nu)$, as a representation of K, is isomorphic to the following direct sum

$$(A.3.17) \qquad \bigoplus_{\pi\in\hat{K}} \pi \otimes \mathrm{Hom}_M(\pi,\delta), \quad \text{[K-Sp Prop 5.1, W2 §8.11].}$$

We shall interpret the operators (A.3.14) in the context of (A.3.17). For each element x of $G = KAN$ we have its Iwasawa decomposition:

$$(A.3.18) \qquad x = k(x)a(x)n(x)$$

Put $\rho = \rho(\underline{n})$ (see "Notation"). For $\pi \in \hat{K}$, $w \in N_K(A)$, $\nu \in \underline{a}^*$ with the real part in the positive Weyl chamber define

$$(A.3.19) \qquad A_\pi(w,\nu) = \int_{w^{-1}Nw \cap \theta N} e^{-(\nu+\rho)}(a(x))\, \pi(k(x))^{-1}\, dx\, \pi(w)^{-1}.$$

Then

$$(A.3.20) \qquad A_\pi(x,\nu)\pi(w) \in \mathrm{Hom}_M(\pi,\pi) \quad \text{[K-Sp Prop 5.2],}$$

and $A(w,\delta,\nu)$, (A.3.15), when restricted to the space of K-finite vectors (A.3.17) coincides with the following direct sum

$$(A.3.21) \qquad \bigoplus_{\pi\in\hat{K}} \mathrm{id}_\pi \otimes r(A_\pi(w,\nu)),$$

where id_π is the identity operator on π and $r(A_\pi(w,\nu))$ means the operator of right multiplication by $A_\pi(w,\nu)$ on the space $\mathrm{Hom}_M(\pi,\delta)$ [K-Sp, Prop 5.2]. Therefore the cocycle relation (A.3.16) translates to

$$(A.3.22) \qquad A_\pi(w_1w_2,\nu) = A_\pi(w_2,\nu)A_\pi(w_1,w_2\nu) \quad \text{if}$$

$$w_1,\ w_2 \in W(A) \text{ and length } (w_1w_2) = \text{length } (w_1) + \text{length } (w_2).$$

Assume now that the conditions of the Unitarizability Criteria (A.3.7) are satisfied and consider the operators $B(P,w,\delta,\nu)$ (A.3.9) from the view point of the K-decomposition (A.3.17).

For a $\pi \in \hat{K}$ define an operator on $\mathrm{Hom}_n(\pi,\delta)$:

(A.3.23) $$B_\pi(w,\delta,\nu) = l(\delta(w))r(A_\pi(w,\nu))$$

where $l(\delta(w))$ is the multiplication on the left by $\delta(w)$. Then $B(P,w,\delta,\nu)$, when restricted to the space of K-finite vectors (A.3.7) coincides with the following direct sum

(A.3.24) $$\bigoplus_{\pi\in\hat{K}} \mathrm{id}_\pi \otimes B_\pi(w,\delta,\nu)$$

Therefore the Unitarizability Criteria (A.3.7) can be expressed as follows:

(A.3.25) The Langlands quotient $J(P,\delta,\nu)$ is unitarizable if and only if

(A.3.26) there is a $w \in W(A)$ with $w^{-1}Pw = \theta P$, $w\delta \cong \delta$, $w\nu = -\bar{\nu}$

and

(A.3.27) the hermitian operators $B_\pi(w,\delta,\nu)$, $\pi \otimes \hat{K}$,

are all positive semidefinite, or all are negative semidefinite.

For any $w \in W(A)$ any $\pi \in \hat{K}$ and any ν in the positive Weyl chamber define

(A.3.28) $$A^\pi(w,\nu) = \int_{w^{-1}Nw \cap \theta N} \exp(-(\nu+\rho))(a(x))\, \pi^c(k(x))dx$$

where $a(x)$, $k(x)$ are defined in (A.3.18) and π^c stands for the contragradient representation to π. Assume that $\dim \delta=1$. Then

(A.3.29) $$r(A_\pi(w,\nu)) = \pi^c(w)\, A^\pi(w,\nu)$$

and if w,δ,ν satisfy (A.3.26) then $\delta(w) \in \mathbf{C}$ and

(A.3.30) $$B_\pi(w,\delta,\nu) = \delta(w)\pi^c(w)A^\pi(w,\nu).$$

(A.3.31) <u>Example</u>. First we compute the determinants of the operators $B_\pi(w,\delta,\nu)$ for $G = Sp(2,\mathbf{R})$. Put

(A.3.32) $\quad w_\theta = w_{e_1-e_2} w_{2e_2} w_{e_1-e_2} w_{2e_2} \quad$ (A.2.5).

Then for the minimal parabolic $P = P_0'$ (2.2.10)

(A.3.33) $\quad w_\theta^{-1} P w_\theta = \theta P,\ w_\theta \delta = \delta$ for $\delta \in \hat{M}$, and $w_\theta \nu = -\nu$ for $\nu \in \underline{\mathbf{a}}^*$.

To satisfy the symmetry condition (A.3.26) of the Unitarizability Criteria we shall assume that

(A.3.34) $\quad \nu = \nu_1 e_1 + \nu_2 e_2;\ \nu_1, \nu_2 \in \mathbb{R};\ \nu_1, \nu_2 > 0;\ \nu_1 - \nu_2 > 0.$

The maximal compact subgroup (2.1.2) of G is isomorphic to K = U(2) via the maps (1.4.18) and (1.4.15). Explicitly

(A.3.35) $\quad O(4) \cap Sp(2,\mathbb{R}) \ni \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \leftrightarrow x + iy \in K.$

In particular

(A.3.36) $\quad w_\theta$ can be represented by $\operatorname{diag}(-i,-i) \in K$, $(i = \sqrt{-1})$

and therefore for $\delta = (\delta_1, \delta_2) \in \hat{M}$ (A.2.4) we have

(A.3.37) $\quad \delta(w_\theta) = 1$ if $\delta_1 = \delta_2$, $\delta(w_\theta) = i$ if $\delta_1 \neq \delta_2$

(because $\delta(w_\theta)^2 = \delta(w_\theta^2) = \delta(-\text{identity})$).

For an irreducible unitary representation $\pi = \pi'_{m,n}$ (2.1.9) of K chose a basis of weight vectors v_j of $\pi^c = \pi'_{-n,-m}$ with respect to the diagonal Cartan subgroup of K = U(2) such that

(A.3.38) $\quad \pi^c(\operatorname{diag}(u_1,u_2))v_j = u_1^{-j-n} u_2^{j-m} v_j \quad (j = 0,1,2,.,m-n).$

(A.3.39) <u>Lemma.</u> The δ-isotypic component of π^c is equal to

(A.3.40) $\quad \sum_{j-m-\delta_2 \in 2\mathbb{Z}} \mathbb{C} v_j \quad$ for $m-n-\delta_1-\delta_2 \in 2\mathbb{Z}$

and to zero otherwise.

Proof: Putting $u_1 = u_2 = -1$ in (A.3.38) we see that the condition (A.3.40) is necessary for the space in question to be non-zero. Taking $u_1 = 1$ and $u_2 = -1$ in (A.3.38) we check that (A.3.40) is correct

Q.E.D.

Let for $x \in \mathbb{R}$ and $n = 1,2,3,4\ldots$,

(A.3.41)
$$\phi(x,0) = \Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{x}{2}\right)\Gamma\left(\tfrac{x+1}{2}\right)^{-1},$$
$$\phi(x,1) = \Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{x+1}{2}\right)\Gamma\left(\tfrac{x+2}{2}\right)^{-1},$$
$$\phi(x,2n) = \phi(x,0)\prod_{j=1}^{n}(x-(2j-1))(x+(2j-1))^{-1},$$
$$\phi(x,2n+1) = \phi(x,1)\prod_{j=1}^{n}(x-2j)(x+2j)^{-1}.$$

For $m \in \mathbf{Z}$ put $\phi(x,m) = \phi(x,|m|)$.

(A.3.42) <u>Lemma.</u> Let $\pi = \pi'_{m,n}$, ν be as in (A.2.14), and $\delta = (\delta_1,\delta_2) \in \hat{M}$. Then, with $\alpha = e_1 - e_2$, the determinant of the operator

(A.3.43) $A^{\pi}(w_\alpha,\nu)$ restricted to the δ-isotypic component of π^c

is

(A.3.44)
$$\prod_{j=0}^{d}\phi(\nu_1-\nu_2,\ m-n-2j)$$

where d is an integer equal to

(A.3.45) $(m-n-1)/2$ for $\delta_1 \neq \delta_2$,

(A.3.46) $(m-n-2)/2$ for $\delta_1 = \delta_2$ and $m-\delta_2 \in 2\mathbf{Z}+1$

(A.3.47) $(m-n)/2$ for $\delta_1 = \delta_2$ and $m-\delta_2 \in 2\mathbf{Z}$.

If $\alpha = 2e_2$ and $m-n-\delta_1-\delta_2 \in 2\mathbf{Z}$, then the determinant of the operator (A.3.43) is

(A.3.48) $\prod_j \phi(\nu_2,m-j)$, where the product is over the integers

$0 \leqslant j \leqslant m-n$ satisfying $j-m-\delta_2 \in 2\mathbf{Z}$.

Proof: We begin with the formulas for the Iwasawa decomposition (A.3.18) of elements of the group $w_\alpha^{-1} N w_\alpha \cap \theta N$: (see [V1, 1.3.6])

(A.3.49)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{k} b^{-1}\begin{bmatrix} 1 & -x \\ x & 1 \end{bmatrix}$$
$$\downarrow a$$
$$\operatorname{diag}(b,b,b^{-1},b^{-1}), \quad b = (1+x^2)^{\frac{1}{2}} \quad \text{for } \alpha = e_1 - e_2;$$

A.3.50)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & x & 0 & 1 \end{bmatrix} \xrightarrow{k} \begin{bmatrix} 1 & 0 \\ 0 & (1-ix)b^{-1} \end{bmatrix}$$
$$\downarrow a$$
$$\operatorname{diag}(1,b,1,b^{-1}), \quad b = (1+x^2)^{1/2} \text{ for } \alpha = 2e \,.$$

By a straightforward computation one checks that for Re $y > 0$,

(A.3.51)
$$\phi(y,n) = \int_{\mathbf{R}} (1+x^2)^{-\frac{1}{2}(1+y)} ((1-ix)(1+x^2)^{-\frac{1}{2}})^n \, dx,$$

where ϕ was defined in (A.3.41) [K-Sp (5.8)].
Since $\rho = \rho(\underline{n})$ is equal to $2e_1 + e_2$ (A.2.1) it follows from the definition (A.3.28) and from the formulas (A.3.50) that the operator

(A.3.52) $A^\pi(w_{2e_2},\nu)$ acts on v_j (A.3.38) via the multiplication by $\phi(\nu_2, j-m)$.

This statement together with the lemma (A.3.39) imply (A.3.48).

Similarly, since

(A.3.53)
$$s^{-1}\begin{bmatrix} 1 & -x \\ x & 1 \end{bmatrix} s = \begin{bmatrix} 1-ix & 0 \\ 0 & 1+ix \end{bmatrix} \quad \text{(for } s \text{ see (3.1.1))},$$

the definition (A.3.28) and the formula (A.3.49) imply that

(A.3.54) $A^\pi(w_{e_1-e_2},\nu)$ acts on $\pi^c(c)v_j$ (A.3.38) via the multiplication by $\phi(\nu_1-\nu_2, m-n-2j)$.

Combining (A.3.54), with the lemma (A.3.39) we get (A.3.44).

Q.E.D.

(A.3.55) Lemma. Under the assumptions of (A.3.42), the determinant $D(\pi,\delta,\nu)$ of the operator

(A.3.56) $A^{\pi}(w_\theta,\nu)$ restricted to the δ-isotypic component of π^c is equal to

$$\prod_{j=0}^{d}(\phi(\nu_1-\nu_2,m-n-2j)\phi(\nu_1+\nu_2,m-n-2j))\times \tag{A.3.57}$$

$$\times \prod_{\substack{j=0\\ j-m-\delta_1\in 2\mathbf{Z}}}^{m-n}\phi(\nu_1,j-m)\times\prod_{\substack{j=0\\ j-m-\delta_2\in 2\mathbf{Z}}}^{m-n}\phi(\nu_2,j-m)$$

for $\delta_1\neq\delta_2$, $d=(m-n-1)/2\in\mathbf{Z}$, and to

$$\prod_{j=0}^{d}(\phi(\nu_1-\nu_2,m-n-2j)\phi(\nu_1+\nu_2,m-n-2j))\times \tag{A.3.58}$$

$$\times\prod_{\substack{j=0\\ j-m-\delta_2\in 2\mathbf{Z}}}^{m-n}(\phi(\nu_1,j-m)(\phi(\nu_2,j-m))$$

for $\delta_1=\delta_2$, $d=(m-n-2)/2\in\mathbf{Z}$ if $m-\delta_2\in 2\mathbf{Z}+1$ and $d=(m-n)/2\in\mathbf{Z}$ if $m-\delta_2\in 2\mathbf{Z}$.

Proof: Put $\alpha=e_1-e_2$ and $\beta=2e_2$. Then (A.3.32) with the cocycle relation (A.3.22) and (A.3.29) imply that

$$A^{\pi}(w_\theta,\nu)=\pi^c(w_\beta w_\alpha w_\beta)^{-1}A^{\pi}(w_\alpha,w_\beta w_\alpha w_\beta\nu)\,\pi^c(w_\beta w_\alpha w_\beta)\times \tag{A.3.59}$$

$$\times\,\pi^c(w_\alpha w_\beta)^{-1}\,A^{\pi}(w_\beta,w_\alpha w_\beta\nu)\,\pi^c(w_\alpha w_\beta)\times$$

$$\times\,\pi^c(w_\beta)^{-1}\,A^{\pi}(w_\alpha,w_\beta\nu)\,\pi^c(w_\beta)\times$$

$$\times\,A^{\pi}(w_\beta,\nu).$$

Let π^c_δ be the δ-isotypic component of π^c. Since w_β acts trivially on $\hat{M}$, (A.3.59) implies that

(A.3.60) $$\det(A^\pi(w_\sigma,\nu)|\pi^c_\delta) = \det(A^\pi(w_\alpha,w_\beta w_\alpha w_\beta\nu)|\pi^c_\delta) \times$$
$$\times \det(A^\pi(w_\beta,w_\alpha w_\beta\nu)|\pi^c_{\delta'}) \times \det(A^\pi(w_\alpha,w_\beta\nu)|\pi^c_\delta) \times$$
$$\times \det(A^\pi(w_\beta,\nu)|\pi^c_\delta)$$

where $\delta' = \delta$ for $\delta_1 = \delta_2$ and $\delta' = w_\alpha\delta$ for $\delta_1 \neq \delta_2$.

This combined with the lemma (A.3.39) completes the proof.

Q.E.D.

(A.3.61) Proposition. The determinant $B(\pi,\delta,\nu)$ of the operator $B_\pi(w_\theta,\delta,\nu)$ (A.3.30) restricted to the δ-isotypic component of π^c is equal to (see (A.3.55) for $D(\pi,\delta,\nu)$)

(A.3.62) $$(-1)^{m(d+1)} D(\pi,\delta,\nu) \quad \text{for } \delta_1 \neq \delta_2,\ d = (m-n-1)/2 \in \mathbf{Z};$$

(A.3.63) $$(-1)^{\delta_1 e} D(\pi,\delta,\nu) \qquad \text{for } \delta_1 = \delta_2,\ e = (m+n)/2,\ (m-n)/2 \in \mathbf{Z}.$$

Proof: Clearly by (A.3.37), (A.3.32) and (A.3.39)

$$\det(\delta(w_\theta)\pi^c(w_\theta)|_{\pi^c_\delta}) = (\delta(w_\theta)(-1)^m(-i)^{m-n})^k$$

where $k = \dim \pi^c_\delta$. This and the lemma (A.3.39) completes the proof.

Q.E.D.

(A.3.64) Theorem. The Langlands quotient $J(P,\delta,\nu)$ (A.3.11) is unitarizable if and only if either

(A.3.65) $$\nu_1,\nu_2 \in \mathbf{R};\ \nu_1 + \nu_2 \leq 1, \text{ and } \sigma_1 = \sigma_2, \text{ or}$$

(A.3.66) $$\nu_1 = 2,\ \nu_2 = 1,\ \sigma_1 = \sigma_2 = 0.$$

In the last case this quotient is a trivial representation.

Proof: We work in the open Weyl chamber (A.3.11). The symmetry criteria (A.3.26) requires that

(A.3.67) $\nu_1, \nu_2 \in \mathbf{R}$

The fact that under the condition (A.3.66) $J(P,\delta,\nu)$ is trivial is well known (recall that $\rho = 2e_1+e_2$) and follows, for example, from [K-S, (3.5)]. A generalization [B-W, Theorem 5.2] of the Howe-Moore theorem [H-M] on vanishing at infinity of the matrix coefficients of a non-trivial, irreducible, unitary representation of a real, simple, algebraic group imply that we may restrict our attention to ν satisfying (A.3.11), (A.3.67) and

(A.3.68) $\nu_1+\nu_2 < 3$ and $\nu_1 < 2$.

It follows from the lemma (A.3.55) that in this region the representation

(A.3.69) $\mathrm{Ind}_P^G(\delta\otimes\nu)$

is reducible only on the following line segments:

(A.3.70) $\nu_1+\nu_2 = 1$ or $\nu_1 = 1$ or $\nu_2 = 1$ or $\nu_1-\nu_2 = 1$ for $\delta = (0,0)$,

(A.3.71) $\nu_1+\nu_2 = 1$ or $\nu_1-\nu_2 = 1$ for $\delta = (1,1)$,

(A.3.72) $\nu_1+\nu_2 = 2$ or $\nu_2 = 1$ for $\delta = (1,0)$,

(A.3.73) $\nu_1+\nu_2 = 2$ or $\nu_1 = 1$ for $\delta = (0,1)$.

The Proposition (A.3.61) and the fact that the function $\Gamma(x)$ (A.3.41) is positive for $x > 0$ imply that the determinant $B(\pi,\delta,\nu)$ (A.3.61) is a positive multiple of

(A.3.74) 1 for $\pi = \pi'_{0,0}$, $\delta = (0,0)$

(A.3.75) $(\nu_1-\nu_2-1)(\nu_1+\nu_2-1)$ for $\pi = \pi'_{1,-1}$, $\delta = (0,0)$

(A.3.76) 1 for $\pi = \pi'_{0,-1}$, $\delta_1 \neq \delta_2$

(A.3.77) -1 for $\pi = \pi'_{1,0}$, $\delta_1 \neq \delta_2$

(A.3.78) $\quad -1 \qquad$ for $\pi = \pi'_{-1,-1}$, $\delta = (1,1)$

(A.3.79) $\quad -(\nu_1-\nu_2-1)(\nu_1+\nu_2-1) \qquad$ for $\pi = \pi'_{0,-2}$, $\delta = (1,1)$

From (A.3.39) we deduce that all the K-types π in (A.3.74),..., (A.3.79) have the property that $\dim(\pi^c_\delta) = 1$. Therefore the Unitarizability Criteria and (A.3.76), (A.3.77) imply that

(A.3.80) $\quad J(P,\delta,\nu)$ is not unitarizable for $\delta_1 \neq \delta_2$.

Similarly (A.3.74), (A.3.75), (A.3.78), (A.3.79) combined with (A.3.68) and the continuity argument [K-Sp §4] imply that

(A.3.81) $\quad$ if $J(P,\delta,\nu)$, with $\delta_1 = \delta_2$, is unitarizable then $\nu_1+\nu \leqslant 1$.

Let $0 < \mu < 1/2$. One can check (easily) using [Vl 4.2.25] and the properties of the principal series of $SL(2,\mathbb{R})$ that

(A.3.82) $\quad \mathrm{Ind}_P^G(\delta\otimes\mu(e_1+e_2))$ is irreducible.

Harish-Chandra's theorem [Vl, 4.1.20] implies that

(A.3.83) $\quad \mathrm{Ind}_P^G\,(\delta \otimes \mu(e_1-e_2))$ is irreducible for all δ.

By double induction [Vl, 4.1.17] (A.3.83) coincides with

(A.3.84) $\quad \mathrm{Ind}_{P_1'}^G(\tau \otimes \nu)$, where $\nu = 0$ and

(A.3.85) $\quad \tau = \mathrm{Ind}_{M_1\cap P}^{M_1}(\delta \otimes \mu(e_1-e_2))$.

Since $M_1 \cong SL^{\pm}(2,\mathbb{R})$, τ is (irreducible and) unitarizable iff $\delta_1 = \delta_2$. Therefore (A.3.84) is unitarizable iff $\delta_1 = \delta_2$ and the continuity argument completes the proof.

Q.E.D.

(A.3.86) $\quad$ <u>Proposition.</u> The Langlands quotient (A.3.12) is unitarizable if and only if $0 < \nu_1 = \nu_2 \leqslant 1/2$ and $\sigma_1 = \sigma_2$.

Proof: The symmetry condition (A.3.26) (with $w = w_\theta$) requires that $\mu = \nu_1 = \nu_2 \in \mathbf{R}$. Since $\rho(\underline{n}_1) = \frac{3}{2}(e_1+e_2)$, Howe-Moore theorem [B-W, theorem 5.2] implies that we may restrict our attention to

(A.3.87) $\quad \mu < \frac{3}{2}.$

Now a straight forward analysis of reducibility of the induced representation (A.3.82) - for example via [V1, 4.2.25] - combined with the theorem (A.3.64) and the continuity argument [K-Sp, §4] implies (A.3.86).

Q.E.D.

(A.3.88) Proposition. The Langlands quotient (A.3.13) is unitarizable if and only if either

(A.3.89) $\quad \sigma_1 = \sigma_2 = 0,\ 0 < \nu_1 \leqslant 1$ and $\nu_2 = 0$, or

(A.3.90) $\quad \sigma_2 = 1,\ 0 < \nu_1 \leqslant 1$ and $\nu_2 = 0.$

Proof: The proof of this proposition under the assumption $\sigma_2 = 0$ is entirely analogous to the above proof of (A.3.86). The case $\sigma_2 = 1$ follows directly from [K-B].

Q.E.D.

The Theorem (2.4.14) follows from (A.3.64), (A.3.86), (A.3.88) by chasing through the dictionary (A.3.10).

§4. THE REPRESENTATIONS INDUCED FROM MAXIMAL PARABOLIC SUBGROUPS.

Since the representations induced from maximal parabolics are understood for any connected semi-simple Lie group other than F_4 or split G_2 [K-B] we are not going to elaborate on this subject here. As a hint for a reader willing to do the computations we mention that one can deduce (2.4.20), (2.4.25), (2.4.29), (2.4.39) from the properties of the (nonunitary) principal series representations of $SL(2,\mathbb{R})$ by rewriting the representations (2.2.16) in terms of, so called, θ-stable data [V1, 6.6.2] and then using [V1, 6.6.15] together with the Vogan-Wallach theorem [V4, W1] on the unitarizability preserving properties of Zuckerman functors. The methods used in [K-B] are different.

APPENDIX B. THE SMOOTH FROBENIUS RECIPROCITY THEOREM.

For reader's convenience we present here an immediate consequence of the theorem 5.3.3.1 in [Wa].

Let G be a Lie group and P a closed subgroup of G with G/P compact. Fix two Frechet spaces V and V'. Let G act on V' and P act on V. We define a subspace

$$\text{(B.1)} \qquad \mathrm{Ind}_P^G(V) \subseteq C^\infty(G,V)$$

consisting of these functions $f \in C^\infty(G,V)$, that

$$\text{(B.2)} \qquad f(xy) = \rho(y)^{-1}\, y^{-1} \cdot f(x) \qquad (x \in G,\ y \in P)\ .$$

Here $\rho(y) = (\Delta_G(y)/\Delta_P(y))^{1/2}$ where Δ_G, Δ_P are modular functions on G and P respectively. The group G acts on the space (B.1) via the left translations:

$$\text{(B.3)} \qquad y \cdot f(x) = f(y^{-1}x) \qquad (x,y \in G)\ .$$

(B.4) <u>Theorem.</u> Under the above assumptions

$$\mathrm{Hom}_P(V', \rho\otimes V) \cong \mathrm{Hom}_G(V', \mathrm{Ind}_P^G(V))$$

where for a P-intertwining map

$$u : V' \to \rho \otimes V$$

the corresponding G-intertwining map is given by

$$\text{(B.5)} \qquad \mathrm{Ind}\, u(x)(v) = u(x^{-1}\cdot v) \qquad (x \in G,\ v \in V')\ .$$

APPENDIX C. THE DETERMINANT REPRESENTATION OF $O(p,q)$ AND THE OSCILLATOR DUALITY CORRESPONDENCE.

Let V be a finite dimensional, real vector space with a symmetric, nondegenerate, bilinear form $(,)$ of signature p,q. Denote by

(C.1) G the isometry group of $(V,(,))$.

Then clearly G is isomorphic to the matrix group $O(p,q)$.

For an integer $m \geq 1$ let

(C.2) X_m be an m-dimensional, real vector space and $X = V \otimes X_m$.

Here, and in the rest of this paper, $\otimes = \otimes_{\mathbf{R}}$. Denote by

(C.3) $S(X)$ the Schwartz space of rapidly decreasing functions on X as usual. Since the group G acts naturally on X we may define the following representation of G on $S(X)$,

(C.4) $\Omega(g)\, f(x) = f(g^{-1}x) \qquad (g \in G,\ f \in S(X),\ x \in X)$,

and by dualization extend it to $S^*(X)$

(C.5) $(\Omega(g)u)(f) = u(\Omega(g^{-1})f) \qquad (g \in G,\ f \in S(X),\ u \in S^*(X))$.

By a standard argument one can show that for $p+q \leq m$
(C.6) there is a non-zero u in $S^*(X)$ such that

$$\Omega(g)u = \det(g)u \qquad (g \in G).$$

Here det is the character of G by which G acts on the top-dimensional component of the exterior algebra of V [J1, pp. 395].

(C.7) Theorem. Assume that $p+q > m$. Then there is no, non-zero, tempered distribution $u \in S^*(X)$ such that

(C.8) $\Omega(g)u = \det(g)\, u \qquad (g \in G)$.

Proof: We begin with a reformulation of this theorem in terms of the theory of the Oscillator Duality Correspondence (Ch. 1). Let X_m^* stand for the dual vector space to X_m (C.2). Define the vector space

(C.9) $V' = X_m \oplus X_m^*$ with a nondegenerate symplectic form

$$(x \oplus x^*, y \oplus y^*)' = x^*(y) - y^*(x) \qquad (x,y \in X_m, x^*,y^* \in X_m^*).$$

Denote by

(C.10) G' the isometry group of $(V',(,)')$.

Let

(C.11) $W = V \otimes V'$ and $\langle,\rangle$ be the symplectic form on W obtained by tensoring the symmetric form $(,)$ on V and the symplectic form $(,)'$ on V'.

Then, since G and G' act in the obvious way on W, preserving the form $\langle,\rangle$ we may identify them with their images in $Sp(W)$, (1.1.1), respectively.

Let

(C.12) $\widetilde{S}p(W)$ denote the metaplectic group (1.2.3),

$$pr : \widetilde{S}p(W) \to Sp(W)$$

the corresponding covering map, and

ω the oscillator representation of $\widetilde{S}p(W)$ (1.2.6).

Choose the Schrödinger model (1.3.22) of ω adapted to the decomposition

$$W = X \oplus Y$$

where X is as in (C.2) and $Y = V \otimes X_m^*$. Then it follows from (1.3.16) that

(C.13) there is a character ch of $\widetilde{G}$ $(= pr^{-1}(G))$ such that the kernel of ch is isomorphic to G and

$$\omega(g)f = ch(g)\, \Omega(pr(g))f \qquad (g \in \widetilde{G},\ f \in S(X)).$$

Define the following, one dimensional, representation of $\widetilde{G}$:

(C.14) $\Pi(g) = ch(g) \det(pr(g)) \qquad (g \in \widetilde{G})$.

Then clearly (C.13) and the Proposition (1.2.19) imply that the Theorem (C.7) is equivalent to the following statement:

(C.15) $\Pi \notin R(\widetilde{G},\omega)$ (1.2.12).

We shall need some additional notation to show (C.15).

Let us choose a decomposition of the space V into an orthogonal sum of anisotropic subspaces V_p and V_q such that the restriction $(,)_p$ of the form $(,)$ to V_p is positive definite and the restriction $(,)_q$ of $(,)$ to V_q is negative definite:

(C.16) $\quad V = V_p \oplus V_q,\ (,) = (,)_p \oplus (,)_q,\ (,)_p > 0,\ (,)_q < 0.$

Let

(C.17) $\quad G_s$ be the isometry group of $(V_s,(,)_s)$, $(s = p,q)$

and

(C.18) $\quad K$ be the subgroup of G consisting of all the elements g in G which preserve the decomposition (C.16) of V.

Then K is a maximal compact subgroup of G and there is an obvious isomorphism from $G_p \times G_q$ onto K:

(C.19) $\quad G_p \times G_q \ni (g_p,g_q) \to g \in K$ with

$$g(v_p \oplus v_q) = g_pv_p \oplus g_qv_q, \quad (v_s \in V_s,\ s = p,q).$$

Let

(C.20) $\quad W_s = V_s \otimes V',\ \langle,\rangle_s = (,)_s \otimes (,)',\ (s = p,q)$, as in (C.11).

Since G_s $(s = p,q)$ and G' act in the obvious way on W_s, preserving the form $\langle,\rangle_s$, we may embed them into $Sp(W_s)$.

Let

(C.21) $\quad G'_s$ be the image of G' in $Sp(W_s)$.

We shall identify G_s and G' with their images in $Sp(W_s)$. The decomposition (C.16) and the definition (C.11) imply that

(C.22) $\quad W = W_p \oplus W_q,\ \langle,\rangle = \langle,\rangle_p \oplus \langle,\rangle_q.$

Thus clearly we have the injection

(C.23) $\quad Sp(W_p) \times Sp(W_q) \ni (g_p, g_q) \to g \in Sp(W)$ with

$$g(w_p \oplus w_q) = g_p(w_p) \oplus g_q(w_q), \ (w_s \in W_s, \ \ s = p,q).$$

It is known, and not hard to check, that

(C.24) the centralizer of K in $Sp(W)$ preserves the decomposition (C.22) and its preimage by (C.23) is equal to $G'_p \times G'_q$ (C.21).

Moreover by

(C.25) composing the injection $G' \to Sp(W)$ with the inverse of (C.23) and with the isomorphism $G'_p \times G'_q \to G' \times G'$ (C.21) we obtain the diagonal embedding

$$G' \ni g \to (g,g) \in G' \times G'.$$

So far we have constructed the following reductive dual pairs

(C.26) $\quad G, G'$ in $Sp(W)$,

G_s, G'_s in $Sp(W_s)$ $(s = p,q)$,

K, $G'_p \times G'_q$ in $Sp(W)$, $(K \cong G_p \times G_q)$.

Let K' be a maximal compact subgroup of G'. Then, as in [H4, I, Prop. 12.3]

(C.27) there is an isomorphism $K' \cong U(m)$.

For $s = p,q$ let $\widetilde{Sp}(W_s)$ be the metaplectic group (1.2.3) with the covering map

(C.28) $\quad pr_s : \widetilde{Sp}(W_s) \to Sp(W_s)$.

Let $(Sp(W_p) \times Sp(W_q))^{\sim}$ denote the preimage under pr (C.12) of the image of $Sp(W_p) \times Sp(W_q)$ in $Sp(W)$ under (C.23). Then a standard argument shows that there is a group homomorphism $\widetilde{id}$ such that the following diagram is commutative:

(C.29)
$$\begin{tikzcd} \widetilde{Sp}(W_p)\times\widetilde{Sp}(W_q) \arrow[r, "\widetilde{id}"] \arrow[d, "pr_p\times pr_q"] & (Sp(W_p)\times Sp(W_q))^{\sim} \arrow[d, "pr"] \\ Sp(W_p)\times Sp(W_q) \arrow[r, "id"'] & Sp(W_p)\times Sp(W_q). \end{tikzcd}$$

Here id stands for the identity map, as usual. Moreover

(C.30) the pull back of $\omega|_{(Sp(W_p)\times Sp(W_q))^{\sim}}$

to $\widetilde{Sp}(W_p)\times\widetilde{Sp}(W_q)$ by $\widetilde{id}$ coincides with $\omega_p \check{\otimes} \omega_q^c$.

Here ω_q^c is the oscillator representation of $\widetilde{Sp}(W_q)$ contragradient to ω_q. Put

(C.31) $\widetilde{G}'_s = pr_s^{-1}(G')$, $\widetilde{K}'_s = pr_s^{-1}(K')$ $(s = p,q)$, and

$\widetilde{K} = pr^{-1}(K)$, $\widetilde{K}' = pr^{-1}(K')$.

We shall need a parametrization of representations of $\widetilde{K}'$, $\widetilde{K}'_s$ $(s = p,q)$ which occur in ω, ω_s respectively. It is known and not hard to check that

(C.32) any of the groups K', K'_s $(s = p,q)$ is either

(C.32.1) connected, or

(C.32.2) is isomorphic to the group $U(m)\times(\mathbf{Z}/2\mathbf{Z})$.

Moreover, by passing to a Fock model (1.4.6), one can check that

(C.33) in the case (C.32.2) for any representation of the group in question ($\widetilde{K}'$, $\widetilde{K}'_p$ or $\widetilde{K}'_q$) its pull back to $U(m)\times(\mathbf{Z}/2\mathbf{Z})$ has the property that the $(\mathbf{Z}/2\mathbf{Z})$ acts non-trivially.

Combining (C.32) and (C.33) we see that

(C.34) any representation which occurs in $R(\widetilde{K}',\omega)$ or $R(\widetilde{K}'_s,\omega_s)$ $(s = p,q)$ (1.2.11) is completely determined by the derived representation of the corresponding lie algebra.

The lie algebras of $\tilde{K}'$, $\tilde{K}'_p$, $\tilde{K}'_q$ are all isomorphic to the lie algebra $\underline{u}(m)$ (C.27). The finite dimensional, irreducible representations of $\underline{u}(m)$ may be parametrized by highest weights with respect to the upper triangular Borel subalgebra of $\underline{u}(m)$. These highest weights in turn correspond to m-tuples

(C.35) $\mu = (\mu_1, \mu_2, .., \mu_m)$, $\mu_j \in \mathbf{C}$, $\mu_j - \mu_k \in Z$, $(j,k = 1\ 2,.,m)$.

Finally we come to the essential part of the proof.

Assume, contrary to (C.15), that

(C.36) $\check{\Pi} \otimes \Pi' \in R(\tilde{G}\cdot\tilde{G}', \omega)$ (1.2.16)

when Π is as in (C.14). Then Π has the unique $\tilde{K}$-type $\pi = \Pi|_{\tilde{K}}$ which must be of lowest degree (1.5.11). Moreoever it follows from [K-Ve, II], (C.24), (C.30), (C.36) and the lemma (1.5.12) that

(C.37) $p, q \leqslant m$ and $\deg \pi = p+q$.

More precisely [K-Ve, II] and (1.4.6) imply that (for $s = p,q$)

(C.38) there is a character ch_s of $\tilde{G}_s$ such that $ch_s^{-1}(1)$ is isomorphic to G_s and if we define

(C.38.1) $\pi_s(g) = ch_s(g) \det (pr_s(g))$ $(g \in \tilde{G}_s)$ then there is a representation Π'_s of $\tilde{G}'_s$ such that

(C.38.2) $\check{\pi}_s \otimes \Pi'_s \in R(\tilde{G}_s \cdot \tilde{G}'_s, \omega_s)$

and the lowest degree $\tilde{K}'_s$-type π'_s of Π'_s has highest weight

$\frac{s}{2}\underbrace{(1,..,1)}_{m} + (\underbrace{1,..\ 1}_{s},\underbrace{0,..,0}_{m-s})$ (C.35).

Let π' be the lowest $\tilde{K}'$-type of Π' (C.36). Combining (C.38.2), (C.30), (C.25) (and the obvious identifications of the lie algebras of $\tilde{K}'$, $\tilde{K}'_p$, $\tilde{K}'_q$) we conclude that

(C.39) the derived representation $d\pi'$ of the lie algebra of $\tilde{K}'$ occurs in the tensor product

$$d\pi_p' \otimes (d\pi_q')^c.$$

It follows from (C.39), (C.38.2) and [Hu, Exercise 12 pp. 142] that

(C.40) the highest weight (C.35) of π' is of the form

$$\frac{p-q}{2}\underbrace{(1,\ldots,1)}_{m} + (\lambda_1,\ldots,\lambda_m)$$

with $|\lambda| \leq 1$ $(j = 1,2,.,m)$.

In particular, again by [K-Ve, II],

(C.41) $$\deg \pi' = \sum_{j=1}^{m} |\lambda_j| \leq m < p+q = \deg \pi \qquad \text{(C.37)}.$$

Clearly the statement (C.41) contradicts Howe's Lemma (1.5.17). Therefore (C.15) holds and we are done.

Q.E.D.

The representation Ω defined in (C.4) depends on the integer m (C.2). To make it explicit we shall write Ω_m for Ω (C.4).

For G defined in (C.1) let

(C.42) $R(G, \Omega_m)$ denote the set of infinitesimal equivalence classes of continuous, irreducible admissible representations of G on locally convex topological vector spaces which can be realized as quotients of S(X) (C.4) by Ω_m (G)-invariant closed subspaces.

The following corollary can be verified by the argument used in the proof of the theorem (3.6.1).

(C.43) Corollary. Fix two integers $m, n \geq 1$ such that $m+n < p+q$. Assume that

(C.44) $\Pi \in R(G, \Omega_m)$, and

(C.45) $\det \otimes \Pi$ is not equivalent to Π.

Then $\det \otimes \Pi \notin R(G, \Omega_n)$.

REFERENCES

[A] J. D. Adams, Discrete spectrum of the reductive dual pair (O(p,q), Sp(2m)), Invent. Math. 74, (1983) 449-475.

[B-K] M.W. Baldoni-Silva, A.W. Knapp, Irreducible unitary representations of some groups of real rank two, Non-Commutative Harmonic Analysis on Lie Groups, Lecture Notes i-Math. No. 1243, 15-36 Springer Verlag 1987.

[B-W] A. Borel, N. Wallach, Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups, Princeton University Press 1980.

[Ca] J. Carmona, Sur la classification des modules admissibles, Non-Commutative Harmonic Analysis and Lie groups, Proceedings Marseille Luming, 1982 #1020.

[D] J. Dieudonné, Treatise on Analysis, Academic Press, New York, London, 1972.

[F] D. Flath, Decomposition of Representations into Tensor Products, Proc. Symp. Pure Math., XXXIII, AMS, Providence, RI, (1980).

[HC] Harish-Chandra, Representations of Semisimple Lie Groups I, Trans. Amer. Math. Soc. 75 (1953), 185-243.

[He] S. Helgason, Differential Geometry and Symmetric Spaces Academic Press, New York, London, 1962.

[H1] R. Howe, Remarks on Classical Invariant Theory, preprint.

[H2] R. Howe, Transcending The Classical Invariant Theory, preprint.

[H3] R. Howe, θ-series and invariant theory, Proc. Symp. Pure Math. XXXIII, A.M.S., Providence, RI, 1979, 275-285.

[H4] R. Howe, manuscript in preparation on dual pairs.

[H5] R. Howe, Dual Pairs in Physics: Harmonic Oscillators, Photons, Electrons, and Singletons, Lectures in Applied Mathematics V-21, A.M.S., Providence, RI, 1985, 179-207.

[H6] R. Howe, On a notion of rank for unitary representations of the classical groups, C.I.M.E. Summer School on Harmonic Analysis, Cortona, 1982.

[H7] R. Howe, Quantum Mechanics and Partial Differential Equations, Journal of Funct. Analysis, 38 (1980), 255-272.

[H-M] R. Howe, C.C. Moore, Asymptotic properties of unitary representations, Journal of Funct. Analysis 32 (1979), 72-96.

[Hu] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer Verlag, 1972.

[J1] N. Jacobson, Basic Algebra I, W. H. Freeman and Company, San Francisco, 1974.

[J2] N. Jacobson, Basic Algebra II, W. H. Freeman and Company, San Francisco, 1980.

[Ke-Na] J. L. Kelley, I. Namioka, Linear Topological Spaces, Springer-Verlag, New York, Heidelberg, Berlin, 1963.

[Ki] A. A. Kiryllov, Elements of the Theory of Representation, Moscow, 1972.

[K] A. W. Knapp, Representation Theory of Semisimple groups - an overview based on examples, Princeton University Press, Princeton, New Jersey, 1986.

[K-B] A. W. Knapp, M. W. Baldoni - Silva, Unitary Representations Induced from Maximal Parabolic Subgroups, Journal of Funct. Anal. 69 (1986), 21-120.

[K-S] A. W. Knapp, E. M. Stein, Intertwining Operators for Semisimple Groups, II, Inventiones Mathematicae, 60, (1980), 9-84.

[K-Sp] A. W. Knapp, B. Speh, Status of Classification of Irreducible unitary Representations, "Harmonic Analysis," Springer Verlag Lecture Notes in Math. 908 (1982), 1-38.

[K-V] A. W. Knapp and David A. Vogan J., Duality Theorems in the Relative Lie Algebra Cohomology, preprint.

[K-Ve] M. Kashiwara, M. Vergne, On the Segal-Shale-Weil representation and harmonic polynomials, Invent. Math. 44, (1978), 1-97.

[K-Z1] A. W. Knapp and G. Zuckerman, Classification Theorems for Representations of Semisimple Lie Groups, Non-Commutative Harmonic Analysis, Springer Verlag Lecture Notes in Math. 587, (1977) 138-159.

[K-Z2] A. W. Knapp and G. Zuckerman, Classification of Irreducible Tempered Representations of Semisimple Groups, Ann. of Math., 116 (1982), 389-455.

[La] R. P. Langlands, On the classification of irreducible representations of real algebraic groups, mimeographed notes, Institute for Advanced study, Princeton, New Jersey, 1973.

[L] S. Lang, $SL_2(\mathbf{R})$, Springer Verlag, New York, Heidelberg, Berlin, Tokyo, 1985.

[Li] J. S. Li, Theta Series and Distinguished Representations for Symplectic Groups, Yale thesis.

[P1] T. Przebinda, On Howes Duality Theorem, to appear in Journal of Functional Analysis, 81 (1988), 160-183.

[P2] T. Przebinda, The Duality Correspondence of Infinitesimal Characters, to appear.

[R] J. Repka, Tensor Products of Holomorphic Discrete Series, Can. J. Math. Vol. XXXI, (No. 4), (1979), 836-844.

[S-W] E. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidian Spaces, Princeton, NY, Princeton University Press, 1971.

[Tr] P. C. Trombi, The Tempered Spectrum of a Real Semisimple Lie Group, Amer. J. Math. 99 (1977), 57-75.

[Vi] N. J. Vilenkin, Special Functions and the Theory of Group Representations, AMS, Providence, Rhode Island, 1968.

[V1] D. A. Vogan, Jr., Representations of Real Reductive Lie Groups, Birkhäuser, Boston-Basel-Stuttgart, 1981.

[V2] D. A. Vogan, Jr., Classifying Representations by Lowest K-types, Lectures in Applied Mathematics Vol. 21 (1985).

[V3] D. A. Vogan, Jr., The Unitary Dual of Gl_n over an Archimedian Field, Invent. Math., Vol. 83 Fasc. 3, (1986), 449-507.

[V4] D. A. Vogan, Jr., Unitarity of certain series of representations, Ann. of Math. 120 (1984), 141-187.

[W1] N. R. Wallach, On the Unitarizability of Derived Functor Modules, Invent. Math. 78 (1984), 131-141.

[W2] N. R. Wallach, Harmonic Analysis on Homogenous Space, Marcel Dekker, New York, 1973.

[Wa] G. Warner, Harmonic Analysis on Semisimple Lie Groups I, Springer Verlag, 1972.

[Y] K. Yosida, Functional Analysis, Springer Verlag, Berlin, Heidelberg, New York, 1980.

[Z] G. Zuckerman, Tensor Products of Finite and Infinite Dimensional Representations of Semisimple Lie Groups, Annals of Math., 106 (1977), 295-308.

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Front Matter (first pages in book, preceding main body of text).

Page i – *Title, *Author's name.

Page iii – Table of contents.

Page iv – *Abstract (at least 1 sentence and at most 300 words).

1980 Mathematics Subject Classification (1985 Revision). This classification represents the primary and secondary subjects of the paper, and the scheme can be found in Annual Subject Indexes of MATHEMATICAL REVIEWS beginning in 1984.

Key words and phrases, if desired. (A list which covers the content of the paper adequately enough to be useful for an information retrieval system.)

Page v, etc. – Preface, introduction, or any other matter not belonging in body of text.

Page 1 – Chapter Title (dropped 1 inch from top line, and centered).

Beginning of Text.

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